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A world population growth model: Interaction with Earth's carrying capacity and technology in limited space

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ABSTRACT

Up to 1900, world population growth over 1500 years fitted the quasi-hyperbolic format $P(t) = a/(D - t)^M$, but this fit projected to infinite population around 2000. The recent slowdown has been fitted only by iteration of differential equations. This study fits the mean world population estimates from CE 400 to present with "tamed quasi-hyperbolic function" $P(t) = A/[\ln(B + e^{(D-t)/\tau})]^M$, which reverts to $P = a/(D - t)^M$ when $t \ll D$. With coefficient values $P(t) = 3.83 \times 10^9/[\ln(1.28 + e^{(1980-t)/22.9})]^{0.70}$, the fit is within ±9%, except in 1200–1400, and projects to a plateau at 10.2 billion. An interaction model of population, Earth's carrying capacity and technological–organizational skills is proposed. It can be approximately fitted with this $P(t)$ and an analogous equation for carrying capacity.

1. Introduction

Fifty years ago, *Science* published a study with the provocative title "Doomsday: Friday 13 November, A.D. 2026" [1]. It fitted world population during the previous two millennia with $P = 179 \times 10^9/(2026.9-t)^{0.99}$. This "quasi-hyperbolic" equation (hyperbolic having exponent 1.00 in the denominator) projected to infinite population in 2026 — and to an imaginary one thereafter. Later growth has fallen short of this equation, calling for a modification that averts "doomsday". The smoothness of world population growth curve since CE 400, with a single inflection point around 2000, suggests that stable long-term factors may be at work, rather than accumulation of random developments. This underlying basis for quasi-hyperbolic pattern and later slowdown needs elaboration. Fits based on iterations of differential equations have been offered [2,3], but no explicit function $P(t)$ like the one above.

Here a modified explicit equation is proposed, which fits the mean world population estimates from CE 400 to present and to foreseeable future. This "tamed quasi-hyperbolic function" fits approximately an interaction model of population, Earth's carrying capacity and technological–organizational skills. This three-factor model combines two earlier ones, one in terms of population and technology only [4–6], and the other in terms of population and Earth's human carrying capacity only [2,7]. Application to other phenomena with apparent asymptotes is briefly discussed, as well as population growth outside the time period considered.

2. Quasi-hyperbolic growth up to 1900

Over the last 1600 years human population has increased 35-fold. Up to the mid-1900s, it grew at an ever-increasing percent rate per year, which the exponential model cannot express. As early as 1951, André de Cailleux [8] noticed that the world population fitted a quasi-hyperbolic equation:

$$P = a/(D-t)^M \quad (1)$$

where a, D and M are constants. In differential form,

$$dP/dt = Ma/(D-t)^{M+1} = \left(M/a^{1/M}\right)P^{1+1/M} \quad (2)$$

and the relative growth rate is simply $dP/Pdt = M/(D - t)$. At asymptote $t = D$, $P \to \infty$. Eq. (1) becomes purely hyperbolic

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when $M = 1$ and exponential when $M \to \infty$. In 1958–1979, this pattern was repeatedly rediscovered or confirmed [1,4,9–12], right at the time when world population began to fall visibly short of this pattern. The estimates for D ranged from 2005 to 2027 [1,4,9–12]. In retrospect these were overestimates, because they included post-1900 data, when the shift away from the quasi-hyperbolic pattern had already set in, however mildly. Correspondingly, exponent M has been also overestimated, ranging from 0.74 up to 1. By habit, CE 1 has most often been taken as the starting point for data fitting, but this was the high point of a previous speedup and leveling-off in world population growth, to be discussed later. A new smooth upward trend in population began around CE 400. The best fit for population estimates from CE 400 to 1900 is close to

$$P_Q = 34.3 \times 10^9/(1980-t)^{0.70} \quad (3)$$

where subscript Q indicates quasi-hyperbolic fit. Mean estimates of world population during these 1500 years (Table 1) agree with Eq. (3) within ±5%, except for spurts in 1200–1300 and 1850, and shortfall in 1400 (Black Death).

Table 1
World population from CE 400 to 2010, in million: mean and range of estimates, and as calculated from quasi-hyperbolic and T-function approaches. Shown in bold are cases where values from Eq. (15) fall outside the range of estimates by more than ±3%. Time intervals are taken so that successive population ratios remain between 1.1 and 1.3.

