Title
Mental models for proportional reasoning

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Abstract

Three studies investigated the role of perceptual and quantitative situational factors on the structure of 5th- and 6th-graders' mental models. A task involved a carton of orange juice made from concentrate and water, and two glasses of different sizes filled from the carton. The children had to predict whether the two glasses would taste the same. We manipulated whether students were presented with physical, diagrammatic, photographic, or textual information. We also manipulated the type of relationship specified between quantities: qualitative, easy numerical, or difficult numerical. We found that for the diagram condition, difficult numerical relationships yielded poor performance, whereas the easy numerical and qualitative relationships yielded excellent performance. In contrast, in the physical condition, the easy numerical relationships yielded poor performance, whereas the difficult numerical and qualitative relationships yielded excellent performance. These and other results are interpreted by developing a sketch of the mental models pre-proportional children construct to reason about this quantitative situation, and describing how situational factors influence the construction of the models. For example, physical features led to models that captured the identity relationship between the juice in the glasses (e.g., the juice came from the same carton) whereas numerical features led to models that captured the relationship between the constituents of concentrate and water in each glass (e.g., within a glass there is more water than concentrate).

To reason with a model means that the forms of mental representation and transformations on those representations capture the structure of the world (e.g., Johnson-Laird, 1983). Using Palmer's (1978) terms, the representation of relations in a model is intrinsic, that is, the structure of the representation preserves the structure of the referent. However, given that any situation has multiple structures, what determines which structure is modeled? For example, consider the task illustrated in Figure 1. The question is whether the two glasses, filled from the same carton, taste the same. The problem has a causal structure: the juice in each glass was poured from the same carton and is therefore the same. The problem also has a quantitative structure: the ratio of concentrate and water is preserved under a transformation of total size. In this paper, we explore whether 5th- and 6th- grade students' pre-proportional reasoning can be understood in terms of mental models. We also develop evidence that the structures the students model are heavily determined by situational factors.

Our interest is in the way situations influence two key features that define the structure of students' pre-proportional models. One feature was whether the model supported quantitative inferences or not. For example, if one reasons that the juice in the two glasses comes from the same source, one is not reasoning from a model of quantity. On the other hand, if one reasons that more orange juice means more orange flavor, one has included quantitative information in the model. The second key feature was whether the students' models partitioned the orange juice into the constituents of water and concentrate. For example, a non-quantitative and partitioned model is implicated if one reasons that the carton was not shaken properly, and therefore one glass received only water. An example of a quantitative and partitioned model would occur if one considered the relative amounts of concentrate and water in each glass. Although the non-quantitative and non-partitioned model of the problem provides a simple solution, it is the partitioned and quantitative models that are necessary for proportional reasoning.

Research by Harel, Behr, Post, and Lesh (in press) used the problem shown in Figure 1a. Their work demonstrated that many 6th-graders decide that the juice in the two glasses would not taste the same, because more juice in one glass means more flavor. One interpretation of this surprising result is that the children ignored the contextual information that would lead to a causal model, in favor of a decontextualized application of mathematical knowledge. An alternative interpretation is that the children attempted to model the quantitative structure of the problem that was made salient by the presence of numerical information; however, they were not able to construct a model that supported reasoning about the complex ratios. If this interpretation is correct, then simplifying the quantitative structure of the problem or making the quantitative structure less salient would improve performance.

We predicted that presentation of physical materials would support the creation of models based primarily on the physical and causal structure of the situation. These models are non-quantitative and do not support inferences involving comparisions among the constituent quantities of concentrate and water. We also predicted that situations that do not involve numerical information would support the creation of models in which quantities are not partitioned into their constituent quantities. On the other hand, we predicted that the inclusion of numerical information would support partitioning a quantity space into its constituent quantities. This is because the numerical information highlights the decomposition of the quantity of orange juice into quantities of water and concentrate. We predicted that for middle-school students complex numerical relationships support a partitioned quantitative space, yet do not support the coordination of the quantitative relationships necessary to make a valid inference. However, simpler numerical relationships would support reasoning about the relationship between constituent quantities.

