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(Ph. D. Thesis)

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ABSTRACT

Unitarity, analyticity, Regge asymptotic behavior, and a resonance approximation are combined to derive a new sum rule. The sum rule is very convergent; the contribution of high-mass resonances is suppressed by a decreasing weight function. The spin-flip and non-spin-flip residues of the ρ meson in the $\pi\pi \to N\bar{N}$ amplitude are evaluated at the mass of the ρ, and in conjunction with the first-moment finite-energy sum rule a calculation of the ρ-meson mass is performed. The results are in good agreement with experiment. The sum rule is then applied to the calculation of the ρ and f_0 resonance parameters in the $\pi\pi \to \pi\pi$ amplitude. A discussion of the behavior of the sum rule near the elastic threshold is also included. This discussion may give some insight into the nature of the approximations involved in the derivation of the sum rule.

INTRODUCTION

Recently the finite-energy sum rules have been applied to the calculation of strong interaction parameters and, to within the errors involved in the approximations, the results have been in agreement with experiment. It has been conjectured that these relations may actually provide a new approach to the bootstrap problem, and a few attempts in formulating such an approach have met with some success.[1,2]

The finite-energy sum rules consist of an infinite set of equations which relate all the positive and negative moments of the discontinuity of the amplitude over a finite energy region to the Regge parameters. In practice, however, only the first few positive moment sum rules have been utilized, because the higher moments emphasize a higher interval of the energy spectrum and the negative-moment sum rules contain the value of the amplitude, or one of its derivatives at some point, as an unknown constant. It would be useful to have sum rules in which the weight function decreases without introducing subtraction constants, since even the low-positive-moment sum rules already put an uncomfortable emphasis on the higher energy behavior of the discontinuity of the amplitude. Furthermore, the finite-energy sum rules used in a bootstrap scheme only provide linear equations for the widths and as such can only determine ratios.

Therefore, it would also be useful to introduce some non-linearity into these equations.

In this paper, we will use the two body unitarity equation in the complex J-plane in addition to analyticity and Regge behavior to derive a sum rule with the decreasing weight functions $Q_J(z)$. In order to apply this sum rule to the calculation of high energy parameters, however, we will have to introduce certain approximations. One of the possible applications of finite-energy sum rules to the boot-strap problem is to use the sum rules for values of energy equal to the masses of the resonances under consideration. The first few sections of this paper are concerned with such a problem. In order to use the sum rule in this context, we will make a small-width approximation (but not the usual zero-width approximation) which we will discuss in detail. By a small-width approximation we mean the width of the resonances we consider are small enough that the Breit-Wigner formula is reasonably accurate, but we do not assume $\mathrm{Im}(\alpha) = 0$.

Finite-energy sum rules in general contain a parameter N, the upper limit of the integral of the imaginary part of the amplitude multiplied by some weight function. In order for these sum rules to be useful in bootstrap-type calculations, N must correspond to the "intermediate energies," so that the integrand may be parametrized by a sum of resonances. For the sum rule presented here, the value of N depends on the magnitude of $\mathrm{Im}(\alpha)$. If N is to correspond to intermediate energies, $\mathrm{Im}(\alpha)$ can not be very small. In the first few sections where we are concerned with bootstrap calculations our results depend crucially on the fact that experimentally, near the mass of many prominent resonances, $\mathrm{Im}(\alpha) \approx 0.1$ and not much smaller. However, 0.1 is small enough to allow neglecting terms of order $\mathrm{Im}(\alpha)$ compared to one, without causing a very large error. Of course, since in these calculations we do not take the limit $\mathrm{Im}(\alpha) \rightarrow 0$, we must be careful that the coefficient of $\mathrm{Im}(\alpha)$ in such terms is not much larger than one. For example, when $\mathrm{Im}(\alpha)$ is multiplied by $2\ln(2N)$, as is the case of this sum rule, nonlinear terms in $\mathrm{Im}(\alpha)$ should also be included. The small-width approximation used here does introduce an intrinsic error of about 15% into our calculations.

In Section II we will give the derivation of the sum rule and discuss the relevant approximations. In Section III we will apply the sum rule to a calculation of the non-spin-flip and spin-flip residues of the ρ trajectory at $t = m_\rho^2$ in the $\pi\pi \rightarrow N\bar{N}$ scattering amplitudes. The nucleon plus all the established πN resonances below 2 GeV c.m. (center of mass) energy constitute the input for the sum rule. The dominant contribution comes from the nucleon. Since the identical model can be applied to the finite-energy sum rules, it is interesting to ask whether this sum rule has dynamical content beyond that contained in the positive-moment sum rules. The numerical calculations indicate that they have very different content, and therefore, our sum rule may be used in addition to the finite-energy sum rules to restrict further the resonance parameters of the model. In this case, we have four equations for three unknowns, so we have a check on the internal consistency of the model. The calculated mass of the ρ is about 900 MeV.

In Section IV a simple model of $\pi\pi \rightarrow \pi\pi$ scattering in which the amplitude is dominated by the ρ and f_0 resonances is discussed. Again reasonable constraints on the resonance parameters can be obtained. Together with the finite-energy sum rules, these equations provide a nearly complete bootstrap system. The slopes of the Regge trajectories [which are arbitrarily set at $1\ (\mathrm{GeV})^{-2}$] and a scale for the resonance widths cannot be determined from the equations. (Our sum rule is not nonlinear enough in $\mathrm{Im}\ (\alpha)$ to obtain an absolute scale for the resonance widths.) For such a simple model, the determined values of the masses and ratio of the widths are quite reasonable.