Year	Estimates of P		P_Q from Eq. (3)	P_T from Eq. (15)	Deviation (%) of mean from Eq. (15)
	Mean	Range			
400	198	190–206	197.8	197.8	+0.1
600	214	200–237	217.5	217.5	−1.6
800	235	220–261	242.5	242.5	−3.2
1000	281	254–310	276.3	276.3	+1.7
1100	310	301–320	297.8	297.8	+4.1
1200	398	360–450	324.2	**324.2**	+22.8
1300	396	360–432	356.9	356.9	+11.0
1400	362	350–374	398.9	**398.9**	−10.2
1500	457	425–500	455.4	455.4	+0.3
1600	544	498–579	536.3	536.1	+1.5
1700	635	603–679	664.2	663.9	−4.6
1750	771	720–824	762.2	761.8	+1.2
1800	941	890–981	904.9	904.5	+4.0
1850	1242	1200–1265	1136.4	**1135.5**	+9.3
1900	1639	1564–1680	1596.4	1583.5	+3.5
1920	1905	1860–1968	1952	1906	−0.1
1940	2313	2300–2340	2593	2401	−3.8
1950	2526	2499–2556	3172	**2748**	−8.8
1960	3035	3023–3042	4213	3185	−4.9
1970	3667	3600–3712	6844	3728	−1.7
1980	4442	4436–4453	∞	4385	+1.3
1990	5278	5260–5290		5147	+2.5
2000	6021	5750–6115		5980	+0.7
2010	6861	6831–6909		6825	+0.5

Mean estimates and ranges of world population are those of mean estimates by 9 sources in Wikipedia, http://en.wikipedia.org/wiki/World_population_estimates, visited 11/9/10: US Census Bureau 2009; Population Reference Bureau 2008; UN Dept. of Econ. and Soc. Aff. 2008; HYDE 2006, A. Maddison 2003; J. H. Tanton 1994; J.-N. Biraben 1980; C. McEvedy and R. Jones 1978, R. Thomlinson 1975; J. D. Durand 1974; Clark 1967. Estimates given with only 100 million precision were omitted. During the period since CE 400, individual estimates deviate from the means by up to 13%. Evaluating the validity of population estimates in centuries past is beyond the scope of this study.

This millennial upward curvatures in log$P(t)$ and the recent downward curvature are quite disparate. To fit them together, one has to consider deep-set factors boosting growth and now slowing it down. Differential equations can be set up, and an iteration process can be used to fit actual population estimates [2,3], but integration into a single explicit equation has been lacking. An ultimate ceiling (U) could easily be inserted into Eq. (2):

$$dP/dt = KP^{1+1/M}(1-P/U). \quad (4)$$

Due to $P^{1/M}$, this is not simple exponential approach to the ceiling, but it satisfies the basic desiderata — quasi-hyperbolic initial growth plus a ceiling. No integration formula is available, however, for non-integer M. The same applies to $dP/dt = a/[(D-t)^2 + c^2]$, also proposed [13]. Attempts have been made to use functions $P(t)$ other than quasi-hyperbolic, as reviewed in [14], but it is hard to match its simplicity and degree of fit prior to 1900.

What could cause the quasi-hyperbolic pattern, and the later slowdown? At least two other factors must interact with population. Some models have focused on technology (T) [4–6], others on Earth's carrying capacity (C) at a given time [2,3]. Space limitations on Earth may impose an ultimate population limit even when technology may uncover new resources. Dry land area is the ultimate resource, upon which most others are predicated. A ceiling at $U = 10$ billion would mean 1.5 ha per person. If evenly spread out, with 940 million placed in Antarctica, humans would stand at about 100 m from their six closest neighbors. Two models are reviewed next, respectively based on technology [4] and on carrying capacity [2]. Weak links are pointed out in each, and a new model subsuming both will be presented.

3. Interaction with technology and carrying capacity

3.1. The population–technology model

Assume endogenous exponential growth of population. Assume the same for "technology" [15], using this term in its broadest meaning, which includes social organization skills as stressed in the world systems literature [16]. Indeed, "skills" might express this broad ability more clearly [12], but we'll stick with the traditional term. Eq. (1) can be derived from interaction between these exponential growths when they reciprocally enhance their rate "constants" [4]. Such interaction might be assumed because more people means more potential innovators, and higher technological–organizational skills increase Earth's carrying capacity and hence make a larger population possible. Assume that interaction terms can be approximated by power functions [4].

Then

$$dP/dt = kT^nP \quad (5)$$

$$dT/dt = hP^mT \quad (6)$$

Eliminating time, P and T are related as $dP/dT = (k/h)P^{1-m}/T^{1-n}$. Integration yields $hnP^m = kmT^n$, when we assume that $T = 0$ when $P = 0$. Inserting $kT^n = hnP^m/m$ into Eq. (5) leads to $dP/dt = (hn/m)P^{m+1}$, which is equivalent to

Eq. (2), with $m = 1/M$. Simplistic as this model [4] is, it does lead to the observed pattern in 400–1900. In a simpler form, with $n = m = 1$, it was devised again two decades later [5,6]. This simpler form imposes $M = 1$ in Eqs. (1) and (2), which restricts fitting options needlessly.

