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Self-Extensions and Prime Factorizations of Quantum Affine Algebras

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Self-Extensions and Prime Factorizations of Representations of Quantum Affine Algebras

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Mathew Arthur Lunde

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ABSTRACT OF THE DISSERTATION

Self-Extensions and Prime Factorizations of Representations of Quantum Affine Algebras

by

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University of California, Riverside, June 2015
Professor Vyjayanthi Chari, Chairperson

It is well known that the category of finite dimensional representations of a quantum affine algebra is not semi-simple. Moreover, the tensor product of irreducible representations remains irreducible generically. This observation leads naturally to the definition of prime objects and the factorization of irreducible objects into irreducible primes. We show that there is an interesting connection between the prime objects and the homological properties of the category: an irreducible representation V of $\hat{\mathbf{U}}_q(\mathfrak{s}l_2)$ is a tensor product of r prime representations if and only if the dimension of the space of self-extensions of V is r. In addition, in the case when V is a tensor power of an irreducible prime module, we give generators and relations for V, as well as classify all self-extensions of V (up to equivalence) in terms of polynomials.

Contents

Introduction

The category $\mathcal{F}$ of finite-dimensional representations of a quantum affine algebra has been intensively studied in recent years. The simple objects were classified in [4, 5], and since then many authors have attempted to understand the structure of these representations. Many important tools have been developed, in particular the theory of q-characters, developed in [11, 12], has given deeper insight into the combinatorial structure of the representations, and a connection to cluster algebras has been established in [13, 17]. It was shown in [8] that the "$q = 1$" limit of certain families of objects in $\mathcal{F}$ give rise to graded representations of the corresponding current algebra. Recently, there has been a lot of work to understand these graded representations of the current algebra in an attempt to understand the representations of the corresponding quantum affine algebra. See for example [9].

It is well known that $\mathcal{F}$ is not a semi-simple category and the tensor product of irreducible representations in $\mathcal{F}$ remains irreducible in general. This leads naturally to the definition of prime objects, that is objects that cannot be decomposed into nontrivial tensor products. More generally, one is interested in the factorization of simple objects into a tensor product of prime representations. When $\mathfrak{g} = \mathfrak{sl}_2$, it was shown in [4] that the finite-dimensional, simple prime objects are precisely the evaluation modules. Outside of the $\mathfrak{sl}_2$-case, a complete classification of the simple prime objects is not known. However, many examples of prime objects are known, for example the Kirillov-Reshetikhin modules, and more generally the minimal affinizations are known to be prime. Other examples of prime objects may be found in [13].

In this dissertation, we continue the investigation of prime objects and prime factorizations initiated in [3], by analyzing the homological properties of $\mathcal{F}$. In [3], it was shown for $\mathfrak{g} = \mathfrak{sl}_2$ that a simple object of $\mathcal{F}$ is prime if and only if

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = 1.$$

Our main result (Theorem 1) generalizes the above statement to prime factorizations. We show that a simple object V of $\mathcal{F}$ is a tensor product of r simple prime representations if and only if

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = r.$$

The proof yields some results that we believe are of independent interest. In addition to proving the above, we give a presentation of V in terms of generators and relations when V is a tensor power of an evaluation module, i.e. a prime power. In particular, we show in this case that V is the quotient of the corresponding local Weyl module obtained by imposing a single linear relation. This linear relation allows us to construct an explicit basis of $\operatorname{Ext}^1_{\mathcal{F}}(V, V)$ as a vector space over $\mathbb{C}$ in this case and give a classification, up to equivalence, of the self-extensions of V in terms of polynomials whose coefficients involve the Eulerian numbers.

The dissertation is organized as follows: In Chapter 1, we introduce the notation and develop the necessary preliminaries used in the dissertation, recall the classification of simple and simple prime objects in $\mathcal{F}$, and recall the definition of and various results about $\mathrm{Ext}^1_{\mathcal{F}}(V,V)$. In Chapter 2 we state the main results of the dissertation. In Chapter 3 we prove an upper bound on the dimension of $\mathrm{Ext}^1_{\mathcal{F}}(V,V)$, namely the dimension of $\mathrm{Ext}^1_{\mathcal{F}}(V,V)$ is less than or equal to the number of prime factors in a prime factorization of V.

In Chapter 4, we prove Theorem 1 in the case that V is a "prime power", i.e. a tensor power of an evaluation module. To do so, we prove the previously mentioned presentation of V in terms of generators and relations (Theorem 2) and give a classification of the self-extensions of V (Theorem 3). Finally, in Chapter 5, for an arbitrary simple object V, we prove a lower bound on the the dimension of $\mathrm{Ext}^1_{\mathcal{F}}(V,V)$, namely the dimension of $\mathrm{Ext}^1_{\mathcal{F}}(V,V)$ is greater than or equal to the number of prime factors in the prime factorization of V.

Chapter 1

Notation and preliminaries

In this chapter we will set notation and define the algebras that we will study, review basic properties of their structure, discuss important subalgebras, review basic facts about their representations and their homological properties.

1.1 Notation

Throughout this dissertation $\mathbb{C}$ (resp. $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$) denotes the set of complex numbers (resp. integers, non-negative integers, positive integers) and $\mathbb{C}^\times$ is the set of non-zero complex numbers. Given an indeterminate u, let $\mathbb{C}[u]$ (resp. $\mathbb{C}(u)$, $\mathbb{C}[[u]]$) be the algebra of polynomials (resp. field of rational functions, ring of formal power series) in u with coefficients in $\mathbb{C}$. We fix $q \in \mathbb{C}^\times$ not a root of unity. For $m \in \mathbb{Z}$, $r, s \in \mathbb{Z}_+$ we define

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [0]! = 1, \quad [r]! = [r][r-1]\cdots[2][1], \quad \begin{bmatrix} r \\ s \end{bmatrix}_q = \frac{[r]!}{[r-s]![s]!}$$

1.2 The quantum affine algebra

Let $\hat{\mathbf{U}}_q(\mathfrak{sl}_2)$ be the associative algebra defined over $\mathbb{C}$ which is generated by the elements $e_i^{\pm}$, $k_i^{\pm 1}$, $i = 0, 1$, with defining relations: for $i, j \in \{0, 1\}$,

$$k_i k_i^{-1} = 1, \quad k_i k_j = k_j k_i, \quad k_i e_j^{\pm} k_i^{-1} = q^{\pm 2} e_j^{\pm}, \quad [e_i^+, e_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q - q^{-1}},$$

$$(e_i^{\pm})^3 - [3](e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3] e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \quad i \neq j. \tag{1.2.1}$$

It is well-known that $\hat{\mathbf{U}}_q(\mathfrak{sl}_2)$ is a Hopf algebra with counit ε, comultiplication Δ, and antipode S, defined on generators as follows: for $i = 0, 1$,

$$\varepsilon(k_i) = 1, \quad \varepsilon(e_i^{\pm}) = 0, \quad \Delta(k_i) = k_i \otimes k_i,$$

$$\Delta(e_i^+) = e_i^+ \otimes 1 + k_i \otimes e_i^+, \quad \Delta(e_i^-) = e_i^- \otimes k_i^{-1} + 1 \otimes e_i^-,$$

$$S(k_i) = k_i^{-1}, \quad S(e_i^+) = -k_i^{-1} e_i^+, \quad S(e_i^-) = -e_i^- k_i.$$

1.3 The quantum loop algebra

The quantum loop algebra $\hat{\mathbf{U}}_q$ is the quotient of $\hat{\mathbf{U}}_q(\mathfrak{sl}_2)$ by the two sided ideal generated by $k_0 k_1 - 1$. It is clearly a Hopf ideal, and thus $\hat{\mathbf{U}}_q$ acquires the structure of a Hopf algebra.

The algebra $\hat{\mathbf{U}}_q$ has an alternate presentation which was given in [10] and is more suited for the study of finite-dimensional representations. Namely $\hat{\mathbf{U}}_q$ is isomorphic to the associative algebra over $\mathbb{C}$ with generators $x_r^{\pm}$, h_s, $k^{\pm 1}$, where $r \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$, and defining relations: for all $r, \ell \in \mathbb{Z}$ and for all $s, s' \in \mathbb{Z} \setminus \{0\}$,

$$kk^{-1} = 1, \quad kh_s = h_s k, \quad h_s h_{s'} = h_{s'} h_s,$$

$$[h_s, x_r^{\pm}] = \pm\frac{1}{s}[2s]x_{r+s}^{\pm}, \quad kx_r^{\pm}k^{-1} = q^{\pm 2}x_r^{\pm},$$

$$x_r^{\pm}x_\ell^{\pm} - q^{\pm 2}x_\ell^{\pm}x_r^{\pm} = q^{\pm 2}x_{r-1}^{\pm}x_{\ell+1}^{\pm} - x_{\ell+1}^{\pm}x_{r-1}^{\pm},$$

$$[x_r^+, x_\ell^-] = \frac{\phi_{r+\ell}^+ - \phi_{r+\ell}^-}{q - q^{-1}},$$

where for $m > 0$, $\phi_{\mp m}^{\pm} = 0$, and for $m \geq 0$, $\phi_{\pm m}^{\pm}$ are defined by the following equality of formal power series in u,

$$\sum_{m\geq 0} \phi_{\pm m}^{\pm} u^m = k^{\pm 1}\exp\left(\pm(q - q^{-1})\sum_{s>0} h_{\pm s}u^s\right).$$

It will often be convenient to use the following convention: for $s \in \mathbb{Z}$, set

$$\phi_s = \begin{cases} \phi_s^+, & s > 0 \\ (k - k^{-1}), & s = 0 \\ -\phi_s^-, & s < 0 \end{cases} \tag{1.3.1}$$

Note that with this definition, we have,

$$[x_\ell^+, x_s^-] = \frac{\phi_{\ell+s}}{q - q^{-1}}, \qquad \text{for all } \ell, s \in \mathbb{Z}$$

The algebra $\hat{\mathbf{U}}_q$ has a $\mathbb{Z}$-grading satisfying

$$\operatorname{gr} x_r^{\pm} = r, \quad \operatorname{gr} h_s = s, \quad \operatorname{gr} \phi_m = m \tag{1.3.2}$$

for $r, m \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$.

1.3.1

We will also need the subalgebra $\mathbf{U}_q$ generated by the elements $e_1^{\pm}, k_1^{\pm 1}$ which we recall is isomorphic to the quantized enveloping algebra associated to $\mathfrak{sl}_2$. Given $a \in \mathbb{C}^{\times}$, let

$\mathrm{ev}_a \colon \hat{\mathbf{U}}_q \to \mathbf{U}_q$ be the homomorphism of algebras given on generators by

$$\mathrm{ev}_a(k) = k_1, \quad \mathrm{ev}_a(x_s^+) = a^s k_1^s e_1^+, \quad \mathrm{ev}_a(x_s^-) = a^s e_1^- k_1^s \quad s \in \mathbb{Z}.$$

Remark 1.3.1. It is important to recall that we are working in the $\mathfrak{sl}_2$-case in this dissertation and outside of the $\mathfrak{sl}_n$-case there is no analogue of the maps ev_a.

1.3.2

Let $\hat{\mathbf{U}}_q^{\pm}$ (resp. $\hat{\mathbf{U}}_q(0)$) be the subalgebra of $\hat{\mathbf{U}}_q$ generated by the elements $\{x_r^{\pm} : r \in \mathbb{Z}\}$ (resp. $\{\phi_m : m \in \mathbb{Z}\}$). We have an isomorphism of vector spaces,

$$\hat{\mathbf{U}}_q = \hat{\mathbf{U}}_q^- \hat{\mathbf{U}}_q(0) \hat{\mathbf{U}}_q^+. \tag{1.3.3}$$

Let $\hat{\mathbf{U}}_q^0$ be the subalgebra of $\hat{\mathbf{U}}_q(0)$ generated by $\{\phi_s : s \in \mathbb{Z} \setminus \{0\}\}$. It is well-known [1, 2] that it is a $\mathbb{Z}$-graded polynomial algebra in these variables. It is clear from the defining relations that it is also the polynomial algebra in the variables $\{h_s : s \in \mathbb{Z} \setminus \{0\}\}$. We will also need a third set of algebraically independent generators for $\hat{\mathbf{U}}_q^0$, originally defined in [4]. We define Λ_m, $m \in \mathbb{Z}$ by the following equality of formal power series in u,

$$\sum_{m \geq 0} \Lambda_{\pm m} u^m = \exp\left(-\sum_{s \geq 1} \frac{h_{\pm s}}{[s]} u^s \right). \tag{1.3.4}$$

Notice, if we set

$$\Phi^{\pm}(u) = \sum_{s \geq 0} \phi_{\pm s}^{\pm} u^s, \quad \Lambda^{\pm}(u) = \sum_{m \geq 0} \Lambda_{\pm m} u^m,$$

then we have the following equality of formal power series in u,

$$\Phi^{\pm}(u) = k^{\pm 1} \frac{\Lambda^{\pm}(q^{\mp 1} u)}{\Lambda^{\pm}(q^{\pm 1} u)}. \tag{1.3.5}$$

1.3.3

The next proposition, proved in [6], gives partial information on the comultiplication of $\hat{\mathbf{U}}_q$ sufficient for our purposes. Let $\Delta : \hat{\mathbf{U}}_q \to \hat{\mathbf{U}}_q \otimes \hat{\mathbf{U}}_q$ be the comultiplication. Let $X_\pm$ be the subspaces of $\hat{\mathbf{U}}_q$ spanned by $x_s^\pm$, $s \in \mathbb{Z}$.

Proposition 1.3.2. *The comultiplication Δ of $\hat{\mathbf{U}}_q$ satisfies, $\Delta(k) = k \otimes k$, and*

(i) Modulo terms in $\hat{\mathbf{U}}_q X_- \otimes \hat{\mathbf{U}}_q X_+^2$,

$$\Delta(x_s^+) = x_s^+ \otimes 1 + k \otimes x_s^+ + \sum_{j=1}^{s} \phi_j \otimes x_{s-j}^+, \quad s \geq 0,$$

$$\Delta(x_{-s}^+) = x_{-s}^+ \otimes 1 + k^{-1} \otimes x_{-s}^+ + \sum_{j=1}^{s-1} \phi_{-j} \otimes x_{-(s-j)}^+, \quad s > 0.$$

(ii) Modulo terms in $\hat{\mathbf{U}}_q X_-^2 \otimes \hat{\mathbf{U}}_q X_+$,

$$\Delta(x_s^-) = x_s^- \otimes k + 1 \otimes x_s^- + \sum_{j=1}^{s-1} x_{s-j}^- \otimes \phi_j, \quad s > 0,$$

$$\Delta(x_{-s}^-) = x_{-s}^- \otimes k^{-1} + 1 \otimes x_{-s}^- + \sum_{j=1}^{s} x_{-(s-j)}^- \otimes \phi_{-j}, \quad s \geq 0.$$

(iii) Modulo terms in $\hat{\mathbf{U}}_q X_- \otimes \hat{\mathbf{U}}_q X_+ + \hat{\mathbf{U}}_q X_+ \otimes \hat{\mathbf{U}}_q X_-$ we have for $s > 0$,

$$\Delta(\phi_{\pm s}^\pm) = \sum_{j=0}^{s} \phi_{\pm(s-j)}^\pm \otimes \phi_{\pm j}^\pm, \quad \Delta(\Lambda_{\pm s}) = \sum_{j=0}^{s} \Lambda_{\pm(s-j)} \otimes \Lambda_{\pm j},$$

$$\Delta(h_{\pm s}) = h_{\pm s} \otimes 1 + 1 \otimes h_{\pm s}$$

□

1.4 Finite dimensional representations

A representation V of $\mathbf{U}_q$ or $\hat{\mathbf{U}}_q$ is said to be of type 1 if

$$V = \bigoplus_{\mu \in \mathbb{Z}} V_\mu, \quad V_\mu = \{v \in V : kv = q^\mu v\} \tag{1.4.1}$$

Let $\mathcal{F}$ be the category of finite-dimensional, type 1 representations of $\hat{\mathbf{U}}_q$. Since the algebra $\hat{\mathbf{U}}_q$ is Hopf algebra, it follows that $\mathcal{F}$ is a tensor category. It is however, not a semi-simple category and one of the goals of this dissertation is to explore the relationship between the tensor structure on $\mathcal{F}$ and its homological properties. To make this precise we have the following definition:

Definition 1. We say that an object V of $\mathcal{F}$ is *prime* if it is either trivial or cannot be written in a nontrivial way as a tensor product $V \cong V_1 \otimes V_2$ where V_1, V_2 are nontrivial objects of $\mathcal{F}$.

Since any object of $\mathcal{F}$ is finite dimensional, clearly any object can be written as a tensor product

$$V \cong V_1 \otimes \cdots \otimes V_r, \quad V_s \text{ prime.} \tag{1.4.2}$$

Definition 2. We call (1.4.2) a *prime factorization* of V with *prime factors* V_s, $1 \leq s \leq r$.

We note that it is far from clear that such a factorization is unique up to a permutation. In fact it is not clear that r is independent of the choice of prime factors. Observe also that if V is irreducible then every prime factor of V is also irreducible. We shall see later in this chapter that these facts are true for $\hat{\mathbf{U}}_q$ (which is the quantized affine algebra associated to $\mathfrak{sl}_2$), but this is not known for the higher rank algebras.