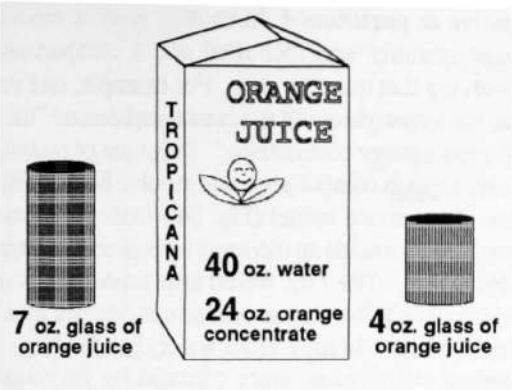

Figure 1a. Diagram for numerical condition.

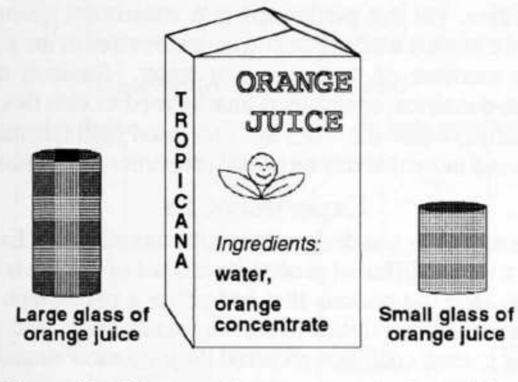

Figure 1b. Diagram for non-numerical condition.

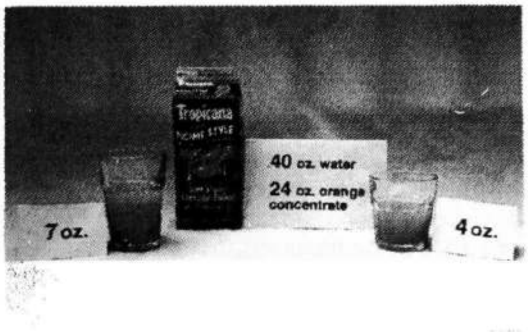

Figure 1c. Photograph for numerical condition.

Experiments 1a & b

Experiment 1a was designed to demonstrate that if the physical structure of a problem is made salient, children will tend to answer the problem correctly. However, if the quantities in a problem are emphasized without a supporting physical model students perform more poorly. The experiment was also used to develop a coding system that would be used to analyze children's explanations. The explanations provided evidence for the use of different types of models in students' reasoning.

Subjects and Design

98 6th-grade students from four classrooms at a Nashville inner-city school participated in a between-subjects, 2 x 2 experiment. A presentation factor had the levels of physical and diagrammatic presentation; a quantitative factor had the levels of non-numerical and numerical information. Subjects were randomly assigned to one of the four cells.

Materials and Procedure

Students in the physical conditions were shown a carton of orange juice and two glasses, whereas students in the diagram conditions were shown a diagram of a carton of orange juice and two glasses on a large poster board (Figure 1a). Cardboard labels were placed beside the glasses and juice carton in the physical conditions.

Procedures for the physical and diagram conditions were identical except that the experimenter poured the orange juice into each glass in the physical conditions. Different experimenters, reading from a script, addressed each of the groups of students as a whole. In the numerical conditions, an experimenter said "Here I have a [drawing of a] carton of orange juice. The juice was made with 40 ounces of water and 24 ounces of orange concentrate. This glass has 4 ounces of orange juice from the carton. And this glass has 7 ounces of orange juice from the carton." In the non-numerical conditions, an experimenter said "Here I have [a drawing of] a carton of orange juice. The juice was made with water and orange concentrate. This small glass has orange juice from the carton. And this large glass has orange juice from the carton."

After the presentation, students received worksheets that asked, "Would the large glass taste the same as the small glass?" and "If they would not taste the same, which one would taste more orangy?" Then the worksheet asked students to explain their answers. Students took approximately 10 minutes to complete the worksheets.

Results and Discussion

Student answers were coded correct if they stated that the two glasses would taste the same, or if they stated that the glasses would taste different because the carton of juice had not been shaken. Table 1 shows that the diagram-numerical students were the most likely to give an incorrect answer; $\chi 2$ (1df, n=98) = 4.37, $p<.05$. One interpretation is that the numbers in the diagram condition led students to construct inadequate quantitative models, whereas in the physical numerical condition, the students attended to the causal structure of the situation. An alternative hypothesis is that the numerical information was simply a distraction to the students. This hypothesis is addressed in Experiment 2.

Table 1. Percentages of correct answers and explanation types (Exp. 1a).