Similarly to P, T also grows quasi-hyperbolically, in this model. The model may seem to presume near-instantaneous technological–organizational interaction worldwide, but this is not so. It would suffice to presume that spread of major innovations among the major population centers in Eurasia takes place within a time appreciably shorter than population doubling time — and this doubling time after CE 400 was 1000 years.

If the number of innovators mattered, $m = 1$ should hold, while if interaction among them mattered, it would be larger. The observed $M = .70$ means $m = 1.43$. The amount of technology is difficult to quantify, except in special circumstances [9,12]. However, T cancels out from the population equation, so we are not forced to measure it. Moreover, transformations of type $T' = \alpha T^{\beta}$ do not alter value of m.

3.2. The population-carrying capacity model

The P–T explanation for the quasi-hyperbolic nature of population growth does not account for the present slowdown, because it ignores carrying capacity. Alternative models, based on carrying capacity but ignoring technology, have addressed this slowdown. In particular, iteration of the following interaction equations for population and carrying capacity leads to a numerical illustration that fits population estimates within ±25% for the period CE 1–1995, except for 1200–1300 [2]. It projects to an ultimate carrying capacity of $U = 17$ billion. In our present notation,

$$dP/dt = kP(C - P) \tag{7}$$

$$dC/dt = (L/P)dP/dt \tag{8}$$

where k and L are constants. (In [2], these are designated as Eqs. (4) and (6), the iteration of which leads to a numerical illustration.) Proceeding beyond [2], we can collapse these two equations into a single one, eliminating C:

$$dP/dt = kP[L\ln(P/U) + U - P] \tag{9}$$

Indeed, it follows from Eqs. (7) and (8) that $dC/dt = kL(C - P)$. Dividing by Eq. (7) yields $dC/dP = LP$, which integrates into $C = L\ln P + G$, where G is integration constant. As $dP/dt = 0$ when population reaches the ultimate capacity U, this constant must be $G = U - L\ln U$. Hence $C = L\ln(P/U) + U$. Eq. (7) then becomes Eq. (9).

For $t < 1900$, $P << U$, and Eq. (9) further simplifies to $dP/dt = kP[L\ln(P/U) + U]$, which differs from Eq. (2) — and the latter fits the pre-1900 population estimates better. Eq. (9) also differs markedly from simple Eq. (4). In the numerical illustration presented [2], carrying capacity exceeds population by a factor of 5 to 6 during 1550–1800. This persistent gap is puzzling, because human populations have been quite able to grow at 2%/year and thus could increase their numbers 6-fold within 90 years, to catch up with carrying capacity. In sum, even the most prominent iteration approaches based on differential equations present difficulties — and they have not led to an explicit equation $P(t)$.

Both models have their weak points. The weak link in the P–T model [4] is Eq. (5), which connects dP/dt directly to technological–organizational skills, without having carrying capacity as an intermediary. The weak link in the P–C model [2] is Eq. (8), which connects dC/dt directly to population (and to dP/dt), without technology as an intermediary. Now we proceed to combine the stronger parts of the two approaches.

3.3. The P–C–T model

Accept Eq. (7) from [2], which is like simple logistic, except that the maximum level C itself varies. Accept Eq. (6) from [4] — more people means more innovators. Instead of linking technology to population growth, as in the debatable Eq. (5), it is linked to growth of carrying capacity, as follows. Higher technological and organizational skills boost carrying capacity, limited by ultimate carrying capacity:

$$dC/dt = gT^{n}(U - C) \tag{10}$$

where g and n are constants. Together, Eqs. (2), (10) and (6) form a causal loop $P \leftarrow C \leftarrow T \leftarrow P\ldots$, where dP/dt depends on C, dC/dt depends on T, and dT/dt depends on P. However, this model cannot be solved directly, except by complex iteration. Before making use of it, we have to reach an approximate solution by other means.

4. Empirical taming of the quasi-hyperbolic equation

Given that the explicit Eq. (1) serves us well up to 1900, it is advisable not to dump it but try to modify it, so as to avoid infinite population growth. This task is mathematically quite difficult — functions with vertical asymptotes resist capping. This goal can be achieved only with a complex format, designated here as "tamed quasi-hyperbolic function" or simply T-function:

$$P = A/[\ln(B + E)]^{M} \tag{11}$$

where

$$E = \exp[(D - t)/\tau] \tag{12}$$

and A, B, M, D and τ are constants. Ungainly and hard to visualize as it is, this is nonetheless the simplest way to impose a cap on runaway growth in Eq. (1). We will first verify that the T-function satisfies the requirements, and then show that it can be made to fit empirical data.