1.5 Irreducible and prime objects of $\mathcal{F}$

We now discuss the classification and construction of the irreducible and prime irreducible objects of $\mathcal{F}$. It is well-known [15] that the irreducible representations of $\mathbf{U}_q$ are indexed by the non-negative integers. Moreover, if we denote by $V(m)$ an irreducible representation of $\mathbf{U}_q$ associated to $m \in \mathbb{Z}_+$, then $\dim_{\mathbb{C}} V(m) = m+1$ and $V(m)$ is prime, i.e., is not isomorphic to a nontrivial tensor product of $\mathbf{U}_q$ representations. It is also known that any finite-dimensional representation of $\mathbf{U}_q$ is completely reducible. It follows that for every $m \in \mathbb{Z}_+$ we get by using $\mathrm{ev}_a : \hat{\mathbf{U}}_q \to \mathbf{U}_q$, $a \in \mathbb{C}^\times$, an $(m+1)$-dimensional, prime nontrivial, irreducible representation, denoted $V(\omega_{m,a})$ defined as follows: $V(\omega_{m,a})$ has a basis $v_m, \dots, v_0$ and the action of $\hat{\mathbf{U}}_q$ is given on generators by,

$$x_s^+ v_i = (aq^{-m+2i+2})^s [i+1] v_{i+1}, \quad x_s^- v_i = (aq^{-m+2i})^s [m-i+1] v_{i-1}, \tag{1.5.1}$$

where $0 \le i \le m$ and $v_{-1} = v_{m+1} = 0$. The action of ϕ_s and h_s for all $s \in \mathbb{Z} \setminus \{0\}$ can be determined using the defining relations. For future use we have,

$$\phi_s v_m = (aq^m)^s (q^m - q^{-m}) v_m, \quad h_s v_m = a^s \frac{[sm]}{s} v_m. \tag{1.5.2}$$

Note that $V(\omega_{0,a})$ and $V(\omega_{m,0})$ are the trivial module and will just be denoted as $\mathbb{C}$. Moreover, $V(\omega_{m,a}) \cong V(\omega_{n,b})$ if and only if $m = n$ and $a = b$.

Since $\hat{\mathbf{U}}_q$ is a Hopf algebra, the vector space dual V^* of an object of $\mathcal{F}$ is also in $\mathcal{F}$, with action given by the the antipode S of $\hat{\mathbf{U}}_q$,

$$(xf)(v) = f(S(x)v), \quad x \in \hat{\mathbf{U}}_q,\ f \in V^*,\ v \in V.$$

It was shown in [4] that the antipode S of $\hat{\mathbf{U}}_q$ satisfies the relation $S \circ \mathrm{ev}_{aq^2} = \mathrm{ev}_a \circ S$, and hence,

$$V(\omega_{m,a})^* \cong V(\omega_{m,aq^2}). \tag{1.5.3}$$

The following result was proved in [4].

Proposition 1.5.1.

(i) Any nontrivial prime irreducible object of $\hat{\mathbf{U}}_q$ is isomorphic to $V(\omega_{m,a})$ for a unique choice of $m \in \mathbb{N}$, $a \in \mathbb{C}^\times$.

(ii) Let V be an irreducible object of $\mathcal{F}$. There exists a unique integer r and unique (up to ordering) elements $(m_s, a_s) \in \mathbb{N} \times \mathbb{C}^\times$, $1 \le s \le r$ with

$$\frac{a_s}{a_\ell} \ne q^{\pm(m_s+m_\ell-2p)}, \quad 0 \le p < \min\{m_s, m_\ell\},$$

such that

$$V \cong_{\hat{\mathbf{U}}_q} V(\omega_{m_1,a_1}) \otimes \cdots \otimes V(\omega_{m_r,a_r}).$$

Moreover, if we set $\mathbf{v} = v_{m_1} \otimes \cdots \otimes v_{m_s}$ and $m = m_1 + \cdots + m_r$, then for all $k \in \mathbb{Z}$, $s \in \mathbb{Z} \setminus \{0\}$, we have

$$x_k^+\mathbf{v} = 0, \qquad k\mathbf{v} = q^m\mathbf{v}, \qquad h_s\mathbf{v} = \sum_{\ell=1}^{r} a_\ell^m \frac{[sm_\ell]}{s}\mathbf{v}, \qquad (x_0^-)^{m+1}\mathbf{v} = 0.$$

□

It is worth isolating the following corollary.

Corollary 1.5.2. *Any nontrivial irreducible representation in $\mathcal{F}$ has a unique factorization (up to permutations) as a tensor product of prime representations, and we can write*

$$V \cong V_1^{\otimes s_1} \otimes \cdots \otimes V_r^{\otimes s_r},$$

where $r \geq 1$ and $V_1, \cdots, V_r$ are non-isomorphic nontrivial prime representations and $s_1, \ldots, s_r \in \mathbb{N}$. □

With the notation of the corollary, we call $V_1, \cdots, V_r$ the *prime factors* of V occurring with multiplicity $s_1, \cdots, s_r$.

1.5.1

Given $(m, a) \in \mathbb{N} \times \mathbb{C}^\times$, define the following polynomial in $\mathbb{C}[u]$,

$$\omega_{m,a}(u) = (1 - aq^{m-1}u)(1 - aq^{m-3}u) \cdots (1 - aq^{-m+1}u). \tag{1.5.4}$$

Notice we have,

$$\omega_{m,a^{-1}}(u) = (-a)^m u^m \omega_{m,a}(u^{-1}). \tag{1.5.5}$$

Let V be any irreducible object of $\mathcal{F}$ generated by a vector $\mathbf{v}$, and let $(m_i, a_i) \in \mathbb{N} \times \mathbb{C}^\times$ for $1 \leq i \leq r$ such that $V \cong V(\omega_{m_1,a_1}) \otimes \cdots \otimes V(\omega_{m_r,a_r})$ as in Proposition 1.5.1. It is clear from the relations in part (ii) of Proposition 1.5.1 that there exist complex numbers d_s, $s \in \mathbb{Z}$ such that $\Lambda_s \mathbf{v} = d_s \mathbf{v}$. The following result was proven in [4] and explains our previous notation.

Proposition 1.5.3. *With the notation above, we have an equalities in $\mathbb{C}[u]$,*

$$\sum_{s \geq 0} d_s u^s = \prod_{j=1}^{r} \omega_{m_j,a_j}(u), \quad \sum_{s \geq 0} d_{-s} u^s = \prod_{j=1}^{r} \omega_{m_j,a_j^{-1}}(u).$$

□

Set $\pi_V(u) = \prod_{i=1}^r \omega_{m_i,a_i}(u)$. Using equation (1.5.5) we see that d_s for $s \leq 0$ is determined by π_V.

In view of Proposition 1.5.3, we shall use the following convention: for all $s \in \mathbb{Z}$, set $\Lambda_s(\pi_V) = d_s$, with d_s as in the proposition. Then we have,

$$\Lambda_s \mathbf{v} = \Lambda_s(\pi_V)\mathbf{v}. \tag{1.5.6}$$

Define $\phi_s(\pi_V)$, $s \in \mathbb{Z}$ similarly.

1.6 Self-extensions in $\mathcal{F}$

We recall the definition of self-extensions of objects of $\mathcal{F}$. We shall always work with the Yoneda Ext-groups. We discuss the main tools we will need and refer the interested reader to [19] for a more detailed explanation.

Given an object U of $\mathcal{F}$ we say that an object V of $\mathcal{F}$ is a self-extension of U if we have a short exact sequence in $\mathcal{F}$ of the form,

$$0 \to U \xrightarrow{\iota} V \xrightarrow{\tau} U \to 0.$$

We say that V is trivial if the sequence splits and say it is nontrivial otherwise. Given self-extensions, V_1 and V_2 of U, we say V_1 and V_2 are equivalent if we have a commutative diagram of the form,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{\iota_1} & V_1 & \xrightarrow{\tau_1} & U & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & U & \xrightarrow{\iota_2} & V_2 & \xrightarrow{\tau_2} & U & \longrightarrow & 0 \end{array} \tag{1.6.1}$$

and we call the set of equivalences classes of the above relation $\mathrm{Ext}^1_{\mathcal{F}}(U,U)$. Given a self-extension V of U, we shall denote the equivalence class of V in $\mathrm{Ext}^1_{\mathcal{F}}(U,U)$ by $[V]$, and the equivalence class of the trivial extension is denoted by $[0]$. Note, if $[V] = [0]$, then

$V \cong U \oplus U$ as $\hat{\mathbf{U}}_q$-modules and the trivial object $\mathbb{C}$ satisfies $\mathrm{Ext}^1_{\mathcal{F}}(\mathbb{C}, \mathbb{C}) = 0$. Notice also, if V_1 and V_2 are self-extensions of U such that $[V_1] = [V_2]$ in $\mathrm{Ext}^1_{\mathcal{F}}(U, U)$, then by (1.6.1) it is clear that $V_1 \cong V_2$ as $\hat{\mathbf{U}}_q$-modules. The set $\mathrm{Ext}^1_{\mathcal{F}}(U, U)$ is an abelian group under the Baer sum and has a natural scalar multiplication and hence is actually a complex vector space.

It is a well known consequence of the adjoint pairs of exact functors,

$$(- \otimes V, - \otimes V^*), \quad (V^* \otimes -, V \otimes -),$$

that for any objects U, V, and W in $\mathcal{F}$, we have natural isomorphisms in the category of $\mathbb{C}$-vector spaces,

$$\mathrm{Ext}^1_{\mathcal{F}}(W \otimes V, U) \cong \mathrm{Ext}^1_{\mathcal{F}}(W, U \otimes V^*), \quad \mathrm{Ext}^1_{\mathcal{F}}(W, V \otimes U) \cong \mathrm{Ext}^1_{\mathcal{F}}(V^* \otimes W, U). \tag{1.6.2}$$

1.6.1

It is easy to construct a self-extension of any object V of $\mathcal{F}$. For $s \in \mathbb{Z}$, let $\hat{\mathbf{U}}_q[s]$ be the s-th graded piece of $\hat{\mathbf{U}}_q$. Given $V \in \mathcal{F}$ define an object $\mathbf{E}(V)$ of $\mathcal{F}$ by requiring that $\mathbf{E}(V) = V \oplus V$ as a vector space and the action of $\hat{\mathbf{U}}_q$ is given by,

$$g_s(v, w) = (g_s v, s g_s v + g_s w), \quad g_s \in \hat{\mathbf{U}}_q[s], \quad v, w \in V \tag{1.6.3}$$

The following was proved in [3, Propositions 3.3, 3.6, 5.4 and Lemma 3.5].

Proposition 1.6.1. *Let V, V' be irreducible objects of $\mathcal{F}$ such that $V \otimes V'$ is irreducible.*

(i) The representation $\mathbf{E}(V)$ defines a non-zero element of $\mathrm{Ext}^1_{\mathcal{F}}(V, V)$.

(ii) If $\tilde{V}$ is a nontrivial self-extension of V, then $\tilde{V} \otimes V'$ is a nontrivial self-extension of $V \otimes V'$.

(iii) If $\tilde{V}_1$ *and* $\tilde{V}_2$ *are nontrivial self-extensions of* V*, then*

$$\tilde{V}_1 \otimes V' \cong_{\hat{\mathbf{U}}_q} \tilde{V}_2 \otimes V' \quad \text{if and only if} \quad \tilde{V}_1 \cong_{\hat{\mathbf{U}}_q} \tilde{V}_2.$$

(iv) Suppose that V *and* V' *are non-isomorphic objects of* $\mathcal{F}$*; then* $\mathbf{E}(V) \otimes V'$ *and* $V \otimes \mathbf{E}(V')$ *define linearly independent elements of* $\mathrm{Ext}^1_{\mathcal{F}}(V \otimes V', V \otimes V')$.

□

We have the following consequence.

Corollary 1.6.2. *Let* V, V' *be irreducible objects of* $\mathcal{F}$ *such that* $V \otimes V'$ *is irreducible. Then, we have an injective map of vector spaces,*

$$p : \mathrm{Ext}^1_{\mathcal{F}}(V, V) \to \mathrm{Ext}^1_{\mathcal{F}}(V \otimes V', V \otimes V'), \quad [V] \mapsto [V \otimes V']$$

Proof. The fact that p is a well-defined and linear follows from the exactness of the functor $(- \otimes V')$ and the vector space structure of $\mathrm{Ext}^1_{\mathcal{F}}(V, V)$. The fact that p is injective now follows immediately from part (ii) of Proposition 1.6.1. □

Chapter 2

Statements of results

In this chapter we present the main results of this dissertation.

2.1 The main result

The main result of this dissertation is:

Theorem 1. *Let V be an irreducible object in $\mathcal{F}$. Then*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = r, \quad \textit{equivalently} \quad \dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(\mathbb{C}, V \otimes V^*) = r$$

if and only if there exists irreducible prime objects of $\mathcal{F}$, V_s, $1 \leq s \leq r$ such that

$$V \cong V_1 \otimes \cdots \otimes V_r$$

Remark 2.1.1. The theorem generalizes the work of [3] in the case of quantum affine $\mathfrak{sl}_2$. It was shown in that paper, that if V is irreducible and had r non-isomorphic prime factors (possibly occurring with multiplicity), then $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) \geq r$. It was also shown that

V was prime and irreducible if and only if $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = 1$. We discuss this further in the rest of this chapter. We also isolate some results that we establish in the course of proving Theorem 1 that we believe are of independent interest.

2.2 Generators and relations of prime powers

It is clear from Proposition 1.6.1(iv) that one of the obstructions to proving Theorem 1 comes from the case $V(\omega_{m,a})^{\otimes r}$; notice that the representation is irreducible by Proposition 1.5.1(ii). A key step in the proof of Theorem 1 is to prove this case. Note that the relations satisfied by the element $\mathbf{v}$ in Proposition 1.5.1(ii) are not a defining set of relations for the irreducible module $V(\omega_{m_1,a_1}) \otimes \cdots \otimes V(\omega_{m_r,a_r})$ and in fact such relations are not known in general. Our next result gives a defining set of relations in the case of $V(\omega_{m,a})^{\otimes r}$. Define an element $\mathbf{x}_a^-(m,r) \in \hat{\mathbf{U}}_q$, by

$$\mathbf{x}_a^-(m,r) = \sum_{s=0}^{r} (-aq^m)^s \binom{r}{s} x_{r-s}^-. \tag{2.2.1}$$

We shall prove,

Theorem 2. *The $\hat{\mathbf{U}}_q$-module $V(\omega_{m,a})^{\otimes r}$ is isomorphic to the $\hat{\mathbf{U}}_q$-module generated by a non-zero vector $\mathbf{v}$ with defining relations: for all $j \in \mathbb{Z}$ and $s \in \mathbb{Z} \setminus \{0\}$,*

$$x_j^+ \mathbf{v} = 0, \qquad k\mathbf{v} = q^{rm}\mathbf{v}, \qquad h_s \mathbf{v} = ra^s \frac{[sm]}{s} \mathbf{v},$$

$$(x_0^-)^{rm+1}\mathbf{v} = 0, \qquad \mathbf{x}_a^-(m,r)\mathbf{v} = 0. \tag{2.2.2}$$

The proof of Theorem 2 will be given in Chapter 4, Section 4.1.

2.3 Self-extensions of prime powers: a classification

Using Theorem 2 we can now construct all of the self-extensions of $V(\omega_{m,a})^{\otimes r}$ (up to equivalence). For the rest of this section, we assume that m, a, r are all fixed. Let $\mathbf{v}$ be as in Theorem 2 and recall for $s \in \mathbb{Z} \setminus \{0\}$,

$$\Lambda_s \mathbf{v} = \Lambda_s(\omega^r_{m,a})\mathbf{v}.$$

Given $c \in \mathbb{C}$, let $\mathbf{D}\colon \mathbb{C}[u] \to \mathbb{C}[u]$ be the operator defined by,

$$\mathbf{D}_c[f] = (cu)\frac{d}{du}f, \quad f \in \mathbb{C}[u].$$

For each $1 \le j < r$, define $\mathbf{Z}_j(u)$ by,

$$\mathbf{Z}_j(u) = \begin{cases} [\omega_{m,aq^2}(u)]^r \mathbf{D}^j_{aq^{m+1}} \left[\frac{1}{1-aq^{m+1}u}\right], & 1 \le j < r, \\ \mathbf{D}_1[\omega_{m,a}(u)], & j = r. \end{cases}$$

Given a vector $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{C}^r$ set,

$$\mathbf{Z}_{\mathbf{c}}(u) = \sum_{j=1}^{r} c_j \mathbf{Z}_j(u).$$

We shall prove,

Theorem 3. *Given any nonzero vector* $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{C}^r$, *there exists a nontrivial self-extension* $V(\mathbf{c})$ *of* $V(\omega_{m,a})^{\otimes r}$ *such that the action of* $\hat{\mathbf{U}}^0_q$ *on* $V(\mathbf{c})_{rm}$ *is determined by the functional equations,*

$$\Lambda^+(u)(v, w) = ([\omega_{m,a}(u)]^r v, \ \ [\omega_{m,a}(u)]^r w + \mathbf{Z}_{\mathbf{c}}(u)v), \quad v, w \in V(\mathbf{c})_{rm}.$$

The equivalence classes $[V(\mathbf{e}_j)]$, $1 \le j \le r$ *form a basis of* $\mathrm{Ext}^1_{\mathcal{F}}(V(\omega_{m,a})^{\otimes r}, V(\omega_{m,a})^{\otimes r})$. *In particular,* $V(\mathbf{c})$ *is uniquely determined (up to isomorphism) by the Baer sum*

$$[V(\mathbf{c})] = \sum_{j=1}^{r} c_j [V(\mathbf{e}_j)],$$

and if V *is any self-extension of* $V(\omega_{m,a})^{\otimes r}$, *then* $V \cong V(\mathbf{c})$ *as* $\hat{\mathbf{U}}_q$*-modules for some* $\mathbf{c} \in \mathbb{C}^r$.

