Title
Time Analyticity of Several Evolutionary Partial Differential Equations

Author
Zeng, Chulan

Publication Date
2022

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Time Analyticity of Several Evolutionary Partial Differential Equations

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Chulan Zeng

June 2022

Dissertation Committee:

Dr. Qi S. Zhang, Chairperson
Dr. James Kelliher
Dr. Zhenghe Zhang

The Dissertation of Chulan Zeng is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

I am grateful to all the people who supported me during the years that I spent completing my PhD program and writing this thesis. First, I would like to express my gratefulness to my advisor Qi S. Zhang, without whose help and directions I would not have been able to accomplish it, and my family, my parents Xingzhi Zeng and Bin He and my sister Yulan Zeng, with whom I have not spent as much time as I would have hopeed to during these last five years. In addition, I would like to thank my wife Hong Tang for being always there.

This work would not have been possible without all the support of my friends. I particularly want to thank Dr. Xin Yang for helping me walk through the process of job searching and thesis formatting. Also, I am grateful for my friends Zijin Li, Michael Mcnulty, Kiyoshi Takeuchi, Weitao Chen, Matthew Overduin, Mingyue Liu, Jun xu, Chenxu He, Chaoping Zhu, Zhixian Zhu who helped me get used to the life in the US. Without their help, it would have been much harder.

I would also like to thank for my collaborators of my math research. In particular, Dr. Hongjie Dong helped me to prove the time analyticity of nonlocal parabolic equations. Without his help, I would have to struggle for much more time, since the original method I took made this problem much more complicated and I failed to overcome the difficulty at first. I also want to thank Dr. Zijin Li, Dr. Xinghong Pan, Dr. Xin Yang and Dr. Na Zhao for sharing their ideas and offering a lot of helpful suggestions during discussion.

Besides, I would like to thank Dr. Jim Kelliher, Dr. Zhenghe Zhang, for serving as my doctoral committee members and providing invaluable suggestions.

I could not finish these acknowledgements without recognizing the work and help of all the staff members of the Department of Mathematics at UCR, particularly Margarita Roman, Melissa Gomez, James Marberry, Crissy Reising, Joyce Sphabmisai, Randy Morgan, for our many spontaneous conversations that have developed into friendship over the years.

To my parents and my brother and my wife, for being always there.

ABSTRACT OF THE DISSERTATION

Time Analyticity of Several Evolutionary Partial Differential Equations

by

Chulan Zeng

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2022
Dr. Qi S. Zhang, Chairperson

This thesis has been on the pointwise time analyticity of several evolutionary partial differential equations, including the heat equation, the biharmonic heat equation, the heat equation with potentials, some nonlinear heat equations and nonlocal parabolic equations.

For the first there equations, we prove if u satisfies some growth conditions in $(x, t) \in \mathrm{M} \times [0, 1]$, then u is analytic in time $(0, 1]$. Here M is R^d or a complete noncompact manifold with Ricci curvature bounded from below by a constant. Then we obtain a necessary and sufficient condition such that $u(x, t)$ is analytic in time at $t = 0$. Applying this method, we also obtain a necessary and sufficient condition for the solvability of the backward equations, which is ill-posed in general. An interesting point is that a solution may be analytic in time even if it is not smooth in the space variable x, implying that the analyticity of space and time can be independent. Actually, for general manifolds, space analyticity may not hold since it requires certain bounds on curvature and its derivatives.

For the nonlinear heat equation with power nonlinearity of order p, we prove that a solution is analytic in time $t \in (0,1]$ if it is bounded in $\mathrm{M} \times [0,1]$ and p is a positive integer. In addition, we investigate the case when p is a rational number with a stronger assumption $0 < C_3 \leq |u(x,t)| \leq C_4$. It is also shown that a solution may not be analytic in time if it is allowed to be 0. As a lemma, we obtain an estimate of $\partial_t^k \Gamma(x,t;y)$ where $\Gamma(x,t;y)$ is the heat kernel on a manifold, with an explicit estimation of the coefficients.

We also investigate pointwise time analyticity of solutions to nonlocal parabolic equations in the settings of R^d and a complete Riemannian manifold M. On one hand, in R^d, we prove that any solution $u = u(t,x)$ to $u_t(t,x) - \mathrm{L}_\alpha^\kappa u(t,x) = 0$, where L_α^κ is a nonlocal operator of order α, is time analytic in $(0,1]$ if u satisfies the growth condition $|u(t,x)| \leq C(1+|x|)^{\alpha-\epsilon}$ for any $(t,x) \in (0,1] \times \mathrm{R}^d$ and $\epsilon \in (0,\alpha)$. We also obtain pointwise estimates for $\partial_t^k p_\alpha(t,x;y)$, where $p_\alpha(t,x;y)$ is the fractional heat kernel. Furthermore, under the same growth condition, we show that the mild solution is the unique solution. On the other hand, in a manifold M, we also prove the time analyticity of the mild solution under the same growth condition and the time analyticity of the fractional heat kernel, when M satisfies the Poincaré inequality and the volume doubling condition. Moreover, we also study the time and space derivatives of the fractional heat kernel in R^d using the method of Fourier transform and contour integrals. We find that when $\alpha \in (0,1]$, the fractional heat kernel is time analytic at $t = 0$ when $x \neq 0$, which differs from the standard heat kernel. As corollaries, we obtain sharp solvability condition for the backward nonlocal parabolic equations and time analyticity of some nonlinear nonlocal parabolic equations with power nonlinearity of order p. These results are related to those in [19] and [58] which deal with

local equations. At last, we get some nowhere-analytic smooth solutions to the heat equation in either half space or whole space.

Contents

Chapter 1

Introduction

1.1 Motivation

The study of analyticity property of solutions to PDEs has been a classical topic. Even though the spatial analyticity is usually true for generic solutions of the heat equation, the time analyticity is harder to prove and is false in general. For instance, it is not hard to construct a solution of the heat equation in a space-time cylinder in the Euclidean setting, which is not time analytic in a sequence of moments. In fact, the time analyticity is not a local property, rather it requires certain boundary or growth conditions on the solutions. There is a vast literature on time-analyticity for the heat equation and other parabolic type equations under various assumptions. See, for example, [46], [35], [23], [21], [55], [19], [61], and [19] and the citations therein. One can also consider solutions in certain L^p spaces with $p \in (1, \infty)$. See [50] for a large class of dissipative equations in the periodic setting. We also mention that in [21], for any bounded domain $\Omega \subset \mathrm{R}^d$ with analytic boundary, the authors proved that any solution of the high order heat equation

$$\begin{cases} u_t + (-\Delta)^m u = 0, \quad \forall (t,x) \in (0,1] \times \Omega, \\ u = Du = \cdots = D^{m-1}u = 0 \text{ on } (0,1] \times \partial\Omega, \ u(0,x) \in L^2(\Omega) \end{cases}$$

is time analytic in $t \in (0,1]$.

Recently new applications of time analyticity are found in control theory and in the study of backward equations which is essential in stochastic analysis and mathematical finance. A fundamental fact in control theory for heat type equations is that if a state is reachable by the free equation then it is reachable by suitable control from any reasonable initial value. The former is equivalent to the solvability of the free backward equation from this state. However this backward solvability question has been vexing the control theory community for years. As a matter of fact, in a recent paper [40], it was written:"However, it is a quite hard task to decide whether a given state is the value at some time of a trajectory of the system without control (free evolution). In practice, the only known examples of such states are the steady states." This problem for the heat equation was solved in [19] not long ago. More precisely, in the paper [19] (see also [61]), it was proved that if a smooth solution of the heat equation in $(-2,0] \times \mathrm{M}$ is of exponential growth of order 2, then it is time analytic in $t \in [-1,0]$. Here M is either the Euclidean space or certain noncompact manifolds. Also, an explicit condition is found on the solvability of the backward heat equation from a given time, which is equivalent to the time analyticity of the solution of the heat equation at that time. Lately, the time analyticity of solutions to the biharmonic heat equation, the heat equation with potentials, and some nonlinear heat equations are proven in [58]. See also [16] for other results about time analyticity of parabolic type differential equations in the half space.

1.2 Differential Equations We Study

In this thesis, we investigate the pointwise time analyticity of five differential equations. The first one is the biharmonic heat equation

$$\partial_t u + \Delta^2 u = 0, \quad \forall (x,t) \in \mathrm{M} \times [0,1]. \tag{1.1}$$

Here and below, M is R^d or a d demensional complete noncompact manifold with Ricci curvature bounded from below by a constant. The second one is the heat equation with potentials

$$\partial_t u(x,t) - \Delta u(x,t) + V(x)u(x,t) = 0, \quad \forall (x,t) \in \mathrm{M} \times [0,1],\ d \geq 3. \tag{1.2}$$

In one case, $V = V(x)$ is a potential function in $L^q(\mathrm{M})$ for some $q \geq 1$, with some growth conditions. In another case, we treat $V(x) \geq 0$. The last equation is some nonlinear heat equations with power nonlinearity of order p where p is some positive rational number,

$$u_t(x,t) - \Delta u(x,t) = u^p(x,t), \quad \forall (x,t) \in \mathrm{M} \times [0,1]. \tag{1.3}$$

The last equation for this part is fractional heat equations. For clarity, we will first treat the fractional heat equations in the setting of R^d, which reads

$$u_t(t,x) - \mathrm{L}^\kappa_\alpha u(t,x) = 0,\ \alpha \in (0,2),\ (t,x) \in [0,1] \times \mathrm{R}^d, \tag{1.4}$$

where L^κ_α is a nonlocal elliptic operator defined as follows.

Definition 1 *We define*

$$\mathrm{L}^\kappa_\alpha f(x) := p.v. \int_{\mathrm{R}^d} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}}\, dz \tag{1.5}$$

where p.v. means the principal value. Here $\kappa = \kappa(x, z)$ *on* $\mathrm{R}^d \times \mathrm{R}^d$ *is a measurable function satisfying that*

$$0 < \kappa_0 \leq \kappa(x, z) \leq \kappa_1, \ \kappa(x, z) = \kappa(x, -z), \tag{1.6}$$

and for a constant $\beta \in (0, 1)$*,*

$$|\kappa(x, z) - \kappa(y, z)| \leq \kappa_2 |x - y|^\beta, \tag{1.7}$$

where κ_0*,* κ_1*, and* κ_2 *are positive constants.*

The fraction Laplacian $(-\Delta)^{\alpha/2}$ is a typical example of L_α^κ. As a special case, we also obtain the time and space derivative estimates of the fractional heat kernel $p_\alpha(t, x)$ of

$$u_t(t, x) + (-\Delta)^{\alpha/2} u(t, x) = 0, \ \alpha \in (0, 2), \ (t, x) \in [0, 1] \times \mathrm{R}^d. \tag{1.8}$$

Our results involve both solutions and fractional heat kernels. We say that a function $p_\alpha(t, x; y)$ is a fractional heat kernel of the equation (1.4) in R^d, if

$$\partial_t p_\alpha(t, x; y) = \mathrm{L}_\alpha^\kappa p_\alpha(t, x; y), \quad \lim_{t \searrow 0} p_\alpha(t, x; y) = \delta(x, y).$$

In [10], it was proved that the fractional heat kernel is unique under the condition that

$$|p_\alpha(t, x; y)| \leq \frac{Ct}{\left(t^{1/\alpha} + |x - y|\right)^{d+\alpha}},$$

for a constant C. In Lemma 65, we improve this uniqueness result by only requiring the growth condition (3.2). The definition of the fractional heat kernel $p_\alpha(t, x; y)$ on a manifold M will be given in Section 3.4.

The last differential equation is the standard heat equation

$$\partial_t u - \Delta u = 0, \quad \forall (x, t) \in \mathrm{R}^+ \times [0, 1],$$

from which we get some solutions that are nowhere analytic in time.

1.3 Organization of This Thesis

In chapter 2, we will investigate the time analyticity of the biharmonic heat equation, the heat equation with potentials and some nonlinear heat equations. We proved that under some growth condition, the solution to the above differential equations are analytic in time. In chapter 3, we will investigate pointwise time analyticity of solutions to fractional heat equations in the settings of R^d and a complete Riemannian manifold M satisfying the standard Conditions (3.7) and (3.8). Chapter 4 is about two bounded solutions to the heat equation in the half plane, which were nowhere analytic in time.

1.4 Notation

Let us collect some frequently used notation.

- If x is in R^d, then $|x| = \sqrt{\sum_{i=1}^d x_i^2}$ and $B_r(x)$ is a ball of radius r centered at x.
- In M, $B(x,r)$ denotes the geodesic ball of radius r centered at x and $|B(x,r)|$ denotes its volume. We define $d(x,y)$ to be the geodesic distance of two points x, $y \in$ M and 0 to be a reference point in M.
- $p_\alpha(t,x;y)$ is the fractional heat kernel of equations (1.4), (1.8), or (3.9), and $E(t,x;y)$ is the heat kernel of the usual heat equation.
- $Q_r(x,t) = B(x,r) \times (t-r^2, t)$ and $Q'_r(x,t) = B(x,r) \times (t-r^4, t)$.

Please note throughout this paper, constant C may be different from case to case.

Chapter 2

Time Analyticity of the Biharmonic Heat Equation, the Heat Equation With Potentials and Some Nonlinear Heat Equations

2.1 Main Results and Outline

Here are the main results of this Chapter. The first one is about the biharmonic heat equation (1.1).

Theorem 2 *Let* M *be a* d *dimensional, complete, noncompact Riemannian manifold such that the Ricci curvature satisfies* $Ric \geq -(d-1)K_0$ *for a nonnegative constant* K_0*.*

Let $u = u(x,t)$ *be a smooth solution of the biharmonic heat equation* (1.1) *on*

$\mathrm{M}\times[0,1]$ *of exponential growth of order* $\frac{4}{3}$, *namely*

$$|u(x,t)| \le A_1 e^{A_2 d^{\frac{4}{3}}(x,0)}, \quad \forall (x,t)\in \mathrm{M}\times[0,1],$$

where A_1 *and* A_2 *are positive constants. Then* u *is analytic in time* $t\in(0,1]$ *with radius of convergence depending only on* t, d, K_0 *and* A_2. *Moreover, if* $t\in(1-\delta,1]$ *for some small* $\delta>0$, *we have*

$$u(x,t)=\sum_{j=0}^{\infty} a_j(x)\frac{(t-1)^j}{j!}$$

with $-\Delta^2 a_j(x)=a_{j+1}(x)$, *and*

$$|a_j(x)| = \left|(-\Delta^2)^j a_0(x)\right| \le A^* A_3^{j+1} j^j e^{2A_2 d^{\frac{4}{3}}(x,0)}, \quad j=0,1,2,\dots$$

where $A_3=A_3(d,K_0,A_2)$ *and* $A^*=A^*(A_1,d,x_0,\mathrm{M})$.

Then we have two main theorems about the heat equation with potentials (1.2). We define the weak solution in the beginning of Section 3.

Theorem 3 *Let* M *be a* d *dimensional, complete, noncompact, smooth Riemannian manifold such that the Ricci curvature satisfies* $Ric \ge -(d-1)K_0$ *for some nonnegative constant* K_0 *and*

$$\inf_{x\in\mathrm{M}} |B(x,1)| > 0.$$

Assume $V=V(x)$ *satisfies the following conditions:*

(1) There exists some $R^*>0$ *such that* $V(\cdot)\in L^q(B(0,R^*))$ *for some* $q>\frac{d}{2}$.

(2) For some constant $C^{**}>0$, *if* $d(x,0)>R^*$, *then* $|V(x)|\le C^{**} d(x,0)^{\alpha}$ *where* $\alpha=\frac{2q-d}{q-1}$ *and* $d>2$.

(3) $V(\cdot)\in L^1(\mathrm{M}\backslash B(0,R^*))$ *and assume* $\|V\|_{L^1(\mathrm{M}\backslash B(0,R^*))}=D^*$.

Let

$$\|V\|_{L^q(B(0,R^*))} = C^*$$

where C^* *is a positive constant and let* $u = u(x,t)$ *be a weak solution of equation* (1.2) *on* $\mathrm{M} \times [0,1]$ *of exponential growth of order* 2, *namely*

$$|u(x,t)| \leq A_1 e^{A_2 d^2(x,0)}, \quad \forall (x,t) \in \mathrm{M} \times [0,1],$$

where A_1 *and* A_2 *are some positive constants. Then* u *is analytic in* $t \in (0, 1/2]$ *with radius of convergence depending only on* t, d, q, K_0, A_2, α *and* C^*.

Moreover, if $t \in (1/2 - \delta, 1/2]$ *for some small* $\delta > 0$, *we have*

$$u(x,t) = \sum_{j=0}^{\infty} a_j(x) \frac{(t - 1/2)^j}{j!}$$

with $(\Delta - V)a_j(x) = a_{j+1}(x)$, *and*

$$|a_j(x)| = \left|(\Delta - V)^j a_0(x)\right| \leq A_1 A_3^{j+1} j^j e^{A_4 d^2(x,0)}, \quad j = 0, 1, 2, \ldots \tag{2.1}$$

where constants $A_3 = A_3(d, q, K_0, A_2, \alpha, C^*)$ *and* $A_4 = A_4(A_2, \alpha, C^{**}, D^*)$.

Here the extra condition $d \geq 3$ can be removed in the case of R^d. We will explain in more detail during the proof.

Theorem 4 *Let* M *be a* d *dimensional, complete, noncompact Riemannian manifold such that the Ricci curvature satisfies* $Ric \geq -(d-1)K_0$ *for some nonnegative constant* K_0.

Let $u = u(x,t)$ *be a weak solution of the heat equation with nonnegative potentials* (1.2) *where* $V = V(x) \geq 0$ *on* $\mathrm{M} \times [0,1]$. *If* u *is of exponential growth of order* 2, *namely*

$$|u(x,t)| \leq A_1 e^{A_2 d^2(x,0)}, \quad \forall (x,t) \in \mathrm{M} \times [0,1],$$

where A_1 and A_2 are positive constants, then u is analytic in $t \in (0,1]$ with radius depending only on t, d, K_0 and A_2.

Moreover, if $t \in (1-\delta, 1]$ for some small $\delta > 0$, we have

$$u(x,t) = \sum_{j=0}^{\infty} a_j(x) \frac{(t-1)^j}{j!}$$

with $(\Delta - V)a_j(x) = a_{j+1}(x)$, and

$$|a_j(x)| = \left| (\Delta - V)^j \, a_0(x) \right| \le A_1 A_5^{j+1} j^j e^{2A_2 d^2(x,0)}, \quad j = 0, 1, 2, \ldots \tag{2.2}$$

where $A_5 = A_5(d, K_0, A_2)$.

We also have two theorems about some nonlinear heat equations with power nonlinearity of order p.

Theorem 5 *Let M be a d dimensional, complete, noncompact Riemannian manifold such that the Ricci curvature satisfies $Ric \ge -(d-1)K_0$ for some nonnegative constant K_0.*

Let $u = u(x,t)$ be a solution to equation (1.3) where p is a positive integer. Suppose u satisfies

$$|u(x,t)| \le C_2 \quad in \quad \mathrm{M} \times [0,1],$$

for some constant C_2. Then u is analytic in time for any $t \in (0,1]$ with radius of convergence independent of x.

Theorem 6 *Let M be the same manifold as Theorem 5 above and $p = q_1/q_2$ for some positive integers q_1 and q_2. Assume that a solution $u = u(x,t)$ to the equation (1.3) satisfies*

$$0 < C_3 \le |u(x,t)| \le C_4 \quad in \quad \mathrm{M} \times [0,1],$$

where C_3, C_4 are some constants. Then u is analytic in time for any $t \in (0,1]$ with radius of convergence independent of x.

Now we give a brief outline of this Chapter. In Section 2.2, we investigate the time analyticity of the biharmonic heat equation (1.1). As a corollary, we obtain a necessary and sufficient condition for the solvability of the backward biharmonic heat equation $\partial_t u - \Delta^2 u = 0$. As another corollary, we also obtain a necessary and sufficient condition under which the solution of (1.1) is analytic in time at initial time $t = 0$. Section 2.3 pertains the time analyticity of the heat equation with potentials (1.2). We use similar methods and obtain similar results as in Section 2.2. We demonstrate some solutions which may not be smooth in space but analytic in time. Finally, Section 2.4 is about the time analyticity of some nonlinear heat equations with power nonlinearity of order p (1.3). We prove that a solution $u = u(x,t)$ of (1.3) is analytic in time $t \in (0,1]$ if it is bounded in $\mathrm{M} \times [0,1]$ and p is a positive integer. In addition, we investigate the case when p is a rational number with a stronger assumption $0 < C_3 \le |u(x,t)| \le C_4$. As necessary lemmas, for any nonnegative integer k, we establish an explicit estimate of $|\partial_t^k \Gamma(x,t;y)|$ where $\Gamma(x,t;y)$ is the heat kernel on a manifold, and a connection between $\partial_t^k(t^k u^p)$ and $\partial_t^k(t^k u)$.

2.2 Biharmonic Heat Equation

We now begin investigating the time analyticity of the biharmonic heat equation (1.1). The main result in this section is Theorem 2. First, we have several remarks about Theorem 2.

Remark 7 *Just note we use the condition that* u *is of exponential growth of order* $\frac{4}{3}$ *in the computation of* $\iint_{\Gamma_k^1} (u(x,t))^2\, dxdt$ *in* (2.18).

Remark 8 *For any smooth solution* $u = u(x,t)$ *of the biharmonic heat equation* (1.1) *and any* $(x_0, t_0) \in \mathrm{M} \times (0,1]$, *actually we can get*

$$|\partial_t^k u(x_0,t_0)| \leq \frac{A^* A_3^{k+1} k^k}{t_0^{k+q/4-d/8}} e^{2A_2 d^{4/3}(x_0,0)},$$

where $q = \left[\frac{d}{2}\right] + 1$ *and* $[\cdot]$ *means the floor function. Thus, we can see at* $t = 0$, *this method fails to prove the time analyticity.*

Remark 9 *Just note the radius of convergence does not depends on* x *because* A_3 *is independent of* x.

Remark 10 *The exponential growth of order* $\frac{4}{3}$ *corresponds to the heat kernel estimate of the biharmonic heat equation* (1.1) *which can be found in*

[1]. Actually, we can expect that the solutions of high order Laplacian heat equation $u_t + (-\Delta)^m u = 0$ *are also analytic in time with exponential growth of order* $\frac{2m}{2m-1}$ *for any integer* $m \geq 1$.

Remark 11 *Now we briefly go over the main idea of the proof of Theorem 2. For any* $(x_0,t_0) \in \mathrm{M} \times (0,1]$ *and positive integer* k, *consider some regions for any* $j = 1, 2, \cdots, k$,

$$\Gamma_j^1 = \left\{ (x,t) | d(x,x_0) < \tfrac{j t_0^{1/4}}{(2k)^{1/4}}, t \in [t_0 - \tfrac{j t_0}{2k}, t_0] \right\},$$

$$\Gamma_j^2 = \left\{ (x,t) | d(x,x_0) < \tfrac{(j+0.5) t_0^{1/4}}{(2k)^{1/4}}, t \in [t_0 - \tfrac{(j+0.5) t_0}{2k}, t_0] \right\}.$$

Immediately $\Gamma_j^1 \subset \Gamma_j^2 \subset \Gamma_{j+1}^1$.

There are three main steps. We have a lemma for each step in the following.

The first step is to prove that for some constant $C = C(d, K_0)$ *and any* $j = 1, 2, \cdots, k$,

$$\iint_{\Gamma_j^1} |u_t(x,t)|^2 dxdt \le \frac{Ck}{t_0} \iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt.$$

The second step is to prove

$$\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt \le \frac{Ck}{t_0} \iint_{\Gamma_{j+1}^1} |u(x,t)|^2 dxdt.$$

Then we can combine the above two inequalities and iterate to deduce

$$\iint_{\Gamma_1^1} |\partial_t^k u(x,t)|^2 dxdt \le \left(\frac{Ck}{t_0}\right)^{2k} \iint_{\Gamma_{k+1}^1} |u(x,t)|^2 dxdt.$$

The last step is to use the mean value inequality to get, for some constant $C = C(d, x_0, \mathrm{M})$,

$$\begin{aligned} |\partial_t^k u(x_0,t_0)|^2 &\le C\left(\frac{k}{t_0}\right)^{1+q/2} \iint_{\Gamma_1^1} |\partial_t^l u(x,t)|^2 dxdt \\ &\le C\left(\frac{k}{t_0}\right)^{1+q/2} \left(\frac{Ck}{t_0}\right)^{2k} \iint_{\Gamma_{k+1}^1} |u(x,t)|^2 dxdt, \end{aligned} \tag{2.3}$$

which is exactly what we want.

2.2.1 Iterated Energy Estimates

Now we begin to estimate the L^2_{loc} norm of $|\partial_t u(x,t)|^2$.

Lemma 12 *For any smooth solution* $u = u(x,t)$ *of the biharmonic heat equation* (1.1) *and any* $l = 1, 2, \cdots, k$, *there exist some constant* C *such that*

$$\iint_{\Gamma_j^1} |\partial_t u(x,t)|^2 dxdt \le \frac{Ck}{t_0} \iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt.$$

Proof. By Theorem 6.33 of the paper [9], there exists some smooth cut-off function $\psi^{(1)}(x,t)$ such that for some constant C,

$$\frac{|\nabla\psi^{(1)}(x,t)|^2}{\psi^{(1)}(x,t)} \le \frac{C\sqrt{k}}{\sqrt{t_0}}, \quad |\partial_t\psi^{(1)}(x,t)| + |\nabla\psi^{(1)}(x,t)|^4 + |\Delta\psi^{(1)}(x,t)|^2 \le \frac{Ck}{t_0}, \tag{2.4}$$

and

$$0 \le \psi^{(1)}(x,t) \le 1,\ \psi^{(1)}(x,t) = 1 \text{ in } \Gamma_j^1,\ \psi^{(1)}(x,t) \text{ is supported in } \Gamma_j^2.$$

As we are doing the biharmonic heat equation instead of the heat equation, we need to have the estimate for $|\Delta\psi^{(1)}(x,t)|^2$ which is why we need to cite the paper [9].

We use ψ instead of $\psi^{(1)}(x,t)$ in this proof for simplicity of notation. By Green's formula, integration by parts and equation (1.1), we find

$$\begin{aligned}
&\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt = -\iint_{\Gamma_j^2} \partial_t u(x,t)\Delta^2 u(x,t)\psi^2 dxdt \\
&= -\iint_{\Gamma_j^2} \Delta u(x,t)\Delta(\partial_t u(x,t)\psi^2) dxdt \\
&= -\iint_{\Gamma_j^2} \Delta u(x,t)\left(\Delta\partial_t u(x,t)\psi^2 + 2\nabla\partial_t u(x,t)\nabla\psi^2 + \partial_t u(x,t)\Delta\psi^2\right) dxdt \\
&= -\frac{1}{2}\iint_{\Gamma_j^2} \partial_t(\Delta u(x,t))^2\psi^2 dxdt - 2\iint_{\Gamma_j^2} \Delta u(x,t)\nabla\partial_t u(x,t)\nabla\psi^2 dxdt \\
&\quad - \iint_{\Gamma_j^2} \Delta u(x,t)\partial_t u(x,t)\Delta\psi^2 dxdt \\
&= \frac{1}{2}\iint_{\Gamma_j^2} (\Delta u(x,t))^2\partial_t\psi^2 dxdt - \frac{1}{2}\int_{B(x_0,\frac{(j+0.5)t_0^{1/4}}{(2k)^{1/4}})} (\Delta u(x,t))^2 dx\Big|_{t=t_0} \\
&\quad - 2\iint_{\Gamma_j^2} \Delta u(x,t)\nabla\partial_t u(x,t)\nabla\psi^2 dxdt \\
&\quad - 2\iint_{\Gamma_j^2} \Delta u(x,t)\partial_t u(x,t)\Delta\psi\psi dxdt - 2\iint_{\Gamma_j^2} \Delta u(x,t)\partial_t u(x,t)|\nabla\psi|^2 dxdt \\
&\le \frac{1}{2}\iint_{\Gamma_j^2} (\Delta u(x,t))^2\partial_t\psi^2 dxdt + 2\iint_{\Gamma_j^2} \partial_t u(x,t)\nabla\Delta u(x,t)\nabla\psi^2 dxdt \\
&\quad + 2\iint_{\Gamma_j^2} \Delta u(x,t)\partial_t u(x,t)\Delta\psi\psi dxdt + 2\iint_{\Gamma_j^2} \Delta u(x,t)\partial_t u(x,t)|\nabla\psi|^2 dxdt.
\end{aligned} \tag{2.5}$$

Next we can use the bounds for the cutoff function ψ and the Cauchy-Schwarz inequality to get:

$$\begin{aligned}
&\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt \\
\leq& \frac{Ck}{t_0}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt + \epsilon\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt \\
&+ \frac{4}{\epsilon}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2|\nabla\psi|^2 dxdt \\
&+ \epsilon\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt + \frac{Ck}{\epsilon t_0}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt \qquad (2.6)\\
&+ \epsilon\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt + \frac{Ck}{\epsilon t_0}\iint_{\Gamma_j^2} |\Delta u(x,t)^2| dxdt \\
=& \frac{Ck}{t_0}(1+\frac{2}{\epsilon})\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt + 3\epsilon\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt \\
&+ \frac{4}{\epsilon}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2|\nabla\psi|^2 dxdt.
\end{aligned}$$

Now we need to get the estimate for the term $\frac{4}{\epsilon}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2|\nabla\psi|^2 dxdt$ as above. For some small positive constants ϵ_2 and ϵ_3,

$$\begin{aligned}
&\frac{4}{\epsilon}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2|\nabla\psi|^2 dxdt \leq \frac{4C\sqrt{k}}{\epsilon\sqrt{t_0}}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2\psi dxdt \\
=& -\frac{4C\sqrt{k}}{\epsilon\sqrt{t_0}}\iint_{\Gamma_j^2} \Delta^2 u(x,t)\Delta u(x,t)\psi dxdt \\
&- \frac{4C\sqrt{k}}{\epsilon\sqrt{t_0}}\iint_{\Gamma_j^2} \nabla\Delta u(x,t)\Delta u(x,t)\nabla\psi dxdt \\
\leq& \frac{2C\sqrt{k}\epsilon_3}{\epsilon\sqrt{t_0}}\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2\psi^2 dxdt + \frac{2C\sqrt{k}}{\epsilon\epsilon_3\sqrt{t_0}}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt \\
&+ \frac{2C\sqrt{k}\epsilon_2}{\epsilon\sqrt{t_0}}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2|\nabla\psi|^2 dxdt + \frac{2C\sqrt{k}}{\epsilon\epsilon_2\sqrt{t_0}}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt.
\end{aligned}$$

Take $\epsilon = 1/8$, $\epsilon_2 = \frac{\sqrt{t_0}}{C\sqrt{k}}$ and $\epsilon_3 = \frac{\sqrt{t_0}}{64C\sqrt{k}}$, we have

$$\begin{aligned} &\frac{4}{\epsilon}\iint_{\Gamma_j^2} |\nabla\Delta u(x,t)|^2 |\nabla\psi|^2 dxdt \\ &\leq \frac{1}{2}\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2 \psi^2 dxdt + \frac{2080C^2k}{t_0}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt. \end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we can get

$$\iint_{\Gamma_j^2} |\partial_t u(x,t)|^2 \psi^2 dxdt \leq \frac{Ck}{t_0}\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt,$$

which finishes the proof of Lemma (12). ■

Now we begin to estimate the L^2_{loc} norm of $|\Delta u(x,t)|^2$. We can get a Caccioppoli type inequality (energy estimate) as follows.

Lemma 13 *For any smooth solution $u = u(x,t)$ of the biharmonic heat equation* (1.1) *and any $l = 1, 2, \cdots, k$, there exist some constant C such that*

$$\begin{aligned} \sup_{t\in(t_0-\frac{(j+1)t_0}{(2k)},t_0)} \int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} u^2(x,t)\psi^2 dx + \iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt \\ \leq \frac{Ck}{t_0}\iint_{\Gamma_{j+1}^1} |u(x,t)|^2 dxdt. \end{aligned} \tag{2.8}$$

Proof. By Theorem 6.33 of the paper [9] again, there exists some smooth cut-off function $\psi^{(2)}(x,t)$ satisfying the condition 2.4 and

$$0 \leq \psi^{(2)}(x,t) \leq 1,\ \psi^{(2)}(x,t) = 1 \text{ in } \Gamma_j^2,\ \psi^{(2)}(x,t) \text{ is supported in } \Gamma_{j+1}^1.$$

We denote the cuf-off function $\psi^{(2)}(x,t)$ by ψ again in this proof for the simplicity of notation. Similar to (2.5) and (2.6), using Green's formula, Cauchy-Schwarz inequality, integration by parts and assumption for the cut-off function ψ, we yield

$$\begin{aligned}
&\iint_{\Gamma_{j+1}^1} (\Delta u(x,t))^2\psi^2 dxdt \\
&\leq \left(1+\frac{2}{\epsilon}\right)\frac{Ck}{t_0}\iint_{\Gamma_{j+1}^1} u^2(x,t)dxdt + 3\epsilon\iint_{\Gamma_{j+1}^1} |\Delta u(x,t)|^2\psi^2 dxdt \\
&+\frac{4}{\epsilon}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2|\nabla\psi|^2 dxdt,
\end{aligned} \tag{2.9}$$

for any small positive constant ϵ.

Next we need to obtain the estimate for the term $\frac{4}{\epsilon}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2|\nabla\psi|^2 dxdt$.

