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UNIVERSITY OF CALIFORNIA

Radiation Laboratory
Berkeley, California

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MESON PRODUCTION

I. MESON PRODUCTION BY MESONS

II. ISOBAR MODEL FOR MESON PRODUCTION IN PROTON-PROTON COLLISIONS

Saul Barshay

(Thesis)

November 29, 1956

MESON PRODUCTION

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MESON PRODUCTION

I. MESON PRODUCTION BY MESONS

II. ISOBAR MODEL FOR MESON PRODUCTION IN PROTON-PROTON COLLISIONS

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ABSTRACT

In Part I, calculation of meson production in a meson-nucleon collision has been carried out for the processes $\pi^+ + p \rightarrow \pi^+ + \pi^+ + n$ and $\pi^+ + p \rightarrow \pi^+ + \pi^0 + p$, by means of the method of Low. Use has been made of an approximate form of the Chew-Low transition matrix for scattering off the energy shell. This matrix exhibits explicitly the resonance behavior of the isotopic spin $3/2$, angular momentum $3/2$ scattering state. At 350-Mev incident-meson kinetic energy, the cross section for π^+ production is 0.86 mb and that for π^0 production is 5.5 mb. The angular distribution of each final-state pion is of the form $A + \cos^2 \theta$. For π^+ production at 250 Mev, $A = 0.64$. The two final-state mesons seem to share the available kinetic energy. The probability of the nucleon's flipping its spin in the process is about 1.5 time the probability for no spin flip.

In Part II, a model is considered for single- and double-pion production which takes place via an intermediate state wherein either one or both of the initial nucleons are excited to the isobaric state of $J = T = 3/2$. The treatment is phenomenological and comparison is made with recent experiments in the 0.5 to 1.5 Bev range. Two striking

features of the experiments, the strong preference for the emission of mesons with kinetic energies of 50 to 150 Mev, and the rapid increase in the two-meson processes at bombarding energies above 1 Bev, are exhibited by the calculation.

I. MESON PRODUCTION BY MESONS

INTRODUCTION

There has been increasing interest in the production of mesons in meson-nucleon collisions at high energies. We have felt that it would be of interest to calculate the production cross section for energies near threshold. There is evidence indicating strong P-wave production of mesons. Such evidence consists of the small cross section for π^0 production in p-p collisions, the rapid increase with energy of the total cross section for π^+ production in these collisions, and the center-of-mass angular distribution for the π^+ of $0.29 + \cos^2 \theta$.^{1,2} The (3/2, 3/2) resonant state in meson-nucleon scattering appears to be of importance in the production process. The hypothesis that meson production takes place through excitation of a nucleon into the (3/2, 3/2) resonant state is capable of explaining the small cross section for π^+ production in n-p collisions relative to that in p-p collisions.³ The hypothesis has also been used to explain the π^+/π^- production ratio in nucleon-nucleus collisions.⁴ We have assumed production from the (3/2, 3/2) state and have investigated the processes $\pi^+ + p \rightarrow \pi^+ + \pi^+ + n$ and $\pi^+ + p \rightarrow \pi^0 + \pi^+ + p$.

The procedure has been to derive the amplitude describing the process by the method of Low.⁵ The method deals with a perturbation-like expansion in terms of exact eigenstates of the total Hamiltonian. This feature makes possible an improvement of the ordinary perturbation-theory treatment, since it is possible to describe part of the process as meson-nucleon scattering and to use here the scattering transition matrix developed by Chew and Low.^{6,7} We will be dealing with a transition matrix for scattering off the energy shell. Chew and Low

have derived an approximate expression for this matrix in a form that exhibits the resonant behavior of its $(3/2, 3/2)$ part. It was this possibility which led to the calculation presented here, in the hope that, in spite of the approximations, the form of this matrix off the energy shell would give reasonable results in the computation of a cross section for meson production. Chew and Low have shown that, on the energy shell, their matrix gives a good description of low-energy scattering.⁶

CALCULATIONS

Following Low,⁵ one starts with Dyson's S matrix between initial and final bare particle states,

$$\langle f | S | i \rangle = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_n \langle \phi_{p'} | a_j(k_1) a_l(k_2) P \left[H_I(t_1) \dots H_I(t_n) \right] a_i^*(k) | \phi_p \rangle , \quad (1)$$

where $\hbar = c = 1$, $x = (\vec{x}, x_0)$, $k = (\vec{k}, \omega)$.

The symbol H_I is the usual interaction Hamiltonian for pseudoscalar meson theory.⁵

Assume $k \neq k_1 \neq k_2$; $p \neq p'$. The $a_i^*(k)$ is commuted through the P bracket to the left and the $a_l(k_2)$ and $a_j(k_1)$ are commuted through to the right, making use of

$$\langle \phi_{p'} | a_i^*(k) = a_l(k_2) | \phi_p \rangle = a_j(k_1) | \phi_p \rangle = 0 . \quad (2)$$

The resulting expression is transformed by making use of the identity

$$\begin{aligned} \langle \phi_{p'} | \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n P \left[H_I(t_1) \dots H_I(t_n) A_i(x) A_j(y) \dots \right] | \phi_p \rangle \\ = \langle \Psi_{p'} | P \left[\widetilde{A}_i(x) \widetilde{A}_j(y) \dots \right] | \Psi_p \rangle , \end{aligned} \quad (3)$$

where on the left side all operators are in the interaction representation and on the right side $\widetilde{A}_i(x)$ has the Heisenberg time dependence,

$$\widetilde{A}_i(\underline{x}, x_0) = e^{iHx_0} A_i(\vec{x}, 0) e^{-iHx_0}, \quad (4)$$

and the $|\Psi_p\rangle$ are exact single-nucleon eigenstates of the total Hamiltonian (free field plus interaction). The S matrix is then

$$\begin{aligned} \langle f | S | i \rangle &= (-i)^3 / (8 \omega \omega_1 \omega_2)^{\frac{1}{2}} \int d^4x d^4y d^4z e^{ikx - ik_1y - ik_2z} \\ &\quad \langle \Psi_{p'} | P \left[\widetilde{A}_i(x) \widetilde{A}_j(y) \widetilde{A}_\ell(z) \right] | \Psi_p \rangle \\ &\quad + (-i)^2 \lambda / (8 \omega \omega_1 \omega_2)^{\frac{1}{2}} \int d^4x d^4y e^{ikx} \left\{ e^{-i(k_1 + k_2)y} \right. \\ &\quad \cdot \langle \Psi_{p'} | P \left[(2\widetilde{\phi}_j(y)\widetilde{\phi}_\ell(y) + \delta_{j\ell} \widetilde{\phi}_m(y)\widetilde{\phi}_m(y)) \widetilde{A}_i(x) \right] | \Psi_p \rangle \\ &\quad + e^{-ik_2x - ik_1y} \langle \Psi_{p'} | P \left[\widetilde{A}_j(y) (2\widetilde{\phi}_i(x)\widetilde{\phi}_\ell(x) + \delta_{i\ell} \widetilde{\phi}_m(x)\widetilde{\phi}_m(x)) \right] | \Psi_p \rangle \\ &\quad \left. + e^{-ik_1x - ik_2y} \langle \Psi_{p'} | P \left[\widetilde{A}_\ell(y) (2\widetilde{\phi}_i(x)\widetilde{\phi}_j(x) + \delta_{ij} \widetilde{\phi}_m(x)\widetilde{\phi}_m(x)) \right] | \Psi_p \rangle \right\} \\ &\quad (-i) \lambda / (8 \omega \omega_1 \omega_2)^{\frac{1}{2}} \int d^4x e^{i(k - k_1 - k_2)x} \langle \Psi_{p'} | 2\widetilde{\phi}_\ell(x) \delta_{ij} | \Psi_p \rangle \\ &= (-i) / (2\omega)^{\frac{1}{2}} \int d^4x e^{ikx} \langle p' k_1 k_2 j \ell | A_i(x) | p \rangle. \end{aligned} \quad (5)$$

The quantity $\langle p' k_1 k_2 j \ell | A_i(x) | p \rangle$ is defined by the above equation for all values of p, p', k_1 , and k_2 . By the commutation

process it may be shown to be identical with

$$\begin{aligned} \langle \phi_{p'} | a_j(k_1) a_l(k_2) \sum_{n=0}^{\infty} (-i)^n / n! \int dt_1 \dots dt_n P \left[H_I(t_1) \dots H_I(t_n) A_i(x) \right] | \phi_p \rangle \\ = \langle \psi_{p', k_1, k_2}^{(-)} | \tilde{A}_i(x) | \psi_p \rangle \end{aligned} \quad (6)$$

On the energy shell, $p' + k_1 + k_2 = p + k$, this quantity is the S matrix with the delta function of energy and momentum left out.

However, off the energy shell, the S matrix is not defined, whereas

$\langle p' k_1 k_2 j l | A_i(x) | p \rangle$ is defined by the integrand on the left side of Eq. (6).

We will be interested in the static limit. We therefore drop terms in λ and consider

$$\begin{aligned} \langle p' k_1 k_2 j l | A_i(x) | p \rangle &= (-i)^2 / (4 \omega_1 \omega_2)^{\frac{1}{2}} \\ &\cdot \int d^3 \vec{z} dz_0 d^3 \vec{y} dy_0 e^{-i(\omega_1 y_0 - \vec{k}_1 \cdot \vec{y} + \omega_2 z_0 - \vec{k}_2 \cdot \vec{z})} \\ &\cdot \langle \psi_{p'} | P \left[e^{i(Hz_0 - \vec{P} \cdot \vec{z})} A_l(0) e^{-i(Hz_0 - \vec{P} \cdot \vec{z})} e^{i(Hy_0 - \vec{P} \cdot \vec{y})} A_j(0) e^{-i(Hy_0 - \vec{P} \cdot \vec{y})} \right. \\ &\quad \left. \times e^{i(Hx_0 - \vec{P} \cdot \vec{x})} A_i(0) e^{-i(Hx_0 - \vec{P} \cdot \vec{x})} \right] | \psi_p \rangle, \end{aligned} \quad (7)$$

where we have utilized invariance under the translation (momentum) operator $\vec{P}$,

$$A_1(\vec{x}) = e^{-i\vec{P}\cdot\vec{x}} A_1(0) e^{i\vec{P}\cdot\vec{x}} \quad (8)$$

We now perform the indicated space and time integrations, setting $\vec{x} = x_0 = 0$ in the final result. This is best done as in the following example. Introducing complete sets of exact eigenstates $|\psi_n\rangle$ and $|\psi_m\rangle$, and considering a particular time ordering $z_0 > y_0 > x_0$, we have

$$\begin{aligned} & -\sqrt{4\omega_1\omega_2} \langle p' k_1 k_2 j \ell | A_1(x) | p \rangle = \\ & \sum_{n,m} \int d^3\vec{z} d^3\vec{y} e^{-i\vec{k}_1\cdot\vec{y} - i\vec{k}_2\cdot\vec{z}} \int_{-\infty}^{\infty} dz_0 \eta_+(z_0 - y_0) e^{i\omega_2 z_0} \\ & \int_{-\infty}^{\infty} dy_0 \eta_+(y_0 - x_0) e^{i\omega_1 y_0} \langle \psi_{p'} | e^{i(Hz_0 - \vec{P}\cdot\vec{z})} A_\ell(0) e^{-i(Hz_0 - \vec{P}\cdot\vec{z})} | \psi_n \rangle \\ & \cdot \langle \psi_n | e^{i(Hy_0 - \vec{P}\cdot\vec{y})} A_j(0) e^{-i(Hy_0 - \vec{P}\cdot\vec{y})} | \psi_m \rangle \\ & \cdot \langle \psi_m | e^{i(Hx_0 - \vec{P}\cdot\vec{x})} A_1(0) e^{-i(Hx_0 - \vec{P}\cdot\vec{x})} | \psi_p \rangle \end{aligned}$$

$$\begin{aligned}
 = & \sum_{n,m} \left(\frac{1}{2\pi i} \right)^2 \int_{-\infty}^{\infty} dz_0 e^{i(\omega_2 + \omega_{p'} - E_n)z_0} \int_{-\infty}^{\infty} \frac{da e^{ia(z_0 - y_0)}}{a - i\epsilon} \\
 & \int_{-\infty}^{\infty} dy_0 e^{i(\omega_1 + E_n - E_m)y_0} \int_{-\infty}^{\infty} \frac{db e^{ib(y_0 - x_0)}}{b - i\epsilon} \\
 & \int d^3\vec{z} d^3\vec{y} e^{i(-\vec{k}_1 - \vec{p}_n + \vec{p}_m) \cdot \vec{y}} e^{i(-\vec{k}_2 - \vec{p}' + \vec{p}_n) \cdot \vec{z}}
 \end{aligned}$$

$$\begin{aligned}
 & \langle \Psi_{p'} | A_\ell(0) | \Psi_n \rangle \langle \Psi_n | A_j(0) | \Psi_m \rangle \langle \Psi_m | A_i(0) | \Psi_p \rangle \\
 = & - \sum_{n,m} \frac{\langle \Psi_{p'} | A_\ell(0) | \Psi_n \rangle \langle \Psi_n | A_j(0) | \Psi_m \rangle \langle \Psi_m | A_i(0) | \Psi_p \rangle \delta(\vec{p}_m - \vec{p}_n - \vec{k}_1) \delta(\vec{p}_n - \vec{p}' - \vec{k}_2)}{(E_n - \omega_2 - \omega_{p'} - i\epsilon)(E_m - \omega_2 - \omega_1 - \omega_{p'} - i\epsilon)} .
 \end{aligned}$$

(9)

Here we have used

$$\begin{aligned}
 \eta_+(x) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{da e^{iax_0}}{a - i\epsilon} \\
 &= 1 \quad x_0 > 0 \\
 &= 0 \quad x_0 < 0 .
 \end{aligned}$$

This procedure is carried out for each time ordering. The full amplitude is then given by

$$\langle p' k_1 k_2 j \ell | A_i(0) | p \rangle = 1/(4 \omega_1 \omega_2)^{\frac{1}{2}}$$

$$\begin{aligned} & \sum_{n,m} \left\{ \frac{\langle \Psi_{p'} | A_\ell(0) | \Psi_n \rangle \langle \Psi_n | A_j(0) | \Psi_m \rangle \langle \Psi_m | A_i(0) | \Psi_p \rangle \delta(\vec{p}_n - \vec{p}' - \vec{k}_2) \delta(\vec{p}_m - \vec{p}_n - \vec{k}_1)}{(E_n - \omega_2 - \omega_{p'} - i\epsilon)(E_m - \omega_2 - \omega_1 - \omega_{p'} - i\epsilon)} \right. \\ & + \frac{\langle \Psi_{p'} | A_j(0) | \Psi_n \rangle \langle \Psi_n | A_i(0) | \Psi_m \rangle \langle \Psi_m | A_\ell(0) | \Psi_p \rangle \delta(\vec{p}_n - \vec{p}' - \vec{k}_2) \delta(\vec{p}_m - \vec{p} + \vec{k}_1)}{(E_n - \omega_1 - \omega_{p'} - i\epsilon)(E_m + \omega_2 - \omega_p - i\epsilon)} \\ & \left. + \frac{\langle \Psi_{p'} | A_i(0) | \Psi_n \rangle \langle \Psi_n | A_j(0) | \Psi_m \rangle \langle \Psi_m | A_\ell(0) | \Psi_p \rangle \delta(\vec{p}_m - \vec{p}_n - \vec{k}_1) \delta(\vec{p}_n - \vec{p} - \vec{k}_2)}{(E_n + \omega_1 + \omega_2 - \omega_p - i\epsilon)(E_m - \omega_p + \omega_2 + i\epsilon)} \right\}, \end{aligned}$$

(10)

plus three additional terms obtained by interchanging the subscripts j and ℓ and 1 and 2.

If the $|\Psi_p\rangle$, $|\Psi_{p'}\rangle$, $|\Psi_n\rangle$, and $|\Psi_m\rangle$ were replaced by bare nucleon spinors, the above equation would yield lowest-order perturbation theory for the six diagrams shown in Fig. 1. In this case, although $|\Psi_p\rangle$ and $|\Psi_{p'}\rangle$ represent physical nucleon states, the $|\Psi_n\rangle$ and $|\Psi_m\rangle$ may be any members of a complete orthonormal set of eigenstates of H , a physical nucleon, or a physical nucleon plus any number of real mesons. Another interesting feature is that although $|\Psi_p\rangle$ is to represent the initial physical proton in our process, and although $A_i(0)$ --since it is derived from an interaction which represents the absorption of

a π^+ by a neutron--contains the operator $\tau_+ = \tau_1 + i\tau_2$, yet $A_i(0) | \Psi_p \rangle \neq 0$. Operation on a bare proton with τ_+ yields zero, but the physical proton is a superposition of a bare-proton amplitude, a bare-neutron-plus-meson amplitude, etc. Equation (11) contains only energy denominators of the form $(E_{\text{intermediate}} - E_{\text{final}})$. Since the physical nucleon self-energy appears in both states, this unobservable quantity does not appear in the equation.

We wish to treat the equation for $\langle p' k_1 k_2 j \ell | A_i(0) | p \rangle$ in the static limit:

$$\begin{aligned} A_i(0) &\longrightarrow \frac{i \sqrt{4\pi} f \vec{\sigma} \cdot \vec{k} \tau_1}{\mu \sqrt{2\omega}} = A_k, \\ A_j(0) &\longrightarrow \frac{-i \sqrt{4\pi} f \vec{\sigma} \cdot \vec{k}_1 \tau_j}{\mu \sqrt{2\omega_1}} = A_{k_1}^*, \\ A_\ell(0) &\longrightarrow \frac{-i \sqrt{4\pi} f \vec{\sigma} \cdot \vec{k}_2 \tau_\ell}{\mu \sqrt{2\omega_2}} = A_{k_2}^*. \end{aligned} \quad (11)$$

Here f is the unrenormalized, unrationalized pseudovector coupling constant. $\tau_i = (\frac{1}{\sqrt{2}} \tau_+, \frac{1}{\sqrt{2}} \tau_-, \tau_3)$. In the double sum over the two intermediate states, we keep those terms in which both intermediate states contain only a physical nucleon, and those terms in which one intermediate state contains only a physical nucleon and the other state contains a nucleon, plus one meson. These terms correspond to the nine diagrams shown in Fig. 2, plus nine more obtained by interchanging k_1 and k_2 .

Let us consider first the process $\pi^+ + p \longrightarrow \pi^+ + \pi^+ + n$.

Since there exist neither doubly charged nor negatively charged physical nucleons, our initial physical proton cannot make a transition to another state of the physical nucleon by absorbing a π^+ or by emitting two successive π^+ . Therefore the matrix elements corresponding to Diagrams (a), (c), (f), and (g) of Fig. 2 vanish.