We remark that $V(0)$ is isomorphic to the trivial self-extension. The proof of Theorem 3 will be given in Chapter 4.

Chapter 3

An upper bound

In this chapter we take the first step toward proving Theorem 1. Namely, our goal in this chapter is to prove the following proposition,

Proposition 3.0.1. *Let* V *be an irreducible object of* $\mathcal{F}$ *with prime factors* V_i*,* $1 \leq i \leq k$ *occurring with multiplicities* $r_i \in \mathbb{Z}_+$*,* $1 \leq i \leq k$*. Then,*

$$\dim \operatorname{Ext}^1_{\mathcal{F}}(V, V) \leq \sum_{i=1}^{k} r_i.$$

3.1 Jordan-Hölder multiplicities

In the case when $r = 1$, Proposition 3.0.1 was proved in [3] and in [14]. We recall the key step of the proof in [14] since this will play an important role in this dissertation. Given $(m, a) \in \mathbb{N} \times \mathbb{C}^\times$, set

$$V(\alpha_{m,a}) = V(\omega_{m+1,aq}) \otimes V(\omega_{m-1,aq}). \tag{3.1.1}$$

It follows from Proposition 1.5.1 that $V(\alpha_{m,a})$ is an irreducible object of $\mathcal{F}$.

Proposition 3.1.1. *Let V be an irreducible object of $\mathcal{F}$. Then,*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(\mathbb{C}, V) = \begin{cases} 1, & \text{if } \ V \cong_{\hat{\mathbf{U}}_q} V(\alpha_{m,a}) \ \text{ for some } \ (m,a) \in \mathbb{N} \times \mathbb{C}^\times, \\ \\ 0, & \text{otherwise.} \end{cases}$$

□

Given objects M and V of $\mathcal{F}$ such that V is irreducible, let $[M : V]$ be the multiplicity of V in a Jordan-Hölder series of M. The following consequence of Proposition 3.1.1 is proved by a straightforward induction on the length of a Jordan-Hölder series for M.

Corollary 3.1.2. *We have*

$$\dim \operatorname{Ext}^1_{\mathcal{F}}(\mathbb{C}, M) \leq \sum_{(m,a)\in\mathbb{N}\times\mathbb{C}^\times} [M : V(\alpha_{m,a})].$$

□

As a consequence of the corollary, we see that Proposition 3.0.1 follows if we prove that

$$\sum_{(m,a)\in\mathbb{N}\times\mathbb{C}^\times} [V \otimes V^*, V(\alpha_{m,a})] \leq r. \tag{3.1.2}$$

Establishing this inequality is our goal in this section from now on.

3.1.1

The first step in understanding the Jordan-Hölder multiplicities in the inequality (5.1.1) is the following proposition proved in [4].

Proposition 3.1.3. *Let $(m,a),(n,b) \in \mathbb{N}\times\mathbb{C}^\times$ and assume that $a/b = q^{\pm(m+n-2p)}$ for some $0 \leq p < \min\{m,n\}$. Then $V(\omega_{m,a}) \otimes V(\omega_{n,b})$ is indecomposable and has a Jordan-Hölder* *series of length two. The non-zero multiplicities are,*

$$[V(\omega_{m,a}) \otimes V(\omega_{n,b}) : V(\omega_{m-p,aq^{-p}}) \otimes V(\omega_{n-p,bq^p})] = 1,$$

$$[V(\omega_{m,a}) \otimes V(\omega_{n,b}) : V(\omega_{p-1,aq^{m-p+1}}) \otimes V(\omega_{m+n-p+1,bq^{-(m-p+1)}})] = 1.$$

Moreover, if U *is isomorphic to the unique irreducible submodule of* $V(\omega_{m,a}) \otimes V(\omega_{n,b})$ *then* U *is isomorphic to the unique irreducible quotient of* $V(\omega_{n,b}) \otimes V(\omega_{m,a})$.

□

The proof of the inequality (5.1.1) in the case when $V = V(\omega_{m,a})$ is then completed as follows. Recall from equation (1.5.3) that $V(\omega_{m,a})^* \cong V(\omega_{m,aq^2})$. Then, Proposition 3.1.3 tells us that the Jordan-Hölder constituents are the trivial representation and $V(\alpha_{m,a})$ each occurring with multiplicity one. Appyling the functor $\mathrm{Ext}^1_{\mathcal{F}}(\mathbb{C}, -)$ to the Jordan-Hölder series of $V \otimes V^*$ and using Proposition 3.1.1 now proves the inequality (5.1.1) in this case.

3.1.2

We need some further results on the Jordan-Hölder multiplicities in the (reducible) tensor products of irreducible prime objects of $\mathcal{F}$.

Lemma 3.1.4. *Suppose that* V_s, $1 \leq s \leq k$ *are irreducible prime representations of* $\hat{\mathbf{U}}_q$ *and let* σ *be any element of the symmetric group on* k*-letters. Then* $V_1 \otimes \cdots \otimes V_k$ *and* $V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(k)}$ *have the same Jordan-Hölder multiplicities.*

Proof. It suffices to show the result when $\sigma = (i, i+1)$ for some $1 \leq i < k$. If $V_i \otimes V_{i+1}$ is irreducible, then it follows from Corollary 1.5.1 that it is isomorphic to $V_{i+1} \otimes V_i$ and hence the Lemma follows. If $V_i \otimes V_{i+1}$ is reducible, then by Proposition 3.1.3 the Jordan-Hölder series of $V_i \otimes V_{i+1}$ and $V_{i+1} \otimes V_i$ are respectively,

$$0 \to U_1 \to V_i \otimes V_{i+1} \to U_2 \to 0, \quad 0 \to U_2 \to V_{i+1} \otimes V_i \to U_1 \to 0. \tag{3.1.3}$$

Let

$$\mathbf{V}_1 = V_1 \otimes \cdots \otimes V_{i-1} \quad \text{and} \quad \mathbf{V}_2 = V_{i+2} \otimes \cdots \otimes V_k$$

Applying the exact functors $(\mathbf{V}_1 \otimes -)$ and $(- \otimes \mathbf{V}_2)$ to the short exact sequences in equation (3.1.3) we see that

$$[\mathbf{V}_1 \otimes U_1 \otimes \mathbf{V}_2 : V] + [\mathbf{V}_1 \otimes U_2 \otimes \mathbf{V}_2 : V]$$

gives the multiplicity of a representation V in a Jordan-Hölder series for both $\mathbf{V}_1 \otimes V_i \otimes V_{i+1} \otimes \mathbf{V}_2$ and $\mathbf{V}_1 \otimes V_{i+1} \otimes V_i \otimes \mathbf{V}_2$. This proves the lemma. □

3.1.3

The following is a straightforward application of Proposition 3.1.3 by using the appropriate tensor functors.

Lemma 3.1.5. *Let $(m_j, a_j) \in \mathbb{N} \times \mathbb{C}^\times$ be distinct elements for $1 \leq j \leq k$. Then, for all $r_j \in \mathbb{N}$, $1 \leq j \leq k$, the module*

$$(V(\omega_{m_1,a_1}) \otimes V(\omega_{m_1,a_1q^2}))^{\otimes r_1} \otimes \cdots \otimes (V(\omega_{m_k,a_k}) \otimes V(\omega_{m_k,a_kq^2}))^{\otimes r_k},$$

has a decreasing filtration where the successive quotients are of the form

$$V(\alpha_{m_1,a_1})^{\otimes p_1} \otimes \cdots \otimes V(\alpha_{m_k,a_k})^{\otimes p_k}$$

where $p_k \in \mathbb{Z}_+$ $1 \le j \le k$ *and we understand that if all the* $p_k = 0$ *then we have the trivial module. Further, the multiplicity of* $V(\alpha_{m_j,a_j})$ *in this filtration is at most* r_j*.* □

3.1.4

Assume the following result for the moment.

Proposition 3.1.6. *Let* $(m_j, a_j) \in \mathbb{N} \times \mathbb{C}^\times$*,* $1 \le j \le k$ *be distinct and let* $r_j \in \mathbb{N}$ $1 \le j \le k$ *be such that* $r_1 + \cdots + r_k \ge 2$*. Assume moreover, that*

$$V(\omega_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\omega_{m_k,a_k})^{r_k}$$

is irreducible. Then, for all $(m, a) \in \mathbb{N} \times \mathbb{C}^\times$*, we have*

$$[V(\alpha_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\alpha_{m_k,a_k})^{r_k} : V(\alpha_{m,a})] = 0.$$

The proof of the inequality (5.1.1) is now completed as follows. Let V be an irreducible object of $\mathcal{F}$ with prime factors $V(\omega_{m_j,a_j})$ occurring with multiplicity $r_j \in \mathbb{N}$ for $1 \le j \le k$. Lemma 3.1.4 implies that $V \otimes V^*$ has the same Jordan-Hölder multiplicities as the module

$$\mathbf{V} = (V(\omega_{m_1,a_1}) \otimes V(\omega_{m_1,a_1q^2}))^{\otimes r_1} \otimes \cdots \otimes (V(\omega_{m_k,a_k}) \otimes V(\omega_{m_k,a_kq^2}))^{\otimes r_k},$$

and hence it suffices to prove that

$$\sum_{(m,a) \in \mathbb{N} \times \mathbb{C}^\times} [\mathbf{V} : V(\alpha_{m,a})] \le (r_1 + \cdots + r_k). \tag{3.1.4}$$

Let $\mathbf{V} = \mathbf{V}_0 \supset \mathbf{V}_1 \supset \cdots \supset \mathbf{V}_s \supset \mathbf{V}_{s+1} = \{0\}$, be the filtration of Lemma 3.1.5. Then

$$[\mathbf{V}, V(\alpha_{m,a})] = \sum_{p=0}^{s} [\mathbf{V}_p/\mathbf{V}_{p+1} : V(\alpha_{m,a})].$$

Using the last statement of Lemma 3.1.5 and Proposition 3.1.6, we have for each $0 \le p \le s$

$$[\mathbf{V}_p/\mathbf{V}_{p+1} : V(\alpha_{m,a})] \le \begin{cases} 0, & (m,a) \ne (m_j, a_j), \quad 1 \le j \le k, \\ \\ r_j, & (m,a) = (m_j, a_j), \end{cases}$$

thus proving the inequality in (3.1.4). The proof of Proposition 3.1.6 will be our goal from now on.

3.2 q-characters

One of the main tools to prove Proposition 3.1.6 is the theory of q-characters. We begin this section by recalling the notation and the main results that we shall need from this theory.

3.2.1

The elements ϕ_s, $s \in \mathbb{Z} \setminus \{0\}$ are commutative and hence, it follows that any object V of $\mathcal{F}$ can be written, as a sum of joint generalized eigenspaces for their action, namely

$$V = \bigoplus_{(\varpi^+, \varpi^-)} V_{(\varpi^+, \varpi^-)}, \quad \varpi^{\pm} = \sum_{s \ge 1} \varpi_s^{\pm} u^s \in \mathbb{C}[[u]], \tag{3.2.1}$$

where

$$V_{(\varpi^+, \varpi^-)} = \{v \in V : \left(\phi_m^{\pm} - \varpi_m^{\pm}\right)^N v = 0, \text{ for some } N = N(m) \in \mathbb{N} \text{ and all } m \in \mathbb{Z} \setminus \{0\}\}. \tag{3.2.2}$$

It was shown in [12] that $V_{(\varpi^+, \varpi^-)}$ is non-zero only if $\varpi^{\pm}(u)$ are actually rational functions in u with $\varpi^{\pm}(0) = 1$; moreover, the function $\varpi^-(u)$ is completely determined by $\varpi^+(u)$

and hence we shall just write V_ϖ, $\varpi \in \mathbb{C}(u)$ for the eigenspaces. Set

$$\mathrm{wt}_\ell V = \{\varpi \in \mathbb{C}(u) : V_\varpi \neq 0\}, \quad \mathrm{wt}_\ell^\pm V = (\mathrm{wt}_\ell V)^{\pm 1} \cap \mathbb{C}[u].$$

Moreover, if M is any subquotient or submodule of V, then $\mathrm{wt}_\ell M \subset \mathrm{wt}_\ell V$. Note that for the trivial module $\mathbb{C}$, we have

$$\mathrm{wt}_\ell \mathbb{C} = \mathrm{wt}_\ell^+ \mathbb{C} = \{1\}.$$

Since $\mathrm{wt}_\ell V \subset \mathbb{C}(u)$ for all objects V of $\mathcal{F}$, we use the standard notions such as the greatest common divisor (gcd), least common multiple(lcm) associated with $\mathbb{C}[u]$ freely; moreover if we write $\varpi = \varpi'/\varpi''$ we shall assume without mention that ϖ' and ϖ'' are coprime.

It follows from this discussion that Proposition 3.1.6 is a consequence of,

Proposition 3.2.1. *Let $(m_j, a_j) \in \mathbb{N} \times \mathbb{C}^\times$, $1 \leq j \leq k$ be distinct and let $r_j \in \mathbb{N}$ $1 \leq j \leq k$ be such that $r_1 + \cdots + r_k \geq 2$. Assume moreover, that*

$$V(\omega_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\omega_{m_k,a_k})^{r_k}$$

is irreducible. Then, for all $(m, a) \in \mathbb{N} \times \mathbb{C}^\times$, we have

$$\mathrm{wt}_\ell V(\alpha_{m,a}) \not\subset \mathrm{wt}_\ell \left(V(\alpha_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\alpha_{m_k,a_k})^{\otimes r_k}\right). \tag{3.2.3}$$

The proof of Proposition 3.2.1 will be given in Section 3.3.

3.2.2

Given $(m, a) \in \mathbb{N} \times \mathbb{C}^\times$, define polynomials in $\mathbb{C}[u]$ by,

$$\omega_{m,a} = (1 - aq^{m-1}u)(1 - aq^{m-3}u) \cdots (1 - aq^{-m+1}u),$$

$$\alpha_{m,a} = \omega_{m+1,aq}\omega_{m-1,aq} = \prod_{s=0}^{m-1} (1 - aq^{m+1-2s}u)(1 - aq^{m-1-2s}u).$$

It is also convenient to adopt the convention that $\omega_{0,a} = \alpha_{0,a} = 1$. The following was proved in [12] and explains our choice of notation.

Proposition 3.2.2. *We have,*

(i) For $(m,a) \in \mathbb{N} \times \mathbb{C}^\times$, we have

$$\mathrm{wt}_\ell\, V(\omega_{m,a}) = \{\omega_{m,a}\alpha^{-1}_{s,aq^{m-s}} : 0 \le s \le m\}, \tag{3.2.4}$$

or equivalently, $\varpi \in \mathrm{wt}_\ell\, V(\omega_{m,a})$ if and only if,

$$\varpi = \frac{(1-aq^{m-2s-1}u)(1-aq^{m-2s-3}u)\cdots(1-aq^{-m+1}u)}{(1-aq^{m+1}u)(1-aq^{m-1}u)\cdots(1-aq^{m-2s+3}u)}$$

for some $0 \le s \le m-1$, where we understand that if $s = 0$ then $\varpi = \omega_{m,a}$ and if $s = m-1$, then $\varpi = (\omega_{m,aq^2})^{-1}$.

(ii) Let M_j, $j = 1, 2$ be finite-dimensional $\mathbf{U}_q(\hat{\mathfrak{g}})$-modules. Then

$$\mathrm{wt}_\ell(M_1 \otimes M_2) = \mathrm{wt}_\ell\, M_1\, \mathrm{wt}_\ell\, M_2 = \mathrm{wt}_\ell\, M_2 \otimes M_1.$$

□

The following corollary is immediate.

Corollary 3.2.3. *We have*

$$\mathrm{wt}_\ell\, V(\alpha_{m,a}) = \{\alpha_{m,a}(\alpha_{s_1,aq^{m+1-s_2}})^{-1}(\alpha_{s_2,aq^{m-1-s_2}})^{-1} : 0 \le s_1 \le m+1, 0 \le s_2 \le m-1\}.$$

□

We note further consequences of the proposition. For all $r \geq 1$, we have

$$\mathrm{wt}_\ell^+ V(\omega_{m,a})^{\otimes r} = \{\omega_{m,a}^r\}, \quad \mathrm{wt}_\ell^- V(\omega_{m,a})^{\otimes r} = \{\omega_{m,aq^2}^r\}, \tag{3.2.5}$$

$$\mathrm{wt}_\ell^+ V(\alpha_{m,a})^{\otimes r} = \omega_{m+1,aq}^r \,\mathrm{wt}_\ell\, V(\omega_{m-1,aq})^{\otimes r} = \left\{ \prod_{j=1}^r \alpha_{m-s_j+1,aq^{-s_j+1}} : 0 \leq s_j \leq m \right\}, \tag{3.2.6}$$

$$\mathrm{wt}_\ell^- V(\alpha_{m,a})^{\otimes r} = \omega_{m+1,aq^3}^r \,\mathrm{wt}_\ell\, V(\omega_{m-1,aq})^{\otimes r} = \left\{ \prod_{j=1}^r \alpha_{m-s_j+1,aq^{s_j+1}} : 0 \leq s_j \leq m \right\}. \tag{3.2.7}$$

Finally, observe that if $\varpi = \varpi'/\varpi'' \in \mathrm{wt}_\ell\, V(\omega_{n_1,b_1}) \otimes \cdots \otimes V(\omega_{n_r,b_r})$ for some $(n_j, r_j) \in \mathbb{N} \times \mathbb{C}^\times$ then

$$\gcd(\omega', \omega_{n_1,b_1} \cdots \omega_{n_r,b_r}) = \omega', \qquad \gcd(\omega'', \omega_{n_1,b_1q^2} \cdots \omega_{n_r,b_rq^2}) = \omega''. \tag{3.2.8}$$

3.2.3

The following elementary lemma (whose proof we include for completeness) will be useful.