By integration by parts and Cauchy-Schwarz inequality, for some small positive constants ϵ_2 and ϵ_3,

$$\begin{aligned}
&\frac{4}{\epsilon}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2|\nabla\psi|^2 dxdt \leq \frac{4C\sqrt{k}}{\epsilon\sqrt{t_0}}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2\psi dxdt \\
&\leq \frac{2C\sqrt{k}\epsilon_2}{\epsilon\sqrt{t_0}}\iint_{\Gamma_{j+1}^1} |\Delta u(x,t)|^2\psi^2 dxdt + \frac{2C\sqrt{k}}{\epsilon\epsilon_2\sqrt{t_0}}\iint_{\Gamma_{j+1}^1} u^2(x,t)dxdt \\
&+\frac{2C\sqrt{k}\epsilon_3}{\epsilon\sqrt{t_0}}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2|\nabla\psi|^2 dxdt + \frac{2C\sqrt{k}}{\epsilon\epsilon_3\sqrt{t_0}}\iint_{\Gamma_{j+1}^1} u^2(x,t)dxdt.
\end{aligned}$$

Take $\epsilon = \frac{1}{8}$, $\epsilon_2 = \frac{\sqrt{t_0}}{128C\sqrt{k}}$ and $\epsilon_3 = \frac{\sqrt{t_0}}{C\sqrt{k}}$, then

$$\begin{aligned}
&\frac{4}{\epsilon}\iint_{\Gamma_{j+1}^1} |\nabla u(x,t)|^2|\nabla\psi|^2 dxdt \\
&\leq \frac{1}{4}\iint_{\Gamma_{j+1}^1} |\Delta u(x,t)|^2\psi^2 + \frac{4128C^2k}{t_0}\iint_{\Gamma_{j+1}^1} u^2(x,t)dxdt.
\end{aligned} \tag{2.10}$$

Plugging (2.10) into (2.9), we can get inequality

$$\iint_{\Gamma_j^2} |\Delta u(x,t)|^2 dxdt \leq \frac{Ck}{t_0}\iint_{\Gamma_{j+1}^1} |u(x,t)|^2 dxdt. \tag{2.11}$$

Besides, we can also see

$$\begin{aligned}
&\partial_t\left(1/2\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} u^2(x,t)\psi^2 dx\right) \\
&= \int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} -\Delta^2 u(x,t)u(x,t)\psi^2 dx + \int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} u^2(x,t)\psi\partial_t\psi dx.
\end{aligned} \tag{2.12}$$

For the term $\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} -\Delta^2 u(x,t)u(x,t)\psi^2 dx$, we have, by integration by parts and assumption for ψ,

$$
\begin{aligned}
&\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} -\Delta^2 u(x,t)u(x,t)\psi^2 dx \\
\leq &\frac{1}{8}\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} (\Delta(u(x,t)\psi))^2 dx + \int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} |\nabla u(x,t)\nabla\psi|^2 dx \\
&+ \frac{Ck}{t_0}\int_{B(x_0,R)} |u(x,t)|^2 dx + \frac{1}{4}\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} (u(x,t)\Delta\psi)^2 dx \\
&+ 5\int_{B(x_0,\frac{(j+1)t_0^{1/4}}{(2k)^{1/4}})} (\Delta u(x,t)\psi)^2 dx.
\end{aligned}
$$

By integration about time in (2.12), using the assumption about ψ and (2.10), (2.11), we can get the (2.8) immediately. ■

2.2.2 Mean Value Inequality for the Biharmonic Heat Equation (1.1)

We also need the following lemma about the mean value inequality.

Lemma 14 *Let* (x_0, t_0) *be any point in* $\mathrm{M} \times (0, 1]$ *and* $u = u(x, t)$ *be any solution to the biharmonic heat equation* (1.1). *Then for some constant* $C_1 = C_1(d, x_0, \mathrm{M})$,

$$\sup_{Q'_r(x_0,t_0)} |u(x,t)|^2 \leq \frac{C_1}{(R-r)^{2q+4}} \iint_{Q'_R(x_0,t_0)} u^2(x,t)dxdt, \tag{2.13}$$

where $q = \left[\frac{d}{2}\right] + 1$ *and* $0 < r < R < 1$.

Remark 15 *Just note here the constant is dependent on* x_0 *and* M. *This is because in the following proof, we need to use the Sobolev inequality, make sure the all the gradients of cut-off function* ψ *below is bounded, and make sure* ∇ *can commute with* Δ. *In* R^d, *due to all of these peoperties satisfied, the constant* C *should be independend of* x_0 *and* M.

Proof. Let $r < R_0 < R_1 < R_2 < R$ where $R - R_2 = R_2 - R_1 = R_1 - R_0 = R_0 - r$ and define a smooth cut-off function $\phi = \phi(x,t)$ which is supported in $Q'_{R_0}(x_0,t_0)$ and $\phi = 1$ in $Q'_r(x_0,t_0)$. Just note because the manifold is smooth in $B(x_0,1)$, for any nonnegative integer k, it holds for some constant $C = C(x_0,k,\mathrm{M})$,

$$|\nabla^k R_m| \leq C(x_0,k,\mathrm{M}),$$

where R_m means the curvature tensor.

Since ϕ is smooth in $B(x_0,1)$, for any positive integer i, there exist some constant $C(x_0,i,\mathrm{M})$ depending on x_0, i and M such that,

$$|\nabla^i \phi^2| \leq \frac{C(x_0,i,\mathrm{M})}{|R_0 - r|^i}, \ |\Delta^i \phi| \leq \frac{C(x_0,2i,\mathrm{M})}{|R_0 - r|^{2i}}$$
$$|\nabla^i \partial_t \phi| \leq \frac{C(x_0,4+i,\mathrm{M})}{|R_0 - r|^{4+i}}, \ \ |\Delta^i \partial_t \phi| \leq \frac{C(x_0,2i+4,\mathrm{M})}{|R_0 - r|^{2i+4}},$$

where ∇ is the covariant derivative and ∇^i means the i-th order covariant derivative. We can also define a smooth cut-off function $\psi = \psi(x,t)$ which is supported in $Q'_{R_1}(x_0,t_0)$ and $\psi = 1$ in $Q'_{R_0}(x_0,t_0)$ satisfying similar condition as above.

Following the method in [15], we can use the Morrey type Sobolev inequality which can be find in Theorem 2.7 of [30], which means there exist some constant $C = C(d,x_0,\mathrm{M})$ that

$$\sup_{B(x_0,R_1)} |u(\cdot,t)\psi| \leq C\|u(\cdot,t)\psi\|_{W^{q,2}(B(x_0,R_1))}.$$

Also, for some constant $C = C(d)$, by the fundamental theorem of calculus, we yield

$$\begin{aligned} \sup_{t\in(t_0-R_1^4,t_0)} |u(x,\cdot)\phi|^2 &\leq \sup_{t\in(t_0-R_1^4,t_0)} \int_{t_0-R_1^4}^{t} \partial_t (u(x,\cdot)\phi)^2 dt \\ &\leq \int_{t_0-R_1^4}^{t_0} |\partial_t u(x,t)|^2 \phi^2 dt + \frac{C}{(R-r)^4} \int_{t_0-R_1^4}^{t_0} |u(x,t)|^2 dt \end{aligned}.$$

Therefore for some $C = C(d, x_0, \mathrm{M})$,

$$\sup_{Q'_r(x_0,t_0)} |u(x,t)|^2 \leq C \sum_{i=0}^{q} \|\nabla^i (\partial_t u(\cdot,\cdot)\psi + \frac{C}{(R-r)^4} u(\cdot,\cdot)\psi)\|_{W^{q,2}(Q_{R_1}(x_0,t_0))}. \tag{2.14}$$

Then we need to apply the well-known Bochner's formula and the related cummutation formula to commute ∇ with Δ and its high order version, see Proposition 3.2.1 of [60] e.g.. Using the above commutation formula,

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla_j f. \tag{2.15}$$

By this formula, for any smooth function f and any cut-off function ψ which is supported in $B(x_0, R_2)$, there exist some constant $C = C(d, K_0)$ such that

$$\begin{aligned}
&\int_{B(x_0,R_2)} (\Delta f)^2 \psi^2 dx = \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_i \nabla_i f \nabla_j \nabla_j f \psi^2 dx \\
&= - \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_j \nabla_i \nabla_i f \nabla_j f \psi^2 dx - \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_i \nabla_i f \nabla_j f \nabla_j \psi^2 dx \\
&= \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_i \nabla_j f \nabla_i \nabla_j f \psi^2 dx + \int_{B(x_0,R_2)} Ric\,(\nabla f, \nabla f)\, \psi^2 dx \\
&\quad + \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_i \nabla_j f \nabla_j f \nabla_i \psi^2 dx - \sum_{i,j=1}^{n} \int_{B(x_0,R_2)} \nabla_i \nabla_i f \nabla_j f \nabla_j \psi^2 dx \\
&\geq 1/2 \int_{B(x_0,R_2)} \left|\nabla^2 f\right|^2 \psi^2 dx - C \int_{B(x_0,R_2)} |\nabla f|^2 \, |\nabla \psi|^2 dx.
\end{aligned} \tag{2.16}$$

By using formula (2.15) and its high order version repeatedly and (2.16) where we separate (R_1, R_2) into q equal parts, we can get for some $C = C(d, x_0, \mathrm{M})$,

$$\begin{aligned}
&\sum_{i=0}^{q} \iint_{Q'_{R_1}(x_0,t_0)} |\nabla^i(\partial_t u(x,t)\psi)|^2 + \frac{C}{(R-r)^4}|\nabla^i(u(x,t)\psi)|^2 dxdt \\
&\le \sum_{i=0}^{[\frac{q}{2}]} \frac{C}{(R-r)^{2q-4i}} \iint_{Q'_{R_2}(x_0,t_0)} |\Delta^i \partial_t u(x,t)|^2 dxdt \\
&+ \sum_{i=0}^{[\frac{q}{2}]} \frac{C}{(R-r)^{2q-4i+4}} \iint_{Q'_{R_2}(x_0,t_0)} |\Delta^i u(x,t)|^2 dxdt \\
&+ \sum_{i=0}^{[\frac{q-1}{2}]} \frac{C}{(R-r)^{2q-4i-1}} \iint_{Q'_{R_2}(x_0,t_0)} |\nabla \Delta^i \partial_t u(x,t)|^2 dxdt \\
&+ \sum_{i=0}^{[\frac{q-1}{2}]} \frac{C}{(R-r)^{2q-4i+3}} \iint_{Q'_{R_2}(x_0,t_0)} |\nabla \Delta^i u(x,t)|^2 dxdt.
\end{aligned} \tag{2.17}$$

By Lemma 12 and Lemma 13, we have for some constant C

$$\iint_{Q'_{R_2}(x_0,t_0)} |u_t(x,t)|^2 dxdt \le \frac{C}{|R-r|^4} \iint_{Q'_{R_3}(x_0,t_0)} |u(x,t)|^2 dxdt.$$

Plugging (2.11) and (2.10) into (2.17), it holds for some constant $C = C(x_0, \mathrm{M})$,

$$\begin{aligned}
&\sum_{i=0}^{q} \iint_{Q'_{R_1}(x_0,t_0)} |\nabla^i(\partial_t(u(x,t)\psi) + u(x,t)\psi)|^2 dxdt \\
&\le \frac{C}{|R-r|^{4+2q}} \iint_{Q'_{R_3}(x_0,t_0)} |u(x,t)|^2 dxdt.
\end{aligned}$$

Plugging into (2.14), we can get (2.13) immediately.

2.2.3 Proof of Theorem 2

Now we are ready to prove Theorem 2. Combining Lemma 12 and Lemma 13, for any $l = 1, 2, \cdots, k$, we yield

$$\iint_{\Gamma_j^1} |\partial_t u(x,t)|^2 dxdt \le \frac{C^2 k^2}{t_0^2} \iint_{\Gamma_{j+1}^1} |u(x,t)|^2 dxdt.$$

Since $\partial_t^l u$ is also a solution of (1.1), by iteration, we have

$$\iint_{\Gamma_1^1} \left(\partial_t^k u(x,t)\right)^2 dxdt \leq \frac{C^2k^2}{t_0^2} \iint_{\Gamma_2^1} \left(\partial_t^{k-1} u(x,t)\right)^2 dxdt$$
$$\leq ... \leq \left(\frac{C^2k^2}{t_0^2}\right)^k \iint_{\Gamma_{k+1}^1} u(x,t)^2 dxdt.$$

Using the mean value inequality (2.13), for some constant $A_3 = A_3(d, K_0, A_2)$ and $A^* = A^*(A_1, d, x_0, \mathrm{M})$,

$$\begin{aligned}
|\partial_t^k u(x_0,t_0)|^2 &\leq C_1 \left(\frac{2k}{t_0}\right)^{\frac{4+2q}{4}} \iint_{Q'_{\left(\frac{t_0}{2k}\right)^{1/4}}(x_0,t_0)} |\partial_t^k u(x,t)|^2 dxdt \\
&\leq C_1 \left(\frac{2k}{t_0}\right)^{\frac{4+2q}{4}} \left(\frac{C^2k^2}{t_0^2}\right)^k \iint_{\Gamma_k^1} (u(x,t))^2 dxdt \\
&\leq C_1 \left(\frac{2k}{t_0}\right)^{\frac{4+2q}{4}} \left(\frac{C^2k^2}{t_0^2}\right)^k \times A_1^2 e^{4A_2 d^{4/3}(x_0,0)} e^k t_0^{1+\frac{d}{4}} \\
&\leq \frac{A^{*2} A_3^{2k+2} k^{2k}}{t_0^{2k+q/2-d/4}} e^{4A_2 d^{4/3}(x_0,0)}.
\end{aligned} \tag{2.18}$$

Thus,

$$|\partial_t^k u(x_0,t_0)| \leq \frac{A^* A_3^{k+1} k^k}{t_0^{k+q/4-d/8}} e^{2A_2 d^{4/3}(x_0,0)}. \tag{2.19}$$

Then we fix a number $R \geq 1$ and let $t \in [1-\delta, 1]$ for some small $\delta > 0$. For any positive integer j, Taylor's theorem implies that

$$u(x,t) - \sum_{i=0}^{j-1} \partial_t^i u(x,1) \frac{(t-1)^i}{i!} = \frac{(t-1)^j}{j!} \partial_t^j u(x,s), \tag{2.20}$$

where $s = s(x,t,j) \in [t,1]$. By (2.19), for sufficiently small $\delta > 0$, the right-hand side of (2.20) converges to 0 uniformly for $x \in B(0,R)$ as $j \to \infty$. Hence

$$u(x,t) = \sum_{j=0}^{\infty} \partial_t^j u(x,1) \frac{(t-1)^j}{j!}$$

i.e., u is analytic in time with radius δ. Denote $a_j = a_j(x) = \partial_t^j u(x,1)$. By (2.19) again, we have

$$\partial_t u(x,t) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{(t-1)^j}{j!} \text{ and } \Delta^2 u(x,t) = \sum_{j=0}^{\infty} \Delta^2 a_j(x) \frac{(t-1)^j}{j!}$$

where both series converge uniformly for $(x,t) \in B(0,R) \times [1-\delta, 1]$. Since u is a solution of the biharmonic heat equation (1.1), it implies $-\Delta^2 a_j(x) = a_{j+1}(x)$ with

$$|a_j(x)| \leq A_1 A_3^{k+1} k^k e^{2A_2 d^{4/3}(x,0)}.$$

This completes the proof of Theorem 2. ■

We can then reach two corollaries similar to Corollary 2.2 and Corollary 2.6 in the paper [19].

Corollary 16 *The Cauchy problem for the backward biharmonic heat equation*

$$\begin{cases} \partial_t u - \Delta^2 u = 0 \\ u(x,0) = a(x) \end{cases} \tag{2.21}$$

has a smooth solution of exponential growth of order $\frac{4}{3}$ *in* $\mathrm{M} \times (0,\delta)$ *for some* $\delta > 0$ *if and only if for any integer* $k \geq 0$*,*

$$|\left(\Delta^2\right)^k a(x)| \leq A_3^{k+1} k^k e^{A_2 d^{\frac{4}{3}}(x_0,0)}, \quad j = 0, 1, 2, \ldots \tag{2.22}$$

where A_2, A_3 *are some positive constants.*

Proof. Suppose (2.21) has a smooth solution of exponential growth of order $\frac{4}{3}$, say $u = u(x,t)$. Then $u(x,-t)$ is a solution of the biharmonic heat equation (1.1) with polynomial growth of order $\frac{4}{3}$. By Theorem 2, (2.22) follows as $(-1)^j(\Delta^2)^j a(x) = a_j(x)$ in the theorem.

On the other hand, suppose (2.22) holds. Then it is easy to check that

$$u(x,t) = \sum_{j=0}^{\infty} (-1)^j (\Delta^2)^j a(x) \frac{t^j}{j!}$$

is a smooth solution of the biharmonic heat equation for $t \in [-\delta, 0]$ with δ sufficiently small. Indeed, the bounds (2.19) guarantee that the above series and the series

$$\sum_{j=0}^{\infty} (-1)^{j+1} (\Delta^2)^{j+1} a(x) \frac{t^j}{j!} \text{ and } \sum_{j=0}^{\infty} (-1)^j (\Delta^2)^j a(x) \frac{\partial_t t^j}{j!}$$

all converge absolutely and uniformly in $B(0,R) \times [-\delta, 0]$ for any fixed $R > 0$. Hence $\partial_t u + \Delta^2 u = 0$. Moreover u has exponential growth of order $\frac{4}{3}$ since

$$|u(x,t)| \leq \sum_{j=0}^{\infty} \left|(\Delta^2)^j a(x)\right| \frac{t^j}{j!} \leq \sum_{j=0}^{\infty} A_3^{j+1} j^j e^{A_2 d^{\frac{4}{3}}(x_0,0)} \frac{t^j}{j!} \leq A_3 e^{A_2 d^{\frac{4}{3}}(x_0,0)}$$

for some A_3 provided that $t \in [-\delta, 0]$ with δ sufficiently small. Thus, $u(x,-t)$ is a solution to the Cauchy problem of the backward biharmonic heat equation (2.21) of exponential growth of order $\frac{4}{3}$. ■

Remark 17 *It is known that generally the Cauchy problem for the backward biharmonic heat equation is not solvable. We can expect this corollary can be used in control theory, Ricci flow, stochastic analysis and some other areas.*

We have another corollary about time analyticity at initial time $t = 0$.

Corollary 18 *For the Cauchy problem for the biharmonic heat equation*

$$\begin{cases} \partial_t u + \Delta^2 u = 0 \\ u(x,0) = a(x). \end{cases} \tag{2.23}$$

It has a smooth solution $u = u(x,t)$ of exponential growth of order $\frac{4}{3}$, which is analytic in time in $\mathrm{M} \times [0, \delta)$ *for some $\delta > 0$ with a radius of convergence independent of x if and only if*

$$|\left(\Delta^2\right)^k a(x)| \leq A_3^{k+1} k^k e^{A_2 d^{\frac{4}{3}}(x_0,0)}, \quad k = 0,1,2,\ldots, \quad j = 0,1,2,\ldots \tag{2.24}$$

where A_2, A_3 are some positive constants.

Proof. Assuming (2.24), it is well-known that the problem (2.23) has a solution

$$u = u(x,t) = \int_{\mathrm{M}} p(x,t;y)a(y)dy,$$

for some $\delta > 0$ and $t \in [0, \delta]$ where $p(x,t;y)$ is the heat kernel for the biharmonic heat equation on M.

By Corollary 16, the following backward problem also has a solution

$$\begin{cases} \partial_t v - \Delta^2 v = 0 \\ v(x,0) = a(x) \end{cases}$$

in $\mathrm{M} \times [0, \delta)$ for some sufficiently small $\delta > 0$. Define the function $U = U(x,t)$ by

$$U(x,t) = \begin{cases} u(x,t), & t \in [0, \delta) \\ v(x,-t), & t \in (-\delta, 0] \end{cases}$$

It is straight forward to check that $U(x,t)$ is a solution of the biharmonic heat equation in $\mathrm{M} \times (\delta, \delta)$.

By the theorem 2, $U(x,t)$ and hence $u(x,t)$ is analytic in time at $t = 0$.

On the other hand, suppose $u(x,t)$ is a solution of the equation (2.23), which is analytic in time at $t = 0$ with a radius of convergence independent of x. Then, by definition, u has a power series expansion in a time interval $(-\delta, \delta)$, for some $\delta > 0$. Hence, (2.24) holds following the proof of Corollary 16. ∎

Remark 19 *Recall the well-known Kovalevskaya counter-example*

$$\begin{cases} \partial_t u - \Delta u = 0, \quad \forall (x,t) \in R \times [0,1] \\ u(x,0) = \frac{1}{1+x^2}, \end{cases}$$

which says there are no analytic solutions in a neighborhood of the origin. We can extend it to the case of the biharmonic heat equation.

Lemma 20 *Any smooth solution to the biharmonic heat equation1.1*

$$\begin{cases} \partial_t u + \Delta^2 u = 0, \quad \forall (x,t) \in R \times [0,1] \\ u(x,0) = \frac{1}{1+x^2}, \end{cases}$$

is not analytic near origin.

Actually, if we have a analytic solution u near original, we can define

$$u(x,t) = \sum_{k,l \geq 0} a_{kl} \frac{t^k}{k!} \frac{x^l}{l!}.$$

By induction, we can prove

$$a(m, 2n) = (-1)^{m+n}(4m+2n)!, \text{ for any nonnegative integers } m, n.$$

Therefore

$$\frac{|a(n,4n)|}{n!(4n)!} = \frac{(8n)!}{n!(4n)!} \to \infty,$$

e solution is not analytic near origin.

This corollary partially solves the problem about time analyticity of the biharmonic heat equation at $t = 0$.

Remark 21 *We can give a non-uniqueness example similar to the well-known non-uniqueness example for heat equation by A.N.Tychonov. To be precise, when* $\mathrm{M} = R^1$, *we can give a* *solution* u *of* (1.1) *which does not satisfy* $|u(x,t)| \leq A_1 e^{A_2|x|^{4/3}}$ *in* $R^1 \times (0,1]$ *and is not analytic at* $t = 0$. *It is*

$$u(x,t) = \sum_{k=0}^{\infty} (-1)^k D_t^k g(t) \frac{x^{4k}}{(4k)!},$$

where

$$g(t) = \begin{cases} e^{-t^{-\alpha}}, & \text{for any} \quad \alpha > 1, t > 0 \\ 0, & t \leq 0. \end{cases}$$

We can prove for some positive constant C,

$$|D_t^k g(t)| \leq \frac{C^k k!}{t^k} e^{-\frac{1}{2t^\alpha}},$$

and therefore by $k!/(4k)! \leq 1/(3k)!$

$$|u(x,t)| \leq \sum_{k=0}^{\infty} \frac{C^k k!}{t^k} e^{-\frac{1}{2t^\alpha}} \frac{x^{4k}}{(4k)!} \leq \sum_{k=0}^{\infty} \frac{C^k}{t^k} e^{-\frac{1}{2t^\alpha}} \frac{x^{4k}}{(3k)!} \leq e^{\left(\frac{Cx^4}{t}\right)^{1/3} - \frac{1}{2t^\alpha}}.$$

This example also shows the non-uniqueness for (1.1) *because obviously we have another solution* $u = 0$.

2.3 Heat Equation With Potentials

In this section, we mainly investigate the time analyticity of the heat equation with potentials (1.2). The main idea of this section is similar to the idea as explained in Remark 11 of Section 2. First, let us define the weak solution.

Definition 22 *We say* $u = u(x,t) \in L^2_{loc}((t_1,t_2), W^{1,2}_{loc}(\mathrm{M}))$ *is a weak subsolution (weak supersolution) to* (1.2) *if it satisfies,*

$$-\int_{t_1}^{t_2} \int_{\mathrm{M}} u(x,t)\partial_t \phi(x,t) dx dt + \int_{t_1}^{t_2} \int_{\mathrm{M}} \nabla u(x,t) \nabla \phi(x,t) dx dt$$
$$+ \int_{t_1}^{t_2} \int_{\mathrm{M}} V(x) u(x,t) dx dt \leq 0 \ (\geq 0),$$

for any nonnegative $\phi \in C_c^\infty(\mathrm{M} \times (t_1, t_2))$.

Especially, if $\phi \in C_c^\infty(\mathrm{M} \times (t_1, t_2))$ and $\phi(\cdot, \frac{t_1+t_2}{2}) = 1$, then we can prove

$$
\begin{aligned}
&-\int_{t_1}^{\frac{t_1+t_2}{2}} \int_{\mathrm{M}} u(x,t)\partial_t\phi(x,t)dxdt + \int_{\mathrm{M}} u(x, \frac{t_1+t_2}{2})dx \\
&+\int_{t_1}^{\frac{t_1+t_2}{2}} \int_{\mathrm{M}} \nabla u(x,t)\nabla\phi(x,t)dxdt + \int_{t_1}^{\frac{t_1+t_2}{2}} \int_{\mathrm{M}} V(x)u(x,t)dxdt \le 0 \ (\ge 0)
\end{aligned}
$$

by testing with $\phi\eta_j$ and taking the limit $j \to \infty$, where $\eta_j = \eta_j(t) \in C_c^\infty(t_1, t_2)$ is a sequence of nonnegative functions satisfying

$$
\lim_{j\to\infty} \eta_j(t) = \chi_{(t_1, \frac{t_1+t_2}{2})} \ a.e.
$$

We say u is a weak solution if it is both a weak subsolution and a weak supersolution.

Now for Theorem 3, we have some remarks first.

Remark 23 *To be more precise, for any* $(x_0, t_0) \in \mathrm{M} \times (0, 1/2]$, *then in Theorem 3, it holds*

$$
|\partial_t^k u(x_0, t_0)| \le \frac{B_1 B_2^{k+1} k^k}{t_0^k} e^{B_3 d^2(x_0, 0)}.
$$

for some constants B_1, B_2 *and* B_3. *Besides, in Theorem 4, it holds*

$$
|\partial_t^k u(x_0, t_0)| \le \frac{B_1 B_2^{k+1} k^k}{t_0^k} e^{B_3 d^2(x_0, 0)}.
$$

for some constants B_1, B_2 *and* B_3.

Remark 24 *By the method of Steklov average, or to be more precise, by Theorem 4.1 of [24] which states if the heat kernel* Γ_V *of* (1.2) *satisfies the* L^2 *Gaussian type upper bound and, for any weak solution* u *of* (1.2), $\partial_t^l u$ *is also a weak solution of* (1.2) *for any* $l = 1, 2, \cdots$. *on the one hand, if* $V \ge 0$ *and if* Γ *is the heat kernel of heat equation on the* *same manifold* M*, then by maximal principle,* $0 \le \Gamma_V \le \Gamma$*, which means* Γ_V *satisfies this Gaussian type upper bound condition considering* (2.46) *and the mean value inequality. On the other hand, if* $V(\cdot) \in L^q(B(0,R^*))$ *for some* $q > \frac{d}{2}$*, it is well known that* Γ_V *also satisfies this Gaussian type upper bound condition. Besides, we can prove* $\partial_t^l u \in L^2_{loc}(\mathrm{M} \times (0,1))$ *by combining* (2.28) *and* (2.33) *next. Therefore,* $\partial_t^l u$ *is locally Hölder continuous, which means* u *is smooth in time.*

Now we begin to investigate Theorem 3.

Remark 25 *In Theorem 3, we have an extra condition* $\inf_{x\in\mathrm{M}} |B(x,1)| > 0$ *to use the Sobolev inequality. To be precise, by Theorem 3.6 of [30], we can see for any* $\lambda \in (0,1)$*,* $q \ge 1$ *and* $\frac{1}{q} \le \frac{1-\lambda}{d}$*, there exists some constant* $C = C(d,\mathrm{M})$ *such that*

$$\|u\|_{C^\lambda(\mathrm{M})} \le C\|u\|_{W_{1,q}(\mathrm{M})}.$$

Also by Proposition 3.7 of [30], if $d > q \ge 1$*, then for* $q^* = \frac{qd}{d-q}$*, we have*

$$\|u\|_{L^{q^*}(\mathrm{M})} \le C\|u\|_{W_{1,q}(\mathrm{M})}.$$

Before embarking on the proof of theorem (3), we need to have some lemmas first. The first one is about the Poincaré inequality which is a result of [5] and we can find it in Theorem 5.6.5 of [38], e.g..

Lemma 26 *Let* M *be a manifold satisfying same conditions as above Theorem 2.*

Then for any $1 \le p < \infty$*, there exists some constant* $C = C(d,p,K_0)$ *such that for any ball* $B(x_0,r) \subset \mathrm{M}$ *where* $0 < r < 4$*,*

$$\int_{B(x_0,r)} |f(x) - f_{B(x_0,r)}|^p dx \le Cr^p \int_{B(x_0,r)} |\nabla f(x)|^p dx, \tag{2.25}$$

where $f_{B(x_0,r)} = \frac{\int_{B(x_0,r)} f(x)dx}{|B(x_0,r)|}$ *is the mean value of* f *in* $B(x_0,r)$.

Using this result, we have the following lemma about the Sobolev inequality:

Lemma 27 *Let* M *be a manifold satisfying the conditions as above Theorem 2.*

Then for any $1 \leq p < \infty$, $f \in C_c^\infty(B(x_0,r))$ *where* $B(x_0,r) \subset \mathrm{M}$ *with* $r \leq 1$, *there exist some constants* $\nu_p > p$ *and* $C = C(d,p,K_0)$ *such that*

$$\left(\int_{B(x_0,r)} |f|^{\frac{p\nu_p}{\nu_p - p}} dx\right)^{\frac{\nu_p - p}{\nu_p}} \leq \frac{Cr^p}{|B(x_0,r)|^{\frac{p}{\nu_p}}} \int_{B(x_0,r)} |\nabla f|^p dx. \tag{2.26}$$

Proof. By Bishop-Gromov volume comparison theorem,

$$|B(x,r)| \leq |B(x,s)| \left(\frac{r}{s}\right)^d \exp\left((d-1)\sqrt{K_0}r\right) \leq C|B(x,s)| \left(\frac{r}{s}\right)^d, \tag{2.27}$$

when $0 < s < r < 4$.

Combine (2.25) and (2.27), we can get (2.26) by Theorem 5.2.6 of [38] immediately.

■

Remark 28 *Here* $\nu_2 = d$ *when* $d > 2$ *and* ν_2 *can be some number which is close to* 2 *when* $d = 1$ *or* $d = 2$. *We use this Sobolev inequality for the mean value inequality in Lemma 31. Unlike Remark 25, this is true for all dimensions but with extra condition* $r \leq 1$.

Now for any $(x_0, t_0) \in \mathrm{M} \times (0, 1/2]$, we introduce some regions similar to [45] first. For any positive integer k and any $j = 1, 2, \cdots, k$,

$H_j^1 = \left\{(x,t)|d(x,x_0) < \frac{j\sqrt{t_0}}{\sqrt{2k}}, t \in [t_0 - \frac{jt_0}{2k}, t_0 + \frac{jt_0}{2k}]\right\}$,

$H_j^2 = \left\{(x,t)|d(x,x_0) < \frac{(j+0.5)\sqrt{t_0}}{\sqrt{2k}}, t \in [t_0 - \frac{(j+0.5)t_0}{2k}, t_0 + \frac{(j+0.5)t_0}{2k}]\right\}$.

So immediately $H_j^1 \subset H_j^2 \subset H_{j+1}^1$.

Then we have the following lemma to estimate $\iint_{H_j^1} |\partial_t u(x,t)|^2 dxdt$.

Lemma 29 *For any $j = 1, 2, \cdots, k$, there exists some positive constant C such that*

$$\begin{aligned}&\iint_{H_j^1} |\partial_t u(x,t)|^2 dxdt \\ &\leq \frac{Ck}{t_0} \iint_{H_j^2} |\nabla u(x,t)|^2 dxdt + \frac{Ck}{t_0} \iint_{H_{j+1}^1} |V(x)||u(x,t)|^2 dxdt.\end{aligned} \tag{2.28}$$

Proof. Let us define a smooth cut-off function $\phi^{(1)}(x,t)$ such that $\phi^{(1)}(x,t) = 1$ in H_j^1 and is supported in H_j^2. We can also suppose there is some constant C such that

$$|\nabla \phi^{(1)}(x,t)|^2 + |\partial_t \phi^{(1)}(x,t)| \leq \frac{Ck}{t_0}.$$

We use $\phi = \phi^{(1)}(x,t)$ below for the simplicity of notation. By assumption of cut-off function ϕ and Cauchy-Schwarz inequality, integration by parts in time,

$$\begin{aligned}&\iint_{H_j^2} |\partial_t u(x,t)|^2 \phi^2 dxdt = \iint_{H_j^2} \partial_t u(x,t)(\Delta u(x,t) - V(x)u(x,t))\phi^2 dxdt \\ &= -\iint_{H_j^2} \nabla \partial_t u(x,t) \nabla u(x,t) \phi^2 dxdt - \iint_{H_j^2} \partial_t u(x,t) \nabla u(x,t) \nabla \phi^2 dxdt \\ &\quad - \iint_{H_j^2} V(x) \partial_t u(x,t) u(x,t) \phi^2 dxdt \\ &\leq \frac{3}{4} \iint_{H_j^2} |\partial_t u(x,t)|^2 \phi^2 dxdt + \frac{Ck}{t_0} \iint_{H_j^2} |\nabla u(x,t)|^2 | dxdt \\ &+ \frac{Ck}{t_0} \iint_{H_j^2} |V(x)||u(x,t)|^2 \phi dxdt.\end{aligned}$$

Then we can get a Caccioppoli type inequality (energy estimate) as below.