We are left with two pairs of diagrams, (d) and (e), and (h) and (i). These correspond, respectively, to resonance scattering in the first and in the second intermediate state. In Fig. 2, the open diagrams, (d) and (h), contain the energy denominators corresponding to the resonating intermediate state. The energy controlling the resonance in Diagram (d) (the singularity introduced by the energy denominator) is the total energy of the incident meson, whereas in Diagram (h) it is the total energy of the second outgoing meson. We have investigated Diagram (d) and its crossed counterpart, Diagram (e), and we find that, since $\omega_1 + \omega_2$ is well above the $(3/2, 3/2)$ resonance energy, the contribution of these diagrams to the matrix element is less than 10% of that from Diagrams (h) and (i). We shall consider only the latter two diagrams.

Our amplitude is then

$$\begin{aligned} \langle \beta \, k_1 \, k_2 | A_k | \alpha \rangle &= \left\{ \frac{\langle \beta | A_{k_1}^* | \gamma \rangle \langle \gamma | A_k | \lambda \rangle}{-\omega_1} \right. \\ &+ \frac{\langle \beta | A_{k_1}^* | \omega' \, \gamma \rangle \langle \omega' \, \gamma | A_k | \lambda \rangle}{\omega' - \omega_1 - i\epsilon} \\ &+ \left. \frac{\langle \beta | A_k | \omega' \, \gamma \rangle \langle \omega' \, \gamma | A_{k_1}^* | \lambda \rangle}{\omega' + \omega_1 + \omega_2} \right\} \frac{\langle \lambda | A_{k_2}^* | \alpha \rangle}{\omega_2} \end{aligned} \quad (12)$$

plus the same expression with 1 and 2 interchanged.

In the new notation, $|\alpha\rangle$ ($|\beta\rangle$) represents the initial (final) physical proton (neutron) state; $|\lambda\rangle$ and $|\gamma\rangle$ represent the four states of the physical nucleon, and these are to be summed over; $|\omega\rangle$ represents the nucleon, one-meson states. One sums over the spin and i spin of the nucleon and the i spin of the meson, and one integrates over the meson momentum from zero up to the cutoff value. The delta functions of the momenta of intermediate and initial and final states are taken into account by integrating over intermediate nucleon momenta, the nucleon in the static limit being infinitely heavy, and hence the absorber of any amount of momentum required for conservation at a vertex.

In Eq. (12), we now neglect the ω_2 in the denominator of the last term in the first parentheses and the ω_1 in the denominator of the same term in the second parentheses. Our approximations are made in the terms coming from the "crossed" diagrams ((i) of Fig. 2). These matrix elements contain no poles and will therefore be small compared with the matrix elements coming from the "open" diagrams ((h) of Fig. 2). More explicitly, the approximation is made in a term of the form

$$\int_{\mu}^{\omega_{\max}} p \omega' d\omega' \frac{\langle \lambda | A_{k_1}^* | \omega' \gamma \rangle \langle \omega' \gamma | A_k | \alpha \rangle}{\omega' + \omega_1 + \omega_2}, \quad (13)$$

where

$$p = (\omega'^2 - \mu^2)^{\frac{1}{2}}.$$

Dropping ω_2 in the denominator gives rise to an error in the integrand of the order of $\omega_2/(\omega' + \omega_1 + \omega_2)$, which, at most, is about $2\mu/\mu + 3\mu = \frac{1}{2}$. The error in the integral is less, since the integral diverges linearly, and hence the main contributions come from ω' , several times μ . This can be seen from the following:

$$\begin{aligned} \langle \omega' \gamma | A_k | \alpha \rangle &= - \sum_n \left\{ \frac{\langle \gamma | A_p^* | n \rangle \langle n | A_k | \alpha \rangle}{\omega' - \omega_1 - i\epsilon} \right. \\ &\quad \left. \frac{\langle \gamma | A_k | n \rangle \langle n | A_p^* | \alpha \rangle}{\omega' + \omega_1} \right\} \\ &\sim \frac{1}{\omega'} \left\{ \langle \gamma | A_p^* | n \rangle \langle n | A_k | \alpha \rangle + \langle \gamma | A_k | n \rangle \langle n | A_p^* | \alpha \rangle \right\}, \end{aligned} \quad (14)$$

where

$$A_p^* = -i \sqrt{4\pi} f \vec{\sigma} \cdot \vec{p} \tau_i / \mu \sqrt{2\omega'}$$

Therefore $\langle \omega' \gamma | A_k | \alpha \rangle \propto 1/\sqrt{\omega'}$, and the above integral is proportional to $\omega_{\max}$.

Our amplitude now has the following form:

$$\begin{aligned} \langle \beta_{k_1 k_2} | A_k | \alpha \rangle &= \left\{ \frac{\langle \beta | A_{k_1}^* | \gamma \rangle \langle \gamma | A_k | \lambda \rangle}{-\omega_1} \right. \\ &\quad + \frac{\langle \beta | A_{k_1}^* | \omega' \gamma \rangle \langle \omega' \gamma | A_k | \lambda \rangle}{\omega' - \omega_1 - i\epsilon} \\ &\quad \left. + \frac{\langle \beta | A_k | \omega' \gamma \rangle \langle \omega' \gamma | A_{k_1}^* | \lambda \rangle}{\omega' + \omega_1} \right\} \frac{\langle \lambda | A_{k_2}^* | \alpha \rangle}{\omega_2}. \end{aligned} \quad (15)$$

The term in parentheses is the one-meson approximation for the Chew-Low scattering matrix, $T_k(k_1)$ (in the notation of Chew⁶). It describes the scattering of a π^+ of momentum (energy) $k(\omega)$ into a π^+ of momentum (energy) $k_1(\omega_1)$, with the physical neutron going from state α to state λ .

A similar equation may be derived for the process $\pi^+ + p \rightarrow \pi^+ + \pi^0 + p$. In this case, the matrix element corresponding to Diagram (f) in Fig. 2 does not vanish. We neglect this diagram, since the energy denominators give rise to no poles. There is, in addition, a nonvanishing matrix element for the Born approximation represented by Diagram (c). The contribution from this diagram is given by

$$\frac{\langle \beta | A_k | \delta \rangle \langle \delta | A_{k_2}^* | \lambda \rangle \langle \lambda | A_{k_1}^* | \alpha \rangle}{\omega_1(\omega_1 + \omega_2)} \quad (16)$$

We neglect ω_1 in the quantity $(\omega_1 + \omega_2)$. The part

$$\frac{\langle \beta | A_k | \delta \rangle \langle \delta | A_{k_2}^* | \lambda \rangle}{\omega_2}$$

is then the "crossed" contribution to the Born approximation in the Chew-Low scattering matrix. The error in the Born term is of the order of $\omega_2/(\omega_1 + \omega_2)$, which is quite small compared with the complete contribution to the matrix element from the Born terms plus the one-meson terms. The Born term can be treated exactly; however, we feel that to do so would alter the cross section by less than 10%.

In Eq. (15), the matrix element $\langle \lambda | A_{k_2}^* | \alpha \rangle$ is of the form

$$\langle \text{physical nucleon} \mid f \vec{\sigma} \cdot \vec{k} \tau_i \mid \text{physical nucleon} \rangle .$$

Evaluating this matrix element is equivalent to taking the vertex operator between bare nucleon states and renormalizing the coupling constant, $f \xrightarrow{7,8} f_r$.

The scattering matrix may be expanded in terms of projection operators for the various states of total isotopic spin and angular momentum. We keep only the $I = 3/2$, $J = 3/2$ part. Then

$$T_k(k_1) = \frac{4 \pi}{\sqrt{4 \omega_k \omega_{k_1}}} k k_1 h_3(k_1) P_3 Q_3 . \quad (17)$$

Here P_3 and Q_3 are projection operators for the $J = 3/2$ and $I = 3/2$ states respectively. P_3 is given explicitly by

$$P_3 = \frac{1}{3kk_1} \left\{ 2 \vec{k} \cdot \vec{k}_1 + i \vec{\sigma} \cdot (\vec{k} \times \vec{k}_1) \right\} . \quad (18)$$

The expression for $h_3(k_1)$ in the "effective range approximation" of Chew and Low is given by

$$h_3(k_1) = \frac{\lambda_3}{\omega_{k_1} (1 - \omega_{k_1}/\omega_3) - i \lambda_3 k_1^3} \quad \text{where} \quad \lambda_3 = 4 f_r^2 / 3 \mu^2 . \quad (19)$$

Here, f_r^2 is the renormalized, unrationalized coupling constant; ω_3 is the "resonance" energy. We take $f_r^2 = 0.08$; $\omega_3 = 2.1$.⁶

The matrix element for $\pi^+ + p \rightarrow \pi^+ + \pi^+ + n$ is now

$$T_k(k_1, k_2) = \sum_{\chi} \langle \chi_f | \frac{4\pi}{2\sqrt{\omega} \omega_1} \left\{ \frac{\lambda_3}{\omega_1(1 - \omega_1/\omega_3) - i \lambda_3 k_1^3} \right\} \\ \left\{ 2\vec{k} \cdot \vec{k}_1 + i \vec{\sigma} \cdot (\vec{k} \times \vec{k}_1) \right\} q_3 | \chi \rangle \langle \chi | \frac{-i \sqrt{4\pi} f_r \vec{\sigma} \cdot \vec{k}_2}{\sqrt{2} \mu \omega_2^{3/2}} \sqrt{2} \chi_- | \chi_i \rangle$$

(20)

plus the same expression with subscripts 1 and 2 interchanged.

The matrix element for π^0 production is given by the above formula with the χ_- in the perturbation term changed to $\chi_3/\sqrt{2}$ when the subscripts 1 and 2 are interchanged.

Here $\chi_i(\chi_f)$ is the initial (final) bare proton (nucleon) spinor, spin up or down. The intermediate bare spinors are summed over spin and charge states.

RESULTS AND DISCUSSION

The results for the square of the matrix element averaged and summed over initial and final spin states follow.

Spin flip

$$\left| T_k(k_1 k_2) \right|^2 = \frac{4(4\pi)^3 f_r^6 (k k_1 k_2)^2}{81 \mu^6(\omega_1 \omega_2 \omega)}$$

$$\left[A_o \left\{ 4 \sin^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_2 \cos^2 \theta_1 \right. \right. \\ \left. \left. + \frac{1}{2} \sin 2\theta_1 \sin 2\theta_2 \cos(\phi_2 - \phi_1) \right\} \right. \\ \left. + B_o \left\{ 1 \rightarrow 2, 2 \rightarrow 1 \right\} \right. \\ \left. + C_o \left\{ 2.5 \sin 2\theta_1 \sin 2\theta_2 \cos(\phi_2 - \phi_1) + 4 \sin^2 \theta_2 \cos^2 \theta_1 \right. \right. \\ \left. \left. + 4 \sin^2 \theta_1 \cos^2 \theta_2 \right\} \right] .$$

No spin flip

$$\left| T_k(k_1 k_2) \right|^2 = \frac{4(4\pi)^3 f_r^6 (k k_1 k_2)^2}{81 \mu^6(\omega_1 \omega_2 \omega)}$$

$$\left[(A_o + B_o) \left\{ 4 \cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_1 \sin^2 \theta_2 \right. \right. \\ \left. \left. - \sin 2\theta_1 \sin 2\theta_2 \cos(\phi_1 - \phi_2) \right\} \right. \\ \left. + C_o \left\{ 8 \cos^2 \theta_1 \cos^2 \theta_2 + 2 \sin^2 \theta_1 \sin^2 \theta_2 \cos 2(\phi_1 - \phi_2) \right. \right. \\ \left. \left. - 2 \sin 2\theta_1 \sin 2\theta_2 \cos(\phi_1 - \phi_2) \right\} \right] ,$$

where, for π^+ production,

$$A_0 = \frac{1}{\omega_2^2 \left\{ \omega_1^2 (1 - \omega_1/\omega_3)^2 + \lambda_3^2 k_1^6 \right\}},$$

$$B_0 = \frac{1}{\omega_1^2 \left\{ \omega_2^2 (1 - \omega_2/\omega_3)^2 + \lambda_3^2 k_2^6 \right\}},$$

$$C_0 = \frac{\omega_1 \omega_2 (1 - \omega_1/\omega_3)(1 - \omega_2/\omega_3) + \lambda_3^2 k_1^3 k_2^3}{(\omega_1 \omega_2) \left\{ \omega_1^2 (1 - \omega_1/\omega_3)^2 + \lambda_3^2 k_1^6 \right\} \left\{ \omega_2^2 (1 - \omega_2/\omega_3)^2 + \lambda_3^2 k_2^6 \right\}}.$$

The coefficients for π^0 production are

$$A_0' = 4.5 A_0,$$

$$B_0' = 2 B_0,$$

$$C_0' = 3 C_0.$$

The differential cross section is given by

$$d\sigma = (k/\omega)^{-1} \left| T_k(k_1 k_2) \right|^2$$

$$\frac{2\pi \delta(\omega - \omega_1 - \omega_2) k_1 k_2 \omega_1 \omega_2 d\omega_1 d\omega_2 d\Omega_1 d\Omega_2}{(2\pi)^6}.$$

Spin flip

$$\sigma = \frac{5}{4} (A + B) + 2C ,$$

No spin flip

$$\sigma = A + B + C . \quad (23)$$

For π^+ production,

$$A(B)(C) = Z \int_{\mu}^T d\omega_1 k_1^3 k_2^3 A_0(B_0)(C_0) ,$$

where

$$Z = \frac{0.7 f_r^6 k}{\mu^6} ,$$

$$\omega_2 = \omega - \omega_1 ,$$

$$k_2 = \sqrt{\omega_2^2 - \mu^2} ,$$

$$T = \text{incident meson kinetic energy} = \omega - \mu .$$

For π^0 production replace A_0 by A_0' , etc.

The angular distributions for each meson are given by the following:

No spin flip

$$\frac{d\sigma}{d(\cos \theta)} = a \cos^2 \theta + b , \quad (24)$$

where

$$a = \frac{3}{8} (A + B) + \frac{3}{2} C ,$$

$$b = \frac{3}{8} (A + B) ;$$

Spin flip

$$\frac{d\sigma}{d(\cos \theta)} = g(\cos^2 \theta) + f ,$$

where

$$g = \frac{3}{4} \left\{ \frac{7}{4} A - \frac{1}{2} B + C \right\}$$

$$f = \frac{3}{4} \left\{ \frac{1}{4} A + B + C \right\} .$$

The integrations were performed numerically. At 250-Mev incident-meson kinetic energy, the differential cross section for π^+ production as a function of the polar angle between either outgoing meson and the incident direction is of the form $0.64 + \cos^2 \theta$. At 350 Mev, the angular distribution is of the form $0.70 + \cos^2 \theta$. The dip at 90° is more pronounced than that obtained by Miyachi⁹ for the angular distribution of the π^+ from the process $\pi^- + p \rightarrow \pi^+ + \pi^- + n$ at 210 Mev. The latter calculation was done by using covariant perturbation theory with pseudovector coupling.

The total cross sections for the two processes are listed below.

Incident-meson kinetic energy (Mev)	Energy above threshold (c.m.) (Mev)	$\sigma(\pi^+ \pi^+)$ (mb)	$\sigma(\pi^+ \pi^0)$ (mb)	$\sigma(\pi^+ \pi^+) \sigma(\pi^+ \pi^0)$ (mb)
250	60	0.071	0.44	0.51
350	130	0.86	5.5	6.4

At 350 Mev, the cross section for production of an additional meson is about 12% of the total cross section.¹⁰ The sharp increase in

cross section at 350 Mev is due to the resonance behavior of the scattering part of the matrix element when one or both of the mesons may come out with kinetic energy approaching the resonance kinetic energy. At 500-Mev incident-meson kinetic energy, the production cross section becomes of the order of the total cross section of 20 mb, indicating the breakdown of this approximation.

Comparison with definite experimental results is not yet possible. However, Blau and Caulton¹¹ have observed inelastic collisions of 500-Mev π^- in emulsions. They have estimated the cross section for the production of an additional charged meson, averaged over $\pi^- - p$ and $\pi^- - n$ collisions, to be between 3.5 and 10 mb. We have taken the value of 3.5 mb and have deduced, on the basis of the isobar model, a lower limit on the production cross section in $\pi^+ - p$ collisions of about 6 mb. Yuan and Lindenbaum¹⁰ have recently indicated, on the basis of their total-cross-section measurements, that meson production in $\pi^+ - p$ collisions is likely to become important above 300 Mev.

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II. ISOBAR MODEL FOR MESON PRODUCTION IN PROTON-PROTON COLLISIONS

INTRODUCTION

Recent experiments on meson production in nucleon-nucleon collisions have thrown some light on the usefulness of the $J = 3/2$, $T = 3/2$ isobar as an intermediate state in high energy processes involving one or more nucleons. At the cosmotron, Yuan and Lindenbaum¹ have observed the energy spectra of positive and negative pions produced in p - Be and p - p collisions at 1 Bev and at 2.3 Bev. At the higher energy, double production appears to predominate. At both 1 and 2.3 Bev, a plot of the relative meson-production cross sections per Mev per unit solid angle versus meson kinetic energy in the nucleon-nucleon center-of-mass system exhibit strong peaks between 100 and 150 Mev. In the case of π^+ production, these peaks are surprisingly similar in shape to the peak that appears in the $\pi^+ - p$ total-interaction cross section when plotted versus meson kinetic energy in the $\pi^+ - p$ center-of-mass system.² At 1 Bev the curve is shifted somewhat toward the lower energies. At 2.3 Bev the curve is considerably broadened as compared to the $\pi^+ - p$ curve. The peaks in the negative pion spectra are markedly similar to that which appears in a plot of the $\pi^- - p$ interaction cross section versus meson kinetic energy. The π^- spectra also show the effects described above, at 1 and 2.3 Bev respectively.