Lemma 3.2.4. *Let $a \in \mathbb{C}^\times$ and $p \in \mathbb{Z}_+$ and suppose that*

$$(1-au)^p \prod_{j=1}^s (1-b_ju)(1-b_jq^2u) = \prod_{j=1}^r (1-c_ju)(1-c_jq^2u), \tag{3.2.9}$$

for some $r, s \in \mathbb{Z}_+$ and $b_j, c_j \in \mathbb{C}^\times$. Then $p = 0$. Equivalently, the element $(1-au)^p$ is not in the multiplicative subgroup of $\mathbb{C}(u)$ generated by the elements $\{\alpha_{n,b} : (n,b) \in \mathbb{N} \times \mathbb{C}^\times\}$.

Proof. We proceed by induction on s, with induction obviously beginning at $s = 0$. Hence we can assume that $b_i, c_j \in aq^{\mathbb{Z}}$ for all $1 \leq i \leq r$ and $1 \leq j \leq s$. Let $m \in \mathbb{Z}$ be maximal so that $(1-aq^{m+2})$ divides the right hand side of equation (3.2.9). Then $(1-aq^m)(1-aq^{m+2})$ occurs on the right hand side of (3.2.9) and hence must also divide the left hand side of (3.2.9). Assume that $m \neq 0, -2$. Since $(1-aq^{m+4})$ does not divide the right hand side of (3.2.9), it follows that there must exist b_j with $b_j = aq^m$ and hence $(1-aq^m)(1-aq^{m+2})$ cancels on both sides of (3.2.9), i.e we get a similar expression with s replaced by $s-1$ and the result follows from the inductive hypothesis.

If $m = 0$, then $(1-aq^2u)$ must divide the left hand side of (3.2.9): i.e, there must exist b_i with $b_i = a$ or $b_i = aq^2$. In the first case the term $(1-au)(1-aq^2u)$ will cancel on both sides of (3.2.9) and induction gives the result again. In the second case, we have the situation where there is no b_i with $b_i = a$, which means that the left hand side of (3.2.9) has a term of the form $(1-aq^2u)(1-aq^4u)$, but this is impossible because $(1-aq^4u)$ does not occur on the right hand side of (3.2.9).

Finally, consider the case $m = -2$. If $p > 0$, then $(1-au)^p$ divides the right hand side of (3.2.9) and we must have $c_i = a$ for at least p values of i and we get an expression

$$\prod_{j=1}^{s}((1-b_ju)(1-b_jq^2u)) = (1-aq^{-2}u)^p \prod_{j=1}^{r-p}(1-c_ju)(1-c_jq^2u)).$$

Since $2r = 2s+p$, we see that $r-p < s$, and we get a contradiction to the inductive hypothesis. Hence $p = 0$ and the proof is complete. $\square$

The following consequence of Corollary 3.2.2(ii) and Lemma 3.2.4 will be crucial in the rest of this section.

$$(1-au)^p \notin \mathrm{wt}_\ell\,(V(\alpha_{n_1,b_1}) \otimes \cdots \otimes V(\alpha_{n_k,b_k})) \quad \text{if} \quad p \in \mathbb{Z} \setminus \{0\}. \tag{3.2.10}$$

3.2.4

Suppose that $(m_j, a_j) \in \mathbb{N} \times \mathbb{C}^\times$, $1 \leq j \leq k$ are distinct and let $r_j \in \mathbb{N}$, $1 \leq j \leq k$, be such that $r_1 + \cdots + r_k \geq 2$. Assume moreover, that

$$V = V(\omega_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\omega_{m_k,a_k})^{r_k}$$

is irreducible. Recall that the conditions given in Proposition 1.5.1 for this module to be irreducible are:

$$a_j/a_s \neq q^{\pm(m_j+m_s-2p)}, \quad 0 \leq p \leq \min\{m_j, m_s\}.$$

This means that we can and will assume without loss of generality (after a relabeling if necessary) that (m_1, a_1) is such that the following hold: for $2 \leq j \leq k$ we have, either $\gcd(\omega_{m_1,a_1}, \omega_{m_j,a_j}) = 1$, or $\gcd(\omega_{m_1,a_1}, \omega_{m_j,a_j}) = \omega_{m_j,a_j}$. Moreover, in the case $\gcd(\omega_{m_1,a_1}, \omega_{m_j,a_j}) = 1$, we may also assume that either $a_1 \notin a_j q^{\mathbb{Z}}$ or that $a_1 = a_j q^{m_1+m_j-2+s}$ where $s \in \mathbb{N} \setminus \{2\}$. Observe that with these choices the element ω_{m_1,a_1} divides $\mathrm{lcm}(\omega_{m_j,a_j} : 1 \leq j \leq k)$.

Set

$$J_1(V) = \{1 \leq j \leq k : \gcd(\omega_{m_1,a_1}, \omega_{m_j,a_j}) = \omega_{m_j,a_j}\},$$

$$J_2(V) = \{1, \cdots, k\} \setminus J_1(V) = \{1 \leq j \leq k : \gcd(\omega_{m_1,a_1}, \omega_{m_j,a_j}) = 1\}.$$

It is trivially checked that for $j \in J_1(V)$ and $j' \in J_2(V)$, we have

$$\gcd(\alpha_{m_j,a_jq^2}, \alpha_{m_{j'},a_{j'}}) = 1 = \gcd(\alpha_{m_j,a_j}, \alpha_{m_{j'},a_{j'}}). \qquad (3.2.11)$$

Consider the modules

$$\mathbf{V}_1 = \bigotimes_{j \in J_1(V)} V(\alpha_{m_j,a_j})^{\otimes r_j}, \quad \mathbf{V}_2 = \bigotimes_{j \in J_2(V)} V(\alpha_{m_j,a_j})^{\otimes r_j}.$$

where we understand that if $J_2(V) = \emptyset$, then $\mathbf{V}_2$ is the trivial module. Using Lemma 3.2.2(ii), we see that if

$$\mathbf{V} = V(\alpha_{m_1,a_1})^{\otimes r_1} \otimes \cdots \otimes V(\alpha_{m_k,a_k})^{\otimes r_k}.$$

then

$$\mathrm{wt}_\ell \mathbf{V} = \mathrm{wt}_\ell \mathbf{V}_1 \, \mathrm{wt}_\ell \mathbf{V}_2. \tag{3.2.12}$$

Proposition 3.2.5. *Retain the notation of this section.*

(i) For all $(\varpi_i'/\varpi_i'') \in \mathrm{wt}_\ell \mathbf{V}_i$, $i = 1, 2$, *we have* $\gcd(\varpi_1'', \varpi_2') = 1$.

(ii) We have

$$\mathrm{wt}_\ell^+ \mathbf{V} \subset \mathrm{wt}_\ell^+ \mathbf{V}_1 \, \mathrm{wt}_\ell \mathbf{V}_2.$$

(iii) For $\varpi^\pm \in \mathrm{wt}_\ell \mathbf{V}_1$,

$$\varpi^+ = \prod_{s=0}^{m_1} (1 - a_1 q^{-m+2s+1} u)^{p_s^+},, \qquad \varpi^- = \prod_{s=0}^{m_1} (1 - a q^{m-2s+3} u)^{p_s^-}$$

where $p_s^\pm \in \mathbb{Z}_+$ *are such that:* $p_0^\pm \geq r_1$, $\sum_{s=1}^{m_1} p_s^\pm > 0$.

(iv) $1 \notin \mathrm{wt}_\ell \mathbf{V}$.

Proof. Using equation (3.2.8), we have

$$\gcd(\varpi_1'', \prod_{j\in J_1} (\alpha_{m_j, a_j q^2})^{r_j}) = \varpi_1'', \qquad \gcd(\varpi_2', \prod_{j \in J_2} (\alpha_{m_j, a_j})^{r_j}) = \varpi_2'.$$

Equation 3.2.11 now forces $\gcd(\varpi_1'', \varpi_2') = 1$ and part (i) is proved.

To prove (ii), given, $\varpi \in \mathrm{wt}_\ell^+ \mathbf{V}$, using equation (3.2.12), we write it as $\varpi = (\varpi_1'/\varpi_1'')(\varpi_2'/\varpi_2'')$, with $(\varpi_i'/\varpi_i'') \in \mathrm{wt}_\ell \mathbf{V}_i$ for $i = 1, 2$. Using part (i) of the proposition we see immediately that $\varpi'' = 1$ and the proof is complete.

To prove part (iii), let $\varpi \in \mathrm{wt}_\ell^+ \mathbf{V}_1$ or equivalently assume that $J_2(V) = \emptyset$ and recall from equation (3.2.10) that $\varpi \neq (1-bu)^s$ for any $(s,b) \in \mathbb{N} \times \mathbb{C}^\times$. Write

$$\varpi = (\varpi_1'/\varpi_1'')(\varpi_2'/\varpi_2''), \;\; (\varpi_1'/\varpi_1'') \in \mathrm{wt}_\ell\, V(\alpha_{m_1,a_1})^{\otimes r_1}, \;\; (\varpi_2'/\varpi_2'') \in \mathrm{wt}_\ell(\otimes_{j=2}^k V(\alpha_{m_j,a_j})^{\otimes r_j}).$$

Since α_{m_j,a_j} divides α_{m_1,a_1} for all $1 \le j \le k$, one sees trivially, using Lemma 3.2.2, that ϖ_j' is a product of terms of the form $(1 - a_1 q^{-m_1+2s+1}u)$, $0 \le s \le m_1$ and ϖ_j'' is a product of terms of the form $(1 - a_1 q^{m_1-2s+3}u)$, $0 \le s \le m_1$. In particular,

$$\gcd(1 - aq^{-m_1+1}u, \varpi_2'') = 1 = \gcd(1 - a_1 q^{m_1+3}u, \varpi_2'), \tag{3.2.13}$$

and hence we must also have

$$\gcd(1 - a_1 q^{m_1+3}u, \varpi_1'') = 1.$$

Since

$$\varpi_1'/\varpi_1'' \in \mathrm{wt}_\ell\, V(\alpha_{m_1,a_1})^{\otimes r_1} = \mathrm{wt}_\ell\, V(\omega_{m_1+1,a_1 q})^{\otimes r_1}\, \mathrm{wt}_\ell\, V(\omega_{m_1-1,a_1 q})^{\otimes r_1},$$

the explicit description of $\mathrm{wt}_\ell\, V(\omega_{m_1,a_1})$ given in Lemma 3.2.2 forces $\varpi_1'' = 1$, i.e.,

$$\varpi_1' \in \mathrm{wt}_\ell\, V(\alpha_{m_1,a_1})^{\otimes r_1}.$$

Equation (3.2.6) gives,

$$\varpi_1' = \prod_{j=1}^{r_1} \alpha_{m_1-s_j+1, a_1 q^{-s_j+1}}, \quad \text{for some} \;\; 0 \le s_j \le m_1.$$

Since $(1 - a_1 q^{-m_1+1}u)^{r_1}$ divides ϖ_1', we get by using the first equality in equation (3.2.13), that,

$$\varpi = (1 - a_1 q^{-m_1+1}u)^{r_1} \varpi_0,$$

for some $\varpi_0 \in \mathbb{C}[u]$ which is a product of terms of the form $(1 - aq^{-m+2s+1}u)$, with $0 \leq s \leq m$. Since $\varpi \neq (1 - a_1 q^{-m_1+1}u)^p$ for any $p \in \mathbb{N}$ by equation (3.2.10), the proof of part (iii) is complete in this case. The case of $\varpi \in \mathrm{wt}_\ell^- \mathbf{V}_1$ is identical and we omit the details.

To prove (iv) assume for a contradiction that $1 \in \mathrm{wt}_\ell \mathbf{V}$, i.e. $1 \in \mathrm{wt}_\ell^+ \mathbf{V}$. By part (ii) of the proposition, we may write $1 = \varpi_1(\varpi_2'/\varpi_2'')$ with $\varpi_1 \in \mathrm{wt}_\ell^+ \mathbf{V}_1$ and $\varpi_2'/\varpi_2'' \in \mathrm{wt}_\ell \mathbf{V}_2$. It is then immediate that we must have $\varpi_1 = \varpi_2''$ and $\varpi_2' = 1$. By part (iii) of the proposition, there exists $s \geq 1$ such that $(1 - aq^{-m+1+2s}u)$ divides ϖ_1; on the other hand it does not divide ϖ_2'' by (3.2.11) and the proof is complete. □

3.2.5

We keep the notation from Section 3.2.4. We now prove,

Lemma 3.2.6. *If $J_2(V) \neq \emptyset$, then $\alpha_{n,b} \notin \mathrm{wt}_\ell^+ \mathbf{V}$ for any $(n, b) \in \mathbb{N} \times \mathbb{C}^\times$.*

Proof. Assume for a contradiction that $\alpha_{m,a} \in \mathrm{wt}_\ell^+ \mathbf{V}$. Using Proposition 3.2.5 we may write

$$\alpha_{n,b} = \prod_{s=0}^{m_1} (1 - aq^{-m_1+1+2s}u)^{p_s} (\varpi'/\varpi'') \tag{3.2.14}$$

where the following hold:

(i) $p_0 \geq r_1$, $\sum_{s=1}^{m_1} p_s > 0$,

(ii) $\varpi'/\varpi'' \in \mathrm{wt}\, \mathbf{V}_2$ and $\varpi' \neq \varpi''$,

(iii) $\gcd(1 - a_1 q^{-m_1+2s+1}u, \varpi'') = 1$, for all $1 \leq s \leq m_1$.

It follows that $\varpi'' = (1 - a_1 q^{-m_1+1}u)^\ell$ for some $\ell \in \mathbb{Z}_+$, with $0 \le \ell \le p_0$ and now Lemma 3.2.4 forces $\varpi' \neq 1$. Hence equation (3.2.14) now becomes,

$$\alpha_{n,b} = (1 - a_1 q^{-m_1+1}u)^{p_0-\ell} \left(\prod_{s=1}^{m_1} (1 - a_1 q^{-m_1+1+2s}u)^{p_s} \right) \varpi', \quad \varpi' \neq 1. \tag{3.2.15}$$

Since

$$\alpha_{n,b} = (1 - bq^{n+1}u)(1 - bq^{n-1}u)^2 \cdots (1 - bq^{-n+3}u)^2(1 - bq^{-n+1}u),$$

we can now draw the following conclusions:

(i) the fact that $p_{s_1} > 0$ for some $s_1 \ge 1$ implies that $(1 - a_1 q^{-m_1+1+2s_1}u)$ divides $\alpha_{n,b}$,

(ii) the fact that $\varpi' \neq 1$ implies that there exists $j \in J_2(V)$ such that $(1 - a_j q^{m_j+1-2s_j}u)$ divides ϖ' and hence also $\alpha_{n,b}$ for some $0 \le s_j \le m_j$.

It follows that $a_1 \in a_j q^{2\mathbb{Z}}$ and hence by the the discussion in Section 3.2.4 we get

$$a_1 = a_j q^{m_1+m_j+2r}, \quad \text{for some } r \ge 2.$$

Hence we have proved that $((1 - a_1 q^{-m_1+1+2s_1}u)(1 - a_1 q^{-m_1-2r-2s_j+1}u)$ divides $\alpha_{n,b}$. The form of $\alpha_{n,b}$ then gives us that $(1 - a_1 q^{-m_1+1+2p}u)^2$ divides $\alpha_{n,b}$ if $-r - s_j < p < s_1$, and we get also that there exists $0 < s_0 \le m_1$ with

$$p_s = \begin{cases} 0, & s > s_0, \\ 1, & s = s_0, \\ 2, & 0 < s < s_0, \\ \ell + 2, & s = 0. \end{cases}$$

In other words, we have shown that if $\alpha_{n,b} \in \mathrm{wt}_\ell^+ \mathbf{V}$, then for some $\ell \geq 0$, the element

$$(1-a_1q^{-m_1+2s_0+1}u)(1-a_1q^{-m_1-1+2s_0}u)^2 \cdots (1-a_1q^{-m_1+3}u)^2(1-a_1q^{-m_1+1}u)^{\ell+2} \in \mathrm{wt}_\ell^+ \mathbf{V}_1,$$

or equivalently,

$$\alpha_{s_0,a_1q^{s_0-m_1}}(1-a_1q^{-m_1+1}u)^{\ell+1} \in \mathrm{wt}_\ell \mathbf{V}_1.$$

Equation (3.2.10) now shows that $(1-a_1q^{-m_1+1}u)^{\ell+1}$ is in the subgroup of $\mathbb{C}(u)$ generated by the elements $\{\alpha_{r,z} : (r,z) \in \mathbb{N} \times \mathbb{C}^\times\}$ contradicting Lemma 3.2.4 and the proof is complete. □

3.3 Proof of Proposition 3.2.1

We now complete the proof of Proposition 3.2.1. Notice that Lemma 3.2.6 proves the proposition if $J_2(V) \neq \emptyset$ and hence we now consider the case when $J_2(V) = \emptyset$.

