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Essays on Decisions Under Limited Attention

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To my family and Z. Chang for all their love and support.

ABSTRACT OF THE DISSERTATION

Essays on Decisions Under Limited Attention

by

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This dissertation consists of three essays that investigate decisions under limited attention. Chapter 1 provides an introduction for this dissertation. In Chapter 2, I study additive representations under idempotent attention. Additive representations suggest that the utility of a menu equals the sum of the utility of alternatives in it, which implicitly assumes that a decision maker (DM) has full attention. However, many theoretical and empirical studies have argued that the DM does not always pay full attention. The purpose of this paper is to extend our understanding of additive representations to the context of limited attention. Specifically, we focus on additive representations when attention rules are idempotent, which are generalizations of attention filters. We propose the characterization of additive representations under idempotent attention as well as special cases of it: additive representations under attention filter. This model can also be applied to simple cardinality-based ordering and Borda scores when the DM has limited attention.

In Chapter 3, I focus on satisficing under idempotent attention. Satisficing choice pattern has been accused of a lack of cognitive ability in perceiving and analyzing alterna-

tives. In an effort to disentangle the limitations of noticing alternatives from other factors, this paper proposes a model of satisficing under limited attention. Our focus centers on the idempotent attention rules, leading to Satisficing under Idempotent Attention (SIA). Our study provides a characterization of SIA as well as a discussion of revealed attention and preference practices. Notably, these outcomes stem from choices made on a subset of menus. The revealed attention and preference, however, remain inherently non-unique. Additionally, considering the idempotent nature of attention filters and competition filters, we also present distinctive characterizations of satisficing under these two attention rules.

Chapter 4 investigates choices under limited attention where attention is formed by interactions between alternatives. Specifically, given a choice problem, the presence of an alternative can trigger or block the attention of other options. An alternative captures the DM's attention when the effects of triggers outweigh the effects of blockers. This paper also provides valuable insights into the concept of signed orders within the decision-making process.

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Chapter 1

Introduction

Many economic theories implicitly assume that individuals pay attention to every item when making decisions. However, numerous theoretical and empirical studies suggest that a decision maker (DM) might overlook some alternatives due to their limited cognitive capacity (cf. Cowan [2001], Cowan et al. [2005], Ok et al. [2015], Masatlioglu et al. [2012]). In such situations, the DM exhibits limited attention. She focuses on some alternatives rather than all options. The three essays in this thesis concentrate on decision-making processes under limited attention.

Chapter 2, titled "Additive Representation under Idempotent Attention," explores a scenario where the DM evaluates a menu by summing the utility of the attended alternatives. To simplify the analysis, we assume (i) each alternative has a nonnegative utility, and (ii) the DM employs an idempotent attention rule. The first assumption implies that the DM can discard any undesirable alternative in a menu. The second assumption restricts the DM's attention: When the DM applies her attention rule to the same menu multiple times, the outcome remains consistent with the initial application. This suggests that the DM perceives a menu and its attended alternatives as identical. Given preferences over menus, our goal is to find axioms that characterize additive representations under idempotent attention. Interestingly, we can confine our analysis to preferences within a subcollection of menus. This subcollection contains all the information necessary to reveal the DM's attention rule and utility function.

Chapter 3, "Satisficing under Limited Attention," investigates satisficing by assuming idempotent attention. When presented with a menu, the DM selects an alternative. Contrary to classical theories where the DM picks the best alternative, satisficing implies she chooses an option that meets her internal criteria (see Simon [1959], Simon [1982], Simon [1997]). From a revealed preferences perspective, satisficing suggests that the DM prefers the chosen alternatives to the unchosen ones. With limited attention, these interpretations only hold among the attended alternatives. As in Chapter 2, we can focus on choices within a subcollection of menus. This allows us to reveal her preferences and attention based on those choices.

Instead of restricting our analysis to a specific attention rule and viewing attention as a "black box," we aim to understand the underlying rationale for attention formation. Chapter 4, "Attention Formation: Triggers and Blockers," examines how a DM's attention to an alternative is influenced by two opposing forces: attention triggers, which attract the DM's focus, and attention blockers, which divert it. When faced with a choice problem, if the power of attention triggers for an alternative surpasses that of attention blockers, the DM pays attention to the alternative; otherwise, she ignores it. We employ signed orders to model the effects of attention triggers and blockers for each alternative. In order to determine her attention, the DM initially maximizes the signed order for each alternative. If the maximizer for an alternative is its attention trigger, she attends to the alternative. Subsequently, she selects the best alternative based on her preferences among those she pays attention to. This model, along with its variations, also provides insights into the "decoy effect."

Chapter 2

Additive Representation under Idempotent Attention

2.1 Introduction

People often need to choose from bundles containing numerous items, such as online streaming, gaming, music, and personal styling services. These bundles exhibit two distinct characteristics: (i) individuals can enjoy multiple items within a bundle; (ii) the sheer number of items can make it challenging for individuals to pay attention to each of them. A natural question arises: What is the utility of a bundle when individuals have limited attention?

Fishburn [1992b] introduces the additive representation (AR) to assign utility to bundles. Under the AR, a decision maker (DM) determines the bundle's utility by summing each item's utility. This utility is referred to as the additive utility of the bundle. The DM prefers the bundle with a larger additive utility between any two bundles.

Additive utility provides a way to evaluate bundles. [1] It is used in various studies in economics such as purely utilitarian social welfare functions, preferences over coalitions (Bogomolnaia and Jackson [2002]), measures of freedom of choice (Pattanaik and Xu [1990]), and Borda scores (Darmann and Klamler [2019]). Though it captures the idea that the utility of a bundle depends on the multiple items it contains, its applicability is based on the assumption of full attention.

Many theoretical studies (Masatlioglu et al. [2012], Manzini and Mariotti [2014], Lleras et al. [2017]) and empirical research (Goeree [2008], Wilson [2008]) argue that limited attention is more realistic. Under this assumption, the majority of them investigate DMs who select the best alternatives from attended alternatives. These behaviors lead to indirect utility representations, where the utility of a choice set is determined by the utility of the best attended alternative in it. However, there is a distinction between indirect utility and additive utility under limited attention. For instance, if a DM only overlooks the second-best alternative in a set S, the additive utility will decrease, while the indirect utility remains unchanged.

Under limited attention, additive utility can naturally be extended to the summation of attended items. For any bundle S, let $\Gamma(S)$ denote the DM's attention on S, where $\Gamma(S)$ is a nonempty subset of S. Under this framework, the DM pays attention to every

[1] Additive utility implies that the DM assigns the same utility to each alternative in every bundle. This indicates that there are no interactions, such as competing and completing, between alternatives.

alternative within $\Gamma(S)$ and ignores those outside it. Here, Γ represents the attention rule. Γ is idempotent when $\Gamma(S) = \Gamma(\Gamma(S))$ for every bundle S.

Idempotent attention rules generalize attention filters (Masatlioglu et al. [2012]). The attention filter proposes that deleting any subset of omitted alternatives in S should not affect $\Gamma(S)$. Put differently, the DM cannot distinguish between bundles lying between $\Gamma(S)$ and S. Idempotent attention rules, in contrast, assume that eliminating all omitted alternatives in S will not affect $\Gamma(S)$. Here, the DM regards S and $\Gamma(S)$ as identical.

This paper characterizes the additive representations under idempotent attention (AR-IA) and the additive representation under attention filter (AR-AF). With idempotent attention rules, we can focus our analysis on a sub-collection of bundles, the image of Γ, rather than the entire collection.

Idempotent attention rules can also be applied to other classical measures under limited attention. Two special cases are introduced: the top k AR and the simple cardinality-based ordering under idempotent attention (SCO-IA). The top k AR assumes that a decision-maker aggregates the utility of the best k alternatives from a bundle, restricting the DM's attention to certain items. The SCO-IA, on the other hand, assumes that the utility of each item is the same, extending the ranking proposed by Pattanaik and Xu [1990], which reflects the degree of freedom of choice for the DM with limited attention.

The structure of this paper is as follows: Section 2 formally introduces the model. Section 3 provides characterizations of AR-IAs and AR-AFs. In Section 4, we discuss topics related to revealed attention. Section 5 covers two applications of the AR-IA. Finally, Section 6 concludes the paper.

2.2 The Model

Let X be a nonempty finite set, and let $\mathcal{X}$ be the collection of all nonempty subsets of X. Here, X represents the set of all alternatives of interest, and $\mathcal{X}$ represents the collection of all bundles.[2] The DM's preference is denoted by a binary relation $\succcurlyeq$ on $\mathcal{X}$. For any two bundles S and T, $S \succcurlyeq T$ means that the DM considers S weakly preferable to T. Strict preference, denoted by $\succ$, and indifference, denoted by $\sim$, are defined as follows: Given $S, T \in \mathcal{X}$, $S \succ T$ if $S \succcurlyeq T$ but not $T \succcurlyeq S$, and $S \sim T$ if $S \succcurlyeq T$ and $T \succcurlyeq S$.

2.2.1 Additive Representations under Limited Attention

Based on Fishburn [1992b], a preference relation $\succcurlyeq$ on $\mathcal{X}$ admits an **Additive Representation (AR)** if there exists $u : X \to \mathbb{R}$ such that for any $S, T \in \mathcal{X}$, $S \succcurlyeq T$ if and only if $\sum_{s \in S} u(s) \geq \sum_{t \in T} u(t)$. The AR implicitly assumes that the DM has full attention, which is not always the case. This paper examines additive representations under limited attention.

The DM's attention can be mathematically defined by mappings that assign each bundle to one of its nonempty subsets. These mappings are called attention rules, and this paper focuses on idempotent attention rules.

Definition 1. *A mapping $\Gamma : \mathcal{X} \to \mathcal{X}$ is an* ***idempotent attention rule*** *if $\Gamma(S) \subseteq S$ and $\Gamma(S) = \Gamma(\Gamma(S))$ for all $S \in \mathcal{X}$.*

[2] In Economics, bundles are often represented as vectors in $\mathbb{R}^n$. If $X = \{x_1, ..., x_{|X|}\}$, then $\mathcal{X}$ is equivalent to $\{0,1\}^{|X|}$. Consequently, a bundle S can be expressed in vector form, where each element corresponds to whether the corresponding element of X is present in the bundle S. For instance, $\{x_2, x_4\}$ in vector form is $(0, 1, 0, 1, ...)$.

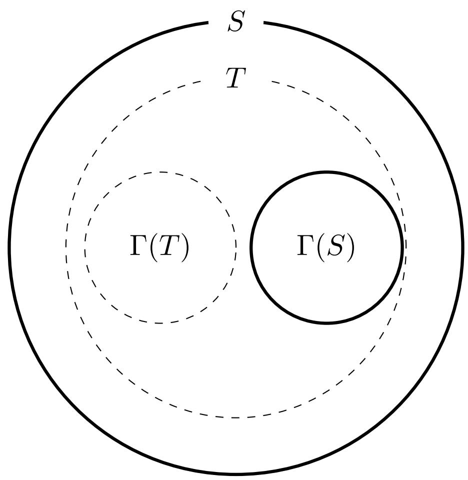

Figure 2.1: Idempotent Attention but not Attention Filter

Idempotent attention rules indicate that the DM focuses on the same alternatives in both S and $\Gamma(S)$. Essentially, the DM is unaware of the ignored alternatives and treats S and $\Gamma(S)$ as identical. Furthermore, idempotent attention rules generalize attention filters introduced in Masatlioglu et al. [2012]. Unlike idempotent attention rules, attention filters assume that the DM considers all sets between $\Gamma(S)$ and S as identical.

Definition 2. *An attention rule Γ is an **attention filter** if for any $S \in \mathcal{X}$, $\Gamma(T) = \Gamma(S)$ whenever $\Gamma(S) \subseteq T \subseteq S$.*

The disparities between idempotent attention rules and attention filters arise from their assumptions regarding overlooked alternatives. Attention filters posit that expanding $\Gamma(S)$ with any subset of overlooked alternatives should not influence the DM's attention on S. Conversely, the idempotent attention rule permits this alteration. In Figure **??**, given any two bundles S and T, the DM pays attention to $\Gamma(S)$ and $\Gamma(T)$, respectively. Attention is altered by adding some omitted alternatives from $T \setminus \Gamma(S)$ to $\Gamma(S)$.

We investigate additive representations under idempotent attention (AR-IAs) and additive representations under attention filters (AR-AFs). For ease of comparison, we restrict our scope to non-negative utility functions by assuming that the DM can freely discard any alternatives with negative utility.[3].

Definition 3. *$\succcurlyeq$ on $\mathcal{X}$ admits an* ***Additive Representation under Idempotent Attention (AR-IA)*** *if there exist an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$, and a function $u : X \to \mathbb{R}^+$ such that for any $S, T \in \mathcal{X}$,*

$$S \succcurlyeq T \iff \sum_{s \in \Gamma(S)} u(s) \geq \sum_{t \in \Gamma(T)} u(t).$$

Given any pair of above Γ and u, we say that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u). Moreover, $\succcurlyeq$ on $\mathcal{X}$ admits an ***Additive Representation under Attention Filter (AR-AF)*** *under (Γ, u) if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u) where Γ is an attention filter.*

If $\succcurlyeq$ on $\mathcal{X}$ admits ARs, AR-AFs and AR-IAs must also be admitted. The converse, however, is not necessarily true.

Example 1. *Let $X = xyz$[4]. $\succcurlyeq$ on $\mathcal{X}$ is $xy \succ xyz \sim xz \sim x \succ yz \sim y \succ z$. It admits an AR-AF when $u(x) = 2, u(y) = 1, u(z) = 0,$ and*

$$\Gamma(S) = \begin{cases} xz & \text{if } S = xyz, \\ S & \text{if } S \neq xyz. \end{cases}$$

[3]For the AR-IA, we can relax the assumption of nonnegative utility. The characterization is provided in the Appendix A.5

[4]Henceafter, we designate sets multipilicatively. For example, $X = \{x, y, z\}$ is denoted as xyz.

However, if it admits an AR, $xy \succ xyz$ implies that $u'(z) < 0$ for any consistent utility function u'.

Due to the distinction between idempotent attention rules and attention filters, certain preferences on $\mathcal{X}$ may admit AR-IAs but not AR-AFs.

Example 2. *Let $X = xyz$. $\succcurlyeq$ on $\mathcal{X}$ is $xy \succ xz \succ xyz \sim x \succ yz \succ y \succ z$. Let $u(x) = 4, u(y) = 2, u(z) = 1$, and*

$$\Gamma(S) = \begin{cases} x & \text{if } S = xyz, \\ \\ S & \text{if } S \neq xyz. \end{cases}$$

$\succcurlyeq$ admits an AR-IA under (Γ, u). However, if $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs under (Γ', u'), $\Gamma'(xyz)$ should be x or xyz. If $\Gamma'(xyz) = x$, then $\Gamma'(xy) = \Gamma'(xz) = x$ which cannot be true. If $\Gamma'(xyz) = xyz$, then $u'(x) + u'(y) + u'(z) = u'(x)$ which suggests that $u'(z) < 0$.[5]

2.3 Characterizations

2.3.1 The Characterization of AR-IA

The AR-IA assigns a positive real number to each bundle. Binary comparisons between bundles are equivalent to real numbers comparisons. As a result, $\succcurlyeq$ must be both complete and transitive.

[5]Despite relaxing the assumption of utility functions, $\succcurlyeq$ still does not admit an AR-AF. Since $xz \succ x$ implies that $u'(x) + u'(z) > u'(x)$, i.e., $u'(z) > 0$.

Axiom 1. ***(WO: Weak Order)*** *$\succcurlyeq$ on $\mathcal{X}$ is complete and transitive.*

Under the AR-IA, the DM should regard S as indifferent to one of its subset $\Gamma(S)$. If S has indifferent proper subsets, it is possible that $\Gamma(S) \subset S$.[6] However, this scenario is invalid for any S that lacks indifferent proper subsets. Such sets are referred to as basic sets.

Definition 4. *$S \in \mathcal{X}$ is **basic** if there is no $T \subset S$ such that $T \sim S$.*

The collection of basic sets is denoted by $\mathcal{B}$. A basic set B is a corresponding basic set of S if $B \subseteq S$ and $S \sim B$. $\mathcal{B}(S)$ refers to the collection of corresponding basic sets of S.

Proposition 1. *If $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), then $\Gamma(B) = B$ for all $B \in \mathcal{B}$.* [7]

Proof. By contradiction, suppose that there is a basic set B such that $B \neq \Gamma(B)$. That is, $\Gamma(B) \subset B$. As Γ is idempotent, we have $\Gamma(B) = \Gamma(\Gamma(B))$. The additive utility of $\Gamma(\Gamma(B))$ equals the additive utility of $\Gamma(B)$. Hence, $B \sim \Gamma(B)$, which implies that $B \notin \mathcal{B}$. $\square$

As basic sets must attract full attention, the AR-IA requires that $\succcurlyeq$ on $\mathcal{B}$ admit ARs. For any $B_1, B_2 \in \mathcal{B}$, $\sum_{b \in B_1} u(b) \geq \sum_{b \in B_2} u(b)$ whenever $B_1 \succcurlyeq B_2$. The utility consistent with the AR-IA must solve the system of inequalities induced by $\succcurlyeq$ on basic sets. The condition for the existence of solutions to a finite system of linear inequalities can be found in Kraft et al. [1959], Scott [1964], Krantz et al. [1971] and Fishburn [1992b].

[6] In this paper, we use the symbols $\subseteq$ and $\subset$ to indicate subset and proper subset, respectively. $T \subset S$ means that T is a proper subset of S.

[7] This Proposition can be generalized to any utility representation under idempotent attention. Formally, we say $\succcurlyeq$ on $\mathcal{X}$ admits a U-representation under idempotent attention if $S \succcurlyeq T \iff U(\Gamma(S)) \geq U(\Gamma(T))$, where Γ is idempotent. Evidently, basic sets must catch full attention under this representation.

Axiom 2. ***(NR: Nonnegative Remainders)*** *There do not exist a positive integer m and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$ where $\mathbb{1}$ is the indicator function.*

NR is a stricter condition than Monotonicity on basic sets: $B_1 \supseteq B_2 \implies B_1 \succcurlyeq B_2$ for any $B_1, B_2 \in \mathcal{B}$. In particular, when $m = 1$, NR combined with the complete preference is equivalent to Monotonicity on basic sets. To illustrate, let $m = 1$. In this case, NR asserts that there is no $B_1, B_2 \in \mathcal{B}$ with $B_2 \succ B_1$ such that $\mathbb{1}_{B_2}(x) \leq \mathbb{1}_{B_1}(x)$ for all $x \in X$. It is equivalent to stating that if $B_2 \subseteq B_1$, then B_2 is not strictly preferred to B_1. By completeness, $B_1 \succcurlyeq B_2$.

However, WO and Monotonocity on $\mathcal{B}$ do not guarantee that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA.

Example 3. *Let $X = xyz$ and $\succcurlyeq$ on $\mathcal{S}$ is $xyz \succ xz \succ xy \succ x \succ yz \succ y \succ z$. Note that all sets are basic, and $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and Monotonicity. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), we then have*

$$xz \succ xy \iff u(x) + u(z) > u(x) + u(y) \iff z \succ y.$$

However, we have $y \succ z$. To check NR, we can consider $\{xz, y\}$ and $\{xy, z\}$. We know that $xz \succ xy$ and $y \succ z$, but x, y, z shows one time in both collections.

NR can be interpreted as a generalization of Monotonicity. Consider any two collections of basic sets $\{S_n\}_{n=1}^m$ and $\{T_n\}_{n=1}^m$. We say that $\{S_n\}_{n=1}^m$ contains less content than $\{T_n\}_{n=1}^m$ if the former collection contains fewer alternative x than the latter for every

$x \in X$. Equivalently, $\sum_{n=1}^{m} \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^{m} \mathbb{1}_{T_n}(x)$ for all $x \in X$. According to Monotonicity, the DM cannot prefer a smaller set. Analogously, NR requires that the DM cannot prefer $\{S_n\}_{n=1}^{m}$ to $\{T_n\}_{n=1}^{m}$, i.e., the preference cannot be $S_n \succcurlyeq T_n$ for all n and $S_n \succ T_n$ for some n.

WO and NR are the axioms for characterizing the AR-IA. These two axioms first ensure the AR on $\mathcal{B}$. We then map every S to one of its corresponding basic sets to construct the idempotent attention rule.

Theorem 1. *$\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and NR.*

Proof. See Appendix A.1. □

2.3.2 The Characterization of AR-AF

Now consider the characterization of AR-AFs. As they are special cases of AR-IAs, we can simplify the analysis by focusing on the requirement for attention filters. [8]

The AR-IA permits us to assign $\Gamma(S) = B$ for a $B \in \mathcal{B}(S)$. On the contrary, for some $\succcurlyeq$ that admits an AR-AF, $\Gamma(S)$ must be non-basic. For instance, in Example 1, if Γ is an attention filter, then $\Gamma(xyz)$ must be xz. Consequently, z must have zero utility.

The set of alternatives with zero utility plays a crucial role in characterizing the AR-IA. If $\Gamma(S)$ is an indifferent non-basic proper subset of S, then the set difference between $\Gamma(S)$ and any of $\Gamma(S)$'s corresponding basic sets must have zero utility. Under the assumption of nonnegative utility, these alternatives must be the least preferred. We refer to them as dominated alternatives and denote them as $MIN(X, \succcurlyeq) := \{x \in X : y \succcurlyeq x \; for \; all \; y \in X\}$.

[8] It is worth noting that this logic also applies to characterizing the AR-IA. WO and NR give us utility functions. The requirement for idempotent attention is $\mathcal{B}(S) \neq \emptyset$ for all $S \in \mathcal{X}$, which is implicitly assumed in the definition of basic sets.

However, the dominated alternatives do not necessarily have zero utility. For instance, in Example 2, the dominated alternative z must have positive utility. We present a condition for determining when dominated alternatives have zero utility.

Condition 1 (ENS: Existence of Null Sets)**.** *There do not exist a positive integer m and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{S_n}(x) > \sum_{x \in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{T_n}(x)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \notin MIN(X, \succcurlyeq)$.*

When $m = 1$, ENS amounts to there not being basic sets S and T with $S \succ T$ such that $T \subset S$ and $S \setminus T \subseteq MIN(X, \succcurlyeq)$. In other words, if the DM strictly prefers a larger bundle, the reason cannot be that the larger bundle contains more dominated alternatives. However, the $m = 1$ case is not sufficient for the existence of null alternatives under the AR-IA.

Example 4. *Let $X = abcd$. The preference on $\mathcal{B}$ is $bcd \succ ab \succ a \succ b \succ c \succ d$.*[9] *This preference satisfies NR. Moreover, there are no basic sets S and T such that $T \subset S$ and $S \setminus T \subseteq MIN(X, \succcurlyeq)$. $bcd \succ ab$ implies that $u(c) + u(d) > u(a)$ for all consistent u. We then have $u(d) > 0$ for all consistent u because $a \succ c$.*

ENS refines the solution sets to the linear inequalities induced by $\succcurlyeq$ on $\mathcal{B}$. It identifies the condition under which the dominated alternatives can have zero utility.

[9] We slightly abuse the concept of basic sets. Because basic sets are defined as sets that satisfy some conditions on preference, the preference given in this example should be understood as a part of the preference on $\mathcal{X}$. In this example, the preference can be extended to the preference on $\mathcal{X}$ in which the sets listed are basic. To see this, for a "non-basic" set S, we can let $S \sim a$ if $a \in S$, $S \sim b$ if $a \notin S$ and $b \in S$, $S \sim c$ if $a, b \notin S$ and $c \in S$, and $S \sim d$ if $a, b, c \notin S$ and $d \in S$.

Proposition 2. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. There exists a consistent u such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$ if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies ENS.*

Proof. See Appendix A.2. $\square$

Let N be the collection of alternatives that may have zero utility, and infer N as the set of null alternatives.

$$N = \begin{cases} MIN(X, \succcurlyeq) & \text{if } \succcurlyeq \text{ satisfies ENS,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Axiom 3. ***(IB: Indifferent Betweenness)*** *For each $S \in \mathcal{X}$ there exist a set $B \in \mathcal{B}(S)$ and a $T \in \mathcal{X}$ where $B \subseteq T \subseteq S$ and $T \setminus B \subseteq N$, such that $T \sim Y \sim S \sim B$ for all Y with $T \subseteq Y \subseteq S$.*

Let $\mathcal{IB}(S)$ be the collection of $T \in \mathcal{X}$ such that: (i) for all Y with $T \subseteq Y \subseteq S$, $T \sim Y \sim S$, and (ii) there is a $B \in \mathcal{B}(S)$ with $B \subseteq T$ such that $T \setminus B \subseteq N$.