These facts may be explained in a qualitative manner as follows. In the collision of the two nucleons, a mechanism whose detailed nature is probably quite complicated operates to form an intermediate state in which either one or both of the nucleons has been raised to

the $(3/2, 3/2)$ isobaric state. If the initial nucleons have momenta $\pm \vec{p}$ in the center-of-mass system and if the excitation process involves a transfer of momentum $\vec{g}$ between the particles, the particles in the intermediate state have momenta $\pm(\vec{p} - \vec{g})$. It is then supposed that these particles separate from the region of strong interaction, and one or both decay from the isobaric state by emission of a pion. The decay of one isobar is to be thought of as being independent of the presence of another isobar or nucleon. This is an abstraction from reality in light of the fact that, from the width of the $\pi^+ - p$ scattering peak, the isobar lifetime appears to be $\sim 10^{-23}$ seconds. It would seem that the intermediate-state particles just have time to separate from a region of radius $\sim 10^{-13}$ cm if they are moving with considerable relative velocity. The excitation may, however, occur near the edge of the region of strong interaction, with not too great a momentum transfer. This would make the separation and independent decay more plausible. The usefulness of this model of the intermediate state rests ultimately on the comparison of its predictions with experiment. The present strong resemblance between the meson-energy spectra and the $\pi - p$ interaction cross sections in the $(3/2, 3/2)$ state as a function of energy speaks well for investigating the predictions of this model. A further reason for looking at the predictive powers of this model is the following. The isobar decay into a meson of energy E_π and a nucleon of energy E_N will be characterized by a function that is essentially the square root of the $\pi - p$ scattering cross section in the $(3/2, 3/2)$ state evaluated at the energy $E_\pi + E_N$. A theoretical expression for this

function has been made available by the work of Chew and Low on the pion-nucleon scattering problem.³ For meson production through the isobaric state, we will use this function evaluated at just those energies in the resonance region where the Chew-Low theory has proved most effective in describing the scattering.

We discuss now briefly the most familiar of the meson-production reactions in nucleon-nucleon collisions at the lower energies, for further evidence of the influence of the strong $(3/2, 3/2)$ -state interaction.⁴ These reactions are $p + p \rightarrow \pi^+ + D$, and $p + p \rightarrow n + p + \pi^+$. The meson comes out predominately in p-states with an angular distribution of $(0.1 \text{ to } 0.3) + \cos^2 \theta$ at energies from 311 to 437 Mev. This is in qualitative agreement with the idea of the reaction proceeding through an intermediate state containing a $(3/2, 3/2)$ isobar and a nucleon. The ratio of S- to P-wave meson emission is about 1/7 in agreement with an estimate of the S-wave emission as a recoil effect. Further evidence for the influence of this intermediate state in single-meson production phenomena at the lower energies is supplied by the ratio of P-wave meson production in reactions in which the isotopic spin of the nucleons changes from 1 to 0, to that in reactions in which the isotopic spin changes from 0 to 1. This ratio is about 0.3. These facts on the low-energy single-production phenomena cannot be interpreted with the aid of a statistical model such as may be applied in the Bev region. Geffen has given a phenomenological treatment of the above two processes to show that the observations may be interpreted without invoking the strong $(3/2, 3/2)$ -state interactions if proper attention is paid to the influence of the nuclear force on the initial and final two-

nucleon wave functions.⁵ However, in view of the evidence in the Bev range for the strong role of the $(3/2, 3/2)$ state--such as the considerably larger amount of double meson production than is predicted by statistical considerations, even when modified for the nuclear-force effects--we prefer to take the above facts as further evidence for the usefulness of this intermediate-state approach, and to investigate the predictions of the model in some further detail.

In Section A under the Calculations, we derive the charge ratios in single and double production on the basis of the model. These were derived originally by Peaslee⁶ and are presented here for clarity in the ensuing discussion. In Section B, we define and briefly derive the fundamental amplitudes to be used in calculating the transition probabilities for meson production. These amplitudes were defined by Austern⁷ and were utilized by him in his study of the photodisintegration of the deuteron on the basis of the isobar model. The reader is referred to his paper for a lucid discussion of the model. In Section C, we derive the cross section for the process $p + p \rightarrow n + p + \pi^+$. The energy spectrum of the emitted pion is of particular interest for comparison with experiment. In Section D, we derive the cross sections for double-pion production and present the energy spectrum of the pions. An estimate of the double-to-single ratio as a function of bombarding energy is given. In Section E, we discuss the modification of the angular distributions of the two pions that would occur if a hypothetical meson-meson interaction were present. The predictions of the model are compared in the Discussion with the recent experiments on meson production in $p - p$ collisions.

CALCULATIONS

A. Charge Ratios in Single and Double Production

The initial state containing two protons has total isotopic spin $I = 1$ and total z -component $I_z = 1$. We expand the state $|I, I_z\rangle = |1, 1\rangle$ in terms of a product of states describing a $I = 3/2$ isobar and a $I = 1/2$ nucleon.

$$\begin{aligned} |1, 1\rangle &= \langle 3/2, 1/2, 3/2, -1/2 | 1, 1\rangle |3/2, 3/2\rangle |1/2, -1/2\rangle \\ &+ \langle 3/2, 1/2, 1/2, 1/2 | 1, 1\rangle |3/2, 1/2\rangle |1/2, 1/2\rangle \\ &= \sqrt{3/4} |3/2, 3/2\rangle |1/2, -1/2\rangle + \sqrt{1/4} |3/2, 1/2\rangle |1/2, 1/2\rangle. \end{aligned} \quad (1)$$

The expansion coefficients are the usual Clebsh-Gordan coefficients in the notation $\langle j_1, j_2, m_1, m_2 | J, m\rangle$. The $I = 3/2$ state is expanded in terms of a product of states describing a $I = 1$ pion and a $I = 1/2$ nucleon.

$$\begin{aligned} |3/2, 3/2\rangle &= \langle 1, 1/2, 1, 1/2 | 3/2, 3/2\rangle |1, 1\rangle |1/2, 1/2\rangle = |\pi^+ p\rangle \\ |3/2, 1/2\rangle &= \langle 1, 1/2, 0, 1/2 | 3/2, 1/2\rangle |1, 0\rangle |1/2, 1/2\rangle \\ &+ \langle 1, 1/2, 1, -1/2 | 3/2, 1/2\rangle |1, 1\rangle |1/2, -1/2\rangle \\ &= \sqrt{2/3} |\pi^0 p\rangle + \sqrt{1/3} |\pi^+ n\rangle \end{aligned} \quad (2)$$

The expansion of the state $|1, 1\rangle$ is then

$$|1, 1\rangle = \sqrt{3/4} |\pi^+ p; n\rangle + \sqrt{1/4} \left\{ \sqrt{2/3} |\pi^0 p; p\rangle + \sqrt{1/3} |\pi^+ n; p\rangle \right\} \quad (3)$$

In Equation (3), a semicolon stands between the pion-nucleon pair resulting from the isobar decay and the second nucleon. The differential cross section is given in terms of differential cross sections for the separate processes by

$$d\sigma = 3/4 d\sigma(\pi^+p;n) + \frac{1}{12} d\sigma(\pi^+n;p) + \frac{2}{12} d\sigma(\pi^0p;p). \quad (4)$$

The total cross section is then

$$\sigma = \frac{9}{12} \sigma(\pi^+p;n) + \frac{1}{12} \sigma(\pi^+n;p) + \frac{2}{12} \sigma(\pi^0p;p) \quad (5)$$

and the π^+/π^0 ratio is 5. It will be noted later that this ratio may be lowered by the suppression of π^0 production due to the Pauli principle. It is important to note that the total cross sections $\sigma(\pi^+p;n)$ and $\sigma(\pi^+n;p)$ represent physically distinguishable processes, in that the $\pi^+ - p$ pair and the $\pi^+ - n$ pair respectively, will show the characteristic Q-value of the isobar decay.

By expanding the state $|1,1\rangle$ in terms of a product of two $I = 3/2$ isobar states and then decomposing the latter into pion-nucleon pairs, we obtain for the total cross section for double production

$$\sigma = \frac{1}{45} \left\{ 18\sigma(\pi^+p; \pi^0n) + 8\sigma(\pi^0p; \pi^+n) + 9\sigma(\pi^+p; \pi^-n) \right. \\ \left. + 8\sigma(\pi^0p; \pi^0p) + 2\sigma(\pi^+n; \pi^+n) \right\}. \quad (6)$$

This gives for total π^+ , π^0 , π^- production the relative weights 13, 14, 3 respectively. A prediction of this model, as seen from Eq. (6), is that in the dominant double production process $p + p \rightarrow \pi^+ + \pi^0 + p + n$, the π^+ should be correlated by Q-value to the proton 9/4 of the time.

B. Fundamental Amplitudes for Calculating Transition Probabilities

Following Austern's method we now define an amplitude $\mathcal{T}$ in the following manner. Consider the scattering of positive P-wave pions by protons, at a total energy E . We assume that the scattering proceeds through an intermediate state that involves formation of the $(3/2, 3/2)$ isobar with energy $E_0 = m + \mu + 0.16 = 1.24$ Bev, where m and μ are the nucleon and meson rest masses respectively. The P-wave part of the incident meson plane wave of momentum $\vec{k}$ may be written as

$$\sum_{m=-1}^{m=1} Y_{1,m}^* (\vec{k} \cdot \vec{z}) Y_{1,m} (\vec{r} \cdot \vec{z}) \quad (7)$$

where $\vec{z}$ is the quantization axis and the $Y_{1,m}$ are normalized spherical harmonics. The $J = 3/2$ part of the product of the above expression with the initial proton spinor with spin z-component s , N^s , is given by

$$\sum_{m=-1}^{m=1} Y_{1,m}^* (\vec{k} \cdot \vec{z}) \langle 1, 1/2, m, s \mid 3/2, m+s \rangle \mid 3/2, m+s \rangle . \quad (8)$$

The definition⁷ of $\mathcal{T}$ is then achieved by stating that a positive-meson plane wave of unit amplitude incident upon a proton forms the $J = 3/2$ isobar with spin z-component $\sigma = m + s$, with the amplitude

$$\mathcal{T}(E) Y_{1,m}^* (\vec{k} \cdot \vec{z}) \langle 1, 1/2, m, s \mid 3/2, m+s \rangle . \quad (9)$$

If the initial pion-nucleon state is not pure $I = 3/2$, say $\pi^+ - n$, then an additional Clebsh-Gordon coefficient appears in the above amplitude, i.e. $\langle 1, 1/2, 1, -1/2 \mid 3/2, 1/2 \rangle$, representing that part of the state that couples into the $I = 3/2$ isobar. Taking the direction $\vec{z}$ as $\vec{k}/|\vec{k}|$, we see the isobar is formed with amplitude

$$\pi(E) Y_{1,0}^* (1) \langle 1, 1/2, 0, s \mid 3/2, s \rangle \quad (10)$$

It then decays into a π^+ of momentum $\vec{k}'$ and a proton of spin z-component s' with an amplitude

$$\sum_{\substack{m'=-1 \\ m'+s'=s}}^{m'=1} \pi(E) Y_{1,m'}^* (\vec{k} \cdot \vec{k}') \langle 1, 1/2, m', s' \mid 3/2, m' + s' \rangle \quad (11)$$

The matrix element for the scattering process is then

$$M = \pi^2(E) Y_{1,0}^* (1) Y_{1,s-s'}^* (\vec{k} \cdot \vec{k}') \langle 1, 1/2, 0, s \mid 3/2, s \rangle \\ \times \langle 1, 1/2, s-s', s' \mid 3/2, s \rangle D^{-1} \quad (12)$$

where D^{-1} is the energy denominator given by

$$D = E - E_0 - i\Gamma/2 \quad (13)$$

As we are doing a phenomenological treatment, we go beyond ordinary perturbation theory and introduce an imaginary part to the isobar energy because of its decay. This is given by⁴

$$\Gamma = \frac{1.36 (k/\mu)^3 \cdot 58 \text{ Mev}}{1 + 0.77 (k/\mu)^2} \quad (14)$$

where k is the meson momentum, $\sqrt{(E - m)^2 - \mu^2}$. By squaring M and summing over final and averaging over initial spin states, we arrive at the scattering cross section in terms of π

$$d\sigma_E(\pi^+ + p \rightarrow \pi^+ + p) = (8\pi)^{-1} \pi^4 \frac{\rho_E(\pi N)}{V_E(\pi)} D^{-2} (3 \cos^2 \frac{\theta}{2} + 1) d\Omega / 4\pi \quad (15)$$

From this, we solve for $\mathcal{M}$ as function of the total cross section

$$\mathcal{M}^4(E) = 4\pi \frac{V_E(\pi)}{\rho_E(\pi N)} \left| D \right|^2 \sigma_E(\pi^+ + p \rightarrow \pi^+ + p). \quad (16)$$

The quantity $V_E(\pi)$ is the incident pion velocity $= k/\omega$ where $\omega = E - m$. $\rho_E(\pi N)$ is the density of final states $= \frac{2}{\pi} k \omega$. We now have one of our fundamental amplitudes evaluated in terms of a cross section. In Austern's work, use was made of this amplitude with the experimental cross section on the right-hand side. In order to perform certain phase-space integrations when investigating meson-production phenomena, we will need a theoretical expression for $\sigma_E(\pi^+ - p)$. For this we will use the form given by the one-meson approximation to the Chew-Low theory³

$$\sigma(E) = 8 \lambda_3^2 k^4 \left\{ \frac{1}{\omega^2 (1 - \omega/\omega_3)^2 + \lambda_3^2 k^6} \right\}, \quad (17)$$

where ω is the meson energy $= \sqrt{k^2 + \mu^2} = E - m$,

$\lambda_3^2 = (16/9)(f^4/\mu^4) = 29.5$ for the pseudovector coupling constant $f^2 = 0.08$, and $\omega_3 = 0.3$ Bev.

We now need to evaluate one more basic amplitude. We recapitulate Austern's argument in brief. Consider the mesonic disintegration of the deuteron, $\pi^+ + D \rightarrow 2p$, at a total energy E . We consider the process to go through an intermediate state involving one isobar and one nucleon. We neglect the energy of relative motion of the particles in the intermediate state, and we consider them to be in an S state. It is then readily seen that, in order to conserve angular momentum

and parity, the final-state protons must be in a 1D_2 state. What we want to do is to relate the amplitude for the transition of an isobar-nucleon system in an S state into a two-nucleon system in the 1D_2 state to the cross section for mesonic disintegration of the deuteron. The $J_z = 1$ spin, isotopic-spin wave function for the deuteron is given by

$$2^{-\frac{1}{2}}(N_1^{\frac{1}{2},\frac{1}{2}} N_2^{\frac{1}{2},-\frac{1}{2}} - N_1^{\frac{1}{2},-\frac{1}{2}} N_2^{\frac{1}{2},\frac{1}{2}}) \quad (18)$$

where the first superscript refers to the z-component of ordinary spin and the second superscript to the z component of isotopic spin. The incident π^+ may now be absorbed by nucleon 1 to form the isobar $X^{\sigma\rho}$ where σ is the z component of spin and ρ the z component of isotopic spin. The intermediate state is then

$$\begin{aligned} & 2^{-\frac{1}{2}} \pi(E) Y_{1,0}(1) \langle 1, 1/2, 0, 1/2 \mid 3/2, 1/2 \rangle \\ & \cdot (X^{\frac{1}{2}, 3/2} N_2^{\frac{1}{2}, -\frac{1}{2}} - \langle 1, 1/2, 1, -1/2 \mid 3/2, 1/2 \rangle X^{\frac{1}{2}, \frac{1}{2}} N_2^{\frac{1}{2}, \frac{1}{2}}) \\ & = \pi(E) (4\pi)^{-\frac{1}{2}} (X^{\frac{1}{2}, 3/2} N_2^{\frac{1}{2}, -\frac{1}{2}} - \sqrt{1/3} X^{\frac{1}{2}, \frac{1}{2}} N_2^{\frac{1}{2}, \frac{1}{2}}) \quad (19) \end{aligned}$$

We now take the $I = 1$, $I_z = 1$ part of this state in isotopic-spin space and the $J = 2$, $J_z = 1$ part in ordinary-spin space. The result is

$$\pi(E) (4\pi)^{-\frac{1}{2}} \mid 2, 1 \rangle \mid 1, 1 \rangle \quad (20)$$

Now the D part of the two-proton plane wave of momentum $\vec{p}$ is

$$\sum_{m=-2}^{m=2} Y_{2,m}^* (\vec{p} \cdot \vec{z}) Y_{2,m} (\vec{r} \cdot \vec{z}) \quad (21)$$

We define the amplitude⁷ T_2 by stating that the two-nucleon plane wave of unit amplitude forms the $J = 2$, $J_z = m$ part of the isobar-nucleon state with amplitude

$$T_2(E) Y_{2,m}^* (\vec{p} \cdot \vec{z}) . \quad (22)$$

With $\vec{z}$ taken as the direction of the incident-meson momentum $\vec{k}/|\vec{k}|$, we get the matrix element for $\pi^+ + D \rightarrow 2p$ from the $J_z = 1$ state of the deuteron given by

$$M_1 = D^{-1}(E - m) \mathcal{T}(E - m) (4\pi)^{-\frac{1}{2}} T_2(E) Y_{2,1}^* (\vec{k} \cdot \vec{p}) . \quad (23)$$

Of course nucleon 2 may absorb the π^+ , and this results in a factor of 2 multiplying the above matrix element. The matrix element for the transition from the $J_z = -1$ state of the deuteron is the same as Eq. (23). The matrix element M_0 for the transition from the $J_z = 0$ state given by

$$2^{-1}(N_1^{\frac{1}{2}, \frac{1}{2}} N_2^{-\frac{1}{2}, -\frac{1}{2}} + N_1^{-\frac{1}{2}, \frac{1}{2}} N_2^{\frac{1}{2}, -\frac{1}{2}} - N_1^{\frac{1}{2}, -\frac{1}{2}} N_2^{-\frac{1}{2}, \frac{1}{2}} - N_1^{-\frac{1}{2}, -\frac{1}{2}} N_2^{\frac{1}{2}, \frac{1}{2}}) \quad (24)$$

may be calculated in the same manner. Adding $2/3$ the square of M_1 to $1/3$ the square of M_0 , multiplying by suitable factors to form a cross section, and integrating over the solid angle of $\vec{p}$, we obtain

$$\sigma_E(\pi^+ + D \rightarrow 2p) = \frac{5}{9\pi} \rho_E(2N)/V_E(\pi) |D|^{-2} \mathcal{T}^2(E - m) T_2^2(E) \quad (25)$$

where $V_E(\pi) = k/\omega$ with $\omega = \sqrt{k^2 + \mu^2} = E - 2m$

$$\rho_E(2N) = (2/\pi)p \frac{E}{2} \text{ with } p = \sqrt{\frac{E^2}{4} - m^2} .$$

We solve this for $T_2(E)$ using Eq. (16) for $\mathcal{M}(E)$

$$\left| T_2(E) \right|^2 = \frac{9\pi}{10\sqrt{2}} \frac{k}{p^2} \left\{ (E - m - E_0)^2 + \frac{\Gamma^2}{4} \right\}^{\frac{1}{2}} \frac{\sigma_E(\pi^+ + D \rightarrow 2p)}{\sigma_{(E-m)}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p)} \quad (26)$$

$$= \frac{9\pi}{10\sqrt{2}} \frac{4}{3k} \left\{ (E - m - E_0)^2 + \frac{\Gamma^2}{4} \right\}^{\frac{1}{2}} \frac{\sigma_E(p + p \rightarrow \pi^+ + D)}{\sigma_{(E-m)}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p)}$$

where we have used the relation from detailed balancing

$$\sigma_E(\pi^+ + D \rightarrow 2p) = \frac{4}{3} \frac{p^2}{k^2} \sigma_E(p + p \rightarrow \pi^+ + D) . \quad (27)$$