Assume for a contradiction that there exists $(n,b) \in \mathbb{N} \times \mathbb{C}^\times$ such that $\alpha_{n,b} \in \mathrm{wt}_\ell^+ \mathbf{V}$ and $\alpha_{n,bq^2} \in \mathrm{wt}_\ell^- \mathbf{V}$. Then, by part (iii) of Proposition 3.2.5, we may write,

$$\alpha_{n,b} = \prod_{s=0}^{m_1}(1-a_1q^{-m+2s+1}u)^{p_s^+}, \qquad \alpha_{n,bq^2} = \prod_{s=0}^{m_1}(1-aq^{m-2s+3}u)^{p_s^-}, \tag{3.3.1}$$

where $p_s^\pm \in \mathbb{Z}_+$ are such that: $p_0^\pm \geq r_1$, $\sum_{s=1}^{m_1} p_s^\pm > 0$. Using the explicit form of $\alpha_{n,b}$ and $\alpha_{n.bq^2}$ and arguing as in the proof of Lemma 3.2.6, we find that

$$p_0^\pm = 1, \quad a_1q^{\pm m_1} = bq^{\pm n}, \text{ and so}, r_1 = 1, \quad (m_1,a_1) = (n,b).$$

In other words we have shown that

$$\mathbf{V} = V(\alpha_{m_1,a_1}) \otimes (\otimes_{s=2}^k V(\alpha_{m_s,a_s})^{\otimes r_s}), \quad \alpha_{m_1,a_1} \in \mathrm{wt}_\ell^+ \mathbf{V}, \quad \alpha_{m_1,a_1q^2} \in \mathrm{wt}_\ell^- \mathbf{V}.$$

Now, using Proposition 3.2.2(ii) along with the fact that if $\varpi'/\varpi'' \in \mathrm{wt}_\ell V(\alpha_{m_s,a_s})$, then

$$\gcd(1-a_1q^{m_1+3}u, \varpi') = 1 = \gcd(1-a_1q^{-m_1+1}u, \varpi''),$$

we may write $\alpha_{m_1,a_1} = \varpi^+\varpi_1$ and $\alpha_{m_1,a_1q^2} = \varpi^-\varpi_2$ with

$$\varpi^\pm \in \mathrm{wt}_\ell^\pm V(\alpha_{m_1,a_1}) \qquad \varpi_j \in \mathrm{wt}_\ell \otimes_{s=2}^k V(\alpha_{m_s,a_s})^{\otimes r_s}, \quad j=1,2.$$

Equations (3.2.6) and (3.2.7) show that $\varpi^+ = \alpha_{m_1-s_1+1,aq^{-s_1+1}}$ for some $0 \le s_1 \le m_1$ and $\varpi^- = \alpha_{m_1-s_2+1,a_1q^{s_2+1}}$ for some $0 \le s_2 \le m_1$. Moreover if $s_1 = 0$ or $s_2 = 0$ we would have $1 \in \mathrm{wt}_\ell \otimes_{s=2}^k V(\alpha_{m_s,a_s})^{\otimes r_s})$ which would contradict Proposition 3.2.5(iv). Hence $s_i \ge 1$ for $i = 1,2$ and we now get,

$$\varpi_1 = (1-a_1q^{m_1+1}u)(1-aq^{m_1+1}u)^2\cdots(1-a_1q^{m_1+3-2s_1}u)^2(1-a_1q^{m_1+1-2s_1}u) \quad (3.3.2)$$

and

$$\varpi_2 = (1-a_1q^{-m_1+3-2s_2}u)(1-a_1q^{-m_1+1-2s_2}u)^2\cdots(1-a_1q^{-m_1+5}u)^2(1-a_1q^{-m_1+3}u) \quad (3.3.3)$$

Since $V_1 = \otimes_{j=2}^k V(\omega_{m_j,a_j})$ is irreducible, the preceding equations along with Lemma 3.2.6 imply that we must have $J_2(V_1) = \emptyset$. On the other hand, we claim also that $J_2(V_1) \neq \emptyset$ which gives the desired contradiction and establishes Proposition 3.2.1.

For the claim, assume without loss of generality (see Section 3.2.4) that for $2 < j \le k$, we have $\gcd(\omega_{m_2,a_2}, \omega_{m_j,a_j}) = 1$, or $\gcd(\omega_{m_2,a_2}, \omega_{m_j,a_j}) = \omega_{m_j,a_j}$ and in the case $\gcd(\omega_{m_2,a_2}, \omega_{m_j,a_j}) = 1$, we have $a_2 = a_jq^{m_2+m_j+2s}$ for some $s \ge 2$. Using equations (3.3.2) and (3.3.3) we see that we must have

$$a_1q^{m_1} = a_2q^{m_2}, \quad a_1q^{-m_1} = a_jq^{-m_j} \quad \text{for some } 2 \le j \le k. \quad (3.3.4)$$

If $j \in J_1(V_1)$, i.e., ω_{m_j,a_j} divides ω_{m_2,a_2}, then the condition for irreducibility of V_1 given in Proposition 1.5.1 and equation (3.3.4) forces $(m_2, a_2) = (m_1, a_1)$ contradicting our assumption that they were distinct. Hence, $j \in J_2(V_1)$ and proves the claim. This completes the proof of Proposition 3.2.1 and thus completes the proof of Proposition 3.0.1.

Chapter 4

Prime Powers

In this chapter, we will prove Theorem 2 and Theorem 3. We note that this will prove Theorem 1 in the case that V is a prime power, i.e $V \cong V(\omega_{m,a})^{\otimes r}$ for some $(m,a) \in \mathbb{N}\times\mathbb{C}^\times$ and $r \in \mathbb{Z}_+$.

4.1 Generators and Relations for $V(\omega_{m,a})^{\otimes r}$

In this section, we prove Theorem 2. We recall here, the statement of Theorem 2 for the readers convenience. From now on, we fix $(m,a) \in \mathbb{N} \times \mathbb{C}^\times$ and also $r \in \mathbb{N}$ and we let $\mathbf{V}(r) = V(\omega_{m,a})^{\otimes r}$. Let $\mathbf{x}_r^- = \mathbf{x}_a^-(m,r)$ be as in equation (2.2.1),

$$\mathbf{x}_r^- = \sum_{s=0}^{r}(-aq^m)^s\binom{r}{s}x_{r-s}^- = \sum_{s=0}^{r-1}(-aq^m)^s\binom{r-1}{s}(x_{r-s}^- - aq^m x_{r-s-1}^-)$$

Let $\tilde{\mathbf{V}}(r)$ be the $\hat{\mathbf{U}}_q$-module generated by an element $\tilde{v}_r$ with defining relations: for all $j \in \mathbb{Z}$ and $s \in \mathbb{Z} \setminus \{0\}$,

$$x_j^+ \tilde{v}_r = 0, \qquad k\mathbf{v} = q^{rm}\tilde{v}_r, \qquad h_s\tilde{v}_r = ra^s \frac{[sm]}{s}\tilde{v}_r, \tag{4.1.1}$$

$$(x_0^-)^{rm+1}\tilde{v}_r = 0, \qquad \mathbf{x}_r^- \tilde{v}_r = 0. \tag{4.1.2}$$

4.1.1

To prove Theorem 2 we must show that

$$\tilde{\mathbf{V}}(r) \cong \mathbf{V}(r).$$

The proof proceeds as follows. Recall from Section 1.5 that $V(\omega_{m,a})$ has a basis $v_0, \ldots, v_m$ with the action of $\hat{\mathbf{U}}_q$ given in equation (1.5.1). We shall first prove that the element $v_m^{\otimes r}$ satisfies the relations in equations (4.1.1) and (4.1.2). Since $\mathbf{V}(r)$ is irreducible, it follows that the assignment $\tilde{v}_r \to v_m^{\otimes r}$ defines a surjective map $\tilde{\mathbf{V}}(r) \to \mathbf{V}(r) \to 0$ of $\hat{\mathbf{U}}_q$-modules and hence

$$\dim \tilde{\mathbf{V}}(r) \geq \dim \mathbf{V}(r) = (m+1)^r.$$

After that we shall prove that we have an equality of dimensions thus completing the proof of Theorem 2.

4.1.2

We proceed by induction on r; with the case $r = 1$ being clear from the explicit action in (1.5.1). Assume the result for $r-1$. The fact that $v_m^{\otimes r} = v_m \otimes v_m^{\otimes r-1}$ satisfies the relations in (4.1.1) and the first relation in (4.1.2) is now immediate from Proposition 1.3.2. To prove that it satisfies the second relation in (4.1.2) we need to understand the comulitpication on the element $\mathbf{x}_r^-$. But before we do this, we collect together some consequences of our inductive hypothesis.

Note first that we have the following easily checked recursive formula:

$$\mathbf{x}_r^- = -\frac{1}{[2]}[h_1, \mathbf{x}_{r-1}^-] - aq^m \mathbf{x}_{r-1}^-. \tag{4.1.3}$$

Using our induction hypothesis that $\mathbf{x}_{r-1}^- v_m^{\otimes r-1} = 0$ and $h_1(v_m^{\otimes r-1}) = (r-1)[m](v_m^{\otimes r-1})$, we see that

$$\mathbf{x}_r^-(v_m^{\otimes r-1}) = 0. \tag{4.1.4}$$

The next consequence is that we have

$$x_0^+ \mathbf{x}_{r-1}^- v_m^{\otimes r-1} = 0.$$

Taking commutators, we get

$$\sum_{s=0}^{r-1} (-aq^m)^s \binom{r-1}{s} \phi_{r-s-1} v_m^{\otimes r-1} = 0. \tag{4.1.5}$$

4.1.3

For $1 \leq s \leq r$, let

$$c_s = (-aq^m)^s \binom{r-1}{s}.$$

Proposition 4.1.1. *Modulo terms in $\hat{\mathbf{U}}_q X_- \otimes \hat{\mathbf{U}}_q X_+$, we have for $r > 1$, we have*

$$\Delta(\mathbf{x}_r^-) = 1 \otimes \mathbf{x}_r^- + \sum_{s=0}^{r-2} c_s (x_{r-s}^- - aq^m x_{r-s-1}) \otimes k + \sum_{s=0}^{r-2} \sum_{j=1}^{r-s-j} c_s (x_{r-s-j}^- - aq^m x_{r-s-j-1}^-) \otimes \phi_j$$
$$+ c_{r-1}(x_1^- \otimes k - aq^m x_0^- \otimes k^{-1}) + x_1^- \otimes \sum_{s=0}^{r-2} c_s \phi_{r-s-1}.$$

Proof. The proof proceeds by an induction on r. If $r = 1$, the result is immediate by using Proposition 1.3.2. The inductive step is a straightforward calculation using equation (4.1.3) and we omit the details. $\square$

Calculating $\Delta(\mathbf{x}_r^-)$ on $v_m \otimes v_m^{r-1}$ using Proposition 4.1.1, we find that the first term on the right hand side of the formula in the preceding proposition is zero by (4.1.4). The next two terms are zero since $(x_s^- - aq^m x_{s-1}^-)v_m = 0$ for all s. To see that the sum of the last two terms is zero we see by using equation (1.5.1), that

$$c_{r-1}(aq^m v_{m-1} \otimes q^{(r-1)m} v_m^{\otimes r-1} - aq^m v_{m-1} \otimes q^{-(r-1)m} v_m^{r-1}) + aq^m v_{m-1} \otimes \sum_{s=0}^{r-2} c_s \phi_{r-s-1} v_m^{r-1}$$

$$= aq^m v_{m-1} \otimes \left[c_{r-1}(q^{(r-1)m} - q^{-(r-1)m}) + \sum_{s=0}^{r-2} c_s \phi_{r-s-1} \right] v_m^{r-1}$$

$$= aq^m v_{m-1} \otimes \left[c_{r-1}\phi_0 + \sum_{s=0}^{r-2} c_s \phi_{r-s-1} \right] v_m^{r-1} = aq^m v_{m-1} \otimes \left[\sum_{s=0}^{r-1} c_s \phi_{r-s-1} \right] v_m^{r-1}$$

The last equality is zero by equation (4.1.5) and we have proved that $\mathbf{V}(r)$ is a quotient of $\tilde{\mathbf{V}}(r)$.

4.1.4

We now prove that

$$\dim \tilde{\mathbf{V}}(r) \leq \dim \mathbf{V}(r),$$

which completes the proof that $\tilde{\mathbf{V}}(r)$ and $\mathbf{V}(r)$ are isomorphic $\hat{\mathbf{U}}_q$-modules. The first step is to understand the classical limit $(q \to 1)$ of $\tilde{\mathbf{V}}(r)$. For this, we let ε be an indeterminate and consider the algebra $\hat{\mathbf{U}}_\varepsilon$ as an algebra over $\mathbb{C}(\varepsilon)$ given by the same generators and relations as $\hat{\mathbf{U}}_q$. Similarly, we let $V_\varepsilon(\omega_{m,a})$ be the $\mathbb{C}(\varepsilon)$ vector space with the action of $\hat{\mathbf{U}}_\varepsilon$

being given by the formulae in equation 1.5.1. The modules $\mathbf{V}_\varepsilon(r)$ and $\tilde{\mathbf{V}}_\varepsilon(r)$ are defined in the obvious way . Setting $\mathbb{A} = \mathbb{C}[\varepsilon, \varepsilon^{-1}]$ we recall [16] that $\hat{\mathbf{U}}_\varepsilon$ admits a free $\mathbb{A}$-submodule denoted $\hat{\mathbf{U}}_\mathbb{A}$ such that we have an isomorphism of $\mathbb{C}(\varepsilon)$-algebras

$$\hat{\mathbf{U}}_\varepsilon \cong \hat{\mathbf{U}}_\mathbb{A} \otimes_\mathbb{A} \mathbb{C}(\varepsilon).$$

It was shown in [8, 7] that the modules $\mathbf{V}_\varepsilon(r)$ admit free $\hat{\mathbf{U}}_\mathbb{A}$-submodules, $\mathbf{V}_\mathbb{A}(r)$ of rank equal to $(m+1)$ and $(m+1)^r$ respectively (we emphasize here that we are assuming that $a \in \mathbb{C}^\times$; the statement need not be true otherwise). Moreover, the arguments in [8, Lemma 4.6] show that $\tilde{\mathbf{V}}_\varepsilon(r)$ also admits a free $\mathbb{A}$-lattice. If q is not a root of unity, we have an isomorphism of algebras and modules over $\mathbb{C}$,

$$\hat{\mathbf{U}}_q \cong \hat{\mathbf{U}}_\mathbb{A} \otimes_\mathbb{A} \mathbb{C}_q, \quad \mathbf{V}(r) \cong \mathbf{V}_\mathbb{A}(r) \otimes_\mathbb{A} \mathbb{C}_q, \quad \tilde{\mathbf{V}}(r) \cong \tilde{\mathbf{V}}_\mathbb{A} \otimes_\mathbb{A} \mathbb{C}_q$$

where $\mathbb{C}_q$ is the irreducible $\mathbb{A}$-module obtained by letting ε acts as q.

4.1.5

Suppose now that we take $q = 1$. It is well-known that the algebra $\hat{\mathbf{U}}_1 = \hat{\mathbf{U}}_\mathbb{A} \otimes_\mathbb{A} \mathbb{C}_q$ is essentially the enveloping algebra of the loop algebra $L(\mathfrak{sl}_2)$. Recall that $\mathfrak{sl}_2$ is the Lie algebra of complex 2×2-matrices of trace zero, with standard basis $x^\pm$, h. As a complex vector space, $L(\mathfrak{sl}_2) = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$, where t is an indeterminate and the commutator is given by:$[x \otimes f, y \otimes g] = [x, y] \otimes fg$, wehere $x, y \in \mathfrak{sl}_2$ and $f, g \in \mathbb{C}[t, t^{-1}]$. Then $\mathbf{U}(L\mathfrak{sl}_2)$ is isomorphic to $\hat{\mathbf{U}}_1$ by the ideal generated by the elements $k^\pm - 1$. Moreover the elements $(x_r^\pm \otimes 1) \in \hat{\mathbf{U}}_1$ map to the elements $x^\pm \otimes t^r$ in $\mathbf{U}(L(\mathfrak{sl}_2)$; a similar statement is true for the element $h_r \otimes 1$. It is also easily checked that the element $\mathbf{x}_r^-$ maps to the element

$x^- \otimes (t-a)^r$ of $L(\mathfrak{sl}_2)$. Finally, $\mathbf{V}_1(r) = \mathbf{V}_{\mathbb{A}}(r) \otimes_{\mathbb{A}} \mathbb{C}_1$, and $\tilde{\mathbf{V}}_1(r)$ are modules for $L(\mathfrak{sl}_2)$ with

$$\dim_{\mathbb{C}} \mathbf{V}_1(r) = (m+1)^r = \mathbf{rk}_{\mathbb{A}} \mathbf{V}_\varepsilon(r), \quad , \quad \dim_{\mathbb{C}} \tilde{\mathbf{V}}_1(r) = \mathbf{rk}_{\mathbb{A}} \tilde{\mathbf{V}}_\varepsilon(r) = \dim_{\mathbb{C}} \tilde{\mathbf{V}}(r).$$

Hence to complete the proof of $\dim \tilde{\mathbf{V}}(r) \le \dim \mathbf{V}(r)$ it suffices to prove that

$$\dim \tilde{\mathbf{V}}_1(r) = (m+1)^r.$$

For this, we set $w_r = \tilde{v}_r \otimes 1 \in \tilde{\mathbf{V}}_1(r)$. The preceding comments show that the element w_r generates $\tilde{\mathbf{V}}_1(r)$ and satisfies the relations,

$$(x^+ \otimes t^k)w_r = 0, \;\; (h \otimes (t-a)^k)w_r = \delta_{k,0}(rm), \;\; (x^- \otimes 1)^{rm+1}w_r = 0, \;\; (x^- \otimes (t-a)^r)w_r = 0.$$

It follows that

$$(\mathfrak{sl}_2 \otimes (t-a)^r \mathbb{C}[t,t^{-1}])w_r = 0.$$

This means that we have an isomorphism of Lie algebras

$$\mathfrak{sl}_2 \otimes \mathbb{C}[t,t^{-1}]/(t-a)^r\mathbb{C}[t,t^{-1}] \cong \mathfrak{sl}_2 \otimes \mathbb{C}[t]/(t-a)^r\mathbb{C}[t],$$

and hence we may regard $\tilde{\mathbf{V}}_1(r)$ as a module for $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$. Consider the pull-back of $\tilde{\mathbf{V}}_1(r)$ via the automorphism of $\mathfrak{sl}_2 \otimes \mathbb{C}[t]$ which maps $x \otimes t^s \to x \otimes (t+a)^s$, $s \in \mathbb{Z}_+$. The pull back is generated by the element w_r with relations,

$$(x^+ \otimes t^k)w_r = 0, \;\; (h \otimes t^k)w_r = \delta_{k,0}rm, \;\; (x^- \otimes 1)^{rm+1}w_r = 0, \;\; (x^- \otimes t^r)w_r = 0.$$

It was shown in [9, Theorem 5(ii)] that the dimension of such a module is at most $(m+1)^r$ which now completes our proof.