Lemma 30 *For any $j = 1, 2, \cdots, k$, there exists some positive constant C such that*

$$\begin{aligned}
&\sup_{t \in (t_0 - \frac{(j+1)t_0}{2k}, t_0 + \frac{(j+1)t_0}{2k})} \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} u^2(x,t)\phi^2 dx \\
&+ \iint_{H_j^2} |\nabla u(x,t)|^2 dxdt + \iint_{H_j^2} |V(x)||u(x,t)|^2 dxdt \le \frac{Ck}{t_0} \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt \\
&+ C \left(C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0, 0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2 \right) \\
&\quad \times \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt.
\end{aligned} \tag{2.29}$$

Proof. Let us define another smooth cut-off function $\phi^{(2)}(x,t)$ such that $\phi^{(2)}(x,t) = 1$ in H_j^2 and is supported in H_{j+1}^1. We can also suppose there is some constant C such that

$$|\nabla \phi^{(2)}(x,t)|^2 + |\partial_t \phi^{(2)}(x,t)| \le \frac{Ck}{t_0}.$$

We use $\phi = \phi^{(2)}(x,t)$ for the simplicity of notation in this proof. By integration by parts, assumption about ϕ and (1.2),

$$\begin{aligned}
&\iint_{H_{j+1}^1} |\nabla u(x,t)|^2 \phi^2 dxdt \le \frac{1}{4} \iint_{H_{j+1}^1} |\nabla u(x,t)|^2 \phi^2 dxdt \\
&+ \frac{Ck}{t_0} \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt + \iint_{H_{j+1}^1} |V(x)||u(x,t)|^2 \phi^2 dxdt.
\end{aligned} \tag{2.30}$$

Now we need to estimate the last term above. By Hölder inequality, interpolation inequality and Sobolev inequality, we know:

$$\begin{aligned}
&\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} V(x)|u(x,t)|^2\phi^2 dx \\
&\le \left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{1/q} \left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} \left(|u(x,t)|^2\phi^2\right)^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \\
&\le C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{1/q} 2\epsilon^2 \left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |\nabla(\phi u(x,t))|^{2^*} dx\right)^{\frac{2}{2^*}} \\
&+ C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{1/q} C(d,q)\epsilon^{\frac{-2d}{2q-d}} \int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |\phi u(x,t)|^2 dx \qquad (2.31)\\
&\le C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{1/q} \\
&\times 2\epsilon^2 \left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |\nabla(\phi u(x,t))|^2 dx + \int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |(\phi u(x,t))|^2 dx\right) \\
&+ C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{1/q} \left(\epsilon^{\frac{-2d}{2q-d}} \int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |\phi u(x,t)|^2 dx\right).
\end{aligned}$$

By taking $\epsilon = \frac{1}{2\sqrt{C}}\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{-\frac{1}{2q}}$ and integrating with respect to time,

$$\begin{aligned}
&\iint_{H_j^2} V(x)|u(x,t)|^2\phi^2 dxdt \le \frac{1}{2}\iint_{H_j^2} |\nabla u(x,t)|^2\phi^2 dxdt + \frac{Ck}{t_0}\iint_{H_j^2} |u(x,t)|^2 dxdt \\
&\quad + C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})} |V(x)|^q dx\right)^{\frac{2}{2q-d}} \iint_{H_j^2} |\phi u(x,t)|^2 dxdt \\
&\le \frac{1}{2}\iint_{H_j^2} |\nabla u(x,t)|^2\phi^2 dxdt + \frac{Ck}{t_0}\iint_{H_j^2} |u(x,t)|^2 dxdt \qquad (2.32)\\
&+ \left(C^{*\,\frac{2q}{2q-d}} + CD^{*\,\frac{2}{2q-d}} C^{**\,\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}}\right)^2\right) \\
&\quad \times \iint_{H_j^2} |\phi u(x,t)|^2 dxdt.
\end{aligned}$$

Plugging into (2.30), we yield ,

$$\iint_{H_j^2} |\nabla u(x,t)|^2 dxdt \leq \frac{Ck}{t_0} \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt$$
$$+ C\left(C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2 \right) \tag{2.33}$$
$$\times \iint_{H_j^2} |\phi u(x,t)|^2 dxdt.$$

Besides, we can also see

$$\partial_t \left(1/2 \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} u^2(x,t)\phi^2 dx \right)$$
$$= \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} u(x,t)(\Delta u(x,t) - V(x)u(x,t))\phi^2 dx$$
$$+ \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} u^2(x,t)\phi\partial_t\phi dx \tag{2.34}$$
$$\leq \frac{1}{2} \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} |\nabla u(x,t)|^2 \phi^2 dx + \frac{Ck}{t_0} \int_{B(x_0, \frac{(j+1)\sqrt{t_0}}{\sqrt{k}})} |u(x,t)|^2 dx$$

By integration by time and (2.33), we can get the (2.29) immediately. ■

Then we need the mean value inequality as follows.

Lemma 31 *Assume* M *is a manifold satisfying same conditions as Theorem 3. Let* $u = u(x,t)$ *be a nonnegative weak subsolution to* (2.33)*. Then for any* $0 < p < \infty$, $0 < r < R < 1$ *and* $(x_0, t_0) \in \mathrm{M} \times (0, 1/2]$,

$$\sup_{Q_r(x_0,t_0)} |u(x,t)|^p \leq C \left(\frac{R^2}{|B(x_0,R)|^{\frac{2}{\nu_2}}} \right)^{\frac{1}{\theta^*-1}}$$
$$\times \left(\frac{1}{|R-r|^2} + C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2 \right)^{\frac{\theta^*}{\theta^*-1}}$$
$$\times \iint_{Q_R(x_0,t_0)} |u(x,t)|^p dxdt,$$

where $\theta^* = 1 + \frac{2}{\nu_2}$*. Here* ν_2 *is defined in Lemma 27.*

Proof. We can prove this one by Moser iteration. By Hölder inequality and Lemma 27, we have for any $w(x) \in C_c^\infty(B(x_0, R))$,

$$\begin{aligned} &\int_{B(x_0,R)} |w(x)|^{2(1+\frac{2}{\nu_2})} dx \\ &\leq \left(\int_{B(x_0,R)} |w(x)|^{\frac{2\nu_2}{\nu_2-2}} dx \right)^{\frac{\nu_2-2}{\nu_2}} \left(\int_{B(x_0,R)} |w(x)|^2 dx \right)^{\frac{2}{\nu_2}} \\ &\leq C \left(\frac{R^2}{|B(x_0,R)|^{\frac{2}{\nu_2}}} \right) \int_{B(x_0,R)} |\nabla w(x)|^2 dx \left(\int_{B(x_0,R)} |w(x)|^2 dx \right)^{\frac{2}{\nu_2}} \end{aligned} \tag{2.35}$$

Let $\psi = \psi(x,t)$ be a standard smooth cut-off function such that $\psi = 1$ in $Q_r(x_0, t_0)$ and is supported in $Q_R(x_0, t_0)$. We can assume $|\nabla\psi|^2 + |\partial_t\psi| \leq \frac{C}{|R-r|^2}$. Then by integration by parts and assumption about ψ and (1.2),

$$\begin{aligned} &\partial_t \left(1/2 \int_{B(x_0,R)} |u(x,t)|^2 \psi^2 dx \right) + \int_{B(x_0,R)} |\nabla(u(x,t)\psi)|^2 dx \\ &\leq \frac{C}{|R-r|^2} \int_{B(x_0,R)} |u(x,t)|^2 dx + 4 \int_{B(x_0,R)} |\nabla u(x,t)|^2 \psi^2 dx \\ &\quad + \int_{B(x_0,R)} |V(x)||u(x,t)|^2 \psi^2 dx. \end{aligned}$$

Combining (2.31) and (2.33), by integrating with respect to time, we yield

$$\begin{aligned} &\sup_{t\in(-R^2,0)} \int_{B(x_0,R)} |u(x,t)|^2 \psi^2 dx + \iint_{Q_R(x_0,t_0)} |\nabla(u(x,t)\psi)|^2 dxdt \\ &\leq C \left(\frac{1}{|R-r|^2} + C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2 \right) \\ &\times \iint_{Q_R(x_0,t_0)} |u(x,t)|^2 dxdt. \end{aligned}$$

Let $E(R) = \left(\frac{R^2}{|B(x_0,R)|^{\frac{2}{\nu}}} \right)$ and

$$F^* = \frac{1}{|R-r|^2} + C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2$$

for simplicity of notataion. From inequality (2.35), we can see

$$\iint_{Q_R(x_0,t_0)} |u(x,t)\psi|^{2\theta^*} dxdt \leq CE(R) \left(F^* \iint_{Q_R(x_0,t_0)} |u(x,t)|^2 dxdt \right)^{\theta^*}. \tag{2.36}$$

Now we have two cases.

Case (1): $p \geq 2$. In this case, we can see $u^{p/2}$ is also a nonnegative subsolution. Therefore, (2.36) yields that:

$$\iint_{Q'_r(x_0,t_0)} (u(x,t))^{p\theta}\, dxdt \leq CE(R)\left(F^* \iint_{Q'_R(x_0,t_0)} (u(x,t))^p dxdt\right)^\theta. \tag{2.37}$$

Set for some positive constant $\delta = \frac{r}{R} < 1$, $\omega_i = \frac{(1-\delta)R}{2^i}$ so that $\sum_1^\infty \omega_i = (1-\delta)R$. Set also $\sigma_0 = R, \sigma_{i+1} = \sigma_i - \omega_i = R - \sum_1^i \omega_j$. Applying (2.37) with $p = p_i = \theta^i, r = \sigma_i, R = \sigma_{i+1}$ we obtain

$$\iint_{Q'_{\sigma_{i+1}}(x_0,t_0)} (u(x,t))^{\theta^{i+1}}\, dxdt \leq CE(R)16^{i\theta}F^{*\theta}\left(\iint_{Q'_{\sigma_i}(x_0,t_0)} (u(x,t))^{\theta^i} dxdt\right)^\theta.$$

Hence, by iteration,

$$\begin{aligned}&\left(\iint_{Q'_{\sigma_{i+1}}(x_0,t_0)} (u(x,t))^{\theta^{i+1}}\, dxdt\right)^{\theta^{-1-i}}\\ &\leq (CE(R))^{\Sigma\theta^{-1-j}}16^{\Sigma j\theta^{-1-j}}F^{*\Sigma\theta^{-j}} \iint_{Q'_R(x_0,t_0)} (u(x,t))^2 dxdt,\end{aligned}$$

where all the summations are taken from 0 to i and we can easily see $\Sigma j\theta^{-1-j}$ converges. Letting i tend to infinity, we obtain

$$\sup_{Q'_r(x_0,t_0)} u^2 \leq C\,(E(R))^{\frac{1}{\theta-1}}\, F^{*\frac{\theta}{\theta-1}} \|u\|^2_{2,Q'_R(x_0,t_0)}. \tag{2.38}$$

Then when $p > 2$, we can see $u^{p/2}$ is also a nonnegative subsolution, so

$$\sup_{Q'_r(x_0,t_0)} u^p \leq C\,(E(R))^{\frac{1}{\theta-1}}\, F^{*\frac{\theta}{\theta-1}} \|u\|^p_{p,Q'_R(x_0,t_0)},$$

which proves (2.13) for the case $p \geq 2$.

Case (2): $\mathbf{0 < p < 2}$.

For this case, we can use the method of [22] or more precisely, Theorem 2.2.3 in the book [38].

Fix $\sigma \in (0,1)$ and set $\rho = \sigma + (1-\sigma)/4$. Then (2.38) applies

$$\sup_{Q'_{\sigma R}(x_0,t_0)} u \le C\,(E(R))^{\frac{1}{2(\theta-1)}} \left(1 + \frac{1}{(\rho R - \sigma R)^4}\right)^{\frac{\theta}{2\theta-2}} \|u\|_{2,Q'_{\rho R}(x_0,t_0)}.$$

Now, as $\|u\|_{2,Q} \le \|u\|_{\infty,Q}^{1-p/2}\|u\|_{p,Q}^{p/2}$ for any parabolic cylinder Q, we get

$$\|u\|_{\infty,Q'_{\sigma R}(x_0,t_0)} \le J\left(1 + \frac{1}{(\rho R - \sigma R)^4}\right)^{\frac{\theta}{2\theta-2}} \|u\|^{1-p/2}_{\infty,Q'_{\rho R}(x_0,t_0)}, \tag{2.39}$$

where $J = C\|u\|^{p/2}_{p,Q'_{\rho R}(x_0,t_0)}\,(E(R))^{\frac{1}{2(\theta-1)}}$.

Fix $\delta = \frac{r}{R}$, $\sigma_0 = \delta R = r$ and $\sigma_{i+1} = \sigma_i + (R-\sigma_i)/4$. Then $R - \sigma_i = (3/4)^i(1-\delta)R$.

Applying the above inequality (2.39) for each i yields

$$\|u\|_{\infty,Q'_{\sigma_i}(x_0,t_0)} \le (4/3)^{\theta i/(2\theta-2)} J F^{*\,\frac{\theta}{2\theta-2}} \|u\|^{1-p/2}_{\infty,Q'_{\sigma_{i+1}}(x_0,t_0)}.$$

Hence by iteration, for $i = 1, 2, \ldots$

$$\begin{aligned}&\|u\|_{\infty,Q'_r(x_0,t_0)}\\ &\le (4/3)^{(\theta/(\theta-2))\sum_0^{i-1} j(1-p/2)^j}\left[JF^{*\,\frac{\theta}{2\theta-2}}\right]^{\sum_0^{i-1}(1-p/2)^j} \|u\|^{(1-p/2)^i}_{\infty,Q'_{\sigma_i}(x_0,t_0)}.\end{aligned}$$

Letting i tend to infinity, we yield,

$$\|u\|_{\infty,Q'_r(x_0,t_0)} \le C\,(E(R))^{\frac{1}{p(\theta-1)}}\, F^{*\,\frac{\theta}{(\theta-1)p}} \|u\|_{p,Q'_R(x_0,t_0)},$$

which proves inequality (2.13) for the case $0 < p < 2$.

■

2.3.1 Proof of Theorem 3

Now we are in a position to prove Theorem 3. Because $\partial_t^l u(x,t)$ is also a weak solution of (1.2) for any $l = 1, 2, \cdots, k$, we can put inequality (2.28) and (2.33) together to obtain,

$$\begin{aligned}
&\iint_{H_j^1} |\partial_t^{k-j+1} u(x,t)|^2 dxdt \le \frac{C^2k^2}{t_0^{2k}} \iint_{H_{j+1}^1} |\partial_t^{k-j} u(x,t)|^2 dxdt \\
&+ \frac{Ck}{t_0}\left(C^{*\frac{2q}{2q-d}} + CD^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^2 \right) \\
&\times \iint_{H_{j+1}^1} |\partial_t^{k-j} u(x,t)|^2 dxdt \\
&\le \left(\frac{C^2k^2}{t_0^2} \right) \iint_{H_{j+1}^1} |\partial_t^{k-j} u(x,t)|^2 dxdt \\
&+ \left(C^{*\frac{4q}{2q-d}} + D^{*\frac{4}{2q-d}} C^{**\frac{4(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^4 \right) \\
&\times \iint_{H_{j+1}^1} |\partial_t^{k-j} u(x,t)|^2 dxdt.
\end{aligned}$$

By iteration,

$$\begin{aligned}
&\iint_{H_1^1} |\partial_t^k u(x,t)|^2 dxdt \\
&\le \prod_{j=1}^{k} \left(\frac{C^2k^2}{t_0^2} + C^{*\frac{4q}{2q-d}} + D^{*\frac{4}{2q-d}} C^{**\frac{4(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}} \right)^4 \right) \\
&\times \iint_{H_{k+1}^1} |u(x,t)|^2 dxdt.
\end{aligned} \tag{2.40}$$

By Lemma 5.2.7 of [38] or the book [51], we see for some constant $D > 0$ and any $0 < r < 1$,

$$|B(x,r)| \le e^{D\frac{d(x,y)}{r}} |B(y,r)|. \tag{2.41}$$

As $|\partial_t^k u|^2$ is a weak subsolution to (1.2), by mean value inequality in Lemma 31, it holds

$$
\begin{aligned}
&|\partial_t^k u(x_0,t_0)|^2 \le Ce^{Dd(x_0,0)} \left(\frac{k}{t_0}\right)^{\frac{d-\nu_2}{2}} \\
&\times \left(\frac{k}{t_0} + C^{*\frac{2q}{2q-d}} + D^{*\frac{2}{2q-d}} C^{**\frac{2(q-1)}{2q-d}} \left(d(x_0,0) + \frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}}\right)^2\right)^{\frac{\nu_2+2}{2}} \\
&\times \iint_{H_1^1} |\partial_t^k u(x,t)|^2 dxdt.
\end{aligned} \tag{2.42}
$$

Combining these two inequalities (2.40) and (2.42), and applying the assumption that u is of exponential growth of order 2, we yield,

$$
|\partial_t^k u(x_0,t_0)|^2 \le \frac{A_1 A_3^{2k+2} k^{2k}}{t_0^{2k}} e^{2A_4 d^2(x_0,0)}.
$$

Just note we put some terms involving $d(x_0,0)$ into $e^{2A_4 d^2(x_0,0)}$.

The proof for the conclusions about $a_j = \partial_t^j u(x,1/2)$ is the same as Theorem 2. In this way, we have completed the proof of Theorem 3. ■

Remark 32 *To see the set of functions satisfying the condition 2.1 is nontrivial when $V(x) = x^2$ in R^d, we give some examples here. For Hermite polynomials*

$$
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2},
$$

and

$$
\psi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x),
$$

it is well-known that $(D^2 - x^2)\psi_n(x) = -(2n+1)\psi_n(x)$ and thus

$$
(D^2 - x^2)^k \psi_n(x) = (-1)^k (2n+1)^k \psi_n(x).
$$

Therefore, $\psi_n(x)$ satisfies the condition 2.1 as $|(D^2 - x^2)^k \psi_n(x)| \le C^k k!$.

Remark 33 *Theorem* (3) *is about the time analyticity when* $t \in (0, 1/2]$*. Because* (1.2) *is a linear equation, it is a natural assumption that* u *is of exponential growth of order 2 in* $t \in [0, 2]$*, a longer time interval, then the solution should be time analytic in* $[0, 1]$*.*

Especially, when $\mathrm{M} = R^d$, there is no necessity to assume $V \in L^1(R^d \ B(0, R^*))$ and $d \geq 3$, instead we have the following corollary:

Corollary 34 *Let* $\mathrm{M} = R^d$*. Assume* $V = V(x)$ *satisfies the following conditions:*

(1) There exists some $R^* > 0$ *such that* $V(\cdot) \in L^q(B(0, R^*))$ *for some* $q > \frac{d}{2}$*.*

(2) For some constant $C^{**} > 0$*, if* $d(x, 0) > R^*$*, then* $|V(x)| \leq C^{**} d(x, 0)^\alpha$ *where* $\alpha = 2 - \frac{2d}{q}$*.*

Let

$$\|V\|_{L^q(B(0,R^*))} = C^*$$

where C^* *is a positive constant and let* $u = u(x, t)$ *be a weak solution of equation* (1.2) *for any dimension* $d \geq 1$ *on* $\mathrm{M} \times [0, 1]$ *of exponential growth of order* 2*, namely*

$$|u(x, t)| \leq A_1 e^{A_2 d^2(x,0)}, \quad \forall (x, t) \in \mathrm{M} \times [0, 1],$$

where A_1 *and* A_2 *are some positive constants. Then* u *is analytic in* $t \in (0, 1/2]$ *with radius of convergence depending only on* t*,* d*,* q*,* K_0*,* A_2*,* α *and* C^**.*

Moreover, if $t \in (1/2 - \delta, 1/2]$ *for some small* $\delta > 0$*, we have*

$$u(x, t) = \sum_{j=0}^{\infty} a_j(x) \frac{(t - 1/2)^j}{j!}$$

with $(\Delta - V) a_j(x) = a_{j+1}(x)$*, and*

$$|a_j(x)| = \left|(\Delta - V)^j a_0(x)\right| \leq A_1 A_3^{j+1} j^j e^{A_4 d^2(x,0)}, \quad j = 0, 1, 2, \ldots$$

where constants $A_3 = A_3(d, q, K_0, A_2, \alpha, C^*)$ *and* $A_4 = A_4(A_2, \alpha, C^{**})$*.*

Proof. The proof is almost the same as Theorem 3. There are just two differences. The first one is to make a little change in (2.32), instead, we yield,

$$
\begin{aligned}
&C\left(\int_{B(x_0,\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}})}|V(x)|^q dx\right)^{\frac{2}{2q-d}}\iint_{H_j^2}|\phi u(x,t)|^2 dxdt\\
&\le C\left(C^{**\frac{2q}{2q-d}}\left(\frac{(j+0.5)\sqrt{t_0}}{\sqrt{k}}+d(x_0,0)\right)^2+C^{*\frac{2q}{2q-d}}\right)\iint_{H_j^2}|\phi u(x,t)|^2 dxdt.
\end{aligned}
$$

The second difference is in (2.31). Instead of Sobolev inequality, we use the Gagliardo-Nirenberg interpolation inequality and Young's inequality directly, which is

$$
\begin{aligned}
&\|u(\cdot,t)\phi\|_{L^{\frac{2q}{q-1}}(R^d)}\le C\|\nabla(u(\cdot,t)\phi)\|_{L^2(R^d)}^{\frac{d}{2q}}\|u(\cdot,t)\phi\|_{L^2(R^d)}^{1-\frac{d}{2q}}\\
&\le \epsilon\|\nabla(u(\cdot,t)\phi)\|_{L^2(R^d)}+C\epsilon^{\frac{-d}{2q-d}}\|u(\cdot,t)\phi\|_{L^2(R^d)}.
\end{aligned}
$$

The rest of the proof is exact same. ■

As a special case, when $V(x)\ge 0$, we need to prove Theorem 4 now.

Remark 35 *In Theorem 3, an interesting property is that the solution $u=u(x,t)$ can be not smooth in x at all. Actually, if $\mathrm{M}=R^d$ and $V(x)=\frac{A}{|x|^2}$ where $A\ge 0$, we have one solution $u(x,t)=|x|^{\alpha(A)}$ where $\alpha(A):=\frac{-(d-2)+\sqrt{(d-2)^2+4A}}{2}$. We can see this solution is not smooth if $\alpha(A)$ is not an integer.*

Similarly, we have a lemma about the mean value inequality using the same proof as in Lemma 31:

Lemma 36 *Assume* M *is a manifold satisfying same conditions as Theorem 4. Then for any nonnegative weak subsolution $u=u(x,t)$ to (1.2) where $V\ge 0$, for any $0<p<\infty$,*

$0 < r < R < 1$ *and* $(x_0, t_0) \in \mathrm{M} \times [-1, 0]$*, there exist some canstant* C *such that:*

$$\sup_{Q_r(x_0,t_0)} |u(x,t)|^p$$
$$\leq C \left(\frac{R^2}{|B(x_0,R)|^{\frac{2}{\nu_2}}} \right)^{\frac{1}{\theta^*-1}} \left(\frac{1}{|R-r|^2} \right)^{\frac{\theta^*}{\theta^*-1}} \iint_{Q_R(x_0,t_0)} |u(x,t)|^p dxdt,$$

where $\theta^* = 1 + \frac{2}{\nu_2}$ *and* ν_2 *is defined in Lemma 27.*

Remark 37 *As a very special example, we get the heat equation with inverse-square potential when* $V(x) = \frac{A}{d(x,0)^2}$*,*

$$\partial_t u(x,t) - \Delta u(x,t) + \frac{Au(x,t)}{d(x,0)^2} = 0, \quad \forall (x,t) \in \mathrm{M} \times [0,1].$$

It is well-konwn that this potential is a borderline one where the regularity theory differs from the standard one. For the regularity and mean value inequality of this equation in R^d*, we can refer to [44], [56] and [43]. Actually, the inverse-square potential term* $\frac{A}{|x|^2}$ *helps with it.*

2.3.2 Proof of Theorem 4

Proof. Now for any $(x_0, t_0) \in \mathrm{M} \times (0, 1]$, we introduce some regions first. For any positive integer k and any $j = 1, 2, \cdots, k$,

$H_j^1 = \left\{ (x,t) | d(x,x_0) < \frac{j\sqrt{t_0}}{\sqrt{2k}}, t \in [t_0 - \frac{jt_0}{2k}, t_0] \right\}$,

$H_j^2 = \left\{ (x,t) | d(x,x_0) < \frac{(j+0.5)\sqrt{t_0}}{\sqrt{2k}}, t \in [t_0 - \frac{(j+0.5)t_0}{2k}, t_0] \right\}$.

So immediately $H_j^1 \subset H_j^2 \subset H_{j+1}^1$.

Denote by $\psi_j^{(1)}(x,t)$ a standard smooth cut-off function supported in H_j^2 such that $\psi_j^{(1)}(x,t) = 1$ in H_j^1 and $|\partial_t \psi_j^{(1)}(x,t)| + |\nabla \psi_j^{(1)}(x,t)|^2 \leq \frac{Ck}{t_0}$ for some constant C.

We denote $\psi = \psi_j^{(1)}(x,t)$ for simplicity of notation below. Then by equation (1.2) and integration by parts,

$$\iint_{H_j^2} (\partial_t u(x,t))^2 \psi^2 dxdt \leq \frac{1}{2} \iint_{H_j^2} |\nabla u(x,t)| \partial_t \psi^2 dxdt$$
$$+ \epsilon_1 \iint_{H_j^2} (\partial_t u(x,t))^2 \psi^2 dxdt + \frac{4}{\epsilon_1} \iint_{H_j^2} |\nabla u(x,t)|^2 |\nabla \psi|^2 dxdt$$
$$+ \frac{1}{2} \iint_{H_j^2} V(x) u^2(x,t) \partial_t \psi^2 dxdt.$$

Using the assumption of ψ and taking $\epsilon_1 = \frac{1}{2}$, we yield

$$\iint_{H_j^1} |\partial_t u(x,t)|^2 dxdt \leq \frac{Ck}{t_0} \left(\iint_{H_j^2} |\nabla u(x,t)|^2 dxdt + \iint_{H_j^2} V(x) u^2(x,t) dxdt \right). \tag{2.43}$$

Define another smooth cut-off function $\psi_j^{(2)}(x,t)$ supported in H_{j+1}^1 such that $\psi_j^{(2)}(x,t) = 1$ in H_j^2. We assume for some constant C, $|\partial_t \psi_j^{(2)}(x,t)| + |\nabla \psi_j^{(2)}(x,t)|^2 \leq \frac{Ck}{t_0}$. We denote $\psi = \psi_j^{(2)}(x,t)$ for simplicity of notation below. Then by equation (1.2),

$$\iint_{H_{j+1}^1} |\nabla u(x,t)|^2 \psi^2 dxdt + \iint_{H_{j+1}^1} V(x) u^2(x,t) \psi^2 dxdt$$
$$\leq \epsilon_2 \iint_{H_{j+1}^1} |\nabla u(x,t)|^2 \psi^2 dxdt + \frac{1}{\epsilon_2} \iint_{H_{j+1}^1} |u(x,t)|^2 |\nabla \psi|^2 dxdt$$
$$+ \frac{1}{2} \iint_{H_{j+1}^1} u^2(x,t) \partial_t \psi^2 dxdt.$$

By the assumption on ψ and taking $\epsilon_2 = \frac{1}{2}$, we can see,

$$\iint_{H_j^2} |\nabla u(x,t)|^2 dxdt + \iint_{H_j^2} V(x) u^2(x,t) dxdt \leq \frac{Ck}{t_0} \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt. \tag{2.44}$$

Combine the inequalities (2.43) and (2.44), we have

$$\iint_{H_j^1} |\partial_t u(x,t)|^2 dxdt \leq \frac{C^2 k^2}{t_0^2} \iint_{H_{j+1}^1} |u(x,t)|^2 dxdt.$$

By Remark 24, $\partial_t^l u$ is also a weak solution of (1.1) for any nonnegative integer l. Thence

$$\iint_{H_1^1} \left(\partial_t^k u(x,t)\right)^2 dxdt \le \frac{C^2k^2}{t_0^2} \iint_{H_2^1} \left(\partial_t^{k-1} u(x,t)\right)^2 dxdt$$
$$\le ... \le \frac{C^{2k}k^{2k}}{t_0^{2k}} \iint_{H_{k+1}^1} u(x,t)^2 dxdt.$$

Therefore, by Lemma 36, (2.41)

$$|\partial_t^k u(x_0,t_0)|^2 \le C\left(\frac{k}{t_0}\right)^{d/2+1} e^{Dd(x_0,0)} \iint_{Q_{k^{-\frac{1}{2}}}(x_0,t_0)} |\partial_t^k u(x,t)|^2 dxdt$$
$$\le C\left(\frac{k}{t_0}\right)^{d/2+1} \left(\frac{Ck}{t_0}\right)^{2k} \iint_{H_{k+1}^1} (u(x,t))^2 \, dxdt \le \frac{A_1^2 A_5^{2k+2} k^{2k}}{t_0^{2k}} e^{4A_2 d^2(x_0,0)}.$$

The rest of the proof is the same as Theorem 2. ■

Remark 38 *To make sure the set of functions satisfying condition 2.2 is nontrivial when $V(x) = \frac{A}{d(x,0)^2}$, we give some examples here. The first one is*

$$a_0(x) = \sum_{j=1}^{\infty} \frac{|x|^{2j}}{((2j)!)^{1+s}},$$

where $s \ge 0$. Now we give a lemma explaining $a_0(x)$ satisfies condition 2.2 in R^d. We can prove the following lemma by induction.

Lemma 39 *Let the space M $= R^d$, then there are two sequences of positive number $a_{j,k}$ and $b_{j,k}$ where j,k are nonnegative integers satisfying*

$$\left(\Delta - \frac{2d}{|x|^2}\right)^k a_0(x) = \sum_{j=k+1}^{\infty} b_{j,k} |x|^{2j-2k}$$

and

$$\Delta^k a_0(x) = \sum_{j=k}^{\infty} a_{j,k} |x|^{2j-2k}.$$

Besides, we have $0 \le b_{j,k} \le a_{j,k}$, and

$$\left|\left(\Delta - \frac{2d}{|x|^2}\right)^k a_0(x)\right| \le \Delta^k a_0(x) \le C^k k! e^{4d|x|^2}.$$

Then we can have another example $a^*(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}|x|^{2j}}{((2j)!)^{1+s}}$, $s > 0$ *which also satisfies the condition* (2.2). *This is because if we let*

$$\left(\Delta - \frac{2}{|x|^2}\right)^k a^*(x) = \sum_{j=k+1}^{\infty} d_{j,k}|x|^{2j-2k},$$

then $d_{j,m+1} = (-1)^{j+1} b_{j,m+1}$ *for any nonnegative integers* j,m.

Especially, we can also prove the functions $|x|^2 cos(|x|)$ *and* $|x| sin(|x|)$ *also satisfies the condition* (2.2) *by the same method.*

We have similar corollaries as Corollary 16 and Corollary 18 using the same proof.

Corollary 40 *Let* $V = V(x)$ *be a potential function satisfying either the conditions in Theorem 4 or* $V(x) \geq 0$. *Then the Cauchy problem for the backward heat equation with potentials*

$$\begin{cases} \partial_t u(x,t) + (\Delta - V(x))u(x,t) = 0 \\ u(x,0) = a(x), \end{cases}$$

has a weak solution of exponential growth of order 2 in $\mathrm{M} \times (0, \delta)$ *for some* $\delta > 0$ *if and only if there exist some constants* A_2, A_3 *satisfying:*

$$\left|(\Delta - V(x))^j\, a(x)\right| \leq A_2^{j+1} j^j e^{A_3 d^2(x,0)}, \quad j = 0, 1, 2, \ldots$$

Corollary 41 *Let* $V = V(x)$ *satisfies the same conditions as Corollary 40 above. Then the Cauchy problem*

$$\begin{cases} \partial_t u(x,t) - (\Delta - V(x))u(x,t) = 0 \\ u(x,0) = a(x) \end{cases}$$

has a weak solution of exponential growth of order 2, which is also analytic in time in $\mathrm{M} \times [0, \delta)$ *for some* $\delta > 0$ *with a radius of convergence independent of* x *if and only if there* *exist some constants* A_2, A_3 *satisfying:*

$$\left|(\Delta - V(x))^j\, a(x)\right| \leq A_2^{j+1} j^j e^{A_3 d^2(x,0)}, \quad j = 0, 1, 2, \ldots$$

2.4 Nonlinear Heat Equations With Power Nonlinearity

This section is about some nonlinear heat equations with power nonlinearity of order p (1.3) where $p \in (0, \infty)$. There are two main theorems 5 and 6 in this section and the main tools to prove them are Lemmas 45 and 46. We first prove the case when the solution u is bounded and p is an integer. Then we turn to the case when $0 < C_3 \leq |u| \leq C_4$ and p is any rational number.

For (1.3), since we assume the solution u is bounded, by standard theory, u is actually smooth. We need a lemma about the time derivative of the heat kernel on M first.

Lemma 42 *Let* M *be the same manifold as Theorem 5 above. Then for any* $x, y \in$ M, $0 < t \leq 1$ *and any nonnegative integer* k, *there exist some constants* C_1 *and* C_5 *depending only on* M *and* d *such that the heat kernel* $\Gamma(x, t; y)$ *of the heat equation*

$$\partial_t u - \Delta u = 0,$$

satisfies the following condition:

$$|\partial_t^k \Gamma(x, t; y)| \leq \frac{C_1^{k+1} k^{k-2/3}}{t^k |B(x, \sqrt{t})|} e^{\frac{-C_5 d(x,y)^2}{t}}. \tag{2.45}$$

Remark 43 *To our best knowledge, up to now, in the literature, one just have*

$$|\partial_t^k \Gamma(x, t; y)| \leq \frac{C(k)}{t^k |B(x, \sqrt{t})|} e^{\frac{-C_5 d(x,y)^2}{t}}$$

in the manifold case, where $C(k)$ is not calculated explicitly. So here we obtain a more accurate result.

Proof of Lemma 42. Fix any $t_0 \in (0,1]$ and $x_0, y_0 \in \mathrm{M}$, we would like to get the estimates of $\partial_t^k \Gamma(x_0, t_0; y_0)$. For any nonnegative integer k and $j = 1, 2, \cdots, k$, we define some space-time domains:

$$M_j^1 = \left\{(x,t) : d(x, x_0) < \frac{j\sqrt{t_0}}{\sqrt{2k}}, t \in \left(t_0 - \frac{jt_0}{2k}, t_0\right)\right\},$$
$$M_j^2 = \left\{(x,t) : d(x, x_0) < \frac{(j+0.5)\sqrt{t_0}}{\sqrt{2k}}, t \in \left(t_0 - \frac{(j+0.5)t_0}{2k}, t_0\right)\right\}.$$

Then $M_j^1 \subset M_j^2 \subset M_{j+1}^1$.

Following the method used in the proof of Theorem 4, for some constant C, it holds

$$\iint_{M_1^1} |\partial_t^k \Gamma(x, t; y_0)|^2 dx dt \le \frac{C^{2k} k^{2k}}{t_0^{2k}} \iint_{M_{k+1}^1} |\Gamma(x, t; y_0)|^2 dx dt. \tag{2.46}$$

Then we need to use the well-known result for the upper bound of the heat kernel which can be found in [42] or [38], which is

$$\Gamma(x, t; y) \le \frac{C_3' e^{\frac{-C_4' d(x,y)^2}{t}}}{|B(x, \sqrt{t})|}, \ \forall x, y \in \mathrm{M} \text{ and } t \in (0, 1],$$

for some constants C_3' and C_4'.