C. Cross-Section Derivation for $p + p \rightarrow \pi^+ + p + n$.

We are now in a position to apply the model to calculate some single and double production processes involving unbound nucleons in the final state. We consider first $p + p \rightarrow \pi^+ + n + p$. The transition is to the 3S_1 state of the final nucleons, because they cannot emerge in a 1S_0 state with a P-wave meson and yet conserve angular momentum and parity. The calculation proceeds in a manner similar to that above. We construct the final n-p states of $S = 1, S_z = \pm 1, 0; I = 0$, and $I_z = 0$, and from these we obtain the intermediate state of isobar and nucleon by coupling the pion to either the proton or the neutron. We then extract the $J = 2, J_z = 0; I = 1$, and $I_z = 1$ part of this state, and project it upon the initial two-proton state with an amplitude T_2' , as only the 1D_2 part of the initial state conserves angular momentum and parity with the intermediate state. The quantity

$|T_2'|^2 = |T_2|^2 |\psi_D(r=0)|^{-2}$ where $\psi_D(r=0)$ is the deuteron space function at $r = 0$. (See Appendix B.) In computing the cross section for this process, we add the cross sections for the separate processes in which the pion is coupled to the proton and to the neutron to form the isobar, instead of adding the amplitudes as in Section B, remembering that these two final states are distinguishable by Q-values measurements. The results for the two cross sections are

$$d\sigma_E(p + p \rightarrow \pi^+ + p + n) = a |T_2'(E)|^2 |\pi(E-m)|^2 \cdot (3 \cos^2 \theta + 1) |D|^{-2} \rho_E(2N\pi)/V_E(\pi) \frac{d\Omega}{4\pi} \quad (28)$$

where $a = 1/8$ when the $\pi^+ - p$ result from the isobar and $a = 1/72$ when the $\pi^+ - n$ result from the isobar. Before using this result, we would like to modify it to take into account somewhat the final state n-p interaction. In Appendix A, it is shown that consideration of the relative motion of the two nucleons in the final state approximately modifies the above matrix element by the multiplicative factor

$$e^{i\delta} \frac{\sin \delta}{q} \int d^3\vec{r} f(r) = \text{constant} \cdot e^{i\delta} \frac{\sin \delta}{q} \quad (29)$$

where δ is the n-p scattering phase shift in the triplet state, q is the magnitude of the relative momentum, and $f(r)$ is a function of the magnitudes of the relative coordinate between the nucleons. We approximate δ by $q \cot \delta = \alpha$ where $\alpha^{-1} = 5.39 \times 10^{-13}$ cm is the triplet-scattering length. The density of final states $\rho_E(2N\pi)$ is given by

$$(2/\pi) k \omega (2\pi)^{-3} 2\pi m^{3/2} \mathcal{E}^{\frac{1}{2}} d\mathcal{E} \quad (30)$$

where $\mathcal{E} = q^2/m$ is the relative energy of the nucleons. Inserting the expressions for T_2 , π , ρ_E into Eq. (28) appropriately modified for the final-state interaction, we obtain

$$\frac{d\sigma_E(p + p \rightarrow n + p + \pi^+)}{\sigma_E(p + p \rightarrow \pi^+ + D)} \propto$$

Eq. (31) cont.

$$\begin{aligned}
 & a \left| \psi_D(r=0) \right|^{-2} \frac{\left\{ (E - 2m - \mathcal{E})^2 - \mu^2 \right\}^{\frac{1}{2}} \left\{ (E - m - \frac{\mathcal{E}}{2} - E_0)^2 + \frac{\Gamma^2}{4} \right\}^{\frac{1}{2}}}{\left\{ (E - 2m)^2 - \mu^2 \right\}^{\frac{1}{2}} \left\{ (E - m - E_0)^2 + \frac{\Gamma^2}{4} \right\}^{\frac{1}{2}}} \\
 & \times \frac{m^{3/2} \mathcal{E}^{\frac{1}{2}} \sigma_{E_1}^{\frac{1}{2}} (\pi^+ + p \rightarrow \pi^+ + p) d\mathcal{E}}{\sigma_{(E-m)}^{\frac{1}{2}} (\pi^+ + p \rightarrow \pi^+ + p) \left\{ \mathcal{E}^2 + m\mathcal{E} \right\}}
 \end{aligned} \tag{31}$$

The proportionality sign indicates that this is the ratio to within a constant factor involving $\int d^3\vec{r} f(r)$. If we call the ratio of $\sigma_E(p + p \rightarrow \pi^+ + n + p)$ to $\sigma_E(p + p \rightarrow \pi^+ + D)$, r , then Eq. (31) determines $r(E)$. We will normalize $r(E)$ to its experimental value at 500 Mev and then compare it with the experiments at higher energies. The quantity $\sigma_{E_1}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p)$ arises from the vertex at which the isobar decays. The energy E_1 is the sum of the pion and nucleon rest masses plus the sum of their kinetic energies. If we denote the pion kinetic energy by t , then $E_1 = \mu + m + t + \frac{\mathcal{E}}{2}$, and t is given by energy conservation, neglecting the recoil of the center-of-mass of the two-nucleon system, $t = E - 2m - \mu - \mathcal{E}$. In the expression for E_1 we have set the nucleon kinetic energy equal to one-half the energy of relative motion of the two nucleons, $\mathcal{E}/2$. This is not exact, but is reasonable, since the massive nucleons tend to account for most of the conservation of momentum, and hence tend to move off in opposite directions with about equal momenta. We now use Eq. (17) giving $\sigma_{E_1}^{\frac{1}{2}}$ as a function of meson energy

$$\omega_1 = E_1 - m = E - 2m - \frac{\mathcal{E}}{2} :$$

$$\sigma_{E_1}^{\frac{1}{2}} (\pi^+ + p \rightarrow \pi^+ + p) = \left\{ (E - m - \frac{\mathcal{E}}{2})^2 - \mu^2 \right\} \cdot \left\{ \frac{8 \lambda_3^2}{\omega_3 (E - 2m - \frac{\mathcal{E}}{2})^2 (1 - \frac{(E - 2m - \mathcal{E}/2)}{\omega_3})^2 + \lambda_3^2 \left\{ (E - 2m - \frac{\mathcal{E}}{2})^2 - \mu^2 \right\}^3} \right\}^{\frac{1}{2}} \quad (32)$$

Inserting this into Eq. (31) and putting $\mathcal{E} = E - 2m - \mu - t$, we obtain the energy spectrum of the produced pion for a fixed value of E . The ratio of the cross sections as a function of E is obtained by integrating over $\mathcal{E}$ for $0 \leq \mathcal{E} \leq E - 2m - \mu$.

D. Cross-Section Derivation for $p + p \rightarrow \pi^+ + \pi^+ + n + n$.

Turning our attention to the double-pion production processes, we will illustrate the method of calculation for the process $p + p \rightarrow \pi^+ + \pi^+ + n + n$. We consider only S states of relative motion for the final two nucleons. The final state of the $2n$ is then the 1S_0 . The production process is now considered to go through an intermediate state involving two $J = 3/2$, $I = 3/2$ isobars. We neglect the energy of motion of the isobars and we consider them to be in an S state. We have two identical fermions in the intermediate state with total $I = 1$, total $I_z = 1$. These are in a symmetric space state and because the total state must be antisymmetric, we see that the total angular momentum of the intermediate state must be $J = 2$ or $J = 0$. Hence the 1S_0 and 1D_2 parts of the incident two-proton state couple into the two-isobar state, with amplitudes that we shall call A_0 and A_2 respectively. In Appendix B it is shown that in the approximation in which we neglect the motion of the intermediate state isobars, the amplitude A_2 is related to the amplitude T_2' derived earlier, by

$$|A_2(E)|^2 \sim \left| \psi_{2X}^{(R)} / \psi_{X,N}^{(R)} \right|^2 |T_2'|^2 = \beta |T_2'(E)|^2 \quad (33)$$

where $\psi_{2X}^{(R)}$ is the two-isobar wave function evaluated at some relative coordinate R characteristic of the interaction region for meson production, $\psi_{X,N}^{(R)}$ is the isobar-nucleon wave function, and the factor β is energy independent. An approximation such as this is of very limited validity. It will be used only to get an idea of the behavior of the two-meson excitation function from 0.8 to

about 1.3 Bev. We are not able to relate A_0 to a simpler reaction, so we leave it as an undetermined parameter. However, the amplitudes A_0 and A_2 bear the following relationships to the S-matrix elements for the corresponding transitions, S_0 and S_2

$$A_0 = S_0, \quad A_2 = (i)^2 \sqrt{5} S_2. \quad (34)$$

$|A_2|^2$ carries a statistical factor of 5 relative to $|A_0|^2$.

The process to be calculated may be represented by the Feynman diagrams in Fig. 1. The energy denominators that will enter into the matrix elements may be read off from the diagrams. The first intermediate state contributes $D^{-1} = (E - 2E_0 - i\Gamma_0)^{-1}$. The second intermediate state contributes $D_{1,2}^{-1} = (E - E_0 - \mathcal{E}/2 - t_{1,2} - \frac{i\Gamma_0}{2})^{-1}$. Here the subscripts 1 and 2 refer to Diagrams a and b and Diagrams c and d respectively. In the former case, the meson $\vec{k}_1$ is emitted first; in the latter case, the meson $\vec{k}_2$ is emitted first. The $t_{1,2}$ refer to the meson kinetic energies. The kinetic energy of the nucleon in the second intermediate state is taken as $\mathcal{E}/2$ where $\mathcal{E}$ is the energy of relative motion of the final two nucleons. The width Γ_0 is given by Eq. (14) with $k = \left\{ [(E - 2m)^2/4] - \mu^2 \right\}^{\frac{1}{2}}$. We must now evaluate the remainder of the matrix element. The wave function for the 1S_0 state of the nucleons and the two P-wave mesons is

$$\begin{aligned} \psi = & 2^{-\frac{1}{2}} (N_1^{\frac{1}{2}, -\frac{1}{2}} N_2^{-\frac{1}{2}, -\frac{1}{2}} - N_1^{-\frac{1}{2}, -\frac{1}{2}} N_2^{\frac{1}{2}, -\frac{1}{2}}) \\ & \cdot \sum_{m, m'} Y_{1, m}^* (\vec{k}_1 \cdot \vec{z}) Y_{1, m'}^* (\vec{k}_2 \cdot \vec{z}) Y_{1, m} (\vec{r}_1 \cdot \vec{z}) Y_{1, m'} (\vec{r}_2 \cdot \vec{z}). \end{aligned} \quad (35)$$

We combine meson $\vec{k}_1$ with nucleon 1 in the $J = T = 3/2$ state, and meson $\vec{k}_2$ with nucleon 2 in this state, with amplitudes $\Pi(E_1)$ and $\Pi(E_2)$ respectively. To take into account Diagrams a and b of Fig. 1 (for a particular time ordering) we also combine meson $\vec{k}_1$ with nucleon 2 and meson $\vec{k}_2$ with nucleon 1. We then take the $J = 2$ and 0, $I = 1, I_z = 1$ parts of this two-isobar state and project them on the initial two-proton state with amplitudes A_2 and A_0 , respectively. The matrix element is then

$$\begin{aligned}
 M(m, m') = & \left[\Pi(E_1) \Pi(E_2) A_2 Y_{1,m}^* (\vec{k}_1 \cdot \vec{z}) Y_{1,m}^* (\vec{k}_2 \cdot \vec{z}) D^{-1} D_1^{-1} \right. \\
 & \cdot 2^{-\frac{1}{2}} \left\{ \langle 1, \frac{1}{2}, m, \frac{1}{2} \mid 3/2, m + \frac{1}{2} \rangle \langle 1, \frac{1}{2}, m', -\frac{1}{2} \mid 3/2, m' - \frac{1}{2} \rangle \right. \\
 & \quad \langle 3/2, 3/2, m + \frac{1}{2}, m' - \frac{1}{2} \mid 2, 0 \rangle + \langle 1, \frac{1}{2}, m', \frac{1}{2} \mid 3/2, m' + \frac{1}{2} \rangle \\
 & \quad \langle 1, \frac{1}{2}, m, -\frac{1}{2} \mid 3/2, m - \frac{1}{2} \rangle \langle 3/2, 3/2, m' + \frac{1}{2}, m - \frac{1}{2} \mid 2, 0 \rangle \\
 & \quad - \langle 1, \frac{1}{2}, m, -\frac{1}{2} \mid 3/2, m - \frac{1}{2} \rangle \langle 1, \frac{1}{2}, m', \frac{1}{2} \mid 3/2, m' + \frac{1}{2} \rangle \\
 & \quad \langle 3/2, 3/2, m - \frac{1}{2}, m' + \frac{1}{2} \mid 2, 0 \rangle - \langle 1, \frac{1}{2}, m', -\frac{1}{2} \mid 3/2, m' - \frac{1}{2} \rangle \\
 & \quad \left. \left. \langle 1, \frac{1}{2}, m, \frac{1}{2} \mid 3/2, m + \frac{1}{2} \rangle \langle 3/2, 3/2, m' - \frac{1}{2}, m + \frac{1}{2} \mid 2, 0 \rangle \right\} \right. \\
 & \cdot \left\{ \langle 1, \frac{1}{2}, 1, -\frac{1}{2} \mid \frac{3}{2}, \frac{1}{2} \rangle^2 \langle 3/2, 3/2, \frac{1}{2}, \frac{1}{2} \mid 1, 1 \rangle \right\} \\
 & + \left[\text{same expression with } D_1^{-1} \rightarrow D_2^{-1} \right] \\
 & + \left[\text{same expression with } A_2 \rightarrow A_0 \text{ and } J = 2 \rightarrow J = 0 \text{ in} \right. \\
 & \quad \left. \text{the Clebsh-Gordan coefficients} \right]
 \end{aligned}
 \tag{36}$$

In the line written out explicitly the first bracket contains the angular-momentum vector addition and the second bracket contains the isotopic-spin vector addition. It should be noted that the antisymmetry of the angular-momentum bracket with respect to the interchanges $m \leftrightarrow m'$, $s = \frac{1}{2} \leftrightarrow s' = -\frac{1}{2}$, which is equivalent to interchanging the spin z components of the two isobars. The total transition amplitude is obtained by summing the expression (36) over the spin z components, σ and σ' , of each intermediate state isobar, subject to $\sigma + \sigma' = J_z = 0$. This is equivalent to summing over m and m' in Eq. (36), subject to $m + m' = 0$. The result for the transition amplitude is

$$\begin{aligned}
 M = & \pi(E_1) \pi(E_2) A_2(E) D^{-1} \left\{ D_1^{-1} + D_2^{-1} \right\} \left\{ 1/3 \sqrt{2/5} \right\} \\
 & \cdot 2^{\frac{1}{2}} (3/4\pi) \left\{ \frac{2}{3} \cos \theta_1 \cos \theta_2 + \frac{1}{3} \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2) \right\} \\
 & + \pi(E_1) \pi(E_2) A_0(E) D^{-1} \left\{ D_1^{-1} + D_2^{-1} \right\} \left\{ 1/3 \sqrt{2/5} \right\} \\
 & \cdot 2^{\frac{1}{2}} (3/4\pi) \left\{ \frac{2}{3} \cos \theta_1 \cos \theta_2 + \frac{2}{3} \sin \theta_1 \sin \theta_2 \cos (\phi_1 - \phi_2) \right\} .
 \end{aligned}
 \tag{39}$$

The amplitude is properly symmetric in the two final-state mesons.

The quantities $\pi(E_1)$ and $\pi(E_2)$ are the isobar decay amplitudes evaluated at the total energies of the resulting pion-nucleon systems,

$E_1 = \mu + m + t_1 + \epsilon/2$, and $E_2 = \mu + m + t_2 + \epsilon/2$, respectively.

The square of the transition amplitude is given by

$$\begin{aligned}
 |M|^2 &= (2/45)(3/4\pi)^2 (2/9) \left| \mathcal{M}(E_1) \right|^2 \left| \mathcal{M}(E_2) \right|^2 |D|^{-2} \\
 &\quad \cdot \left\{ |D_1|^{-2} + |D_2|^{-2} + 2 \operatorname{Re} (D_1 D_2)^{-1} \right\} \\
 &\quad \cdot \left[|A_2|^2 \left\{ 4 \cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2(\phi_1 - \phi_2) \right. \right. \\
 &\quad \left. \left. + \sin(2\theta_1) \sin(2\theta_2) \cos(\phi_1 - \phi_2) \right\} \right. \\
 &\quad \left. |A_0|^2 \left\{ 4 \cos^2 \theta_1 \cos^2 \theta_2 + 4 \sin^2 \theta_1 \sin^2 \theta_2 \cos^2(\phi_1 - \phi_2) \right. \right. \\
 &\quad \left. \left. - 2 \sin(2\theta_1) \sin(2\theta_2) \cos(\phi_1 - \phi_2) \right\} \right. \\
 &\quad \left. + 2 |A_2| |A_0| \cos \Delta \left\{ -4 \cos^2 \theta_1 \cos^2 \theta_2 + 2 \sin^2 \theta_1 \sin^2 \theta_2 \right. \right. \\
 &\quad \left. \left. \cdot \cos^2(\phi_1 - \phi_2) \right. \right. \\
 &\quad \left. \left. + \frac{1}{2} \sin(2\theta_1) \sin(2\theta_2) \cos(\phi_1 - \phi_2) \right\} \right]
 \end{aligned}
 \tag{40}$$

Here $\Delta = \delta_0 - \delta_2$, the difference in the phases of the amplitudes A_0 and A_2 . The differential cross section is obtained by multiplying Eq. (40) by $2\pi \varrho_E(2\pi, 2N)/V_E(N)$ where

$$\varrho_E(2\pi, 2N) = (2/\pi)^2 k_1 \omega_1 d\omega_1 k_2 \omega_2 d\omega_2 (2\pi)^{-2} m^{3/2} \varepsilon^{\frac{1}{2}} d\varepsilon d\Omega_1 d\Omega_2.
 \tag{41}$$

The salient features of the angular distribution of either meson are best seen by integrating over one of the solid-angle elements. The angular dependence is then given by

$$\begin{aligned} \sigma(\theta) \propto & |A_2|^2 \{ 3 \cos^2 \theta + 1 \} + 8 |A_0|^2 \\ & + 4 |A_2| |A_0| \cos \Delta \{ 1 - 3 \cos^2 \theta \} . \end{aligned} \quad (42)$$

Little can be said about the interference term because the phase factor, $\cos \Delta$, is not given by this analysis. We see that if $A_0 = 0$, then angular distribution of each meson is given by the familiar $1 + 3 \cos^2 \theta$ characteristic of emission from a $J = 3/2$ state. On the other hand, if $A_2 = 0$, the angular distribution of each meson is isotropic, as it must be for any initial state with $J = 0$. Neglecting the interference term and using Eq. (34) to write $\sigma(\theta)$ in terms of the S-matrix elements, we have

$$\sigma(\theta) \propto 5 |S_2|^2 + 8 |S_0|^2 + 15 |S_2|^2 \cos^2 \theta . \quad (43)$$

If $|S_0| \ll |S_2|$, the angular distribution can be somewhat more isotropic than that which is characteristic of the $J = 3/2$ state.

