① HIFAR Note 486

Rough Estimate of Emittance Growth From Magnetic Field Errors

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Background

Each beam of a heavy ion driver will pass through about 1000 magnetic quadrupoles in the linac. These are multichannel structures ($N_{beam} \approx 16-192$) which will never be a perfect transport system. It is assumed here that errors of quadrupole strength, alignment, or an unwanted dipole component are compensated by rematching of the beam envelope and steering. Unwanted magnetic field multipoles, sextupole and higher, are not compensated and will cause emittance growth, halos, and even particle loss. It is frequently asked "what is the magnetic field tolerance for design and manufacture?". No general answer has ever been available except for, eg an intuitively based "about .1%". However, a partial justification for this answer can be easily derived from a simple consideration of the effect of the unwanted multipoles, <u>assuming they are random from one magnet to the next</u>. Equation (13), derived below, relates emittance growth to an effective rms measure of the random field multipoles.

Multipole Fields

The transverse field components in a quadrupole can be derived with sufficient accuracy from the z-component of a vector potential (A_z). Denoting the unwanted, random part by δA_z, we have inside the vacuum aperture

$$\nabla_\perp^2 \, \delta A_z \simeq 0 \qquad (1)$$

Solution of eqn(1) using cylindrical coordinates (r, θ, z) gives multipoles:

$$\delta A_z = \sum_{n=3}^{\infty} A_n \, r^n \cos n(\theta - \varphi_n) \, . \qquad (2)$$

Here A_n and φ_n are functions of z within the magnet channel, but we will replace them by their mean values over z. A beam ion recieves an extra kick in transverse momentum from a single magnet:

$$\underset{\sim}{\delta P_\perp} = \int dt\, q e v \hat{e}_z \times \underset{\sim}{\delta B_\perp}$$

$$= q e (\eta L) \underset{\sim}{\nabla_\perp} \delta A_z \quad . \qquad (3)$$

Here we used

$$\underset{\sim}{\delta B_\perp} = \underset{\sim}{\nabla_\perp} \times \delta A_z \hat{e}_z = -\hat{e}_z \times \underset{\sim}{\nabla_\perp} \delta A_z , \qquad (4)$$

$v = \beta c$ is the ion's speed, and the effective field length (ηL) is the product of the occupancy factor (η) and the lattice half period length (L). In cylindrical coordinates the magnetic field components are

$$\underset{\sim}{\delta B_\perp} = -\sum_{3}^{\infty} A_n\, n\, r^{n-1} \Big[\hat{e}_r \sin n(\theta - \varphi_n) + \hat{e}_\theta \cos n(\theta - \varphi_n) \Big] \qquad (5)$$

Thus the absolute value of any multipole of $\underset{\sim}{\delta B_\perp}$ is independent of θ:

$$\delta B_n = |A_n|\, n\, r^{n-1} \quad . \qquad (6)$$

Low order multipoles associated with the quadrupolar layout of wire are assumed to be eliminated by design.

These are the potentially large repeating amplitudes with $n = 6, 10, 14, \ldots$. Any systematic presence of these fields is not considered here. Fringe field multipoles and pseudo multipoles are another non-random feature of the focal system. Their characterization and effect on dynamics is an outstanding issue for simulation which goes well beyond the treatment in this note.

Emittance Increase

Since the transverse kicks ($\delta \underset{\sim}{P}_{\perp}$) are random, their averaged effect on emittance should increase only as the square root of the number of magnets. We therefore calculate the mean increase in squared emittance per magnet. Let the beam be matched and centered in the quadrupole system, with mean edge radius $\bar{a}$. Then the average increase in normalized edge (squared) emittance is

$$\delta(\epsilon_n^2) = 16(\beta\gamma)^2 \left(\overline{x^2}\right) \overline{(\delta x')^2}$$

$$= 16(\beta\gamma)^2 \left(\frac{\bar{a}^2}{4}\right) \frac{\overline{(\delta P_x)^2}}{p^2} = \frac{2\bar{a}^2}{m^2c^2} \overline{|\delta \underset{\sim}{P}_{\perp}|^2} \qquad (7)$$