At $t << D$, $E >> B$, and Eq. (11) reverts to Eq. (1), with $a = A\tau^{M}$. At $t >> D$, $E \rightarrow 0$, and P approaches a ceiling at $U = A/[\ln B]^{M}$, provided that $B > 1$. For $B = 1$, P would shift to exponential growth. Decreasing exponential $E = e^{(D-t)/\tau}$ is the central link between P and t.

The differential form of Eq. (11) is

$$dP/dt = \left[MU(\ln B)^M/\tau\right]E/\left\{(B+E)[\ln(B+E)]^{M+1}\right\} \tag{13}$$

Relative growth rate has a simpler form:

$$(dP/dt)/P = [M/\tau]E/[(B+E)\ln(B+E)]. \tag{14}$$

At maximum dP/dt, we have $E(M + 1) = B\ln(B + E)$, and at maximum relative rate, $E = B\ln(B + E)$. These forms are hardly intuitive; they just reflect the simplest explicit equation $P(t)$ that puts a lid on quasi-hyperbolic growth, which is relentlessly propelled by technological–organizational factors. Is this model stable against population overshooting the limit U? In the present form, this is not clear. This is clarified in Section 6.

The crucial question is how well Eq. (11) can fit actual population estimates. Table 1 shows the fit with constant values $A = 3.83$ billion, $B = 1.28$, $D = 1980$, $\tau = 22.9$ years and $M = 0.70$, which are close to optimal. The resulting equation is

$$P_T = 3.83 \times 10^9/[\ln(1.28+E)]^{0.70} \text{ where } E = e^{(1980-t)/22.9} \tag{15}$$

and subscript T indicates a fit with T-function. As t increases, P_T approaches $U = 10.197 \times 10^9$. When $t << D$, this equation reverts to Eq. (3), with $a = 34.3$ billion (years)$^{.70}$. For $t < 1900$, Eq. (15) falls short of the quasi-hyperbolic Eq. (3) by less than 0.8%. For $t > 1900$, the shortfall grows rapidly. The relative rate of growth is

$$dP/P_T dt = .03057E/[(1.28+E)\ln(1.28+E)]. \tag{16}$$

The range of $E(t)$ is enormous. In CE 400, $E = 9 \times 10^{29}$; in 1980, $E = 1$; and in 2100, $E = .005$.

Mean estimates of world population during the last 1600 years agree with Eq. (15) within ±5% (Table 1), with the following exceptions. A surge in 1200–1300 preceded collapse by 1400 (Black Death). A surge in 1850 was followed by slowdown in 1950, after the two world wars (Table 1). When measured in terms of shift in years (rather than % population), the actual population in 1850 was ahead of the model by 16 years, while trailing it by 5 years in 1950. During the last 40 years the agreement has been within ±3% of the population or a time shift of ±2 years.

The constant τ reflects the reaction time of population growth to closeness of U. If new births caused P to exceed U, the excess would make itself felt the hard way when these babies begin to require full amounts of resources, about 20 years after birth. Therefore, it is not surprising that the empirical value of τ, 22.9 years, is close to the time for reaching adulthood.

No curve with a single inflection point could fit the mean population estimates markedly closer. In comparison, the aforementioned numerical illustration [2] for iteration of differential equations deviates from mean estimates by more than ±20% for most of CE 400–1600. The main distinction, however, is not in having a better fit to data but in having an explicit function $P(t)$.

Within the observed fluctuation range of ±6% since 1900, Eq. (15) projects to a population ceiling at $U = 10.2 \pm .6$ billion. It implies a top growth rate of 85.0 million/year, reached in 2002, at $P = 6.15$ billion — 60% of the ultimate limit. The top relative rate of 1.645%/year was reached in 1977–78, at $P = 4.20$ billion — 41% of U. (Higher relative rates actually observed around 1960 reflect catching up after the losses of WWII.)

Projections of Eq. (15) for the century ahead (Table 2) slightly exceed the mean estimates based on current demographic data, but only by up to 2%. At very large t, Eq. (15) can be approximated by exponential approach to ceiling, but this approximation fits well only after 2100, when $P > 0.99U$ (Table 2):

$$(U-P)/U = [M/(B\ln B)]e^{(D-t)/\tau} = 2.215e^{(1980-t)/22.9}. \tag{17}$$

Attempting to project an upper limit from early growth data can lead to major errors. Even while world population growth curve seems to show a definite inflection point around 2000, this caution still applies. Estimates based on current demographic data hesitate going beyond 2050. Estimates of Earth's ultimate carrying capacity have fluctuated wildly, with medians of lowest and highest estimates at 7.7 and 12 billion, respectively [2]. The projection to a ceiling of 10.2 billion depends on the continuing effect of broad factors that have prevailed during the last 1600 years. A look into more distant past offers a cautionary note. Prior to CE 400, Eq. (15) falls short of world population estimates from CE 200 back to 200 BCE, while increasingly exceeding them in the more remote past (Table 3). It looks as if some basic factor, which later remained stable from CE 400 to at least 2010, had shifted around CE 200.