We now discuss the energy spectrum of the mesons and the total cross section as a function of energy on the basis of Eq. (40). For simplicity, the angular integrations are performed with A_0 set equal to zero. Substituting from Eqs. (16), (26), (33), (35), (36), and (49), into Eq. (40), we obtain

$$d^2 \sigma(p + p \rightarrow \pi^+ + \pi^+ + n + n) / d\varepsilon dt_1 \propto$$

$$\begin{aligned} & \beta \left| \Psi_D(r=0) \right|^2 \left\{ t_1(t_1 + 2\mu)(E - 2m - 2\mu - t_1 - \varepsilon)(E - 2m - t_1 - \varepsilon) \varepsilon \right\}^{\frac{1}{2}} \\ & \cdot \left\{ (E - 2m)^2 - \mu^2 \right\}^{-\frac{1}{2}} \left\{ (E - m - E_0)^2 + \frac{\Gamma^2}{4} \right\}^{\frac{1}{2}} \left\{ (E - 2E_0)^2 + \Gamma_0^2 \right\}^{-1} \\ & \cdot \left[\left\{ (E - E_0 - t_1 - \varepsilon/2 - \mu - m)^2 + \frac{\Gamma_0^2}{4} \right\}^{-1} \right. \\ & \quad + \left\{ (-E_0 + m + \mu + t_1 + \varepsilon/2)^2 + \frac{\Gamma_0^2}{4} \right\}^{-1} \\ & \quad + 2 \left\{ (E - E_0 - t_1 - \varepsilon/2 - \mu - m)(-E_0 + m + \mu + t_1 + \varepsilon/2) + \frac{\Gamma_0^2}{4} \right\} \\ & \quad \cdot \left\{ (E - E_0 - t_1 - \varepsilon/2 - \mu - m)^2 + \frac{\Gamma_0^2}{4} \right\}^{-1} \\ & \quad \cdot \left\{ (-E_0 + m + \mu + t_1 + \varepsilon/2)^2 + \frac{\Gamma_0^2}{4} \right\}^{-1} \\ & \cdot \left\{ (t_1 + \mu + m + \varepsilon/2 - E_0)^2 + \frac{\Gamma_1^2}{4} \right\}^{\frac{1}{2}} \\ & \cdot \left\{ (E - t_1 - \varepsilon/2 - \mu - m - E_0)^2 + \frac{\Gamma_2^2}{4} \right\}^{\frac{1}{2}} \\ & \cdot \sigma_{E_1}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p) \sigma_{E_2}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p) \\ & \cdot \sigma_E(p + p \rightarrow \pi^+ + D) / \sigma_{(E-m)}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p) \end{aligned}$$

In this expression $\sigma_{E_{1,2}}(\pi^+ + p \rightarrow \pi^+ + p)$ is given by Eq. (17)

with $E_1 = t_1 + \mathcal{E}/2 + \mu + m$, $E_2 = E - m - \mu - t_1 - \mathcal{E}/2$ and

$$k_1^2 = (t_1 + \mathcal{E}/2)(t_1 + \mathcal{E}/2 + 2\mu)$$

$$k_2^2 = (E - 2m - 2\mu - t_1 - \mathcal{E}/2)(E - 2m - t_1 - \mathcal{E}/2)$$

The $\Gamma_{1,2}$ are defined by Eq. (14) with $k = k_{1,2}$, respectively.

The quantities Γ and Γ_0 are defined by Eq. (14) with

$$k = \left\{ (E - 2m)^2 - \mu^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad k = \left\{ [(E - 2m)/4] - \mu^2 \right\}^{\frac{1}{2}},$$

respectively.

The energy spectra are obtained by integrating over $\mathcal{E}$ for $0 \leq \mathcal{E} \leq E - 2m - 2\mu - t_1$, and plotting the resulting function vs t_1 for given values of the total energy E . Finally one integrates over t_1 for $0 \leq t_1 \leq E - 2m - 2\mu$ to obtain $\sigma(E)$. It should be noted that in the energy region above 1 Bev, where double-meson production begins to occur to some extent, $\sigma(p + p \rightarrow \pi^+ + D)$ is essentially energy-independent. The right-hand side of Eq. (44) should then give the shape of the double-production cross section as a function of the total energy E . For this purpose

$\sigma_{(E-m)}^{\frac{1}{2}}(\pi^+ + p \rightarrow \pi^+ + p)$ will be evaluated from the experimental work of Yuan and Lindenbaum.² As a final test of the analysis, the ratio of double to single production as a function of energy may be obtained by using Eqs. (31) and (44).

Normalizing this ratio to its experimental value at 1.5 Bev., we may calculate it at lower and at higher energies and compare the results with those of recent experiments at around 800 Mev and 2 Bev.

In a similar manner we may calculate the processes

$$\begin{aligned}
 (a) \quad & p + p \longrightarrow p + p + \pi^0 + \pi^0 \\
 (b) \quad & p + p \longrightarrow p + p + \pi^+ + \pi^+ \\
 (c) \quad & p + p \longrightarrow n + p + \pi^+ + \pi^0
 \end{aligned}
 \tag{45}$$

whose relative weights were given in Section A. In connection with Process (c), which is predicted by the model to be the dominant one, and which, experimentally, seems to be so, we note that the transition amplitude into the 3S_1 state of the final nucleons tends to be suppressed. This condition results from several factors. The total mesonic isotopic spin must be one. In the approximation in which we may consider the center-of-mass of the two-meson system to be in an S-state, the relative orbital angular momentum of the two mesons gives their total angular momentum in the total center-of-mass system. However, the odd-orbital states are forbidden by parity conservation, and the even orbital states are forbidden by the requirement of symmetry for the two-meson wave function. The effect implies that deuteron formation in the final state of reaction (c) should be suppressed. Such an absence of deuteron formation is not contradicted by the present preliminary experimental results.

E. Modification of the Angular Distribution for Two Pions Produced
by a Meson-Meson Interaction.

In making this phenomenological analysis of the meson-production problem, we have so far taken into account the strong pion-nucleon interaction in the $J = I = 3/2$ state and the nucleon-nucleon final-state interaction in the S state. There is also the possibility that a meson-meson interaction may affect the momentum and angular distributions of the final-state pions. In this section, we introduce a hypothetical effect of a particular form, and observe its consequences for the angular correlation of the two mesons. In the same manner in which we separate out the relative motion of the two final-state nucleons from the matrix element, and replace the plane wave by a wave function of the form $e^{i\delta} \sin \delta f(r)/q$ for an S state, we may separate out the relative motion of the two final-state mesons and describe the interaction in terms of a similar modification of the matrix element, where δ is now the meson-meson S-wave phase shift and q is the relative momentum of the two mesons. This separation of the nucleon-nucleon and meson-meson final-state effects is the simplest manner of observing the modifications brought about by each. In reality, the two effects may interfere, and one may be obscured by the other.

For the meson-meson phase shift, we choose a Breit-Wigner form

$$\sin \delta = \frac{\Gamma/2}{\{E - 2\mu - E_r\} + i\Gamma/2} \quad (46)$$

where $\mathcal{E}_r$ is the resonant energy in the meson-meson center-of-mass system, Γ is the resonance width, and $\mathcal{E}$ is the total energy of the two mesons in their center-of-mass system. The $\mathcal{E}$ is given in terms of the meson momenta in the total center-of-mass system, k_1 and k_2 , and the angle between them, $\theta_{1,2}$, by

$$4\mathcal{E}^2 = \left\{ \omega_1 + \omega_2 \right\}^2 - k_1^2 - k_2^2 - 2k_1 k_2 \cos \theta_{1,2} \quad (47)$$

where

$$\omega_{1,2} = \left\{ k_{1,2}^2 + \mu^2 \right\}^{\frac{1}{2}}.$$

Also

$$q = \left\{ k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta_{1,2} \right\}^{\frac{1}{2}}.$$

The modification of the angular distribution is then in the factor $\sin^2 \mathcal{S} / q^2$. The angular function of the double-production cross section is given by Eq. (40) in terms of the polar angles of each meson with respect to the direction of the incident nucleon.

$$\begin{aligned} \sigma(\theta_1 \phi_1, \theta_2 \phi_2) \propto & \left\{ 4 \cos^2 \theta_1 \cos^2 \theta_2 + \sin^2 \theta_1 \sin^2 \theta_2 \cos^2(\phi_1 - \phi_2) \right. \\ & \left. + \sin(2\theta_1) \sin(2\theta_2) \cos(\phi_1 - \phi_2) \right\} d\Omega_1 d\Omega_2. \end{aligned} \quad (48)$$

An exercise in spherical trigonometry allows one to transform this distribution into a function of the polar angle, θ_1 , of one meson with respect to the incident direction, and the polar angle, θ , of a second meson with respect to the first.

The result is

$$\sigma(\theta_1, \theta) \propto \left\{ 4 \cos^4 \theta_1 \cos^2 \theta + \sin^4 \theta_1 \cos^2 \theta + 2.5 \sin^2 \theta_1 \cos^2 \theta_1 \sin^2 \theta + \sin^2 \theta_1 (2 \cos^2 \theta - \sin^2 \theta) \right\} d\Omega_1 d\Omega.$$

In Fig.2 we plot this function, as well as the function modified by $\sin^2 \delta/q^2$, vs. θ for $\theta_1 = 45^\circ$, with parameters, and $\mathcal{E}_r = 50$ Mev and $\Gamma = 10$ Mev, for several pairs of meson-momenta values. Because the mesons account for a portion of the momentum conservation, there is a kinematic tendency for angles $\theta > 90^\circ$. This effect has not been taken into account in the figure.

DISCUSSION

Figures 3 to 8 contain the energy spectra for the pion produced in the process $p + p \rightarrow n + p + \pi^+$. The final state nucleon-nucleon interaction is included in the 450 Mev spectrum, but is neglected in the spectra at higher energies. This final-state interaction should be important only where $\hbar/q \gg a$,⁸ where q is the relative momentum of the final nucleons and a is the radius of the region of the primary interaction (excitation of the isobar). Experiment seems to indicate for the proton a region of very strong interaction of radius $\sim 0.5 \times 10^{-13}$ cm,⁹ surrounded by a region of weaker interaction of $\sim 1 \times 10^{-13}$ cm.⁹ Blokhintsev¹⁰ has termed the former region the kernel of the nucleon, and the latter region, the meson shell. In nucleon-nucleon collisions, one may then speak of interactions between the meson shells, between a meson shell and a kernel, and between the kernels. Barring a strong meson-meson effect the first mentioned interaction, which is of the longest range, is probably not responsible for the excitation of the isobaric states. However, the interaction between the meson shell of one nucleon and the kernel of the second may account for this excitation. The region of the primary interaction may therefore be $\sim 1 \times 10^{-13}$ cm. The final state nucleon-nucleon interaction should play a decreasing role in the reaction as the bombarding energy is raised and q may be $\sim 2 \mu$. The curve labeled (b) in Fig. 7 and the curve in Fig. 8 represent the meson spectra when the final nucleons are in a P state. These are obtained by replacing $\epsilon^{\frac{1}{2}}$ in the phase space of the final nucleons by $\epsilon^{3/2}$. In Figs. 6 and 8 the experimental histograms at 810 Mev¹¹ and

1.5 BeV¹² respectively are superposed on the theoretical spectra. Agreement, especially at the lower energy, appears to be fair. At 1.5 BeV a certain amount of double production may be included in the histogram.¹² Further experimental evidence on the shape of the energy spectra in the region 0.5 to 1 BeV is supplied by the Russian experiments on $p + p \rightarrow n + p + \pi^+$ at 560 and 660 MeV.¹³ The mean energies of the experimental spectra were 82 and 110 MeV respectively. These are to be compared with the peak energies in Figs. 4 and 5 of about 75 and 100 MeV respectively. Figures 9 to 12 contain the energy spectra for the pions produced in the processes $p + p \rightarrow 2N + 2\pi$ with the final nucleons in an S state. These spectra all exhibit pronounced peaks at relatively low meson kinetic energies. Such a marked preference for the emission of low-energy mesons is indeed one of the striking features of the experimental situation in the BeV range.^{1,14} Another striking feature of the experiments, the rapid increase above 1 BeV of the double-meson production processes,^{14,15} is evidenced in Fig. 14, where we have plotted a rough estimate of the two-meson excitation function versus bombarding energy.

The situation as to the charge ratios has been covered in some detail in the recent series of papers by the workers at the Cosmotron.¹⁴ We only mention that the charge ratios in double production are not yet well established. In single production, the isobar model predicts a π^+/π^0 ratio of 5. The experimental ratio at present is between 5 and 17. In connection with this ratio, it should be remembered that the process $p + p \rightarrow p + p + \pi^0$ is forbidden by the requirements of angular-momentum and parity conservation and the Pauli Principle

whenever the meson is in a P state and the final nucleons are in an S state. The process will be suppressed during that portion of the time in which the final nucleons are in an S state.

Figure 13 shows a rough estimate of the ratio

$$\sigma(p + p \rightarrow n + p + \pi^+) / \sigma(p + p + \pi^+ + D)$$

versus bombarding energy. The ratio is normalized to 2 at 500 Mev. Below 500 Mev the ratio is a very sensitive function of the final state nucleon-nucleon interaction. Above 500 Mev it should be less sensitive for reasons given earlier. The Russian measurement of ~ 3.5 at 660 Mev and the Brookhaven measurement of 84 at 810 Mev would indicate that the ratio takes a huge jump in an energy interval of only ~ 150 Mev. The curve is in better accord with the Birmingham¹⁶ value of ~ 9 at 660 Mev. At the higher energies the ratio is probably badly underestimated because we have included only S states for the nucleons in the unbound reaction.

Little can be said as to meson-meson effects between two final-state low-energy pions. A marked forward-backward asymmetry in the angle between the mesons might be detectable if enough events could be observed. No such correlation was observed in the few double-production events analyzed in the early Brookhaven work.¹⁷

In conclusion, we may say that the current experiments on meson production in the Bev range seem to strongly indicate an important role is being played by the $J = I = 3/2$ isobar. This is particularly true when examining meson-energy spectra and Q-values between final meson-nucleon pairs. The mechanism of excitation is unknown, but it

seems to involve a sort of peripheral collision in which the final nucleons retain much of their incident momenta. The calculations presented here are meant to give a rough idea of how far this model can go in correlating the data on meson production.

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APPENDIX A

We here separate the nucleon-nucleon final-state interaction from the meson-production matrix element. If we consider single-meson production, the part of the total transition amplitude in which we are interested is the matrix element that describes the transition from the isobar-nucleon intermediate state, $|\psi_{X,N}\rangle$, to the meson-two-nucleon final state, $|\psi_{N_1,N_2,\pi}\rangle$. We shall call the operator that brings about this transition U . It operates on the isobar with momentum $\vec{s}$ and energy E_s to form a meson of momentum $\vec{k}$ and energy ω and a nucleon of momentum $\vec{p}_1$ and energy E_1 . The second nucleon has momentum $\vec{p}_2$ and energy E_2 in the final state, and $\vec{p}'$ and E' in the intermediate state. With all particles in plane-wave states the matrix element is the following

$$\begin{aligned} \langle \psi_{N_1,N_2,\pi} | U | \psi_{X,N} \rangle &= \\ &= \int d^3\vec{r} d^3\vec{r}' \langle \psi_{N_1,N_2,\pi} | \vec{r} \rangle \langle \vec{r} | U | \vec{r}' \rangle \langle \vec{r}' | \psi_{X,N} \rangle \\ &= \int d^3\vec{r}_\pi d^3\vec{r}_{N_1} d^3\vec{r}_{N_2} d^3\vec{r}' d^3\vec{r}_X e^{-i(\vec{k}\cdot\vec{r}_\pi + \vec{p}_1\cdot\vec{r}_1 + \vec{p}_2\cdot\vec{r}_2)} \\ &\quad \cdot e^{i(\vec{s}\cdot\vec{r}_X + \vec{p}'\cdot\vec{r}')} \delta(\vec{r}_1 - \vec{r}_X) \delta(\vec{r}_\pi - \vec{r}_X) \delta(\vec{r}_2 - \vec{r}') \langle U \rangle \\ &= (2\pi)^6 \delta(\vec{p}_2 - \vec{p}') \delta(\vec{s} - \vec{p}_1 - \vec{k}) \langle U \rangle . \end{aligned}$$

In the second line, the notation for the position vectors of the particles is self-evident. The δ functions arise from the definition of an

operator U in the $\vec{r}$ representation and the assumed local nature of the interaction. The quantity $\langle U \rangle$ is the isobar-decay operator between initial and final states after the space dependence has been extracted. This quantity is, in general, a function of $\vec{k}, \omega, \vec{p}_1, E_1, \vec{s}, E_s$ and, possibly, even the spin orientation of the isobar and the resulting nucleon. In this work the isobar notion has been neglected, and we have considered this function to be given by our quantity $\mathcal{M}(E)$, with $E = \omega + E_1$, times a P wave spherical harmonic for the meson. The relative motion of the final-state nucleons may be separated out from Eq. (1) by writing the matrix element in terms of $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$, $\vec{r} = \vec{r}_1 - \vec{r}_2$; $\vec{P} = \vec{p}_1 + \vec{p}_2$, and $\vec{q} = (\vec{p}_1 - \vec{p}_2)/2$. The factor $e^{-i\vec{q} \cdot \vec{r}}$, which represents the plane-wave relative motion of the particles, is then replaced by an S state wave function of the form

$$\psi(r) \propto e^{i\delta} \frac{\sin \delta}{q} f(r) .$$

The matrix element now becomes the following:

$$\begin{aligned} & (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}'_1 - \vec{p}'_2) \int d^3\vec{r} d^3\vec{r}_1 d^3\vec{r}_2 d^3\vec{r}'_1 d^3\vec{r}'_2 \\ & \cdot e^{-i\vec{P} \cdot (\vec{r}_1 + \vec{r}_2)/2} e^{-i\vec{k} \cdot \vec{r}} e^{i\delta} \sin \delta/q \\ & \cdot f(|\vec{r}_1 - \vec{r}_2|) e^{i(\vec{s} \cdot \vec{r}_X + \vec{p}'_1 \cdot \vec{r}'_1)} \delta(\vec{r}_1 - \vec{r}_X) \delta(\vec{r}_2 - \vec{r}') \delta(\vec{r}_X - \vec{r}') \langle U \rangle \\ & = (2\pi)^3 \delta(\vec{p}_2 - \vec{p}'_1) e^{i\delta} \sin \delta/q \int d^3\vec{R} d^3\vec{r} e^{i(\vec{s} - \vec{k} - \vec{p}_1) \cdot \vec{R}} \\ & \cdot e^{i(\vec{s} - \vec{k} - \vec{p}_2) \cdot (\vec{r}/2)} f(r) \langle U \rangle \\ & = (2\pi)^6 \delta(\vec{p}_2 - \vec{p}'_1) \delta(\vec{s} - \vec{k} - \vec{p}_1) e^{i\delta} \sin \delta/q \langle U \rangle \\ & \cdot \int f(r) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r} . \end{aligned} \tag{2}$$