[Profile ellipticity is ignored]

Here we have used $x' = P_x/P = P_x/\beta\gamma mc$. The transverse mean is taken over the beam profile, i.e. $0<r<\bar{a}$, and the longitudinal mean has already been assumed in the definition of $\delta \underset{\sim}{B}_\perp$. We have from eqns (2) – (5):

$$|\delta \underset{\sim}{P}_\perp|^2 = (qe\eta L)^2 |\underset{\sim}{\nabla}_\perp \delta A_z|^2 = (qe\eta L)^2 |\delta \underset{\sim}{B}_\perp|^2, \tag{8}$$

$$\overline{|\delta \underset{\sim}{P}_\perp|^2} = (qe\eta L)^2 \int_0^{\bar{a}} dr \frac{2r}{\bar{a}^2} \sum_3^{\infty} A_n^2 n^2 r^{2n-2}$$

$$= (qe\eta L)^2 \sum_3^{\infty} A_n^2 n \bar{a}^{2n-2}. \tag{9}$$

[the cross terms between multipoles in eqn (8) are eliminated in the average over θ]. This expression is conveniently written using the magnetic field multipoles $\delta B_n(r)$ evaluated at $\bar{a}$; using eqn (6) we have

$$\overline{|\delta \underset{\sim}{P}_\perp|^2} = (qe\eta L)^2 \sum_3^{\infty} [\delta B_n(\bar{a})]^2 / n. \tag{10}$$

The value of $\delta B_n(\bar{a})$ is in turn expressed as a fraction of the quadrupole field evaluated at $\bar{a}$:

$$B_{quad}(\bar{a}) = B'\bar{a}, \qquad (11)$$

where B' is the design gradient, We have

$$\overline{|\delta_{\sim} P_\perp|^2} = (qe\eta L B'\bar{a})^2 \sum_3^\infty \frac{1}{n}\left[\frac{\delta B_n(\bar{a})}{B'\bar{a}}\right]^2. \qquad (12)$$

From eqn (7) the <u>expected</u> (mean) emittance increase is then

$$\delta(\epsilon_n^2) = \frac{2\bar{a}^2}{m^2c^2}(qe\eta L B'\bar{a})^2 \sum_3^\infty \frac{1}{n}\left[\frac{\delta B_n(\bar{a})}{B'\bar{a}}\right]^2. \qquad (13)$$

<u>Sample Case</u>

Consider the representative magnetic quadrupole:

$$\left.\begin{aligned} \eta L &= .5\ m \\ \bar{a} &= .02\ m \\ B'\bar{a} &= 1.5\ T \\ m &= 133\ m_0 \\ q &= 1 \end{aligned}\right\} C_s^+$$

$m_0 = 1\,amu = 931.5\ MeV/c^2$

Note for the atomic mass unit m_0:

$$\frac{m_0 c}{e} = \frac{m_0 c^2}{ce} = \frac{931.5\times10^6}{2.998\times10^8} = 3.107\ T\text{-}m.$$

Then eqn (13) gives

$$\delta(\epsilon_n^2) = \left[\frac{2 \times (.02)^2}{(133 \times 3.107)^2} \times (.5)^2 \times (1.5)^2\right] \sum_3^\infty \frac{1}{n}\left[\frac{\delta B_n(\bar{a})}{B'\bar{a}}\right]^2$$

$$= \left(2.635 \times 10^{-9}\ m^2\text{-}r^2\right) \sum \frac{1}{n}\left[\frac{\delta B_n(\bar{a})}{B'\bar{a}}\right]^2 . \quad (14)$$

Suppose the dimensionless sum on the right of eqn(14) is 10^{-6}; this corresponds roughly to the intuitive .1% field error mentioned above. Then for 1000 identical quadrupoles we get

$$\sum_1^{1000} \delta(\epsilon_n^2) = 2.635 \times 10^{-9} \times 10^3 \times 10^{-6}$$

$$= 2.635 \times 10^{-12} = \left(1.623 \times 10^{-6}\ m\text{-}r\right)^2 \quad (15)$$