This is puzzling. While the period from CE 1 to 400 saw the gradual demise of Han empire in the east, Rome in the west, and Andhra and Kushan in the south of Eurasia, this period of contraction in organized civilization does not stand out by various pertinent measures used in world systems literature [16]. Yet the world population in CE 1 to 400 does show something more than just an inflection point in a rising curve: It shows 400 years of standstill, with mean estimates 214 million in CE 1 and 198 million in 400 (Tables 1 and 3). This suggests that population growth prior to stabilization around CE 1 might be fitted separately with Eq. (11), with D around CE 100. The projections beyond 2015, based on Eq. (15), apply only to the extent there is no major shift in basic underlying factors, comparable to what seems to have taken place around CE 1 to 200. Fitting the pre-400 population estimates and pondering the potential implications for distant future requires a separate study.

Table 2
World population from CE 2020 to 2100, in billion: mean estimates and ranges, and as calculated from Eq. (15) and from its final exponential approximation, Eq. (17).

Year	Estimates of P		P_T from Eq. (15)	Deviation (%) of mean from Eq. (15)	P from Eq. (17)
	Mean	Range			
2020	7.70	7.56–7.67	7.616	+1.1	6.261
2030	8.24	8.20–8.31	8.301	−0.7	7.655
2040	8.77	8.75–8.80	8.852	−0.9	8.533
2050	9.23	9.15–9.35	9.268	−0.4	9.137
2100	–		10.080		10.077
+∞			10.197		10.197

Mean estimates and ranges of world population — see Table 1.

Table 3
World population from 10,000 BCE to CE 200, in million: geometric mean estimates and ranges, and as calculated from Eq. (15).

Year	Estimates of P		P_T from Eq. (15)	Deviation by factor
	Mean	Range		
−10,000	3.6	1–10	47	13×
−8000	6.3	5–8	54	9×
−5000	7	5–10	70	10×
−3000	14	–	88	6×
−2000	27	–	104	4×
−1000	50	–	127	2.5×
−500	100	–	144	1.4×
−200	186	150–230	158	0.8×
1	214	150–300	169	0.8×
+200	220	190–256	181	0.8×

Mean estimates and ranges of world population — see Table 1.

5. Broader implications

How stable are the constants in Eq. (15), and what do they mean? A comparable degree of fit can be obtained with slightly different combinations of constants, depending on which period one wishes to fit with the least error. In particular, the following set reduces deviation in the 1900s to ±5%, but increases it to 12% in 1850: $A = 3.80$ billion, $B = 1.27$, $D = 1980$, $\tau = 21.8$ years and $M = 0.69$, leading again to U = 10.2 billion. It seems hard to shift D even by a single year or use a combination of constants leading to U lower than 10.0 billion or higher than 10.4 billion.

While it is convenient to deal in terms of constant A, it is a derivative constant. Two basic constants characterize the initial unlimited quasi-hyperbolic growth:

- M, steepness of quasi-hyperbolic growth;
- D, date of asymptote.

Two further basic constants characterize the slowdown to ultimate limit:

- U, ultimate population carrying capacity of Earth;
- τ, time of reaction to closeness of U.

A fifth basic constant, B, bridges the two sets and determines how early on the brakes are applied to the quasi-hyperbolic growth. Constants A and a are convenient, but they have no direct substantive meaning. Constant a in Eq. (1) results from combining almost all substantive constants: $a = U(\tau \ln B)^M$, omitting only D, while the constant A in Eq. (11) is $A = U(\ln B)^M$. In terms of the basic constants, Eq. (11) could be rewritten in a possibly more elegant form as

$$P = U[\ln B/\ln(B+E)]^M, \text{where}\, E = \exp[(D-t)/\tau]. \quad (18)$$

This format shows clearly that $B > 1$ is needed so as to keep P real.

As approaches to a ceiling go, the present one displays a remarkably sudden slowdown in growth. Space limitation usually makes itself felt long before the ceiling is reached, slowing down further growth. In simple logistic growth, it takes as much time to go from 10 to 25% of the maximum as from 75 to 90%. For Eq. (15), in contrast, the former time span is 120 years (1822 to 1942), while the latter is compressed to 26 years (2019 to 2045). An expansion that gradually picked up speed over 1500 years comes to a screeching halt. This is so because still-growing technological–organizational capability keeps counteracting space limitation, pushing carrying capacity up until its ultimate ceiling is almost reached. The growth patterns of carrying capacity and technological–organizational skills themselves are addressed in Section 6.