4.2 Local and Global Weyl Modules

We will require the use of the local and global Weyl modules, originally defined in [8]. In this section, we recall their definitions and summarize their important properties. We use the approach developed in [18].

4.2.1

Given $m \in \mathbb{Z}_+$, the global Weyl module $W(m)$ is the $\hat{\mathbf{U}}_q$-module generated by a vector w_m with defining relations,

$$kw_m = q^m w_m, \quad x_r^+ w_m = 0, \quad (x_0^-)^{m+1} w_m = 0,$$

for all $r \in \mathbb{Z}$. If $m \neq 0$, then $W(m)$ is an infinite-dimensional type 1 module, and $W(0)$ is the trivial module. Notice also that if $V = V(\omega_{m_1,a_1}) \otimes \cdots \otimes V(\omega_{m_k,a_k})$ is irreducible such that $m = m_1 + \cdots m_k$, then the relations in Proposition 1.5.1 show that V is a quotient of $W(m)$.

4.2.2

We may regard $W(m)$ as a right $\hat{\mathbf{U}}_q^0$-module by setting

$$(uw_m)\phi_s = u\phi_s w_m, \quad u \in \hat{\mathbf{U}}_q, \quad s \in \mathbb{Z} \setminus \{0\}.$$

Consider the annihilating ideal of w_m in $\hat{\mathbf{U}}_q$,

$$\mathrm{Ann}(w_m) = \{x \in \hat{\mathbf{U}}_q^0 : w_m x = 0\} = \{x \in \hat{\mathbf{U}}_q^0 : xw_m = 0\},$$

and let $\mathbf{A}_m$ be the quotient of $\hat{\mathbf{U}}_q^0$ by $\mathrm{Ann}(w_m)$. Then, $W(m)$ is a $(\hat{\mathbf{U}}_q, \mathbf{A}_m)$-bimodule. Moreover, the subspace $W(m)_m$ is also a left $\mathbf{A}_m$-module and we have an isomorphism of $\mathbf{A}_m$-bimodules,

$$W(m)_m \cong \mathbf{A}_m.$$

If we regard $\hat{\mathbf{U}}_q^0$ as the polynomial algebra in the variables Λ_r, $r \in \mathbb{Z} \setminus \{0\}$, then it is known that $\mathbf{A}_m$ is the quotient of $\hat{\mathbf{U}}_q^0$ obtained by setting,

$$\Lambda_r{=}0 \text{ for } |r| > m \quad \text{and} \quad (\Lambda_m \Lambda_{-s} - \Lambda_{m-s}) = 0 \text{ for } 0 \leq s \leq m. \tag{4.2.1}$$

In particular, if we let $\bar{\Lambda}_s$ be the image of Λ_s in $\mathbf{A}_m$, then we have

$$\mathbf{A}_m \cong \mathbb{C}[\bar{\Lambda}_1, \ldots, \bar{\Lambda}_m, \bar{\Lambda}_m^{-1}].$$

4.2.3

Denote by $\mathbf{A}_m$-mod the category of finitely generated left $\mathbf{A}_m$-modules, and for an object M of $\mathbf{A}_m$-mod, define

$$\mathbf{W}_m M = W(m) \otimes_{\mathbf{A}_m} M.$$

We have an isomorphism in $\mathbf{A}_m$-mod,

$$(\mathbf{W}_m M)_m = W(m)_m \otimes_{\mathbf{A}_m} M \cong M.$$

If V is a quotient of $W(m)$, then the $\hat{\mathbf{U}}_q^0$-action on V_m descends to $\mathbf{A}_m$ since

$$uV_m = 0 \quad \text{for} \quad u \in \mathrm{Ann}(w_m).$$

It is simple to check, [18, Proposition 3.6] that V is a quotient of $\mathbf{W}_m V_m$. The following was proved in [3],

Proposition 4.2.1. *For objects M, N of $\mathbf{A}_m$-mod and $f \in \mathrm{Hom}_{\mathbf{A}_m}(M,N)$, the assignment*

$$\mathbf{W}_m M = W(m) \otimes_{\mathbf{A}_m} M, \quad \mathbf{W}_m f = 1 \otimes f,$$

defines an exact functor $\mathbf{W}_m \colon \mathbf{A}_m\text{-mod} \to \mathcal{F}$. Moreover, M is indecomposable in $\mathbf{A}_m$-mod if and only if $\mathbf{W}_m M$ is indecomposable in $\mathcal{F}$. □

4.2.4

Let $V = V(\omega_{m_1,a_1}) \otimes \cdots \otimes V(\omega_{m_k,a_k})$ be an irreducible object in $\mathcal{F}$, generated by $\mathbf{v}$ and set $\pi = \omega_{m_1,a_1} \cdots \omega_{m_k,a_k}$ so that for $s \in \mathbb{Z}$,

$$\phi_s \mathbf{v} = \phi_s(\pi)\mathbf{v}.$$

Define $\Lambda_s(\pi)$ similarly for $s \in \mathbb{Z}$. Let $\mathbb{C}(\pi)$ be the 1-dimensional quotient of $\hat{\mathbf{U}}_q^0$ defined by taking the quotient by the ideal generated by the elements

$$\{\Lambda_s - \Lambda_s(\pi) : s \in \mathbb{Z} \setminus \{0\}\}$$

or equivalently

$$\{\phi_s - \phi_s(\pi) : s \in \mathbb{Z} \setminus \{0\}\}.$$

It is clear from equation (4.2.1) that $\mathbb{C}(\pi)$ is a $\mathbf{A}_m$-module where $m = m_1 + \cdots + m_k$. The local Weyl module $W(\pi)$ is given by

$$W(\pi) = \mathbf{W}_m \mathbb{C}(\pi), \quad w_\pi = w_m \otimes 1.$$

Alternatively, $W(\pi)$ is the quotient of $W(m)$ obtained by imposing the additional relations: for all $s \in \mathbb{Z} \setminus \{0\}$,

$$(\Lambda_s - \Lambda_s(\pi))w_m = 0 \quad \text{or equivalently} \quad (\phi_s - \phi_s(\pi))w_m = 0$$

It is clear from Theorem 2 that V as above is a quotient of $W(\pi)$. In fact, V is the unique irreducible quotient of $W(\pi)$ by the maximal submodule not containing $W(\pi)_m = \mathbb{C}w_\pi$.

4.2.5

Let V be as in the previous section. For a $\hat{\mathbf{U}}_q$-module W, such that W is a quotient of $W(m)$, let W^m be the unique maximal submodule of W such that

$$W^m \cap W_m = 0.$$

Recall from the previous section that

$$W(\pi)/W(\pi)^m \cong V(\pi).$$

The following was proved in [3],

Proposition 4.2.2. *We have,*

(i) Let V' be any self-extension of V. The restriction $V' \to V'_m$ induces an injective map of vector spaces

$$\mathrm{Ext}^1_{\mathcal{F}}(V,V) \to \mathrm{Ext}^1_{\mathbf{A}_m}(\mathbb{C}(\pi),\mathbb{C}(\pi)) \cong \mathrm{Ext}^1_{\mathcal{F}}(W(\pi),W(\pi))$$

(ii) Let W be a nontrivial self-extension of $W(\pi)$,

$$0 \to W(\pi) \xrightarrow{\iota} W \xrightarrow{\tau} W(\pi) \to 0$$

such that $\tau(W^m) = W(\pi)^m$, then $V' = W/W^m$ is a nontrivial self-extension of V. In particular, $V'_m \cong_{\mathbf{A}_m} W_m$.

$\square$

The following is immediate from part (i) of Proposition 4.2.2,

Corollary 4.2.3. *Let V^j, $1 \leq j \leq s$ be nontrivial self-extensions of V. If the equivalence classes $[V_m^j]$, $1 \leq j \leq s$ are linearly independent in $\operatorname{Ext}^1_{\mathbf{A}_m}(\mathbb{C}(\pi), \mathbb{C}(\pi))$, then the equivalence classes $[V^j]$, $1 \leq j \leq s$ are linearly independent in $\operatorname{Ext}^1_{\mathcal{F}}(V, V)$.* $\square$

4.2.6

We retain the notation from the previous section. We shall need the following result concerning self-extensions of the $\mathbf{A}_m$-module $\mathbb{C}(\pi)$. Recall that $\mathbb{C}(\pi)$ is the one dimensional $\mathbf{A}_m$-module generated by a vector v_π satisfying, for all $s \in \mathbb{Z} \setminus \{0\}$,

$$\Lambda_s v_\pi = \Lambda_s(\pi) v_\pi. \tag{4.2.2}$$

Given a self-extension U of $\mathbb{C}(\pi)$ it is clear that $\dim_{\mathbb{C}} U = 2$. We now prove,

Lemma 4.2.4. *Let U and U_i, $1 \leq i \leq j$ be self-extensions of $\mathbb{C}(\pi)$ and suppose there exists $c_i \in \mathbb{C}$ for $1 \leq i \leq j$ such that $[U] = \sum_{i=1}^j c_i [U_i]$ in $\operatorname{Ext}^1_{\mathbf{A}_m}(\mathbb{C}(\pi), \mathbb{C}(\pi))$. Then, the action of $\hat{\mathbf{U}}_q^0$ is given by: for $s \in \mathbb{Z} \setminus \{0\}$,*

$$\Lambda_s(v, w) = \left(\Lambda_s(\pi) v, \Lambda_s(\pi) w + \sum_{i=1}^j c_i z_{i,s} v \right), \quad v, w \in \mathbb{C}(\pi). \tag{4.2.3}$$

for some $z_{i,s} \in \mathbb{C}$, $1 \leq i \leq j$.

Proof. Let

$$\rho : \hat{\mathbf{U}}_q^0 \to \operatorname{End}(\mathbb{C}(\pi)), \quad \rho_i : \hat{\mathbf{U}}_q^0 \to \operatorname{End}(U_i)$$

be the representations corresponding to $\mathbb{C}(\pi)$ and U_i, $1 \le i \le j$ respectively. Since $U_i = \mathbb{C}(\pi) \oplus \mathbb{C}(\pi)$ as a vector space, ρ_i can be given by the operator,

$$\rho_i(\Lambda_s) = \begin{pmatrix} \rho(\Lambda_s) & 0 \\ f_i(\Lambda_s) & \rho(\Lambda_s) \end{pmatrix}, \quad s \in \mathbb{Z} \setminus \{0\}, \tag{4.2.4}$$

where $f_i(\Lambda_s) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(\pi), \mathbb{C}(\pi))$. Since $\mathbb{C}(\pi)$ is one dimensional and generated by v_π, $f_i(\Lambda_s)$ is completely determined by the image of v_π and we must have,

$$f_i(\Lambda_s)(v_\pi) = z_{i,s} v_\pi \quad \text{for some } z_{i,s} \in \mathbb{C}.$$

It now follows by equation (4.2.4) that the action of $\hat{\mathbf{U}}_q^0$ on U_i is given by: for $s \in \mathbb{Z} \setminus \{0\}$,

$$\Lambda_s(v, w) = (\Lambda_s(\pi) v, \Lambda_s(\pi) w + z_{i,s} v), \quad v, w \in \mathbb{C}(\pi).$$

The result now follows from the definition of the Baer sum and the lemma is proved. $\square$

4.3 Self-extensions of $V(\omega_{m,a})^{\otimes r}$: proof of Theorem 3

In this section we will prove Theorem 3. We fix $(m, a, r) \in \mathbb{N} \times \mathbb{C}^\times \times \mathbb{N}$ and recall from Section 1.5.1 that $V(\omega_{m,a})$ has a basis $v_0, \ldots, v_m$ with the action of $\hat{\mathbf{U}}_q$ given by equations (1.5.1) and (1.5.2). Set,

$$\mathbf{V}(r) = V(\omega_{m,a})^{\otimes r}, \quad \mathbf{v}_r = v_m^{\otimes r},$$

and recall from Proposition 1.5.1 that $\mathbf{V}(r)$ is generated as a $\hat{\mathbf{U}}_q$-module by $\mathbf{v}_r$. Set

$$\pi(u) = [\omega_{m,a}(u)]^r$$

so that we have,

$$\Lambda_s \mathbf{v}_r = \Lambda_s(\pi) \mathbf{v}_r, \quad s \in \mathbb{Z} \setminus \{0\}.$$

Let $W(rm)$ and $W(\pi)$ be the global and local Weyl modules respectively defined in Section 4.2. Recall also that $\mathbb{C}(\pi)$ is the one dimensional $\mathbf{A}_{rm}$-module generated by a non-zero vector v_π satisfying,

$$\Lambda_s v_\pi = \Lambda_s(\pi) v_\pi, \quad s \in \mathbb{Z} \setminus \{0\}\,. \tag{4.3.1}$$

Finally, recall that given a $\hat{\mathbf{U}}_q$-module U that is a quotient of $W(rm)$ we have U^{rm} is the unique maximal submodule of U such that

$$U^{rm} \cap U_{rm} = 0.$$

4.3.1

We recall for the readers convenience

$$\mathbf{A}_{rm} = \mathbb{C}[\bar{\Lambda}_1, \ldots, \bar{\Lambda}_{rm}, \bar{\Lambda}_{rm}^{-1}]$$

and we have a homomorphism of algebras $\hat{\mathbf{U}}_q^0 \to \mathbf{A}_{rm}$ given on generators by,

$$\Lambda_s \mapsto \begin{cases} 0, & s > rm \\ \bar{\Lambda}_s, & 0 < s \leq rm \\ \bar{\Lambda}_{rm+s}\bar{\Lambda}_{rm}^{-1}, & -rm \leq s \leq 0. \end{cases}$$

Set $\bar{\Lambda}_0 = 1$ and,

$$\bar{\Lambda}(u) = \sum_{s=0}^{rm} \bar{\Lambda}_s u^s.$$

Recall also that we have

$$\omega_{m,a}(u) = (1 - aq^{m-1}u)(1 - aq^{m-3}u) \cdots (1 - aq^{-m+1}u).$$

4.3.2

For integers $0 \le k \le j$, the Eulerian numbers $\left\langle {j \atop k} \right\rangle$ are defined by,

$$\left\langle {j \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i \binom{j+1}{i} (k+1-i)^j. \tag{4.3.2}$$

We note that $\left\langle {j \atop j} \right\rangle = 0$ for $j > 1$ and we have $\left\langle {j \atop 0} \right\rangle = 1 = \left\langle {j \atop j-1} \right\rangle$ for all $j \ge 0$. It is a well known fact we have for $j > 0$,

$$\sum_{k=0}^{j-1} \left\langle {j \atop k} \right\rangle = j! \tag{4.3.3}$$

It is also well known that for $j > 0$ we have,

$$\sum_{s \ge 0} s^j u^s = \left[u \frac{d}{du} \right]^j \left(\frac{1}{1-u} \right) = \frac{1}{(1-u)^{j+1}} \sum_{k=0}^{j-1} \left\langle {j \atop k} \right\rangle u^{k+1}. \tag{4.3.4}$$

Given $c \in \mathbb{C}$, let $\mathbf{D} \colon \mathbb{C}[u] \to \mathbb{C}[u]$ be the operator defined by,

$$\mathbf{D}_c[f] = (cu) \frac{d}{du} f, \quad f \in \mathbb{C}[u].$$

For each $1 \le j < r$, define $\mathbf{Z}_j(u)$ by,

$$\mathbf{Z}_j(u) = \begin{cases} [\pi(q^2 u)]^r \mathbf{D}^j_{aq^{m+1}} \left[\frac{1}{1-aq^{m+1}u} \right], & 1 \le j < r, \\ \mathbf{D}_1[\pi(u)], & j = r. \end{cases} \tag{4.3.5}$$

Using the explicit form of $\omega_{m,a}(u)$, we see that we have,

$$\gcd \left([\pi(q^2 u)]^r, (1 - aq^{m+1}u)^r \right) = (1 - aq^{m+1}u)^r.$$

In particular, using equation (4.3.4) we see that $\mathbf{Z}_j(u) \in \mathbb{C}[u]$ for all $1 \le j \le r$ and,

$$\deg \mathbf{Z}_j(u) = \begin{cases} rm - 1, & 1 \le j < r \\ rm, & j = r. \end{cases} \tag{4.3.6}$$

We also note that it is clear form the definition that the polynomials $\mathbf{Z}_j(u)$, $1 \leq j \leq r$ have no constant term.