WO, NR, and IB can characterize the AR-AF.[10] WO and NR characterize the AR-IA, while IB emphasizes the requirements of attention filters. For AR-AFs, we only need to select the pairs of (Γ, u) where Γ is an attention filter from all consistent pairs of AR-IAs. For any S, if a $B \in \mathcal{B}(S)$ satisfies that $T \sim B \sim S$ for all $B \subseteq T \subseteq S$, then we can let $\Gamma(S) = B$. If no such basic set of S exists, IB implies that there is an indifferent subset T of S such that T only contains additional null alternatives compared to one of S's corresponding basic sets. We then can let $\Gamma(S) = T$.

[10]The checking of independence of these three axioms is attached in Appendix A.4.

Theorem 2. $\succcurlyeq$ *on* $\mathcal{X}$ *admits an AR-AF if and only if* $\succcurlyeq$ *on* $\mathcal{X}$ *satisfies WO, NR and IB.*

Proof. See Appendix A.3. $\square$

2.3.3 Uniqueness of Representations

For both AR-IAs and AR-AFs, multiple pairs of (Γ, u) can represent the same $\succcurlyeq$. For example, if u is a consistent utility function of the AR-IA or AR-AF under Γ, then (cu, Γ) with $c > 0$ will also be a consistent pair of the AR-IA or AR-AF.[11]

In this section, we examine the uniqueness of utility functions and attention rules for $\succcurlyeq$ on $\mathcal{X}$ that admit either AR-IAs or AR-AFs.[12] Henceforth, we use the notation $\{(\Gamma_i, u_i)\}_{i\in I_{\succcurlyeq}}$ and $\{(\Gamma_j, u_j)\}_{j\in J_{\succcurlyeq}}$ to represent the collection of consistent pairs for the AR-IA and the AR-AF, respectively.

The utility function is a solution to the system of linear inequalities induced by the preference on $\mathcal{B}$. The collection of solutions $\mathcal{U}_{\succcurlyeq}$ is defined as follows:

$$\mathcal{U}_{\succcurlyeq} = \left\{ u \in \mathbb{R}_+^X : \sum_{b\in B_1} u(b) \geq \sum_{b\in B_2} u(b) \text{ for all } B_1, B_2 \in \mathcal{B} \text{ with } B_1 \succcurlyeq B_2 \right\}. \tag{2.1}$$

The following Corollary follows directly from Proposition 1 and Theorem 1.

Corollary 3. *Assume* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs. For any* $u : X \to \mathbb{R}^+$*, there is an idempotent attention rule* Γ_i *such that* $(\Gamma_i, u) \in \{(\Gamma_i, u_i)\}_{i\in I_{\succcurlyeq}}$ *if and only if* $u \in \mathcal{U}_{\succcurlyeq}$*.*

For the AR-AF, even though the utility of S equals to every $B \in \mathcal{B}(S)$, there is no attention filter Γ_j such that $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j\in J_{\succcurlyeq}}$ for some $u \in \mathcal{U}_{\succcurlyeq}$. Consider

[11] If the converse holds, the utility function is said to be unique under similarity transformation (see Fishburn [1992b]). In Appendix A.5, we provide several results related to it.

[12] In Appendix A.5, we also investigate the uniqueness of the consistent pairs (Γ, u).

the Example 1, where the collection of basic sets is $\{xy, x, y, z\}$. It is easy to verify that $u(x) = 3$, $u(y) = 2$, and $u(z) = 1$ is a utility function in $\mathcal{U}_{\succcurlyeq}$. However, there is no attention filter Γ_j such that (Γ_j, u) forms a consistent pair of the AR-AF, as we cannot identify a suitable image for $\Gamma_j(xyz)$.

Corollary 4. *Assume $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. For a $u : X \to \mathbb{R}^+$, if there is an attention filter Γ_j such that $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ then $u \in \mathcal{U}_{\succcurlyeq}$. The converse holds when $N = \emptyset$.*

Proof. The first half of the statement is straightforward to illustrate. For the second half of the statement, assume that $N = \emptyset$. For any $S \in \mathcal{X}$, there is a $B \in \mathcal{B}(S)$ such that for all Y with $B \subseteq Y \subseteq S$, $Y \sim B \sim S$. By the similar construction as in the proof of Theorem 2, we can have an attention filter Γ_j where $\Gamma_j(\mathcal{X}) = \mathcal{B}$. Further, $(\Gamma_j, u) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ for all $u \in \mathcal{U}_{\succcurlyeq}$. $\square$

When all consistent pairs share the same attention rule, we can identify the DM's attention for sure. We refer to this scenario as the attention rule is unique.

Definition 5. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. The idempotent attention rule Γ_i is **unique** if $\Gamma_i = \Gamma_{i'}$ for all $i, i' \in I_{\succcurlyeq}$. Similarly, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. The attention filter Γ_j is **unique** if $\Gamma_j = \Gamma_{j'}$ for all $j, j' \in J_{\succcurlyeq}$.*

In the AR-IA, we can treat a corresponding basic set B of S as $\Gamma(S)$. Furthermore, if S contains an alternative x where $u(x) = 0$ and $x \notin B$, then there exists a consistent idempotent attention rule Γ' where $\Gamma'(S) = B \cup \{x\}$. The uniqueness of idempotent attention rules is determined by both the number of corresponding basic sets and the existence of null alternatives.

Proposition 3. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The idempotent attention is unique if and only if*

(i) for every $S \in \mathcal{X}$ there is a unique $B \in \mathcal{B}$ such that $B \subseteq S$ and $B \sim S$, i.e., $|\mathcal{B}(S)| = 1$ for all $S \in \mathcal{X}$.

(ii) $\succcurlyeq$ on $\mathcal{X}$ does not satisfy ENS, i.e., $N = \emptyset$.

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ satisfy (i) and (ii). Take any $S \in \mathcal{X}$; if S is basic, $\Gamma_i(S) = S$ for all $i \in I$. If S is not basic, then there is a $\Gamma_i(S) = B$ where $\{B\} = \mathcal{B}(S)$. By contradiction, suppose that there is another Γ'_i where $\Gamma'_i(S) = T$ where $T \neq B$. By (i), we have $T \notin \mathcal{B}$ which implies that $N \neq \emptyset$. For the converse direction, suppose that the idempotent attention rule is unique. (i) is obvious from the proof of Theorem 1. Suppose that (ii) does not hold, then we know that there is an $i \in I$ such that $u_i(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. Take any $y \neq x$, we can let $\Gamma'_i(xy) = xy$ and $\Gamma'_i(S) = \Gamma_i(S)$ for all $S \neq xy$. It is clear that (Γ'_i, u_i) is still a consistent AR-IA pair. □

$\mathcal{IB}(S)$ plays a similar role in AR-AFs as $\mathcal{B}(S)$ does for AR-IAs. A similar result also holds for AR-AFs.

Proposition 4. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF. The attention filter is unique if and only if*

(i) for any S there is a unique basic set B such that $T \sim S \sim B$ for all T with $B \subseteq T \subseteq S$.

(ii) $\succcurlyeq$ on $\mathcal{X}$ does not satisfy ENS, i.e., $N = \emptyset$.

Proof. By the proof of Theorem 2, the attention filter Γ_j is unique if and only if $|\mathcal{IB}(S)| = 1$ for all $S \in \mathcal{X}$. The sufficiency of (i) and (ii) are relatively straightforward. For the necessity, we can assume that $|\mathcal{IB}(S)| = 1$ for all $S \in \mathcal{X}$. We only need to prove the necessity of (ii). By contradiction, suppose that $N = MIN(X, \succcurlyeq)$. Take any $x, y \in X$ where $x \in MIN(X, \succcurlyeq)$. We have either $x \sim xy$ or $y \sim xy$ which suggests that $|\mathcal{IB}(xy)| > 1$. $\square$

2.3.4 Comments

In the AR-IA, we can also drop the nonnegative utility function assumption. We still require the induced system of linear inequalities to have a solution when characterizing AR-IAs. The NR axiom can be replaced by the finite cancellation axiom (see Kraft et al. [1959], Scott [1964], Krantz et al. [1971], and Fishburn [1992b]) in characterizing the AR-IA (see Appendix A.5). However, in the AR-AF, we need to characterize potential null sets. The assumption of the nonnegative utility function simplifies our characterization of the AR-IA by describing it in IB.

The AR is widely used, but it has limitations in capturing interactions between alternatives such as complements and substitutes. However, it is critical to note that when $\succcurlyeq$ is complete and transitive, characterizing any utility representation under idempotent attention is mathematically equivalent to characterizing it on basic sets under full attention. As a result, the AR can be replaced by other representations under idempotent attention, and the idea of characterization will be the same.

The idempotent attention rule is a generalization of attention filter (Masatlioglu et al. [2012]), competition filter (Lleras et al. [2017]), and path independent consideration (Lleras et al. [2021]). However, not all the attention rules used in past papers are idempotent. For example, Geng [2022] introduces the weak competition filter: For any $|S| > 1$, if

$x \in \Gamma(S)$, there is a T with $T \subset S$ and $|T| = |S| - 1$ such that $x \in \Gamma(T)$. An idempotent attention rule is not necessarily a weak competition filter.

Interpreting AR-IA as intuitive probability under idempotent attention is interesting. As in Kraft et al. [1959], let X represent the collection of incompatible events, and $\succcurlyeq$ denote the weak likely binary relation. Fix any $u \in \mathcal{U}_{\succcurlyeq}$ with $u \neq 0$, we define $p(x)$ as $\frac{u(x)}{\sum_{x \in X} u(x)}$ for each $x \in X$. Under this framework, the AR-IA reflects the probability measure when the DM has limited attention.

2.4 Revealed Attention

When $\succcurlyeq$ admits AR-IAs or AR-AFs, we can reveal the DM's attention by analyzing the preference relation. We also can gain insight into how alternatives interact to shape the DM's attention by comparing revealed attention between these two representations.

2.4.1 Revealed Attention under AR-IAs

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Multiple consistent pairs of (Γ, u) can represent the same preference. The DM's attention rule can be any one of them. Fortunately, the DM's attention is associated with basic sets and null alternatives. Any corresponding basic set B of S, as well as $T \supset B$ that only contains additional null alternatives, can be considered as the DM's attention on S (see Example 1).

Proposition 5. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. For any $S \in \mathcal{X}$, $\Gamma_i(S) = T$ for some $i \in I_{\succcurlyeq}$ if and only if $T \sim B$ and $T \setminus B \subseteq S \cap N$ for some $B \in \mathcal{B}(S)$.*

Proof. The only if part is easy to show. For the if part, take any $T \in \mathcal{X}$ such that $T \sim B$ and $T \setminus B \subseteq S \cap N$ for a $B \in \mathcal{B}(S)$. Without loss of generality, suppose that $T \in \mathcal{B}^c$. By the proof of Theorem 1, there is $(\Gamma_i, u_i) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ where $\Gamma(\mathcal{X}) = \mathcal{B}$ such that $\Gamma_i(S) = B$, and $u_i(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. Let's consider

$$\Gamma_i'(Y) = \begin{cases} T & \text{if } Y = T \text{ or } S, \\ \\ \Gamma_i(Y) & \text{otherwise.} \end{cases}$$

Let's first verify that Γ_i' is an idempotent attention rule. We first notice that $S \in \mathcal{B}^c$ as $T \in \mathcal{B}^c$. Therefore, there is no $Y \in \mathcal{X}$ such that $\Gamma_i(Y) = S$. Take any $Y \in \mathcal{X}$, if $Y = T$ *or* S, then $\Gamma_i'(Y) = \Gamma_i'(\Gamma_i'(Y))$. If $Y \neq T$ or $Y \neq S$, then $\Gamma_i'(\Gamma_i'(Y)) = \Gamma_i'(\Gamma_i(Y)) = \Gamma_i(Y) = \Gamma_i'(Y)$. Hence, Γ_i' is an idempotent attention rule. It is easy to verify that $(\Gamma_i', u_i) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$. □

To obtain an accurate welfare implication, we use the same definition of revealed attention as in Masatlioglu et al. [2012]. That is, under what conditions must an alternative and a bundle attract full attention?

Definition 6. *Suppose that the* $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs. An alternative* s ***catches attention under AR-IAs*** *in* S *if* $s \in \Gamma_i(S)$ *for all* $i \in I_{\succcurlyeq}$*. A bundle* S ***catches full attention under AR-IAs*** *if* s *catches attention in* S *for all* $s \in S$*, i.e.,* $\Gamma_i(S) = S$ *for all* $i \in I_{\succcurlyeq}$*.*

For the AR-IA, the DM's attention on S can be restricted to $\mathcal{B}(S)$. When an alternative $s \in S$ belongs to every corresponding basic set, it must attract attention. In addition, a set catches the full attention only when it is a basic set.

Corollary 5. *An alternative s catches attention under AR-IAs in S if and only if $s \in \bigcap_{B \in \mathcal{B}(S)} B$.*

Proof. Let $\bar{\mathcal{B}}(S) := \{T \subseteq S : \exists B \in \mathcal{B}(S) \text{ s.t. } T \sim B \text{ and } T \setminus B \subseteq N\}$. By Proposition 5, an alternative s catches attention under AR-IAs in S if and only if $s \in \bigcap_{T \in \bar{\mathcal{B}}(S)} T$. If $T \in \bar{\mathcal{B}}(S)$ is not a basic set, then there is a $B \in \mathcal{B}(S)$ such that $T \setminus B \subseteq S \cap N$. As a result, $s \in \bigcap_{T \in \bar{\mathcal{B}}(S)} T$ if and only if $s \in \bigcap_{B \in \mathcal{B}(S)} B$. $\square$

Corollary 6. *A bundle S catches full attention under AR-IAs if and only if S is basic.*

Proof. By Proposition 1, basic sets catch full attention under AR-IAs. Suppose that S catches full attention, and S is nonbasic. Then, there is a corresponding basic set B of S that $S \setminus B \subseteq N$. As a result, there is an idempotent attention Γ consistent with AR-IAs such that $\Gamma(S) = B$. $\square$

2.4.2 Revealed Attention under AR-AFs

Now assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. As shown in the proof of Theorem 2, we can associate $\Gamma(S)$ to any $T \in \mathcal{IB}(S)$.

Proposition 6. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. For any $S \in \mathcal{X}$, $\Gamma_j(S) = T$ for some $j \in J_{\succcurlyeq}$ if and only if $T \in \mathcal{IB}(S)$.*

Proof. The if part is trivial. For another direction, take any $(\Gamma_j, u_j) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ where $\Gamma_j(S) = T$. If $T \in \mathcal{B}$, it is done. If $T \notin \mathcal{B}$, then there is a $B \in \mathcal{B}(S)$ such that $T \setminus B \subseteq N$. Combining with the fact that $T \sim Y \sim S$ for all Y with $T \subseteq Y \subseteq S$, $T \in \mathcal{IB}(S)$. $\square$

We use a similar criterion for revealing attention under AR-AFs.

Definition 7. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. An alternative s* ***catches attention under AR-AFs*** *in S if $s \in \Gamma_j(S)$ for all $j \in J_{\succcurlyeq}$. A bundle S* ***catches full attention under AR-AFs*** *if s catches attention in S for all $s \in S$, i.e., $\Gamma_j(S) = S$ for all $j \in J_{\succcurlyeq}$.*

For any $S \in \mathcal{X}$, $\mathcal{IB}(S)$ plays a similar role in AR-AFs as $\mathcal{B}(S)$ in AR-IAs.

Corollary 7. *An alternative s catches attention in S under AR-AFs if and only if $s \in \bigcap_{T \in \mathcal{IB}(S)} T$.*

For any bundle S, we know $\mathcal{IB}(S)$ contains all the candidates for the DM's attention on S when Γ is an attention filter. When we cannot find a proper subset T of S such that all sets lying between them are indifferent, then $\mathcal{IB}(S)$ must equal $\{S\}$. We refer to such sets as weakly basic sets.

Definition 8. *A bundle S is* ***weakly basic*** *if there is no $T \subset S$ such that $T \sim Y \sim S$ for all Y with $T \subseteq Y \subseteq S$.*

Weakly basic sets must catch full attention under AR-AFs. The converse is also true.

Corollary 8. *A bundle S catches full attention under AR-AFs if and only if S is weakly basic.*

Proof. Take any bundle S where it catches full attention under AR-AFs. By contradiction, we can assume that S is not weakly basic. Since S is not weakly basic, there is a proper subset T of S that $T \in \mathcal{IB}(S)$ by IB axiom. Hence, we know there is a Γ'_j that is consistent with AR-AFs such that $\Gamma'_j(S) = T$. For the converse direction, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF under (Γ_j, u_j). By contradiction, suppose that $\Gamma_j(S) = T$ where $T \subset S$, and S is weakly basic. Then, we know that $\Gamma_j(Y) = T$ for any Y with $T \subseteq Y \subseteq S$ implies that $T \sim Y \sim S$. S, as a result, is not weakly basic. $\square$

2.4.3 Implications of Attention Formation

In contrast to the attention filter, eliminating some omitted alternatives may indeed affect the DM's attention under idempotent attention. For instance, as illustrated in Figure 2.1, the alternatives in $S \setminus T$ make the DM pay attention to certain alternatives in $S \setminus \Gamma(S)$ and ignore $\Gamma(S)$.

Definition 9. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. For any $S, T, Y \in \mathcal{X}$ with $Y, T \subset S$, Y* ***obstructs the DM's attention*** *on T in S if for all $i \in I$, $T \cap \Gamma_i(S) = \emptyset$ and $T \subseteq \Gamma_i(S \setminus Y)$.*

Similar to revealing attention, we can focus on basic sets. The following corollary is a direct result of Proposition 5.

Corollary 9. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. For any $S, T, Y \in \mathcal{X}$ with $Y, T \subset S$, Y obstructs the DM's attention on T in S if and only if $T \cap \bar{B} = \emptyset$ for all $\bar{B} \in \bar{\mathcal{B}}(S)$, and $T \subseteq B'$ for all $B' \in \mathcal{B}(S \setminus Y)$.*

The effect of obstruction offers valuable insights into bundle design. Given any S, suppose Y obstructs T's attention. When a provider aims to highlight the existence of T within S, rather than offering a larger bundle S, he can provide a smaller bundle $S \setminus Y$. Furthermore, if Y hinders the DM's attention on T in every bundle, providers seeking to direct the DM's focus towards T should refrain from providing Y.

2.4.4 Comments

The practice of revealed attention provides a reliable way to infer DMs' attention for each bundle. It should be noted that both Propositions 5 and Proposition 6 could provide potential representations of attention rules. Based on them, we can construct the weakest criteria to reveal attention under AR-IAs and AR-AFs. Specifically, for the AR-IA, an alternative s may attract the DM's attention when and only when it belongs to T for some $T \in \bar{\mathcal{B}}(S)$. The AR-AF counterpart can be obtained by replacing $\bar{\mathcal{B}}(S)$ with $\mathcal{IB}(S)$. These analyses enable us to compare attention interpersonally. The formal discussion can be found in Appendix A.5.

2.5 Applications of the AR-IA

The AR-IA describes the aggregation behavior of the limitedly attentive DM. It is sometimes necessary to impose additional restrictions in practice. This section delves into two of them: First, the DM selects the best k alternatives from a bundle; Second, the utility function is constant.

2.5.1 Top k Alternatives

Γ can also be viewed as a selection process. The idempotence property of Γ implies that, in terms of preferences, the bundle S is indifferent to $\Gamma(S)$. When $\Gamma(S)$ represents the best alternative in S, this interpretation aligns with the indirect utility representation. Moreover, it is feasible to extend this framework to accommodate multiple alternatives, which has been studied in many papers. For example, Eliaz et al. [2011] develops a model for choosing two alternatives, and Chambers and Yenmez [2018] studies the behavior of selecting the top k alternatives.

Definition 10. *$\succcurlyeq$ on $\mathcal{X}$ admits a **Top k AR** if there is a utility function $u : X \to \mathbb{R}^+$ such that for all $S, T \in \mathcal{X}$*

$$S \succcurlyeq T \iff \max_{\substack{S' \subseteq S \\ |S'| \leq k}} \sum_{s \in S'} u(s) \geq \max_{\substack{T' \subseteq T \\ |T'| \leq k}} \sum_{t \in T'} u(t).$$

Given a permutation of $X = \{x_1, \ldots, x_{|X|}\}$ where $\{x_i\} \succcurlyeq \{x_j\}$ for all $i \leq j$. Every $S \in \mathcal{X}$ can be considered as a subsequence of X, i.e., $S = \{x_{S_1}, \ldots, x_{S_{|S|}}\}$.

Axiom 4. ***(k-Cutoff)*** *For any S with $|S| \leq k$, there is a $B \in \mathcal{B}(S)$ such that $S \setminus B \subseteq N$. Moreover, for any S with $|S| > k$, S is indifferent to $\{x_{S_n}\}_{n=i}^{k}$.*

It is impossible to distinguish whether the DM selects the top k alternatives or only pays attention to them under the AR-IA framework. Consequently, the top k AR is a particular case of AR-IAs.

Proposition 7. *$\succcurlyeq$ on $\mathcal{X}$ admits a Top k AR if and only if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA, and $\succcurlyeq$ on $\mathcal{X}$ satisfies k-Cutoff.*

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a top k AR under u. Consider an attention rule

$$\Gamma(S) = \begin{cases} S & \text{if } |S| \leq k' \\ \{x_{S_n}\}_{n=1}^{k} & \text{if } |S| > k. \end{cases}$$

Γ is idempotent. Consequently, (Γ, u) is a consistent pair of AR-IA, and $u \in \mathcal{U}_{\succsim}$. k-Cutoff is true. For the converse, when $N = \emptyset$, it is obvious. When $N \neq \emptyset$, we know that there is a u that solves the system of linear inequalities induced by $\succsim$ on $\bar{\mathcal{B}}$ such that $u(x) = 0$ for all $x \in MIN(X, \succsim)$. By k-Cutoff, for any S there is a $S' \in \bar{B}$ such that $S \sim S'$ and $|S'| = k$. As a result, u also solves the system of inequalities induced by $\succsim$ on $\{|S| \leq k : S \in \mathcal{X}\}$. $\square$

2.5.2 Freedom of Choice

Under the AR-IA, the DM prefers a larger bundle over a smaller one. An interpretation of this situation, as argued in Sen [1988], is that the DM prefers flexibility or freedom of choice, which indirect utility ranking neglects.

Sen [1988] introduces the intrinsic importance of freedom, emphasizing that it should depend solely on preference for freedom. In Pattanaik and Xu [1990], a simple cardinality-based ordering (SCO) is introduced, in which a set S has a greater degree of freedom than a set T if $|S| \geq |T|$. They characterize the SCO with three axioms. One of the axioms is known as Indifference between No-choice Situation (INS) [13].

Axiom 5. ***(INS: Indifference between No-choice Situation)*** *For all $x, y \in X$, $\{x\} \sim \{y\}$.*

We consider the SCO under the idempotent attention.

Definition 11. *$\succsim$ on $\mathcal{X}$ is an* ***SCO under idempotent attention (SCO-IA)*** *if there is an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$ such that for any $S, T \in \mathcal{X}$, $S \succsim T$ if and only if $|\Gamma(S)| \geq |\Gamma(T)|$.*

[13] The other two axioms are (1) Strict Monotonicity (SM): For all distinct $x, y \in X$, $xy \succ x$; and (2) Independence (IND): For all $S, T \in \mathcal{X}$, and $x \in X \setminus (S \cup T)$, $S \succsim T$ if and only if $S \cup \{x\} \succsim T \cup \{x\}$.

Limited attention may cause the DM to overlook some states when it comes to the instrumental value of freedom. When X corresponds to states, omitting some items is equivalent to omitting some states.

Theorem 10. *$\succcurlyeq$ on $\mathcal{X}$ is an SCO-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and satisfies INS.*

Proof. Assume that $\succcurlyeq$ on $\mathcal{X}$ is an SCO-IA. By the definition of idempotent attention rules, $\Gamma(x) = x$ for all $x \in X$. For any $x, y \in X$, $|\Gamma(x)| = |x| = |y| = |\Gamma(y)|$ which implies that $x \sim y$. We can let $u : X \to \mathbb{R}^+$ be a constant function such that $u(x) = 1$ for all $x \in X$. Take any $S, T \in \mathcal{X}$, we then have that $S \succcurlyeq T \iff |\Gamma(S)| \geq |\Gamma(T)| \iff \sum_{s\in\Gamma(S)} u(s) \geq \sum_{t\in\Gamma(T)} u(t)$. For the converse direction, assume that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and $\succcurlyeq$ satisfies INS. We first know that there is a pair of an idempotent attention rule Γ and a utility function u where $u(x) = 1$ for all $x \in X$. Hence, $S \succcurlyeq T \iff \sum_{s\in\Gamma(S)} u(s) \geq \sum_{t\in\Gamma(T)} u(t) \iff |\Gamma(S)| \geq |\Gamma(T)|$. $\square$

As long as $\succcurlyeq$ on $\mathcal{X}$ satisfies INS, all alternatives are considered to provide the same degree of freedom by the DM. Klemisch-Ahlert [1993] investigates the case where alternatives may possess different degrees of freedom under full attention. This version of measuring freedom of choice is identical to the AR-IA when the DM's attention rule is idempotent.