	Diagram		Physical	
	Number (n=27)	No-Num (n=22)	Number (n=23)	No-Num (n=26)
Correct	29.6%	68.2%	60.9%	57.7%
Explanations*				
NQ& NP	16.7%	38.5%	29.0%	30.3%
Q& NP	56.7%	46.2%	41.9%	48.5%
NQ& P	3.3%	15.4%	9.7%	9.1%
Q& P	23.3%	0.0%	19.4%	12.1%

* NQ = Non-quantitative, NP = Non-Partitioned,
Q = Quantitative, P = Partitioned.

Student explanations were coded as evidence for the use of different types of models. These models are delineated in Figure 2 with a graphical notation. For each model it is indicated whether it supports quantitative reasoning, and whether the total quantities of orange juice have been decomposed into constituent quantities of concentrate and water. Each of the models is described briefly below with representative student explanations.

Non-quantitative & non-partitioned. Non-quantitative and non-partitioned models capture the physical or causal properties of a situation (Fig. 2a). For example, one student wrote that the glasses would taste the same because "the orange juice was poured out of the same carton." Similarly, one could reason that both glasses contain the same kind of orange juice, focusing on an identity relationship (Fig. 2b). As one student wrote, "they both have the same orange juice and they can't change by themselves." Using either of these models, one has not made a quantitative comparison but instead reasoned about the invariant physical properties of the situation.

Quantitative & non-partitioned. Some students made explicit judgements that quantity is irrelevant in determining a quality such as taste. For example, "No matter how much you have in the glasses they will taste the same." This type of model (Fig. 2c) does not support reasoning about the the constituent quantities of water and concentrate that underlie proportional reasoning, yet takes into account the different quantities of orange juice in the two glasses. Some students reasoned that an increase in the measure of a quantity always results in an increase of the measure of a quality associated with that quantity (Fig. 2d). For example, one student wrote that "the 7 oz. glass has more so it will taste stronger." Again, this model captures only the total quantity of orange juice, not the constituent quantities of water and concentrate.

Non-quantitative & partitioned. Some students combined aspects of reasoning about the physical properties of the orange juice and one of the constituent quantities. For example, one student stated that the large glass would taste stronger because "it was poured last and the pulp is at the bottom." In this model (Fig. 2e), pouring plays a causal role in determining the quality of "oranginess", at the same time constituent quantities are distinguished in the carton. In a second model of this type (Figure 2f), students identify that the two glasses have "the same ingredients"; however, no quantitative statements are made.

Quantitative & partitioned. In another type of model, or constitutent quantity was identified and a comparison wa made involving that quantity only. For example, one studei wrote that the larger glass will taste stronger because "the larg glass has more orange concentrate." This type of model (Fi 2g) does not support comparisons that involve both constituer quantities. In a related model (Fig. 2h), students considere both ingredients but made an incorrect inference. For exampl one student wrote, "The 7 oz. would taste more orangy than oz. because it would have more orange concentrate than the oz. and the 4 oz. would have more water than the 7 oz."

All student explanations were captured by the models i Figures 2a - 2f except for one uninterpretable explanatior Table 1 shows that the most common type of model wa quantitative, but not partitioned into constituent quantitie These are models that take into account the sizes of the glasse but not amounts of concentrate or water. Students in th diagram-numerical condition primarily used models that wer quantitative, while the other students used both quantitativ models and those that rely on causal properties of the situatior

Experiment 1b

Experiment 1b was designed to replicate and extend Experi ment 1a with a different procedure and set of materials. 23 students received packets that had either a photograph, dia gram or textual description of the orange juice task. Half of th students in each condition received the numerical version an half received the non-numerical version. Based on the result of Experiment 1a, we predicted that students in the tw photograph conditions and the diagram non-numerical condi tion would outperform the students in the other three condi tions. We believed that the lack of a visual structure in the tw text conditions would diminish the chances that student would model the physical structure of the problem. As show in Table 2, these expectations were generally borne out wit 70.6% of the former group making correct responses compare to 57.4% correct responses in the latter group; $\chi 2$ (1df, n=234 = 4.43, $p < .05$. Also as predicted, the photograph and diagran non-numerical groups were more likely to use the non-quanti fied and non-partitioned models; 33.6% vs. 20.9%, $\chi 2 = 4.78$ $p < .05$. For this age group, these models almost invariably le to correct solutions (96.9%) compared to the quantitativ models (51.8%); $\chi 2 = 41.1$, $p < .001$.