4.3.3

Let $\mathbf{e}_j$, $1 \leq j \leq r$ be the standard basis of $\mathbb{C}^r$, i.e $\mathbf{e}_j = (e_1, \cdots, e_r)$ with $e_i = \delta_{i,j}$. For each $1 \leq j \leq r$ and we define the $\mathbf{A}_{rm}$-module $\mathbf{C}(\mathbf{e}_j)$ by,

$$\mathbf{C}(\mathbf{e}_j) \cong \mathbb{C}(\pi) \oplus \mathbb{C}(\pi),$$

as a vector space and the action of $\mathbf{A}_{rm}$ is given by the functional equation,

$$\bar{\Lambda}(u)(v, w) = (\pi(u)v, \ \ \pi(u)w + \mathbf{Z}_j(u)v), \quad v, w \in \mathbb{C}(\pi), \tag{4.3.7}$$

in the sense that the action of $\bar{\Lambda}_s$, $1 \leq s \leq rm$ is determined by the coefficient of u^s on the right-hand side of equation 4.3.7. We note it is easily seen that the coefficient c_j of u in $\mathbf{Z}_j(u)$ is nonzero for all $1 \leq j < r$, and using the action in equation (4.3.7), we see that,

$$(\bar{\Lambda}_1 - \Lambda_1(\pi))(v_\pi, 0) = c_j(0, v_\pi). \tag{4.3.8}$$

Setting,

$$\mathbf{v}_j = (v_\pi, 0), \quad 1 \leq j < r,$$

we have $\mathbf{C}(\mathbf{e}_j)$ is generated as a $\mathbf{A}_{rm}$-module by $\mathbf{v}_j$ for all $1 \leq j < r$. Moreover, it is clear from the actions in equations (1.6.3) and (4.3.7) that we have an isomorphism of $\mathbf{A}_{rm}$-modules,

$$\mathbb{C}(\mathbf{e}_r) \cong \mathbf{E}(\mathbf{V}(r))_{rm}.$$

4.3.4

Clearly, for each $1 \leq j \leq r$ we have a self-extension in $\mathbf{A}_{rm}$-mod,

$$0 \to \mathbb{C}(\pi) \to \mathbf{C}(\mathbf{e}_j) \to \mathbb{C}(\pi) \to 0.$$

Recall from Proposition 4.2.1 that we have the functor $\mathbf{W}_{rm} \colon \mathbf{A}_{rm}\text{-mod} \to \mathcal{F}$ given on objects by ,

$$\mathbf{W}_{rm} M = W(rm) \otimes_{\mathbf{A}_{rm}} M, \quad M \in \operatorname{Ob} \mathbf{A}_{rm}\text{-mod}$$

For all $1 \leq j \leq r$, set

$$V(\mathbf{e}_j) = \begin{cases} \mathbf{W}_{rm}\mathbb{C}(\mathbf{e}_j)/(\mathbf{W}_{rm}\mathbb{C}(\mathbf{e}_j))^{rm}, & 1 \leq j < r, \\ \mathbf{E}(\mathbf{V}_r), & j = r. \end{cases}$$

We shall prove,

Proposition 4.3.1. *With the notation above, the equivalence classes* $[V(\mathbf{e}_j)]$, $1 \leq j \leq r$ *form a basis of* $\operatorname{Ext}^1_{\mathcal{F}}(\mathbf{V}(r), \mathbf{V}(r))$.

Assuming we have proven the proposition, the proof of Theorem 3 proceeds as follows. It is clear that given a nonzero vector $\mathbf{c} = (c_1, \ldots, c_r) \in \mathbb{C}^r$ we can produce a self-extension $V(\mathbf{c})$ as a representative of the equivalence class of the Baer sum

$$[V(\mathbf{c})] = \sum_{j=1}^{r} c_j [V(\mathbf{e}_j)],$$

in $\operatorname{Ext}^1_{\mathcal{F}}(\mathbf{V}(r), \mathbf{V}(r))$. It is also clear by construction that $V(\mathbf{c})_{rm}$ is isomorphic to the $\mathbf{A}_{rm}$-module given by a representative of the Baer sum,

$$\sum_{j=1}^{r} c_j [\mathbf{C}(\mathbf{e}_j)]$$

in $\mathrm{Ext}^1_{\mathbf{A}_{rm}}(\mathbb{C}(\pi), \mathbb{C}(\pi))$. Using Lemma 4.2.4 and equation (4.3.7), we see that the action of $\mathbf{A}_{rm}$ on $V(\mathbf{c})_{rm}$ is given by the functional equation,

$$\bar{\Lambda}(u)(v,w) = \left(\pi(u)v, \;\; \pi(u)w + \sum_{j=1}^{r} c_j \mathbf{Z}_j(u)v \right), \quad v, w \in \mathbb{C}(\pi).$$

Since we have,

$$\mathbf{Z}_{\mathbf{c}}(u) = \sum_{j=1}^{r} c_j \mathbf{Z}_j(u), \tag{4.3.9}$$

the proof of Theorem 3 is complete. The proof of Proposition 4.3.1 will be given in Section 4.3.8.

4.3.5

Our first step in the proof of Proposition 4.3.1 is the following.

Proposition 4.3.2. *The equivalence classes* $[\mathbf{C}(\mathbf{e}_j)]$, $1 \le j \le r$, *are linearly independent in* $\mathrm{Ext}^1_{\mathbf{A}_{rm}}(\mathbb{C}(\pi), \mathbb{C}(\pi))$.

Proof. Suppose that there exists $d_j \in \mathbb{C}$, $1 \le j \le r$, such that the Baer sum,

$$\sum_{j=1}^{r-1} d_j [\mathbf{C}(\mathbf{e}_j)] = [0] = [\mathbb{C}(\pi) \oplus \mathbb{C}(\pi)] \tag{4.3.10}$$

Let U be a representative of the equivalence class above. Then, it is clear from Lemma 4.2.4 and equation (4.3.7) that the action of $\mathbf{A}_{rm}$ on U is given by the functional equation,

$$\bar{\Lambda}(u)(v,w) = \pi(u)(v,w) + \sum_{j=1}^{r} d_j \mathbf{Z}_j(u)(0,v), \quad v, w \in \mathbb{C}(\pi). \tag{4.3.11}$$

In particular, since by equation (4.3.10) we have $U \cong \mathbb{C}(\pi) \oplus \mathbb{C}(\pi)$ as $\mathbf{A}_{rm}$-modules we must have

$$\sum_{j=1}^{r} d_j \mathbf{Z}_j(u) = 0, \tag{4.3.12}$$

Since by equation (4.3.6)

$$\deg \mathbf{Z}_r(u) > \deg \mathbf{Z}_j(u) \quad \text{for all} \quad 1 \leq j < r,$$

we must have $d_r = 0$. For $1 \leq j < r$, using the explicit form of $\mathbf{Z}_j(u)$, and setting $aq^{m+1} = 1$ for clarity, equation (4.3.12) is equivalent to,

$$d_{r-1} \sum_{k=0}^{r-2} \left\langle {r-1 \atop k} \right\rangle u^{k+1} = (1-u) \sum_{j=1}^{r-2} \sum_{k=0}^{j-1} d_j (1-u)^{r-j-2} \left\langle {j \atop k} \right\rangle u^{k+1}.$$

In particular, either $d_{r-1} = 0$ or we have,

$$(1-u) \quad \text{divides} \quad \sum_{k=0}^{r-2} \left\langle {r-1 \atop k} \right\rangle u^{k+1}.$$

However, if the latter holds, then it would imply that,

$$\sum_{k=0}^{r-2} \left\langle {r-1 \atop k} \right\rangle = 0,$$

contradicting equation 4.3.3, and thus $d_{r-1} = 0$. Repeating this argument for all $1 \leq j \leq r-2$ shows that $d_j = 0$ for all $1 \leq j \leq r$ and completes the proof. □

4.3.6

We shall need the following elementary lemma,

Lemma 4.3.3. *For all* $0 \leq j < r$ *and* $\ell \in \mathbb{Z}$ *we have,*

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} (s+\ell)^j = 0.$$

Proof. We first assume that $\ell = 0$. The case when $r = 1$ is trivial. We proceed by induction on $r > 1$. Assume we have shown the result for $k < r$. If $j = 0$ the result is clear. We have for $j > 0$,

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^j = \sum_{s=0}^{r-1}(-1)^{r-1-s}\binom{r}{s+1}(s+1)^j = r\sum_{s=0}^{r-1}(-1)^{r-1-s}\binom{r-1}{s}(s+1)^{j-1}$$

and thus,

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^j = r\sum_{k=0}^{j-1}\binom{j-1}{k}\sum_{s=0}^{r-1}(-1)^{r-1-s}\binom{r-1}{s}s^k.$$

The result follows from the inductive hypothesis. For $\ell \neq 0$ we have for all $1 \leq j < r$,

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}(s+\ell)^j = \sum_{k=0}^{j}\binom{j}{k}\ell^{j-k}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}s^k.$$

The last sum in the equation above is zero by the $\ell = 0$ case above completing the proof.

□

Using the functional equation,

$$\Phi^{\pm}(u) = k^{\pm 1}\frac{\Lambda^{\pm}(q^{\mp 1}u)}{\Lambda^{+}(q^{\pm 1}u)},$$

and equation (4.3.4), for $1 \leq j < r$ it is clear that the action given in equation (4.3.7) is equivalent to the $\hat{\mathbf{U}}_q^0$ action given by: for $s \in \mathbb{Z}$,

$$\phi_s(v,w) = (\phi_s(\pi)v, \phi_s(\pi)w + z_{j,s}v), \quad v, w \in \mathbb{C}(\pi), \tag{4.3.13}$$

where for $1 \leq j < r$,

$$z_{j,s} = s^j(aq^m)^s, \quad s \in \mathbb{Z}.$$

In particular, we see from Lemma 4.3.3 that for all $1 \leq j < r$ and $\ell \in \mathbb{Z}$,

$$\sum_{s=0}^{r}(-aq^m)^{r-s}\binom{r}{s}z_{j,s+\ell} = (aq^m)^{r+\ell}\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}(s+\ell)^j = 0. \tag{4.3.14}$$

4.3.7

For $1 \le j \le r$ set

$$W(\mathbf{e}_j) = \mathbf{W}_{rm}\mathbf{C}(\mathbf{e}_j) = W(rm) \otimes_{\mathbf{A}_{rm}} \mathbf{C}(\mathbf{e}_j), \quad \mathbf{w}_j = w_{rm} \otimes \mathbf{v}_j.$$

For all $1 \le j \le r$, set,

$$\tilde{\mathbf{w}}_j = (\bar{\Lambda}_1 - \Lambda_1(\pi))\mathbf{w}_j.$$

Recall from Section 4.2 that $W(rm)$ is a right $\hat{\mathbf{U}}_q^0$-module via the action,

$$(uw_{rm})x = (xu)w_{rm}, \quad u \in \hat{\mathbf{U}}_q, \quad x \in \hat{\mathbf{U}}_q^0,$$

and if V is a quotient of $W(rm)$ then the $\hat{\mathbf{U}}_q^0$ action descends to $\mathbf{A}_{rm}$. Hence for $x \in \hat{\mathbf{U}}_q^0$ we have

$$xW(\mathbf{e}_j) = W(rm)x \otimes_{\mathbf{A}_{rm}} \mathbf{C}(\mathbf{e}_j) = W(rm) \otimes_{\mathbf{A}_{rm}} x\mathbf{C}(\mathbf{e}_j). \tag{4.3.15}$$

By Proposition 4.3.2 and Proposition 4.2.1, for each $1 \le j \le r$, we have a non-split short exact sequence of $\hat{\mathbf{U}}_q$-modules,

$$0 \to W(\pi) \xrightarrow{\iota_j} W(\mathbf{e}_j) \xrightarrow{\tau_j} W(\pi) \to 0. \tag{4.3.16}$$

Let $\mathbf{x}_r^- = \mathbf{x}_r^-(m, a)$ be as in equation (2.2.1) and recall also since

$$\mathbf{V}(r) \cong W(\pi)/W(\pi)^{rm}$$

it follows from the defining relations of $\mathbf{V}(r)$ given in Theorem 2 that,

$$W(\pi)^{rm} \cong \hat{\mathbf{U}}_q(\mathbf{x}_r^- w_\pi). \tag{4.3.17}$$

We now prove,

Proposition 4.3.4. *For each* $1 \le j < r$*, we have* $\tau_j(W(\mathbf{e}_j)^{rm}) = W(\pi)^{rm}$*. In particular, for each* $1 \le j < r$*,*

$$V(\mathbf{e}_j) = W(\mathbf{e}_j)/(W(\mathbf{e}_j))^{rm},$$

gives a non-trivial element of $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{V}_r, \mathbf{V}_r)$*.*

Proof. Fix $1 \le j < r$. Using equation (4.3.17), to prove that $\tau_j(W(\mathbf{e}_j)^{rm}) = W(\pi)^{rm}$, it suffices to show that the submodule generated by $\mathbf{x}_r^- \mathbf{w}_j$ is contained in $W(\mathbf{e}_j)^{rm}$. To prove this, it suffices to show that,

$$(\hat{\mathbf{U}}_q \mathbf{x}_r^- \mathbf{w}_j)_{rm} = 0.$$

The subspace $(\hat{\mathbf{U}}_q \mathbf{x}_r^- \mathbf{w}_j)_{rm}$ is the $\hat{\mathbf{U}}_q^0$- submodule generated by the elements $x_\ell^+ \mathbf{x}_r^- \mathbf{w}_j$ for all $\ell \in \mathbb{Z}$. Using the relation,

$$(q - q^{-1})[x_\ell^+, x_s^-] = \phi_{\ell+s}, \quad s, \ell \in \mathbb{Z},$$

since $x_\ell^+ \mathbf{w}_j = 0$ for all $\ell \in \mathbb{Z}$, we have,

$$(q - q^{-1}) x_\ell^+ \mathbf{x}_r^- \mathbf{w}_j = \sum_{s=0}^{r} (-aq^m)^{r-s} \binom{r}{s} \phi_{s+\ell} \mathbf{w}_j, \tag{4.3.18}$$

Using the action given in equation (4.3.13), the right hand side of equation (4.3.18) is equivalent to,

$$\sum_{s=0}^{r} (-aq^m)^{r-s} \binom{r}{s} \phi_{s+\ell}(\pi) \mathbf{w}_j + \sum_{s=0}^{r} (-aq^m)^{r-s} \binom{r}{s} z_{j,s+\ell} \tilde{\mathbf{w}}_j. \tag{4.3.19}$$

The second sum in equation (4.3.19) is zero by equation (4.3.14). Using the defining relations of $\mathbf{V}(r)$ in Theorem 2 we have,

$$0 = (q - q^{-1})[x_\ell^+, \mathbf{x}_r^-] \mathbf{v}_r = \sum_{s=0}^{r} (-aq^m)^{r-s} \binom{r}{s} \phi_{s+\ell} \mathbf{v}_r.$$

In particular,

$$\sum_{s=0}^{r}(-aq^m)^{r-s}\binom{r}{s}\phi_{s+\ell}(\pi) = 0, \tag{4.3.20}$$

showing the first sum in equation (4.3.19) is also 0 as desired. The second statement in the proposition is now immediate from part (ii) of Proposition 4.2.2. □

4.3.8

We now complete the proof of Proposition 4.3.1 and thus complete the proof of Theorem 3. Recall that we have,

$$V(\mathbf{e}_j) = \begin{cases} W(\mathbf{e}_j)/W(\mathbf{e}_j)^{rm}, & 1 \le j < r \\ \mathbf{E}(\mathbf{V}(r)), & j = r. \end{cases}$$

Proposition 4.3.4 and part (i) of Proposition 1.6.1 show that for each $1 \le j \le r$, $[V(\mathbf{e}_j)]$ is a nontrivial element of $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{V}(r), \mathbf{V}(r))$. Using Proposition 3.0.1, to prove Proposition 4.3.1 it suffices to show $[V(\mathbf{e}_j)]$, $1 \le j \le r$ are linearly independent in $\mathrm{Ext}^1_{\mathcal{F}}(\mathbf{V}(r), \mathbf{V}(r))$, and by Corollary 4.2.3 it actually suffices to show $[V(\mathbf{e}_j)_{rm}]$ for $1 \le j \le r$ are linearly independent in $\mathrm{Ext}^1_{\mathbf{A}_{rm}}(\mathbb{C}(\pi), \mathbb{C}(\pi))$. Proposition 4.3.2 now completes the proof.