2.6 Concluding Remarks

This paper investigates the additive representation in the context of limited attention. With a focus on the idempotent attention rules, we characterize the AR-IA and its special cases AR-IAs.

Idempotent attention rules separate the DM's attention and aggregation procedures. Both procedures relate to a subcollection of $\mathcal{X}$, namely basic sets. In general, $\mathcal{B}$ can be understood as the image of attention rules when $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Furthermore, the collection of consistent utility functions is determined by $\succcurlyeq$ on $\mathcal{B}$.

Chapter 3

Satisficing under Limited Attention

3.1 Introduction

The classical theory of choice proposes that decision makers (DMs) choose the ”best” alternative from a given set of choices. In real life scenarios, however, the best alternative may not always be the one chosen. Simon [1959] argues: “ The entrepreneur may not care to maximize, but may simply want to earn a return that he regards as satisfactory.”

Satisficing, as described in Simon [1997], refers to the selection of alternatives that meet or exceed the criteria instead of selecting the best alternative. Simon [1997] ascribe satisficing to “the limits of human cognitive capacity for discovering alternatives, computing their consequences under certainty or uncertainty, and making comparisons among them.”

The lack of cognitive abilities has been used as motivation in previous research related to satisficing. It has been demonstrated in the research on intransitive indifference (for example, see Luce [1956], Fishburn [1975], Aleskerov et al. [2007], and Manzini and

Mariotti [2012]) that DMs are incapable of noticing small differences in utility between two alternatives. As outlined in Tyson [2008], DMs are unable to recognize some preferences among alternative options due to their limitations in cognitive ability. According to Frick [2016], DMs have limited cognitive resources, causing them to perceive coarse preferences in a large menu.

The reasons for this can be divided into two categories: the ability to perceive alternatives and the ability to analyze them. Simon [1982] uses an example of chess players to illustrate it. Chess players may notice a limited number of moves at any given time. Analyzing them sequentially, they select a move once it meets their objectives. The process of analyzing moves involves the complexity of computing, which is difficult to model. DMs' limited ability to perceive alternatives, however, can be modeled separately from the classical satisfying model.

This limited ability of DMs to recognize alternatives is referred to as limited attention. Assuming limited attention, satisficing becomes that a DM pays attention to a nonempty subset of the choice set, and selects options that meet or exceed certain criteria.

Satisficing under limited attention (SLA) is comprised of three components: attention, preferences, and criteria. Let Γ be the attention rule of the DM. It maps every choice set to one of its nonempty subsets. The DM pays attention to $\Gamma(S)$ given a choice set S. The preference is a binary relation $\succcurlyeq$ on alternatives which is assumed to be a linear order, i.e., complete, transitive and antisymmetric. The DM has a criterion for every choice set S, denoted by $\theta(S) \in \Gamma(S)$. We refer to θ as a threshold function. If the DM's behavior is an

SLA, the chosen alternatives should catch the attention and meet or exceed the criterion. Mathematically, for S, the chosen alternatives should in $\{x \in \Gamma(S) : s \succcurlyeq \theta(S)\}$.

Suppose that the set of alternatives is finite. Imagine that we can observe the DM's choice for every choice set as an outsider. The DM's choice in S is denoted by $C(S)$, which is a nonempty subset of S. Since the DM can select multiple alternatives, C is a choice correspondence. It follows that a choice correspondence can be rationalized by an SIA if and only if there exists a tuple $\langle \Gamma, \succcurlyeq, \theta \rangle$ such that $C(S) = \{x \in \Gamma(S) : s \succcurlyeq \theta(S)\}$ for every choice set S.

In the absence of any further restrictions, every choice correspondence can be rationalized by an SLA. Similar situations can be found in Masatlioglu et al. [2012], Lleras et al. [2017] and Lleras et al. [2021]. The papers impose a number of restrictions on attendance rules [1]. We also investigate SLA with respect to a special attention rule, idempotent attention rules (see Li [2024]). According to Li [2024], an attention rule Γ is idempotent if $\Gamma(S) = \Gamma(\Gamma(S))$ for every choice set S. This implies that the DM believes that S and $\Gamma(S)$ are the same choice set. As the logic of choices within identical sets should be the same, we should ensure that the thresholds of S and $\Gamma(S)$ are the same, i.e., $\theta(S) = \theta(\Gamma(S))$. This model is referred to as satisficing under idempotent attention (SIA).

We provide two methods of characterizing the SIA: the acyclicity of a binary relation and the WARP-like axiom. The former characterization drives from an equivalent statement of satisficing: for a choice set, the picked alternatives should be better than the unchosen ones. It turns out that the acyclicity of this "revealed" binary relation charac-

[1] These new attention rules are known as attention filters (see Masatlioglu et al. [2012]), competition filters (see Lleras et al. [2017]), and path independent consideration (see Lleras et al. [2021]), respectively.

terizes the satisficing. As well, in the SIA, the chosen options should be better than the alternatives that attract attention but are not chosen.

The WARP-like axiom for satisficing suggests that every menu contains the "best" alternative, and if a non-best alternative is chosen when the best one is available, then the best alternative must be chosen. As an analogy, the WARP-like axion for the SIA requires that the best alternative be selected if the DM pays attention to it.

For characterizing the SIA, the remaining question is how to determine the DM's attention rule. According to the idempotent attention rule, the choice in S is the same as the choice in $\Gamma(S)$. In other words, if some nonempty set of S serves as the $\Gamma(S)$, they must exhibit the same choices. When a set does not have any proper subsets exhibiting the same choice, the DM should pay full attention to it. We borrow the concept from Li [2024], and refer to these sets as basic sets. Consequently, every set S has a corresponding basic set B, which can be roughly interpreted as $\Gamma(S)$, i.e., $\Gamma(S) = B$.

Inspired by Bernheim and Rangel [2007] and Bernheim and Rangel [2009], we study the revealed SIA pairs, i.e., $\langle \Gamma, \succcurlyeq, \theta \rangle$ with the consideration of welfare implications. Similar to Masatlioglu et al. [2012], we use the most stringent criterion for revealing SIA pairs to provide the safest option for welfare policies.

It is critical to emphasize that every choice correspondence that can be rationalized by an SIA must have at least two consistent SIA pairs. This is primarily due to the non-unique nature of revealed preferences. It also results in the inability to identify the DM's choice procedure under the SIA in a unique manner.

Attention filters and competition filters are special cases of idempotent attention rules (see Li [2024]). The satisficing under attention filter (SAF) and satisficing under competition filters (SCF) are therefore special cases of the SIA. In this paper, we demonstrate that the SAF, as well as the SCF, can both be characterized by two distinct systems of axioms. These axioms are based on the same intuitions as those in the SIA. The distinctions between these three models are driven by attention formation procedures.

The reaming paper is arranged as follows. The SIA will be formally introduced in Section 2. In Section 3, we provide two characterizations of the SIA as well as the results of revealed SIA pairs. In Section 4, we characterize two special cases of the SIA: SAF and SCF. Section 5 concludes this paper.

3.2 The Model

Let X be a finite set, and $\mathcal{X}$ be the collection of nonempty subsets of X. We refer to each element in X as an alternative and each element in $\mathcal{X}$ as a menu. A mapping $C : \mathcal{X} \to \mathcal{X}$ is a choice correspondence if $C(S) \subseteq S$ for all $S \in \mathcal{X}$. An outside researcher can observe the choice made by a decision maker (DM) for each menu, i.e., C.

In the satisficing model, DMs have a specific criterion in mind. Given any menu S, they only choose the alternatives that are "better" than the criterion. We generalize the idea of criteria as a mapping $\theta : \mathcal{X} \to X$. Further, we say θ is a threshold function if $\theta(S) \in S$ for all $S \in \mathcal{X}$.

When DMs are limitedly attentive, they pay attention to some (not necessarily all) alternatives in each menu. When they use the same procedure in the satisficing model to make decisions, they should use it with the alternatives that capture their attention. We refer to this process as satisficing under limited attention. To describe that DMs has limited attention, we define a mapping $\Gamma : \mathcal{X} \to \mathcal{X}$. We say Γ is an attention rule if $\Gamma(S) \subseteq S$ for all $S \in \mathcal{X}$.

Definition 12. *A choice correspondence C is a* ***satisficing under limited attention (SLA)*** *if there is a tuple $\langle \Gamma, \succcurlyeq, \theta \rangle$ such that $C(S) = \{s \in \Gamma(S) : s \succcurlyeq \theta(S)\}$ where*

(1) $\Gamma : \mathcal{X} \to \mathcal{X}$ is an attention rule;

(2) $\succcurlyeq$ is a linear order, i.e., $\succcurlyeq$ is complete, transitive, and antisymmetric;

(3) $\theta : \mathcal{X} \to X$ is a threshold function with $\theta(S) = \theta(\Gamma(S)) \in \Gamma(S)$ for all $S \in \mathcal{X}$.

In this framework, there is another way to interpret the decision under satisficing: The chosen alternatives should be better than the unchosen ones. Under the assumption of limited attention, it can be interpreted as if the DM pays attention to x and y in a menu S, $x \in C(S)$ while $y \notin C(S)$ implies that $x \succ y$.

Proposition 8. *C can be rationalized by an SLA if and only if there is a linear preference $\succcurlyeq$ and an idempotent attention rule Γ such that for all $S \in \mathcal{X}$*

$$C(S) = \begin{cases} \{s \in \Gamma(S) : s \succ x \ for\ all\ x \in \Gamma(S) \setminus C(S)\} & if\ \Gamma(S) \setminus C(S) \neq \emptyset, \\ \Gamma(S) & if\ \Gamma(S) \setminus C(S) = \emptyset. \end{cases}$$

Proof. Suppose $\succcurlyeq$ can be rationalized by an SLA under $\langle \Gamma, \succcurlyeq, \theta \rangle$. Fix any S, and take any $x \in C(S)$. When $C(S) = \Gamma(S)$, by the SIA, we have that $\theta(S) = \min(\Gamma(S), \succcurlyeq)$. When

$C(S) \neq \Gamma(S)$, we know that $x \succ \theta(S) \succ y$ for all $y \in \Gamma(S) \setminus C(S)$. Take any $s \in \Gamma(S)$ with $s \succ x$ for all $x \in \Gamma(S) \setminus C(S)$. If $s = \theta(S)$, then $s \in C(S)$. If $s \neq \theta(S)$ and $\theta(S) \succ s$, then $s \notin C(S)$. We then know that $s \succcurlyeq \theta(S)$, which implies that $s \in C(S)$.

For the converse direction, suppose that there is a linear preference $\succcurlyeq$ and an idempotent attention rule Γ such that $C(S)$ is given as in this Proposition. Let $\theta(S) = min(C(S), \succcurlyeq)$ for all $S \in \mathcal{X}$. It is evident that $s \in \Gamma(S) \cap C(S)$ implies that $s \succcurlyeq \theta(S)$ for all S. □

Similar to the situation in Masatlioglu et al. [2012], if attention rules are not further restricted, every choice correspondence is an SLA.

Proposition 9. *Every choice correspondence C is an SLA.*

Proof. Fix an arbitrary linear order $\succcurlyeq$ on X. Let $\Gamma(S) = C(S)$, and $\theta(S) = min(C(S), \succcurlyeq)$. Take any $x \in C(S)$, we know that $\theta(S) = min(C(S), \succcurlyeq) \succcurlyeq x$, which implies that $C(S) \subseteq \{x \in \Gamma(S) : x \succcurlyeq \theta(S)\}$. The converse direction is obvious. □

Among all potential restrictions on attention rules, we focus on the family of idempotent attention rules.

Definition 13. *(Idempotent Attention Rule) An attention rule Γ is an idempotent attention rule if $\Gamma(S) = \Gamma(\Gamma(S))$ for all $S \in \mathcal{X}$.*

Li [2024] points out that idempotent attention rules are generalizations of attention filters (cf. Masatlioglu et al. [2012]), competition filters (cf. Lleras et al. [2017]), and path independent consideration from Lleras et al. [2021]. It is implicit in idempotent attention rules that DMs are not aware of omitted alternatives on a menu S. Additionally, they believe that S and $\Gamma(S)$ are identical.

3.3 Satisficing under Idempotent Attention

The satisficing under idempotent attention can be achieved by imposing idempotence on the attention rule of the SLA.

Definition 14. *(SIA) A choice correspondence C is a **satisficing under idempotent attention (SIA)** if $C(S) = \{s \in \Gamma(S) : s \succcurlyeq \theta(S)\}$ where*

1. *Γ is an idempotent attention rule.*

2. *$\succcurlyeq$ is a linear order, i.e., $\succcurlyeq$ is complete, transitive, and antisymmetric.*

3. *$\theta : \mathcal{X} \to X$ is a threshold function such that $\theta(S) = \theta(\Gamma(S)) \in \Gamma(S)$ for each $S \in \mathcal{X}$.*

We infer $\langle \Gamma, \succcurlyeq, \theta \rangle$ as an SIA pair that is consistent with C.

When we assume that DMs' attention rules are idempotent, there are some choice correspondences that cannot be rationalized by an SIA.

Example 5. *Let $X = abcd$, and the choice correspondence is given as*

$\lvert S\rvert \geq 2$	*abcd*	*abc*	*abd*	*acd*	*bcd*	*ab*	*ac*	*ad*	*bc*	*bd*	*cd*
C	*abc*	*ab*	*a*	*ad*	*bd*	*a*	*a*	*d*	*b*	*b*	*c*

Suppose that $\langle \Gamma, \theta, \succcurlyeq \rangle$ is a consistent SLA. Since $C(abcd) = abc \neq C(abc)$, thus $\Gamma(abcd) = abcd$ which implies that $c \succ d$. Similarly, $\Gamma(acd) = acd$. However, $C(acd) = ad$ implies that $d \succ c$.

3.3.1 The Characterization of SIA

According to idempotent attention rules, the DM pays attention to $\Gamma(S)$ for a menu S, all the alternatives in $\Gamma(S)$ should catch the DM's attention when $\Gamma(S)$ is displayed. Due to this, the DM's attention can be reduced to one of its subsets for each menu. Moreover, since $\theta(S) = \theta(\Gamma(S))$, $C(S) = C(\Gamma(S))$ when C can be rationalized by an SIA. The reduction procedure is not applicable to a set S if it does not have a proper subset in which the DM makes the same choice. We refer to S as a basic set.

Definition 15. *(Basic Sets) A set $S \in \mathcal{X}$ is **basic** if there is no $T \subset S$ such that $C(T) = C(S)$.*

The collection of basic sets is determined by the choice correspondence of the DM. The collection of basic sets with respect to C is referred to as $\mathcal{B}_C$. Usually, when investigating the choices of the DM and there is no misrepresentation, we use $\mathcal{B}$ as the collection of basic sets.

Each menu S has a basic set B with $B \subseteq S$ where $C(S) = C(B)$. This B is referred to as a corresponding basic set of S, and the collection of corresponding basic sets of S is denoted as $\mathcal{B}(S)$. If we want to interpret a $B \in \mathcal{B}(S)$ as $\Gamma(S)$ as the intuition in defining basic sets, B must catch the DM's full attention due to the requirement of idempotent attention rules.

Proposition 10. *If C can be rationalized by an SIA under $\langle \Gamma, \succcurlyeq, \theta \rangle$, then $\Gamma(B) = B$ for all $B \in \mathcal{B}$.*

Proof. Suppose C is an SIA. Take any consistent pair $\langle \Gamma, \theta, \succcurlyeq \rangle$, and any $B \in \mathcal{B}$. By contradiction, suppose that $\Gamma(B) \subset B$. Notice that, $\theta(\Gamma(B)) = \theta(B)$ because $\Gamma(B) = \Gamma(\Gamma(B))$. We have

$$\begin{aligned} C(\Gamma(B)) &= \{x \in \Gamma(\Gamma(B)) : x \succcurlyeq \theta(\Gamma(B))\} \\ &= \{x \in \Gamma(B) : x \succcurlyeq \theta(B)\} \\ &= C(B). \end{aligned}$$

Hence, B is not basic. That's a contradiction. $\square$

When we interpret the DMs' choice on basic sets as mentioned in Proposition 8, $x, y \in B$, $x \in C(B)$ and $y \notin C(B)$ indicates that x is strictly preferred to y.

Definition 16. *For any $x, y \in X$, we say $x\ P\ y$ if there is a basic set B with $x, y \in B$ such that $x \in C(B)$ and $y \notin C(B)$.*

When C can be rationalized by an SIA, $x\ P\ y$ must imply that x is strictly preferred to y. Due to this, for every linear preference $\succcurlyeq$ that is consistent with the SIA, $P \subseteq \succcurlyeq$. Assuming that $x\ P\ y$ and $y\ P\ z$, we can infer $x\ P\ z$ despite the absence of B where $x \in C(B)$ and $z \notin C(B)$. In a different perspective, if there is a B where the DM's choice shows that $z\ P\ x$, then C cannot be rationalized by an SIA since $\succcurlyeq$ exhibits a cycle.

Aleskerov et al. [2007] and Tyson [2008] provide a characterization of satisficing. Specifically, the DM should prefer the chosen alternative over the unchosen alternative, and this binary relation should be acyclic. In the same manner, the acyclicity of P ensures a satisficing on $\mathcal{B}$.

The SIA is also characterized by the acyclicity of P. Due to the fact that $\mathcal{B}(S) \neq \emptyset$, S can be associated with one of its corresponding basic sets by setting $\Gamma(S) = B$. It is possible to make this construction an idempotent attention rule by arbitrary permutations of $\mathcal{B}$. The linear preference in this SIA may be any completion of P. Once the $\succcurlyeq$ has been determined, would the threshold function be $min(C(S), \succcurlyeq)$ for all $S \in \mathcal{X}$.

Lemma 1. *C is an SIA if and only if P is acyclic.*

Proof. We first suppose that C is an SLA under $\langle \Gamma, \succcurlyeq, \theta \rangle$. Since $\Gamma(B) = B$ for all $B \in \mathcal{B}$. By contradiction, suppose there is a collection $\{x_i\}_{i=1}^k$ where $x_i \in X$ for all i such that $x_1 \ P \ x_2 \ P, ..., P \ x_k \ P \ x_1$. Hence, we know there is a collection of basic sets $\{B_i\}_{i=1}^k$ such that $x_i \in C(B_i)$ and $x_{i+1} \in B_i \setminus C(B_i)$ for all $i < k$, and $x_k \in C(B_k)$ and $x_1 \in B_k \setminus C(B_k)$. As a result, $x_1 \succcurlyeq x_2, ..., \succcurlyeq x_k \succcurlyeq x_1$.

Now supposes P is acyclic. Let $\{B_i\}_{i=1}^n$ be the collection of basic sets. Let P_R be the transitive closure of P, $\overline{P_R}$ be a completion of P_R. Let $\Gamma(S) = B_{min\{i:\ B_i \subseteq S \ and \ C(B_i)=C(S)\}}$, and $\theta(S) = min(C(S), \overline{P_R})$. We then claim that $C(S) = \{x \in \Gamma(S) : \ x \succcurlyeq \overline{P_R} \ \theta(S)\}$. Take any $x \in C(S)$, we then know that $x \in B_{min\{i:\ B_i \subseteq S \ and \ C(B_i)=C(S)\}} = \Gamma(S)$, and $x \ \overline{P_R} \ \theta(S)$. For the converse, take any S and $y \in \{x \in \Gamma(S) : \ x \ \overline{P_R} \ \theta(S)\}$. By contradiction, suppose that $y \notin C(S)$, we know that $y \in B_{min\{i:\ B_i \subseteq S \ and \ C(B_i)=C(S)\}} = \Gamma(S)$ and $y \notin C(\Gamma(S))$. That's a contradiction. □

According to classical revealed preference theory, a choice function is rationalizable when it satisfies the Weak Axiom of Revealed Preference (WARP). Masatlioglu et al. [2012] states the WARP as:

> "For any nonempty S, there exists $x^* \in S$ such that for any T including x^*, if $C(T) \in S$; then $C(T) = x^*$."

The intuition behind WARP is that the DM's choice should be consistent with some maximization behavior. The x^* is the "best" alternative in S. As a rule, rational DMs should not choose alternatives that are less desirable than the best alternative. [2] From the standpoint of revealing preferences, the chosen alternative should be the best option from the menu. The DM should, of course, prefer it to other alternatives that have not been chosen.

As part of the satisficing framework, DMs do not necessarily need to engage in maximization behaviors. As opposed to selecting the best option in every menu, they tend to set a criterion θ and choose the options that meet or exceed that criterion. Despite the differences in intuition between classical choice and satisficing models, they share a similar implication regarding preferences: the chosen alternative should be better than the unchosen alternatives. The satisficing model under full attention can be characterized by the **WARP-S:** For any nonempty S, there exists $x^* \in S$ such that for any T including x^*, if $C(T) \cap S \neq \emptyset$, then $x^* \in C(T)$. [3]

In WARP-S, if an inferior alternative is chosen when there is a better option, then the better option should also be selected. There is an additional requirement for the DM's attention when limited attention is present. Specifically, if x is better than y and both x and y are available, if y is chosen, the DM must choose x if x catches the attention of the DM.

[2]This version of WARP is of classical choice theory with linear preferences. The best alternatives in each menu are not necessarily unique if the preference is a weak order. DMs' choices are described as choice correspondences in this case. The corresponding WARP becomes: For any $S, T \in \mathcal{X}$, if $C(S) \cap T \neq \emptyset$; then $C(T) \cap S \subseteq C(S)$. Two different versions of WARP have similar meanings. This paper focuses on the version provided by Masatlioglu et al. [2012], which provides a suitable benchmark for comparison.

[3]Please see Appendix A.7 for details.

Axiom 6. ***(WARP-IA)*** *For every $S \in \mathcal{X}$, there is an $x^* \in S$ such that for any basic set B if $C(B) \cap S \neq \emptyset$ and $x^* \in B$ then $x^* \in C(B)$.*

WARP-IA generalizes the idea of WARP-S to cases of limited attention. When we relax the requirement of basic sets in WARP-IA, WARP-IA and WARP-S are identical. A DM considers all alternatives in a basic set when $\succcurlyeq$ can be rationalized by an SIA based on Proposition 10. Therefore, when the DM chooses a less desirable alternative in B, we can infer that better alternatives should also be selected.

The x^* in S mentioned in WARP-IA could be interpreted as the most preferred alternative in S with respect to the DM's preference. Moreover, since the DM's attention on S is one of the corresponding basic sets of S, when using WARP-IA we can focus on the collection of $\mathcal{B}$ rather than $\mathcal{X}$. Consequently, WARP-IA is equivalent to the acyclicity of P.

Lemma 2. *P is acyclic if and only if C satisfies WARP-IA.*

Proof. Suppose that P is acyclic. We know that C can be rationalized by an SIA pair $\langle \Gamma, \theta, \succcurlyeq \rangle$. Take any $S \in \mathcal{X}$, and let $x^* = max(S, \succcurlyeq)$. If B is basic, then $\Gamma(B) = B$. Since $C(B) \cap S \neq \emptyset$, we know that $x^* \succcurlyeq y$ for all $y \in C(B) \cap S$. Therefore $x^* \in C(B)$.

For the converse direction, we prove it by the contrapositive statement. Suppose that P is cyclical. There is a $z_1 \in X$ such that $z_1 \; P_R \; z_1$ for a finite collection of alternatives

$\{z_j\}_{j=1}^m$. Let $\{Z_j\}_{j=1}^m$ be the collection of basic sets where

$$z_1, z_2 \in Z_1,\ z_1 \in C(Z_1),\ \textit{and}\ z_2 \notin C(Z_1);$$

$$z_2, z_3 \in Z_2,\ z_2 \in C(Z_2),\ \textit{and}\ z_3 \notin C(Z_3);$$

$$\dots$$

$$z_1, z_m \in Z_m,\ z_m \in C(Z_m),\ \textit{and}\ z_1 \notin C(Z_m).$$

Now, consider $Z = \{z_j\}_{j=1}^m$. For $z_1 \in Z$, there is a basic set Z_m with $z_1 \in Z_m$ such that $C(Z_m) \cap S \neq \emptyset$ and $z_1 \notin Z_m$. For $z_j \in Z$ where $j > 1$, there is a basic set Z_{j-1} with $z_j \in Z_{j-1}$ such that $C(Z_{j-1}) \cap S \neq \emptyset$ and $z_j \notin Z_{j-1}$. $\square$

Lemma 1 and Lemma 2 suggest that WARP-IA characterizes the SIA.

Theorem 11. *A choice correspondence C is an SIA if and only if C satisfies WARP-IA.*

3.4 Revealed SIA Pairs

A choice correspondence C might be rationalized by multiple SIA pairs. The situation arises from the fact that there are multiple candidates for linear preferences and idempotent attention rules.