Comparing this result with the last line of Eq. (1), we see that the matrix element is modified by the factor $e^{i\delta} \frac{\sin \delta}{q} f(q)$ where

$$f(q) = \int f(r) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r} \sim \int f(r) d^3\vec{r} = \text{constant.}$$

APPENDIX B

In Section B, we defined an amplitude $T_2(E)$ for the transition from a 1D_2 two-nucleon state of energy, E , to an isobar-nucleon state at rest. This amplitude was defined in terms of the cross section, $\sigma_E(\pi^+ + D \rightarrow 2p)$, and the previously defined amplitude, $\mathcal{T}(E)$, for creation of the isobar. In order to modify $T_2(E)$ so that we may use it in calculating $\sigma_E(p + p \rightarrow \pi^+ + n + p)$ we need simply to observe that, by defining it in terms of the deuteron mesodisintegration, we include in it a factor of the deuteron wave function evaluated at the origin, $\psi_D(r=0)$. This is simply because the production of an isobar-nucleon intermediate state involves a matrix element of the form

$$\langle \pi \psi_D | U | \psi_{X,N} \rangle = \int d^3\vec{p} \langle \pi \psi_D | \vec{p} \rangle \langle \vec{p} | U | X,N \rangle \quad (1)$$

where the integration is over the momentum distribution of the deuteron. The matrix element $\langle \vec{p} | U | \psi_{X,N} \rangle$ is simply $\mathcal{T}(E')$ at the energy E' in the center-of-mass system of the incident pion and the nucleon within the deuteron that absorbs it to form the isobar. If we neglect the deuteron momentum distribution, Eq. (1) becomes

$$\mathcal{T}(E) \int d^3\vec{p} \langle \psi_D | \vec{p} \rangle = \mathcal{T}(E) \psi_D(r=0) \quad (2)$$

where E is now the energy of the pion-nucleon system considering the nucleon to be at rest. This is a good approximation for the high energies, E , in which we will be interested. For use in calculating the unbound reaction, we define $|T_2'(E)|^2 = |T_2(E)|^2 |\psi_D(r=0)|^{-2}$.

We now show that the amplitude $T_2'(E)$ can be of use in describing the transition from the two-nucleon state of energy, E , to the two-isobar state, in the approximation that the isobars are at rest. The operator, T' , may be defined formally in terms of a potential operator, V , by the relation

$$\langle u_{X,N}^{E'} | T' | u_{2N}^E \rangle = \langle \psi_{X,N}^{E'} | V | u_{2N}^E \rangle. \quad (3)$$

The $|u\rangle$ are plane-wave eigenstates of the free-field Hamiltonian, H_0 ,

$$H_0 |u^E\rangle = E |u^E\rangle. \quad (4)$$

The $|\psi\rangle$ are eigenstates of the total Hamiltonian, H :

$$H |\psi^{E'}\rangle = (H_0 + V) |\psi^{E'}\rangle = E' |\psi^{E'}\rangle. \quad (5)$$

They satisfy the Lippman-Schwinger equation,

$$\psi^E = u^E + (E - H_0 - i\epsilon)^{-1} V \psi^E. \quad (6)$$

Introducing an $\vec{r}$ representation, we obtain

$$\begin{aligned} \langle u_{X,N}^{E'} | T' | u_{2N}^E \rangle &= T'(E, E') \\ &= \int d^3\vec{r} d^3\vec{r}' \langle \psi_{X,N}^{E'} | \vec{r} \rangle \langle \vec{r} | V | \vec{r}' \rangle \langle \vec{r}' | u_{2N}^E \rangle. \end{aligned} \quad (7)$$

In the center-of-mass system, $\vec{r}$ and $\vec{r}'$ are the relative position coordinates of the two particles. The interaction is assumed to be local:

$$\langle \vec{r} | v | \vec{r}' \rangle = v(r) \delta(\vec{r} - \vec{r}') , \quad (8)$$

$$\begin{aligned} T'(E, E') &= \int d^3r \langle \psi_{X,N}^{E'}(r) | v(r) | u_{2N}^E(r) \rangle \\ &\sim \psi_{X,N}^{E'}(R) \int \langle v(r) \rangle e^{i\vec{p} \cdot \vec{r}} d^3\vec{r} . \end{aligned}$$

We have taken out of the integral the isobar-nucleon wave function and have evaluated it at a relative coordinate, R , characteristic of the region in which $|\psi_{X,N}^{E'}(R)|^2$ is large. In addition, we have written the two-nucleon plane wave explicitly where $\vec{p}$ is the relative momentum. The quantity $\langle v(r) \rangle$ is the potential operator between the two states after the position dependence of these states has been extracted. This is in general a function of $E, E', \vec{p}$ and $\vec{p}'$ (The relative momentum of the isobar and nucleon). If we consider the isobar and nucleon at rest, $\vec{p}' = 0$, $E' = E_0 + m$.

In this approximation, the right-hand side of Eq. (8) defines a function of E ,

$$T'(E) = \psi_{X,N}^{E_0+m}(R) f(E) . \quad (9)$$

A similar argument leads to the amplitude for the transition from the two-nucleon state to the two-isobar state in the approximation in which the isobars are at rest:

$$A(E) = \psi_{2X}^{2E_0}(R) f'(E) . \quad (10)$$

We have

$$|f(E) - f'(E)| \sim \left| (E_0 - m)/E_0 \right| \approx 0.3/1.24 \approx 0.24 ; \quad (11)$$

therefore

$$A(E) \sim \psi_{2X}^{2E_0}(R) f(E) \quad (12)$$

and

$$|A(E)|^2 \sim \left| \frac{\psi_{2X}^{2E_0}(R)}{\psi_{X,N}^{E_0+m}(R)} \right|^2 |T'(E)|^2 \quad (13)$$

The separation into amplitudes for various total J states is obtained by writing

$$T' = \sum_J T'_J P_J, \quad (14)$$

$$V = \sum_J V_J P_J,$$

where the P_J are projection operators. An attempt to improve this crude approximation may be made by constructing an energy-dependent wave function for the isobar-nucleon state. For example, if the isobar-nucleon system is considered to be in an S state characterized by a scattering length of $a \sim 0.5$ to 1.0×10^{-13} cm, we may approximate

$$|\psi_{X,N}^E(R)|^2 \text{ by } f(R)/(\alpha^2 + q^2),$$

where q is the relative momentum. The relative momentum may be determined by considering the isobar-nucleon system to be approximately on the energy shell or by assigning a certain average momentum transfer to the excitation process.

In the next appendix we show that one can somewhat correct the angular distributions of this simple model for the motion of the intermediate-state particles.

APPENDIX C

Consider the double production process,

$$p + p \rightarrow 2\pi^+ + 2n, \quad (1)$$

at an energy E . We have up to now considered the process as going through an intermediate state involving two isobars at rest. Actually these isobars move relative to each other with a momentum $2\vec{s}$ in their center-of-mass system. The momentum $\vec{s}$ is given in terms of the momenta of the final-state pions and nucleons by

$$\vec{s} = \vec{k}_1 + \vec{p}_1 = -(\vec{k}_2 + \vec{p}_2),$$

where $\vec{k}_{1,2}$ and $\vec{p}_{1,2}$ are the pion and nucleon momenta of the correlated pairs. If the isobars are in a P state, the above reaction will take place from the $^3P_{0,1,2}$ states of the initial protons into the P states of the final two neutrons. The creation of the two-isobar state is described by the amplitudes $A_{^3P_0}$, $A_{^3P_1}$, and $A_{^3P_2}$, whose possible dependence on the motion of the isobars we continue to neglect. The decay of the two-isobar system at rest was described by the amplitudes $\pi(E_1)$ and $\pi(E_2)$ times P-wave spherical harmonics for each meson, where E_1 and E_2 were total center-of-mass energies of the pion-nucleon system resulting from the decays. We will neglect the effect of the isobar motion on the amplitudes π ; however, we may modify the matrix element by adding a P-wave harmonic that describes the relative motion of the two pion-nucleon systems, $Y_{1,n}(\vec{s} \cdot \vec{z})$. Because the massive nucleons usually carry considerable more momentum than the pions, we may approximate $\vec{s}$, the intermediate-isobar momentum, by $\vec{p}$, the final-state nucleon

momentum. We may compute the modified angular angular distributions as before. For example, for the Process (1), the final state, with nucleon spin z component, $S_z = 1$, is

$$(N_1^{\frac{1}{2} -\frac{1}{2}} N_2^{\frac{1}{2} -\frac{1}{2}}) \sum_{m,m',n} Y_{1,m}^*(\vec{k}_1 \cdot \vec{z}) Y_{1,m}(\vec{r}_1 \cdot \vec{z}) Y_{1,m}^*(\vec{k}_2 \cdot \vec{z}) Y_{1,m}(\vec{r}_2 \cdot \vec{z}) \\ Y_{1,n}^*(\vec{p} \cdot \vec{z}) Y_{1,n}(\vec{r} \cdot \vec{z}) . \quad (2)$$

We perform the vector addition, combining the meson-nucleon systems into isobars with amplitudes $\mathcal{M}$, then adding the two isobar angular momenta, and finally combining this with the orbital angular momentum of the isobars to a total $J = 0, 1$, and 2 with the amplitudes $A_{3P_{0,1,2}}$ respectively. Computing the matrix for $S_z = 0$, and weighting the absolute square of the matrix elements for $S_z = 1$ and $S_z = 0$ with 2 and 1 respectively, we obtain, for the angular part of the cross section from the 3P_0 state of the initial protons, after integration over azimuthal angles,

$$\sigma(\theta_1, \theta_2, \theta) \propto \quad (3) \\ \left\{ 2 \cos^2 \theta_1 \sin^2 \theta_2 + 2 \cos^2 \theta_2 \sin^2 \theta_1 \right. \\ \left. + 8 \cos^2 \theta_1 \cos^2 \theta_2 + 16 \sin^2 \theta_1 \sin^2 \theta_2 \right\} \cos^2 \theta \\ + \left\{ 32 \cos^2 \theta_1 \cos^2 \theta_2 + 11 \sin^2 \theta_1 \sin^2 \theta_2 \right. \\ \left. + \cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_2 \sin^2 \theta_1 \right\} \sin^2 \theta .$$

From the 3P_2 state, we have

$$\begin{aligned} \sigma(\theta_1, \theta_2, \theta) \propto & \left\{ 32 \cos^2 \theta_1 \cos^2 \theta_2 + 64 \sin^2 \theta_1 \sin^2 \theta_2 \right. \\ & \left. + 8 \cos^2 \theta_1 \sin^2 \theta_2 + 8 \cos^2 \theta_2 \sin^2 \theta_1 \right\} \cos^2 \theta \\ & + \left\{ \cos^2 \theta_1 \sin^2 \theta_2 + \cos^2 \theta_2 \sin^2 \theta_1 \right. \\ & \left. + 32 \cos^2 \theta_1 \cos^2 \theta_2 + 11 \sin^2 \theta_1 \sin^2 \theta_2 \right\} \sin^2 \theta \end{aligned} \quad (4)$$

The angle θ is the polar angle of the nucleon relative momentum with respect to the incident-proton direction in the total center-of-mass system. The angles θ_1 and θ_2 are the polar angles of the two mesons with respect to this direction, also in the total center-of-mass system, if one neglects the transformation from the rest system of each isobar. By integration over the meson angles in Eqs. (3) and (4), we can get an idea of the angular distribution of the final-state nucleons that could arise from inclusion of the P-wave motion of the intermediate-state isobars. For the transition from the 3P_0 state, we have

$$2\pi \frac{d\sigma(\theta)}{d\Omega} \propto \cos^2 \theta + \sin^2 \theta ; \quad (5)$$

from the 3P_2 state, we have the familiar

$$2\pi \frac{d\sigma(\theta)}{d\Omega} \propto 3 \cos^2 \theta + 1 . \quad (6)$$

Because the transition amplitudes from $^3P_{0,1,2}$ states give rise to interference terms that depend on their relative phases, a more detailed account of the excitation of the isobaric states will be necessary in order to obtain quantitative angular distributions. However the observed forward-backward preference for the nucleons is not beyond the reach of the model.

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ILLUSTRATIONS

Part I

Fig. 1. Feynman diagrams for meson production in the Born approximation.

Fig. 2. Feynman diagrams for meson production in the one-meson approximation.

Part II

Fig. 1. Feynman graphs for double-meson production through an intermediate two-isobar state.

Fig. 2. (a) Plot of Eq. (49) versus θ with $\theta_1 = 45^\circ$. (b) Plot of Eq. (49) times $\sin^2 \delta/q^2$ versus θ with $\theta_1 = 45^\circ$, $k_1 = 20$ Mev/c, $k_2 = 30$ Mev/c, $\mathcal{E}_T = 50$ Mev, $\sqrt{s} = 10$ Mev. (c) Same as (b) with $k_1 = 70$ Mev/c, $k_2 = 50$ Mev/c. (d) Same as (b) with $k_1 = 150$ Mev/c, $k_2 = 130$ Mev/c.

Fig. 3. Energy spectra for the meson produced in the reaction $p + p \longrightarrow \pi^+ + n + p$ at 450 Mev. The final-state n-p force is included in terms of the low-energy triplet scattering length.

Fig. 4. Energy spectra for the meson produced in the reaction $p + p \longrightarrow \pi^+ + n + p$ at 565 Mev.

Fig. 5. Energy spectra for the meson produced in the reaction $p + p \longrightarrow \pi^+ + n + p$ at 680 Mev.

Fig. 6. Energy spectra for the meson produced in the reaction $p + p \longrightarrow \pi^+ + n + p$ at 795 Mev. Histogram is from the Brookhaven experiment at 810 ± 100 Mev.

Fig. 7. Energy spectra for the meson produced in the reaction $p + p \longrightarrow \pi^+ + n + p$ at 1.015 Bev. Curve (a) nucleons are in an S state. Curve (b) nucleons are in a P state.

Fig. 8. Energy spectra for the meson produced in the reaction $p + p \rightarrow \pi^+ + n + p$ at 1.51 Bev. The nucleons are in a P state. Histogram is from the Brookhaven experiment at 1.5 ± 0.1 Bev.

Fig. 9. Energy spectra of the mesons produced in $p + p \rightarrow 2N + 2\pi$ at 1.015 Bev with final nucleons in an S state.

Fig. 10. Energy spectra of the mesons produced in $p + p \rightarrow 2N + 2\pi$ at 1.27 Bev with final nucleons in an S state.

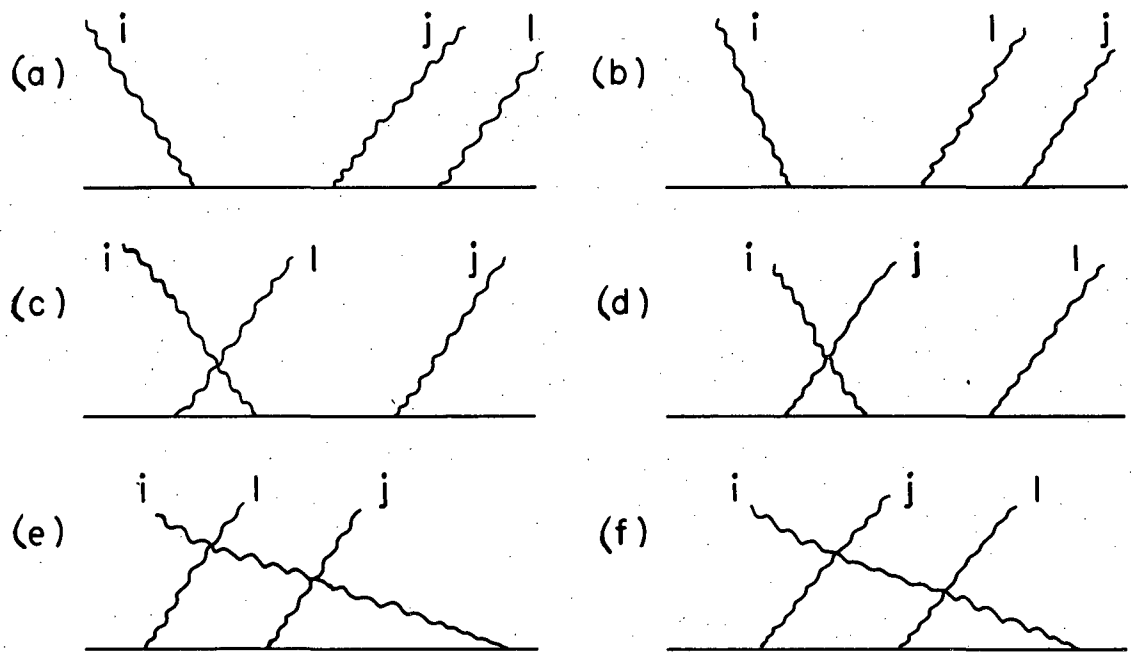
Fig. 11. Energy spectra of the mesons produced in $p + p \rightarrow 2N + 2\pi$ at 1.51 Bev with final nucleons in an S state.

Fig. 12. Energy spectra of the mesons produced in $p + p \rightarrow 2N + 2\pi$ at 2.01 Bev with final nucleons in an S state.

Fig. 13. Ratio $\sigma(p + p \rightarrow \pi^+ + n + p) / \sigma(p + p \rightarrow \pi^+ + D)$ as a function of bombarding energy. The ratio is normalized to 2 at 500. The triangles indicate the Russian points, the circle indicates the Birmingham point. The remaining points are American data from Chicago, Berkeley, and Brookhaven.