As technology keeps increasing, it can eventually put pressure on the ultimate carrying capacity. Having enough knowledge and resources, the people can conceivably populate the sea floor and Antarctic, build floating islands, settle other planets, etc. This could lead to a renewed spurt of population growth, beyond 10 billion. In the short run, however, increasing technology may have to race against dwindling conventional resources so as to keep Earth's carrying capacity from dropping.

Quasi-hyperbolic relationships have been noted for various other phenomena related to population and technology, prior to 1970s. Most data fits have been restricted to integer values of M — 1 or 2. The estimated world GDP in 1990 international dollars, from CE 1 to 1973, has been fitted as $g = 17.3 \times 10^{18}/(2005.6\text{-}t)^2$, with $M = 2$ imposed [17]. Later figures fall short of this projection. World literacy percentage up to 1980 has been observed to follow $l = 3669/(2040 - t)^2$, again with $M = 2$ [18] — until it is capped by the obvious limit of 100%. The same limit applies to percentage of people living in cities above a quarter million, fitted as $u = 403/(1990 - t)$; note that here $M = 1$ was chosen [18]. The size of the largest single settlement was again fitted with $M = 2$: $s = 0.104 \times 10^9/(2040 - t)^2$ [18]. Here the limit is world population.

The following questions arise in all these cases. At the quasi-hyperbolic stage, what are the values of exponent M and asymptote date D, once one accepts non-integer values of M and omits the post-1900 data, which may already be affected by slowdown? Within the error range of the data, could the single value $D = 1980$ apply to all these phenomena? Including the recent slowdown stage, would the tamed quasi-hyperbolic function, Eq. (11), fit the data, and with which constant values? For percentages literate and "megaurban", the fitting is simplified because the maximum is known, and for the largest settlement, the ceiling on world population imposes a ceiling. At its broadest, the T-function may be of use for all phenomena that start out quasi-hyperbolically but then are bound to level off.

6. Inserting the tamed quasi-hyperbolic function into the theoretical model

Eqs. (7), (10) and (6) express a causal loop $P \leftarrow C \leftarrow T \leftarrow P$…., where dP/dt depends on C, dC/dt depends on T, and dT/dt depends on P. This theoretical model is easier to handle when expressed in terms of relative growth rates:

$$dP/Pdt = k(C-P) \quad (19)$$

$$dC/Cdt = gT^n(U/C-1) \quad (20)$$

$$dT/Tdt = hP^m \quad (21)$$

Note first that this system is stable at population limit U. If a random fluctuation makes P exceed U, then dP/Pdt in Eq. (19) becomes negative, bringing P back toward U.

We will focus on Eq. (19), which looks like simple logistic equation, except that Earth's carrying capacity C keeps growing. Given that the tamed quasi-hyperbolic function P_T in Eq. (15) fits the population data, this P_T must be *close* to the solution of the model above, if this model applies. Hence also dP/Pdt must fit Eq. (16). Both P_T and dP/P_Tdt are shown in Table 4, along with the intervening variable E, which decreases over time from close to 10^{30} in CE 400 down to 1 in 1980 and to .005 by 2100.

We could insert the values of P_T and dP/P_Tdt from empirical Eqs. (15) and (16) into Eq. (19) and use it to calculate the corresponding C, provided that we knew the value of constant k. Surprisingly, k can be uniquely determined as follows. At $t >> D$, replace $dP/Pdt = k(C - P)$ with $dP/Pdt = k(U - P)$. Then $k = (dP/Pdt)/(U - P)$, where P and dP/Pdt are supplied by Eqs. (15) and (16). As $t \to \infty$, we find that $k \to .00428/10^9$.

(Maybe not by chance, $kU\tau = 1.00$. This means that, when t is measured in units of 22.9 years and P in fractions of U, Eq. (19) simplifies into $dP/Pdt = (C - P)$, without any constant. The same can be done with simple logistic equation, when using k resulting *from the logistic fit itself*. Here, however, the value of k is imposed *from the outside*.)

Now carrying capacity can be calculated on the basis of population:

$$C_P = P_T + (dP/P_Tdt)/\left(.00428 \times 10^{-9}\right), \tag{22}$$

with P_T and dP/P_Tdt supplied by Eqs. (15) and (16). It turns out that the resulting values of C, from CE 400 to 2100, can be fitted with the T-function, within 2% (Table 4):

$$C_T = 10.11 \times 10^9/[\ln(2.69 + E)]^{.83}, \tag{23}$$

where E has the value given in Eq. (16) – the same as for population. The population and carrying capacity calculated from the T-function, P_T and C_T, are not exact solutions of $dP/Pdt = k(C - P)$. Yet, with P_T given, a corresponding function C_T can be found so as to satisfy this equation within 2%, while population grows from 0.2 billion to 10 billion. It is hard to envisage how this is possible, but it is.