Example 6. *Let $X = xyz$. Consider the choice correspondence C and idempotent attention rules*

$\lvert S\rvert \geq 2$	xyz	xy	xz	yz
C	x	xy	x	y
Γ_1	x	xy	x	y
Γ_2	xz	xy	xz	y
Γ_3	xz	xy	xz	yz
Γ_4	xyz	xy	xz	yz

Consider two preferences on X: $x \succ_1 y \succ_1 z$ and $x \succ_2 z \succ_2 y$. Let $\theta_i(S) = min(C(S), \succ_i)$ for $i = 1, 2$. There are multiple SIA pairs to represent C such as $\langle \Gamma_1, \succcurlyeq_1, \theta_1 \rangle$, $\langle \Gamma_2, \succcurlyeq_1, \theta_1 \rangle$, $\langle \Gamma_3, \succcurlyeq_2, \theta_2 \rangle$ $\langle \Gamma_4, \succcurlyeq_2, \theta_2 \rangle$.

Obviously, when the preference is fixed, only some of the idempotent attention rules are consistent with the SIA. Given the DM's preference as $\succcurlyeq_1$, all idempotent attention rules are plausible, while when the DM's preference is $\succcurlyeq_2$, the Γ_4 is not consistent.

Similarly, when the idempotent attention rule is determined, some of the preference relations and corresponding threshold functions will no longer be plausible. For example, given Γ_2, the DM should strictly prefer x to z. Meanwhile, if the idempotent attention rule is Γ_4, she should strictly prefer x the most followed by y, and z the least.

Let $\langle \Gamma_i, \succcurlyeq_i, \theta_i \rangle_{i \in I}$ be the collection of all consistent SIA pairs for the choice correspondence C. Some linear orders on X should never coincide with $\succcurlyeq_i$ for some $i \in I$. For instance, in Example 5, *abcd* is a basic set which implies that $c\ P\ d$. Therefore, any linear order $\succcurlyeq$ that includes $d \succ c$ cannot be considered consistent. Meanwhile, in Example

6, when the preference is fixed in an SIA pair, the choice of threshold functions and the idempotent attention rules are limited.

Afterwards, we analyze the consistent preferences, then examine the consistent idempotent attention rules, and finally examine the consistent threshold functions.

3.4.1 Revealed Preference under the SIA

Multiple preference relations are consistent with the same C. There are even some that are controversial. For instance, in Example 6 $\succcurlyeq_1$ and $\succcurlyeq_2$ are all consistent linear orders, whereas $y \succ_1 z$ and $z \succ_1 y$ are controversial.

The implications of these two contradictory predictions for welfare policy are controversial. Despite the fact that policymakers do not intend to harm the DM, if they falsely believe one preference rather than another, the policy may adversely affect the DM. Inspiring by Bernheim and Rangel [2007] and Bernheim and Rangel [2009], we follow Masatlioglu et al. [2012] and use the most stringent criteria to reveal the DM's attention.

Definition 17. *Given any $x, y \in X$, x is* ***revealed to be preferred to*** *y if $x \succcurlyeq_i y$ for all $i \in I$.*

According to the proof Lemma 1, any completion of P_R, the transitive closure of P, can be a consistent linear order. As a result, P_R captures the revealed preference.

Theorem 12. *Suppose C admits an SIA. For any $x, y \in X$, x is revealed to be preferred to y if and only if $x \ P_R \ y$.*

Proof. Suppose that C admits an SIA. Take any $x, y \in X$ with $x \ P_R \ y$, by the proof of Lemma 1 we know that $x \succ_i y$ for all $i \in I$. For the converse direction, suppose that x

is revealed to be preferred to y, i.e., $x \succsim_i y$ for all $i \in I$. By contradiction, suppose that $\neg x\ P_R\ y$. If $y\ P_R\ x$, then $y \succ_i x$ for all $i \in I$. If y and x are P_R incomparable, then there is a completion of P_R such that $y\ \overline{P_R}\ x$. As a result, there is $i \in I$ such that $y \succ_i x$. $\square$

According to Theorem 12, there is an immediate relationship between consistent linear orders and P_R.

Corollary 13. *If C is an SIA, then $\bigcap_{i \in I} \succsim_i = P_R$.*

Every $\succsim_i$ represents a potential underlying preference used by the DM. It can be any completion of P_R. It should be noted, however, that uniqueness of preference is never true.

Proposition 11. *Suppose that $|X| \geq 2$. If C is an SIA, then there exists a pair of $x, y \in X$ such that x, y are P_R incomparable. In other words, if C is an SIA, then $|I| \geq 2$.*

Proof. Suppose that C is an SIA. For any $S \in \mathcal{X}$, if $|C(S)| = 1$, then we can let $\Gamma(S) = C(S)$. Consequently, there is no information for revealed preference. We can focus on the collection of basic sets where $|C(B)| \geq 2$. By contradiction, suppose that every pair of elements in X is P_R comparable. Take the top 2 elements of X with respect to P_R, and denote them as x and y. Let's consider all basic sets that contain x and y. We then know that $\{x, y\} \subseteq C(B)$ for all these kinds of basic sets. As a result, x and y are P_R incomparable. $\square$

The uniqueness of revealed preferences is impeded by two factors. As a first point, satisficing differs from classical utility maximization. By satisficing, the DM selects alternatives that meet or exceed the criteria. The best alternative should be selected, but not all chosen alternatives are necessarily the most desirable. Consider the case in which the DM always chooses x whenever x is available. The same is true for y as well. In this manner, we are able to prove that x and y are all (at least) better than the rest of the alternatives while we are unable to determine the preference order between x and y by analyzing the revealed preferences.

Second, idempotent attention rules make it more difficult to analyze revealed preferences. With full attention, it is possible to conclude that the selected alternatives should be superior to the non-selected ones. As a result, determining the best alternatives is ambiguous. When the DM is limited in their attention, we may not be able to distinguish between the worst alternatives and the best because the unchosen alternatives may either be dominated by other alternatives or neglected by the DM.

Example 7. *Let $X = xyz$, and the choice correspondence C is given as*

$\|S\| \geq 2$	xyz	xy	xz	yz
C	xy	xy	x	z
Γ_1	xy	xy	x	z
Γ_2	xyz	xy	x	z
θ	y	y	x	z

Consider two distinct linear orders on X: $x \succ_1 y \succ_1 z$ and $z \succ_1 x \succ_1 y$. It is observed that $\langle \Gamma_1, \succcurlyeq_1, \theta \rangle$ and $\langle \Gamma_2, \succcurlyeq_2, \theta \rangle$ are all consistent pairs of SIA. It is difficult, however, to tell whether z is the best or worst, which makes the practice of revealed attention in this paper more convincing.

3.4.2 Revealed Attention under SIA

In the same manner, we examine the revealed attention by adopting the most stringent criterion.

Definition 18. *For any $S \in \mathcal{X}$, x is* ***revealed to catch attention in*** *S if $x \in \Gamma_i(S)$ for all $i \in I$, and $x \in S$ is* ***revealed to catch no attention in*** *S if $x \notin \Gamma_i(S)$ for all $i \in I$. Meanwhile, S is* ***revealed to catch full attention*** *if x is revealed to catch attention in S for all $x \in S$, i.e., $\Gamma_i(S) = S$ for all $i \in I$.*

For the SIA, we can associate S with one of its corresponding basic sets B, and interpret B as the $\Gamma(S)$. As shown in Example 7, the DM's attention on S is not necessarily basic. Intuitively, we can throw any "dominated" alternatives in any corresponding basic sets, and the new set is still consistent with one of the idempotent attention rules. Let us denote the undominated alternatives in S as $U^c(S, P_R) = \{s \in S : \nexists\ x \in C(S)\ s.t.\ s\ P_R\ x\}$.

Lemma 3. *Suppose that C admits an SIA. For any $T \subseteq S$, there is a Γ_i such that $\Gamma_i(S) = T$ if and only if $C(T) = C(S)$ and $T \setminus C(S) \subseteq U^c(S, P_R)$.*

Proof. Suppose that C admits an SIA. By the proof of Lemma 1, there is an $i \in I$ such that $\Gamma_i(\mathcal{X}) = \mathcal{B}$, and the corresponding consistent pair of the SIA is $\langle \Gamma_i, \succsim_i, \theta_i \rangle$. Take any $T \subseteq S$ with $C(T) = C(S)$ and $T \setminus C(S) \subseteq U^c(S, P_R)$. Without loss of generality, we can assume that $S, T \notin \mathcal{B}$ and $T \subset S$. For any $x \in X$, let us define $x^{\uparrow} =: \{y \in X : y\ P_R\ x\}$ and $x^{\downarrow} =: \{y \in X : x\ P_R\ y\}$. Moreover, let $x^{\uparrow c}$ and $x^{\downarrow c}$ as the complements of $x^{\uparrow}$ and $x^{\downarrow}$, repectively. Let us consider the collection $\bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$. Since $t^{\downarrow c}$ contains either P_R dominating or incomparable alternatives with respect to t, we can find a linear completion of P_R on this collection such that t is worse than every $t' \in t^{\downarrow c}$. Let us denote it as $\overline{P_R}'$. Moreover, we can extend this binary relation by assuming that $x \ \overline{P_R}' \ y$ for $x \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ and $y \in \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$. Let us define a new binary relation

$$x \succcurlyeq_i' y := \begin{cases} x \ \overline{P_R}' \ y & if \ (x,y) \in \overline{P_R}', \\ x \succcurlyeq_i y & otherwise. \end{cases}$$

We then want to show that $\succcurlyeq_i'$ is a linear order. Obviously, $\succcurlyeq_i'$ is complete and antisymmetric. Take any $x, y, z \in Z$ with $x \succcurlyeq_i' y$ and $y \succcurlyeq_i' z$. It is clear that is $x, y, z \in \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ or $x, y, z \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, $x \succcurlyeq_i' z$. When $x, y \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ and $z \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, $x \succcurlyeq_i z$. When $x \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ and $y, z \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, $x \succcurlyeq_i z$. Now, let us consider

$$\Gamma_i'(Y) = \begin{cases} T & if \ Y = S \ or \ T, \\ \Gamma_i(Y) & otherwise. \end{cases}$$

Obviously, Γ_i' is idempotent. Let $\theta_i'(S) = min(C(S), \succcurlyeq_i')$. We want to show that $\langle \Gamma_i', \succcurlyeq_i', \theta_i' \rangle$ is a consistent pair of the SIA. Take any $(x, y) \in P_R$, we want to show that $(x, y) \in \succcurlyeq_i'$. If $x \in \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, then $y \in \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$. As a result, $x \ \overline{P_R}' \ y$ which implies that $x \succcurlyeq_i' y$. If $x \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ and $y \in \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, then $x \succcurlyeq_i' y$. If $x \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$ and $y \in X \setminus \bigcup_{t \in T \setminus C(S)} t^{\downarrow c}$, then $x \succcurlyeq_i y$, which suggests that $x \succcurlyeq_i' y$. Consequently, for any $Y \neq S \ or \ T$, $C(Y) = \{s \in \Gamma_i(Y) : y \succcurlyeq_i \theta_i(Y)\} = \{s \in \Gamma_i'(Y) : y \succcurlyeq_i' \theta_i'(Y)\}$. We then only need to check the cases of $C(S)$ and $C(T)$. Since T is not basic, we know that there is a $B \in \mathcal{B}(T)$ such that $\Gamma_i(T) = B$. Moreover, $C(T) = \{t \in B : t \succcurlyeq_i \theta_i(B)\} = C(B)$. Take any $t \in C(T)$, we know that $t \succcurlyeq_i' \theta_i'(T)$ by the definition of θ_i'. Take any $t \in T$ with $t \succcurlyeq_i' \theta_i'(T)$, by contradiction, suppose that $t \notin C(T)$. As a result, $t \in T \setminus C(T) = T \setminus C(S)$. Therefore, for all $x \in C(S) = C(T) = C(B)$, we have either $x \ P_R \ t$ or x and t are P_R-incomparable. Hence, by the definition of $\succcurlyeq'_i$, we know that $x \succ'_i t$ which implies that $\theta'_i(T) \succ'_i t$.

For the converse direction, suppose that there is a Γ_i such that $\Gamma_i(S) = T$. Thus, $C(S) = \{s \in T : s \succcurlyeq_i \theta_i(S)\} = C(T)$. By contradiction, suppose that $T \setminus C(S)$ is not a subset of $U^c(S, P_R)$. That is, there is a $t \in T \setminus C(S)$ such that there is a $x \in C(S)$ such that $t \ P_R \ x$ which implies that $t \succ_i x$. Therefore, $t \in C(S)$. $\square$

The DM's attention for a menu S can be assigned to one of its subsets T with $C(S) = C(T)$. Moreover, T is the union of a corresponding basic set of S and a subset of $U^c(S, P_R)$. An alternative that attracts the attention of the DM in S must exist in every corresponding basic set. In contrast, an alternative does not draw any attention in S, suggesting it is not in any $B \in \mathcal{B}(S)$ and dominates some of the chosen alternatives.

Theorem 14. *Suppose that C is an SIA. An alternative x is revealed to catch attention in S if and only if $x \in B$ for every $B \in \mathcal{B}(S)$. An alternative $x \in S$ is revealed to catch no attention in S if and only if for all $T \subseteq S$ that including x, either $C(T) \neq C(S)$ or $T \setminus C(S) \not\subseteq U^c(S, P_R)$.*

Proof. For the first half of the statements, suppose that x is revealed to catch attention in S. That is, $x \in \Gamma_i(S)$ for all $i \in I$. As we know that for any $B \in \mathcal{B}(S)$ there is a Γ_i such that $\Gamma_i(S) = B$, which implies that $x \in B$ for all $B \in \mathcal{B}(S)$. For the converse direction, suppose that $x \in B$ for every $B \in \mathcal{B}(S)$. By contradiction, supposes that there is an $i \in I$ such that $x \notin \Gamma_i(S)$. Since $C(S) = C(\Gamma_i(S))$, $\Gamma_i(S) \notin \mathcal{B}$. Take any $B' \in \mathcal{B}(\Gamma_i(S))$, we know that $B \subset \Gamma_i(S) \subseteq S$ and $C(B') = C(\Gamma_i(S)) = C(S)$. Hence, $B' \in \mathcal{B}(S)$ and $x \notin B'$. The second half of the statement is a direct result of Lemma 3. □

When analyzing revealed attention on menus, we seek to identify the characteristics of the menus that the DM pays full attention to. As shown in Proposition 10, basic sets must catch the DM's full attention. As well, the converse is true.

Theorem 15. *Suppose that C is an SIA. S is revealed to catch full attention if and only if S is basic.*

Proof. We only need to show that if S is revealed to catch full attention, then $S \in \mathcal{B}$. By contradiction, suppose that $S \in \mathcal{B}^c$. Then, there is a $B \in \mathcal{B}(S)$. Based on the proof of Lemma 1, there is an $i \in I$ such that $\Gamma_i(S) = B$. □

3.4.3 Revealed Threshold under SIA

Threshold functions are the remaining part of revealing the SIA pair. We also employ the most stringent condition as usual.

Definition 19. *Suppose that C can be rationalized by an SIA. An alternative $x \in S$ is* ***revealed to be the threshold of*** *S if $\theta_i(S) = x$ for all $i \in I$.*

When the idempotent attention rule Γ_i and the linear preference rule $\succcurlyeq_i$ are given, the least preferred chosen alternative serves as the threshold for every menu. The threshold function θ_i will then be uniquely pinned down under these circumstances.

Proposition 12. *If C can be rationalized by an SIA under $\langle \Gamma_i, \succcurlyeq_i, \theta_i \rangle$, then $\theta_i(S)$ must be $\min(C(S), \succcurlyeq_i)$ for all $S \in \mathcal{X}$.*

Proof. Suppose that C can be rationalized by an SIA under $\langle \Gamma_i, \succcurlyeq_i, \theta_i \rangle$. By contradiction, suppose that $\theta_i \neq \min(C(S), \succcurlyeq_i)$. If $\theta_i(S) \succ_i \min(C(S), \succcurlyeq_i)$, then $\min(C(S), \succcurlyeq_i) \notin C(S)$. If $\min(C(S), \succcurlyeq_i) \succ_i \theta_i(S)$, then $\theta_i(S) \in C(S)$. □

Due to the fact that every completion of P_R can be regarded as a consistent linear preference, the threshold function also varies when the completion of P_R changes. If an alternative $x \in C(S)$ dominates another chosen alternative, then it cannot be a candidate for $\theta_i(S)$ for every $i \in I$. Those candidates should be alternatives in $MIN(C(S), P_R) := \{x \in C(S) : \nexists y \in C(S) \ s.t. \ x \ P_R \ y\}$.

Lemma 4. *Suppose that C can be rationalized by an SIA. For any $x \in X$ and $S \in \mathcal{X}$, $\theta_i(S) = x$ for some $i \in I$ if and only if $x \in MIN(C(S), P_R)$.*

Proof. Suppose that C can be rationalized by an SIA, and take any $x \in S \in \mathcal{X}$. Assume that there is an $i \in I$ such that $\theta_i(S) = x$. By contradiction, suppose that $x \notin MIN(C(S), P_R)$, we then know that there is $y \in C(S)$ such that $x \ P_R \ y$. By Corollary 13, we know that $x \succcurlyeq_i y$ for all $i \in I$. Since $C(S) \subseteq \Gamma_i(S)$ for all S, $y \notin C(S)$. For the converse direction, suppose that $x \in MIN(C(S), P_R)$, we know that there is a linear order $\succcurlyeq_i$ which is a linear completion of P_R such that $y \succcurlyeq_i x$ for all $y \in C(S)$. By the proof of Theorem 1, we know that $\succcurlyeq_i$ is a consistent preference for the SIA. Based on Proposition 12, the corresponding $\theta_i(S) = x$. □

As every alternative in $MIN(C(S), P_R)$ can be a candidate for the threshold used in S, it follows that the revealed threshold of S is related to the items in $MIN(C(S), P_R)$.

Theorem 16. *Suppose that C can be rationalized by an SIA. An alternative $x \in S$ is revealed to be the threshold of S if and only if $y \; P_R \; x$ for every $y \in C(X)$ with $y \neq x$.*

3.5 On the Variations of Idempotent Attention Rules

It is important to note that attention rules are unique features of research on choices under limited attention. As far as we know, they all have some restrictions on attention formation. A popular attention rule is the attention filter, which was first used in Masatlioglu et al. [2012]. Lleras et al. [2017] proposes an attention rule known as competition filters. Li [2024] combines attention filters with competition filters to form an attention rule called path independent consideration.

According to Li [2024], attention filters, competition filters, and path independent consideration are all special cases of idempotent attention rules. It should be noted, however, that not all attention rules in the previous research are idempotent. For example, the shortlisting procedure with capacity-k (see Geng and Özbay [2021]) and weak competition filters (see Geng [2022]). The following section considers two special cases of the SIA: the satisficing under attention filter and the satisficing under competition filter.

3.5.1 Attention Filter

Attention filters mean that the DM is unable to distinguish any set between S and $\Gamma(S)$. Formally,

Definition 20. *An attention rule Γ is an **attention filter** if $\Gamma(T) = \Gamma(S)$ for every $S, T \in \mathcal{X}$ with $\Gamma(S) \subseteq T \subseteq S$.*

As with idempotent attention rules, attention filters can be interpreted similarly. When the DM is faced with a menu T that lies between S and $\Gamma(S)$. For the DM, T, S, and $\Gamma(S)$ are all identical. We will now examine satisficing under attention filters.

Definition 21. *A choice correspondence C is a **satisficing under attention filter (SAF)** if C is an SIA with $\langle \Gamma, \succsim, \theta \rangle$ where Γ is an attention filter.*

The distinction between the SIA and the SAF comes from the differences between idempotent attention rules and attention filters. In idempotent attention rules, $\Gamma(S) = \Gamma(S \setminus (S \setminus \Gamma(S)))$ while attention filters require more. Specifically, $\Gamma(S) = \Gamma(S \setminus Y)$ for all $Y \subseteq S \setminus \Gamma(S)$. Therefore, when an alternative x is removed from a menu S and the DM's attention on $S \setminus x$ changes, x must attract attention in S. If x is not chosen by the DM, then all the alternatives in S should be preferred over x. We can define this binary relation similarly to the SIA.

Definition 22. *For any $x, y \in X$, $x\ P^{AF}\ y$ if there is a set $S \in \mathcal{X}$ such that $x \in C(S)$, $y \notin C(S)$, and $C(S) \neq C(S \setminus y)$.*

Similar as the SIA, the acyclicity of P^{CF} characterizes the SAF.

Lemma 5. *C can be rationalized by an SAF if and only if P^{AF} is acyclic.*

Proof. We first show the only if part. Assume that C can be rationalized by an SAF. By contradiction, suppose that P^{AF} is cyclic, i.e., there is a distinct pair of x and y such that $x\ P_R^{AF}\ y$ and $y\ P_R^{AF}\ x$ where P_R^{AF} is the transitive closure of P^{AF}. We also know that x must be revealed to be preferred to y, and y is revealed to be preferred to x. Since x and y are distinct items, this observation contradicts the preference in any SAF is a linear order.

For the converse direction, assume that P^{AF} is acyclic. Let $\succsim^{AF}$ be any completion of P^{AF}. Consider the following attention rule Γ^{AF} and threshold function θ^{AF}:

$$\theta^{AF}(S) = \min\left(C(S), \succsim^{AF}\right) \ , and \ \Gamma^{AF}(S) = \{C(S)\} \cup \{s \in S : \theta^{AF}(S) \succsim^{AF} s\}.$$

Notice that $C(S) = \{x \in \Gamma^{AF}(S) : x \succsim^{AF} \theta(S)\}$ for all $S \in \mathcal{X}$. We then want to show that C can be rationalized under the SIA by $\langle \Gamma^{AF}, \theta^{AF}, \succsim^{AF} \rangle$. We first show that $C(S \setminus y) = C(S)$ whenever $y \notin \Gamma^{AF}(S)$. By contradiction, suppose there is a $y \notin \Gamma^{AF}(S)$ such that $C(S) \neq C(S \setminus y)$. Take any $x \in C(S)$, we have $x \ P^{AF} \ y$ because $y \notin C(S)$ and $C(S) \neq C(S \setminus y)$. Consequently, we know that $\theta^{AF}(S) \succsim^{AF} y$ for $\succsim^{AF}$ is a completion of P^{AF} which suggests that $y \in \Gamma^{AF}(S)$. Hence, we know that $C(S \setminus y) = C(S)$ whenever $y \notin \Gamma^{AF}(S)$. Thus, we have $\theta^{AF}(S \setminus y) = \theta^{AF}(S)$ whenever $y \notin \Gamma^{AF}(S)$. Finally,

$$\begin{aligned}
\Gamma^{AF}(S) = \Gamma^{AF}(S) &= \{C(S)\} \cup \{s \in S : \theta^{AF}(S) \succsim^{AF} s\} \\
&= \{C(S \setminus y)\} \cup \{s \in S : \theta^{AF}(S \setminus y) \succsim^{AF} s\} \\
&= \{C(S \setminus y)\} \cup \{s \in S \setminus y : \theta^{AF}(S \setminus y) \succsim^{AF} s\} \\
&= \Gamma^{AF}(S \setminus y).
\end{aligned}$$

Since Γ^{AF} is an attention filter, we can keep iteration to show that $\theta^{AF}(S) = \theta^{AF}(\Gamma^{AF}(S))$. As a result, C can be rationalized by $\langle \Gamma^{AF}, \theta^{AF}, \succsim^{AF} \rangle$ under the SAF. □

Similarly to the SIA, it is also possible to form a WARP-like axiom analogously to that in Masatlioglu et al. [2012].

Axiom 7. ***(WARP-AF)*** *For every $S \in \mathcal{X}$ there is an $x^* \in S$ such that for every T if $C(T) \cap S \neq \emptyset$ and $C(T \setminus x^*) \neq C(T)$, then $x^* \in C(T)$.*

x^* can still be interpreted as the "best" alternatives in S. Due to the fact that C is a choice correspondence in our framework, our WARP-like axiom modifies WARP(LA) in Masatlioglu et al. [2012] by allowing $C(T) \cap S \neq \emptyset$. As it turns out, the acyclicity of P^{CF} is equivalent to the WARP-AF.

Lemma 6. *P^{AF} is acyclic if and only if C satisfies WARP-AF.*

Proof. If P^{AF} is acyclic, then C can be rationalized by a consistent SLA pair $\langle \Gamma, \succcurlyeq, \theta \rangle$. Let $x^* = \max(S, \succcurlyeq)$ for all $S \in \mathcal{X}$. Take any T where $C(T) \cap S \neq \emptyset$ and $C(T \setminus x^*)$. If $x^* \in C(T)$, we are done. If there is a $y \in C(T) \cap S$, we have $x^* \succcurlyeq y$. Because$C(T \setminus x^*) \neq C(T)$ implies that $x^* \in \Gamma(T)$, we have $x^* \in C(T)$.