4.3.9

Before ending this chapter, we shall collect some results about the polynomials $\mathbf{Z_c}(u)$, $\mathbf{c} = (c_1, \dots, c_r) \in \mathbb{C}^r$, that will be used in the next chapter. Recall that we have,

$$\Lambda^{\pm}(u) = \sum_{s \ge 0} \Lambda_{\pm s} u^s.$$

It is clear from the relations of $\mathbf{A}_{rm}$ and Proposition 1.5.3 that given $\mathbf{c} \in \mathbb{C}^r$, the action of $\mathbf{A}_{rm}$ on $V(\mathbf{c})_{rm}$ can be extended to $\hat{\mathbf{U}}_q^0$ by,

$$\Lambda^{\pm}(u)(v,w) = \left(\pi^{\pm}(u)v, \pi^{\pm}(u)w + \mathbf{Z}_{\mathbf{c}}^{\pm}(u)v\right), \quad v,w \in \mathbb{C}(\pi), \tag{4.3.21}$$

where $\mathbf{Z}_{\mathbf{c}}^{+}(u) = \mathbf{Z}_{\mathbf{c}}(u)$, $\pi^{+}(u) = \pi(u)$ and,

$$\mathbf{Z}_{\mathbf{c}}^{-}(u) = (-a)^{rm}u^{rm}\mathbf{Z}_{\mathbf{c}}^{+}(u^{-1}), \quad \pi^{-}(u) = (-a)^{rm}u^{rm}\pi(u^{-1}) = [\omega_{m,a^{-1}}(u)]^r. \tag{4.3.22}$$

It is clear using the explicit form of $\mathbf{Z}_j^{+}(u)$ for $1 \leq j < r$ that we have,

$$\gcd\left(\mathbf{Z}_j^{+}(u), (1 - aq^{-m+1}u)\right) = 1.$$

Moreover, since $\omega_{m,a}(u)$ has distinct roots and,

$$\gcd(\omega_{m,a}(u), (1 - aq^{-m+1}u)) = (1 - aq^{-m+1}u),$$

we have that,

$$\gcd(\mathbf{D}_1\omega_{m,a}(u), (1 - aq^{-m+1}u)) = 1.$$

Since $\mathbf{D}_1[\pi(u)] = [\omega_{m,a}(u)]^{r-1}\mathbf{D}_1[\omega_{m,a}(u)]$, we see that,

$$\gcd\left(\mathbf{Z}_r^{+}(u), (1 - aq^{-m+1}u)^r\right) = (1 - aq^{-m+1}u)^{r-1}.$$

In particular, the above discussion shows that if $\mathbf{c} \in \mathbb{C}^r$, then

$$(1 - aq^{-m+1}u)^r \quad \text{does not divide} \quad \mathbf{Z}_{\mathbf{c}}^{+}(u). \tag{4.3.23}$$

A similar argument shows,

$$(1 - a^{-1}q^{m-1}u)^r \quad \text{does not divide} \quad \mathbf{Z}_{\mathbf{c}}^{-}(u). \tag{4.3.24}$$

Chapter 5

A Lower Bound

In this chapter, we will complete the proof of Theorem 1.

5.1 Proof of Theorem 1

Assume for the moment we have proven the following result,

Proposition 5.1.1. *Let V be an irreducible object of $\mathcal{F}$ which has r prime factors V_i with multiplicities s_i, $1 \leq i \leq r$. Then,*

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = \sum_{i=1}^{r} s_i.$$

Proposition 5.1.1 clearly proves one direction of Theorem 1. For the other direction, let V be an irreducible object of $\mathcal{F}$ such that $\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) = r$. Proposition 1.5.2 shows that V is isomorphic to a (unique up to order) tensor product of s irreducible prime objects of $\mathcal{F}$ for some $s \in \mathbb{Z}_+$. By Proposition 5.1.1 we must have $s = r$ completing the proof.

We note that Proposition 5.1.1 follows from Theorem 3 when $r = 1$. Moreover, by Proposition 3.0.1 we see that it suffices to prove

$$\dim_{\mathbb{C}} \operatorname{Ext}^1_{\mathcal{F}}(V, V) \geq \sum_{i=1}^{r} s_i. \tag{5.1.1}$$

Establishing this inequality will be our goal for the rest of the chapter.

5.1.1

Before proceeding with the proof of Proposition 5.1.1 we fix notation that will be used throughout this chapter. Let V be as in Proposition 5.1.1. By Corollary 1.5.2, for each $1 \leq i \leq r$ there exists unique (up to order) $(m_i, a_i) \in \mathbb{N} \times \mathbb{C}^{\times}$ such that $V_i \cong V(\omega_{m_i,a_i})$, and

$$V \cong V_1^{\otimes s_1} \otimes \cdots \otimes V_r^{\otimes s_r}$$

For each $1 \leq i \leq r$, set

$$\tilde{V}_i = V_1^{\otimes s_1} \otimes \cdots \otimes V_{i-1}^{\otimes s_{i-1}} \otimes V_{i+1}^{\otimes s_{i+1}} \otimes \cdots \otimes V_r^{\otimes s_r}.$$

We note that by Proposition 1.5.1, $V \cong V_i^{\otimes s_i} \otimes \tilde{V}_i$ since V is irreducible. Recall that $V(\omega_{m_i,a_i})$ has a basis $v_{m_i}, \ldots, v_0$ and that $V(\omega_{m_i,a_i})^{\otimes s_i}$ is generated as a $\hat{\mathbf{U}}_q$-module by $v_{m_i}^{\otimes s_i}$. For each $1 \leq i \leq r$, set

$$v_i = v_{m_i}^{\otimes s_i}, \quad \tilde{v}_i = v_{m_1}^{\otimes s_1} \otimes \cdots \otimes v_{m_{i-1}}^{\otimes s_{i-1}} \otimes v_{m_{i+1}}^{\otimes s_{i+1}} \otimes \cdots \otimes v_{m_r}^{\otimes s_r} \tag{5.1.2}$$

For $1 \leq i \leq r$, set

$$\pi(u) = \prod_{j=1}^{r} [\omega_{m_j,a_j}]^{s_j}, \quad \pi_i(u) = [\omega_{m_i,a_i}]^{s_i}, \quad \tilde{\pi}_i(u) = \prod_{\substack{j=1 \\ j \neq i}}^{r} [\omega_{m_j,a_j}]^{s_j} \tag{5.1.3}$$

so that for each $s \in \mathbb{Z} \setminus \{0\}$ and $1 \leq i \leq r$ we have,

$$\Lambda_s(v_i \otimes \tilde{v}_i) = \Lambda_s(\pi)(v_i \otimes \tilde{v}_i), \quad \Lambda_s v_i = \Lambda_s(\pi_i) v_i, \quad \Lambda_s \tilde{v}_i = \Lambda_s(\tilde{\pi}_i) \tilde{v}_i.$$

5.1.2

The following elementary result will be useful in the proof of Proposition 5.1.1. We include the proof for completeness.

Lemma 5.1.2. *With V as above, there exists $1 \leq k \leq r$ such that for all $1 \leq j \neq k \leq r$, one of the following hold,*

$$a_k q^{m_k} \neq a_j q^{m_j} \quad \textit{or} \quad a_k q^{-m_k} \neq a_j q^{-m_j}.$$

Proof. Suppose for a contradiction that for all $1 \leq k \leq r$ there exists $1 \leq j \neq k \leq r$ such that

$$a_k q^{m_k} = a_j q^{m_j} \quad \text{and} \quad a_k q^{-m_k} = a_j q^{-m_j}. \tag{5.1.4}$$

It follows that we must have $(m_k, a_k) = (m_j, a_j)$ contradicting our assumption that the prime factors of V were distinct. This completes the proof.

□

5.1.3

Our first step in the proof of Proposition 5.1.1 is,

Proposition 5.1.3. *There exists a unique map of vector spaces,*

$$f \colon \bigoplus_{i=1}^{r} \mathrm{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}) \longrightarrow \mathrm{Ext}^1_{\mathcal{F}}(V, V),$$

given by,

$$([U_1], \dots, [U_r]) \mapsto \sum_{i=1}^{r} [U_i \otimes \tilde{V}_i], \quad [U_i] \in \mathrm{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}).$$

Proof. By Corollary 1.6.2, for each $1 \leq i \leq r$, we have an injective map of vector spaces

$$p_i : \operatorname{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}}(V, V)$$

given by

$$[U_i] \mapsto [U_i \otimes \tilde{V}_i], \quad [U_i] \in \operatorname{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i})$$

For each $1 \leq i \leq r$, let

$$\iota_i : \operatorname{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}) \longrightarrow \bigoplus_{i=1}^{r} \operatorname{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}),$$

be the natural inclusions in the category of $\mathbb{C}$-vector spaces. It follows from the universal property of coproducts in the category of $\mathbb{C}$-vector spaces that there exists a unique map of vector spaces ,

$$f\colon \bigoplus_{i=1}^{r} \operatorname{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i}) \longrightarrow \operatorname{Ext}^1_{\mathcal{F}}(V, V),$$

such that $f \circ \iota_i = p_i$ for all $1 \leq i \leq r$. In particular, using the definition of p_i for $1 \leq i \leq r$, and the linearity and uniqueness of f it follows that,

$$f([U_1], \ldots, [U_r]) = \sum_{i=1}^{r} p_i([U_i]) = \sum_{i=1}^{r} [U_i \otimes \tilde{V}_i]. \tag{5.1.5}$$

as desired. □

5.1.4

Assume for the moment we have shown the following result,

Proposition 5.1.4. *If U_i is a non-trivial self-extension of $V_i^{\otimes s_i}$ for all $1 \leq i \leq r$, then $\left\{[U_i \otimes \tilde{V}_i] : 1 \leq i \leq r\right\}$ is a linearly independent subset of $\operatorname{Ext}^1_{\mathcal{F}}(V, V)$.*

The proof of Proposition 5.1.1 is completed as follows. Clearly, it suffices to show that the map f in Proposition 5.1.3 is injective. For $1 \le i \le r$, let U_i be self-extensions of $V_i^{\otimes s_i}$ such that in $\mathrm{Ext}^1_{\mathcal{F}}(V, V)$ we have,

$$f([U_1], \dots, [U_r]) = \sum_{i=1}^{r} [U_i \otimes \tilde{V}_i] = 0.$$

By Proposition 5.1.4 we must have that $[U_i] = 0$ in $\mathrm{Ext}^1_{\mathcal{F}}(V_i^{\otimes s_i}, V_i^{\otimes s_i})$ completing the proof. Proving Proposition 5.1.4 will be our goal from now on.

5.1.5

Fix $1 \le i \le r$, and let $\mathbf{c}_i = (c_{i,1}, \dots, c_{i,s_i})$ be a non zero vector in $\mathbb{C}^{s_i}$. Recall from Theorem 3 that there exists a unique (up to equivalence) self-extension $V(\mathbf{c}_i)$ of $V_i^{\otimes s_i}$ such that the action of $\mathbf{A}_{s_i m_i}$ on $V(\mathbf{c}_i)_{s_i m_i}$ is given by the functional equation,

$$\bar{\Lambda}(u)(v, w) = (\pi_i(u)v, \pi_i(u)w + \mathbf{Z}_{\mathbf{c}_i}(u)v), \quad v, w \in V(\mathbf{c}_i)_{s_i m_i}. \tag{5.1.6}$$

where $\mathbf{Z}_{\mathbf{c}_i}(u)$ is as in Theorem 3. Set

$$\mathbf{Z}_i(u) = \mathbf{Z}_{\mathbf{c}_i}(u)\tilde{\pi}(u), \quad \mathbf{Z}_i^-(u) = \mathbf{Z}_{\mathbf{c}_i}^-(u)\tilde{\pi}^-(u), \tag{5.1.7}$$

where $\mathbf{Z}_{\mathbf{c}_i}^-(u)$ is defined in equation (4.3.22) and

$$\tilde{\pi}^-(u) = \prod_{\substack{j=1 \\ j \ne i}}^{r} [\omega_{m_j, a_j^{-1}(u)}]^{s_j}.$$

Set

$$\mathbf{V}(\mathbf{c}_i) = V(\mathbf{c}_i) \otimes \tilde{V}_i,$$

Finally, set $m = s_1m_1 + \cdots + m_rs_r$, and notice that $\dim_{\mathbb{C}} \mathbf{V}(\mathbf{c}_i)_m = 2$ and $\mathbf{V}(\mathbf{c}_i)_m$ is spanned by vectors of the form,

$$(v, w) \otimes \tilde{v}_i, \quad (v, w) \in V(\mathbf{c}_i)_{s_im_i},$$

with $\tilde{v}_i$ defined in equation (5.1.2). We now prove,

Lemma 5.1.5. *With the notation above, for each $1 \le i \le r$, the action of $\mathbf{A}_m$ on $\mathbf{V}(\mathbf{c}_i)_m$ is given by,*

$$\Lambda(u)((v, w) \otimes \tilde{v}_i) = (\pi(u)v, \pi(u)w + \mathbf{Z}_i(u)v) \otimes \tilde{v}_i, \quad v, w \in V(\mathbf{c}_i)_{s_im_i}.$$

Proof. Fix $1 \le i \le r$. Using the comultiplication formulae in Proposition 1.3.2 it is clear that the comultiplication of $\bar{\Lambda}(u)$ is given by the functional equation,

$$\Delta(\bar{\Lambda}(u)) = \bar{\Lambda}(u) \otimes \bar{\Lambda}(u)$$

Thus, using equation (5.1.6) and Proposition 1.5.3 we have for $(v, w) \otimes \tilde{v}_i \in \mathbf{V}(\mathbf{c}_i)_m$,

$$\Delta(\bar{\Lambda}(u))((v, w) \otimes \tilde{v}_i) = \bar{\Lambda}(u)(v, w) \otimes \bar{\Lambda}(u)\tilde{v}_i = (\pi_i(u)v, \pi_i(u)w + \mathbf{Z}_{\mathbf{c}_i}(u)v) \otimes \tilde{\pi}_i(u)\tilde{v}_i$$

Since $\pi(u) = \pi_i(u)\tilde{\pi}_i(u)$ and $\mathbf{Z}_i(u) = \mathbf{Z}_{\mathbf{c}_i}(u)\tilde{\pi}_i(u)$ by equations (5.1.3) and (5.1.7) respectively, the proof is complete. □

5.2 Proof of proposition 5.1.1

We now prove Proposition 5.1.4 and thus complete the proofs of Proposition 5.1.1 and Theorem 1. By Theorem 3, if U_i is a self-extension of $V_i^{\otimes s_i}$, then as $\hat{\mathbf{U}}_q$-modules, U_i is isomorphic to $V(\mathbf{c}_i)$ for some $\mathbf{c}_i \in \mathbb{C}^{s_i}$. Thus, we must show that $\{[\mathbf{V}(\mathbf{c}_i)] : 1 \le i \le r\}$ as defined in the previous section are linearly independent in $\mathrm{Ext}^1_{\mathcal{F}}(V, V)$. The case when $r = 1$ is given by Theorem 3.

We proceed by induction on $r > 1$. Suppose we have proven the result for all $s < r$. Let $\mathbf{E}$ be a trivial self-extension of $\mathbb{C}(\pi)$ such that in $\mathrm{Ext}^1_{\mathbf{A}_m}(\mathbb{C}(\pi), \mathbb{C}(\pi))$ we have the Baer sum,

$$[\mathbf{E}] = \sum_{i=1}^{r} d_i [\mathbf{V}(\mathbf{c}_i)_m] = [0].$$

for some $d_i \in \mathbb{C}$. In particular, we have an isomorphism of $\mathbf{A}_m$-modules, $\mathbf{E} \cong \mathbb{C}(\pi) \oplus \mathbb{C}(\pi)$. Using Lemma 5.1.2, notice we can assume, after possible reordering that, for all $1 < j \leq r$,

$$a_1 q^{-m_1} \neq a_j q^{-m_j} \quad \text{or} \quad a_1 q^{m_1} \neq a_j q^{m_j}.$$

Suppose first that we have

$$a_1 q^{-m_1} \neq a_j q^{-m_j} \quad \text{for all } 1 < j \leq r. \tag{5.2.1}$$

Using the $\mathbf{A}_m$-action in Lemma 5.1.5 and Lemma 4.2.4, it follows that we must have,

$$\sum_{i=1}^{r} d_i \mathbf{Z}_i(u) = 0.$$

In particular, using equation (5.1.7) we have,

$$d_1 \mathbf{Z}_{\mathbf{c}_1}(u) \tilde{\pi}_1(u) = -\sum_{i=2}^{r} d_i \mathbf{Z}_{\mathbf{c}_i}(u) \tilde{\pi}_i(u). \tag{5.2.2}$$

Notice that $\pi_1(u)$ divides every term in the sum on the right hand side of equation (5.2.2). In particular, we must have that, $(1 - a_1 q^{-m_1+1} u)^{s_i}$ divides the right hand side of equation (5.2.2), and thus must also divide $\mathbf{Z}_{\mathbf{c}_1}(u)\tilde{\pi}_1(u)$. However, using the definition of $\tilde{\pi}_1(u)$ in equation (5.1.3) and our assumption in equation (5.2.1) above, this implies that $(1 -$

$a_1 q^{-m_1+1} u)^{s_i}$ divides $\mathbf{Z}_{\mathbf{c}_1}(u)$ or $d_1 = 0$. Since the former contradicts equation (4.3.23) we must have $d_1 = 0$. It now follows from the inductive hypothesis that $d_i = 0$ for all $1 \leq i \leq r$. If instead the second statement in the equation proceeding equation 5.2.1 holds, a similar argument using the polynomials $\mathbf{Z}_i^-(u)$ and equation (4.3.24) completes the proof. We omit the details.

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