Now we have two cases.

Case (1): $d(y_0, x_0) \le \sqrt{4kt_0}$.

In this case, using (2.41)

$$\frac{C^{2k} k^{2k}}{t_0^{2k}} \iint_{M_{k+1}^1} |\Gamma(x, t; y_0)|^2 dx dt$$
$$\le \frac{C^{2k+1/2} k^{2k} e^{\frac{D(k+1)^2 t_0}{2kt_0}}}{t_0^{2k-1} |B(x_0, \sqrt{t_0})|} \le \frac{C^{2k+1} k^{2k+1}}{t_0^{2k-1} |B(x_0, \sqrt{t_0})|} e^{\frac{-C_5 d(x_0, y_0)^2}{t_0}},$$

for some constant C.

Case (2): $d(y_0, x_0) > \sqrt{4kt_0}$.

In this case, because $d(x, x_0) < \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}}$, $\frac{\sqrt{2}-1}{\sqrt{2}} < \frac{d(x,y_0)}{d(x_0,y_0)} < 2$. Therefore,

$$\begin{aligned}
&\frac{C^{2k}k^{2k}}{t_0^{2k}} \iint_{M_{k+1}^1} |\Gamma(x,t;y_0)|^2 dxdt \\
&\leq \frac{C^{2k}k^{2k}t_0|B(x_0, \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}})|e^{\frac{2D(k+1)^2t_0}{2kt_0}}}{t_0^{2k}|B(x_0,\sqrt{t_0})|^2} e^{\frac{-(3-2\sqrt{2})C_4' d(x_0,y_0)^2}{2t_0}} \\
&\leq \frac{C^{2k+1/2}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{\frac{-C_5 d(x_0,y_0)^2}{t_0}} \leq \frac{C^{2k+1}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{\frac{-C_5 d(x_0,y_0)^2}{t_0}}.
\end{aligned}$$

Combine the above two cases,

$$\iint_{M_1^1} |\partial_t^k \Gamma(x,t;y_0)|^2 dxdt \leq \frac{C^{2k+1}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{\frac{-C_5 d(x_0,y)^2}{t_0}}. \tag{2.47}$$

Then we recall a well-known parabolic mean value inequality which can be found, for instance, in Theorem 14.7 of [48]. To be more precise, by the method of Lemma 31, for any $0 < p < \infty$ and $0 < r < R < 1$, any nonnegative subsolution $u = u(x,t)$ of the heat equation satisfies

$$\begin{aligned}
&\sup_{Q_r(x_0,t_0)} u(x,t)^p \\
&\leq C \left(\frac{R^2}{|B(x_0,R)|^{\frac{2}{\nu_2}}} \right)^{\frac{1}{\theta^*-1}} \left(\frac{1}{|R-r|^2} \right)^{\frac{\theta^*}{\theta^*-1}} \iint_{Q_R(x_0,t_0)} u(x,t)^p dxdt,
\end{aligned}$$

where $\theta^* = 1 + \frac{2}{\nu_2}$ and ν_2 is defined in (2.26). Let $u(x,t) = |\partial_t^k \Gamma(x,t;y_0)|^2$, $p = 1$, $r = 0$ and $R = \sqrt{t_0}/\sqrt{2k}$, we can see

$$\begin{aligned}
|\partial_t^k \Gamma(x_0,t;y_0)|^2 &\leq \frac{Ck}{\left|B\left(x_0, \sqrt{t_0}/\sqrt{2k}\right)\right| t_0} \iint_{Q_{\sqrt{t_0}/\sqrt{2k}}(x_0,t_0)} (\partial_t^k \Gamma(x,t;y_0))^2 dxdt \\
&\leq \frac{Ck^{d/2+1}}{\left|B\left(x_0,\sqrt{t_0}\right)\right| t_0} \iint_{Q_{\sqrt{t_0}/\sqrt{2k}}(x_0,t_0)} (\partial_t^k \Gamma(x,t;y_0))^2 dxdt,
\end{aligned} \tag{2.48}$$

where we have used the Bishop-Gromov volume comparison theorem in the last inequality. By (2.46),(2.47) and (2.48), we see

$$(\partial_t^k \Gamma(x_0,t_0;y_0))^2 \le \frac{C^{2k+2}k^{2k+d/2+2}}{t_0^{2k}|B(x_0,\sqrt{t_0})|^2} e^{\frac{-C_5 d(x_0,y_0)^2}{t_0}}.$$

Thus,

$$|\partial_t^k \Gamma(x_0,t_0;y_0)| \le \frac{C_1^{k+1}k^{k-2/3}}{t_0^{k}|B(x_0,\sqrt{t_0})|} e^{\frac{-C_5 d(x_0,y_0)^2}{t_0}},$$

for some C_1 large enough, which finishes the proof of Lemma 42. ∎

Remark 44 *By the estimate of the time derivative of heat kernel $\Gamma(x,t;y)$, we can see the solution $u = u(x,t)$ of heat equation $u_t - \Delta u = 0$ is analytic in time if u is of exponential growth of order 2 directly.*

Let $\binom{n}{i_1,i_2,\cdots,i_k} := \frac{n!}{i_1!i_2!\cdots(n-i_1-i_2-\cdots-i_k)!}$. Then we have a lemma which will be used frequently.

Lemma 45 *For any integers $n > 1$ and $k > 1$, there exists some constant $C = C(k)$ such that,*

$$\sum_{\Sigma_{m=1}^k i_m < n, i_m > 0} \binom{n}{i_1,i_2,\cdots,i_k} i_1^{i_1-2/3} i_2^{i_2-2/3} \times \cdots$$
$$\times (n-i_1-i_2-\cdots-i_k)^{n-i_1-i_2-\cdots-i_k-2/3} \le Cn^{n-2/3}.$$

This lemma is just an extension of the Lemma 3.2 of [19] and we can prove it by the induction method and the Stirling formula.

Proof.

$$\sum_{\Sigma_{m=1}^{k} i_m < n, i_m > 0} \binom{n}{i_1, i_2, \cdots, i_k} i_1^{i_1 - 2/3} i_2^{i_2 - 2/3} \times \cdots$$

$$\times (n - i_1 - i_2 - \cdots - i_k)^{n - i_1 - i_2 - \cdots - i_k - 2/3}$$

$$= \sum_{i_1=1}^{n} \binom{n}{i_1} i_1^{i_1 - 2/3} \sum_{\Sigma_{m=2}^{k} i_m < n - i_1, i_m > 0} \binom{n - i_1}{i_2, \cdots, i_k} i_2^{i_2 - 2/3} \times \cdots$$

$$\times (n - i_1 - i_2 - \cdots - i_k)^{n - i_1 - i_2 - \cdots - i_k - 2/3}$$

$$\leq C \sum_{i_1=1}^{n-1} \binom{n}{i_1} i_1^{i_1 - 2/3} (n - i_1)^{n - i_1 - 2/3} \leq C n^{n-2/3} \sum_{i_1=1}^{n-1} \frac{n^{7/6}}{i_1^{7/6} (n - i_1)^{7/6}}$$

$$\leq C n^{n-2/3} \sum_{i_1=1}^{n-1} \left(\frac{1}{i_1} + \frac{1}{n - i_1} \right)^{7/6} \leq C n^{n-2/3}.$$

■

Then we have the following lemma to connect $\partial_t^n (t^n u^p)$ and $\partial_t^n (t^n u)$ for any positive integer n.

Lemma 46 *Let $f_1(t)$,$f_2(t)$,$\cdots$,$f_k(t)$ be smooth functions. For any nonnegative integer n, we have*

$$\begin{aligned} &\partial_t^n (t^n f_1(t) f_2(t) \cdots f_k(t)) \\ =& \sum_{m=0}^{k-1} (-1)^m \frac{n!}{(n-m)!} \binom{k-1}{m} \sum_{i_l \geq 0} \binom{n-m}{i_1, i_2, \cdots, i_{k-1}} \\ &\partial_t^{i_1} (t^{i_1} f_1(t)) \cdots \partial_t^{i_{k-1}} (t^{i_{k-1}} f_{k-1}(t)) \partial_t^{n-m-\Sigma_{l=1}^{k-1} i_l} (t^{n-m-\Sigma_{l=1}^{k-1} i_l} f_k(t)). \end{aligned}$$

Here for $\binom{n}{i_1, i_2, \cdots, i_k}$ we always assume $\sum_{l=1}^{k} i_l \leq n$.

Proof. We can prove it by induction using Lemma 3.3 of [19].

Remark 47 *Especially, when $f_1 = f_2 = \cdots = f_k = f$, it holds*

$$\begin{aligned}
&\partial_t^n(t^n f^k(t)) \\
&= \sum_{m=0}^{k-1} (-1)^m \frac{n!}{(n-m)!} \binom{k-1}{m} \sum_{i_l \geq 0} \binom{n-m}{i_1, i_2, \cdots, i_{k-1}} \\
&\partial_t^{i_1}(t^{i_1} f(t)) \cdots \partial_t^{i_{k-1}}(t^{i_{k-1}} f(t)) \partial_t^{n-m-\Sigma_{l=1}^{k-1} i_l}(t^{n-m-\Sigma_{l=1}^{k-1} i_l} f(t)).
\end{aligned} \tag{2.49}$$

Moreover, when $f_i(t) = f(t)^{\frac{1}{k}}$ for any $i = 1, \cdots, k$, we have

$$\begin{aligned}
&k f(t)^{\frac{k-1}{k}} \partial_t^n(t^n f(t)^{\frac{1}{k}}) \\
&= \partial_t^n(t^n f(t)) - \sum_{m=1}^{k-1} (-1)^m \frac{n!}{(n-m)!} \binom{k-1}{m} \sum_{i_l \geq 0} \binom{n-m}{i_1, i_2, \cdots, i_{k-1}} \\
&\partial_t^{i_1}(t^{i_1} f(t)^{\frac{1}{k}}) \cdots \partial_t^{i_{k-1}}(t^{i_{k-1}} f(t)^{\frac{1}{k}}) \partial_t^{n-m-\Sigma_{l=1}^{k-1} i_l}(t^{n-m-\Sigma_{l=1}^{k-1} i_l} f(t)^{\frac{1}{k}}) \\
&- \sum_{\substack{n > i_l \geq 0 \\ \Sigma_{l=1}^{k-1} i_l > 0}} \binom{n}{i_1, i_2, \cdots, i_{k-1}} \\
&\partial_t^{i_1}(t^{i_1} f(t)^{\frac{1}{k}}) \cdots \partial_t^{i_{k-1}}(t^{i_{k-1}} f(t)^{\frac{1}{k}}) \partial_t^{n-\Sigma_{l=1}^{k-1} i_l}(t^{n-\Sigma_{l=1}^{k-1} i_l} f(t)^{\frac{1}{k}}).
\end{aligned} \tag{2.50}$$

We first establish the following proposition before embarking on the proof of Theorem 5.

Proposition 48 *Under the conditions of Theorem 5 above, for any integer $n \geq 1$, it holds*

$$\|\partial_t^n (t^n u(\cdot, t))\|_{L^\infty(\mathrm{M})} \leq N^{n-1/2} n^{n-2/3} \tag{2.51}$$

for some sufficiently large constant $N \geq 1$.

Proof. By induction and by lemma 2.45, there exist some constant C_1 such that for any integer $k > 1$,

$$\left\| \partial_t^k \left(t^k \Gamma(\cdot, t) \right) \right\|_{L^1(\mathrm{M})} \leq C_1^{k+1} k^{k-2/3}.$$

We shall prove the proposition inductively. As u is a solution, we have

$$u(x,t) = \int_{\mathrm{M}} \Gamma(x,t;y)u(y,0)dy + \int_0^t \int_{\mathrm{M}} \Gamma(x,t-s;y)u^p(y,s)dyds,$$

as a consequence,

$$\begin{aligned} &\partial_t^n(t^n u(x,t)) \\ &= \int_{\mathrm{M}} \partial_t^n(t^n\Gamma(x,t;y))u(y,0)dy + \partial_t^n(\int_{\mathrm{M}}\int_0^t t^n\Gamma(x,t-s;y)u^p(y,s)dyds) \\ &:= I_1 + I_2. \end{aligned} \tag{2.52}$$

It holds

$$|I_1| \le C_2 C_1^{n+1} n^{n-2/3} \le N^{n-2/3} n^{n-2/3} \tag{2.53}$$

for sufficiently large N.

To estimate I_2, similar to the inequality (3.7) from the paper [19], we yield

$$\begin{aligned} I_2 &= \sum_{k=0}^{n} \binom{n}{k} \partial_t^n \int_0^t \int_{\mathrm{M}} \left((t-s)^k \Gamma(x,t-s;y)\right)\left(s^{n-k}u^p(y,s)\right) dyds \\ &= \sum_{k=0}^{n} \binom{n}{k} \partial_t^{n-k} \int_0^t \int_{\mathrm{M}} \partial_t^k \left((t-s)^k \Gamma(x,t-s;y)\right)\left(s^{n-k}u^p(y,s)\right) dyds \\ &= \sum_{k=0}^{n} \binom{n}{k} \partial_t^{n-k} \int_0^t \int_{\mathrm{M}} \partial_s^k \left(s^k \Gamma(x,s;y)\right)\left((t-s)^{n-k}u^p(y,t-s)\right) dyds \\ &= \sum_{k=0}^{n} \binom{n}{k} \int_0^t \int_{\mathrm{M}} \partial_s^k \left(s^k \Gamma(x,s;y)\right) \partial_t^{n-k}\left((t-s)^{n-k}u^p(y,t-s)\right) dyds. \end{aligned} \tag{2.54}$$

Using Lemma 45 and equality 2.49, the it holds by induction

$$|\partial_t^n\left(t^n(u^p(x,t))\right)| \le pC_2^{p-1}\left|\partial_t^n\left(t^n u(x,t)\right)\right| + N^{n-3/4}n^{n-2/3},$$

and for $k = 1, \ldots, n-1$

$$\left|\partial_t^k\left(t^k(u^p(x,t))\right)\right| \le N^{k-1/3}k^{k-2/3}.$$

Following the similar procedure as in the paper [19], we have

$$
\begin{aligned}
|I_2| \leq & \int_0^t C_1^{n+1} n^{n-2/3} C_2^p + \sum_{k=1}^{n-1} \binom{n}{k} C_1^{k+1} k^{k-2/3} \cdot N^{n-k-1/3} (n-k)^{n-k-2/3} \\
& + C \left(p C_2^{p-1} \left\| \partial_t^n \left((t-s)^n u(\cdot, t-s) \right) \right\|_{L^\infty} + N^{n-3/4} n^{n-2/3} \right) ds \\
\leq & N^{n-2/3} n^{n-2/3} t + C p C_2^{p-1} \int_0^t \left\| \partial_s^n \left(s^n u(\cdot, s) \right) \right\|_{L^\infty} ds
\end{aligned} \tag{2.55}
$$

for sufficiently large N depending on C_1, C_2,p, d and K_0.

Combining the estimates of I_1 (2.53) and I_2 (2.55), we can get (2.51) by applying Gronwall's inequality and finish the proof of the proposition. ■

Now we begin the proof of the theorem 5.

2.4.1 Proof of Theorem 5

This part is the same as [19]. We just copy it down here for the convenience of reading.

Note that

$$
\partial_t^n \left(t^k u \right) = n \partial_t^{n-1} \left(t^{k-1} u \right) + t \partial_t^n \left(t^{k-1} u \right).
$$

Taking $k = n$, we obtain

$$
\sup_{t \in (0,1]} \left\| t \partial_t^n \left(t^{n-1} u(\cdot, t) \right) \right\|_{L^\infty(\mathrm{M})} \leq N^n (1 + 1/N) n^n.
$$

By induction,

$$
\sup_{t \in (0,1]} \left\| t^n \partial_t^n u(\cdot, t) \right\|_{L^\infty(\mathrm{M})} \leq N^n (1 + 1/N)^n n^n = (N+1)^n n^n.
$$

The theorem is proved. ■

To prove Theorem 6, we also have a proposition first using Lemmas 45 and 46.

Proposition 49 *Under the conditions of Theorem 6 above, for any integer* $n \geq 1$*, we have*

$$\left|\partial_t^n \left(t^n u(x,t)\right)\right| \leq N^{n-1/2} n^{n-2/3},$$

for some sufficiently large constant N*.*

Proof. We shall prove the proposition inductively. First, we can get equality (2.52) in the same way. Then similar to inequality (2.53), we see

$$|I_1| \leq N^{n-2/3} n^{n-2/3},$$

for sufficiently large N.

By equality (2.50) and Lemma 45, we can prove by induction, for any $k = 1, 2, \cdots, n-1$

$$\left|\partial_t^k (t^k u(x,t)^{1/q_2})\right| \leq N^{k-5/12} k^{k-2/3},$$

and

$$\left|\frac{\partial_t^n (t^n u(x,t)^{1/q_2})}{u(x,t)^{1/q_2}}\right| \leq \frac{1}{q_2}\left|\frac{\partial_t^n (t^n u(x,t))}{u(x,t)}\right| + N^{n-19/24} n^{n-2/3}.$$

To be more precise, if we assume for any $l = 1, 2, \cdots, k-1$

$$\left|\partial_t^l (t^l u(x,t)^{1/q_2})\right| \leq N^{l-5/12} l^{l-2/3},$$

then

$$kC_3^{\frac{q_2-1}{q_2}}|\partial_t^k(t^k u(x,t)^{\frac{1}{q_2}})|$$

$$\leq |\partial_t^k(t^k u(x,t))| + \sum_{m=1}^{q_2-1}\frac{k!}{(k-m)!}\binom{q_2-1}{m}\sum_{i_l\geq 0}\binom{k-m}{i_1,i_2,\cdots,i_{q_2-1}}$$

$$N^{i_1-5/12}i_1^{i_1-2/3}\cdots N^{i_{q_2-1}-5/12}i_{q_2-1}^{i_{q_2-1}-2/3}N^{k-m-\Sigma_{l=1}^{q_2-1}i_l-5/12}$$

$$\times(k-m-\Sigma_{l=1}^{q_2-1}i_l)^{k-m-\Sigma_{l=1}^{q_2-1}i_l}$$

$$+\sum_{\substack{k>i_l\geq 0\\ \Sigma_{l=1}^{q_2-1}i_l>0}}\binom{k}{i_1,i_2,\cdots,i_{q_2-1}}N^{i_1-5/12}i_1^{i_1-2/3}\cdots N^{i_{q_2-1}-5/12}i_{q_2-1}^{i_{q_2-1}-2/3}$$

$$N^{k-\Sigma_{l=1}^{q_2-1}i_l-5/12}(k-m-\Sigma_{l=1}^{q_2-1}i_l)^{k-m-\Sigma_{l=1}^{q_2-1}i_l}$$

$$\leq |\partial_t^m(t^m u(x,t))| + N^{k-1/2}.$$

Therefore by equality (2.49) and Lemma 45, we can prove by induction that for any $k=1,2,\cdots,n-1$

$$\left|\partial_t^k(t^k u(x,t)^{q_1/q_2})\right| \leq N^{k-1/3}k^{k-2/3},$$

and

$$\left|\frac{\partial_t^n(t^n u(x,t)^{q_1/q_2})}{u(x,t)^{q_1/q_2}}\right| \leq q_1\left|\frac{\partial_t^n(t^n u(x,t)^{1/q_2})}{u(x,t)^{1/q_2}}\right| + N^{n-3/4}n^{n-2/3}$$

$$\leq \frac{q_1}{q_2}\left|\frac{\partial_t^n(t^n u(x,t))}{u(x,t)}\right| + N^{n-3/4}n^{n-2/3},$$

for some constant N large enough.

Therefore by (2.54),

$$
\begin{aligned}
|I_2| \leq & \int_0^t C_4 C_1^{n+1} n^{n-2/3} \\
& + C \left(\frac{q_1}{q_2} \left| \frac{\partial_t^n ((t-s)^n u(\cdot, t-s))}{u(\cdot, t)^{1-q_1/q_2}} \right|_{L^\infty(\mathrm{M})} + C_4^{q_1/q_2} N^{n-3/4} n^{n-2/3} \right) \\
& + \sum_{k=1}^{n-1} \binom{n}{k} C_1^{k+1} k^{k-2/3} \cdot N^{n-k-1/3} (n-k)^{n-k-2/3} ds \\
\leq & N^{n-2/3} n^{n-2/3} t + C \int_0^t \| \partial_s^n (s^n u(\cdot, s)) \|_{L^\infty} ds,
\end{aligned}
$$

for sufficiently large N depending on C_1, C_3, C_4, p, d and K_0. Using the estimates of I_1, I_2 above and Gronwall's inequality, we can finish the proof of Proposition 49. ■

With this proposition at hand, we can prove the Theorem 6 immediately.

2.4.2 Proof of Theorem 6

The proof is exactly the same as the last part of the proof of Theorem 5.

Remark 50 *For the case when $0 < p < 1$, we can have a particular solution*

$$
u(x,t) = \begin{cases} \left((1-p)(t-\frac{1}{2})\right)^{\frac{1}{1-p}} & \text{when} \quad \frac{1}{2} < t < 1 \\ 0 & \text{when} \quad 0 \leq t \leq \frac{1}{2}, \end{cases}
$$

which is not analytic at $t = \frac{1}{2}$. We can use this example to say that u may not be allowed to be 0 to get the time analyticity conclusion.

Remark 51 *For the time analyticity at $t = 0$, according to the paper [39], even for some polynomial functions $f(u)$, the formal solutions for $\partial_t u(x,t) - \Delta u(x,t) = f(u)$ are not in general analytic at $t = 0$ even if the initial condition is analytic.*

Remark 52 *It is maybe true that the conclusion in Theorem 6 can be extended to all the real number p.*

Chapter 3

Time Analyticity for the Nonlocal Parabolic Equations

3.1 Main Results and Outline

The next four theorems are the main results of this chapter. The first one is a time analyticity result in the case of R^d.

Theorem 53 *(a) Let $p_\alpha(t,x;y)$ be the heat kernel of equation (1.4). Then there exists a positive constant C such that for any $t \in (0,1]$ and any nonnegative integer k,*

$$|\partial_t^k p_\alpha(t,x;y)| \leq \frac{C^{k+1} k^k}{t^{k-1}} \frac{1}{\left(t^{1/\alpha} + |x-y|\right)^{d+\alpha}}. \tag{3.1}$$

(b) Assume that $u = u(t,x)$ is a solution to (1.4) *with polynomial growth of order $\alpha - \epsilon$, i.e.,*

$$|u(t,x)| \leq C_1 \left(1 + |x|^{\alpha-\epsilon}\right), \quad \forall (t,x) \in [0,1] \times \mathrm{R}^d,\ 0 < \alpha < 2,\ \epsilon \in (0,\alpha) \tag{3.2}$$

for a positive constant C_1*. Then*

$$u(t,x) = \int_{\mathrm{R}^d} p_\alpha(t, x-y) u(0,y)\, dy$$

is the unique smooth solution with initial data $u(0,\cdot)$*. Moreover,* u *is time analytic for any* $t \in (0,1]$ *with the radius of convergence being independent of* x*.*

(c) For any $t \in (1-\delta, 1]$ *with a small* $\delta > 0$*, we have*

$$u(t,x) = \sum_{j=0}^{\infty} a_j(x) \frac{(t-1)^j}{j!},$$

where $a_0(x) = u(1,x)$*,* $a_{j+1}(x) = \mathrm{L}_\alpha^\kappa a_j(x)$*,*

$$|a_j(x)| = \left| \left(\mathrm{L}_\alpha^\kappa\right)^j a_0(x) \right| \le C_1 C_2^j j^j \left(1 + |x|^{\alpha-\epsilon}\right), \quad j = 0, 1, 2, \ldots,$$

and C_2 *is a positive constant.*

Remark 54 *The estimate* $|a_j(x)|$ *in part (c) of this theorem will be used for the solvability of the backward nonlocal parabolic equations and the time analyticity at* $t = 0$ *in the last section.*

Remark 55 *From the proof of this theorem, for a constant* $C > 0$*, we have*

$$|\partial_t^k u(t,x)| \le \frac{C^{k+1} k^k}{t^{k-1}} \Big(\frac{1 + |x|^{\alpha-\epsilon}}{t} + \frac{1}{t^{\varepsilon/\alpha}} \Big), \; \forall t \in (0,1] \tag{3.3}$$

under the growth condition (3.2)*.*

Now let us focus on the heat kernel of the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ in R^d. Recall that the fractional heat kernel $p_\alpha(t,x)$ for $u_t + (-\Delta)^{\alpha/2} u(t,x) = 0$ is given by

$$p_\alpha(t,x) = C(d,\alpha) \int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x} d\xi, \tag{3.4}$$

which can be deduced by the Fourier transform.

Theorem 56 *The following statements are true for the fractional heat kernel $p_\alpha(t, x)$ when $t \geq 0$.*

(a) For any $\alpha > 0$ and for any positive integer k, there exist positive constants C, C_1, and C_2 such that

$$|\partial_t^k p_\alpha(t,x)| \leq \min\left\{ \frac{C_1 C_2^{k\alpha} (k\alpha)^{k\alpha}}{|x|^{k\alpha+d}}, \frac{C}{t^{k+d/\alpha}} \Gamma\left(\frac{k\alpha + d}{\alpha}\right) \right\}, \tag{3.5}$$

which implies that p_α is of Gevrey class in time of order α when $x \neq 0$ and p_α is analytic in time when $t > 0$. Moreover, if $0 < \alpha \leq 1$ and $x \neq 0$, then p_α is analytic in time for all $t \geq 0$. Here Γ is the gamma function.

(b) For any $\alpha > 0$ and for any positive integer k,

$$|\partial_x^k p_\alpha(t,x)| \leq \min\left\{ \frac{C_1 C_2^{k+\alpha} (k+\alpha)^{k+\alpha} t}{|x|^{\alpha+k+d}}, \frac{C}{t^{(k+d)/\alpha}} \Gamma\left(\frac{k + d}{\alpha}\right) \right\}, \tag{3.6}$$

which implies that p_α is analytic in space at $|x| \neq 0$. Especially, when $t \neq 0$, p_α is of Gevrey class with order $1/\alpha$ in space for any x.

Part (a) of the theorem shows that for any $\alpha \in (0, 1]$, the fractional heat kernel is time analytic down to $t = 0, x \neq 0$, which is not true for the standard heat kernel.

By the above Theorem 56, we have

Corollary 57 *If the unique smooth solution $u = u(t, x)$ to the fractional heat equation* (1.8) *satisfies the growth condition* (3.2) *for some $\alpha \in [1, 2)$, then it is analytic in space for any $(t, x) \in (0, 1] \times \mathrm{R}^d$. Moreover, when $\alpha \in (0, 1)$, u is of Gevrey class of order $1/\alpha$ in space for any $(t, x) \in (0, 1] \times \mathrm{R}^d$.*

The last two theorems of the paper are in the setting of a complete Riemannian manifold M. We impose the following two standard conditions on M:

Condition (1): There exists a constant $C_0 > 0$ such that for any ball $B(x_0, r)$, $x_0 \in \mathrm{M}$, $r > 0$, and $f \in C^\infty(B(x_0, r))$,

$$\int_{B(x_0,r)} |f - f_{B(x_0,r)}|^2 \, dx \leq C_0 r^2 \int_{B(x_0,r)} |\nabla f|^2 \, dx, \tag{3.7}$$

where

$$f_{B(x_0,r)} := \frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} f \, dx.$$

Condition (2): There exists a constant $C^* > 0$ such that for any ball $B(x, r)$, $x \in \mathrm{M}$, and $r > 0$,

$$|B(x, 2r)| \leq C^* |B(x, r)|. \tag{3.8}$$

The first condition is the Poincaré inequality. The second one is the doubling property of the measure.

We aim to investigate the pointwise time analyticity of solutions to

$$\partial_t u(t, x) - \mathrm{L}^\alpha u(t, x) = 0, \ \alpha \in (0, 2), \ (t, x) \in [0, 1] \times \mathrm{M}, \tag{3.9}$$

where L^α is defined as follows. Let Δ be the Laplace operator on M generating a Markov semigroup P_t which has a density $E(t, x; y)$ i.e. the heat kernel of the standard heat equation on M. Consider the α-stable subordination of P_t,

$$P_t^\alpha := \int_0^\infty P_s \, \mu_t^\alpha(ds), \ t \geq 0,$$

where μ_t^α is a probability measure on $[0, \infty)$ with the Laplace transform

$$\int_0^\infty e^{-\lambda s} \, \mu_t^\alpha(ds) = e^{-t\lambda^\alpha}, \ \lambda \geq 0.$$

Then L^α is the infinitesimal generator of P_t^α.

In particular, we will also study the fractional heat kernel $p_\alpha(t,x;y)$ and its high order time derivatives $\partial_t^k p_\alpha(t,x;y)$.

Theorem 58 *Let* M *be a* $d-$*dimensional complete Riemannian manifold satisfying conditions* (3.7) *and* (3.8) *and* $u = u(t,x)$ *be a mild solution to equation* (3.9)*, i.e.,*

$$u(t,x) = \int_{\mathrm{M}} p_\alpha(t,x;y)u(0,y)\,dy. \tag{3.10}$$

Assume that u *is of polynomial growth of order* $(\alpha-\epsilon)$ *at* $t=0$*, i.e., for a constant* $C>0$*,*

$$|u(0,x)| \le C(1+d(x,0)^{\alpha-\epsilon}),\ \ 0<\epsilon<\alpha,\ \ x\in \mathrm{M}. \tag{3.11}$$

Then for a constant $C>0$*, it holds that*

$$|\partial_t^k u(t,x)| \le \frac{C^{k+1}k^k}{t^{k-1}}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{t}+\frac{1}{t^{\epsilon/\alpha}}\right), \forall (t,x)\in(0,\infty)\times \mathrm{M}, \tag{3.12}$$

which implies that u *is time analytic in* $(0,\infty)\times\mathrm{M}$ *with the radius of convergence independent of* x*.*

We also obtain the time analyticity of the fractional heat kernel in the manifold setting.

Theorem 59 *Let* M *be a* $d-$*dimensional complete Riemannian manifold satisfying conditions* (3.7) *and* (3.8)*. Then for any* $t\in(0,\infty)$*, there exist positive constants* C_1 *and* C_2 *such that the fractional heat kernel* $p_\alpha(t,x;y)$ *satisfies:*

$$\frac{C_1 t}{(d(x,y)^\alpha+t)|B(x,d(x,y)+t^{1/\alpha})|} \le p_\alpha(t,x;y) \le \frac{C_2 t}{(d(x,y)^\alpha+t)|B(x,d(x,y)+t^{1/\alpha})|}. \tag{3.13}$$

Moreover, for any integer $k\ge 0$*, there exists a constant* $C>0$ *such that*

$$|\partial_t^k p_\alpha(t,x;y)| \le \frac{C^{k+1}k!}{t^{k-1}}\frac{1}{(d(x,y)^\alpha+t)|B(x,d(x,y)+t^{1/\alpha})|}. \tag{3.14}$$

Here we remark that (3.13) is more or less known and our main contribution is (3.14).

Remark 60 *It is an interesting question whether the uniqueness result still holds in the manifold case under the same growth condition. In the proof of Lemma 65, we use* (1.5) *as an explicit formula for* L_α^κ *in* R^d*. However, in* M*, we do not have such a formula for* L^α *in* (3.9)*. Therefore, the proof in Lemma 65 does not work in this case.*

Now we give an outline of the rest of this paper. In Section 3.2, we investigate the pointwise time analyticity of a solution of (1.4) in the setting of R^d and prove Theorem 53. In Section 3.3, by using the Fourier transform and contour integrals, we derive some estimates of the fractional heat kernel $p_\alpha(t, x)$, which implies Theorem 56 and Corollary 57. In Section 3.4, we turn to the setting of a manifold and obtain similar results, Theorems 58 and 59. In the proof, we use the subordination relation (3.58) and the estimates for the standard heat kernel. Section 3.5 is devoted to some corollaries. One of them is about a necessary and sufficient condition for the solvability of the backward nonlocal parabolic equations. Another corollary gives a necessary and sufficient condition under which solutions to (1.4) or (3.9) are time analytic at initial time $t = 0$. Also for the nonlinear differential equation (3.85) with power nonlinearity of order p, we prove that a solution $u = u(t, x)$ is time analytic in $t \in (0, 1]$ if it is bounded in $[0, 1] \times \mathrm{M}$ and p is a positive integer.

3.2 Nonlocal Parabolic Equations in R^d

In this section, we prove Theorem 53 in the setting of R^d. First, in Subsection 3.2.1, we prove that the fractional heat kernel p_α and the mild solution $u = u(t, x)$ to (1.4), i.e. (3.10), are analytic in time. Next, we prove that u is the unique smooth solution in

Subsection 3.2.2. Finally, we finish the proof of Theorem 53 in Subsection 3.2.3. The proof is divided into several lemmas for easy reading.

3.2.1 Time Analyticity of the Fractional Heat Kernel p_α and Mild Solutions

Lemma 61 *Assume that $\kappa(\cdot,\cdot)$ satisfies (1.6) and (1.7). Then (3.1) is true. Moreover, if the mild solution*

$$u = u(t,x) = \int_{\mathrm{R}^d} p_\alpha(t,x;y)u(0,y)\,dy$$

is of polynomial growth of order $\alpha - \epsilon$ as in (3.2), then (3.3) holds.