For the converse direction, we prove it by showing its contrapositive statement. Suppose that P^{AF} is cyclic, we then know that there is a $z_1 \in X$ such that $z_1 \ P_R^{AF} \ z_1$, i.e., there are finite collections $\{z_j\}_{j=1}^m$ $\{Z_j\}_{j=1}^m$ where $z_j \in Z_j$ for each j such that

$$z_1 \in C(Z_1), C(Z_1) \neq C(Z_1 \setminus z_2) \textit{ and } z_2 \notin C(Z_1),$$

$$z_2 \in C(Z_2), C(Z_2) \neq C(Z_2 \setminus z_3) \textit{ and } z_3 \notin C(Z_2),$$

$$\ldots$$

$$z_m \in C(Z_m), C(Z_m) \neq C(Z_m \setminus z_1) \textit{ and } z_1 \notin C(Z_m).$$

Let $Z = \{z_j\}_{j=1}^m$. For any $z_1 \in Z$ there is a set Z_m such that $C(Z_m) \cap S \neq \emptyset$, $C(Z_m \setminus z_1) \neq C(Z_m)$, and $z_1 \notin C(Z_m)$. For any $z_j \in Z$ with $j \geq 2$, there is a set Z_{j-1} such that $C(Z_{j-1}) \cap S \neq \emptyset$, $C(Z_{j-1} \setminus z_j) \neq C(Z_{j-1})$. $\square$

Consequently, the SAF can be characterized by the WARP-AF as well.

Theorem 17. *A choice correspondence C is an SAF if and only if C satisfies WARP-AF.*

3.5.2 Competition Filter

Lleras et al. [2017] defines competition filters, which suggest that if an alternative catches the DM's attention in a menu S, it will also do so in all subsets of S that include x.

Definition 23. *An attention rule is a competition filter if $x \in \Gamma(S) \cap T$ implies that $x \in \Gamma(T)$ for all $T, S \in \mathcal{X}$ with $T \subseteq S$.*

In a similar manner to the SAF, satisficing under competition filter can be defined.

Definition 24. *A choice correspondence C is a* ***satisficing under competition filter (SCF)*** *if C is an SIA with $\langle \Gamma, \succsim, \theta \rangle$ where Γ is a competition filter.*

Competition filters indicate that, if the DM pays attention to some alternatives in a larger menu, these alternatives, if available in a smaller menu, should attract the DM's attention. This is because perceiving alternatives in a more complex environment should be more difficult. The challenge is to determine how to infer the DM's attention from a menu. One trivial observation is that the chosen alternatives must catch the DM's attention. Thus, we can define a binary relation based on it.

Definition 25. *For any $x, y \in X$, $x \ P^{CF} \ y$ if there are sets $T, S \in \mathcal{X}$ with $\{x, y\} \subseteq T \subseteq S$ such that $y \in C(S)$, $x \in C(T)$ and $y \notin C(T)$.*

The acyclicity of P^{CF} characterizes the SCF.

Lemma 7. *A choice correspondence C is an SCF if and only if P^{CF} is acyclic.*

Proof. Assume that C can be rationalized by an SCF pair $\langle \Gamma, \theta, \succcurlyeq \rangle$. By contradiction, suppose that there is a $z_1 \in X$ such that $z_1 \ P_R^{CF} \ z_1$ where P_R^{CF} is the transitive closure of P^{CF}. Take any $x \ P^{CF} \ y$, we know that $x \succcurlyeq y$ by the definition of P^{CF}. Since $\succcurlyeq$ is a linear order, we know that $z_1 \ P_R^{CF} \ z_1$ cannot be true.

Suppose that P^{CF} is acyclic for the converse direction. We then can fix any linear order $\succcurlyeq^{CF}$ who is a completion of P_R^{CF}. Consider

$$\Gamma^{CF}(T) = \{x \in T : x \in C(S) \ for \ some \ S \supseteq T\} \ and \ \theta^{CF}(T) = min(C(T), \succcurlyeq^{CF}).$$

It is clear that Γ^{CF} is a competition filter. We then want to show that $\langle \Gamma^{CF}, \succcurlyeq^{CF}, \theta^{CF} \rangle$ is a consistent SAF pair, i.e., $C(T) = \big\{ t \in \Gamma^{CF}(T) : t \succcurlyeq^{CF} \theta^{CF}(T) \big\}$. Take any $t \in C(T)$, we know that $t \in \Gamma^{CF}(T)$ and $t \succcurlyeq^{CF} \theta^{CF}(T)$ by the definitions. For the converse direction, take any $t \in \Gamma^{CF}(T)$ and $t \succcurlyeq^{CF} \theta^{CF}(T)$. By contradiction, suppose that $t \notin C(T)$. We know that there is a set $S \supset T$ such that $x \in C(S)$, which implies that $\theta^{CF}(T) \ P^{CF} \ x$. Hence $\theta^{CF}(T) \succcurlyeq^{CF} x$. $\square$

The WARP-like axiom for the SCF also can be formed as in the SIA and the SAF.

Axiom 8. ***(WARP-CF)*** *For every $S \in \mathcal{X}$, there is a $x^* \in S$ such that for any T with $x^* \in T$, if $C(T) \cap S \neq \emptyset$ and $x^* \in C(Y)$ for some $Y \supseteq T$ then $x^* \in C(T)$.*

Similarly, x^* can be interpreted as the "best" alternatives in S. When an alternative in S is chosen in T, and x^* catches the DM's attention in T, the x^* should be chosen in T. The acyclicity of $P^C F$ is equivalent to WARP-CF.

Lemma 8. *P^{CF} is acyclic if and only if C satisfies WARP-CF.*

Proof. We first suppose that P^{CF} is acyclic. We then know that C is an SCF under a consistent pair $\langle \Gamma, \theta, \succcurlyeq \rangle$. Take any $S \in \mathcal{X}$, and $x^* = max(S, \succcurlyeq)$. For any $T \in \mathcal{X}$, since there is a $Y \supseteq T$ such that $x^* \in C(Y)$, $x^* \in \Gamma(T)$. Because $C(T) \cap S \neq \emptyset$, we know that $x^* \in C(T)$.

For the converse direction, we show it by showing the contrapositive statement. Suppose that P^{CF} is acyclic. There is a finite sequence of $\{z_j\}_{j=1}^m$ with

$$z_1 \ P^{CF} \ z_2, ..., \ P^{CF} \ z_m \ P^{CF} \ z_1.$$

That is there are finite collections of menus $\{Z_j\}_{j=1}^m$ and $\{Z_j'\}_{j=1}^m$ such that

$$\{z_1, z_2\} \subseteq Z_1 \subseteq Z_1', \ z_2 \in C(Z_1'), \ z_1 \in C(Z_1), \ and \ z_2 \notin C(Z_1),$$

$$\{z_2, z_3\} \subseteq Z_2 \subseteq Z_2', \ z_3 \in C(Z_2'), \ z_2 \in C(Z_2), \ and \ z_3 \notin C(Z_2),$$

$$\ldots$$

$$\{z_m, z_1\} \subseteq Z_m \subseteq Z_m', \ z_1 \in C(Z_m'), \ z_m \in C(Z_m), \ and \ z_1 \notin C(Z_m).$$

Let's consider $Z = \{z_j\}_{j=1}^m$. For z_1, there is a $Z_m \subseteq Z'_m$ such that $z_1 \in C(Z'_m)$, $C(Z_m) \cap Z \neq \emptyset$ and $z_1 \notin C(Z_m)$. For any z_j with $j > 1$, there is a $Z_{j-1} \subseteq Z'_{j-1}$ such that $z_j \in C(Z'_{j-1})$, $C(Z_{j-1}) \cap Z \neq \emptyset$ and $z_j \notin C(Z_{j-1})$. □

Obviously, we can characterize the SCF by WARP-CF.

Theorem 18. *A choice correspondence C is an SCF if and only if C satisfies WARP-CF.*

3.6 Conclusion

We propose a satisficing model under idempotent attention in this paper. To characterize the SIA, we take perspectives from both the acyclicity condition of some binary relation and the WARP-like axiom.

SIA separates limited attention from limited cognitive abilities, which have long been considered as factors leading to satisficing or bounded rationality (see Simon [1997], Tyson [2008], Frick [2016] et.al.). In particular, we consider a family of idempotent attention rules used in Li [2024], which is a generalization of attention filters (see Masatlioglu et al. [2012]), competition filters (Lleras et al. [2017]), and path independent consideration (see Lleras et al. [2021]). Two systems of characterization have been provided for both the SAF and SCF as special cases.

We analyze the revealed SIA pairs under the strictest conditions to determine the welfare implications of the SIA. Unfortunately, if a choice correspondence is an SIA, then it must have at least two consistent SIA pairs. This negative result is due to the inability to determine the DM's preference based on the choice correspondence.

Chapter 4

Attention Formation: Triggers and Blockers

4.1 Introduction

Limited attention has been widely recognized as a crucial factor in decision-making processes (cf. Masatlioglu et al. [2012], Manzini and Mariotti [2014], and Lleras et al. [2017], Cattaneo et al. [2020]). When faced with a choice problem, a decision maker (DM) with limited attention selectively concentrates on a subset of options within the choice set. Existing research explores limited attention by focusing on given attention rules rather than the procedures for forming attention. For instance, Masatlioglu et al. [2012] investigates cases where the removal of alternatives that fail to capture attention has no impact on the DM's attention. Additionally, within a stochastic framework, Manzini and Mariotti [2014] assumes that an alternative catches the DM's attention with a fixed probability for every menu, and focuses on the context-independent random attention model.

Many studies have suggested that the DM's attention on an alternative is influenced by the existence of other alternatives (for example, Eliaz and Spiegler [2011], Ok et al. [2015], Brady and Rehbeck [2016], Heiss et al. [2021],). Eliaz and Spiegler [2011] discusses how an alternative provided by one firm, "attention grabbers", can strategically shift the DM's attention towards its own service, diverting attention away from competitors. Additionally, Heiss et al. [2021] document the presence of attention triggers: the existence of an alternative makes the DM pay attention to another option.

These findings suggest that the DM pays attention on an alternative because another alternative exists. On the other hand, the DM pays no attention to some alternatives since the attention is blocked by some alternatives. We refer to alternatives that prevent the DM from noticing an alternative as this alternative's attention blockers. Intuitively, whether an alternative catches the DM's attention depends on the relative forces between attention triggers and attention blockers.

Attention triggers have a positive effect on attracting attention, while attention blockers have a negative effect. These opposing forces present a significant challenge to formalizing attention formation. Our paper uses signed orders to model attention triggers and blockers simultaneously.

Signed orders were first introduced by Fishburn [1992a] to bridge preferences over a set of alternatives and its power sets, linking preferences over alternatives to preferences over menus. Signed orders introduced the concept of reflection, where a reflection of an alternative can be considered as its "opposite number." Adding a reflection of an alternative to a menu has the same effect as eliminating the alternative, making it a suitable tool for depicting the opposite effects of attention triggers and blockers.

This paper employs signed orders in an indirect utility fashion. Specifically, given a menu S, for any $x \in S$ there is the most significant alternative. This maximizes the signed order for x in S. If the significant alternative is an attention trigger, the DM pays attention to x. Otherwise, x is ignored. The collection of signed orders for every alternative in S determines the collection of attended alternatives in S. The DM, as suggested in studies on limited attention, chooses the best among the attended alternatives.

This paper also considers two extreme cases: (i) The effect of attention triggers always dominates that of attention blockers, and (ii) attention blockers always dominate. In the first case, the DM pays attention to an alternative in S if there is an attention trigger for it in S. In contrast, in the latter case, the DM ignores an alternative due to the presence of attention blockers. They are closely related to the "decoy effect" found in Huber et al. [1982].

This paper is organized as follows: Section 2 formally introduces the model. Section 3 provides characterizations of the model. In Section 4, we discuss the two special cases mentioned above. Finally, Section 5 concludes the paper.

4.2 Preliminaries

Let X denote a finite set of mutually exclusive alternatives of interest. We define $\mathcal{X}$ as the collection of all nonempty subsets of X. Throughout this paper, elements in $\mathcal{X}$

are referred to as menus, and elements in X are called alternatives. We use S, T, Y, Z and w, x, y, z to represent them, respectively.

A binary relation $\mathbf{R}$ on X is a nonempty subset of $X \times X$. Instead of $(x, y) \in \mathbf{R}$, we write it as $x \mathbf{R} y$. $\mathbf{R}$ is complete if either $x \mathbf{R} y$ or $y \mathbf{R} x$ for any $x, y \in X$; transitive if $x \mathbf{R} y$ and $y \mathbf{R} z$ implies that $x \mathbf{R} z$ for any $x, y, z \in X$; antisymmetric if $x \mathbf{R} y$ implies that $\neg y \mathbf{R} x$ for any $x, y \in X$. $\mathbf{R}$ is a weak order when it is complete and transitive. A weak order is linear when it is antisymmetric.

For each $x \in X$, let x^* be the reflection of x. The collection of alternatives' reflections is denoted as X^*. A weak order $\mathbf{R}$ on $X \cup X^*$ is a signed order when $x \mathbf{R} y$ if and only if $y^* \mathbf{R} x^*$.

A choice function, denoted as $c : \mathcal{X} \to X$, maps each set $S \in \mathcal{X}$ to an element $c(S) \in S$. The chosen alternative $c(S)$ represents the DM's selection when confronted with S. We introduce $\Gamma : \mathcal{X} \to \mathcal{X}$ as an attention rule employed by the DM. For any menu $S \in \mathcal{X}$, $\Gamma(S)$ represents the alternatives she pays attention to in S. When the DM has limited attention, she chooses the best alternative among the feasible options that capture her attention.

The DM's preference is denoted as a linear order $\succ$ on X. For any x, y, $x \succ y$ indicates that the DM prefers x over y. The choice function is rational under limited attention if there is a $(\Gamma, \succ)$ such that $c(S) = \max(\Gamma(S), \succ)$ for every $S \in \mathcal{X}$.

Attention is determined by interactions between alternatives. For every pair of $x, y \in X$, y may serve as an attention trigger (or blocker): The existence of y makes the DM notice (or ignore) x. Every S that contains x, therefore, has two opposite effects on the attention of x. The DM pays attention to x in S only when the effect of triggers dominates the effect of blockers.

These interactions are represented by a collection of signed orders $\{\trianglerighteq_x\}_{x\in X}$, such that, for any $x, y \in X \cup X^*$, $x \trianglerighteq_x y$ and $y \trianglerighteq_x x$ if and only if y is the reflection of x. We slightly abuse the definition of signed orders by referring to them in this paper as those with the above restrictions. $y \trianglerighteq_x x$ means that y serves as an attention trigger for x. When $z^* \trianglerighteq_x y \trianglerighteq_x x$, it implies that the effect of z outweighs the effect of y on x. The effect of a menu on x depends on the largest effect of alternatives. For instance, if $z^* \trianglerighteq_x y \trianglerighteq_x x \trianglerighteq_x y^* \trianglerighteq_x z$, then x catches no attention when $\{x, y, z\}$ is presented.

Assume that attention formation only works for menus that contain more than three alternatives. [1] Given the collection of signed orders, attention on every menu can be uniquely pinned down. The choice should be the best alternative among the attended ones.

Definition 26. *A choice function $c : \mathcal{X} \to X$ is a* ***choice with attention formation (CAF)*** *if there exists a linear order $\succ$ and a collection of signed orders $\{\trianglerighteq_x\}_{x\in X}$, such that $c(S) = \max(\Gamma(S), \succ)$ where*

$$\Gamma(S) = \begin{cases} S & \text{if } |S| \leq 2; \\ \{x \in S : \max(S, \trianglerighteq_x) \in X\} & \text{if } |S| > 2. \end{cases}$$

CAF introduces a behavioral postulate to the DM. When confronted with a menu S where $|S| \geq 3$, the DM's attention on x is driven by the content of S. If the most

[1] Many research on consideration sets and reference points investigate this situation. For example, research on the "attraction effect" (e.g. Huber et al. [1982], Ok et al. [2015]) suggests that adding a decoy option makes the DM pay attention to the target option.

significant alternative relative to x is an attention trigger for x, x should attract the DM's attention. Otherwise, she pays no attention to x.

Example 8. *Let $X = \{w, x, y, z\}$, and part of the choice function is given below.*

$\lvert S\rvert > 2$	$\{w,x,y,z\}$	$\{w,x,y\}$	$\{w,x,z\}$	$\{w,y,z\}$	$\{x,y,z\}$
C	x	w	y	w	x

Suppose that $w \succ x \succ y \succ z$, which can be induced by all binary choices. Since $c(\{w, x, y, z\}) = x$ and $w \succ x$, one of x, y, z must block the DM's attention on w, and it dominates other alternatives' effects. Since $c(\{w, x, y\}) = w$, z must block the DM's attention, and this effect will dominate the effect of y on w. w should be blocked in $\{w, y, z\}$ while $c(\{w, y, z\}) = w$. Therefore, this choice function is not a CAF.

4.3 Characterization

Only when a menu contains at least three alternatives, do attention triggers and blockers play their roles. Therefore, every choice in doubletons should be made via the underlying preferences.

Axiom 9. ***(No-Cycle)*** *For any $x, y, z \in X$, if $c(\{x, y\}) = x$ and $c(\{y, z\}) = y$, then $c(\{x, z\}) = x$.*

The DM's behavior will be affected by their attention when menus contain more than two alternatives. Let $\mathcal{M}$ be the collection of such menus, i.e., $\mathcal{M} := \{S \in \mathcal{X} : |S| \geq 3\}$.

Given any menu $S \in \mathcal{M}$, the chosen alternative must be noticed by the DM. In other words, the most significant alternative in S relative to $c(S)$ is an attention trigger. If the DM also chose $c(S)$ in a different menu $T \in \mathcal{M}$, she should notice the existence of $c(S)$ in $S \cup T$ since the most significant alternative in S for $c(S)$ is also an attention trigger.

Similarly, if the DM chooses a less desirable alternative than x in S and T when x is available because she pays no attention to x, then she pays no attention to x when $S \cup T$ is presented.

Definition 27. *For any $S \in \mathcal{M}$, we say S is **positively x-covered** by a collection of menus $\{S_i\}_{i \in I}$ where $S_i \in \mathcal{M}$, if (i) $\bigcup_{i \in I}(S_i \cup \{x\}) = S \cup \{x\}$ and (ii) $c(S_i \cup \{x\}) = x$ for each i. Similarly, S is **negatively x-covered** by a collection of menus $\{S_i\}_{i \in I}$ where $S_i \in \mathcal{X}$, if (i) $\bigcup_{i \in I}(S_i \cup \{x\}) = S \cup \{x\}$ and (ii) $c(S_i \cup \{x\}) \neq c(\{c(S_i \cup \{x\}), x\})$ for each i.*

Clearly, the DM must pay (no) attention to x whenever S is (negatively) positively x-covered.

Axiom 10. ***(Positive Contraction)*** *For any $Y, S \in \mathcal{M}$ with $Y \subseteq S$, if S is positively x-covered and Y is negatively x-covered, then $c(Z \cup \{x\}) = c(\{c(Z \cup \{x\}), x\})$ for every Z with $S \setminus Y \subseteq Z \subseteq S$.*

When the DM pays attention to x in $S \cup \{x\}$, and she fails to pay attention to x in $Y \cup \{x\} \subseteq S \cup \{x\}$, the most significant attention triggers of x should be in S but not in T. When the DM faces any choice problem $Z \cup \{x\}$ where Z lies between S and $S \setminus Y$, x must attract her attention in $Z \cup \{x\}$. Consequently, the chosen alternative must be better than x if it is a different alternative.

Axiom 11. ***(Negative Contraction)*** *For any $Y, S \in \mathcal{M}$ with $Y \subseteq S$, if S is negatively x-covered and Y is positively x-covered, then $c(Z \cup \{x\}) \neq x$ for every Z with $S \setminus Y \subseteq Z \subseteq S$.*

Similarly, consider a situation where the DM pays no attention to x when $S \cup \{x\}$ is presented. However, she pays attention to x when $Y \cup \{x\} \subseteq S \cup \{x\}$ is given. For any Z with $S \setminus Y \subseteq Z \subseteq S$, the most magnificent alternatives relative to x in S still work. As a result, the DM pays no attention to x in $Z \cup \{x\}$, and x cannot be chosen.

CAF can be characterized by the above three axioms.

Theorem 19. *A choice function c satisfies No-Cycle, Positive Contraction, and Negative Contraction if and only if c is a CAF.*

Proof. See Appendix A.9 □

The effects of attention triggers and blockers are represented as signed orders. Intuitively, Positive Contraction and Negative Contraction can be summarized as a condition related to signed orders. If S is positively x-covered, the DM pays attention to x in $S \cup \{x\}$. As a result, S cannot be negatively x-covered, i.e., the DM pays no attention to x in $S \cup \{x\}$.

Proposition 13. *A choice function c satisfies Positive Contraction, and Negative Contraction if and only if there is no $S \in \mathcal{X}$ such that S is simultaneously positively x-covered and negatively x-covered.*

Proof. See Appendix A.8 □

4.4 Triggers versus Blockers

The decoy effect is a well-documented found in many studies. It suggests that adding a decoy option to a choice problem can make the DM choose the target options (cf. Huber et al. [1982]). An explanation is that the decoy option serves as a reference point, which makes the DM notice the existence of the target option (see Ok et al. [2015]).

In our framework, the decoy option serves as an attention trigger for the target option. It overcomes any disadvantages in paying attention to the target option. In other words, it suggests that the existence of attention triggers can overcome any negative effects from attention blockers. In this section, we explore this and various other special cases of CAF by specifying the relative strengths between attention triggers and attention blockers.

4.4.1 Triggers Dominate Blockers

In the extreme case where the strength of attention triggers outweighs the strength of attention blockers for any alternative x, the DM's attention on x will be captured by the existence of attention triggers.

Definition 28. *A choice function c is a* ***choice under attention triggers (C-AT)*** *if there is a linear order $\succ$ and a collection of signed order $\{\trianglerighteq_x\}_{x\in X}$ such that $c(S) = \max(\Gamma(S), \succ)$, where*

$$\Gamma(S) = \{x \in S : \exists\ y \in S\ s.t.\ y \rhd_x x\}$$

for every $S \in \mathcal{X}$.

For C-AT, we relaxed the assumption that attention triggers and blockers only work for menus with three or more alternatives. We can easily identify attention triggers for any alternatives based on binary choices.

Definition 29. *Take any distinct alternatives $x, y \in X$, a binary relation R^t is defined as: $x \; R^t \; y$ if there are two menus $T \subseteq S$ such that $c(S) = x$ and $c(T) = y$.*

For any $x, y \in S$ and $T \subseteq S$, $c(T) = y$ means that there is an alternative $z \in T$ that works as an attention trigger for y. y then catches the DM's attention when S is presented. However, the DM chooses x in S, which suggests that x is preferred to y.

Axiom 12. ***(R^t-acyclicity)*** *R^t is acyclic.*

The DM chooses an alternative x in S because it is the best among the attended alternatives. If there is an alternative $y \in S$ such that $y \succ y$, then the DM should choose x when $\{x, y\}$ is presented. If it is the best in S, then the existence of an attention trigger y for x suggests that $c(\{x, y\}) = x$.

Axiom 13. ***(Binary Consistency)*** *For all $S \in \mathcal{X}$ with $c(S) = x$, there is a $y \in S$ such that $c(\{x, y\}) = x$.*

R^t-acyclicity and Binary Consistency are the axioms we need to characterize C-AF.

Theorem 20. *A choice function c is a C-AT if and only if c satisfies R^t-acyclicity and Binary Consistency.*

Proof. For the necessity part, suppose that c is a C-AT under $(\{\trianglerighteq\}_{x \in X}, \succ)$. Take any $T \subseteq S$ with $c(S) = x$ and $c(T) = y$. Since there is a $z \in T \subseteq S$ such that $z \; \trianglerighteq_y \; y$, $y \in \Gamma(S)$. As a result, $x \succ y$. Since $\succ$ is linear, R^t should be acyclic. For Binary Consistency, take any

S with $c(S) = x$. Let $x^{\uparrow} := \{y \in X : y \succ x\}$ and $x^{\downarrow} := \{z \in X : x \succ z\}$. If $x^{\uparrow} \cap S = \emptyset$, $c(\{x, w\}) = w$ for every $w \in S$ implies that $x \trianglerighteq_x w$. As a result, $s \notin \Gamma(S)$, and $c(S) \neq x$. In this case, we have Binary Consistency. If $x^{\uparrow} \cap S \neq \emptyset$, we can take any $y \in x^{\uparrow} \cap S$. If $c(\{x, y\}) = y$, we know that x serves as an attention trigger for y. As a result, $x, y \in \Gamma(S)$ and $y \succ x$ suggest that the DM cannot choose x when S is given.