Experiment 2

Experiment 2 was designed to develop evidence on th interaction between numerical situations and quantitative mod els. Greeno (1991) claims that models involving specifi multiplicative factors or divisors are easier to construct thar

Table 2. Percentages of correct answers and NP & NQ explanations (Exp. 1b).

	Text		Diagram		Photo	
	Number	No-Num	Number	No-num	Number	No-num
Correct	55.3%	65.6%	51.3%	75.6%	65.0%	71.1%
NQ & NP*	32.3%	15.6%	27.6%	42.4%	34.2%	50.0%

* Percentages of explanations within a condition. NQ = Non-quantitative, NP = Non-Partitioned.

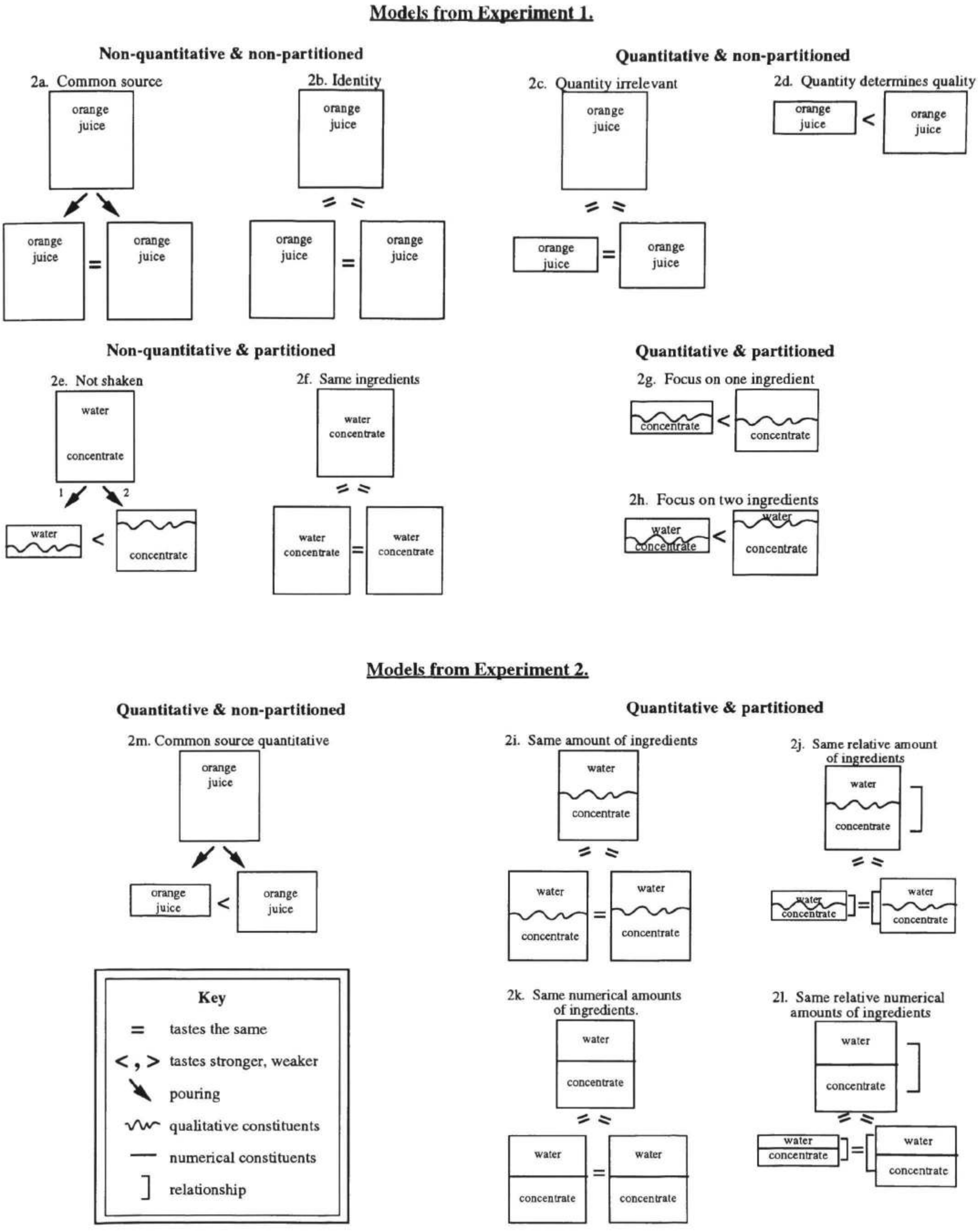

Figure 2. Students models.

those that do not involve specific values, and that models involving doubling and halving are probably available to many people. In line with this argument, Spinillo and Bryant (1991) have shown that judgements about proportionality involving the "half" boundary are easier for children. Therefore, in the diagram condition we expected that simpler numbers would enable students to recognize a relationship between constituent quantities, such as 1/2 and 1/2. This would allow them to map the relationship onto the glasses and construct "correct" quantitative and partitioned models.