Proof. From [11, (1.8), (1.14), and (1.10)], there exist constants C_1 and C_2 such that for any $t \in (0,1]$ and $x, y \in \mathrm{R}^d$,

$$\frac{C_1 t}{\left(t^{1/\alpha} + |x-y|\right)^{d+\alpha}} \le p_\alpha(t,x;y) \le \frac{C_2 t}{\left(t^{1/\alpha} + |x-y|\right)^{d+\alpha}} \tag{3.15}$$

and

$$|\partial_t p_\alpha(t,x;y)| \le \frac{C_2}{\left(t^{1/\alpha} + |x-y|\right)^{d+\alpha}}. \tag{3.16}$$

Thus the conclusions of the lemma are true for $k = 1$. Now we proceed by induction. For any integer $k > 1$, we assume that

$$|\partial_t^{k-1} p_\alpha(t,x;y)| \le \frac{C^k (k-1)^{k-1}}{t^{k-2}} \frac{1}{\left(t^{1/\alpha} + |x-y|\right)^{d+\alpha}}, \ t \in (0,1].$$

Without loss of generality, we may assume that $C_2 \le C^{1/2}$. Using the semigroup property and (3.16), for any $t \in (0,1]$ and $\tau \in (0,t)$, we know that

$$\partial_t^k p_\alpha(t,x;y) = \int_{\mathrm{R}^d} \partial_t p_\alpha(t-\tau,x;z)\partial_\tau^{k-1} p_\alpha(\tau,z;y)\,dz.$$

Therefore, by (3.16) and the inductive assumption, it holds that

$$|\partial_t^k p_\alpha(t,x;y)| \le \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \int_{\mathrm{R}^d} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-z|\big)^{d+\alpha}} \frac{1}{\big(\tau^{1/\alpha}+|y-z|\big)^{d+\alpha}}\,dz. \tag{3.17}$$

Then for any $t \in (0,1]$, we take $\tau = \frac{(k-1)t}{k}$.

On one hand, if $t > |x-y|^\alpha$, then we have

$$\begin{aligned}
|\partial_t^k p_\alpha(t,x;y)| &\le \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\tau^{(d+\alpha)/\alpha}} \int_{\mathrm{R}^d} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-z|\big)^{d+\alpha}}\,dz \\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\tau^{(d+\alpha)/\alpha}} \frac{1}{t-\tau} \\
&\le \frac{C^{k+7/8}k^k}{t^{k-1}} \frac{1}{t^{(d+\alpha)/\alpha}} \le \frac{C^{k+1}k^k}{t^{k-1}} \frac{1}{\big(t^{1/\alpha}+|x-y|\big)^{d+\alpha}}
\end{aligned} \tag{3.18}$$

provided that C is sufficiently large.

On the other hand, if $t < |x-y|^\alpha$, by (3.17) and

$$\mathrm{R}^d \subset \left\{z : |x-z| \ge \frac{|x-y|}{2}\right\} \cup \left\{z : |y-z| \ge \frac{|x-y|}{2}\right\},$$

we have

$$
\begin{aligned}
&|\partial_t^k p_\alpha(t,x;y)| \\
&\le \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \int_{\{z:|x-z|\ge|x-y|/2\}} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-z|\big)^{d+\alpha}} \frac{1}{\big(\tau^{1/\alpha}+|y-z|\big)^{d+\alpha}}\,dz \\
&+ \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \int_{\{z:|y-z|\ge|x-y|/2\}} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-z|\big)^{d+\alpha}} \frac{1}{\big(\tau^{1/\alpha}+|y-z|\big)^{d+\alpha}}\,dz \\
&\le \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-y|/2\big)^{d+\alpha}} \int_{\{z:|x-z|\ge|x-y|/2\}} \frac{1}{\big(\tau^{1/\alpha}+|y-z|\big)^{d+\alpha}}\,dz \\
&+ \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\big(\tau^{1/\alpha}+|x-y|/2\big)^{d+\alpha}} \int_{\{z:|y-z|\ge|x-y|/2\}} \frac{1}{\Big((t-\tau)^{1/\alpha}+|x-z|\Big)^{d+\alpha}}\,dz \\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\big((t-\tau)^{1/\alpha}+|x-y|/2\big)^{d+\alpha}} \frac{1}{\tau} \\
&+ \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}} \frac{1}{\big(\tau^{1/\alpha}+|x-y|/2\big)^{d+\alpha}} \frac{1}{t-\tau}.
\end{aligned}
\tag{3.19}
$$

Noting $\tau = \frac{(k-1)t}{k}$ and $t < |x-y|^\alpha$, by (3.19), we can see that

$$
|\partial_t^k p_\alpha(t,x;y)| \le \frac{C^{k+7/8}k^k}{t^{k-1}} \frac{1}{|x-y|^{d+\alpha}} \le \frac{C^{k+1}k^k}{t^{k-1}} \frac{1}{\big(t^{1/\alpha}+|x-y|\big)^{d+\alpha}}. \tag{3.20}
$$

The combination of (3.18) and (3.20) completes the induction and gives (3.1).

Next we prove (3.3). We claim that

$$
u(t,x) = \int_{\mathrm{R}^d} p_\alpha(t,x;y)u(0,y)\,dy, \tag{3.21}
$$

the proof of which is postponed to the next subsection. Then we have

$$
\partial_t^k u(t,x) = \int_{\mathrm{R}^d} \partial_t^k p_\alpha(t,x;y)u(0,y)\,dy.
$$

This together with (3.1) implies that

$$
\begin{aligned}
|\partial_t^k u(t,x)| &\le \int_{\mathrm{R}^d} |\partial_t^k p_\alpha(t,x;y)||u(0,y)|\,dy \\
&\le \int_{\mathrm{R}^d} \frac{C^{k+1}k^k}{t^{k-1}} \frac{1}{\big(t^{1/\alpha}+|x-y|\big)^{d+\alpha}}(1+|y|^{\alpha-\epsilon})\,dy \\
&\le \int_{\mathrm{R}^d} \frac{C^{k+1}k^k}{t^{k-1}} \frac{1}{\big(t^{1/\alpha}+|x-y|\big)^{d+\alpha}}(1+|x|^{\alpha-\epsilon}+|x-y|^{\alpha-\epsilon})\,dy \\
&\le \frac{C^{k+1}k^k}{t^{k-1}}\Big(\frac{1+|x|^{\alpha-\epsilon}}{t}+\frac{1}{t^{\varepsilon/\alpha}}\Big),
\end{aligned}
$$

i.e., u is time analytic when $t \in (0,1]$. ∎

3.2.2 Uniqueness of Solutions

In this subsection, we prove that the mild solution

$$
u(t,x) = \int_{\mathrm{R}^d} p_\alpha(t,x;y)u(0,y)\,dy
$$

in Theorem 53 is unique among smooth solutions under the growth condition (3.2). This will imply (3.21). The proof is based on Propositions 3.4 and 3.5 in [14], which we recall here for the reader's convenience. The idea is that once a solution is in C^γ with a small $\gamma \in (0,1)$, then it is in C^α with $\alpha \in [1,2)$.

The first lemma is about the case when $\alpha \in (1,2)$.

Lemma 62 (Proposition 3.4 in [14]) *Let $\omega_f(\cdot)$ be a modulus of continuity of a function $f = f(t,x)$ in $Q_{3/4}(1,x_0)$, that is*

$$
|f(t,x)-f(t',x')| \le \omega_f(\max\{|x-x'|,|t-t'|^{1/\alpha}\}),\ \forall (t,x),(t',x') \in Q_{3/4}(1,x_0),
$$

where $Q_r(t,x) = (t-r^\alpha,t)\times B_r(x)$. Assume that u is a smooth solution to

$$
u_t(t,x) - \mathrm{L}_\alpha^\kappa u(t,x) = f(t,x),\ \alpha \in (1,2),\ (t,x) \in [0,1]\times \mathrm{R}^d,
$$

and $u \in C^{\gamma}([0,1] \times \mathrm{R}^d)$ *for some* $\gamma \in (0,1)$*. Then it holds that*

$$\begin{aligned} &[u]^x_{\alpha;Q_{1/2}(1,x_0)} + [Du]^t_{(\alpha-1)/\alpha,Q_{1/2}(1,x_0)} + \|\partial_t u\|_{L^\infty(Q_{1/2}(1,x_0))} \\ &\leq C\|u\|_{\gamma/\alpha,\gamma;[0,1]\times\mathrm{R}^d} + C\sum_{k=1}^{\infty}\omega_f(2^{-k}) \end{aligned}$$

for a constant $C > 0$*. Here*

$$[u]^x_{\alpha;Q_{1/2}(1,x_0)} := \sup_{t\in(1-(1/2)^\alpha,1)} [u(t,\cdot)]_{C^\alpha(B_{1/2}(x_0))},$$

$$[Du]^t_{(\alpha-1)/\alpha,Q_{1/2}(1,x_0)} := \sup_{x\in B_{1/2}(x_0)} [Du(\cdot,x)]_{C^{(\alpha-1)/\alpha}((1-(1/2)^\alpha,1))},$$

and $\|u\|_{\gamma/\alpha,\gamma;[0,1]\times\mathrm{R}^d}$ *is the* $C^{\gamma/\alpha,\gamma}_{t,x}$ *norm in* $[0,1]\times\mathrm{R}^d$*.*

The second lemma is about the case when $\alpha = 1$.

Lemma 63 (Proposition 3.5 in [14]) *Assume that* u *is a smooth solution to*

$$u_t(t,x) - \mathrm{L}^\kappa_\alpha u(t,x) = f(t,x),\ \alpha = 1,\ (t,x) \in [0,1]\times\mathrm{R}^d,$$

and $u \in C^{\gamma}([0,1] \times \mathrm{R}^d)$ *for some* $\gamma \in (0,1)$*. Then it holds that*

$$[Du]_{L^\infty(Q_{1/2}(1,x_0))} + \|\partial_t u\|_{L^\infty(Q_{1/2}(1,x_0))} \leq C\|u\|_{\gamma,\gamma;[0,1]\times\mathrm{R}^d} + C\sum_{k=1}^{\infty}\omega_f(2^{-k})$$

for a constant $C > 0$*.*

The proof of the uniqueness starts with the following lemma.

Lemma 64 *Assume that* $\kappa(\cdot,\cdot)$ *satisfies* (1.6) *and* (1.7)*. For equation* (1.4)*, suppose that a smooth solution* $u = u(t,x)$ *is of polynomial growth of order* $\alpha - \epsilon$*, i.e.,*

$$|u(t,x)| \leq C_1\left(1 + |x|^{\alpha-\epsilon}\right), \forall (t,x) \in [0,1]\times\mathrm{R}^d,\ \alpha \in [1,2),\ \epsilon \in (0,\alpha). \tag{3.22}$$

Then for a constant $C > 0$ and for any $x_0 \in R^d$, it holds that

$$[u]^x_{1;Q_{1/2}(1,x_0)} \leq C\left(1 + |x_0|^{\alpha-\epsilon}\right), \ \epsilon > 0, \tag{3.23}$$

where

$$[u]^x_{1;Q_{1/2}(1,x_0)} := \sup_{t\in(1-(1/2)^\alpha,1)} \|u(t,\cdot)\|_{Lip(B_{1/2}(x_0))}$$

and Lip means the Lipschitz norm.

Proof. From Proposition 2.4 of [18] or Theorem 7.1 of [52], there is a small constant $\gamma \in (0,1)$ such that

$$[u]_{\gamma/\alpha,\gamma;Q_{7/8}(1,0)} \leq C\|u\|_{L^\infty((0,1);L_1(\omega_\alpha))}, \tag{3.24}$$

where $\omega_\alpha = \frac{1}{1+|x|^{d+\alpha}}$ and

$$\|u\|_{L^\infty((0,1);L_1(\omega_\alpha))} = \sup_{t\in(0,1)} \int_{\mathrm{R}^d} \frac{|u(t,x)|}{1+|x|^{d+\alpha}}\,dx.$$

By (3.24), the growth condition (3.22), and the space translation $x \to x+x_0$ for any $x_0 \in \mathrm{R}^d$, we have

$$\begin{aligned}[u]_{\gamma/\alpha,\gamma;Q_{7/8}(1,x_0)} &\leq C \sup_{t\in(0,1)} \int_{\mathrm{R}^d} \frac{|u(t,x+x_0)|}{1+|x|^{d+\alpha}}\,dx \\ &\leq C\int_{\mathrm{R}^d} \frac{(1+|x|^{\alpha-\epsilon}+|x_0|^{\alpha-\epsilon})}{1+|x|^{d+\alpha}}\,dx \leq C(1+|x_0|^{\alpha-\epsilon}).\end{aligned} \tag{3.25}$$

The next step is to prove

$$[u]^x_{\alpha;Q_{5/8}(1,x_0)} \leq C(1+|x_0|)^{\alpha-\epsilon}. \tag{3.26}$$

We modify the proof of Theorem 1.1 of [14].

Take a cut-off function $\eta = \eta(t,x) \in C_0^\infty(Q_{7/8}(1,x_0))$ satisfying $\eta = 1$ in $Q_{5/6}(1,x_0)$ and $\|\partial_t^j D^i \eta\|_{L^\infty} \leq C$ when $i \in \{0,1,2\}$ and $j \in \{0,1\}$.

Let $(t,x),(t',x')$ be two points in $Q_{3/4}(1,x_0)$ and let $v(t,x):=u(t,x)\eta(t,x)$. Then in $Q_{3/4}(1,x_0)$,

$$\partial_t v=\eta\partial_t u+\partial_t\eta u=\eta L^{\kappa}_{\alpha}u+\partial_t\eta u=L^{\kappa}_{\alpha}v+h+\partial_t\eta u, \tag{3.27}$$

where

$$h=\eta L^{\kappa}_{\alpha}u-L^{\kappa}_{\alpha}v=p.v.\int_{R^d}\frac{\xi(t,x,y)\kappa(x,y)}{|y|^{d+\alpha}}\,dy$$

and

$$\xi(t,x,y)=u(t,x+y)(\eta(t,x)-\eta(t,x+y)). \tag{3.28}$$

We are going to apply Lemma 62 or Lemma 63 to (3.27) in $Q_{3/4}(1,x_0)$ and obtain corresponding estimates (3.26) in $Q_{5/8}(1,x_0)$. To this end, we only need to estimate the Hölder semi-norm of h in $Q_{3/4}(1,x_0)$.

First, when $|y|\le 5/6-3/4=1/12$, by (3.28), we have

$$\xi(t,x,y)=\xi(t',x',y)=0. \tag{3.29}$$

By the assumptions on η and (3.28), it holds that

$$|\xi(t',x',y)|\le\begin{cases}C|u(t',x'+y)|, & |y|\ge 1\\ C|u(t',x'+y)||y|, & 1/12<|y|<1.\end{cases} \tag{3.30}$$

Now by the triangle inequality, we deduce that

$$\begin{aligned}&|h(t,x)-h(t',x')|\\&\le\underbrace{\int_{R^d}\frac{|(\xi(t,x,y)-\xi(t',x',y))\kappa(x,y)|}{|y|^{d+\alpha}}\,dy}_{I}\\&\quad+\underbrace{\int_{R^d}\frac{|\xi(t',x',y)(\kappa(x',y)-\kappa(x,y))|}{|y|^{d+\alpha}}\,dy}_{II}.\end{aligned} \tag{3.31}$$

By using (1.7), (3.22), (3.29), and (3.30), we have

$$\begin{aligned}
II &\le \int_{|y|\in(1/12,1)} \frac{C|u(t',x'+y)||y|\kappa_2|x-x'|^\beta}{|y|^{d+\alpha}}\,dy + \int_{|y|>1} \frac{C|u(t',x'+y)|}{|y|^{d+\alpha}}\kappa_2|x-x'|^\beta\,dy \\
&\le \int_{|y|\in(1/12,1)} \frac{C(1+|x_0|^{\alpha-\epsilon}+|y|^{\alpha-\epsilon})|x-x'|^\beta}{|y|^{d+\alpha-1}}\,dy \\
&\quad + \int_{|y|>1} \frac{C(1+|x_0|^{\alpha-\epsilon}+|y|^{\alpha-\epsilon})}{|y|^{d+\alpha}}|x-x'|^\beta\,dy \le C(1+|x_0|^{\alpha-\epsilon})|x-x'|^\beta.
\end{aligned} \tag{3.32}$$

Now we estimate I. When $1/12 \le |y| < 2$, by the fundamental theorem of calculus, we have

$$\xi(t,x,y)-\xi(t',x',y) = -y\int_0^1 \big(u(t,x+y)D\eta(t,x+sy) - u(t',x'+y)D\eta(t',x'+sy)\big)\,ds.$$

Therefore, by (3.22), (3.25), and the triangle inequality, it holds that

$$\begin{aligned}
&\left|\xi(t,x,y)-\xi\left(t',x',y\right)\right| \\
&\le |y|\int_0^1 \left|u(t,x+y)-u(t',x'+y)\right|\left|D\eta(t',x'+sy)\right|\,ds \\
&\quad + |y|\int_0^1 |u(t,x+y)|\left|D\eta(t,x+sy)-D\eta(t',x'+sy)\right|\,ds \\
&\le C|y|\left|u(t,x+y)-u(t',x'+y)\right| + C|y||u(t,x+y)|\left(|x-x'|+|t-t'|\right) \\
&\le C|y|(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|^\gamma+|t-t'|^{\gamma/\alpha}\right) + C|y|(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|+|t-t'|\right).
\end{aligned} \tag{3.33}$$

When $|y| \ge 2$, by (3.28) and (3.25), we have

$$\begin{aligned}
&\left|\xi(t,x,y)-\xi\left(t',x',y\right)\right| = \left|u(t,x+y)-u\left(t',x'+y\right)\right| \\
&\le C(1+|x_0|^{\alpha-\epsilon}+|y|^{\alpha-\epsilon})\left(|x-x'|^\gamma+|t-t'|^{\gamma/\alpha}\right).
\end{aligned} \tag{3.34}$$

Thus, by (1.6), (3.33), (3.34), and (3.29), we infer that

$$\begin{aligned} I &\leq \int_{|y|\in(1/12,2)} \frac{C|y|(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|^\gamma+|t-t'|^{\gamma/\alpha}\right)}{|y|^{d+\alpha}}\,dy \\ &\quad+\int_{|y|\in(1/12,2)} \frac{C|y|(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|+|t-t'|\right)}{|y|^{d+\alpha}}\,dy \\ &\quad+\int_{|y|>2} \frac{C(1+|x_0|^{\alpha-\epsilon}+|y|^{\alpha-\epsilon})\left(|x-x'|^\gamma+|t-t'|^{\gamma/\alpha}\right)}{|y|^{d+\alpha}}\,dy \\ &\leq C(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|^\gamma+|t-t'|^{\gamma/\alpha}\right). \end{aligned} \tag{3.35}$$

Plugging (3.32) and (3.35) into (3.31), we deduce that

$$|h(t,x)-h(t',x')| \leq C(1+|x_0|^{\alpha-\epsilon})\left(|x-x'|^{\gamma'}+|t-t'|^{\gamma'/\alpha}\right),$$

where $\gamma'=\min\{\gamma,\beta\}$, which implies that we can take the modulus of continuity as

$$\omega_h(r)=C(1+|x_0|^{\alpha-\epsilon})r^{\gamma'}$$

for any $r\in(0,1)$. According to Lemma 62, it follows that

$$\sum_{k=1}^{\infty}\omega_h\left(\frac{3}{2^{k+1}}\right) \leq \sum_{k=1}^{\infty} C(1+|x_0|^{\alpha-\epsilon})\left(\frac{3}{2^{k+1}}\right)^{\gamma'} \leq C(1+|x_0|^{\alpha-\epsilon}). \tag{3.36}$$

Now we consider two cases.

Case (1): $\alpha\in(1,2)$. In this case, we apply Lemma 62 to (3.27) in $Q_{3/4}(1,x_0)$ with a scaling argument. From (3.25) and (3.36), we have

$$\begin{aligned} [v]^x_{\alpha;Q_{5/8}(1,x_0)} &\leq C\|v\|_{L^\infty([0,1]\times\mathrm{R}^d)}+C[v]_{\gamma/\alpha,\gamma;[0,1]\times\mathrm{R}^d}+C\sum_{k=1}^{\infty}\omega_h\left(\frac{3}{2^{k+1}}\right) \\ &\leq C\|u\|_{L^\infty(Q_{7/8}(1,x_0))}+C[u]_{\gamma/\alpha,\gamma;Q_{7/8}(1,x_0)}+C(1+|x_0|^{\alpha-\epsilon}) \leq C(1+|x_0|^{\alpha-\epsilon}), \end{aligned}$$

by noting that $v=0$ outside of $Q_{7/8}(1,x_0)$. Because $\eta=1$ in $Q_{5/8}(1,x_0)$, we get (3.26) immediately.

Case (2): $\alpha = 1$. In this case, we apply Lemma 63 with a scaling argument. Using (3.25) and (3.36), we have

$$\|Dv\|_{L^\infty(Q_{5/8}(1,x_0))} \le C\|v\|_{L^\infty([0,1]\times \mathrm{R}^d)} + C[v]_{\gamma,\gamma;[0,1]\times\mathrm{R}^d} + C\sum_{k=1}^{\infty}\omega_h\left(\frac{3}{2^{k+1}}\right)$$

$$\le C\|u\|_{L^\infty(Q_{7/8}(1,x_0))} + C[u]_{\gamma,\gamma;Q_{7/8}(1,x_0)} + C(1+|x_0|^{\alpha-\epsilon}) \le C(1+|x_0|^{\alpha-\epsilon}),$$

which implies (3.26) again.

Finally, by the interpolation inequality, (3.26), and (3.22), we arrive at

$$[u]^x_{1;Q_{1/2}(1,x_0)} \le C[u]^x_{\alpha;Q_{5/8}(1,x_0)} + C\|u\|_{L^\infty(Q_{5/8}(1,x_0))} \le C(1+|x_0|)^{\alpha-\epsilon},$$

which finishes the proof. ■

Now we are ready to prove the uniqueness part of the theorem, which is stated as follows.

Lemma 65 *Assume that $\kappa(\cdot,\cdot)$ satisfies* (1.6) *and* (1.7)*. Then there is an unique smooth solution $u = u(t,x)$ to* (1.4) *satisfying the initial data $u(0,\cdot)$ and the polynomial growth condition* (3.2)*, which is given by*

$$u(t,x) = \int_{\mathrm{R}^d} p_\alpha(t,x;y)u(0,y)\,dy,\ \forall (t,x)\in(0,1]\times\mathrm{R}^d.$$

Proof. By linearity, we just need to prove that if a smooth solution u satisfies (3.2) and $u(0,x) = 0$, then $u \equiv 0$.

Fix $(t_0,x_0)\in(0,1]\times\mathrm{R}^d$. By shifting the coordinates, we may assume $x_0 = 0$ and it suffices to prove $u(t_0,0) = 0$. Now let $L^* = (L^\kappa_\alpha)^*$ be the adjoint operator of L^κ_α and let $p^*_\alpha(t,x;s,y)$ be the heat kernel of L^*, which by definition, satisfies

$$\begin{cases} \partial_t p^*_\alpha(t,x;s,y) - L^* p^*_\alpha(t,x;s,y) = 0,\ t > s \text{ and } x,y\in\mathrm{R}^d \\ p^*_\alpha(s,x;s,y) = \delta(x,y). \end{cases} \tag{3.37}$$

Because the heat kernels of L_α^κ and L^* are independent of time, we have

$$p_\alpha(t,x;s,y) = p_\alpha(t-s,x;0,y), \quad p_\alpha^*(t,x;s,y) = p_\alpha^*(t-s,x;0,y). \tag{3.38}$$

It is also known that

$$p_\alpha(t,x;s,y) = p_\alpha^*(t,y;s,x),\ t \geq s, \tag{3.39}$$

which can be seen as follows. For any $t_0, s_0 \in (0,1)$ with $s_0 \leq t_0$, using (3.37) and (3.38), we have

$$\begin{aligned}
&\int_{s_0}^{t_0}\int_{\mathrm{R}^d} L_\alpha^\kappa p_\alpha(t,z;s_0,y)p_\alpha^*(t_0,z;t,x)\,dzdt \\
&= \int_{s_0}^{t_0}\int_{\mathrm{R}^d} L_\alpha^\kappa p_\alpha(t-s_0,z;0,y)p_\alpha^*(t_0-t,z;0,x)\,dzdt \\
&= \int_{s_0}^{t_0}\int_{\mathrm{R}^d} \partial_t p_\alpha(t-s_0,z;0,y)p_\alpha^*(t_0-t,z;0,x)\,dzdt \\
&= p_\alpha(t_0-s_0,x;0,y) - p_\alpha^*(t_0-s_0,y;0,x) + \int_{s_0}^{t_0} p_\alpha(t-s_0,z;0,y)\partial_t p_\alpha^*(t_0-t,z;0,x)\,dzdt.
\end{aligned}$$

By the definition of the adjoint operator, (3.37), and (3.38), we reach (3.39). The integrations above are justified due to known decay estimates of p_α and p_α^*.

Then we take a cut-off function $\eta = \eta(x) \in C_c^\infty(B_2(0))$ such that for a constant C,

$$\eta = 1 \text{ in } B_1(0) \quad \text{and} \quad |D\eta| + |D^2\eta| \leq C. \tag{3.40}$$

We test (1.4) with $p_\alpha^*(t_0-t,x;0,0)\eta(x/R)$ and use (3.37) to get that

$$\begin{aligned}
0 &= \int_0^{t_0}\int_{\mathrm{R}^d} u_t(t,x)p_\alpha^*(t_0-t,x;0,0)\eta(x/R)\,dxdt \\
&\quad - \int_0^{t_0}\int_{\mathrm{R}^d} L_\alpha^\kappa u(t,x)p_\alpha^*(t_0-t,x;0,0)\eta(x/R)\,dxdt \\
&= u(t_0,0) + \int_0^{t_0}\int_{\mathrm{R}^d} u(t,x)(\partial_t p_\alpha^*)(t_0-t,x;0,0)\eta(x/R)\,dxdt \\
&\quad - \int_0^{t_0}\int_{\mathrm{R}^d} L_\alpha^\kappa u(t,x)p_\alpha^*(t_0-t,x;0,0)\eta(x/R)\,dxdt.
\end{aligned}$$

Therefore, using (3.37) and the definition of the adjoint operator, we infer that

$$
\begin{aligned}
&u(t_0,0) = \\
&\int_0^{t_0}\int_{\mathrm{R}^d} L_\alpha^\kappa(u(t,x))(p_\alpha^*(t_0-t,x;0,0)\eta(x/R)) - p_\alpha^*(t_0-t,x;0,0)L_\alpha^\kappa\,(u(t,x)\eta(x/R))\,dxdt \\
&= p.v. \int_0^{t_0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} \frac{u(t,x+z)p_\alpha^*(t_0-t,x;0,0)\,(\eta(x/R)-\eta((x+z)/R))\,\kappa(x,z)}{|z|^{d+\alpha}}\,dzdxdt \\
&= p.v. \underbrace{\int_0^{t_0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d} \frac{u(t,y)p_\alpha^*(t_0-t,x;0,0)(\eta(x/R)-\eta(y/R))\kappa(x,y-x)}{|x-y|^{d+\alpha}}\,dydxdt}_{J_1},
\end{aligned}
\tag{3.41}
$$

where we took $z = y - x$ in the last step. In the sequel, we omit $p.v.$ when there is no confusion.

Next, we aim to show that $J_1 \to 0$ as $R \to \infty$, treating the cases $\alpha < 1$ and $\alpha \geq 1$ separately.

Case (1): $\boldsymbol{\alpha < 1}$. This case is simpler since the singularity in the integrand is weaker. Using (3.2), (1.6), (3.39), and (3.40), we have

$$
\begin{aligned}
J_1 &= \int_0^{t_0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d\setminus B_R(x)} \frac{u(t,y)p_\alpha^*(t_0-t,x;0,0)(\eta(x/R)-\eta(y/R))\kappa(x,y-x)}{|x-y|^{d+\alpha}}\,dydxdt \\
&\quad + \int_0^{t_0}\int_{\mathrm{R}^d}\int_{B_R(x)} \frac{u(t,y)p_\alpha^*(t_0-t,x;0,0)(\eta(x/R)-\eta(y/R))\kappa(x,y-x)}{|x-y|^{d+\alpha}}\,dydxdt \\
&\leq C\int_0^{t_0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d\setminus B_R(x)} \frac{p_\alpha(t_0-t,0;0,x)}{|x-y|^{d+\alpha}}(1+|y|^{\alpha-\epsilon})\,dydxdt \\
&\quad + \frac{C}{R}\int_0^{t_0}\int_{\mathrm{R}^d}\int_{B_R(x)} \frac{p_\alpha(t_0-t,0;0,x)}{|x-y|^{d+\alpha-1}}(1+|y|^{\alpha-\epsilon})\,dydxdt \\
&\leq C\int_0^{t_0}\int_{\mathrm{R}^d}\int_{\mathrm{R}^d\setminus B_R(x)} \frac{p_\alpha(t_0-t,0;0,x)}{|x-y|^{d+\alpha}}(1+|x|^{\alpha-\epsilon}+|x-y|^{\alpha-\epsilon})\,dydxdt \\
&\quad + \frac{C}{R}\int_0^{t_0}\int_{\mathrm{R}^d}\int_{B_R(x)} \frac{p_\alpha(t_0-t,0;0,x)}{|x-y|^{d+\alpha-1}}(1+|x|^{\alpha-\epsilon}+|x-y|^{\alpha-\epsilon})\,dydxdt \\
&\leq C\int_0^{t_0}\int_{\mathrm{R}^d} p_\alpha(t_0-t,0;0,x)\left(\frac{1}{R^\epsilon}+\frac{1+|x|^{\alpha-\epsilon}}{R^\alpha}\right)dxdt \to 0 \text{ as } R\to\infty,
\end{aligned}
$$

where for the last step, we used (3.15) and

$$\begin{aligned}&\int_{\mathbb{R}^d} p_\alpha(t_0-t,0;0,x)(1+|x|^{\alpha-\epsilon})\,dx\\&\le \int_{\mathbb{R}^d} \frac{C(t_0-t)}{\left((t_0-t)^{1/\alpha}+|x|\right)^{d+\alpha}}(1+|x|^{\alpha-\epsilon})\,dx \le C\left(1+(t_0-t)^{1-\epsilon/\alpha}\right).\end{aligned} \tag{3.42}$$

Case (2): $\alpha \ge 1$. In this case, by the substitution $z \to -z$ in the second line of (3.41), we have

$$J_1 = \int_0^{t_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{u(t,x-z)p_\alpha^*(t_0-t,x;0,0)\,(\eta(x/R)-\eta((x-z)/R))\,\kappa(x,z)}{|z|^{d+\alpha}}\,dzdxdt.$$

where we used $\kappa(x,z)=\kappa(x,-z)$ in the last equation. Then by

$$\begin{aligned}&u(t,x+z)\left(\eta\left(\frac{x}{R}\right)-\eta\left(\frac{x+z}{R}\right)\right)+u(t,x-z)\left(\eta\left(\frac{x}{R}\right)-\eta\left(\frac{x-z}{R}\right)\right)\\&=(u(t,x-z)-u(t,x+z))\left(\eta\left(\frac{x}{R}\right)-\eta\left(\frac{x-z}{R}\right)\right)\\&\quad-u(t,x+z)\left(\eta\left(\frac{x+z}{R}\right)-2\eta\left(\frac{x}{R}\right)+\eta\left(\frac{x-z}{R}\right)\right),\end{aligned}$$

we can write

$$\begin{aligned}J_1 =&\\ &\underbrace{\frac{1}{2}\int_0^{t_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{(u(t,x-z)-u(t,x+z))\left(\eta(\frac{x}{R})-\eta(\frac{x-z}{R})\right)\kappa(x,z)p_\alpha^*(t_0-t,x;0,0)}{|z|^{d+\alpha}}\,dzdxdt}_{J_2}\\&+\underbrace{\frac{1}{2}\int_0^{t_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{-u(t,x+z)\left(\eta(\frac{x+z}{R})-2\eta(\frac{x}{R})+\eta(\frac{x-z}{R})\right)\kappa(x,z)p_\alpha^*(t_0-t,x;0,0)}{|z|^{d+\alpha}}\,dzdxdt}_{J_3}.\end{aligned}$$

For the term J_3, by (3.2), (3.39), and (3.40), we deduce

$$\begin{aligned}|J_3| &\le C\int_0^{t_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B_R(0)}\frac{p_\alpha(t_0-t,0;0,x)}{|z|^{d+\alpha}}(1+|x|^{\alpha-\epsilon}+|z|^{\alpha-\epsilon})\,dzdxdt\\&\quad+\frac{C}{R^2}\int_0^{t_0}\int_{\mathbb{R}^d}\int_{B_R(0)}\frac{p_\alpha(t_0-t,0;0,x)}{|z|^{d+\alpha-2}}(1+|x|^{\alpha-\epsilon}+|z|^{\alpha-\epsilon})\,dzdxdt\\&\le C\int_0^{t_0}\int_{\mathbb{R}^d}p_\alpha(t_0-t,0;0,x)\left(\frac{1}{R^\epsilon}+\frac{1+|x|^{\alpha-\epsilon}}{R^\alpha}\right)dxdt \to 0 \text{ as } R\to\infty,\end{aligned}$$

where we used (3.42) in the last step.