This contribution to ϵ_n^2 has already used up much of the "emittance budget" for a typical driver. More generally, for a total of N_m magnets we have

$$\sum_1^{N_m} \delta(\epsilon_n^2) = \left(1.623 \times 10^{-6}\ m\text{-}r\right)^2 \left(\frac{N_m}{1000}\right)\left(\frac{133}{M/qM_0}\right)\left(\frac{\eta L}{.5m}\right)^2 \times$$

$$\times \left(\frac{B'\bar{a}}{1.5T}\right)^2 \times \sum_3^\infty \frac{1}{n}\left(\frac{\delta B_n(\bar{a})}{10^{-3} B'\bar{a}}\right)^2 . \quad (16)$$

Sample Case Continued

Is it reasonable to assume all magnets are identical (except for the random multipoles)? This is a question of optimization of driver design, and some variation among magnets is expected, particularly at low energy. However the invariant magnet "sample case" computed above essentially transports a beam of constant line charge density with L increasing as the square root of kinetic energy. This is conceptually attractive and a simple model for the high energy portion of a driver. The overall final parameters might be, for example:

$$\left.\begin{array}{l}\text{Final Driver}\\ \text{Energy}\end{array}\right\} W = 5.0 \text{ MJ},$$

$$\left.\begin{array}{l}\text{Final Kinetic}\\ \text{Energy } (Cs^+)\end{array}\right\} T = 2.5 \text{ GeV},$$

$$\text{Total Charge} = \frac{We}{T} = 2\times10^{-3} \text{ Coulomb},$$

$$\beta\gamma = \sqrt{\frac{2T}{mc^2} + \left(\frac{T}{mc^2}\right)^2} = .2019,$$

$$\gamma = \sqrt{1+(\beta\gamma)^2} = 1.0202,$$

$\beta = (\beta\gamma)/\gamma = .1979$,

$(B\rho) = \beta\gamma m/qe = 83.43$ T-m,

Undepressed Tune $\sigma_0 = 80^\circ$,

Depressed Tune $\sigma \simeq 0$,

$$B' = \frac{B'a}{a} = \frac{1.5}{.02} = 75 \text{ T/m},$$

From $\eta L = .5$ m and

$$\cos\sigma_0 \cong 1 - \frac{(1-\frac{2}{3}\eta)}{2}\left[\frac{\eta B' L^2}{(B\rho)}\right]^2 :$$

$$\begin{cases} \eta = .1649 \\ L = 3.032 \text{ m} \end{cases}$$

Transported dimensionless perveance:

$$Q \cong \frac{a^2}{(\eta L)^2}\, 2(1-\cos\sigma_0) = 1.80\times 10^{-5}$$

Current per beam:

$$I = \frac{4\pi\epsilon_0 m c^3 (\beta\gamma)^3 Q}{2qe} = 305.7 \text{ Amps},$$

Assuming there are $N_{beam} = 48$ beams:

Pulse Duration $\tau_p = \dfrac{\text{total charge}}{N_{beam} I} = .136\ \mu s$

Pulse Length $= \tau_p \beta c = 8.086\ m$

Line Charge Density $\lambda = \dfrac{I}{\beta c} = 5.15\ \mu C/m$

There should be no difficulty fitting a quadrupole with effective field length $\eta L = .5\ m$ into the $\sim 3.0\ m$ half period at 2.5 GeV. But at 1/9th final energy ($T = 278$ MeV), we expect $L \approx 3/\sqrt{9} = 1.0\ m$, and things could bebe tight. The sample case worked in this note would then apply to the final 90% of the acceleration. The single magnet formula eqn(13) would be applicable to any of the magnetic quadrupoles. For example, at lower energy the magnets might be made with ηL as short as $.25\ m$, compensated by increasing B' to 150 T/m, while keeping $\bar{a} = .02\ m$.