There are a series of models that did not appear in Experiments 1a &b that we thought would be facilitated by the use of simpler numbers. These are shown in the lower right corner of Figure 2. Using the model in Figure 2i, one reasons that there would be the same amount of ingredients in both glasses. This model contains quantities that have been partitioned, but the constituent quantities themselves are equated rather than the relationship between the quantities. Using the model in Figure 2j, one reasons that the two glasses have the same relative amount of ingredients. This model contains quantities that have been partitioned, as well as the relationship between those partitions. Using the model in Figure 2k, one attempts to reason numerically about the relative amounts of ingredients, stating that the two glasses have the same numerical amounts of ingredients as in the carton. There is recognition that numerically something remains the same about the constituent quantities in the three containers; however, one directly imports the quantities themselves, rather than the relationship between the quantities. In the model 2l, one makes a correct numerical statement about the relative amounts of water and concentrate in the two glasses. One has formed a numerical relationship between the constituent quantities in the carton and mapped this onto both glasses.

In this study, we used simple numerical relationships rather than qualitative relationships. Our expectation was that numerical simplicity would yield different models in the physical condition than those proposed above for the diagram condition. In the physical condition, we expected the easy numbers and the physical format to elicit conflicting models. The prior experiments show that students who view more realistic formats tend not to construct a model of the relationship between the concentrate and water in the carton. This can become a problem if students model the quantitative structure relating the sizes of the glasses. We invited this model by making one glass double the size of the other, a relationship we thought would be easier to model than 7 to 4. If students model this quantitative relationship, they will have a quantitative difference between the sizes of the two glasses and a homogeneous quantity in the carton. As a result of the non-partitioned model of the juice, their models would not include a means for equating the glasses because they do not include a relationship between concentrate and water. As a result, when they reason with their models they will be more likely to conclude that a difference in quantity of juice leads to a difference in flavor. This model is represented as Figure 2m.

Unlike the prior studies, we asked students to mathematically prove their answers, with the idea that the form of their proofs would further illuminate their models. We also asked the students to re-evaluate their answers after doing the mathematical work. If the students had a robust quantitative model, then their confidence in their original answer should not decrease. However, if the students had weak or non-existent quantitative models, then the mathematics should draw them toward the inequality of the two glasses.

Subjects and Design

75 6th-grade students from four classrooms at a Nashville suburban school participated in a between-subjects, 2 x 2 experiment. A presentation factor had the levels of physical and diagrammatic presentation. A numerical difficulty factor had the levels of easy and hard. Student assignment to condition was counter-balanced by class.

Materials and Procedure

The hard-numerical conditions involved the same numbers as in the first two experiments. The easy-numerical conditions involved 20 ounces of water and 20 ounces of concentrate, with 2 and 4 ounce glasses. An experimenter presented the physical and diagram materials as in Experiment 1a. A worksheet was then given to each student. Students chose whether the glasses would taste the same or different. Additionally, they were asked to rate how sure they were of their answer from 1 (very unsure) to 5 (very sure). Students were then asked to "Show why your answer is correct using arithmetic". They were then asked to chose again whether glasses would taste the same or different, and to rate how confident they were in their answers. Students took 20 minutes to complete the questionnaire.

Results and Discussion

Table 3 shows the percentage of correct answers for each condition before and after students attempted a mathematical solution. The results were analyzed in a multivariate analysis of variance. Confidence ratings were given a negative value for different responses and a positive value for same responses. Students' confidence scores on the first and second answer served as dependent measures with format and numerical difficulty as crossed independent variables. Over both responses, there is a format by number interaction in which the diagram-easy and the physical-hard problems yielded the most correct responses; $F(1,71)=8.76$, $p<.01$. There was also a presentation format by trial interaction whereby the physical conditions dropped in accuracy more than the diagram conditions; $F(1,71)=7.97$, $p<.01$.