Finally, we estimate J_2. When $\alpha > 1$, by (3.2), (3.23), and (3.42), we have

$$\begin{aligned}
|J_2| \le\ & C \int_0^{t_0} \int_{\mathrm{R}^d} \int_{\mathrm{R}^d \setminus B_R(0)} \frac{p_\alpha(t_0 - t, 0; 0, x)}{|z|^{d+\alpha}} (1 + |x|^{\alpha-\epsilon} + |z|^{\alpha-\epsilon})\, dz dx dt \\
& + \frac{C}{R^2} \int_0^{t_0} \int_{\mathrm{R}^d} \int_{B_R(0)} \frac{p_\alpha(t_0 - t, 0; 0, x)}{|z|^{d+\alpha-2}} (1 + |x|^{\alpha-\epsilon})\, dz dx dt \\
& + \underbrace{\frac{C}{R} \int_0^{t_0} \int_{\mathrm{R}^d} \int_{B_R(0) \setminus B_R(0)} \frac{p_\alpha(t_0 - t, 0; 0, x)}{|z|^{d+\alpha-1}} (1 + |x|^{\alpha-\epsilon} + |z|^{\alpha-\epsilon})\, dz dx dt}_{J_4} \\
\le\ & C \int_0^{t_0} \int_{\mathrm{R}^d} p_\alpha(t_0 - t, 0; 0, x) \left(\frac{1}{R^\epsilon} + \frac{1 + |x|^{\alpha-\epsilon}}{R^\alpha} \right) dx dt \\
& + \frac{C}{R} \int_0^{t_0} \int_{\mathrm{R}^d} p_\alpha(t_0 - t, 0; 0, x) \big((1 - R^{1-\alpha})(1 + |x|^{\alpha-\epsilon}) + (R^{1-\epsilon} - 1) \big)\, dx dt \\
\to\ & 0 \quad \text{as } R \to \infty.
\end{aligned}$$

When $\alpha = 1$, we only need to estimate J_4 slightly differently. By (3.42),

$$J_4 \le \frac{C}{R} \int_0^{t_0} \int_{\mathrm{R}^d} p_1(t_0 - t, 0; 0, x) \big(\ln(R)(1 + |x|^{1-\epsilon}) + (R^{1-\epsilon} - 1) \big)\, dx \to 0 \quad \text{as } R \to \infty.$$

Combining these two cases and plugging into (3.41), we get $u(t_0, 0) = 0$, which finishes the proof. ■

3.2.3 Completion of Proof of Theorem 53

Proof. We have proved part (a) and (b) of Theorem 53 in Lemmas 61 and 65. Thus it remains to show part (c). First we fix a number $R \ge 1$ and let $x \in B_R(0)$, $t \in [1-\delta, 1]$ for some small $\delta > 0$. For any positive integer j, Taylor's theorem implies that

$$u(t, x) - \sum_{i=0}^{j-1} \partial_t^i u(1, x) \frac{(t-1)^i}{i!} = \frac{(t-1)^j}{j!} \partial_t^j u(s, x), \tag{3.43}$$

where $s = s(x,t,j) \in [t,1]$. By (3.3), for sufficiently small $\delta > 0$, the right-hand side of (3.43) converges to 0 uniformly with respect to $x \in B_R(0)$ as $j \to \infty$. Hence,

$$u(t,x) = \sum_{j=0}^{\infty} \partial_t^j u(1,x) \frac{(t-1)^j}{j!}$$

i.e., u is analytic in time with radius δ. Denote $a_j = a_j(x) = \partial_t^j u(1,x)$. By (3.3) again, we have

$$\partial_t u(t,x) = \sum_{j=0}^{\infty} a_{j+1}(x) \frac{(t-1)^j}{j!} \quad \text{and} \quad \mathrm{L}_\alpha^\kappa u(t,x) = \sum_{j=0}^{\infty} \mathrm{L}_\alpha^\kappa a_j(x) \frac{(t-1)^j}{j!},$$

where both series converge uniformly with respect to $(t,x) \in [1-\delta, 1] \times B_R(0)$. Since u is a solution of (1.4), this implies that $\mathrm{L}_\alpha^\kappa a_j(x) = a_{j+1}(x)$ with

$$|a_j(x)| \le C^{j+1} j^j (1 + |x|^{\alpha-\epsilon}).$$

This completes the proof of Theorem 53. ■

3.3 Fractional Heat Kernel Estimates on R^d

In this section, we estimate the time and space derivatives of the fractional heat kernel $p_\alpha(t,x)$ for (1.8). The main tools are the Fourier transform and contour integrals. We first state and prove the following lemma, which is needed for the proof of Theorem 56 and Corollary 57.

Lemma 66 *(a) If $\alpha > 0$, $\beta \ge 0$, and $t \ge 0$, there exist constants C, C_1, and C_2 such that*

$$\Big| \int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x} |\xi|^\beta d\xi \Big| \le \min\left\{ \frac{C_1 C_2^\beta \beta^\beta}{|x|^{\beta+d}}, \frac{C}{t^{(\beta+d)/\alpha}} \Gamma\left(\frac{\beta+d}{\alpha}\right) \right\}, \tag{3.44}$$

where Γ is the gamma function.

(b) Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_d)$ where β_j is a nonnegative integer with $j \in \{1, 2, \ldots, d\}$, then we have

$$\Big|\int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x} \xi^{\boldsymbol{\beta}} d\xi\Big| \le \min\left\{\frac{C_1 C_2^{\alpha+|\boldsymbol{\beta}|}(\alpha + |\boldsymbol{\beta}|)^{\alpha+|\boldsymbol{\beta}|} t}{|x|^{\alpha+|\boldsymbol{\beta}|+d}}, \frac{C}{t^{(|\boldsymbol{\beta}|+d)/\alpha}}\Gamma\left(\frac{|\boldsymbol{\beta}|+d}{\alpha}\right)\right\}, \quad (3.45)$$

where $\xi^{\boldsymbol{\beta}} = \xi_1^{\beta_1}\xi_2^{\beta_2}\cdots\xi_d^{\beta_d}$ and $|\boldsymbol{\beta}| := \sum_{k=1}^{d} \beta_k$.

Remark 67 *When $t = 0$, the integrals in (3.44) and (3.45) can be understood as the limit as $t \searrow 0$.*

Proof of Lemma 66. The bound $\frac{C}{t^{(\beta+d)/\alpha}}\Gamma\left(\frac{\beta+d}{\alpha}\right)$ on the right-hand side of (3.44) is easily obtained as follows

$$\Big|\int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x} |\xi|^\beta d\xi\Big| \le \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha}|\xi|^\beta d\xi = \frac{C}{t^{(\beta+d)/\alpha}}\Gamma\left(\frac{\beta+d}{\alpha}\right).$$

Similarly, the bound $\frac{C}{t^{(|\boldsymbol{\beta}|+d)/\alpha}}\Gamma\left(\frac{|\boldsymbol{\beta}|+d}{\alpha}\right)$ on the right-hand side of (3.45) holds because

$$\Big|\int_{\mathbb{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x} \xi^{\boldsymbol{\beta}} d\xi\Big| \le \int_{\mathbb{R}^d} e^{-t|\xi|^\alpha}|\xi|^{|\boldsymbol{\beta}|} d\xi = \frac{C}{t^{(|\boldsymbol{\beta}|+d)/\alpha}}\Gamma\left(\frac{|\boldsymbol{\beta}|+d}{\alpha}\right).$$

We shall use the technique of contour integrals to obtain the first bounds in (3.44) and (3.45), respectively. To simplify the calculation, without loss of generality, by rotating the coordinates, we assume that $x = (\frac{|x|}{\sqrt{d}}, \frac{|x|}{\sqrt{d}}, \ldots, \frac{|x|}{\sqrt{d}})$.

For any point $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ and for any $j \in \{1, 2, \ldots, d\}$, we consider ξ_j as a complex number with modulus η_j and argument (angle) ψ_j. For a large $R > 0$ and $\phi := \min\{\pi/16, \pi/(16\alpha)\}$, consider the regions in the complex plane:

$$\Gamma_R^{(1)} = \left\{\eta_0 e^{i\psi} \big|\ \eta_0 \in (0, R), \psi \in [0, \phi]\right\},$$

$$\Gamma_R^{(2)} = \left\{\eta_0 e^{i\psi} \big|\ \eta_0 \in (0, R), \psi \in [\pi - \phi, \pi]\right\},$$

and denote

$$C_R^{(1)} = \left\{ Re^{i\psi} \middle|\ \psi \in [0,\phi] \right\} \text{ and } C_R^{(2)} = \left\{ Re^{i\psi} \middle|\ \psi \in [\pi-\phi,\pi] \right\}.$$

We calculate the contour integrals of the functions $e^{-t|\xi|^\alpha} e^{i\xi x}|\xi|^\beta$ and $e^{-t|\xi|^\alpha} e^{i\xi x}\xi^{\boldsymbol{\beta}}$ on the boundaries of the sectors $\Gamma_R^{(1)}$ and $\Gamma_R^{(2)}$. For the term $|\xi|^a$ in the above two functions, where $a=\alpha$ or β, we extend it to be a holomorphic function

$$\left(\sum_{k=1}^{d} \xi_k^2\right)^{a/2} \quad \text{in } \mathrm{C}^d,$$

which needs to be specified by choosing suitable branches. On one hand, when $\mathrm{Re}(\xi_j) > 0$, we select the branch so that the function $w = z^{a/2}$ maps the sector with angles $[0, 2\phi]$ to the sector with angles $[0, a\phi]$. On the other hand, when $\mathrm{Re}(\xi_j) < 0$, we make the function $w = z^{a/2}$ map the sector with angles $[-2\phi, 0]$ to the sector with angles $[-a\phi, 0]$.

The main idea is to use the contour integrals to equate the integrals on the rays $\psi_j = 0, \pi$ and the integrals on the rays $\psi_j = \phi, \pi-\phi$, respectively. The following are some preliminary calculations on the rays $\psi_j = \frac{\pi}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_j)) \left(\frac{\pi}{2} - \phi\right)$ and the arcs $C_R^{(1)}$ or $C_R^{(2)}$, respectively. Here $\mathrm{sgn}\ (\cdot)$ is the sign function.

First, we consider the case when ξ_j's are on the rays $\psi_j = \frac{\pi}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_j)) \left(\frac{\pi}{2} - \phi\right)$, where we can write $\xi_j = \eta_j \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_j)) \left(\frac{\pi}{2} - \phi\right) i\right)$ with $\eta_j \in [0, R]$. In this case, for any fixed $\xi_k \in \Gamma_R^{(1)} \cup \Gamma_R^{(2)}$, where $k \in \{1, 2, \ldots, d\}$, we have

$$\left(\sum_{k=1}^{d} \xi_k^2\right)^{a/2} = \left(e^{2\mathrm{sgn}\ (\mathrm{Re}(\xi_j)) i\pi\phi} \eta_j^2 + \sum_{k\neq j} \xi_k^2\right)^{a/2}, \tag{3.46}$$

where $a = \alpha$ or β, and

$$e^{i\xi x} = \exp\left(i \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_j)) \left(\frac{\pi}{2} - \phi\right) i\right) \eta_j \frac{|x|}{\sqrt{d}} + \sum_{k\neq j} i\xi_k \frac{|x|}{\sqrt{d}}\right). \tag{3.47}$$

Notice that if $\psi_k = \frac{\pi}{2} - \text{sgn }(\text{Re}(\xi_k))\left(\frac{\pi}{2} - \phi\right)$ for all $k \in \{1, 2, \ldots, d\}$, it holds that

$$\left(\sum_{k=1}^{d} \xi_k^2\right)^{a/2} = \left(\sum_{k=1}^{d} \eta_k^2 e^{2\text{sgn }(\text{Re}(\xi_k))i\pi\phi}\right)^{a/2} \tag{3.48}$$

and

$$e^{i\xi x} = \exp\left(i\sum_{k=1}^{d} \exp\left(\frac{\pi i}{2} - \text{sgn }(\text{Re}(\xi_k))\left(\frac{\pi}{2} - \phi\right) i\right)\eta_k \frac{|x|}{\sqrt{d}}\right). \tag{3.49}$$

Next, we treat the case when ξ_j is on the arc $C_R^{(1)}$ or $C_R^{(2)}$, respectively.

By the definition of the regions $\Gamma_R^{(1)}$ and $\Gamma_R^{(2)}$, for any fixed $\xi_k \in \Gamma_R^{(1)} \cup \Gamma_R^{(2)}$, where $k \neq j$ and $\psi_j \in [0, \phi] \cup [\pi - \phi, \pi]$, the angle between $R^2 e^{2i\psi_j}$ and $\sum_{k \neq j} \xi_k^2$ is less than $\pi/2$, so we have

$$\left|R^2 e^{2i\psi_j} + \sum_{k \neq j} \xi_k^2\right| \geq |R^2 e^{2i\psi_j}|. \tag{3.50}$$

Moreover, since $|\arg(\xi_k^2)| \leq 2\phi$ for any $k \neq j$, where $\arg(\cdot)$ is the argument (angle), it follows that

$$\left|\arg\left(R^2 e^{2i\psi_j} + \sum_{k \neq j} \xi_k^2\right)\right| \leq 2\phi.$$

This together with (3.50) implies that

$$R^\alpha \cos(\alpha\phi) \leq \text{Re}\Big(R^2 e^{2i\psi_j} + \sum_{k \neq j} \xi_k^2\Big)^{\alpha/2}. \tag{3.51}$$

Now we show that the integral of $e^{-t\left(\sum_{k=1}^{d} \xi_k^2\right)^{\alpha/2}} e^{i\xi x}\left(\sum_{k=1}^{d} \xi_k^2\right)^{\beta/2}$ on the arc $C_R^{(1)}$ or $C_R^{(2)}$ tends to 0 as R tends to infinity.

On the arc $C_R^{(1)}$, we can write $\xi_j = Re^{i\psi_j}$, where $\psi_j \in [0, \phi]$. By (3.46), (3.47), and (3.51), we have

$$\begin{aligned}
&\lim_{R\to\infty} \left| \int_{C_R^{(1)}} e^{-t\left(\sum_{k=1}^{d} \xi_k^2\right)^{\alpha/2}} e^{i\xi x} \left(\sum_{k=1}^{d} \xi_k^2\right)^{\beta/2} d\xi_j \right| \\
&\le \lim_{R\to\infty} \int_0^{\phi} \left| \exp\left(-t(R^2 e^{2i\psi_j} + \sum_{k\neq j} \xi_k^2)^{\alpha/2}\right)\right| \left| \exp\left(iRe^{i\psi_j}\frac{|x|}{\sqrt{d}} + \sum_{k\neq j} i\xi_k \frac{|x|}{\sqrt{d}}\right)\right| \\
&\quad \times \left| \left(R^2 e^{2i\psi_j} + \sum_{k\neq j} \xi_k^2\right)^{\beta/2} \right| \left| iRe^{i\psi_j} \right| d\psi_j \\
&\le C \lim_{R\to\infty} \int_0^{\phi} e^{-tR^{\alpha}\cos(\alpha\phi)} \left(R^{\beta} + \left(\sum_{k\neq j} |\xi_k|^2\right)^{\beta/2}\right) R\, d\psi_j = 0
\end{aligned} \tag{3.52}$$

for any fixed $\xi_k \in \Gamma_R^{(1)} \cup \Gamma_R^{(2)}$, where $k \neq j$.

Similarly, on the arc $C_R^{(2)}$, where $\xi_j = Re^{i\psi_j}$ and $\psi_j \in [\pi - \phi, \pi]$, we have

$$\lim_{R\to\infty} \left| \int_{C_R^{(2)}} e^{-t\left(\sum_{k=1}^{d} \xi_k^2\right)^{\alpha/2}} e^{i\xi x} \left(\sum_{k=1}^{d} \xi_k^2\right)^{\beta/2} d\xi_j \right| = 0 \tag{3.53}$$

for any fixed $\xi_k \in \Gamma_R^{(1)} \cup \Gamma_R^{(2)}$, where $k \neq j$.

Combining (3.52) and (3.53) implies that we can apply contour integral to ξ_j if $\xi_k \in \Gamma_R^{(1)} \cup \Gamma_R^{(2)}$ for all $k \neq j$. Therefore, by (3.46), (3.47), (3.48), (3.49), (3.52), and (3.53), using d times of contour integrals, we infer that

$$
\begin{aligned}
&\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}|\xi|^\beta \, d\xi \\
&= \sum_{\mathrm{sgn}\ (\xi_1)=\pm 1} \int_{\mathrm{R}^{d-1}} \int_0^\infty \exp\left(-t\left(e^{2i\mathrm{sgn}\ (\xi_1)\phi}\eta_1^2 + \sum_{k=2}^d \xi_k^2\right)^{\alpha/2}\right) \\
&\quad \times \exp\left(i \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_1))\left(\frac{\pi}{2} - \phi\right) i\right)\eta_1 \frac{|x|}{\sqrt{d}} + \sum_{k=2}^d i\xi_k \frac{|x|}{\sqrt{d}}\right) \\
&\quad \times \left(e^{2i\mathrm{sgn}\ (\xi_1)\phi}\eta_1^2 + \sum_{k=2}^d \xi_k^2\right)^{\beta/2} \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_1))\left(\frac{\pi}{2} - \phi\right) i\right) d\eta_1 d\xi_2 \cdots d\xi_d \quad (3.54) \\
&= \cdots = \sum_{\mathrm{sgn}\ (\xi_1)=\pm 1} \cdots \sum_{\mathrm{sgn}\ (\xi_d)=\pm 1} \int_{\mathrm{R}_1^d} \exp\left(-t\left(\sum_{k=1}^d e^{2i\mathrm{sgn}\ (\xi_k)\phi}\eta_k^2\right)^{\alpha/2}\right) \\
&\quad \times \exp\left(i \sum_{k=1}^d \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_k))\left(\frac{\pi}{2} - \phi\right) i\right)\eta_k \frac{|x|}{\sqrt{d}}\right) \\
&\quad \times \left(\sum_{k=1}^d e^{2i\mathrm{sgn}\ (\xi_k)\phi}\eta_k^2\right)^{\beta/2} \prod_{k=1}^d \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_k))\left(\frac{\pi}{2} - \phi\right) i\right) d\eta,
\end{aligned}
$$

where R_1^d stands for the first quadrant of R^d and $d\eta = d\eta_1 d\eta_2 \cdots d\eta_d$. Plugging

$$
\mathrm{Re}\left(\sum_{k=1}^d e^{2i\mathrm{sgn}\ (\xi_k)\phi}\eta_k^2\right)^{\alpha/2} \geq |\eta|^\alpha \cos(\alpha\phi),
$$

and

$$
\left|\exp\left(i \sum_{k=1}^d \exp\left(\frac{\pi i}{2} - \mathrm{sgn}\ (\mathrm{Re}(\xi_k))\left(\frac{\pi}{2} - \phi\right) i\right)\eta_k \frac{|x|}{\sqrt{d}}\right)\right| = \exp\left(-\sum_{k=1}^d \sin(\phi)\eta_k \frac{|x|}{\sqrt{d}}\right)
$$

into (3.54), we have

$$
\begin{aligned}
&\left|\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}|\xi|^\beta d\xi\right| \leq 2^d \int_{\mathrm{R}_1^d} e^{-t|\eta|^\alpha \cos(\alpha\phi)} e^{-\sum_{k=1}^d \sin(\phi)\eta_k|x|/\sqrt{d}} |\eta|^\beta d\eta \\
&\leq C \int_{\mathrm{R}_1^d} e^{-t|\eta|^\alpha \cos(\alpha\phi)} e^{-\sum_{k=1}^d \sin(\phi)\eta_k|x|/\sqrt{d}} \sum_{k=1}^d \eta_k^\beta d\eta \\
&\leq C \sum_{k=1}^d \int_0^\infty e^{-t|\eta_k|^\alpha \cos(\alpha\phi)} e^{-\sin(\phi)\eta_k|x|/\sqrt{d}} \eta_k^\beta d\eta_k \prod_{i\neq k} \int_{\mathrm{R}_1^{d-1}} e^{-\sin(\phi)\eta_i|x|/\sqrt{d}} d\eta_i \qquad (3.55) \\
&\leq \frac{C}{|x|^{d-1}} \int_0^\infty e^{-t\rho^\alpha \cos(\alpha\phi)} e^{-\sin(\phi)\rho|x|/\sqrt{d}} \rho^\beta d\rho = \frac{C}{|x|^{d-1}} \times I,
\end{aligned}
$$

where

$$I=\int_0^\infty e^{-t\rho^\alpha\cos(\alpha\phi)}e^{-\rho|x|/\sqrt{d}}\rho^\beta d\rho\le\int_0^\infty e^{-\rho|x|/\sqrt{d}}\rho^\beta d\rho\le\frac{C^\beta}{|x|^{\beta+1}}\Gamma(\beta+1).$$

Therefore, we infer that

$$\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}e^{i\xi x}|\xi|^\beta d\xi\Big|\le\frac{C_1C_2^\beta\beta^\beta}{|x|^{\beta+d}}\tag{3.56}$$

for some constants C_1 and C_2, which is the first part on the right-hand side of (3.44).

Finally, we prove (3.45), which is a consequence of the following Claim.

Claim 68 *For any* $\boldsymbol{\beta}=(\beta_1,\dots,\beta_d)$, *where* β_i *is a nonnegative integer, there exists a constant* $C>0$ *such that*

$$\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}e^{i\xi x}\xi^{\boldsymbol{\beta}}d\xi\Big|\le C^{|\boldsymbol{\beta}|+\alpha+1}\frac{(\alpha+|\boldsymbol{\beta}|)^{\alpha+|\boldsymbol{\beta}|}t}{|x|^{\alpha+|\boldsymbol{\beta}|+d}}.$$

We prove this claim by induction. When $|\boldsymbol{\beta}|=0$, by integration by parts with respect to ξ_1, we see that

$$\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}e^{i\xi x}d\xi\Big|=\frac{\alpha\sqrt{d}t}{|x|}\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}\frac{\xi_1}{i|\xi|^{2-\alpha}}e^{i\xi x}d\xi\Big|.$$

Then using the method of contour integrals similarly to (3.55), we find that

$$\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}\frac{\xi_1}{i|\xi|^{2-\alpha}}e^{i\xi x}d\xi\Big|\le\frac{C}{|x|^{\alpha+d-1}},$$

which implies

$$\Big|\int_{\mathrm{R}^d}e^{-t|\xi|^\alpha}e^{i\xi x}d\xi\Big|\le\frac{Ct}{|x|^{\alpha+d}}.$$

Without loss of generality, we assume that $\beta_1>0$. For any positive integer k, we assume that Claim 68 is true for any $|\boldsymbol{\beta}|<k$. When $|\boldsymbol{\beta}|=k$, by integration by parts with respect

to ξ_1, the induction assumption and (3.56), it holds that

$$
\begin{aligned}
&|\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}\xi^{\boldsymbol{\beta}} d\xi| \\
&\leq |\frac{\sqrt{d}}{|x|}\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}\frac{\beta_1}{i\xi_1}\xi^{\boldsymbol{\beta}} d\xi| + \frac{t\alpha\sqrt{d}}{|x|}|\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}\frac{\xi_1}{|\xi|^{2-\alpha}}\xi^{\boldsymbol{\beta}} d\xi| \\
&\leq \frac{\sqrt{d}}{|x|}C^{\alpha+|\boldsymbol{\beta}|-1}\frac{(\alpha+|\boldsymbol{\beta}|-1)^{\alpha+|\boldsymbol{\beta}|-1}t}{|x|^{\alpha+|\boldsymbol{\beta}|-1+d}} + \frac{t\alpha\sqrt{d}}{|x|}\frac{C_1C_2^{\alpha+|\boldsymbol{\beta}|-1}(\alpha+|\boldsymbol{\beta}|-1)^{\alpha+|\boldsymbol{\beta}|-1}}{|x|^{\alpha+|\boldsymbol{\beta}|+d-1}} \\
&\leq C^{\alpha+|\boldsymbol{\beta}|+1}\frac{(\alpha+|\boldsymbol{\beta}|)^{\alpha+|\boldsymbol{\beta}|}t}{|x|^{\alpha+|\boldsymbol{\beta}|+d}}.
\end{aligned}
$$

Thus, we finished the proof of Claim 68 and therefore completed the proof of Lemma 66.

■

Now we are ready to embark on the proof of Theorem 56.

Proof. By (3.4), the heat kernel $p_\alpha(t,x)$ of the fractional heat equation (1.8) satisfies

$$|\partial_t^k p_\alpha(t,x)| = C(d,\alpha)\big|\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}|\xi|^{\alpha k} d\xi\big|,$$

which implies (3.5) by part (a) of Lemma 66. From the first bound $\frac{C_1C_2^{k\alpha}(k\alpha)^{k\alpha}}{|x|^{k\alpha+d}}$ in (3.5), we see that p_α is of Gevrey class in time of order α when $x \neq 0$. By the second bound $\frac{C}{t^{k+d/\alpha}}\Gamma\left(\frac{k\alpha+d}{\alpha}\right)$ in (3.5), p_α is analytic in time when $t > 0$.

Furthermore, for any positive integer k, by (3.4), we have

$$|\partial_x^k p_\alpha(t,x)| \leq C(d,\alpha)\sum_{|\boldsymbol{k}|=k}|\partial_x^{\boldsymbol{k}} p_\alpha(t,x)| = C(d,\alpha)\sum_{|\boldsymbol{k}|=k}\big|\int_{\mathrm{R}^d} e^{-t|\xi|^\alpha} e^{i\xi x}\xi^{\boldsymbol{k}} d\xi\big|,$$

where $\boldsymbol{k} = (k_1,\dots,k_d)$, $\xi^{\boldsymbol{k}} = \xi_1^{k_1}\dots\xi_d^{k_d}$, and we sum over all the $\boldsymbol{k}$ satisfying $|\boldsymbol{k}| = k$. By (3.6) and the fact that we have $\binom{k+d-1}{d-1}$ choices of $\boldsymbol{k}$ satisfying $|\boldsymbol{k}| = k$, we infer that

$$|\partial_x^k p_\alpha(t,x)| \leq C(d,\alpha)\binom{k+d-1}{d-1}\min\left\{\frac{C_1C_2^{\alpha+k}(\alpha+k)^{\alpha+k}t}{|x|^{\alpha+k+d}}, \frac{C}{t^{(k+d)/\alpha}}\Gamma\left(\frac{k+d}{\alpha}\right)\right\},$$

which implies (3.6) for a sufficiently large constant C_2. By the bound $\frac{C_1 C_2^{k+\alpha}(k+\alpha)^{k+\alpha} t}{|x|^{\alpha+k+d}}$ in (3.6), p_α is analytic in space at $|x| \neq 0$. By the other bound $\frac{C}{t^{(k+d)/\alpha}} \Gamma\left(\frac{k+d}{\alpha}\right)$ in (3.6), p_α is of Gevrey class with order $1/\alpha$ in space when $t > 0$ for any $x \in \mathrm{R}^d$. ■

Remark 69 *Theorem 56 is consistent with the fact that the heat kernel of the heat equation* $\partial_t u - \Delta u = 0$ *is of Gevrey class of order 2 at* $t = 0$. *Besides, when* $\alpha = 1$, *it is well known that* $p_1(t, x) = \frac{Ct}{(t^2+|x|^2)^{(d+1)/2}}$. *By a direct computation, we see that* $p_1(t, x)$ *satisfies all the results in Theorem 56.*

We end this section by proving Corollary 57.

Proof. By Theorem 53 and the growth condition (3.2), we know that there is an unique solution to (1.8):

$$u(t, x) = \int_{\mathrm{R}^d} p_\alpha(t, x-y) u(0, y)\, dy.$$

Therefore, by (3.6) and (3.2), we infer that

$$\begin{aligned}
&|\partial_x^k u(t, x)| \leq \int_{\mathrm{R}^d} |\partial_x^k p_\alpha(t, x-y)||u(0, y)|\, dy \\
&\leq \int_{B_1(x)} \frac{C}{t^{(k+d)/\alpha}} \Gamma\left(\frac{k+d}{\alpha}\right) C_1(1 + |y|^{\alpha-\epsilon})\, dy \\
&\quad + \int_{\mathrm{R}^d \setminus B_1(x)} \frac{C_1 C_2^{k+\alpha}(k+\alpha)^{k+\alpha} t}{|x-y|^{\alpha+k+d}} C_1(1 + |y|^{\alpha-\epsilon})\, dy \\
&\leq \frac{C(1 + |x|^{\alpha-\epsilon})}{t^{(k+d)/\alpha}} \Gamma\left(\frac{k+d}{\alpha}\right) + \int_{\mathrm{R}^d \setminus B_1(x)} \frac{C^{k+\alpha+1}(k+\alpha)^{k+\alpha} t}{|x-y|^{\alpha+d}} (1 + |x|^{\alpha-\epsilon} + |x-y|^{\alpha-\epsilon})\, dy \\
&\leq \frac{C(1 + |x|^{\alpha-\epsilon})}{t^{(k+d)/\alpha}} \Gamma\left(\frac{k+d}{\alpha}\right) + C^{k+\alpha+2}(k+\alpha)^{k+\alpha}(1 + |x|^{\alpha-\epsilon}) t,
\end{aligned}$$

which implies that u is analytic in space when $\alpha \in [1, 2)$ and u is of Gevrey class of order $1/\alpha$ in space when $\alpha \in (0, 1)$. ■

3.4 Fractional Heat Equation on a Manifold

In this section, we prove Theorems 58 and 59 in the setting of M, which is a $d-$dimensional, complete Riemannian manifold.

First we recall a well known lemma.

Lemma 70 *Assume that Condition* (3.8) *is satisfied. Then for any* $D > 0$, $\beta \geq 0$, *and* $t > 0$, *there exists a positive constant* C *such that*

$$\int_{\mathrm{M}} \frac{e^{-\frac{Dd(x,y)^2}{t}}}{|B(x,\sqrt{t})|} d(x,y)^{\beta}\, dy \leq Ct^{\beta/2}. \tag{3.57}$$

Proof. We give the proof for completeness. By Condition (3.8), we have

$$\begin{aligned}
&\int_{\mathrm{M}} \frac{e^{-Dd(x,y)^2/t}}{|B(x,\sqrt{t})|} d(x,y)^{\beta}\, dy \\
&= \int_{B(x,\sqrt{t})} \frac{e^{-Dd(x,y)^2/t}}{|B(x,\sqrt{t})|} d(x,y)^{\beta}\, dy + \int_{\mathrm{M}\setminus B(x,\sqrt{t})} \frac{e^{-Dd(x,y)^2/t}}{|B(x,\sqrt{t})|} d(x,y)^{\beta}\, dy \\
&\leq Ct^{\beta/2} + \sum_{k=1}^{\infty} \int_{2^{k-1}\sqrt{t}\leq d(x,y)\leq 2^k\sqrt{t}} \frac{e^{-Dd(x,y)^2/t}}{|B(x,\sqrt{t})|} d(x,y)^{\beta}\, dy \\
&\leq Ct^{\beta/2} + \sum_{k=1}^{\infty} \frac{|B(x,2^k\sqrt{t})|}{|B(x,\sqrt{t})|} e^{-D(2^{k-1})^2} (2^k\sqrt{t})^{\beta} \\
&\leq Ct^{\beta/2} + \sum_{k=1}^{\infty} C^{*k} e^{-D(2^{k-1})^2} (2^k\sqrt{t})^{\beta} \leq Ct^{\beta/2},
\end{aligned}$$

where C^* is the constant in Condition (3.8). ■

We are ready to prove Theorem 58.

3.4.1 Proof of Theorem 58

Proof. It is well known that there is a connection between the heat kernel $E(t,x;y)$ and the fractional heat kernel $p_\alpha(t,x;y)$, which can be found, for instance, in [5], i.e.,

$$p_\alpha(t,x;y) = \int_0^{\infty} E(s,x;y)\eta_t(s)\, ds,$$

where $\eta_t(s)$ is a density function of μ_t^α satisfying

$$\eta_t(s) = t^{-2/\alpha}\eta_1(t^{-2/\alpha}s).$$

Therefore,

$$p_\alpha(t,x;y) = \int_0^\infty E(s,x;y)t^{-2/\alpha}\eta_1(t^{-2/\alpha}s)\,ds = \int_0^\infty E(t^{2/\alpha}s,x;y)\eta_1(s)\,ds. \tag{3.58}$$

It is also known that there exists a constant C such that

$$0 \le \eta_1(s) \le Cs^{-1-\alpha/2}e^{-s^{-\alpha/2}}, \tag{3.59}$$

which can be found, for instance, in Theorem 3.1 of [5], Theorem 37.1 in [13], or Lemma 1 of [29].