The model suggests that, in CE 400, Earth's carrying capacity was about 1.5 times the actual population (Table 4). Thereafter, the C/P ratio increased, as technological–organizational skills boosted carrying capacity faster than population could catch up. The peak ratio, 2.26, was reached around 1920. Thereafter, C/P has been gradually decreasing toward 1, as carrying capacity approaches its ultimate limit and population is catching up.

In sum, what has been achieved? If we assume that the model $dP/Pdt = k(C - P)$ applies, along with $P = A/[\ln(B + E)]^M$ and $C = A'/[\ln(B' + E)]^{M'}$, where $E = \exp[(D - t)/\tau]$, then we find that all 9 constants in these equations can be calculated in a unique way, on the basis of mean estimates for world population, CE 400 to 2000. Deviations from internal consistency do not exceed 2%, suggesting that the model has some merit.

Using a similar approach, Eqs. (20) and (21) enable us to estimate the pattern of T from known P and C. This is more involved and remains to be done. To close the causal loop, the format of T derived from Eq. (20) must satisfy Eq. (21). Preliminary results suggest that this is easy for $t < 1900$, but disagreements arise for $t > 1950$. Fine-tuning of constant values may or may not lead to agreement.

7. Conclusions

Does the world human population growth follow a millennial pattern determined by deep-set demographic and technological–organizational givens, or has it been a random path, with no guidance for future? An attempt at a logical model needs to include at least three factors: exponential growth propensity of any biological species under steady favorable conditions, limits imposed by Earth's carrying capacity, and – specific to humans – impact of technological–organizational change on the two other factors.

Table 4
Carrying capacity (C) calculated from $dP/Pdt = k(C - P)$, for given population (P), and fitted to the tamed quasi-hyperbolic function.

Year	P_T in 10^9	E	dP/P_Tdt	C_P in 10^9	C_P/P	C_T in 10^9	Error (%)
400	.1978	9.213×10^{29}	.000443	.3013	1.52	.3010	−0.1
1000	.2763	3.851×10^{18}	.000714	.4431	1.60	.4474	+1.0
1200	.3242	6.202×10^{14}	.000897	.5351	1.65	.5407	+1.1
1400	.3989	9.991×10^{10}	.001207	.6809	1.71	.6915	+1.6
1600	.5361	1.6093×10^{7}	.001842	.9665	1.80	.9822	+1.6
1800	.9045	2592	.00389	1.8064	2.00	1.826	+1.1
1850	1.1355	292.0	.00538	2.391	2.11	2.389	−0.1
1900	1.5835	32.90	.00833	3.530	2.23	3.514	−.04
1920	1.906	13.737	.01032	4.317	**2.26**	4.303	−0.4
1940	2.401	5.736	.01283	5.399	2.25	5.395	−0.1
1960	3.185	2.395	.01531	6.762	2.12	6.752	−0.1
1980	4.385	1	**.01627**	8.186	1.87	8.103	+1.4
2000	5.980	.4175	.01421	9.300	1.56	9.109	+2.1
2020	7.616	.17443	.00987	9.903	1.30	9.691	+1.0
2040	8.852	.07280	.00544	10.124	1.14	9.976	+1.5
2050	9.268	.04704	.00383	10.155	1.10	10.053	+1.0
2100	10.080	.00530	.000502	10.197	1.01	10.182	−0.1
+∞	10.19725	0	0	10.19725	1	10.199	0

Population from tamed quasi-hyperbolic fit $P_T = 3.83 \times 10^9/[\ln(1.28 + E)]^{0.70}$ where $E = \exp[(1980 - t)/22.9]$; hence $dP/P_Tdt = .03057E/[(1.28 + E)\ln(1.28 + E)]$. C calculated from P: $C_P = P_T + [dP/P_Tdt]/k$ where $k = .00428/10^9$ person-year. Tamed quasi-hyperbolic fit to C_P: $C_T = 10.11 \times 10^9/[\ln(2.69 + E)]^{.83}$. In bold: peak values of dP/P_Tdt and C_P/P. Peak C-P = 3.80 billion, around 1980.

The earliest modeling attempts were disappointing. A simple logistic fit by Pearl and Reed [19] in 1924 confidently projected to a leveling-off at 2 billion, a number surpassed almost immediately. But theirs was a purely biological model, ignoring the technological–organizational impact. They also swept under the carpet the huge discrepancy for pre-1800 estimates, which would have shown up in a logarithmic graph.