For the sufficient part. Fix an arbitrary permutation on X and let $X := \{x_n\}_{n=1}^{N}$. Let's define

$$f_x(x_n) = \begin{cases} \frac{1}{n} & \text{if } c(\{x, x_n\}) = x, \\ \\ -\frac{1}{n} & \text{otherwise.} \end{cases}$$

It is clear that f_x is a bijection and $f_x(x_n) \neq 0$ for any $n \leq N$. Let

$$g_x(x_n) = \begin{cases} 0 & \text{if } x = x_n, \\ \\ f_x(x_n) & \text{otherwise.} \end{cases}$$

Clearly, g_x is a bijectition and $g_x(x_n) = 0$ only when $x = x_n$. Let P^t be a linear completion of R^t, and $G(S) = \{x \in S : f_x(y) > 0 \text{ for some } y \in S\}$. We want to show that $c(S) = \max(G(S), P^t)$.

Clearly, $x \in G(S)$. For every $y \in G(S)$, we know that there is an $z \in S$ such that $c(\{y, z\}) = y$. As a result, $x \; P^t \; y$. Let $y \trianglerighteq_x z$ if $f_x(y) \geq f_x(z)$. It is clear that $\trianglerighteq_x$ is a signed order induced by f_x, and c is a C-AF under $(\{\trianglerighteq_x\}_{x \in X}, P^t)$. $\square$

4.4.2 Blockers Dominate Triggers

In the case where attention blockers have a dominant strength over attention triggers, regardless of the number of attention triggers present for x in the menu S, the presence of even a single attention blocker will result in the DM not paying any attention to x in S.

Definition 30. *A choice function C is a* ***choice under attention blockers (C-AB)*** *if there is a linear order $\succ$ and a collection of signed order $\{\trianglerighteq_x\}_{x \in X}$ such that $c(S) = \max(\Gamma(S), \succ)$, where*

$$\Gamma(S) = \{x \in S : \nexists\ y \in S \ s.t.\ x \triangleright_x y\}$$

for every $S \in \mathcal{X}$.

An alternative catches the DM's attention if there are no attention blockers for it in the given menu. Observe that if there are no attention blockers for x in a menu S, then the DM pays attention to x in any subset of S.

Definition 31. *Take any two distinct alternatives $x, y \in X$, a binary relation R^b is defined as: $x\ R^b\ y$ if there exists an $S \in \mathcal{X}$ with $C(S) = y$ and a $T \subseteq S$ with $y \in T$ such that $C(T) = x$.*

For any $y \in T \subseteq S$, if the DM chooses y in S, then the DM's attention on y is not blocked by any alternatives in S. As a result, she pays attention on y when T is presented. The chosen alternative x in T must be better than x.

Axiom 14. ***(R^b-acyclicity)*** *R^b is acyclic.*

Consider a situation where the DM chooses y when a better alternative x exists. The DM should not pay attention to x, which suggests that there is an attention blocker for x in the given choice problem.

Axiom 15. ***(Global Blocking)*** *For all $x\ Tr(R^b)\ y$, if there is an $S \in \mathcal{X}$ such that $c(S) = y$ and $x \in S$, then there is a $z \in S$ such that $c(T) \neq x$ for all T with $z \in T$.*

The absence of attention blockers for x in a set T can guarantee the DM pays attention to x in every subset of T. The DM's attention on $S \cup T$ is a subset of the union of her attention on S and T. Consequently, if she chooses x is both S and T, she must choose x in $S \cup T$.

Axiom 16. ***(Expansion)*** *For all $S, T \in \mathcal{X}$, if $c(S) = c(T) = x$, then $c(S \cup T) = x$.*

Theorem 21. *A choice function C is a C-AB if and only if c satisfies R^b-acyclicity, Global Blocking, and Expansion.*

Proof. For the necessity part, we first consider $y \in T \subseteq S$ with $c(S) = y$ and $c(T) = x$. We then know that there is no $z \in S$ such that $y \trianglerighteq_y z$. y should catch the DM's attention in T, which implies that $x \succ y$. R^b, therefore, is acyclic. Take any $x, y \in S$ where $x\ Tr(R^b)\ y$ and $c(S) = y$, we know that $x \succ y$ and $c(S) \neq x$ suggest that there is a $z \in S$ such that $y \trianglerighteq_y z$. As a result, x cannot be chosen whenever z exists. For Expansion, take any $S, T \in \mathcal{X}$ where $c(S) = c(T)$. We first want to show that $\Gamma(S \cup T) \subseteq \Gamma(S) \cup \Gamma(T)$. If $x \in \Gamma(S \cup T)$, then we know that there is no $z \in S \cup T$ such that $x \trianglerighteq_x z$, we then know that $x \in \Gamma(S) \cup \Gamma(T)$. If

$c(S) = c(T) = x$, then we know that $x \succ y$ for every $y \in \Gamma(S) \cup \Gamma(T)$. As a result, $x \succ z$ for every $z \in \Gamma(S \cup T)$, and $c(S \cup T) = x$.

For the sufficient part, fix an arbitrary permutation on X and let $X := \{x_n\}_{n=1}^N$. Let's define

$$f_x(x_n) = \begin{cases} -\frac{1}{n} & \text{if } c(Y) \neq x \text{ for every } x, x_n \in Y, \\ \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

It is clear that f_x is a bijection and $f_x(x_n) \neq 0$ for any $n \leq N$. Let

$$g_x(x_n) = \begin{cases} 0 & \text{if } x = x_n, \\ \\ f_x(x_n) & \text{otherwise.} \end{cases}$$

Let P^b be a linear completion of R^b, and $G(S) = \{x \in S : \nexists y \in S \text{ s.t. } g_x(y) < 0\}$. We want to show that $c(S) = \max(G(S), p^b)$. We first know that $c(S) \in G(S)$ because there are no attention blockers for x in S. If $y \in G(S)$, we know that for every $z_1 \in S$, there is an T_{z_1} that contains y and z such that $c(T_{z_1}) = y$. By Expansion, $c(\bigcup_{z \in S} T_z) = y$, which suggests that $x \; R^b \; y$. Let $\trianglerighteq_x$ be the signed order induced by g_x. c is a C-AB under $(\{\trianglerighteq_x\}_{x \in X}, P^b)$. $\square$

4.5 Conclusion

This paper investigates attention formation in the choice theoretical framework. Rather than specifying attention rules as previous research has done, our paper assumes that the DM's attention on an alternative in a menu depends on the relative strength of its attention triggers and blockers. Specifically, if the most significant alternative serves as an attention trigger for an alternative in a given menu, then the DM pays attention to this alternative. When the most significant alternative for it is an attention blocker, the DM pays no attention to it. Similar to previous papers on choices under limited attention, the DM chooses the best alternative among the attended ones.

Attention triggers and blockers bring a context dependence in forming one's attention. One way to introduce these effects is by adopting signed orders. By maximizing the signed order for an alternative x, if the most significant is a reflection, then the DM's attention on x is blocked. Otherwise, the DM pays attention to this x. Instead of using signed orders in extending DM's preferences on the set of alternatives to its power sets as suggested in Fishburn [1992a], we introduced them in the first stage of decision-making, providing a fresh perspective on their usage.

Chapter 5

Conclusions

This dissertation comprises three essays, each focusing on a special case of decision-making under limited attention. By concentrating on the theoretical framework, we establish the axiomatic foundations for various decisions under limited attention.

Chapter 1 extends our understanding of additive representation from full attention to limited attention. Assuming idempotent attention rules, we show that FC on the collection of basic sets is necessary and sufficient for the AR-IA. Moreover, the DM's attention rule and utility function are intrinsically correlated to basic sets. This model also offers valuable insights into intuitive probability and freedom of choice under limited attention.

Chapter 2 investigates satisficing behavior under limited attention. When attention rules are idempotent, the focus can again be on the collection of basic sets. For each basic set, chosen alternatives are better than those not chosen. This enables us to characterize the model from both perspectives of revealed preferences and choice behaviors.

Chapter 3 aims to unpack the “black box” of limited attention by introducing the concepts of attention triggers and blockers in the attention formation process. Given their opposing effects, we employ signed orders in the theoretical model, which presents a fresh usage of them. The presence of attention triggers also provides insight into the ”decoy effect,” connecting research on limited attention to reference points.

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Appendix A

Appendix for Chapter 2

A.1 Proof of Theorem 1

For the only if part, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits the AR-IA under (Γ, u). Let $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m$ be two collections of basic sets such that the DM prefers S_n over T_n for all $n \leq m$, and strictly prefers S_n to T_n for some $n \leq m$. The AR-IA implies that $\sum_{n=1}^m \sum_{s \in T_n} u(s) > \sum_{n=1}^m \sum_{t \in T_n} u(t)$ because basic sets catch full attention. However, if $\sum_{n=1}^m \mathbb{1}_{S_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$, i.e., the collection of S_n contains less x than the collection of T_n for all $x \in X$, then $\sum_{n=1}^m \sum_{s \in T_n} u(s) < \sum_{n=1}^m \sum_{t \in T_n} u(t)$. Therefore, NR is necessary for the AR-IA.

We now move on to the sufficiency of WO and NR.

Step 1. Constructing the utility function by solving the system of linear inequalities induced by the $\succcurlyeq$ on $\mathcal{B}$. Let $\mathcal{B} = \{B_n\}_{n=1}^m$. If $B \sim B'$ for all $B, B' \in \mathcal{B}$, then it is obvious that the system of linear inequalities has a solution. We then can assume that there exists at least one pair of $B_n, B_{n'} \in \mathcal{B}$ such that $B_n \succ B_{n'}$. For any

$B_n \succ B_{n'}$, we have $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) > 0$. Similarly, $B_n \sim B_{n'}$ suggests that $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) = 0$, and $B_n \succcurlyeq B_{n'}$ suggests that $\sum_{x \in B_n} u(x) - \sum_{x \in B_{n'}} u(x) \geq 0$. Moreover, we also need a restriction on u, i.e., $u(x) \geq 0$ for all $x \in X$. We then consider the solution to the above system of linear inequalities.

By Kraft et al. [1959], the above system of linear inequalities has a solution if and only if there are no $m_1 > 0$ pairs of $B_n \succ B_{n'}$, $m_2 \geq 0$ pairs of $B_n \sim B_{n'}$, and $m_3 \geq 0$ pairs of $B_n \succcurlyeq B_{n'}$ such that the collection of B_n has less x than the collection of $B_{n'}$ for all $x \in X$. Hence, the system of linear inequalities has a solution if and only if $\succcurlyeq$ on $\mathcal{B}$ is a weak order and satisfies nonnegative remainders.

Step 2. Constructing Γ by using basic sets. Let $\mathcal{S}(B_1) = \{S : B_1 \subseteq S \textit{ and } B_1 \sim S\}$ and $\mathcal{S}(B_i) = \{S : B_i \subseteq S,\ B_i \sim S,\ \textit{and } S \notin \mathcal{S}_{B_{i-1}}\}$ for all $i > 1$. For every $S \in \mathcal{S}(B_i)$, let $\Gamma(S) = B_i$. It's clear that Γ is idempotent attention.

Take any pair of (Γ, u) constructed above. For any $S, T \in \mathcal{X}$ with $S \succcurlyeq T$, we know that there is a pair of basic sets B_n and $B_{n'}$ such that $S \in \mathcal{S}(B_n)$ and $T \in \mathcal{S}(B_m)$. Therefore,

$$S \succcurlyeq T \iff B_n \succcurlyeq B_{n'} \iff \sum_{x \in B_n} u(x) \geq \sum_{y \in B_{n'}} u(y) \iff \sum_{x \in \Gamma(S)} u(x) \geq \sum_{y \in \Gamma(T)} u(y).$$

A.2 Proof of Proposition 2

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Let $\{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ be the collection of corresponding idempotent attentions and utility functions.

We first prove the only if part of this proposition. By contradiction, assume that there is a positive integer m and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \notin MIN(X, \succcurlyeq)$, and $\sum_{x\in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{S_n}(x) > \sum_{x\in MIN(X,\succcurlyeq)} \sum_{n=1}^m \mathbb{1}_{T_n}(x)$. Then,

$$\begin{aligned}
\sum_{n=1}^m \sum_{s\in S_n} u(s) &= \sum_{n=1}^m \sum_{s\in S_n\cap MIN(X,\succcurlyeq)} u(s) + \sum_{n=1}^m \sum_{s\in S_n\setminus MIN(X,\succcurlyeq)} u(s) \\
&> \sum_{n=1}^m \sum_{t\in T_n} u(t) \\
&= \sum_{n=1}^m \sum_{t\in T_n\cap MIN(X,\succcurlyeq)} u(t) + \sum_{n=1}^m \sum_{t\in T_n\setminus MIN(X,\succcurlyeq)} u(t).
\end{aligned}$$

Therefore,$\sum_{n=1}^m \sum_{s\in S_n\cap MIN(X,\succcurlyeq)} u(s) > \sum_{n=1}^m \sum_{t\in T_n\cap MIN(X,\succcurlyeq)} u(t)$. We then have $u(x) > 0$ for all $x \in MIN(X, \succcurlyeq)$.

For the converse direction, suppose that there is a $u \in \mathcal{U}_{\succcurlyeq}$ such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. We then consider the system of linear inequalities we introduced in the proof of Theorem 1 combining with the restriction that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. By Kraft et al. [1959], this linear system has a solution if and only if there are no $m_1 > 0$ pairs of $B_n \succ B_{n'}$, $m_2 \geq 0$ pairs of $B_n \sim B_{n'}$, $m_3 \geq 0$ pairs of $B_n \succcurlyeq B_{n'}$, and $|MIN(X, \succcurlyeq)| > 0$ restrictions on $u(x) = 0$ where $x \in MIN(X, \succcurlyeq)$ such that the collection of B_n only contains more dominated alternatives than the collection of $B_{n'}$. Hence, we have ENS.

A.3 Proof of Theorem 2

Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-AF under (Γ, u). We first know that (Γ, u) is also a pair of AR-IAs for $\succcurlyeq$ on $\mathcal{X}$. Therefore, we know that $\succcurlyeq$ on $\mathcal{X}$ satisfies WO and NR by Theorem 1.

We then only need to prove that $\succcurlyeq$ on $\mathcal{X}$ satisfies IB. Take any $S \in \mathcal{X}$, we can get $\Gamma(S)$ from the attention filter Γ. We first know that $\Gamma(Y) = \Gamma(S)$ for all $\Gamma(S) \subseteq Y \subseteq S$. If $\Gamma(S)$ is basic, it is clear that $\Gamma(S)$ itself can serve as the corresponding T and B in IB, and $\Gamma(S) \setminus \Gamma(S) = \emptyset \subseteq N$. Moreover, $\sum_{s \in \Gamma(S)} u(s) = \sum_{s \in \Gamma(\Gamma(S))} u(s)$ which implies that $S \sim \Gamma(S)$. We then have IB.

If $\Gamma(S)$ is non-basic, we know that there is a corresponding basic set B of $\Gamma(S)$ such that $B \sim \Gamma(S)$. Hence, $\sum_{s \in \Gamma(S) \setminus B} u(s) = 0$. As a result, $u(s) = 0$ for all $s \in \Gamma(S) \setminus B$. By Proposition 2, we have $\Gamma(S) \setminus B \subseteq N$. We can let $\Gamma(S)$ be the corresponding T of S in IB, and the $\succcurlyeq$ on $\mathcal{X}$ satisfies IB.

We now suppose that $\succcurlyeq$ on $\mathcal{X}$ satisfies WO, NR, and IB. We know that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs by Theorem 1. We then need to consider the construction of an attention filter. Let $\{T_i\}_{i=1}^m = \bigcup_{s \in \mathcal{X}} \mathcal{IB}(S)$, and

$$\mathcal{B}(T,S) := \begin{cases} \{Y : T \subseteq Y \subseteq S\} & \text{if all } Y \text{ inbetween } T \text{ and } S \text{ are indifferent,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Let's also define

$$\mathcal{UB}(T_1) := \{S : S \in \mathcal{X} \text{ } \textit{such that } \emptyset \neq \mathcal{B}(T_1, S)\},$$

and for all $i > 1$, we denote

$$\mathcal{UB}(T_i) := \left\{ S : S \in \mathcal{X} \setminus \bigcup_{j<i} \mathcal{UB}(A_j) \text{ such that } \emptyset \neq \mathcal{B}(T_i, S) \right\}.$$

Claim 1. $\{\mathcal{UB}(T_i)\}_{i=1}^m$ *forms a partition of* $\mathcal{X}$.

Proof. We first show that for any set S, there is a $i \leq m$ such that $S \in \mathcal{UB}(T_i)$. Take any $S \in \mathcal{X}$, we know that there exists at least one of T_i such that $T_i \in \mathcal{IB}(S)$. We then can denote the $\mathcal{IB}(S)$ as a finite subsequence $\{T_{i_j}\}_{j=1}^n$ of $\{T_i\}_{i=1}^m$. We then claim that $S \in \mathcal{UB}(T_{i_1})$. To show this, we only need to show that $S \notin \mathcal{UB}(T_i)$ where $i < i_1$. By contradiction, suppose that $S \in \mathcal{UB}(T_i)$ for some $i < i_1$. Then, we know that $T_i \in \mathcal{IB}(S)$ which is a contradiction.

We then show that for any $T_i, T_j \in \{T_i\}_{i=1}^m$ where $i \neq j$, $\mathcal{UB}(T_i) \cap \mathcal{UB}(T_j) = \emptyset$. Without loss of generality, we can assume that $i > j$. By contradiction, suppose that there is a set S such that $S \in \mathcal{UB}(T_i) \cap \mathcal{UB}(T_j)$, we then know that $S \in \mathcal{B}(T_j, S)$ which implies that $S \notin \mathcal{X} \setminus \bigcup_{j<i} \mathcal{UB}(T_j)$. Hence, $S \notin \mathcal{UB}(T_i)$.

In conclusion, $\{\mathcal{UB}(T_i)\}_{i=1}^m$ forms a partition of $\mathcal{X}$. $\square$

Let's define a function $\Gamma : \mathcal{X} \to \mathcal{X}$ by $\Gamma(S) = T_i$ if $S \in \mathcal{UB}(T_i)$. We then claim that Γ is an attention filter. Take any $S \in \mathcal{X}$, and suppose that $\Gamma(S) = T_i$. Take any Y with $T_i \subseteq Y \subseteq S$, we want to show that $\Gamma(Y) = T_i$. We first know that $Y \sim S$. We then only need to show that $Y \notin \mathcal{UB}(T_j)$ for any $j < i$. By contradiction, if $Y \in \mathcal{UB}(T_j)$ for some $j < i$, then we know that $S \in \mathcal{UB}(T_j)$. It is a contradiction. Therefore, Γ is an attention filter.

We now want to show that there is a utility function u such that $S \succcurlyeq T \iff \sum_{s\in S} u(s) \geq \sum_{t\in T} u(t)$. Take any $S \succcurlyeq Y$, we know that there are two basic sets T_S and T_Y such that $\Gamma(S) = T_S$ and $\Gamma(Y) = T_Y$. By IB, there is a basic set B_1 where $B_1 \subseteq T_S$ and $B_1 \sim T_S$ such that $T_S \setminus B_1 \subset N$. Similarly, we can get a basic set B_2 for T_Y. For all u that consist of AR-IAs, we have

$$
\begin{aligned}
S \succcurlyeq Y &\iff T_S \succcurlyeq T_Y \\
&\iff B_1 \succcurlyeq B_2 \\
&\iff \sum_{b\in B_1} u(b) \geq \sum_{b\in B_2} u(b) \\
&\iff \sum_{s\in \Gamma(S)} u(s) \geq \sum_{y\in \Gamma(Y)} u(y).
\end{aligned}
$$

A.4 Independence between WO, NR and IB

Completeness and NR imply transitivity on $\mathcal{B}$ (see Fishburn [1992b]). However, they can not characterize the AR-IA.

Example 9. *Let $X = xy$, and the $\succcurlyeq$ on $\mathcal{X}$ is $x \succ y$, $xy \sim x$, $y \succ xy$. This preference is complete and nontransitive. Also, $\mathcal{B} = \{x, y\}$, and it satisfies NR. However, if it admits AR-IA under (Γ, u), then $\Gamma(xy) = x$ or xy. When $\Gamma(xy) = x$, the AR-IA suggests that $xy \succ y$. When $\Gamma(xy) = xy$, it suggests that $xy \succ xy$.*

The Example 2 provides a preference relation $\succcurlyeq$ on $\mathcal{X}$ that admits AR-IAs but not AR-AFs. To see the $\succcurlyeq$ provided in Example 2 violates IB, let us consider xyz. First, we know that $N = \emptyset$. xyz is not basic suggests that $\mathcal{IB}(xyz) = \emptyset$.

We now want to show that WO and IB together cannot characterize AR-AFs.

Example 10. *Let $X = xy$, and the $\succcurlyeq$ on $\mathcal{X}$ is $x \succ xy \succ y$. We know that $N = \emptyset$, and all the sets are basic. Then it satisfies WO and IB. However, it does not admit AR-AFs because $x \succ xy \succ y \iff u(y) < 0$.*

A.5 More Appendix for Chapter 2

This appendix contains five parts. First, Appendix A.5.1 provides the characterization of the AR-IA without assuming positive utility functions. Second, in Appendix A.5.2, we discuss the uniqueness of utility functions under similarity transformation and the uniqueness of consistent pairs. Third, the practices of revealed attention under AR-IAs and AR-AFs are introduced using the weakest criteria in Appendix A.6. Fourth, we present several results regarding interpersonal comparisons of attention capacity in Appendix A.6.1. Finally, we introduce the Borda-IA as an application of the AR-IA in Appendix A.6.2

A.5.1 AR-IAs without $u \geq 0$

The paper assumes that $u \geq 0$. As a result of this assumption, we can characterize alternatives with zero utility and state the IB axiom in a straightforward manner. Despite its absence, we are still able to characterize the AR-IA.

Definition 32. *$\succcurlyeq$ on $\mathcal{X}$ admits a* ***General Additive Representation under Idempotent Attention (General AR-IA)*** *if there exist an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$,* *and a function $u : X \to \mathbb{R}$ such that for any $S, T \in \mathcal{X}$,*

$$S \succcurlyeq T \iff \sum_{s \in \Gamma(S)} u(s) \geq \sum_{t \in \Gamma(T)} u(t).$$

Given any pair of above Γ and u, we say that $\succcurlyeq$ on $\mathcal{X}$ admits a General AR-IA under (Γ, u).

Of course, we still need WO.

Axiom 17. *(WO: Weak Order) $\succcurlyeq$ on $\mathcal{X}$ is complete and transitive.*

Based on the proof of the characterization of the AR-IA, we can associate a set S to one of its corresponding basic sets to construct the idempotent attention rule. Moreover, the basic sets must catch the DM's full attention. Consequently, we only need to ensure that the system of inequalities induced by the $\succcurlyeq$ on $\mathcal{B}$ has a solution. The condition still follows Kraft et al. [1959], Scott [1964], Krantz et al. [1971] and Fishburn [1992b], which is known as Finite Cancellation or Strong Additivity.

Axiom 18. *(FC: Finite Cancellation) There do not exist a positive integer m and $\{S_n\}_{n=1}^m$, $\{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$ where $\mathbb{1}$ is the indicator function.*

FC follows a similar logic to NR. According to the FC axiom, if two subcollections of basic sets contain exactly the same content, it cannot be the case that the DM prefers one over the other. Obviously, it imposes a consistent condition on aggregating preference on X.

Theorem 22. *$\succcurlyeq$ on $\mathcal{X}$ admits a General AR-IA if and only if it satisfies WO and FC.*

Proof. There is no doubt that WO is necessary. In order to demonstrate the necessity of FC, assume that $\succcurlyeq$ on $\mathcal{X}$ admits a General AR-IA under (Γ, u). By contradiction, assume that FC does not hold. Fix a positive integer m and $\{S_n\}_{n=1}^m$, $\{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \succcurlyeq T_n$ for all n, and $S_n \succ T_n$ for some n. We then know that $\sum_{n=1}^m \sum_{x \in S_n} u(x) > \sum_{n=1}^m \sum_{x \in T_n} u(x)$. However, $\sum_{n=1}^m \sum_{x \in S_n} u(x) = \sum_{n=1}^m \sum_{x \in T_n} u(x)$ when $\sum_{n=1}^m \mathbb{1}_{S_n}(x) = \sum_{n=1}^m \mathbb{1}_{T_n}(x)$ for all $x \in X$.

To ensure sufficiency, we can follow the proof of characterization of the AR-IA to construct the idempotent attention rule Γ with $\Gamma(\mathcal{X}) = \mathcal{B}$. FC guarantees the AR on $\mathcal{B}$, which follows directly from Lemma 0 in Kraft et al. [1959]. □

A.5.2 More Results about the Uniqueness of (Γ, u)

As we mentioned above, if u is a consistent utility function for the AR-IA (AR-AF), then cu with $c > 0$ is also a consistent utility function for the representation.[1] It is natural to ask whether for any two consistent utility functions u and v for the AR-IA (AR-AF), there is a $c > 0$ such that $u = cv$. If it is true, then say that the utility function is unique up to similarity transformation (Fishburn [1992b]).