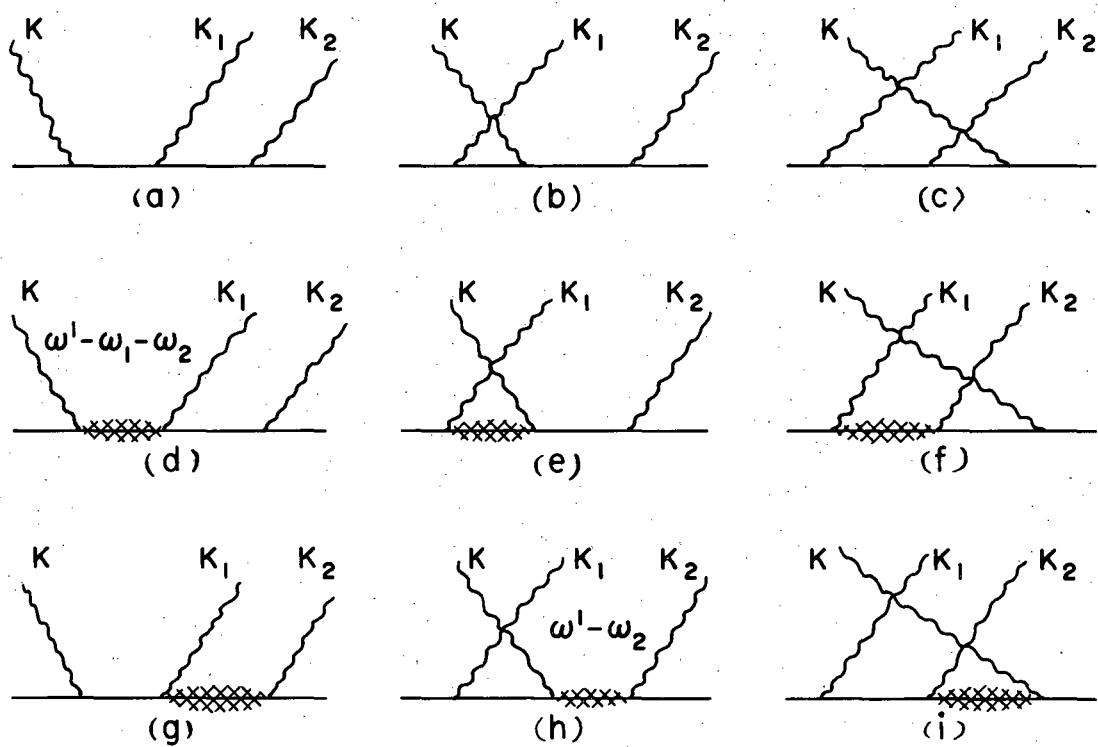
Fig. 14. Two-pion excitation function versus bombarding energy. The curvature of the upper portion of the solid curve is probably due to the neglect of the energy dependence of the factor $|\psi_{X,N(R)}|^{-2}$ which is not justified at the higher bombarding energies. The dashed line is an extrapolation of the essentially linear portion of the solid curve.

Fig. 15. Ratio of two-pion to one-pion cross sections as a function of bombarding energy. The ratio is normalized to 30% at 1.5 Bev. The dashed line is an extrapolation of the essentially linear portion of the solid curve.



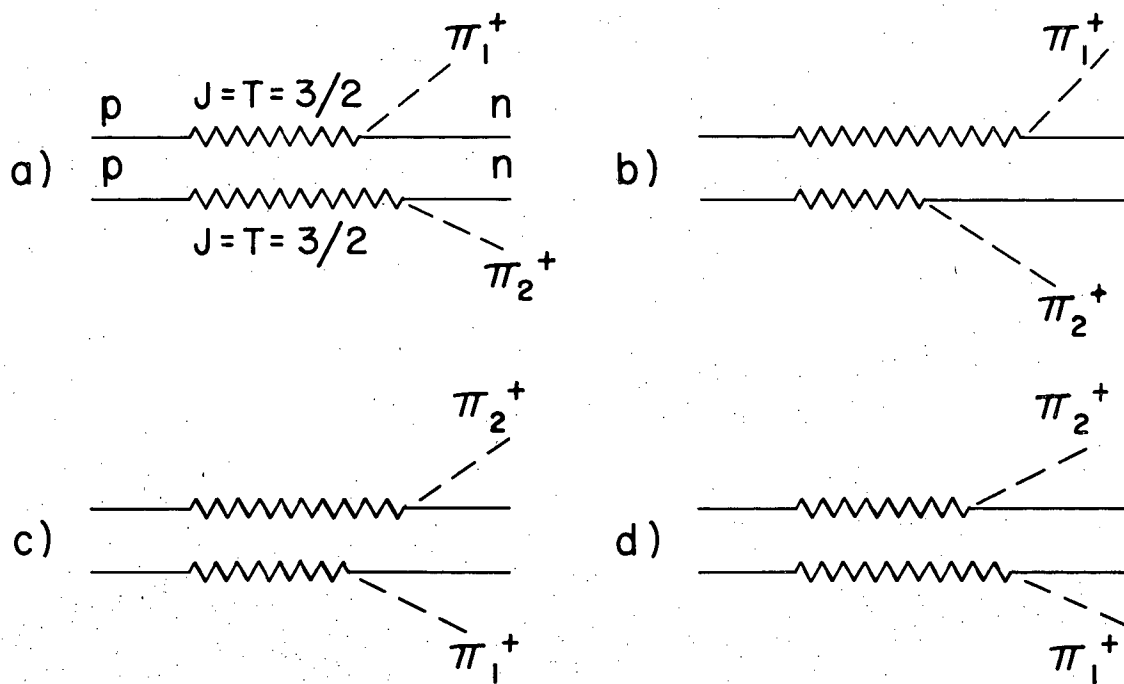
MU-12591

Part I Fig. 1.



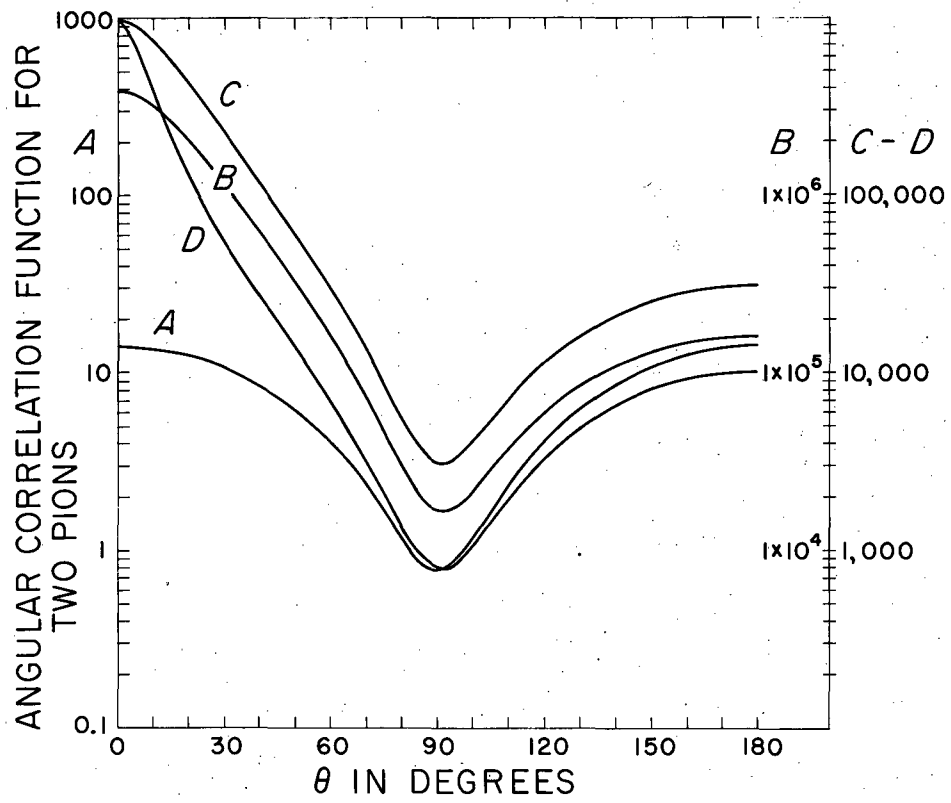
MU-12592

Part I Fig. 2.



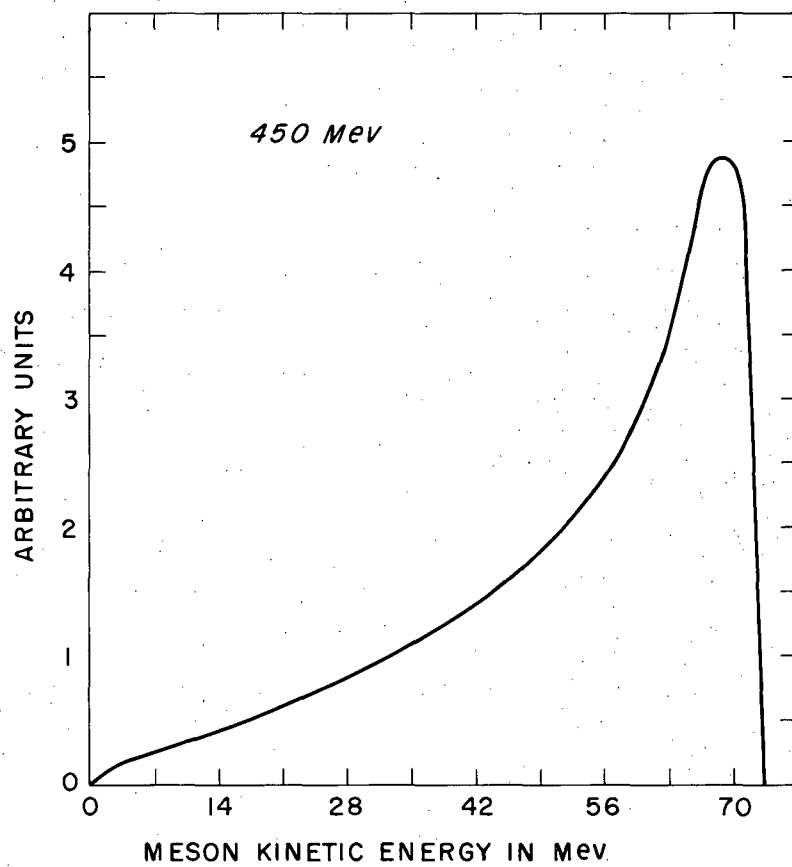
MU-12550

Part II Fig. 1.



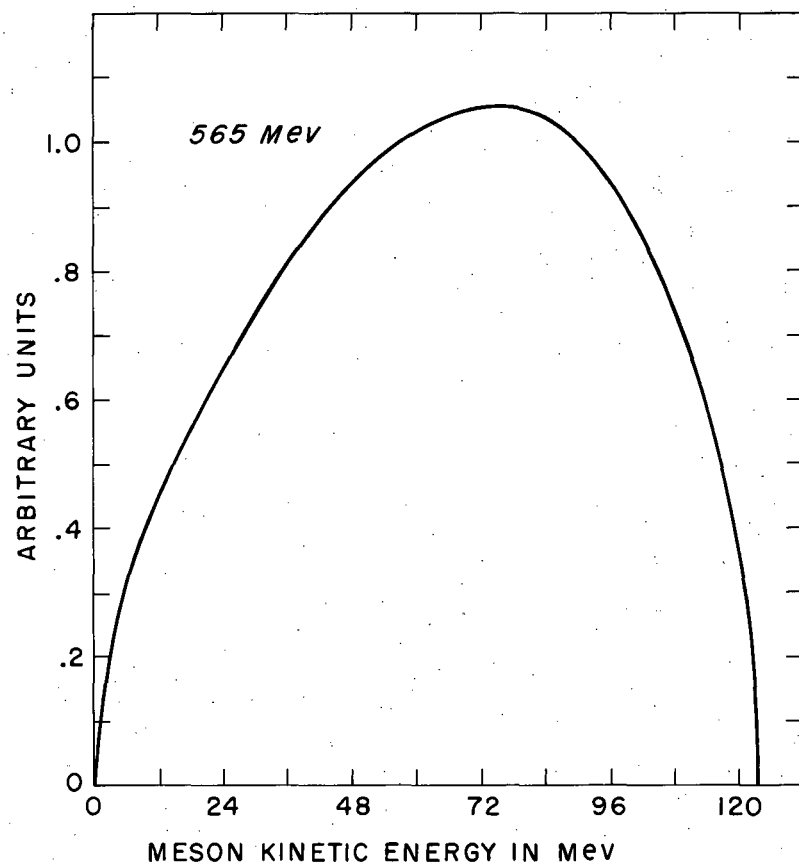
MU-12551

Part II Fig. 2.



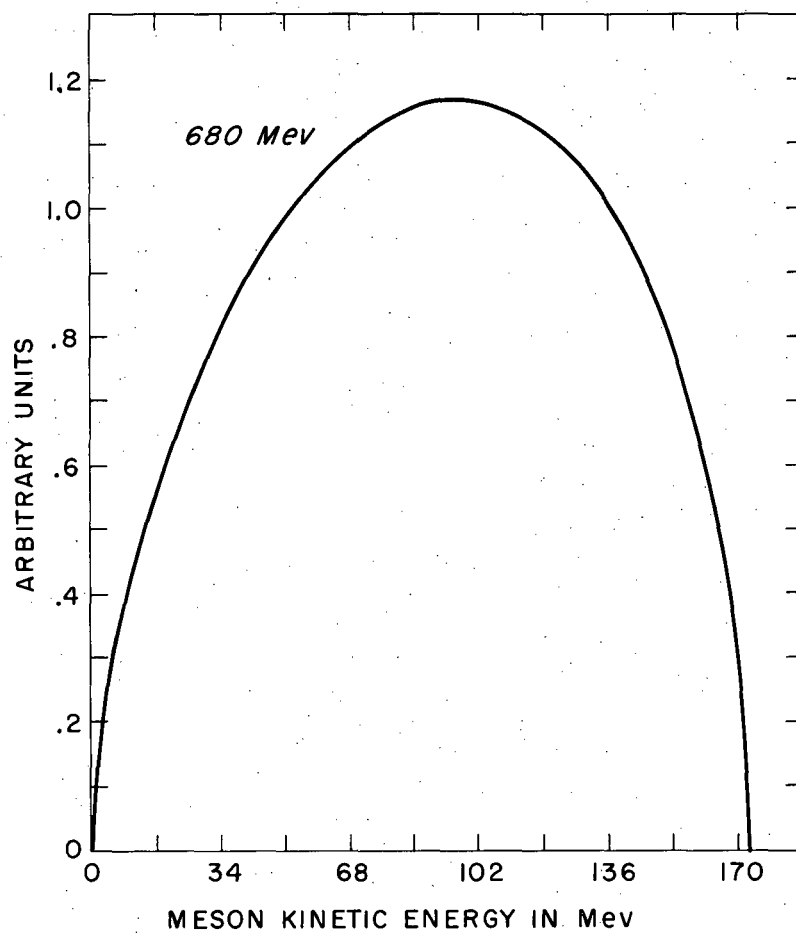
MU-12549

Part II Fig. 3.



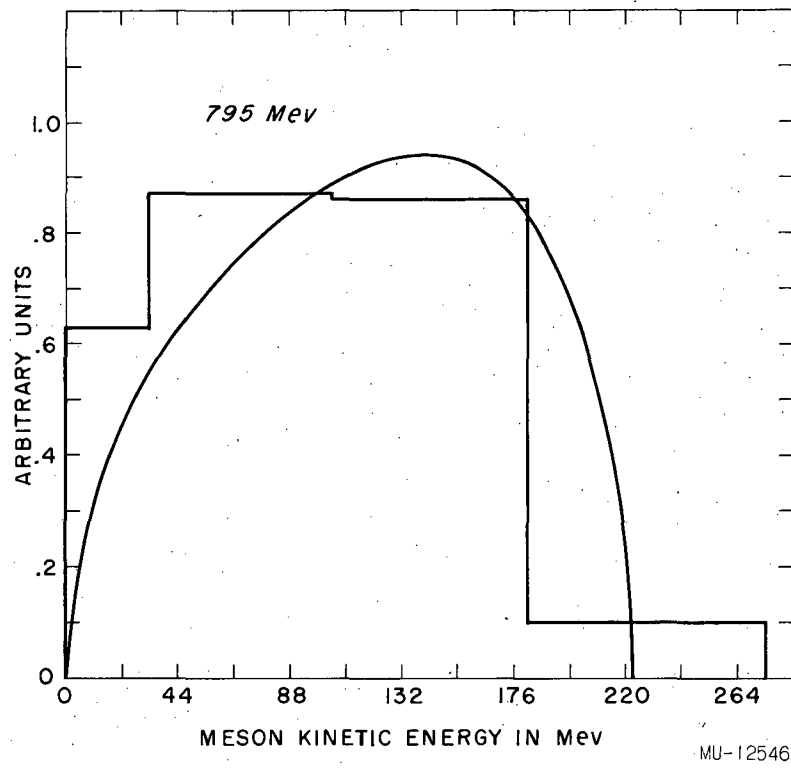
MU-12548

Part II Fig. 4.