A quasi-hyperbolic pattern yielded a remarkably good fit for the last 2 millennia. Moreover, it could be derived from population–technology interaction. However, it did not include carrying capacity and was leading to a manifestly impossible future projection: infinitely large population at a finite time. Capping such a growth, the way simple logistic model caps exponential growth, proved mathematically elusive. Attempts to do so resulted in differential equations that could not be integrated except by iteration.

7.1. The first major result of this study is an explicit equation $P(t)$ to express empirically world population growth since CE 400

This tamed quasi-hyperbolic function fits the mean world population estimates about as well as any curve with a single inflection point could. Short of major shifts in interaction between population, technological–organizational skills, and Earth's carrying capacity, the millennial trend projects toward a stable population of 10 billion by 2100.

A maximum around 10 billion is of course also the standard UN prediction, which is based on present age structure and fertility trends. So what does the tamed quasi-hyperbolic equation add to what we already think we know? The point is that the equation produces the same ceiling as the UN prediction, from utterly different type of data — total world populations going back almost two millennia. Recent population data beyond 2000 could be omitted from fitting the equation, and the result would still be the same. Such agreement between completely different micro and macro approaches lends credibility to both.

The equation presented enhances the prospect that world population is subject to deep-set factors. At the very least, it offers a close fit to the complex actual pattern, from initial slow growth to steep acceleration up to the 20th century and the present sharp deceleration.

Why start at CE 400? The data impose it. The T-function with the given constant values does not apply before this date, at which another tamed quasi-hyperbolic cycle may have reached its ceiling. This possibility remains to be investigated separately. The same applies to distant future, where continuing technological–organizational change may start a new cycle of population growth, less restricted by dry land area on Earth.

7.2. The second major result of this study is fusing two previous theoretical models, one based on technology alone, and the other on carrying capacity alone

If this model were limited to offering unsolvable differential equations for a causal loop $P \leftarrow C \leftarrow T \leftarrow P\ldots$, it would remain an intellectual exercise. But we further show that the tamed quasi-hyperbolic equation developed for population supplies a fairly good *approximate* solution for the theoretical model. Surprisingly, it also leads to a complementary explicit equation for Earth's carrying capacity — and possibly for the technological–organizational factor. This concordance between the empirical equation for population and the theoretical interaction model lends credibility to both.

While the T-function does impose a ceiling, it does not cast any light on the specific process that slows down population growth when the ultimate limit is being approached. The specific factors for present slowdown include the loop technology → higher living standard → reduced birth rate, largely through artificial contraception. These are recent aspects of technology, playing little role prior to 1900. Addressing the difficult mathematical task of capping the asymptote-bound growth, the present approach involves no such substantive features expressly. This is a macro approach. Hence it is remarkable that the T-function devised can be fitted to the actual path of slowdown, up to now — and into foreseeable future.

Projections are not predictions. Temporary spurts in population growth, like those in 1200 and 1850, cannot be excluded, nor can population reduction scenarios. Black Death followed the unusual speedup in growth around 1200, and one may wonder whether sudden increase in population density was a factor. Modern technology speeds up the spread of infections, and conflicts over increasingly scarce resources may become unpredictable. An increasing rate of technological change will require ever more people to retrain ever more frequently during their lifetimes, and a psychological limit may set in.

When viewed against this background, the fit with tamed quasi-hyperbolic function represents conservative middle grounds, based on fitting long-term historical data to interaction of basic factors in limited space. It offers a useful check on extrapolations based on current demographic data. Approximate agreement with a theoretical interaction model adds to the value of both.

Two reservations should be spelled out. First, attempting to project an upper limit from early growth data can lead to major errors. Even while world population growth curve seems to show a definite inflection point around 2000, this caution still applies.

Second, the population–technology-carrying capacity interaction model presented here is extremely general, lacking direct and specific causal factors. One may argue that the long sweep of history has washed out specific but ephemeral causal factors that enter over the course of centuries. Still, it's an act of faith to presume that the combined outcome of specific causal factors would amount to a smooth impact of generalized “technology” and “carrying capacity”. The alternative view is that the population growth pattern is random, so that distant past offers no guidance for future. Given the remarkable smoothness of the growth curve, over 1600 years, this view also involves an act of faith. The present study has investigated a possible interaction mechanism, if any long-term pattern should exist.

Even if such a pattern has existed in the past, it need not continue to hold. A major shift in ultimate carrying capacity can take place, as seems to have happened around CE 400. Wide-scale colonization of space beyond Earth certainly would boost ultimate carrying capacity. On the other hand, individual civilizations have collapsed before, and having now a single world-wide civilization makes humankind singularly vulnerable.

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