Then for any $t > 0$, by (3.10) and (3.58), it holds that

$$u(t,x) = \int_{\mathrm{M}}\int_0^\infty E(t^{2/\alpha}s,x;y)\eta_1(s)u(0,y)\,dsdy. \tag{3.60}$$

By Theorem 5.4.12 of [38], Conditions (3.7) and (3.8) imply that there exist constants C, d_1, d_2, D_1, and D_2 such that

$$\frac{d_1e^{-D_1d(x,y)^2/t}}{|B(x,\sqrt{t})|} \le E(t,x;y) \le \frac{d_2e^{-D_2d(x,y)^2/t}}{|B(x,\sqrt{t})|}, \tag{3.61}$$

and

$$|\partial_t E(t,x;y)| \le \frac{C}{t}\frac{e^{-D_2d(x,y)^2/t}}{|B(x,\sqrt{t})|}. \tag{3.62}$$

From (3.11), (3.60), (3.61), (3.57), and (3.59), we infer that

$$
\begin{aligned}
|u(t,x)| &\le \int_{\mathrm{M}}\int_0^\infty |E(t^{2/\alpha}s,x;y)|\eta_1(s)|u(0,y)|\,dsdy \\
&\le C\int_{\mathrm{M}}\int_0^\infty \frac{e^{-D_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|}\eta_1(s)(1+d(x,0)^{\alpha-\epsilon}+d(x,y)^{\alpha-\epsilon})\,dsdy \\
&\le C(1+d(x,0)^{\alpha-\epsilon})\int_0^\infty \eta_1(s)\,ds + C\int_0^\infty \eta_1(s)(t^{2/\alpha}s)^{(\alpha-\epsilon)/2}\,ds \\
&\le C(1+d(x,0)^{\alpha-\epsilon})\int_0^\infty \eta_1(s)\,ds + Ct^{\frac{\alpha-\epsilon}{\alpha}}\int_0^\infty s^{-1-\alpha/2}e^{-s^{-\alpha/2}}s^{(\alpha-\epsilon)/2}\,ds \\
&\le C(1+d(x,0)^{\alpha-\epsilon}) + Ct^{(\alpha-\epsilon)/\alpha}.
\end{aligned}
$$

For any integer $k>0$, we proceed by induction. First, we assume it is true that

$$
|\partial_t^{k-1}u(t,x)| \le \frac{C^k(k-1)^{k-1}}{t^{k-2}}\left(\frac{(1+d(x,0)^{\alpha-\epsilon}}{t}+\frac{1}{t^{\epsilon/\alpha}}\right). \tag{3.63}
$$

Then for any $t>0$, by (3.10) and (3.58), it holds that

$$
\partial_t^k u(t,x;y) = \int_{\mathrm{M}}\int_0^\infty \partial_t E((t-\tau)^{2/\alpha}s,x;y)\eta_1(s)\partial_\tau^{k-1}u(\tau,y)\,dsdy,\ \forall\tau\in(0,t). \tag{3.64}
$$

By (3.64), (3.63), and (3.62), we have

$$
\begin{aligned}
&|\partial_t^k u(t,x;y)| \\
&\le \int_{\mathrm{M}}\int_0^\infty \frac{2s}{\alpha}(t-\tau)^{2/\alpha-1}\frac{C}{(t-\tau)^{2/\alpha}s}\frac{e^{-D_2 d(x,y)^2/((t-\tau)^{2/\alpha}s)}}{|B(x,\sqrt{(t-\tau)^{2/\alpha}s})|}\eta_1(s)|\partial_t^{k-1}u(\tau,y)|\,dsdy \\
&\le \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}(t-\tau)}\int_{\mathrm{M}}\int_0^\infty \frac{e^{-D_2 d(x,y)^2/((t-\tau)^{2/\alpha}s)}}{|B(x,\sqrt{(t-\tau)^{2/\alpha}s})|}\eta_1(s)\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{\tau}+\frac{1}{\tau^{\epsilon/\alpha}}\right)dsdy \\
&\quad + \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-1}(t-\tau)}\int_{\mathrm{M}}\int_0^\infty \frac{e^{-D_2 d(x,y)^2/((t-\tau)^{2/\alpha}s)}}{|B(x,\sqrt{(t-\tau)^{2/\alpha}s})|}\eta_1(s)d(x,y)^{\alpha-\epsilon}\,dsdy \\
&:= I_1 + I_2,
\end{aligned} \tag{3.65}
$$

where we used the triangle inequality in the second inequality. By (3.57) and (3.59), we have

$$
\begin{aligned}
I_1 &= \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-2}(t-\tau)}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{\tau}+\frac{1}{\tau^{\epsilon/\alpha}}\right)\int_0^\infty\int_{\mathrm{M}}\frac{e^{-D_2 d(x,y)^2/((t-\tau)^{2/\alpha}s)}}{|B(x,\sqrt{(t-\tau)^{2/\alpha}s})|}\eta_1(s)\,dyds\\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}(t-\tau)}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{\tau}+\frac{1}{\tau^{\epsilon/\alpha}}\right)\int_0^\infty \eta_1(s)\,ds\\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}(t-\tau)}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{\tau}+\frac{1}{\tau^{\epsilon/\alpha}}\right),
\end{aligned}
\tag{3.66}
$$

and

$$
\begin{aligned}
I_2 &= \frac{C^{k+1/2}(k-1)^{k-1}}{\tau^{k-1}(t-\tau)}\int_0^\infty\int_{\mathrm{M}}\frac{e^{-D_2 d(x,y)^2/((t-\tau)^{2/\alpha}s)}}{|B(x,\sqrt{(t-\tau)^{2/\alpha}s})|}d(x,y)^{\alpha-\epsilon}\eta_1(s)\,dyds\\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-1}(t-\tau)}\int_0^\infty\left((t-\tau)^{2/\alpha}s\right)^{(\alpha-\epsilon)/2}s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds\\
&\le \frac{C^{k+7/8}(k-1)^{k-1}}{\tau^{k-1}(t-\tau)^{\epsilon/\alpha}}.
\end{aligned}
\tag{3.67}
$$

Now we set $\tau = \frac{(k-1)t}{k}$. Consequently, by plugging (3.66) and (3.67) into (3.65), we conclude that

$$
\begin{aligned}
&|\partial_t^k u(t,x;y)|\\
&\le \frac{C^{k+3/4}(k-1)^{k-1}}{\tau^{k-2}(t-\tau)}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{\tau}+\frac{1}{\tau^{\epsilon/\alpha}}\right)+\frac{C^{k+7/8}(k-1)^{k-1}}{\tau^{k-1}(t-\tau)^{\epsilon/\alpha}}\\
&\le \frac{C^{k+1}k^k}{t^{k-1}}\left(\frac{1+d(x,0)^{\alpha-\epsilon}}{t}+\frac{1}{t^{\epsilon/\alpha}}\right),
\end{aligned}
$$

which gives (3.12) immediately. ■

The proof of Theorem 59 is divided into two parts: the proof of (3.13) and the proof of (3.14). We start with the first part in the following subsection.

3.4.2 Proof of (3.13) in Theorem 59

Proof. By Condition (3.8), it is well known that when $r \le s$,

$$
|B(x,r)| \ge \frac{1}{C^*}\left(\frac{r}{s}\right)^{\log_2 C^*}|B(x,s)|. \tag{3.68}
$$

See, for example, Remark 4.2.2 of [60].

Therefore, by (3.58), (3.61), (3.59), and (3.68), we have

$$
\begin{aligned}
&p_\alpha(t,x;y)\\
&\le \int_0^1 \frac{Ce^{-D_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds + \int_1^\infty \frac{Ce^{-D_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds\\
&= \int_0^1 \frac{Ce^{-D_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,t^{1/\alpha})|}\frac{|B(x,t^{1/\alpha})|}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds\\
&\quad + \int_1^\infty \frac{Ce^{-D_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds\\
&\le \int_0^1 \frac{C}{|B(x,t^{1/\alpha})|}\frac{C^*}{s^{\log_2 C^*/2}} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds + \int_1^\infty \frac{C}{|B(x,t^{1/\alpha})|} s^{-1-\alpha/2}e^{-s^{-\alpha/2}}\,ds\\
&\le \frac{C}{|B(x,t^{1/\alpha})|}.
\end{aligned}
\tag{3.69}
$$

If $d(x,y) \ge t^{1/\alpha}$, letting $\xi = \frac{st^{2/\alpha}}{d(x,y)^2}$, again by (3.58), (3.61), (3.59), and (3.68), we get

$$
\begin{aligned}
p_\alpha(t,x;y) &\le \int_0^\infty \frac{Ce^{-D_2/\xi}}{|B(x,\sqrt{\xi}d(x,y))|}\left(\frac{d(x,y)^2\xi}{t^{2/\alpha}}\right)^{-1-\alpha/2}\frac{d(x,y)^2}{t^{2/\alpha}}d\xi\\
&= \frac{Ct}{d(x,y)^\alpha}\int_0^1 \frac{e^{-D_2/\xi}}{|B(x,\sqrt{\xi}d(x,y))|}\xi^{-1-\alpha/2}d\xi\\
&\quad + \frac{Ct}{d(x,y)^\alpha}\int_1^\infty \frac{e^{-D_2/\xi}}{|B(x,\sqrt{\xi}d(x,y))|}\xi^{-1-\alpha/2}d\xi\\
&\le \frac{Ct}{d(x,y)^\alpha}\int_0^1 \frac{e^{-D_2/\xi}}{|B(x,d(x,y))|}\frac{|B(x,d(x,y))|}{|B(x,\sqrt{\xi}d(x,y))|}\xi^{-1-\alpha/2}d\xi\\
&\quad + \frac{Ct}{d(x,y)^\alpha}\int_1^\infty \frac{e^{-D_2/\xi}}{|B(x,d(x,y))|}\xi^{-1-\alpha/2}d\xi\\
&\le \frac{Ct}{d(x,y)^\alpha}\int_0^1 \frac{e^{-D_2/\xi}}{|B(x,d(x,y))|(\sqrt{\xi})^{\log_2 C^*}}\xi^{-1-\alpha/2}d\xi + \frac{Ct}{d(x,y)^\alpha|B(x,d(x,y))|}\\
&\le \frac{Ct}{d(x,y)^\alpha|B(x,d(x,y))|}.
\end{aligned}
\tag{3.70}
$$

Thus, we proved the upper bound in (3.13).

Now we show the lower bound in (3.13). By Theorem 3.1 of [5], there exists a constant $s_0 = s_0(\alpha)$ such that

$$\eta_1(s) \geq \frac{\alpha s^{-1-\alpha/2}}{4\Gamma(1-\alpha/2)}, \ \forall s > s_0. \tag{3.71}$$

Without loss of generality, we assume that $s_0 \geq 1$ in the sequel. Then we consider two cases.

When $t^{1/\alpha} \geq d(x,y)$, by (3.58), (3.61), (3.71), and (3.68), it holds that

$$\begin{aligned}
&p_\alpha(t,x;y) = \int_0^\infty E(t^{2/\alpha}s, x; y)\eta_1(s)\, ds \\
&\geq \int_{s_0}^\infty \frac{Cd_1 e^{-D_1 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}\, ds = \int_{s_0}^\infty \frac{Cd_1 e^{-D_1 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,t^{1/\alpha})|} \frac{|B(x,t^{1/\alpha})|}{|B(x,\sqrt{t^{2/\alpha}s})|} s^{-1-\alpha/2}\, ds \\
&\geq e^{\frac{-D_1}{s_0}} \int_{s_0}^\infty \frac{Cd_1}{|B(x,t^{1/\alpha})|} \frac{1}{C^* s^{\log_2 C^*/2}} s^{-1-\alpha/2}\, ds \geq \frac{C}{|B(x,t^{1/\alpha})|}.
\end{aligned} \tag{3.72}$$

When $t^{1/\alpha} < d(x,y)$, letting $\xi = \frac{st^{2/\alpha}}{d(x,y)^2}$, again by (3.58), (3.61), (3.71), and (3.68), we have

$$\begin{aligned}
p_\alpha(t,x;y) &\geq \int_{s_0}^\infty \frac{Cd_1 e^{-D_1/\xi}}{|B(x,\sqrt{\xi}d(x,y))|} \left(\frac{d(x,y)^2\xi}{t^{2/\alpha}}\right)^{-1-\alpha/2} \frac{d(x,y)^2}{t^{2/\alpha}} d\xi \\
&\geq \frac{Ct}{d(x,y)^\alpha} \int_{s_0}^\infty \frac{e^{-D_1/\xi}}{|B(x,d(x,y))|} \frac{|B(x,d(x,y))|}{|B(x,\sqrt{\xi}d(x,y))|} \xi^{-1-\alpha/2} d\xi \\
&\geq \frac{Ct}{d(x,y)^\alpha} \int_{s_0}^\infty \frac{e^{-D_1/s_0}}{|B(x,d(x,y))|(\sqrt{\xi})^{\log_2 C^*}} \xi^{-1-\alpha/2} d\xi \\
&\geq \frac{Ct}{d(x,y)^\alpha |B(x,d(x,y))|}.
\end{aligned} \tag{3.73}$$

Combining (3.72) and (3.73), we reach (3.13). ■

Now in order to prove (3.14), we establish an estimate for high-order time derivatives of the heat kernel $E(t,x;y)$ first.

Lemma 71 *Let* M *be a* $d-$*dimensional complete Riemannian manifold satisfying Conditions* (3.7) *and* (3.8). *Then for any* $x, y \in$ M, $t > 0$, *and any nonnegative integer* k, *there* *exist positive constants C_1 and C_2 such that the heat kernel $E(t,x;y)$ of the heat equation*

$$\partial_t u - \Delta u = 0$$

satisfies

$$|\partial_t^k E(t,x;y)| \leq \frac{C_1^{k+1} k^{k-2/3}}{t^k |B(x,\sqrt{t})|} e^{-C_2 d(x,y)^2/t}.$$

Remark 72 *To our best knowledge, up to now, in the literature, one can only find the coarser bounds*

$$|\partial_t^k E(t,x;y)| \leq \frac{C(k)}{t^k |B(x,\sqrt{t})|} e^{-C_2 d(x,y)^2/t}$$

in the manifold case, where $C(k)$ is not explicitly calculated. See, for instance, Theorem 5.4.12 in [38]. Here we obtain a more precise result.

Proof. The proof is similar to Lemma 4.1 of [58]. However, since we have different conditions here and we have the estimate of $\partial_t^k E(t,x;y)$ for all time $t > 0$ instead of $t \in (0,1]$, the proof is a bit different. We present the proof here for the reader's convenience.

Fix any $t_0 > 0$ and $x_0, y_0 \in \mathrm{M}$. For any nonnegative integer k and $j = 1, 2, \ldots, k+1$, we define

$$M_j^1 = \left\{ (t,x) : t \in \left(t_0 - \frac{j t_0}{2k}, t_0 \right), d(x,x_0) < \frac{j\sqrt{t_0}}{\sqrt{2k}} \right\},$$

$$M_j^2 = \left\{ (t,x) : t \in \left(t_0 - \frac{(j+0.5) t_0}{2k}, t_0 \right), d(x,x_0) < \frac{(j+0.5)\sqrt{t_0}}{\sqrt{2k}} \right\}.$$

Then $M_j^1 \subset M_j^2 \subset M_{j+1}^1$.

Following the proof of Lemma 4.1 of [58], for a constant C, we have

$$\iint_{M_1^1} |\partial_t^k E(t,x;y_0)|^2 \, dxdt \leq \frac{C^{2k} k^{2k}}{t_0^{2k}} \iint_{M_{k+1}^1} |E(t,x;y_0)|^2 \, dxdt. \tag{3.74}$$

Now to estimate the right-hand side of (3.74), we have two cases.

Case 1: $d(x_0, y_0) \le \sqrt{4kt_0}$. In this case, we need to use a well-known result which can be found, for instance, in Lemma 5.2.7 of [38]: under Condition (3.8), for a constant C, we have

$$|B(x,r)| \le e^{Cd(x,y)/r}|B(y,r)|, \;\forall x, y \in \mathrm{M} \text{ and } r > 0. \tag{3.75}$$

By (3.61), (3.68), and (3.75), it holds that

$$\begin{aligned}
&\frac{C^{2k}k^{2k}}{t_0^{2k}} \iint_{M^1_{k+1}} |E(t,x;y_0)|^2\, dxdt \le \frac{C^{2k+1/2}k^{2k}|B(x_0, \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}})|}{t_0^{2k-1} \min\limits_{x\in B(x_0,(k+1)\sqrt{t_0}/\sqrt{2k})} |B(x,\sqrt{t_0})|^2} \\
&= \frac{C^{2k+1/2}k^{2k}}{t_0^{2k-1}} \frac{|B(x_0, \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}})|}{|B(x_0,\sqrt{t_0})|^2} \frac{|B(x_0,\sqrt{t_0})|^2}{\min\limits_{x\in B(x_0,(k+1)\sqrt{t_0}/\sqrt{2k})} |B(x,\sqrt{t_0})|^2} \\
&\le \frac{C^{2k+3/4}k^{2k}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} \left(\frac{k+1}{\sqrt{2k}}\right)^{log_2 C^*} \exp\left(\frac{2C(k+1)}{\sqrt{2k}}\right) \le \frac{C^{2k+1}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{-C_2 d(x_0,y_0)^2/t_0}
\end{aligned}$$

for a constant C_2, where we used the condition $d(y_0, x_0) \le \sqrt{4kt_0}$ in the last inequality.

Case 2: $d(x_0, y_0) > \sqrt{4kt_0}$. In this case, because $d(x, x_0) < \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}}$ in M^1_{k+1}, by the triangle inequality, we have $\frac{\sqrt{2}-1}{\sqrt{2}} < \frac{d(x,y_0)}{d(x_0,y_0)} < 2$. Therefore, by (3.61), (3.68), and (3.75), it holds that

$$\begin{aligned}
&\frac{C^{2k}k^{2k}}{t_0^{2k}} \iint_{M^1_{k+1}} |E(t,x;y_0)|^2\, dxdt \\
&\le \frac{C^{2k}k^{2k}t_0|B(x_0, \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}})|}{t_0^{2k} \min\limits_{x\in B(x_0,(k+1)\sqrt{t_0}/(2\sqrt{k}))} |B(x,\sqrt{t_0})|^2} e^{-(3-2\sqrt{2})D_2 d(x_0,y_0)^2/(2t_0)} \\
&\le \frac{C^{2k+1/2}k^{2k}}{t_0^{2k-1}} \frac{|B(x_0, \frac{(k+1)\sqrt{t_0}}{\sqrt{2k}})|}{|B(x_0,\sqrt{t_0})|^2} \frac{|B(x_0,\sqrt{t_0})|^2}{\min\limits_{x\in B(x_0,(k+1)\sqrt{t_0}/(2\sqrt{k}))} |B(x,\sqrt{t_0})|^2} e^{-C_2 d(x_0,y_0)^2/t_0} \\
&\le \frac{C^{2k+3/4}k^{2k}}{t_0^{2k-1}} \frac{1}{|B(x_0,\sqrt{t_0})|} \left(\frac{k+1}{\sqrt{2k}}\right)^{log_2 C^*} \exp\left(\frac{C(k+1)}{\sqrt{k}}\right) e^{-C_2 d(x_0,y_0)^2/t_0} \\
&\le \frac{C^{2k+1}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{-C_2 d(x_0,y_0)^2/t_0}
\end{aligned}$$

for a constant C_2.

Combining the above two cases, we get

$$\iint_{M_1^1} |\partial_t^k E(t,x;y_0)|^2 \, dxdt \le \frac{C^{2k+1}k^{2k+1}}{t_0^{2k-1}|B(x_0,\sqrt{t_0})|} e^{-C_2 d(x_0,y_0)^2/t_0}. \tag{3.76}$$

Now we recall a well-known parabolic mean value inequality, which can be found, for instance, in Theorem 14.7 of [48] or Theorem 5.2.9 of [38]. For $0 < r < R < 1$, any nonnegative subsolution $u = u(t,x)$ of the heat equation satisfies

$$\sup_{Q_r(t_0,x_0)} u(t,x) \le C \left(\frac{R^2}{|B(x_0,r)|^{2/\nu}} \right)^{\nu/2} \left(\frac{1}{|R-r|^2} \right)^{(\nu+2)/2} \iint_{Q_R(t_0,x_0)} u(t,x) \, dxdt,$$

where $\nu > 2$ is a constant and $Q_r(t,x) = (t-r^2,t) \times B(x,r)$. Letting $u(t,x) = |\partial_t^k E(t,x;y_0)|^2$, $r \searrow 0$, and $R = \sqrt{t_0/(2k)}$, using (3.68), we see that

$$\begin{aligned} |\partial_t^k E(t_0,x_0;y_0)|^2 &\le \frac{Ck}{\left|B\left(x_0,\sqrt{t_0/(2k)}\right)\right| t_0} \iint_{Q_{\sqrt{t_0/(2k)}}(t_0,x_0)} (\partial_t^k E(t,x;y_0))^2 \, dxdt \\ &= \frac{Ck}{|B(x_0,\sqrt{t_0})| t_0} \frac{|B(x_0,\sqrt{t_0})|}{\left|B\left(x_0,\sqrt{t_0/(2k)}\right)\right|} \iint_{Q_{\sqrt{t_0/(2k)}}(t_0,x_0)} (\partial_t^k E(t,x;y_0))^2 \, dxdt \\ &\le \frac{Ck\left(\sqrt{2k}\right)^{log_2(C^*)}}{\left|B(x_0,\sqrt{t_0})\right| t_0} \iint_{Q_{\sqrt{t_0/(2k)}}(t_0,x_0)} (\partial_t^k E(t,x;y_0))^2 \, dxdt. \end{aligned} \tag{3.77}$$

By (3.76) and (3.77), we obtain

$$|\partial_t^k E(t_0,x_0;y_0)|^2 \le \frac{C^{2k+2}k^{2k+1+\log_2(C^*)/2}}{t_0^{2k}|B(x_0,\sqrt{t_0})|^2} e^{-C_2 d(x_0,y_0)^2/t_0}.$$

Thus,

$$|\partial_t^k E(t_0,x_0;y_0)| \le \frac{C_1^{k+1}k^{k-2/3}}{t_0^k|B(x_0,\sqrt{t_0})|} e^{-C_2 d(x_0,y_0)^2/t_0}$$

for a sufficiently large constant C_1, which finishes the proof of Lemma 71. ■

To prove the time analyticity of the heat kernel $p_\alpha(t,x;y)$, we use the following result.

Lemma 73 ([37] Proof of Proposition 1.4.2) *Suppose that $f = f(x)$ is real analytic at $x_0 \in \mathrm{R}$, which satisfies near x_0,*

$$|f^{(k)}(x)| \le C_1 \frac{k!}{R^k}, \quad \forall \text{ integer } k \ge 0.$$

Assume that $g = g(x)$ is real analytic at $f(x_0) \in \mathrm{R}$ which satisfies near $f(x_0)$,

$$|g^{(k)}(y)| \le C_3 \frac{k!}{S^k}, \quad \forall \text{ integer } k \ge 0.$$

Here R and S are positive constants. Then $h(x) = g(f(x))$ is analytic near x_0 and satisfies

$$|h^{(k)}(x_0)| \le \frac{C_1 C_3}{S + C_1} \frac{k!(1 + C_1/S)^k}{R^k}, \quad \forall \text{ integer } k \ge 0.$$

Now we are ready to prove (3.14) and thus completes the proof of Theorem 59.

3.4.3 Proof of (3.14) in Theorem 59

Proof. By (3.58), we have

$$\partial_t^n p_\alpha(t, x; y) = \int_0^\infty \partial_t^n E(t^{2/\alpha} s, x; y) \eta_1(s)\, ds. \tag{3.78}$$

We write $E(t^{2/\alpha}s, x; y) = E(t, x; y) \circ (t^{2/\alpha}s) = g(t) \circ f(t)$, where $g(t) := E(t, x; y)$ and $f(t) := t^{2/\alpha}s$. Then by Lemma 71, for a constant $C^{(1)} > 0$,

$$|\partial_t^k g(t)| \le \frac{(C^{(1)})^k k!}{t^k |B(x, \sqrt{t})|} e^{-C_2 d(x,y)^2/t}, \quad \forall \text{ integer } k \ge 0.$$

Let $C_3 = \frac{e^{-C_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x, \sqrt{t^{2/\alpha}s})|}$ and $S = t^{2/\alpha}s/C^{(1)}$. For $f(t)$, it holds that

$$|f^{(k)}(t)| \le \frac{(C^{(2)})^k k! t^{2/\alpha} s}{t^k}, \quad \forall \text{ integer } k \ge 0$$

for a constant $C^{(2)} > 0$. Let $C_1 = t^{2/\alpha}s$ and $R = t/C^{(2)}$. Then by Lemma 73, we have for a constant $C > 0$,

$$|\partial_t^k E(t^{2/\alpha}s, x; y)| \le \frac{C_1 C_3}{S + C_1} \frac{k!(1 + C_1/S)^k}{R^k} \le \frac{C^k k!}{t^k} \frac{e^{-C_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x, \sqrt{t^{2/\alpha}s})|}.$$

Therefore, by (3.78), we deduce that

$$|\partial_t^k p_\alpha(t,x;y)| \le \int_0^\infty \frac{C^k k!}{t^k} \frac{e^{-C_2 d(x,y)^2/(t^{2/\alpha}s)}}{|B(x,\sqrt{t^{2/\alpha}s})|} \eta_1(s)\, ds.$$

By the same calculations as (3.69) and (3.70), we deduce (3.14) immediately. ■

3.5 Corollaries on Backward and Other Equations

In this last section, we present four corollaries, whose statements and proofs are similar to the corresponding results in [19] and [58].

First we consider the Cauchy problem for the backward nonlocal parabolic equations

$$\begin{cases} \partial_t u + \mathrm{L}_\alpha^\kappa u = 0, \ \forall x \in \mathrm{R}^d \\ u(0,x) = a(x) \end{cases} \tag{3.79}$$

with $\kappa(\cdot,\cdot)$ satisfying (1.6) and (1.7).

Corollary 74 *Equation* (3.79) *has a smooth solution* $u = u(t,x)$ *of polynomial growth of order* $\alpha - \epsilon$ *in* $(0,\delta) \times \mathrm{R}^d$ *for some* $\delta > 0$*, i.e.,*

$$|u(t,x)| \le C(1+|x|^{\alpha-\epsilon}), \ 0<\epsilon<\alpha, \ (t,x) \in (0,\delta) \times \mathrm{R}^d, \tag{3.80}$$

if and only if

$$|\left(\mathrm{L}_\alpha^\kappa\right)^k a(x)| \le A_1^{k+1} k^k \left(1+|x|^{\alpha-\epsilon}\right), \quad k = 0,1,2,\ldots \tag{3.81}$$

where A_1 *is a positive constant.*

Proof.

On one hand, suppose that (3.79) has a smooth solution of polynomial growth of order $\alpha - \epsilon$, say $u = u(t,x)$. Then $u(-t,x)$ is a solution of the nonlocal parabolic equations with polynomial growth of order $\alpha-\epsilon$. By Theorem 53 and (3.80), (3.81) follows immediately.

On the other hand, suppose that (3.81) holds. Then it is easy to check that

$$u(t,x)=\sum_{j=0}^{\infty}(\mathrm{L}_{\alpha}^{\kappa})^{j}a(x)\frac{t^{j}}{j!}$$

is a smooth solution of the fraction heat equation for $t\in(-\delta,0]$ with δ sufficiently small. Indeed, the bounds (3.81) guarantee that the above series and the series

$$\sum_{j=0}^{\infty}(\mathrm{L}_{\alpha}^{\kappa})^{j+1}a(x)\frac{t^{j}}{j!}\quad\text{and}\quad\sum_{j=0}^{\infty}(\mathrm{L}_{\alpha}^{\kappa})^{j}a(x)\frac{\partial_{t}t^{j}}{j!}$$

all converge absolutely and uniformly in $[-\delta,0]\times B_{R}(0)$ for any fixed $R>0$. Hence, $\partial_{t}u-\mathrm{L}_{\alpha}^{\kappa}u=0$. Moreover, u has polynomial growth of order $\alpha-\epsilon$ since

$$|u(t,x)|\leq\sum_{j=0}^{\infty}\left|(\mathrm{L}_{\alpha}^{\kappa})^{j}a(x)\right|\frac{t^{j}}{j!}\leq\sum_{j=0}^{\infty}A_{1}^{j+1}j^{j}\left(1+|x|^{\alpha-\epsilon}\right)\frac{t^{j}}{j!}\leq A_{1}\left(1+|x|^{\alpha-\epsilon}\right)\tag{3.82}$$

provided that $t\in[-\delta,0]$ with δ sufficiently small. Thus, $u(-t,x)$ is a solution to the Cauchy problem of the backward nonlocal parabolic equations (3.79) of polynomial growth of order $\alpha-\epsilon$. ■

We have another corollary below about the forward Cauchy problem for the nonlocal parabolic equations

$$\begin{cases}\partial_{t}u-\mathrm{L}_{\alpha}^{\kappa}u=0,\ \forall x\in\mathrm{R}^{d}\\[2ex] u(0,x)=a(x).\end{cases}\tag{3.83}$$

The main point is the analyticity of solutions down to the initial time.

Corollary 75 *Equation* (3.83) *has a smooth solution $u=u(t,x)$ of polynomial growth of order $\alpha-\epsilon$, which is time analytic in $[0,\delta)$ for some $\delta>0$ with the radius of convergence* *independent of x if and only if*

$$|\,(\mathrm{L}_\alpha^\kappa)^k\, a(x)| \le A_1^{k+1} k^k \left(1+|x|^{\alpha-\epsilon}\right), \quad k=0,1,2,\dots \tag{3.84}$$

for a positive constant A_1.

Proof. On one hand, assuming (3.84), we can see

$$u^*(t,x) = \sum_{j=0}^{\infty} (\mathrm{L}_\alpha^\kappa)^j a(x) \frac{t^j}{j!}$$

is a smooth solution to (3.83) for $t \in [0,\delta)$ with δ sufficiently small. Moreover, if δ is sufficiently small, u^* has polynomial growth of order $\alpha-\epsilon$ by (3.82), so u^* is the unique solution to (3.83) by part (b) of Theorem 53.

By Corollary 74, the backward problem (3.79) has a smooth solution $v = v(t,x)$ in $[0,\delta)\times \mathrm{R}^d$. Define the function $U = U(t,x)$ by

$$U(t,x) = \begin{cases} u^*(t,x), & t \in [0,\delta) \\ v(-t,x), & t \in (-\delta, 0]. \end{cases}$$

It is straight forward to check that $U(t,x)$ is a solution of the nonlocal parabolic equations in $(-\delta,\delta)\times\mathrm{R}^d$. By Theorem 53, $U(t,x)$ and hence $u(t,x)$ is time analytic at $t=0$ for some $\delta > 0$.

On the other hand, suppose that $u = u(t,x)$ is a solution of the equation (3.83), which is analytic in time at $t=0$ with the radius of convergence independent of x. Then, by definition, u has a power series expansion in a time interval $(-\delta,\delta)$, for some $\delta > 0$. Hence (3.84) holds following the proof of Corollary 74. ∎

Remark 76 *Since we have not proved the solution to* (3.9) *is unique, the proofs of the above two corollaries cannot be applied to the manifold case. Therefore, we just restrict the above two corollaries to the case of* R^d.

For the following two corollaries, the operator L is either L_α^κ on R^d, or L^α on M. For convenience of notation, let X be either R^d or M satisfying Conditions (3.7) and (3.8).

Then similar to Theorems 1.4 and 1.5 in [58], we have the following two corollaries.

Corollary 77 *Let p be a positive integer and consider the equation*

$$u_t(t,x) - \mathrm{L}u(t,x) = u^p(t,x) \quad \text{in } (0,1] \times X \tag{3.85}$$

with the initial data $u(0,\cdot)$. Assume that $u = u(t,x)$ is a mild solution, i.e.,

$$u(t,x) = \int_X p_\alpha(t,x;y)u(0,y)\,dy + \int_0^t \int_X p_\alpha(t-s,x;y)u^p(s,y)\,dyds$$

and there exists a constant C_2 such that

$$|u(t,x)| \leq C_2,\ \forall (t,x) \in [0,1] \times X.$$

Then u is time analytic in $t \in (0,1]$ and the radius of convergence is independent of x.

Proof. From (3.1) or (3.14), we see by iteration that

$$\|\partial_t^k p_\alpha(t,x,\cdot)\|_{L^1(X)} \leq C^{k+1/2}k^{k-2/3}t^{-k},\ \forall \text{ integer } k \geq 0, \tag{3.86}$$

and thus, by the Leibniz rule, it holds that

$$\|\partial_t^k (t^k p_\alpha(t,x,\cdot))\|_{L^1(X)} \leq C^{k+1}k^{k-2/3},\ \forall \text{ integer } k \geq 0 \tag{3.87}$$

for a sufficient large constant C.

The rest of the proof is the same as that of Theorem 1.4 in [58]. ∎

Corollary 78 *For the equation (3.85) with p being any positive rational number, assume that $u = u(t,x)$ is a mild solution and there exist constants C_1 and C_2 such that*

$$0 \leq C_1 \leq |u(t,x)| \leq C_2,\ \forall (t,x) \in [0,1] \times X.$$

Then u is time analytic in $t \in (0,1]$ and the radius of convergence is independent of x.

Proof. We also have (3.86) and (3.87). Then the rest of the proof is the same as that of Theorem 1.5 in [58]. ■

Remark 79 *It is unclear to us whether a similar result holds when p is an irrational number as we are unable to get an appropriate relation between $\partial_t^n(t^n u)$ and $\partial_t^n(t^n u^p)$, where n is any positive integer. When $p = q_1/q_2$ is a rational number, in Lemma 4.5 of [58], the author used $\partial_t^n(t^n u^{1/q_2})$ as a bridge between $\partial_t^n(t^n u)$ and $\partial_t^n(t^n u^{q_1/q_2})$. Moreover, Lemma 73 cannot be used directly here. In fact, for any integer $k > 0$, if we assume that*

$$|t^n \partial_t^n u| \leq N^n n! \quad \forall \text{ positive integer } n \leq k$$

for a constant $N > 0$, then by Lemma 73, we get

$$|t^k \partial_t^k u^p| \leq N^{k+1/2} k! \left(1 + \frac{1}{\min |u|}\right)^k,$$

which cannot be used to obtain a positive radius of convergence.

Chapter 4

Smooth Solutions to the Heat Equation Which are Nowhere Analytic in Time

4.1 Introduction

The study of the existence of nowhere-analytic smooth functions has a rich history (see e.g. [2]) since the pioneering works du Bois-Reymond[20], Lerch[41] and Cellerier[8]. Later, many other examples were found with different methods, see e.g. [4, 25, 47, 54]. For the heat equation, the space analyticity of the classical solution in a space-time domain is usually expected as a consequence of parabolic regularity. But the time analyticity is more delicate and is not true in general, see e.g. the well-known examples in Kowalevsky[36] and Tychonoff[53]. Under extra assumptions, however, many time-analyticity results for the heat equation, Navier-Stokes equations, and some other parabolic equations may still be justified, see e.g. [55, 21, 23, 35, 46].

Recently, in [61, 19], it was discovered that for any complete and noncompact Riemannnian manifold M whose Ricci curvature is bounded from below, solutions to the heat equation on M with exponential growth of order 2 are analytic in time. In particular, as a corollary to Theorem 2.1 in [19], for any time interval $(a, b] \subseteq \mathrm{R}$, if u is a smooth solution to the heat equation $\partial_t u - \partial_x^2 u = 0$ on $\mathrm{R} \times (a, b]$ that satisfies for two positive constants A_1 and A_2,

$$|u(x,t)|e^{-A_2 x^2} \leq A_1, \quad \forall\,(x,t) \in \mathrm{R} \times (a,b], \tag{4.1}$$

then u must be time analytic in $t \in (a, b]$. The growth restraint (4.1) is sharp due to the Tychonoff's non-uniqueness example with suitable modifications (e.g. see Remark 2.3 in [19] for more details). Later, similar phenomena were also found in other types of PDEs[58, 17] and in domains with boundary[16]. In particular, by denoting $\mathrm{R}^+ = (0, \infty)$, Theorem 2.1 in [16] implies that for any time interval $(a, b] \subseteq \mathrm{R}$, if v is a smooth solution to the heat equation on $\mathrm{R}^+ \times (a, b]$ with the Dirichlet boundary condition $v(0, t) = 0$ and with the growth constraint $|v(x, t)|e^{-A_2 x^2} \leq A_1$ for any $(x, t) \in \mathrm{R}^+ \times (a, b]$, then v is time analytic in $t \in (a, b]$.