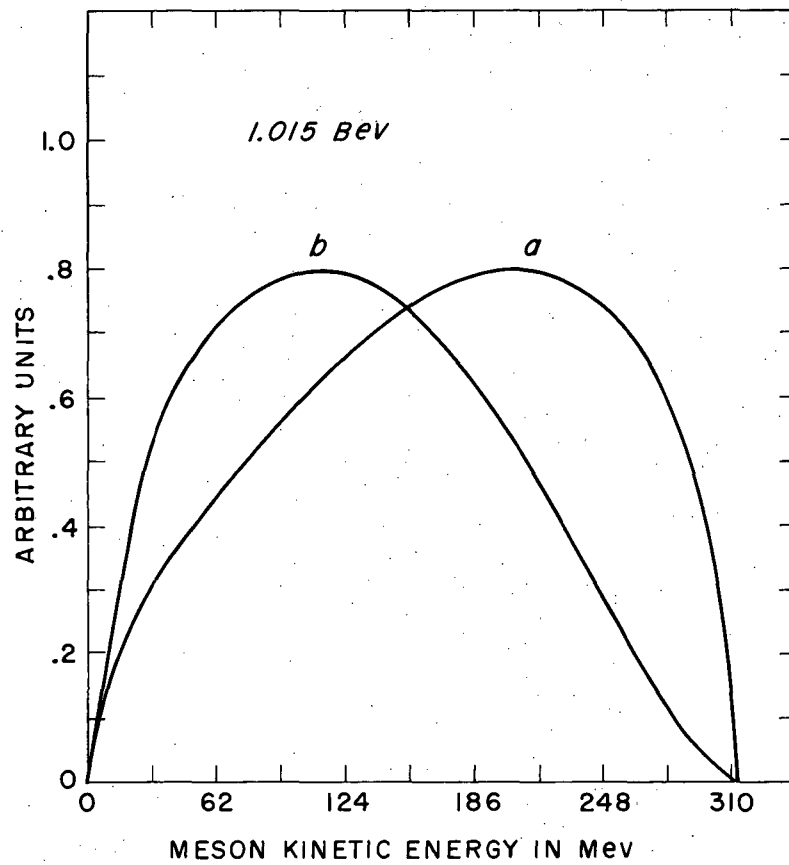


MU-12547

Part II Fig. 5

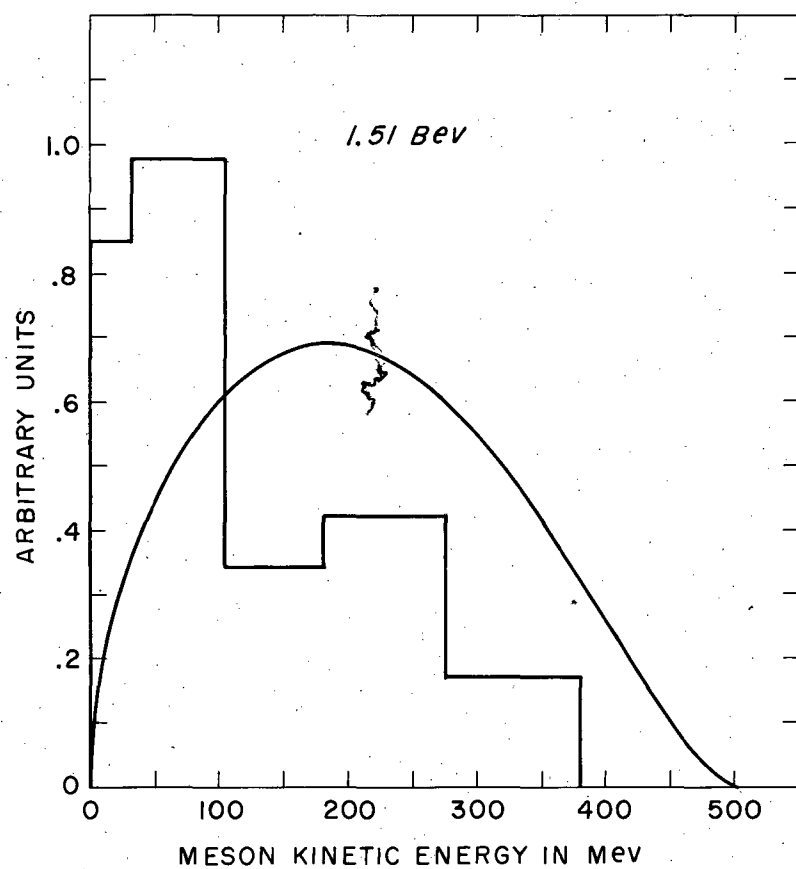


Part II Fig. 6.



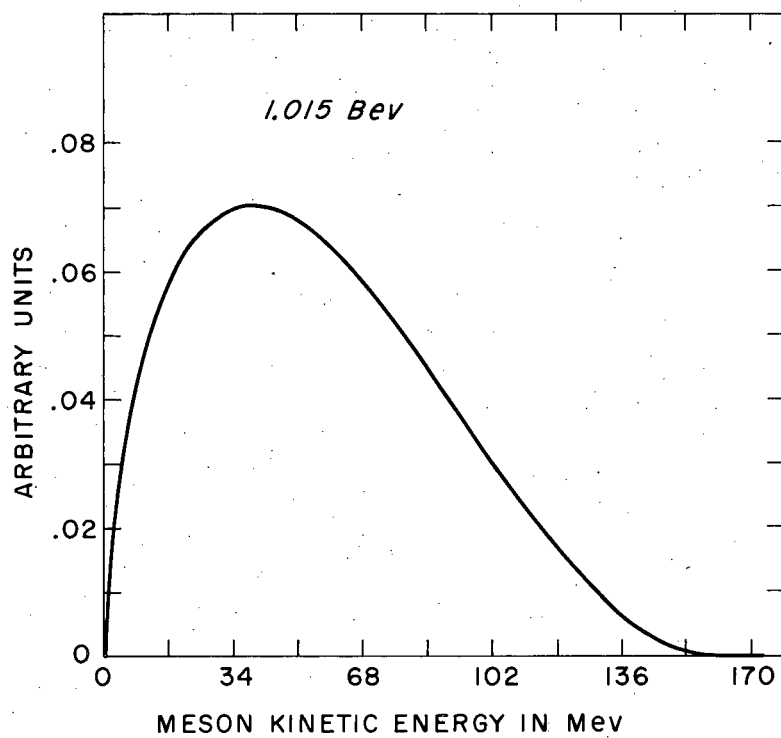
MU-12545

Part II Fig. 7.



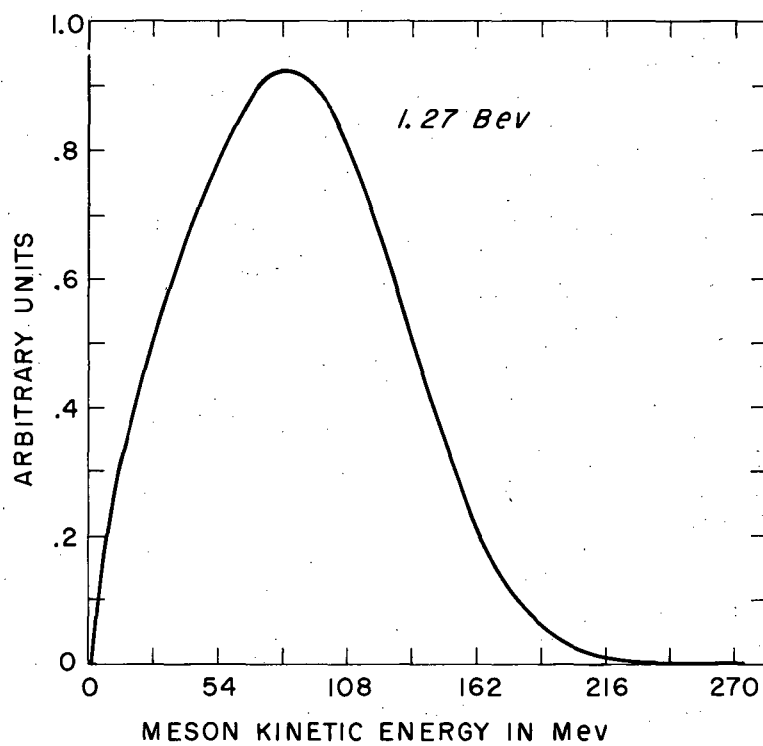
MU-12544

Part II Fig. 8.



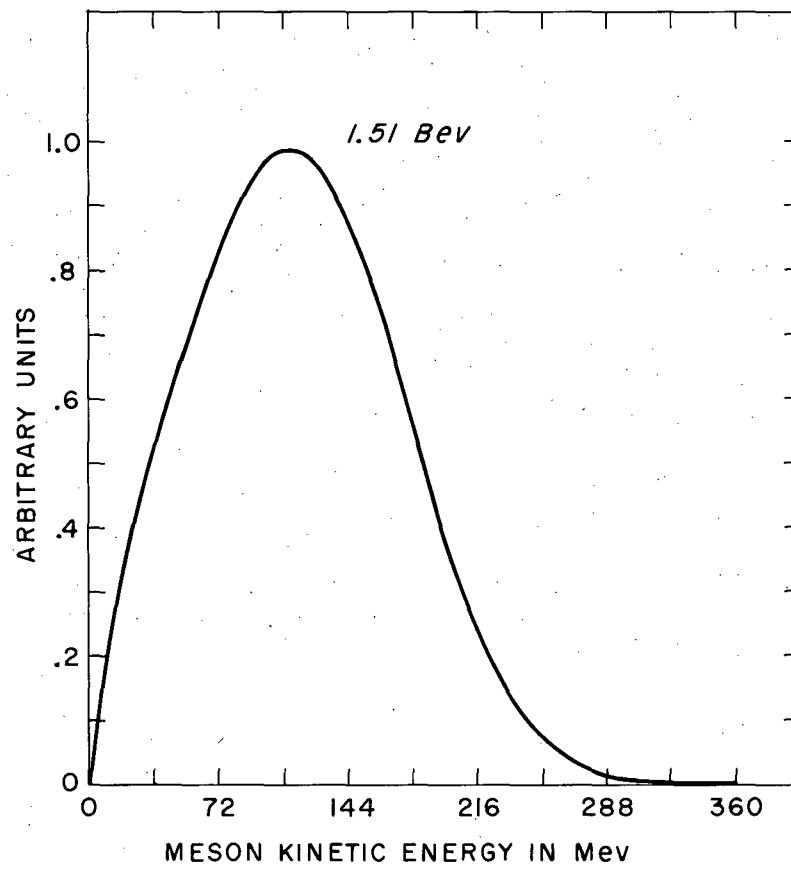
MU-12543

Part II Fig. 9.



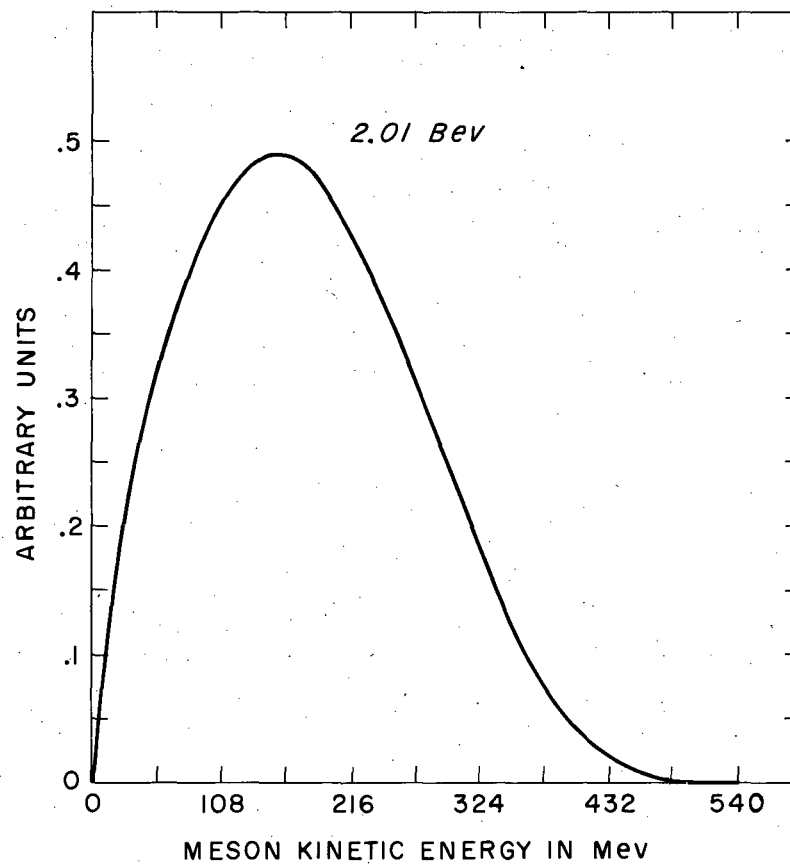
MU-12542

Part II Fig. 10.



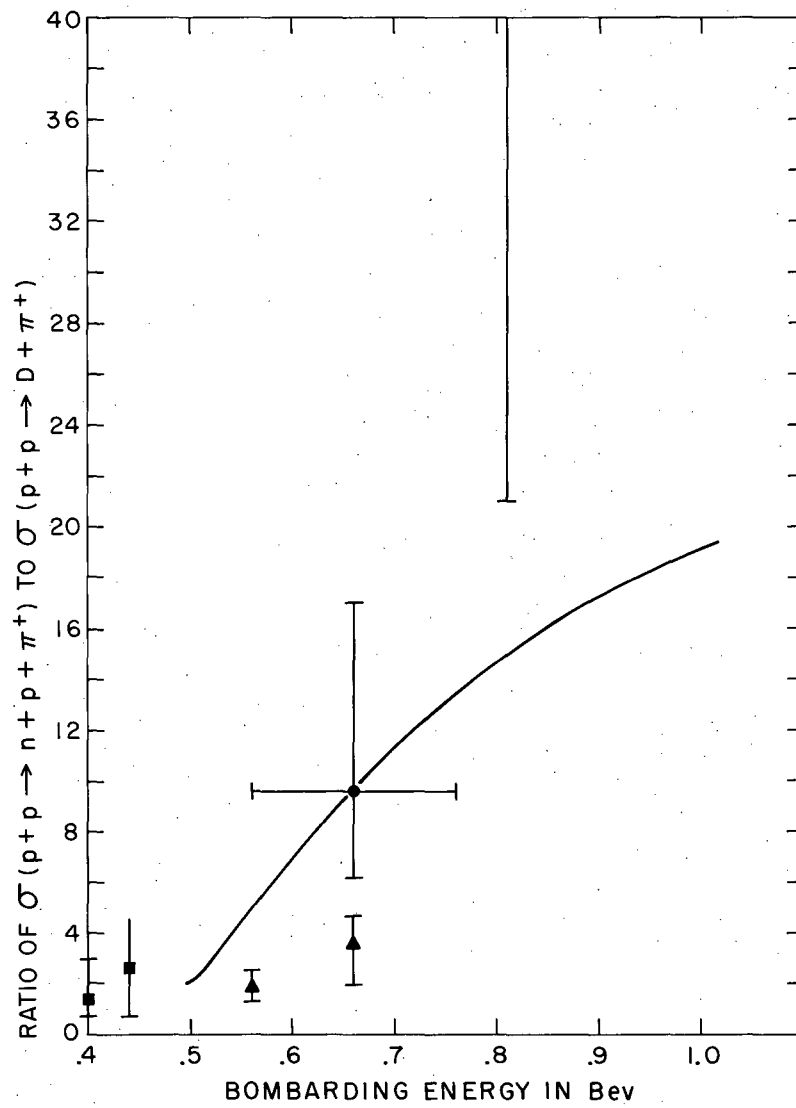
MU-12541

Part II Fig. 11.



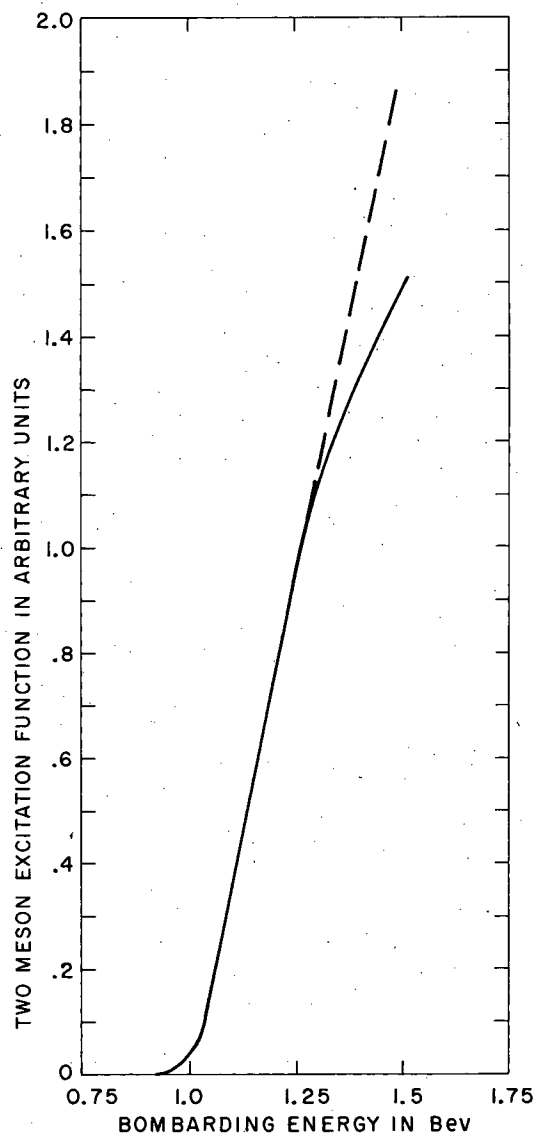
MU-12540

Part II Fig. 12.



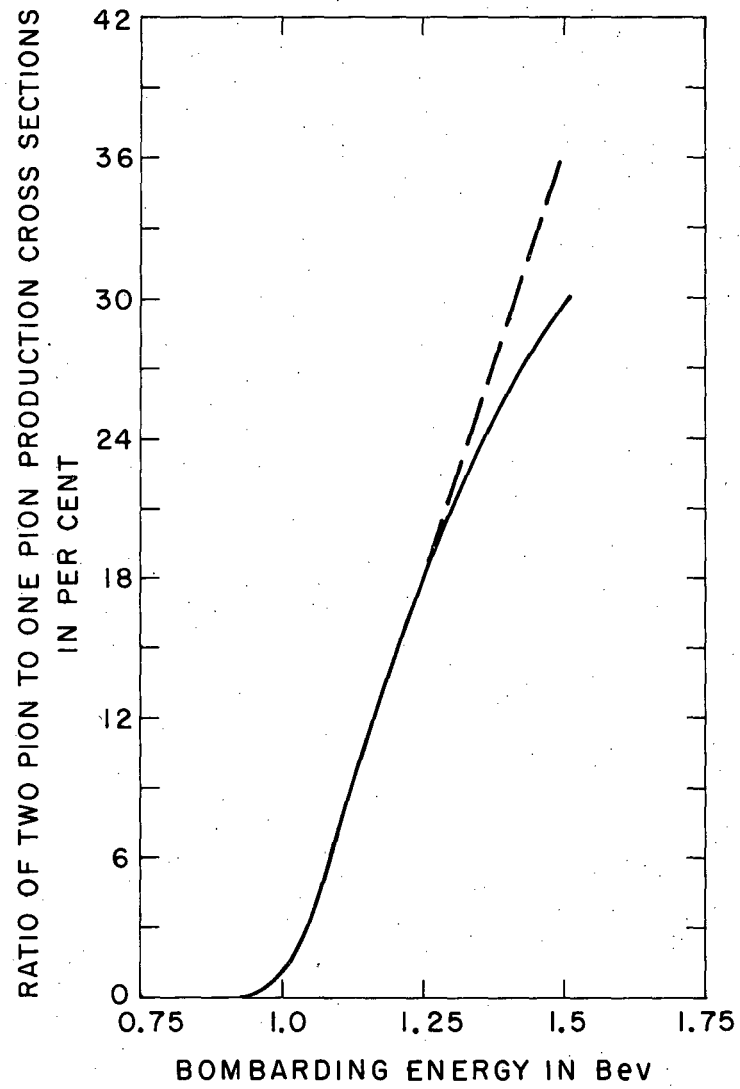
MU-12539

Part II Fig. 13.



MU-12538

Part II Fig. 14



MU-12537

Part II. Fig. 15.