Definition 33. *Assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. We say the utility function is **unique up to similarity transformation under the AR-IA** if for any $(\Gamma_{i_1}, u), (\Gamma_{i_2}, v) \in \{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ there is a $c > 0$ such that $u = cv$. Similarly, Assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs. We say the utility function is **unique up to similarity transformation under the AR-AF** if for any $(\Gamma_{j_1}, u), (\Gamma_{j_2}, v) \in \{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ there is a $c > 0$ such that $u = cv$.*

[1]This implies that $\{(\Gamma_i, u_i)\}_{i \in I_{\succcurlyeq}}$ and $\{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$ must contain infinitely many consistent pairs.

For both AR-IAs and AR-AFs, the utility function is not necessarily unique up to similarity transformation.

Example 11. *Let* $X = xyz$*. The preference* $\succcurlyeq$ *on* $\mathcal{X}$ *is* $xyz \succ xy \sim x \succ yz \sim y \succ xz \sim z$*. It is clear that* $\mathcal{B} = \{xyz, x, y, z\}$*. We consider two pairs of attention rules and utility functions. Let* $u_1(x) = 3, u_1(y) = 2, u_1(z) = 1,$ *and*

$$\Gamma_1(S) = \begin{cases} x & \textit{if } S = xy, \\ y & \textit{if } S = yz, \\ z & \textit{if } S = xz, \\ S & \textit{otherwise.} \end{cases}$$

Let $u_2(x) = 2, u_2(y) = 1, u_2(z) = 0,$ *and*

$$\Gamma_2(S) = \begin{cases} x & \textit{if } S = xy, \\ y & \textit{if } S = yz, \\ xz & \textit{if } S = xz, \\ S & \textit{otherwise.} \end{cases}$$

Clearly, $\succcurlyeq$ *on* $\mathcal{X}$ *admits AR-IAs and AR-AFs, and* (Γ_1, u_1) *and* (Γ_2, u_2) *can both be consistent pairs. However,* $\succcurlyeq$ *on* $\mathcal{X}$ *cannot be represented by* (Γ_2, u_1) *under the AR-IA and the AR-AF.*

We also cannot find a positive real number c such that $u_1 = cu_2$. If such a c exists, it requires that $u_1(z) = cu_2(z)$ which is impossible.

Proposition 14. *If the utility function is unique up to similarity transformation under the AR-IA, then $N = \emptyset$.*

Proof. Assume that the utility function is unique up to similarity transformation under the AR-IA. By Claim 2, we know that $Rank(A) = |X/_{\sim}| - 1$. By contradiction, suppose that $N = \emptyset$. Then, there is a $u \in \mathcal{U}_{\succcurlyeq}$ such that $u(x) = 0$ for all $x \in MIN(X, \succcurlyeq)$. As a result, $u(y) = 0$ for all $y \in X$. Notice that it suggests that $x \sim y$ for all $x, y \in \mathcal{X}$. Moreover, $\mathcal{B} = \{S \in \mathcal{X} : \ |S| = 1\}$. In this situation, we can let $v(x) = 1$ for all $x \in X$. It is clear that $v \in \mathcal{U}_{\succcurlyeq}$ while there is no $c > 0$ such that $u = cv$. □

Suppose that the utility function is unique up to similarity transformation. Fix any two consistent utility functions u, v with $u = cv$ where $c > 0$. For any arbitrary $x \in X$, we know that $u(x) = cv(x)$. In view of the fact that u and v are strictly positive utility functions, $c = \frac{u(x)}{v(x)}$. As a result, $u(y) = cv(y) = \frac{v(y)}{v(x)}u(x)$ for all $y \in X$. Hence, $u(x)$ and $u(y)$ have a proportional relationship for any $x, y \in X$.

Condition 2. *(Proportion) For any $x, y \in X$ with $x \succcurlyeq y$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $0 < \sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$.*

The Condition 2 ensures that there is a proportional relationship between the utility of any two alternatives. It turns out to be the necessary and sufficient condition for the utility function to be unique up to similarity transformation under the AR-IA. Furthermore, this condition is also valid for the case of AR-AFs.

Theorem 23. *Assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs (AR-AFs). The utility function is unique up to similarity transformation if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies Proportion.*

Proof. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs. Let $\mathcal{U}_{\succcurlyeq}$ be the collection of consistent utility functions.

We begin with proving the if part. Consider the equivalent class $[x] = \{y \in X : x \sim y\}$, we know we can partition X by the equivalent classes. Now, let $X/_{\sim} = \{x_i\}_{i=1}^{I}$ where $x_i \succ x_j$ for all $i < j$. Suppose that for any $x_i, x_j \in X$ with $x_i \succcurlyeq x_j$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $\sum_{n=1}^m \mathbb{1}_{S_n}(x_i) \geq \sum_{n=1}^m \mathbb{1}_{T_n}(x_j)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. Since $\succcurlyeq$ on $\mathcal{X}$ admits the AR-IA under (Γ, u), we know that there is a pair of real numbers c_i, c_j such that $c_i u(x_i) = c_j u(x_j)$ where $c_i > c_j$. By induction, we have $c_1 u(x_1) = c_2 u(x_2) = \cdots = c_I u(x_I)$ where $c_1 > c_2 > \cdots > c_I$. Furthermore, for any other AR-IA pair (Γ', v), we still have $c_1 v(x_1) = c_2 v(x_2) = \cdots = c_I v(x_I)$ where $c_1 > c_2 > \cdots > c_I$. Hence, $u = \frac{u(x_1)}{v(x_1)} v$.

For the converse direction, suppose that $\mathcal{U}_{\succcurlyeq}$ is up to under similarity transformation. We first know that if $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA under (Γ, u), then (Γ, cu) also be a consistent AR-IA pair for any $c > 0$. Consequentially, if $\mathcal{U}_{\succcurlyeq} \neq \emptyset$, then $|\mathcal{U}_{\succcurlyeq}| = \infty$.

Let's consider the linear system induced by $\sim$ on $\mathcal{B}$. If $B \sim B'$, we know $\sum_{b \in B} u(b) - \sum_{b'} u(b') = 0$. the system then can be written as $Au(X) = 0$ where A is the coefficient matrix and $u(X) = (u(x_1), ..., u(x_I))^T$.

Claim 2. *If the utility function is unique up to similarity transformation, then* $Rank(A) = |X/_\sim| - 1$.

Proof. By contradiction, suppose that the linear system induced by $\sim \cap(\mathcal{B} \times \mathcal{B})$ does not have $|X/_\sim| - 1$ linearly independent equations. We know the linearly independent equations must be smaller than $|X/_\sim| - 1$, otherwise $u(x) = 0$ for all x. In this case, $u(x) = 1$ for all x still represents $\succcurlyeq$ while the utility is not unique under similarity transformation. Now, suppose that $Rank(A) = |X/_\sim| - k$ where $k > 1$. The solution to $Au(X) = 0$ then has k parameters. In other words, fix any $(u(x_1), .., u(x_k)) = (a_1, ..., a_k)$, $u(x_{k+i})$ should be a linear combination of $(a_1, ..., a_k)$ for any $i \leq I - k$. Suppose that $(u(x_1), .., u(x_k)) = (1, ..., k)$ and $(v(x_1), .., v(x_k)) = (3, ..., k + 3)$, we know there is no $c > 0$ such that $u = cv$. □

By Gaussian elimination, if $Rank(A) = |X/_\sim| - 1$, then the last nonzero row of A can be reduced as $(0, ..., 0, c_i, c_j)$ which implies that $c_i u(x_i) = c_j u(x_j)$ for arbitrary x_i, x_j. Moreover, if $x_i \succcurlyeq x_j$, then $c_i \leq c_j$. Thus, for any $x, y \in X$ with $x \succcurlyeq y$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $0 < \sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x) \leq \sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. When it comes to the uniqueness of (Γ, u), we can combine the uniqueness of utility functions and the uniqueness of attention rules. □

The proportional relationship between the utility of alternatives must be determined by the indifference relationship on $\mathcal{B}$. Intuitively, the linear system induced by $\sim \cap(\mathcal{B} \times \mathcal{B})$ must have one parameter. In other words, for the $x \succ y$ in the statement of Theorem 23, if $u(x)$ is fixed as some positive number, then $u(y)$ must be $cu(x)$ where $c = \frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)}$.

Example 12. *Let $X = xyz$, the DM's preference on $\mathcal{X}$ is $xyz \sim z \sim xy \succ x \sim xz \sim yz \sim y$. Notice that $\mathcal{B} = \{xy, x, y, z\}$. We want to show that $u(z) = 2u(x)$ for all $u \in \mathcal{U}$. Consider, $\{xy, x\}\{z, y\}$, we know that $xy \sim z$ and $x \sim y$. Both collections contain one y. The $\{xy, x\}$ collection contains two x, and no z. The latter collection contains one z and no x. Clearly, if $u(x)$ is fixed, then $u(z) = 2u(x)$. Furthermore, given another $v \in \mathcal{U}_{\succcurlyeq}$, we also have $v(z) = 2v(x) = 2v(y)$. Let $c = \frac{v(x)}{u(x)}$, we then know that $v = cu$. The utility function is unique up to similarity transformation.*

Definition 34. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA (AR-AF). The AR-IA (AR-AF) representation is unique if the attention rule is unique and the utility function is unique up to similarity transformation.*

In light of the requirement that the utility function be unique, it is evident that there are no null alternatives. $N = \emptyset$, as the requirement of the uniqueness of attention rules, is redundant for characterizing the uniqueness of (Γ, u).

Corollary 24. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The AR-IA representation is unique if and only if*

(i) $\succcurlyeq$ on $\mathcal{X}$ satisfies the Proportion condition;

(ii) for every $S \in \mathcal{X}$ there is a unique $B \in \mathcal{B}$ such that $B \subseteq S$ and $B \sim S$, i.e., $|\mathcal{B}(S)| = 1$ for all $S \in \mathcal{X}$.

Corollary 25. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. The AR-IA representation is unique if and only if*

(i) $\succcurlyeq$ on $\mathcal{X}$ satisfies the Proportion condition;

(ii) for any S there is a unique basic set B such that $T \sim S \sim B$ for all T with $B \subseteq T \subseteq S$.

A.6 Revealed Attention

The paper analyzes the revealed attention by imposing stringent criteria for both AR-IAs and AR-AFs. It is also possible to investigate them by imposing the weakest criteria. The combination of these two practices gives us a comprehensive picture of how DMs form their attention.

Revealed Attention under AR-IAs: Suppose that the $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs, and let $\{(\Gamma_i, u_i)\}_{i\in I_{\succcurlyeq}}$ be the collection of all consistent pairs of idempotent attention rule and utility function. When one of the consistent idempotent attention rules suggests that the DM will pay attention to s in S, an alternative s might catch the DM's attention. Formally,

Definition 35. *An alternative s **weakly catches attention under AR-IAs** in S if $s \in \Gamma_i(S)$ for some $i \in I_{\succcurlyeq}$.*

Let $\bar{\mathcal{B}}(S) := \{T \subseteq S : \exists B \in \mathcal{B}(S)\ s.t.\ T \sim B \text{ and } T \setminus B \subseteq N\}$. We know that every $T \in \bar{\mathcal{B}}(S)$ can be the DM's attention on S. Accordingly, if $s \in T$ for some $T \in \bar{\mathcal{B}}(S)$, then s should weakly catch the attention of AR-IAs.

Corollary 26. *An alternative s weakly catches attention under AR-IAs in S if and only if there is a $T \in \bar{\mathcal{B}}(S)$ such that $s \in T$.*

Proof. As we know, if $T \in \bar{\mathcal{B}}(S)$, then there is a Γ_i with $\Gamma_i(S) = T$ for some $i \in I_{\succcurlyeq}$. The Corollary 26 is a direct result of this observation. □

In a similar manner, we can form the weakest criterion for a menu S that catches attention.

Definition 36. *A menu S **weakly catches full attention under AR-IAs** if $\Gamma_i(S) = S$ for some $i \in I_{\succcurlyeq}$.*

Under AR-IAs, the DM considers all alternatives in any basic set. The DM may pay attention to all alternatives in S if a set S is composited with one of its corresponding basic sets and some null alternatives.

Proposition 15. *A menu S weakly catches full attention under AR-IAs if and only if there is a corresponding basic set B of S such that $S \setminus B \subseteq N$.*

Proof. Suppose that a menu S weakly catches full attention under AR-IAs, and the consistent idempotent attention and utility function are Γ and u, respectively. If S is basic, we are done. If S is nonbasic, then for any $B \in \mathcal{B}(S)$, we have

$$\sum_{s \in \Gamma(S)} u(s) - \sum_{b \in B} u(b) = \sum_{s \in S} u(s) - \sum_{b \in B} u(b) = \sum_{s \in S \setminus B} u(s) = 0.$$

That is, $S \setminus B \subseteq N$. For the converse direction, it is clear that $S \setminus B = N$ for some $B \in \mathcal{B}(S)$ implies that $S = B \cup O$ for some $B \in \mathcal{B}(S)$ and $O \subseteq S \cap N$. Hence, there is an idempotent attention Γ that consistents with AR-IAs such that $\Gamma(S) = S$. $\square$

Revealed Attention under AR-AFs: We now assume that $\succcurlyeq$ on $\mathcal{X}$ admits AR-AFs, and the collection of consistent pairs of attention filer and utility function is denoted as $\{(\Gamma_j, u_j)\}_{j \in J_{\succcurlyeq}}$. We can define an alternative that weakly catches attention under AR-AFs in the same manner as AR-AFs.

Definition 37. *An alternative s* ***weakly catches attention under AR-AFs*** *in S if $s \in \Gamma_j(S)$ for some $j \in J_{\succcurlyeq}$.*

We can construct a consistent attention filter by associating S to a $T \in \mathcal{IB}(S)$.

Proposition 16. *An alternative s weakly catches attention in S under AR-AFs if and only if there is a $T \in \mathcal{IB}(S)$ such that $s \in T$.*

Proof. We know that for any $T \in \mathcal{IB}(S)$, there is an attention filter that is consistent with AR-AFs such that $\Gamma(S) = T$. Take any attention filter Γ that is consistent with AR-AFs and a set S such that $\Gamma(S) \notin \mathcal{IB}(S)$, we then know that there is a Y where $\Gamma(S) \subseteq Y \subseteq S$ such that Y is not indifferent to S, which suggests that Γ is not an attention filter. As a result, the theorem is relatively straightforward. □

The weakest criterion of revealed attention can also be applied to sets under AR-AFs.

Definition 38. *A menu S* ***weakly catches full attention under AR-AFs*** *if $\Gamma_j(S) = S$ for some $j \in J_{\succcurlyeq}$.*

We know that any $T \in \mathcal{IB}(S)$ may serve as $\Gamma_j(S)$ for some $j \in J_{\succcurlyeq}$. As a result, if $S \in \mathcal{IB}(S)$, then S should weakly attract the full attention of the DM under AR-AFs. This property can be further investigated by observing that the set difference between S and any of its corresponding basic sets is a subset of N. As a result, this characterization is identical to that found in AR-IAs.

Proposition 17. *A menu S weakly catches full attention under AR-AFs if and only if there is a corresponding basic set B of S such that $S \setminus B \subseteq N$.*

Proof. Suppose that a menu S weakly catches full attention under AR-AFs, and (Γ_j, u_j) the consistent pair of AR-AF. We then know $\Gamma_j(S) = S$. If S is basic, then $S \setminus S = \emptyset \subseteq N$. If S is not basic, then there is a corresponding basic set B of S such that $\sum_{s \in S \setminus B} u_j(s) = 0$, which implies that $S \setminus B \subseteq N$. For the converse direction, taking a $B \in \mathcal{B}(S)$ such that $S \setminus B \subseteq N$. Then, we know that $S \in \mathcal{IB}(S)$, which suggests that there is an attention filer Γ'_j that is consistent with AR-AFs such that $\Gamma'_j(S) = S$. $\square$

A.6.1 Relative Attention

One of the reasons for DMs' limited attention is their limited cognitive capacity. DMs' cognitive abilities can be inferred from basic sets when we interpret basic sets as the images of their attention rule.

In order to compare interpersonal attention, we need to know which alternatives attract the DMs' attention for each menu. When their attention rules are idempotent, it is equivalent to finding the relationship between the corresponding basic sets of each menu. Intuitively, when DM 1 is more attentive than DM 2, then, given any menu, an alternative that attracts DM 2's attention should catch DM 1's attention as well. Occasionally, we are unable to pin down the attention of the DMs. We then use the most stringent condition for this comparison.

Suppose that two DMs' preference $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ both admit AR-IAs, and denote $\{(\Gamma_{i_1}, u_{i_1})\}_{i_1 \in I_{\succcurlyeq_1}}$ and $\{(\Gamma_{i_2}, u_{i_2})\}_{i_2 \in I_{\succcurlyeq_2}}$ as the collection of the consistent pairs of AR-IA for DM 1 and DM 2, respectively.

Definition 39. *Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ admit AR-IAs. $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if given any $i_1 \in I_{\succcurlyeq_1}$ and $S \in \mathcal{X}$, $\Gamma_{i_2}(S) \subseteq \Gamma_{i_1}(S)$ for all $i_2 \in I_{\succcurlyeq_2}$.*

Under AR-AFs, relative attention can be defined similarly. Let's denote the consistent pairs of AR-AFs as $\{(\Gamma_{j_1}, u_{j_1})\}_{j_1 \in J_{\succcurlyeq_1}}$ and $\{(\Gamma_{j_2}, u_{j_2})\}_{j_2 \in J_{\succcurlyeq_2}}$ for DM 1 and DM 2 respectively.

Definition 40. *Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ admit AR-AFs. $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if given any $j_1 \in J_{\succcurlyeq_1}$ and $S \in \mathcal{X}$, $\Gamma_{j_2}(S) \subseteq \Gamma_{j_1}(S)$ for all $j_2 \in J_{\succcurlyeq_2}$.*

By $\mathcal{B}_{\succcurlyeq_1}$ and $\mathcal{B}_{\succcurlyeq_2}$ we respectively denote the collection of basic sets of $\succcurlyeq_1$ and $\succcurlyeq_2$. Since $\mathcal{B}_{\succcurlyeq_1} \subseteq \Gamma_{i_1}(\mathcal{X})$ for all $i_1 \in I_{\succcurlyeq_1}$, we can focus on the $\mathcal{B}_{\succcurlyeq_1}$. Similarly, we denote $\bar{\mathcal{B}}_{\succcurlyeq_1}(S)$ and $\bar{\mathcal{B}}_{\succcurlyeq_2}(S)$ as $\bar{\mathcal{B}}(S)$ for $\succcurlyeq_1$ and $\succcurlyeq_2$ respectively. If, given any menu S, an alternative $s \in T$ for some $T \in \bar{\mathcal{B}}_{\succcurlyeq_2}(S)$, then s must be included in all the corresponding basic sets of S in under $\succcurlyeq_1$.

Corollary 27. *Suppose that $\succcurlyeq_1$ and $\succcurlyeq_2$ on $\mathcal{X}$ admit AR-IAs. $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succcurlyeq_2}(S)} T' \subseteq B$ for all $B \in \mathcal{B}_{\succcurlyeq_1}(S)$.*

Proof. By Corollary 26, $\succcurlyeq_1$ is more attentive than $\succcurlyeq_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succcurlyeq_2}(S)} T' \subseteq T$ for all $T \in \bar{\mathcal{B}}_{\succcurlyeq_1}(S)$. Since $\mathcal{B}_{\succcurlyeq_1}(S) \subseteq \bar{\mathcal{B}}_{\succcurlyeq_1}(S)$ for any S, $\bigcup_{T' \in \bar{\mathcal{B}}_{\succcurlyeq_2}(S)} T' \subseteq T$ for all $T \in \bar{\mathcal{B}}_{\succcurlyeq_1}(S)$ if and only if $\bigcup_{T' \in \bar{\mathcal{B}}_{\succcurlyeq_2}(S)} T' \subseteq B$ for all $B \in \mathcal{B}_{\succcurlyeq_1}(S)$. $\square$

When DMs' attention rules are attention filters, given any menu S, their attention to S is an element in $\mathcal{IB}(S)$. Let $\mathcal{IB}_{\succcurlyeq_1}(S)$ and $\mathcal{IB}_{\succcurlyeq_2}(S)$ denote the $\mathcal{IB}(S)$ for DM 1 and DM 2 respectively. In this case, the relative attention can be understood as follows: If an alternative x is included in T' where $T' \in \mathcal{IB}_{\succsim_2}(S)$, then x should also be included in T for all $T \in \mathcal{IB}_{\succsim_1}(S)$.

Corollary 28. *Suppose that $\succsim_1$ and $\succsim_2$ on $\mathcal{X}$ admit AR-AFs. $\succsim_1$ is more attentive than $\succsim_2$ if and only if, for any $S \in \mathcal{X}$, $\bigcup_{T' \in \mathcal{IB}_{\succsim_2}(S)} T' \subseteq T$ for all $T \in \mathcal{IB}_{\succsim_1}(S)$.*

Proof. Since $\Gamma_{j_1}(S) = T$ for some $T \in \mathcal{IB}_{\succsim_1}(S)$ and $\Gamma_{j_2}(S) = T'$ for some $T' \in \mathcal{IB}_{\succsim_2}(S)$, the statement holds. $\square$

The definition of relative attention allows for the heterogeneity of alternatives in the formation of attention. When some alternatives are objectively easier to catch the attention of DMs than others, relative attention can be considered as an alternative to alternative comparison. When the attention-forming procedures are homogeneous between alternatives, each alternative has the same level of difficulty in attracting DMs' attention. Similar interpretations can be found in Geng and Özbay [2021] and Geng [2022]. In this scenario, we only need to consider the cardinalities of the sets in $\bar{\mathcal{B}}(S)$ and $\mathcal{IB}(S)$ for the AR-IA and AR-AF, respectively.

A.6.2 An Application of the AR-IA: Borda-IA

According to the AR-IA, the DM aggregates utility to form preferences over menus. Depending on the individual, the utility function in the AR-IA varies. However, utility functions are often specified in practice, such as the Borda score. Suppose that $|X/_{\sim}| = k$, by using the Borda score, the DM assigns $k - i + 1$ for the i-th preferred alternatives in X (see, e.g., Baigent and Xu [2004] and Darmann and Klamler [2019]).[2] We may define the

[2] Another Borda score is used in previous literature, where the DM assigns $k - i$ for the i-th preferred alternative. This situation results in $k - 1$ being given to the most preferred alternative, and 0 being given

Borda score as a function $b : X \to \mathbb{N}$. In cases where alternatives are mutually compatible in each menu, the DM is able to add the Borda scores of the alternatives in each menu to form $\succcurlyeq$. [3]

If the DM is using the Borda score to evaluate menus and has an idempotent attention rule, then $\succcurlyeq$ should admit the AR-IA under (Γ, b).

Definition 41. *$\succcurlyeq$ on $\mathcal{X}$ admits a **Borda score under idempotent attention (Borda-IA)** if there is an idempotent attention rule $\Gamma : \mathcal{X} \to \mathcal{X}$ such that*

$$S \succcurlyeq T \iff \sum_{s \in \Gamma(S)} b(s) \geq \sum_{t \in \Gamma(T)} b(t)$$

where $b : X \to \mathbb{N}$ is the Borda score.

A Borda-IA is an AR-IA in which the Borda score b is used as the utility function. A partition of X can be formed by the inverse image of b. When $|X/_{\sim}| = k$, $b^{-1}(k)$ denotes the most preferred alternatives. Similarly, $b^{-1}(1)$ represents the collection of least preferred alternatives, i.e., $MIN(X, \succcurlyeq)$. For simplicity, we denote $b^{-1}(0) = x_0 = \emptyset$. The Borda score implies that $x_i \in b^{-1}(i)$ should be indifferent to $x_{i-1}x_1$ with $x_{i-1} \in b^{-1}(i-1)$ and $i \geq 1$ if the DM pays full attention to both x_i and $x_{i-1}x_1$. Therefore, for some $S \in \mathcal{X}$, suppose that $x_i, x_j \in S$ and $x_{i+1}, x_{j-1} \notin S$. When the DM pays full attention, the preference should exhibit $S \sim (S \setminus x_i x_j) \cup x_{i+1} \cup x_{j-1}$.

to the least preferred alternative. For simplicity, we will focus on the Borda score in which all alternatives possess strict positive utility.

[3] Alcantud and Arlegi [2008] interprets b as a "categorizing" function. Specifically, the DM partitions the alternative set X based on a number of characteristics. The DM has a linear preference for the collection of characteristics represented by the function b. As a result, the value of a menu is determined by adding the value of its characteristics together.