Table 3. Percentages of correct answers and NP & NQ explanations (Exp. 2).

	Diagram		Physical	
	Hard (n=19)	Easy (n=19)	Hard (n=19)	Easy (n=18)
Correct 1st	68.4%	89.5%	94.7%	66.7%
Correct 2nd	47.4%	89.5%	52.6%	38.9%
NQ & NP*	21.1%	21.1%	47.4%	16.7%

* Coding does not consider students' mathematical work.
NQ = Non-quantitative, NP = Non-Partitioned.

These results can be explicated by considering the models students constructed in each condition. The strong initial performance in the physical-hard condition replicates the first two studies in which we also found that a large percentage of the students employed non-partitioned and non-quantitative models. However, students' mathematical work undermined their judgment of the equivalence of the juice in the two glasses. The mathematics required inferences involving the constituent quantities of concentrate and water which are not supported by a physical model. As further evidence for this conjecture, it may be noted that the diagram-hard and physical-hard conditions had nearly equivalent percentages on their second answers. This is because, in both cases, the directive to solve the problem arithmetically led students to construct similar models of the complex quantities of the problem.

The diagram-easy condition presents the first indication of quantitative proportional reasoning. A typical move was to point out that 20 and 20 make a balance of 1/2 and 1/2. The symmetry of the relationship made it easy to map the relationship onto the 4 and 2 ounce glasses. The fact that the percentages of correct answers did not change on the second trial shows that students had initially constructed quantitative models that could support explicit quantitative manipulation.

Here we see that students can construct models of proportional relations if the ratios are of simple, specific quantities. The role of specific numbers and numerical relationships supports the claim that students use quantified models in their reasoning. Had they simply been applying a set of rules with general application, specific values should not have led to superior performance when students considered constituent quantities. However, in Experiments 1a & b we found that solutions that included constituent quantities were generally incorrect, even when there were no difficult quantitative relationships specified in the problem (e.g., diagram-qualitative condition of Experiment 1b). These results implicate specific models of specific situations rather than generally applicable rules.

Of the models we observed, the only type that supports a judgment about proportionality were the quantitative and partitioned models. Using these models, students can compare the relationship between the relative amounts of concentrate and water in the glasses and carton, thus creating a second order relationship, or a "relations between relations" (Inhelder and Piaget, 1958). In the models we found, students were making part-part comparisons between concentrate and water, rather than calculating the proportion of either concentrate or water to the total amount of orange juice. Previous work has shown that children are able to use part-part relations to make proportional judgements earlier than part-whole relations (Noelting, 1980a; Noelting, 1980b; Spinillo & Bryant, 1991). A useful next step in our research would be to determine what situations could lead children to compare a part to the whole.

An intriguing result from Experiment 2 is the poor performance of the students in the physical-easy condition. Our hypothesis was that the physical presentation of the problem would yield a non-partitioned representation of the problem. However, we also thought that the simple 4 ounce to 2 ounce relationship between the glasses would support the construction of a quantitative model of the two glasses. This presents a problem in that the students would have a specific quantitative difference between the sizes of the glasses, but without the ratio of the constituents they would have no way to maintain their equivalence. As a result, the students would be more likely to assert that more juice means more flavor. If we consider errors, we find that 58.3% of the errors of the physical-easy students were based on the claim that more juice yields more flavor. In contrast, combining errors in the other conditions (the n's are too small to treat separately), we find that only 25% of the errors were based on this misconception. This difference is significant; $Z=2.6$, $n=32$, $p<.01$.

Conclusion

In the current studies there were several results that point toward the deployment of mental models tied to specific situations. The regularity of inferences in response to situational factors indicates that subjects were not simply missing a set of rules necessary for proportional reasoning. The strong results from the diagram-easy number condition of Experiment 2 also show that these situational factors (viz. numerical information) were not merely distracting students or leading them down mindless paths of symbol pushing. The support of specific values in the diagram-easy condition also implicates models in that the specific values allow students to construct determinate structures that support specific transformations. The effect of combining easy numbers with the physical situation also implicates mental models under our interpretation. In this condition, student performance was poor, as we predicted. Under a rule-based interpretation, this would be a unlikely prediction because the student would presumably have two rules supporting a "same taste" answer: a causal rule that worked when there were no numbers, and a quantitative rule that worked when there were simple numbers. Given these findings, we believe there is a place for a mental model approach in understanding the development of proportional reasoning.

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