The study of analyticity of solutions to PDEs has both a long history, see e.g. the famous Cauchy-Kowalevsky theorem in [7, 36], and many applications, such as the time reversibility, the solvability of backward equations and the control theory. One particular application is about control problems involving heat type equations. For these problems, it is well known that the set of reachable states, though hard to describe exactly, is just a little larger than the set of those that can be reached by the free heat flow. But until the papers [61, 19], it's even not clear how to characterize the latter in general (see. e.g. the comment on page 1 in [40]). For a precise characterization of the reachable states by the free heat flow, see Corollary 2.2 and Remark 2.5 in [19]. Later, an explicit formula was derived for the control function by representing solutions with power series in time thanks to the time analyticity, see Theorem 2.1 in [59].

In this paper, however, we discover that the time analyticity is hopeless for general boundary conditions or without suitable growth conditions. More precisely, we construct solutions to the heat equation on the half space-time plane $\{x \geq 0,\, t \in \mathrm{R}\}$ that satisfy the growth condition (4.1) but are nowhere analytic in time. As a byproduct, we also find a solution to the heat equation on the whole space which is nowhere analytic in time and almost satisfies the growth condition (4.1). This example will demonstrate the sharpness of the growth condition (4.1) even if the solution is only required to be analytic in time at a single point.

Denote the space-time domain Ω_1 as

$$\Omega_1 = \mathrm{R}^+ \times \mathrm{R}. \tag{4.2}$$

We will construct two bounded solutions to the heat equation on $\overline{\Omega}_1$ which are nowhere analytic in time. Our first example (4.4) can be regarded as an extension to the space-time case of du Bois-Reymond [20], which itself is based on the Weierstrass function: a continuous but nowhere differentiable trigonometric series on R. Our second example (4.5) takes advantage of the heat kernel Φ on R, defined as in (4.3), and the method of the condensation of singularities [28, 6].

$$\Phi(x,t) = \begin{cases} (4\pi t)^{-\frac{1}{2}} \exp\big(-\frac{x^2}{4t}\big) & \text{if} \quad x \in \mathrm{R},\ t > 0, \\ 0 & \text{if} \quad x \in \mathrm{R},\ t \leq 0. \end{cases} \tag{4.3}$$

Although the construction of u_2 is direct via the method of condensation of singularities, we remark that the method of constructing u_1 in (4.4) by the Weierstrass type functions may be more flexible to study other evolutionary PDEs such as the Schrödinger equation and the wave equation.

Theorem 80 *Define two functions* $u_1, u_2 : \overline{\Omega}_1 \to \mathrm{R}$ *by*

$$u_1(x,t) = \sum_{k=1}^{\infty} e^{-2^k} e^{-2^k x} \sin\big(2^{2k+1} t - 2^k x\big), \tag{4.4}$$

$$u_2(x,t) = \sum_{k=1}^{\infty} 2^{-k} \Phi(x+1, t - r_k), \tag{4.5}$$

where $\{r_k\}_{k=1}^{\infty}$ *is an enumeration of all the rational numbers. Then for* $i = 1, 2$, $u_i \in C^\infty(\overline{\Omega}_1) \cap L^\infty(\Omega_1)$ *and* u_i *satisfies the heat equation on* $\overline{\Omega}_1$*. However, for any fixed* $x_0 \in [0, \infty)$*, the function* $u_i(x_0, \cdot)$ *is nowhere analytic in* $t \in \mathrm{R}$.

The functions in Theorem 80 are only defined on $\overline{\Omega}_1$. If we want to construct smooth solutions to the heat equation on the whole plane $\Omega_2 := \mathrm{R} \times \mathrm{R}$, then the solutions have to break the growth constraint (4.1). In addition, it is well-known that this growth constraint is sharp for everywhere time-analyticity. More precisely, for any $\delta > 0$, there exists a solution to the heat equation on Ω_2 which grows slower than $e^{A_2|x|^{2+\delta}}$ but is not time analytic at some point. Then it is interesting to investigate the following question: *Is the growth condition (4.1) sharp for somewhere time-analyticity?* The next result, which is inspired by (4.4), gives a positive answer to this question.

Theorem 81 *Let $\Omega_2 = \mathrm{R} \times \mathrm{R}$ and $\epsilon \in (0,1)$. Define $w_\epsilon : \Omega_2 \to \mathrm{R}$ by*

$$w_\epsilon(x,t) = \sum_{k=1}^{\infty} e^{-2^{(1+\epsilon)k}} e^{-2^k x} \sin\big(2^{2k+1}t - 2^k x\big). \tag{4.6}$$

Then $w_\epsilon \in C^\infty(\Omega_2)$ and w_ϵ satisfies the heat equation on Ω_2. However, for any fixed $x_0 \in \mathrm{R}$, the function $w_\epsilon(x_0, \cdot)$ is nowhere analytic in $t \in \mathrm{R}$. Meanwhile, there exist positive constants A_1 and A_2, which only depend on ϵ, such that

$$\sup_{x,t \in \mathrm{R}} |w_\epsilon(x,t)| \exp\big(- A_2 |x|^{1+\frac{1}{\epsilon}}\big) \le A_1. \tag{4.7}$$

Remark 82 *For any $\epsilon \in (0,1)$, the function w_ϵ in the above theorem, when restricted to $\overline{\Omega}_1$ where $x \ge 0$, is also a bounded nowhere time-analytic solution to the heat equation on $\overline{\Omega}_1$. Furthermore, for any $\delta > 0$, by choosing $\epsilon = \frac{1}{1+\delta}$, w_ϵ is bounded by $A_1 e^{A_2|x|^{2+\delta}}$ but is nowhere time-analytic on Ω_2.*

4.2 Proofs of Theorems 80 and 81

4.2.1 Proof of Theorem 80

- We first study u_1. It is straightforward to check that $u_1 \in C^\infty(\overline{\Omega}_1) \cap L^\infty(\Omega_1)$ and u_1 satisfies the heat equation on $\overline{\Omega}_1$. Next, for any fixed $x_0 \ge 0$, we define

  $$h(t) = u_1(x_0, t), \quad \forall\, t \in \mathrm{R}.$$

  Then it reduces to prove that h is not analytic at any point $t_0 \in \mathrm{R}$. By Cauchy-Hadamard theorem, it suffices to show

  $$\limsup_{n \to \infty} \left(\frac{|h^{(n)}(t_0)|}{n!} \right)^{\frac{1}{n}} = \infty. \tag{4.8}$$

For any integer $m \geq 1$, the $(2m)^{th}$ and the $(2m+1)^{th}$ derivatives of h at t_0 can be written as

$$\begin{aligned} h^{(2m)}(t_0) &= (-1)^m \sum_{k=1}^{\infty} e^{-2^k(1+x_0)} 2^{2m(2k+1)} \sin\Big(2^{2k+1}t_0 - 2^k x_0\Big), \\ h^{(2m+1)}(t_0) &= (-1)^m \sum_{k=1}^{\infty} e^{-2^k(1+x_0)} 2^{(2m+1)(2k+1)} \cos\big(2^{2k+1}t_0 - 2^k x_0\big). \end{aligned}$$

For any $N \in \mathrm{Z}^+$, there exists a unique $m_N \in \mathrm{Z}^+$ such that

$$2^N(1+x_0) \leq 4m_N < 2^N(1+x_0) + 4. \tag{4.9}$$

Define $F_N : \mathrm{Z}^+ \to \mathrm{R}^+$ as

$$F_N(k) = e^{-2^k(1+x_0)} 2^{2m_N(2k+1)}. \tag{4.10}$$

Then

$$\begin{aligned} h^{(2m_N)}(t_0) &= (-1)^{m_N} \sum_{k=1}^{\infty} F_N(k) \sin\Big(2^{2k+1}t_0 - 2^k x_0\Big), \\ h^{(2m_N+1)}(t_0) &= (-1)^{m_N} \sum_{k=1}^{\infty} 2^{2k+1} F_N(k) \cos\Big(2^{2k+1}t_0 - 2^k x_0\Big). \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} \big|h^{(2m_N)}(t_0)\big| &\geq F_N(N)\big|\sin\big(2^{2N+1}t_0 - 2^N x_0\big)\big| - \sum_{k \neq N} F_N(k), \\ \big|h^{(2m_N+1)}(t_0)\big| &\geq 2^{2N+1} F_N(N)\big|\cos\big(2^{2N+1}t_0 - 2^N x_0\big)\big| - \sum_{k \neq N} 2^{2k+1} F_N(k). \end{aligned} \tag{4.11}$$

Since $|\sin(\theta)| + |\cos(\theta)| \geq 1$ for any $\theta \in \mathrm{R}$, adding the two inequalities in (4.11) yields

$$\big|h^{(2m_N)}(t_0)\big| + \big|h^{(2m_N+1)}(t_0)\big| \geq F_N(N) - 4\Big(\sum_{k \neq N} 2^{2k} F_N(k)\Big). \tag{4.12}$$

By direct computation, it follows from (4.10) that for any $k \geq 1$,

$$\frac{F_N(k+1)}{F_N(k)} = \frac{2^{4m_N}}{e^{2^k(1+x_0)}} = \exp\big[4m_N \ln 2 - 2^k(1+x_0)\big]. \tag{4.13}$$

For any fixed N, thanks to the choice (4.9) of m_N and the fact that $\frac{1}{2} < \ln 2 < 1$, $F_N(N)$ is the largest term in the sequence $\{F_N(k)\}_{k\geq 1}$. Moreover, when N is large enough, $F_N(N)$ is much larger than the other terms in the sequence $\{F_N(k)\}_{k\geq 1}$. Actually, it is not difficult to find a positive constant N_0, which only depends on x_0, such that

$$\sum_{k\neq N} 2^{2k} F_N(k) \leq \frac{1}{100} F_N(N), \quad \forall\, N \geq N_0. \tag{4.14}$$

Plugging (4.14) into (4.12) leads to

$$\left|h^{(2m_N)}(t_0)\right| + \left|h^{(2m_N+1)}(t_0)\right| \geq \frac{1}{2} F_N(N), \quad \forall\, N \geq N_0. \tag{4.15}$$

By (4.10),

$$F_N(N) = e^{-2^N(1+x_0)} 2^{2m_N(2N+1)} \geq e^{-2^N(1+x_0)} (2^N)^{4m_N}.$$

Reorganizing (4.9) gives rise to

$$\frac{4(m_N-1)}{1+x_0} < 2^N \leq \frac{4m_N}{1+x_0}. \tag{4.16}$$

Consequently,

$$F_N(N) \geq e^{-4m_N} \left(\frac{4(m_N-1)}{1+x_0}\right)^{4m_N} \geq 2\left(\frac{m_N-1}{1+x_0}\right)^{4m_N}.$$

Thus, for any $N \geq N_0$, it follows from (4.15) that

$$\left|h^{(2m_N)}(t_0)\right| + \left|h^{(2m_N+1)}(t_0)\right| \geq \left(\frac{m_N-1}{1+x_0}\right)^{4m_N}.$$

As a result,

$$\frac{\left|h^{(2m_N)}(t_0)\right| + \left|h^{(2m_N+1)}(t_0)\right|}{(2m_N+1)!} \geq \left[\frac{(m_N-1)^2}{(1+x_0)^2(2m_N+1)}\right]^{2m_N}. \tag{4.17}$$

Since $m_N \to \infty$ as $N \to \infty$, then (4.8) follows immediately from (4.17).

- Now we consider u_2. Although the expression (4.5) looks complicated, the conclusion follows directly from an elegant result in [54].

Lemma 83 *([54]) Let φ be a bounded C^∞ function which is analytic on $\mathrm{R} \setminus \{0\}$ but not analytic at 0. Assume there are positive constants δ_0, A and L such that for any $|t| > A$,*

$$\sup_{n\geq 0} \frac{\left|\partial_t^n \varphi(t)\right|}{n!} \delta_0^n < L. \tag{4.18}$$

Let $\{a_k\}_{k\geq 1}$ be a sequence of non-zero real numbers such that $\sum_{k=1}^{\infty} |a_k| < \infty$. Let $\{r_k\}_{k\geq 1}$ be an enumeration of all the rational numbers. Define a function $f : \mathrm{R} \to \mathrm{R}$ by

$$f(t) = \sum_{k=1}^{\infty} a_k \varphi(t - r_k).$$

Then $f \in C^\infty(\mathrm{R})$ but f is nowhere analytic on R.

Next, we will apply Lemma 83 to prove the desired result for u_2. First, we recall that the heat kernel Φ is defined as in (4.3). In addition, by noticing $x + 1$ is away from 0 for any $x \geq 0$, we know for any integer $n \geq 1$, there exists some constant $M_n > 0$ such that

$$|\partial_x^n \Phi(x+1,t)| + |\partial_t^n \Phi(x+1,t)| \leq M_n, \quad \forall\ x \geq 0,\ t \in \mathrm{R}.$$

As a result, $u_2 \in C^\infty(\overline{\Omega}_1) \cap L^\infty(\Omega_1)$ and u_2 satisfies the heat equation on $\overline{\Omega}_1$ since

$$\begin{aligned}(\partial_t - \partial_x^2)u_2 &= (\partial_t - \partial_x^2)\left(\sum_{k=1}^{\infty} 2^{-k}\Phi(x+1, t-r_k)\right) \\ &= \sum_{k=1}^{\infty} 2^{-k}(\partial_t - \partial_x^2)\Phi(x+1, t-r_k) = 0.\end{aligned}$$

Then for any fixed $x_0 \geq 0$, define

$$\varphi(t) = \Phi(x_0 + 1, t), \quad \forall\, t \in \mathrm{R}.$$

According to classical estimates on the heat kernel Φ (see e.g. formula (3.3) in [33]), there exists some constant $C > 0$ such that for any $n \in \mathrm{N}$,

$$|\partial_x^n \Phi(x,t)| \leq \frac{C^n n^{n/2}}{t^{(n+1)/2}}\, e^{-\frac{x^2}{8t}}, \quad \forall\, x \in \mathrm{R},\ t > 0.$$

Consequently,

$$|\partial_t^n \Phi(x,t)| = |\partial_x^{2n} \Phi(x,t)| \leq \frac{2^n C^{2n} n^n}{t^{n+\frac{1}{2}}}\, e^{-\frac{x^2}{8t}}, \quad \forall\, x \in \mathrm{R},\ t > 0.$$

In particular, there exists some constant $C_1 > 0$ such that for any $n \in \mathrm{N}$,

$$|\partial_t^n \varphi(t)| = |\partial_t^n \Phi(x_0 + 1, t)| \leq \frac{C_1^n n^n}{t^{n+\frac{1}{2}}}\, e^{-(x_0+1)^2/(8t)}, \quad \forall\, t > 0. \tag{4.19}$$

So by choosing $A = 1$ and $\delta_0 = \frac{1}{2C_1}$, it follows from (4.19) that for any $t > A$,

$$\frac{|\partial_t^n \varphi(t)|}{n!}\, \delta_0^n \leq \frac{C_1^n n^n}{n!} \frac{1}{(2C_1)^n} = \frac{n^n}{n!} \frac{1}{2^n}.$$

Thanks to the Sterling formula, we conclude that there exists some constant $L > 0$ such that

$$\frac{|\partial_t^n \varphi(t)|}{n!}\, \delta_0^n < L, \quad \forall\, t > A. \tag{4.20}$$

Noticing that $\partial_t^n \varphi(t) = 0$ for any $t < 0$, so (4.20) is also valid for $|t| > A$. Therefore, (4.18) is justified for φ. Finally, in Lemma 83, by setting $a_k = \frac{1}{2^k}$, we conclude that the function

$$u_2(x_0, \cdot) = \sum_{k=1}^{\infty} \frac{1}{2^k} \Phi(x_0 + 1, \cdot - r_k) = \sum_{k=1}^{\infty} a_k \varphi(\cdot - r_k)$$

is nowhere analytic in $t \in \mathrm{R}$.

4.2.2 Proof of Theorem 81

Fix any $\epsilon \in (0,1)$, it is readily seen that $w_\epsilon \in C^\infty(\overline{\Omega}_2)$ and w_ϵ satisfies the heat equation on $\overline{\Omega}_2$.

Next, for any fixed $x_0 \in \mathrm{R}$, we will prove that the function $w_\epsilon(x_0,\cdot)$ is nowhere analytic on R. Define $h_\epsilon(t) = w_\epsilon(x_0,t)$ for $t \in \mathrm{R}$. Then it reduces to prove h_ϵ is not analytic at any point $t_0 \in \mathrm{R}$. By Cauchy-Hadamard theorem, it suffices to show

$$\limsup_{n\to\infty} \left(\frac{|h_\epsilon^{(n)}(t_0)|}{n!}\right)^{\frac{1}{n}} = \infty. \tag{4.21}$$

The proof of (4.21) is similar to that for the function u_1 in Theorem 80, so we will only sketch the process. For any large N such that $2^{\epsilon N} \geq 2+|x_0|$, there exists a unique $m_N \in \mathrm{Z}^+$ such that

$$\big(2^{\epsilon N} + x_0\big)2^N \leq 4m_N < \big(2^{\epsilon N} + x_0\big)2^N + 4. \tag{4.22}$$

Define $F_N : \mathrm{Z}^+ \to \mathrm{R}^+$ as

$$F_N(k) = e^{-2^{(1+\epsilon)k}} e^{-2^k x_0} 2^{2m_N(2k+1)}. \tag{4.23}$$

Then similar to (4.12), we have

$$\big|h_\epsilon^{(2m_N)}(t_0)\big| + \big|h_\epsilon^{(2m_N+1)}(t_0)\big| \geq F_N(N) - 4\Big(\sum_{k\neq N} 2^{2k} F_N(k)\Big).$$

Thanks to the choice (4.22) of m_N, it is not difficult to find a positive constant N_0, which only depends on x_0 and ϵ, such that

$$\sum_{k\neq N} 2^{2k} F_N(k) \leq \frac{1}{100} F_N(N), \quad \forall\, N \geq N_0. \tag{4.24}$$

Hence,

$$\big|h_\epsilon^{(2m_N)}(t_0)\big| + \big|h_\epsilon^{(2m_N+1)}(t_0)\big| \geq \frac{1}{2} F_N(N), \quad \forall\, N \geq N_0. \tag{4.25}$$

Reorganizing (4.22) leads to

$$\frac{4(m_N-1)}{2^{\epsilon N}+x_0} < 2^N \le \frac{4m_N}{2^{\epsilon N}+x_0}. \tag{4.26}$$

Based on (4.25) and (4.23), for any $N \ge N_0$,

$$\begin{aligned}\left|h_\epsilon^{(2m_N)}(t_0)\right| + \left|h_\epsilon^{(2m_N+1)}(t_0)\right| &\ge \frac{1}{2} e^{-2^{(1+\epsilon)N}} e^{-2^N x_0} 2^{2m_N(2N+1)} \\ &\ge e^{-2^N(2^{\epsilon N}+x_0)} (2^N)^{4m_N}.\end{aligned}$$

Then it follows from (4.26) that

$$\left|h_\epsilon^{(2m_N)}(t_0)\right| + \left|h_\epsilon^{(2m_N+1)}(t_0)\right| \ge e^{-4m_N} \left(\frac{4(m_N-1)}{2^{\epsilon N}+x_0}\right)^{4m_N} \ge \left(\frac{m_N-1}{2^{\epsilon N}+x_0}\right)^{4m_N}.$$

As a result,

$$\frac{\left|h_\epsilon^{(2m_N)}(t_0)\right| + \left|h_\epsilon^{(2m_N+1)}(t_0)\right|}{(2m_N+1)!} \ge \left[\frac{(m_N-1)^2}{(2^{\epsilon N}+x_0)^2(2m_N+1)}\right]^{2m_N}. \tag{4.27}$$

When $N \to \infty$, it follows from (4.22) that $m_N \to \infty$ and

$$\frac{m_N}{(2^{\epsilon N}+x_0)^2} \sim 2^{(1-\epsilon)N} \to \infty.$$

Then (4.21) follows from (4.27).

Finally, we need to establish the growth constraint (4.7). Fix $0 < \epsilon < 1$. Then it suffices to find constants A_1 and A_2, which only depend on ϵ, such that for any $x \ge 0$,

$$\sum_{k=1}^{\infty} \exp\left(-2^{(1+\epsilon)k} + 2^k x\right) \le A_1 \exp\left(A_2\, x^{1+\frac{1}{\epsilon}}\right).$$

Define $g_k(x) = \exp[2^k(x - 2^{\epsilon k})]$ for any $k \ge 1$. Then it reduces to justify

$$\sum_{k=1}^{\infty} g_k(x) \le A_1 \exp\left(A_2\, x^{1+\frac{1}{\epsilon}}\right). \tag{4.28}$$

If $0 \leq x \leq 100$, then it is readily seen that the above series in (4.28) is uniformly bounded by a constant B_1 which only depends on ϵ, so it reduces to consider the case when $x > 100$. Define $K \in \mathrm{Z}^+$ to be the unique positive integer such that

$$\frac{1}{\epsilon}\log_2\left(\frac{x}{2^{1+\epsilon}-1}\right) \leq K < 1 + \frac{1}{\epsilon}\log_2\left(\frac{x}{2^{1+\epsilon}-1}\right), \tag{4.29}$$

which can be rewritten as

$$\frac{x}{2^{1+\epsilon}-1} \leq 2^{\epsilon K} < \frac{2^{\epsilon} x}{2^{1+\epsilon}-1}. \tag{4.30}$$

In addition, since $x > 100$ and $0 < \epsilon < 1$, $K \geq 4/\epsilon > 4$. By direct computation,

$$\frac{g_{k+1}(x)}{g_k(x)} = \exp\left(2^k\left[x - 2^{\epsilon k}(2^{1+\epsilon}-1)\right]\right). \tag{4.31}$$

- **Case 1: Estimate of $\sum_{k=1}^{K} g_k(x)$.**

  For any $1 \leq k \leq K-1$, it follows from (4.31) and (4.30) that $g_{k+1}(x) \geq g_k(x)$. As a consequence,

$$\sum_{k=1}^{K} g_k(x) \leq K g_K(x) \leq K \exp(2^K x). \tag{4.32}$$

  Since $x > 100$ and $K > 4$, then $K \leq \exp(2^K x)$. Moreover, we can see from (4.32) and (4.30) that

$$\sum_{k=1}^{K} g_k(x) \leq \exp\left[2\left(\frac{2^{\epsilon} x}{2^{1+\epsilon}-1}\right)^{\frac{1}{\epsilon}} x\right] = \exp\left(B_2\, x^{1+\frac{1}{\epsilon}}\right), \tag{4.33}$$

  where B_2 is a constant which only depends on ϵ.

- **Case 2: Estimate of $\sum_{k=K+1}^{\infty} g_k(x)$.**

  For any $k \geq K$, it follows from (4.31) and (4.30) that $g_{k+1}(x) \leq g_k(x)$. In particular, by choosing $k = K$ and recalling (4.33), we know

$$g_{K+1} \leq g_K(x) \leq \exp\left(B_2\, x^{1+\frac{1}{\epsilon}}\right). \tag{4.34}$$

From (4.30), we have $x \leq (2^{1+\epsilon} - 1)2^{\epsilon K}$. Plugging this inequality into (4.31) yields

$$\frac{g_{k+1}(x)}{g_k(x)} \leq \exp\big[-(2^{1+\epsilon} - 1)2^k(2^{\epsilon k} - 2^{\epsilon K})\big].$$

So for any $k \geq K+1$,

$$\frac{g_{k+1}(x)}{g_k(x)} \leq \exp\big[-(2^{1+\epsilon} - 1)2^k(2^{\epsilon(K+1)} - 2^{\epsilon K})\big] \leq \exp\big[-2^k(2^{\epsilon} - 1)\big].$$

This implies that for any $k \geq K+1$,

$$\begin{aligned} g_{k+1}(x) &= \Bigg(\prod_{i=K+1}^{k} \frac{g_{i+1}(x)}{g_i(x)}\Bigg) g_{K+1}(x) \\ &\leq \exp\big[-(2^{\epsilon} - 1)2^{K+1}(2^{k-K} - 1)\big] g_{K+1}(x) \\ &\leq \exp\big[-(2^{\epsilon} - 1)(2^{k-K} - 1)\big] g_{K+1}(x) \end{aligned}$$

Thus, by setting $j = k - K$ and adding j from 1 to ∞,

$$\sum_{k=K+2}^{\infty} g_k(x) \leq g_{K+1}(x) \sum_{j=1}^{\infty} \exp\big[-(2^{\epsilon} - 1)(2^j - 1)\big] = B_3\, g_{K+1}(x), \tag{4.35}$$

where B_3 is a positive constant which only depends on ϵ.

Combining (4.33), (4.34) and (4.35) together leads to the desired estimate (4.28).

Chapter 5

Conclusions

On one hand, under a growth condition, we proved the pointwise time analyticity of several evolutionary partial differential equations. On the other hand, it may not true if the growth condition fails, or if we put non-analytic condition on the boundary as shown in Chapter 4. We also proved that the growth condition is sharp by showing that if the growth condition of the biharmonic heat equation fails, we can construct two solutions, one of which is not analytic in time. Surprisingly, for linear differential equations discussed above, we obtain a necessary and sufficient condition such that the solution is time analytic at $t = 0$. We also obtain a necessary and sufficient condition for the solvability of the backward equations. However, we failed to get a similar condition for the nonlinear heat equations, which we will need to do further research in the future. Moreover, for the nonlinear equations with power nonlinearity of order p, we only proved the time analyticity if the solution is bounded. We also only proved the case when p is a rational number, even if we assume the solution is bounded from below and above. Therefore, some further research is still needed to see if we can remove the bounded condition or extend the rational p to all real number.

Bibliography

[1] G. Barbatis and E. B. Davies. Sharp bounds on heat kernels of higher order uniformly elliptic operators. *J. Operator Theory*, 36(1):179–198, 1996.

[2] G. G. Bilodeau. The origin and early development of nonanalytic infinitely differentiable functions. *Arch. Hist. Exact Sci.*, 27(2):115–135, 1982.

[3] K. Bogdan, A. Stós, and P. Sztonyk. Harnack inequality for stable processes on d-sets. *Studia Math.*, 158(2):163–198, 2003.

[4] E. Borel. Sur quelques points de la théorie des fonctions. *Ann. Sci. École Norm. Sup. (3)*, 12:9–55, 1895.

[5] P. Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[6] G. Cantor. Ueber ein neues und allgemeines Condensationsprincip der Singularitäten von Functionen. *Math. Ann.*, 19(4):588–594, 1882.

[7] A. Cauchy. Mémoire sur lemploi du calcul des limites dans lintégration des équations aux dérivées partielles. *Comptes rendus*, XV:44–59, 1842.

[8] C. Cellerier. Note sur les principes fondamentaux de l'analyse. *Bull. Sci. Math.*, 14:142–160, 1890.

[9] Jeff Cheeger and Tobias H. Colding. Lower bounds on Ricci curvature and the almost rigidity of warped products. *Ann. of Math. (2)*, 144(1):189–237, 1996.

[10] Z. Chen and J. Wang. Perturbation by non-local operators. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(2):606–639, 2018.

[11] Z. Chen and X. Zhang. Heat kernels and analyticity of non-symmetric jump diffusion semigroups. *Probab. Theory Related Fields*, 165(1-2):267–312, 2016.

[12] R. B. Darst. Most infinitely differentiable functions are nowhere analytic. *Canad. Math. Bull.*, 16:597–598, 1973.

[13] G. Doetsch. *Introduction to the Theory and Application of the Laplace Transformation.* Springer, 1974.

[14] H. Dong, T. Jin, and H. Zhang. Dini and Schauder estimates for nonlocal fully nonlinear parabolic equations with drifts. *Anal. PDE*, 11(6):1487–1534, 2018.

[15] H. Dong and D. Kim. On the L_p-solvability of higher order parabolic and elliptic systems with BMO coefficients. *Arch. Ration. Mech. Anal.*, 199(3):889–941, 2011.

[16] H. Dong and X. Pan. Time analyticity for inhomogeneous parabolic equations and the Navier-Stokes equations in the half space. *J. Math. Fluid Mech.*, 22(4):Paper No. 53, 20, 2020.

[17] H. Dong, C. Zeng, and Q. S. Zhang. Time analyticity for fractional heat equations. *Submitted, arXiv:2108.01128.*

[18] H. Dong and H. Zhang. On Schauder estimates for a class of nonlocal fully nonlinear parabolic equations. *Calc. Var. Partial Differential Equations*, 58(2):Paper No. 40, 42, 2019.

[19] H. Dong and Q. S. Zhang. Time analyticity for the heat equation and Navier-Stokes equations. *J. Funct. Anal.*, 279(4):108563, 15, 2020.

[20] P. du Bois-Reymond. Ueber den Gültigkeitsbereich der Taylor'schen Reihenentwickelung. *Math. Ann.*, 21(1):109–117, 1883.

[21] L. Escauriaza, S. Montaner, and C. Zhang. Analyticity of solutions to parabolic evolutions and applications. *SIAM J. Math. Anal.*, 49(5):4064–4092, 2017.

[22] Mariano Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[23] Y. Giga. Time and spatial analyticity of solutions of the Navier-Stokes equations. *Comm. Partial Differential Equations*, 8(8):929–948, 1983.

[24] Qi H. and Laurent-Coste L. Time regularity for local weak solutions of the heat equation on local dirichlet spaces. *arXiv:1912.12998.*

[25] J. Hadamard. Essai sur l'etude des fonctions donnees par leur developpement de taylor. *J. Math. Pures Appl.*, 8(4):101–186, 1924.

[26] F. Han and B. Hua. Uniqueness class of the wave equation on graphs. *Submitted, arXiv:2009.12793.*

[27] F. Han, B. Hua, and L. Wang. Time analyticity of ancient solutions to the heat equation on graphs. *Submitted, arXiv:1911.02280.*

[28] H. Hankel. Untersuchungen über die unendlich oft oscillirenden und unstetigen Functionen. *Math. Ann.*, 20(1):63–112, 1882.

[29] J. Hawkes. A lower Lipschitz condition for the stable subordinator. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 17:23–32, 1971.

[30] E. Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, volume 5 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999.

[31] E. Holmgren. Sur les solutions quasianalytiques de l'equation de la chaleur. *Ark. Mat.*, 18(9):1–9, 1924.

[32] F. John. *Partial differential equations*, volume 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, fourth edition, 1982.

[33] C. Kahane. On the spatial analyticity of solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 33:386–405, 1969.

[34] S. S. Kim and K. H. Kwon. Smooth (C^∞) but nowhere analytic functions. *Amer. Math. Monthly*, 107(3):264–266, 2000.

[35] G. Komatsu. Global analyticity up to the boundary of solutions of the Navier-Stokes equation. *Comm. Pure Appl. Math.*, 33(4):545–566, 1980.

[36] S. Kowalevsky. Zur Theorie der partiellen Differentialgleichung. *J. Reine Angew. Math.*, 80:1–32, 1875.

[37] S. G. Krantz and H. R. Parks. *A primer of real analytic functions*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[38] Saloff-Coste L. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[39] G. Ł ysik and S. Michalik. Formal solutions of semilinear heat equations. *J. Math. Anal. Appl.*, 341(1):372–385, 2008.

[40] C. Laurent and L. Rosier. Exact controllability of semilinear heat equations in spaces of analytic functions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 37(4):1047–1073, 2020.

[41] M. Lerch. Ueber die Nichtdifferentiirbarkeit gewisser Functionen. *J. Reine Angew. Math.*, 103:126–138, 1888.

[42] P. Li and S.T. Yau. On the parabolic kernel of the Schrödinger operator. *Acta Math.*, 156(3-4):153–201, 1986.

[43] Z. Li and X. Pan. Some remarks on regularity criteria of axially symmetric Navier-Stokes equations. *Commun. Pure Appl. Anal.*, 18(3):1333–1350, 2019.

[44] Z. Li and Q. S. Zhang. Regularity of weak solutions of elliptic and parabolic equations with some critical or supercritical potentials. *J. Differential Equations*, 263(1):57–87, 2017.

[45] F. Lin and Q. S. Zhang. On ancient solutions of the heat equation. *Comm. Pure Appl. Math.*, 72(9):2006–2028, 2019.

[46] K. Masuda. On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation. *Proc. Japan Acad.*, 43:827–832, 1967.

[47] D. Morgenstern. Unendlich oft differenzierbare nicht-analytische Funktionen. *Math. Nachr.*, 12:74, 1954.

[48] Li P. *Geometric analysis*, volume 134 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012.

[49] A. Pringsheim. Zur Theorie der Taylor'schen Reihe und der analytischen Functionen mit beschränktem Existenzbereich. *Math. Ann.*, 42(2):153–184, 1893.

[50] K. Promislow. Time analyticity and Gevrey regularity for solutions of a class of dissipative partial differential equations. *Nonlinear Anal.*, 16(11):959–980, 1991.

[51] R. Schoen and S.T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, With a preface translated from the Chinese by Kaising Tso.

[52] R. W. Schwab and L. Silvestre. Regularity for parabolic integro-differential equations with very irregular kernels. *Anal. PDE*, 9(3):727–772, 2016.

[53] A. Tychonoff. Théorèmes d'unicité pour l'équation de la chaleur. *Mat. Sb.*, 42(2):199–216, 1935.

[54] P. Walczak. A proof of some theorem on the C^∞-functions of one variable which are not analytic. *Demonstratio Math.*, 4:209–213, 1972.

[55] D. V. Widder. Analytic solutions of the heat equation. *Duke Math. J.*, 29:497–503, 1962.

[56] B. Wong and Q. S. Zhang. Refined gradient bounds, Poisson equations and some applications to open Kähler manifolds. *Asian J. Math.*, 7(3):337–364, 2003.

[57] X. Yang, C. Zeng, and Q. S. Zhang. Smooth solutions to the heat equation which are nowhere analytic in time. *Submitted, arXiv:2109.07014*.

[58] C. Zeng. Time analyticity of the biharmonic heat equation, the heat equation with potentials and some nonlinear heat equations. *Commun. Pure Appl. Anal.*, 21(3):749–783, 2022.

[59] Q. S. Zhang. A formula for backward and control problems of the heat equation. *Submitted, arXiv:2005.08375.*

[60] Q. S. Zhang. *Sobolev inequalities, heat kernels under Ricci flow, and the Poincaré conjecture*. CRC Press, Boca Raton, FL, 2011.

[61] Q. S. Zhang. A note on time analyticity for ancient solutions of the heat equation. *Proc. Amer. Math. Soc.*, 148(4):1665–1670, 2020.