Definition 42. *For any $S \in \mathcal{X}$, $x_i \in b^{-1}(i) \cap S$, $x_j \in b^{-1}(j) \cap S$, $x_{i+1} \in b^{-1}(i+1) \cap S^c$, and $x_{j-1} \subseteq b^{-1}(j-1) \cap S^c$ where $i < k$ and $j \geq 1$, $(S \setminus x_i x_j) \cup (x_{i+1} \cup x_{j-1})$ is a* ***basic equivalent transformation*** *of S. Let us denote it as $S \cong (S \setminus x_i x_j) \cup (x_{i+1} \cup x_{j-1})$. A menu T is an* ***equivalent transformation*** *of S if $S\ Tran(\cong)\ T$ where $Tran(\cong)$ is the transitive closure of $\cong$. Let us denote it as $\simeq$.*

Through the equivalent transformation of S, we can transform it into a menu T with the same Borda score. Instead of limited attention, this transformation is determined by the Borda score. In light of this, even though the DM is limitedly attentive, $\succcurlyeq$ on $\Gamma(\mathcal{X})$ should satisfy the monotonicity axiom.

Axiom 19. *(EM: Extended Monotonocity) For any $S, T \in \mathcal{X}$ and $B_1, B_2 \in \mathcal{B}$, if $B_1 \simeq S \supseteq T \simeq B_2$, then $B_1 \succcurlyeq B_2$.*

The EM axiom posits an extra additive requirement for the AR-IA. For two basic sets B_1 and B_2, if an equivalent transformation of B_1 is a superset of an equivalent transformation of B_2, then the DM should prefer B_1 to B_2. It turns out that the requirement of the Borda-IA can be characterized by the EM axiom.

Theorem 29. *$\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA if and only if $\succcurlyeq$ on $\mathcal{X}$ satisfies EM.*

Proof. We first notice that $\simeq$ is reflexive. Let us first permutate X by using b, and denote this permutation by π. To be specific,

$$X = \left\{x_{m_1}, ..., x_{m_{|b^{-1}(m)|}}, ..., x_{1_1}, ..., x_{1_{|b^{-1}(1)|}}\right\} = \left\{x_{\pi_1}, ..., x_{\pi_{|b^{-1}(m)|}}, ..., x_{\pi_{|X|}}\right\}$$

where $x_{i_\alpha} \in b^{-1}(i)$ for all $1 \leq \alpha \leq |b^{-1}(i)|$. We then can consider X a finite sequence with permutation π, i.e., $X = X_\pi$.

Claim 3. *For any $S \in \mathcal{X}$, there is a $T \in \mathcal{X}$ with $S \simeq T$ such that either $|T| = 1$ or the first $|T| - 1$ elements are $\{x_{\pi_i}\}_{i=1}^{|T|-1}$ when $|T| \geq 2$.*

Proof. Take any $S \in \mathcal{X}$, assume that S is a subsequence of X_π. Suppose that S starts being different from X_π at some n-th element, and list the remaining part of S as $\{s_n, ..., s_{n+k}\}$. If S is identical to the first $|S|$ or $|S| - 1$ elements of X_π, then we are done. We, therefore, assume that $k \geq 1$. By the equivalent transformation, we also can assume that the DM strictly prefers $x_{\pi_{|s|-k}}$ over s_n. This suggests that $s_n \notin b^{-1}(m)$. Suppose that $s_n \in b^{-1}(i)$ where $i < m$ and $s_{n+k} \in b^{-1}(j)$ where $j \geq 1$. By the equivalent transformation, we know that $S \simeq S \setminus s_n s_{n+k} \cup x_{i+1} \cup x_{j-1}$ where $x_{i+1} \in b^{-1}(i+1)$ and $x_{j-1} \in b^{-1}(j-1)$. We can assume that $x_{i+1} = x_\pi$

*where_* is the smallest number of all $x_{\pi_\alpha} \in b^{-1}(i+1)$. We can assume the same thing for x_{j-1}, and denote $S \setminus s_n s_{n+k} \cup x_{i+1} \cup x_{j-1}$ as S^1. If S^1 satisfies the requirement for the claim, we are done. If not, we can repeat the previous steps until the requirement is fulfilled. □

Furthermore, for any $S \in \mathcal{X}$, let T be the equivalent transformation that meets the condition, and suppose that the $|T|$-th element is different from $x_{\pi_{|T|}}$. We can make one more step of the equivalent transformation by giving the lowest available index for the $|T|$-th elements in its equivalent class. If T coincides with the first $|T|$-th element of X_π, we can leave it as T. In this way, we can transform any S into a unique T^1. Let's denote this T^1 as S_π, i.e., $S \simeq S_\pi$.

Corollary 30. *For any $S, T \in \mathcal{X}$, $S \simeq T$ if and only if $\sum_{s\in S} b(s) = \sum_{t\in T} b(t)$.*

Proof. By the previous Claim, we know that there is a S_π and T_π such that $S \simeq S_\pi$ and $T \simeq T_\pi$. Moreover, $\sum_{s\in S_\pi} b(s) = \sum_{t\in T_\pi} b(t)$. Since $S_\pi = T_\pi$, we know that $\sum_{s\in S} b(s) = \sum_{t\in T} b(t)$. For the converse direction, take any $S \in \mathcal{X}$, we know that $S \simeq S_\pi$. Hence, $\sum_{s\in S} b(s) = \sum_{s\in S_\pi} b(s)$. Similarly, there is a T_π for T. Since $\sum_{s\in S} b(S) = \sum_{s\in S_\pi} b(s)$, $\sum_{s\in S_\pi} b(s) = \sum_{t\in T_\pi} b(t)$. Hence, $S_\pi = T_\pi$ which implies that $S \simeq T$. □

Now, we can prove the Theorem 29. Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA. Clearly, $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA. Moreover, take any $B_1, B_2 \in \mathcal{B}$, and $S, T \in \mathcal{X}$ such that $B_1 \simeq S \supseteq T \simeq B_2$. We then have that $\sum_{x\in\Gamma(B_1) b(x)} = \sum_{x\in B_1} b(x) = \sum_{s\in S} b(s)$. Similar equality also holds between B and T. Since $S \supseteq T$, $\sum_{s\in S} b(s) \geq \sum_{t\in T} b(t)$. As a result $\sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y)$ which implies that $B_1 \succcurlyeq B_2$.

For the converse direction, suppose that $\succcurlyeq$ on $\mathcal{X}$ admits an AR-IA and satisfies EM. We know that there is an idempotent attention Γ that is consistent with the AR-IAs such that $\Gamma(\mathcal{X}) = \mathcal{B}$. We then can focus on this Γ. By contradiction, suppose that (Γ, b) is not a consistent pair of the Borda-IA, i.e., there is a pair of $B_1, B_2 \in \mathcal{B}$ such that we cannot have $B_1 \succcurlyeq B_2 \iff \sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y)$. That is, $\sum_{x\in\Gamma(B_1)} b(x) \geq \sum_{y\in\Gamma(B_2)} b(y) \implies B_2 \succ B_1$. We first know that there are $B_{1\pi}, B_{2\pi}$ with $B_1 \simeq B_{1\pi}$ and $B_2 \simeq B_{2\pi}$ such that $\left| B_{j\pi} \setminus \bigcup_{i=1}^{|B_{j\pi}|} x_{\pi_i} \right| \leq 1$ for $j = 1, 2$. We can assume that $B_{1\pi} \neq B_{2\pi}$ Since $\sum_{t\in B_{1\pi}} b(t) \geq \sum_{t\in B_{2\pi}} b(t)$, we know that $|B_{1\pi}| \geq |B_{2\pi}|$. We need to consider the following cases:

Case 1: $\left| B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1} \right| = 1$ and $\left| B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1} \right| = 1$. If $MIN(B_{1\pi}, \succcurlyeq) = x_i \succ MIN(B_{2\pi}, \succcurlyeq) = x_j$, then $B_{1\pi} \setminus x_i \cup x_{i-j} x_j \simeq B_{1\pi} \supseteq B_{2\pi}$. Hence, $B_2 \succ B_1$ violates EM.

If $MIN(B_{2\pi}, \succcurlyeq) = x_j \succ MIN(B_{1\pi}, \succcurlyeq) = x_i$, then we have $|B_{1\pi}| - |B_{2\pi}| \geq 1$. Moreover, $B_{1\pi} \setminus x_{\pi_{|B_{1\pi}|-1}} \cap x_j x_{j'} \simeq B_{1\pi} \supseteq B_{2\pi}$ where $x_{j'} \in b^{-1}(j - b(x_{\pi_{|B_{1\pi}|-1}}))$. Clearly, it also violates EM.

Case 2: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 0$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 0$. We then know that $MIN(B_{1\pi}, \succcurlyeq) = x_i \succ MIN(B_{2\pi}, \succcurlyeq) = x_j$. Following the same procedure, we can check that this violates EM.

Case 3: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 1$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 0$. We then have $B_{1\pi} \supseteq B_{2\pi}$. This case also violates EM.

Case 4: $\left|B_{1\pi} \setminus \bigcup_{i=1}^{|B_{1\pi}|} x_{\pi_1}\right| = 0$ and $\left|B_{2\pi} \setminus \bigcup_{i=1}^{|B_{2\pi}|} x_{\pi_1}\right| = 1$.We then have $B_{1\pi} \supseteq B_{2\pi}$. This case also violates EM. □

As a special evaluation rule, the Borda score indicates that there is a unit increment in utility between x_{i-1} and x_i. In the characterization of the Borda-IA, the condition for $b \in \mathcal{U}_{\succcurlyeq}$ is presented. Further, given the DM's preference for $\mathcal{X}$, when is it appropriate to say that the DM must use the Borda score in evaluating each menu?

Definition 43. *Suppose that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA. The DM is* ***revealed to use Borda score*** *if for all $u \in \mathcal{U}_{\succcurlyeq}$, and $x, y \in X$, $\frac{u(x)}{b(x)} = \frac{u(y)}{b(y)}$.*

If the DM must use the Borda score to evaluate menus, then any consistent utility function should be produced by multiplying a positive number by b. It should be related to the uniqueness of the similarity transformation for the utility function.

Theorem 31. *The DM is revealed to use the Borda score if and only if for any $x \in b^{-1}(i), y \in MIN(X, \succcurlyeq)$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where*

$S_n \sim T_n$ *for all* n*, such that* $\frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)} = i$*, and* $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ *for other* $z \in X$*.*

Proof. Assume that $\succcurlyeq$ on $\mathcal{X}$ admits a Borda-IA, and the DM is revealed to use the Borda score. Take any $u, v \in \mathcal{U}_{\succcurlyeq}$ and $x, y \in X$, we know that $\frac{u(x)}{u(y)} = \frac{b(x)}{b(y)} = \frac{v(x)}{v(y)}$. It suggests that $u(x) = \frac{u(y)}{v(y)} v(x)$. Hence, the utility function is unique under similarity transformation. By the proof of Theorem 23, we can show this direction.

For the converse direction, we can assume that for any $x \in b^{-1}(k), y \in MIN(X, \succcurlyeq)$, there is a positive integer m, and $\{S_n\}_{n=1}^m, \{T_n\}_{n=1}^m \subseteq \mathcal{B}$ where $S_n \sim T_n$ for all n, such that $\frac{\sum_{n=1}^m \mathbb{1}_{S_n}(x) - \sum_{n=1}^m \mathbb{1}_{T_n}(x)}{\sum_{n=1}^m \mathbb{1}_{T_n}(y) - \sum_{n=1}^m \mathbb{1}_{S_n}(y)} = k$, and $\sum_{n=1}^m \mathbb{1}_{S_n}(z) = \sum_{n=1}^m \mathbb{1}_{T_n}(z)$ for other $z \in X$. Since $\succcurlyeq$ on $\mathcal{X}$ admits AR-IAs, for any $u \in \mathcal{U}_{\succcurlyeq}$, we have $\frac{u(x_i)}{u(x_1)} = i$, as a result, $\frac{u(x_i)}{u(x_1)} = \frac{b(x_i)}{b(x_1)} = i$. It suggests that $\frac{u(x)}{b(x)} = \frac{u(y)}{b(y)}$ for all $x, y \in \mathcal{X}$. □

A.7 Characterization of Satisficing under Full Attention

We first introduce the formal definition of satisficing (under full attention).

Definition 44. *A choice correspondence* C *is a* ***satisficing (under full attention)*** *if there is a linear order* $\succcurlyeq$ *and a threshold function* $\theta : \mathcal{X} \to X$ *such that* $C(S) = \{s \in S : s \succcurlyeq \theta(S)\}$ *where* $\theta(S) \in S$ *for all* $S \in \mathcal{X}$*.*

According to the satisfaction model, the chosen alternative should be better than the unchosen alternative. We can define it as a binary relation.

Definition 45. *For any* $x, y \in X$*,* $x \; P^s \; y$ *if there is an* $S \in \mathcal{X}$ *such that* $x \in C(S)$ *and* $y \in S \setminus C(S)$*.*

As shown in Aleskerov et al. [2007] and Tyson [2008], satisficing can be described as the following Lemma. We still show the proof in detail despite the slight difference in our settings.

Lemma 9. *A choice correspondence can be rationalized by a satisficing model if and only if P^s is acyclic.*

Proof. Take an arbitrary choice correspondence C, and suppose that it can be rationalized by a satisficing model under $\langle \succcurlyeq, \theta \rangle$. By contradiction, suppose that there is a finite collection of alternatives $\{x_i\}_{i=1}^k$ such that $x_1\ P^s\ x_2\ P^s, ..., P^s\ x_k\ P^s x_1$. There is a collection of sets $\{S_i\}_{i=1}^k$ such that $x_i \in C(S_i)$ and $x_{i+1} \in S_i \setminus C(S_i)$ for all $i < k$, and $x_k \in C(S_k)$ and $x_1 \in B_k \setminus C(B_k)$. As a result, $\succcurlyeq$ is not linear.

For the converse direction, suppose that P^s is acyclic. Let P^s_R be the transitive closure of P^s, $\overline{P^s_R}$ be a completion of P^s_R. Let $f(S) = min\{C(S), \overline{P^s_R}\}$. It is evident that $x \in C(S)$ implies that $x\ \overline{P^s_R}\ f(S)$. Suppose that there is a $t \in S$ with $t\ \overline{P^s_R}\ f(S)$ such that $t \notin C(S)$. We then know that $f(S)\ P^s\ t$. Consequently, $C(S) = \left\{ s \in S : s\ \overline{P^s_R}\ f(S) \right\}$. $\square$

We remain to show that WARP-S is equivalent to the acyclicity of P^s.

Lemma 10. *P^s is acyclic if and only if C satisfies WARP-S.*

Proof. Suppose that P^s is acyclic. By Lemma 9, C is admits a satisficing under $\langle \succcurlyeq, \theta \rangle$. Take any $S \in \mathcal{X}$, let $x^* = max(S, \succcurlyeq)$. Fix any $T \in \mathcal{X}$ with $x^* \in T$, if $t \in C(T) \cap S$, then $t \succcurlyeq \theta(T)$. Also, $t \in S$ suggests that $x^* \succcurlyeq t \succcurlyeq \theta(S)$. As a result, $x^* \in C(T)$.

For the converse direction, suppose that P is cyclical, i.e., there is a collection of $\{S_i\}_{i=1}^k$

such that

$$s_1, s_2 \in S_1, s_1 \in C(S_1),\ and\ s_2 \notin C(S_1);$$

$$s_2, s_3 \in S_2, s_2 \in C(S_2),\ and\ s_3 \notin C(S_2);$$

$$\ldots$$

$$s_1, s_k \in S_k, s_k \in C(S_k),\ and\ s_1 \notin C(S_k).$$

Let $S = \{s_i\}_{i=1}^k$. We cannot find a corresponding x^* in WARP-S for S. □

The following theorem is a direct result of Lemma 9 and Lemma 10.

Theorem 32. *A choice correspondence C admits satisficing if and only if it satisfies WARP-S.*

A.8 Proof of Proposition 13

Suppose that S is simultaneously positively and negatively x-covered. By Positive and Negative Contraction, we know that $c(Z \cup \{x\}) = c(\{c(Z \cup \{x\}), x\}) \neq x$ for every $Z \subseteq S$ with $Z \in \mathcal{X}$. As a result, S cannot be positively and negatively x-covered.

For the converse direction, suppose that S cannot be positively and negatively x-covered at the same time. If S is positively x-covered, then $c(S \cup x) = c(\{c(Z \cup \{x\}), x\})$. Similarly, if S is negatively x-covered, then $c(S \cup x) \neq x$. Assume that S is positively x-covered and there is a $Y \subset S$ such that Y is negatively x covered. By contradiction, suppose there is a Z with $S \setminus Y \subseteq Z \subseteq S$ such that $c(Z \cup \{x\}) \neq c(\{c(Z \cup \{x\}), x\})$. Z and Y then forms a negative x-cover of S. Similarly, Assume that S is negatively x-covered and there is a $Y \subset S$ such that Y is positively x covered. By contradiction, suppose there is a Z with $S \setminus Y \subseteq Z \subseteq S$ such that $c(Z \cup \{x\}) = x$. Z and Y then forms a positive x-cover of S.

A.9 Proof of Theorem 19

For the necessity part, assume that c is a CAF under $(\{\trianglerighteq\}_{x\in X}, \succ)$, and Γ is the induced attention rule. Since $\Gamma(S) = S$ for all S with $|S| \leq 2$, No-Cycle is required. Take any S that is positively x-covered by $\{S_i\}_{i\in I}$, since $c(S_i \cup \{x\}) = x$ for all i, $\max(S_i, \trianglerighteq_x) \in X$ for every $i \in I$. As a result, $\max(S_i, \trianglerighteq_x) \in X$. Meanwhile, if S is negatively x-covered, $\max(S, \trianglerighteq_x) \in X^*$. Therefore S cannot be positively x-covered and negatively x-covered at the same time. By Proposition 13, c satisfies Positive and Negative Contraction.

For the sufficient part, given any permutation on X, let $X = \{x_n\}_{n=1}^N$ where $|X| = N$. Let us define $x\ P\ y$ if $c(\{x, y\}) = x$. Due to No-cycle, P is a linear order on X. Starting from x_1, let us define:

$$x_1^{+0} = \{S \in \mathcal{X} : c(S) = x_1\};$$

$$x_1^{-0} = \{S \in \mathcal{X} : x_1\ P\ c(S)\};$$

$$X_1^{+0} = \bigcup_{S \in x_1^{+0}} S;$$

$$X_1^{-0} = \bigcup_{S \in x_1^{-0}} S;$$

$$X_1^0 = X_1^{+0} \cup X_1^{-0}.$$

Consider a real function $f^0_{x_1} : X \to \mathbb{R}$ with $f_{x_1}(x_n) = 0$ for all $x_n \in X$. We want to construct $\trianglerighteq_{x_1}$ through a modification of $f^0_{x_1}$.

Let

$$f^1_{x_1}(x_n) = \begin{cases} -\left(6N - 1 - \frac{1}{n+1}\right) & \text{if } (f^0_{x_1})^{-1}(0) \setminus X^0_1; \\ 6N - 2 - \frac{1}{n+1} & \text{if } x_n \in X^{+0}_1 \setminus X^{-0}_1; \\ -\left(6N - 2 - \frac{1}{n+1}\right) & \text{if } x_n \in X^{-0}_1 \setminus X^{+0}_1; \\ f^0_{x_1}(x_n) & \text{Otherwise.} \end{cases}$$

Let

$$x^{+1}_1 = \{S \in x^{+0}_1 : S \cap \{x_n \in X : f^1_{x_1}(x_n) \neq 0\} = \emptyset\};$$

$$x^{-1}_1 = \{S \in x^{-0}_1 : S \cap \{x_n \in X : f^1_{x_1}(x_n) \neq 0\} = \emptyset\};$$

$$X^{+1}_1 = \bigcup_{S \in x^{+1}_1} S;$$

$$X^{-1}_1 = \bigcup_{S \in x^{-1}_1} S;$$

$$X^1_1 = X^{+1}_1 \cup X^{-1}_1.$$

Moreover, let

$$f^2_{x_1}(x_n) = \begin{cases} -\left(6N - 3 - \frac{1}{n+1}\right) & \text{if } (f^1_{x_1})^{-1}(0) \setminus X^1_1; \\ 6N - 4 - \frac{1}{n+1} & \text{if } x_n \in X^{+1}_1 \setminus X^{-1}_1; \\ -\left(6N - 4 - \frac{1}{n+1}\right) & \text{if } x_n \in X^{-1}_1 \setminus X^{+1}_1; \\ f^1_{x_1}(x_n) & \text{Otherwise.} \end{cases}$$

We then define $f^{m+1}_{x_1}(x_n)$ recursively. Formally, let

$$x^{+m}_1 = \{S \in x^{+(m-1)}_1 : S \cap \{x_n \in X : f^m_{x_1}(x_n) \neq 0\} = \emptyset\};$$

$$x^{-m}_1 = \{S \in x^{-(m-1)}_1 : S \cap \{x_n \in X : f^m_{x_1}(x_n) \neq 0\} = \emptyset\};$$

$$X^{+m}_1 = \bigcup_{S \in x^{+m}_1} S;$$

$$X^{-m}_1 = \bigcup_{S \in x^{-m}_1} S;$$

$$X^m_1 = X^{+m}_1 \cup X^{-m}_1.$$

Let

$$f^{(m+1)}_{x_1}(x_n) = \begin{cases} -\left(6N - (2(m+1) - 1) - \frac{1}{n+1}\right) & \text{if } (f^m_{x_1})^{-1}(0) \setminus X^m_1; \\ 6N - 2(m+1) - \frac{1}{n+1} & \text{if } x_n \in X^{+m}_1 \setminus X^{-m}_1; \\ -\left(6N - 2(m+1) - \frac{1}{n+1}\right) & \text{if } x_n \in X^{-m}_1 \setminus X^{+m}_1; \\ f^m_{x_1}(x_n) & \text{Otherwise.} \end{cases}$$

This procedure stops when $f_{x_1}^M = f_{x_1}^{M-1}$ for some $M \in \mathbb{N}$. For every $1 \leq m \leq M$, we know that x_n either in $X_1^{+(m-1)}$, $X_1^{-(m-1)}$, or $\left(f_{x_1}^m\right)^{-1}(0)$. Moreover, for every x_n there is a m_{x_n} such that $x \in X_1^{m_{x_n}}$ and $x \notin X_1^{m_{x_n}+1}$. As a result, $f_{x_1}^M$ is a bijection function.

It is clear that $1 \leq M \leq N+1$. For every $1 \leq m \leq M$, $6N - 2m - \frac{1}{n+1} \geq 6N - 2(N+1) - \frac{1}{2} = 4N - \frac{5}{2} > 0$. Consider a function:

$$g_{x_1}(x_n) = \begin{cases} f_{x_1}^M(x_n) & \text{if } x_n \neq x_1; \\ \\ 0 & \text{if } x_n = x_1. \end{cases}$$

It is easy to verify that $g_{x_1}(x_n)$ is a bijection function, and $g_{x_1}(x_n) = 0$ only when $n = 1$.

We can get a collection of real functions $\{g_{x_n}\}_{n=1}^N$ by applying the previous procedure to every $x_n \in X$. For any $S \in \mathcal{X}$, let $G(S) := \{x \in S : g_x(\arg\max_S |g_x|) \geq 0\}$. We then want to show that $c(S) = \max(G(S), P)$.

We first need to show that $c(S) \in G(S)$. Since S is positively $c(S)$-covered, S cannot be negatively $c(S)$-covered. Therefore, there is an $m \leq M_{c(S)}$ such that $S \subseteq c(S)^{+n}$ for all $n \leq m$, and $S \not\subseteq c(S)^{+(m+1)}$. Based on the construction of $f_{c(S)}$, we know that $g_{c(S)}(\arg\max_S |g_{c(S)}|) > 0$, i.e., $c(S) \in G(S)$.

We then want to show that if for every $y \in S$ with $y\ P\ c(S)$, then $y \notin G(S)$. Under this circumstance, we know that there is an $0 \leq m_1 \leq M_y$ such that $S \subseteq X_y^{-m}$ for every $0 \leq m \leq m_1$. As a result, $\arg\max_S g_y \in X^*$, which means that y catches no attention in S.

We then only need to construct $\trianglerighteq_x$. Let's define $x \ \trianglerighteq_z \ y$ if $g_z(x) \geq g_z(y)$, and $y^* \ \trianglerighteq_z \ x^*$ whenever $x \ \trianglerighteq_z \ y$. Since g_z is a bijection, $x \ \trianglerighteq_z \ y$ and $x \ \trianglerighteq_z \ y$ only when $x = y =$

$z = z^*$. Consequently, $\trianglerighteq_z$ is a signed. We can construct $\{\trianglerighteq_x\}_{x \in X}$ with similar procedures. Clearly, $G(S) = \{x \in S : \max(S, \trianglerighteq_x) \in X\}$, and c is a CAF under $(\{\trianglerighteq_x\}_{x \in X}, P)$.