In Section V we will discuss the behavior of the sum rule near threshold. We will see how a background term becomes increasingly more important as we approach the threshold and how the approximations which hold for $\mathrm{Im}(\alpha) \approx 0.1$ break down as $\mathrm{Im}(\alpha) \rightarrow 0$. We will discuss the possibility of calculating the background term from an N/D model and thus using the sum rule to calculate residue functions over a wider range of their argument.

II. DERIVATION AND DISCUSSION OF THE SUM RULE

Let $A(t,z)$ be the amplitude for the elastic scattering of two spinless particles for which t is the square of the c.m. energy and z is the cosine of the scattering angle in the t-channel. We are considering the spinless , elastic problem to simplify the discussion of this section. In the next section we will generalize our results to the case of inelastic amplitudes with spin (for example $\pi\pi \to N\bar{N}$) . The variable t, however, is restricted to the region between the lowest threshold and the next important one throughout this paper. The asymptotic behavior of $A(t,z)$ as $z \to \infty$ is assumed dominated by the leading t-channel Regge pole which we denote by $R(t,z)$. Although the functional from of the Regge term is somewhat ambiguous, we will require that $R(t,z)$ reduce to the correct resonance formula when $\alpha(t)$ is near an even (or odd) integer. Other modifications of $R(t,z)$ do not affect this derivation, and we choose the explicit form

$$R(t,z) = -\pi(2\alpha+1)\,\beta(t)\,\frac{P_{\alpha(t)}(-z) \pm P_{\alpha(t)}(z)}{2\sin\pi\alpha(t)} \qquad (1)$$

where refer to signature.

The partial wave unitarity equation is written as,

$$a_J(t) - a_J^*(t) = 2i\,\rho(t)\,a_J(t)\,a_J^*(t). \qquad (2a)$$

This equation can be continued to the complex J-plane. For t real and above the threshold , $a_J(t)$ and $a_J^*(t)$ are contitued into the functions $a^{\pm}(J,t)$ and $a'^{\pm}(J,t) \equiv a^{\pm *}(J^*,t)$. Equation (2a) becomes

$$a^{\pm}(J,t) - a'^{\pm}(J,t) = 2i\rho\, a^{\pm}(J,t)\, a'^{\pm}(J,t) \qquad (2b)$$

where $\pm$ again refer to signature, and t is in its physical region. The resonance pole discussed above also corresponds to a pole of a (J,t) at $J=\alpha$. If Im(α) is small compared to the distance to other singularities, then $a'^{\pm}(J,t)$ may also be represented by a single pole plus a background of order Im(α). Equation (2) then implies the relation [3]

$$\rho\beta = \mathrm{Im}(\alpha) + O[\mathrm{Im}(\alpha)]^2 \qquad (3)$$

Since Im(α) is real, the phase of β is of order Im(α),

$$\mathrm{Im}(\beta) = O[\mathrm{Im}(\alpha)]^2 \quad . \qquad (4)$$

Let $\alpha(t)$ be the position of the leading Regge pole. $a'^{\pm}(J,t)$ has a pole at $J=\alpha^*(t)$. With the Froissart-Gribov definition of $a^{\pm}(J,t)$, Eq. (2) implies

$$-1/2i\, \rho(t) = \lim_{J\to\alpha^*} \frac{1}{\pi}\int_{z_0}^{\infty} dz\; A_s^{\pm}(t,z)\; Q_J(z), \qquad (5)$$

where

$$A_s^{\pm}(t,z) = [A(t,z+i\epsilon) - A(t,z-i\epsilon)]/2i$$
$$\pm [A(t,-z+i\epsilon) - A(t,-z-i\epsilon)]/2i \qquad (6)$$

Since theintegral in Eq. (5) is divergent for $\mathrm{Re}(\alpha) \geqslant \mathrm{Re}(J)$, it should be evaluated for $\mathrm{Re}(\alpha) < \mathrm{Re}(J)$ and then continued to $J = \alpha^*$.

If $A^{\pm}(t,z)$ is approximated to any preassigned accuracy by its leading Regge trajectory for $z \geqslant N$, we may rewrite Eq. (5) as

$$-1/2i\rho = \frac{1}{\pi} \int_{z_0}^{N} dz \, A_s^{\pm}(t,z) \, Q_{\alpha^*}(z)$$

$$\lim_{J \to \alpha^*} \frac{1}{\pi} \int_{N}^{\infty} dz \, R_s^{\pm}(t,z) \, Q_J(z) + \frac{1}{\pi} \int_{N}^{\infty} dz \, \bar{R}(t,z) \, Q_{\alpha^*}(z) \tag{7}$$

where $\bar{R}$ is the contribution of the other J-plane singularities.

We will show in the rest of this section that the real part of Eq. (7) may be applied as a finite-energy sum rule to the calculation of some high energy parameters. The imaginary part of this equation also constitutes a constraint, but with a limited knowledge of $A_s(t,z)$ this constraint turns out to be not very useful. As mentioned in the introduction, one possible application of Eq. (7) to bootstrap-like calculations is to consider the sum rule at the mass of a resonance $(t = M^2)$. In this and the next two sections we address ourselves to such a problem. In order to put Eq. (7) into a useful form however, we will make certain approximations. The first approximation involves the quantity N and is similar to the approximations made in finite-energy sum rules. In effect, Eq. (7) is expanded in powers of (1/N) (often fractional powers) and the higher order terms in (1/N) are then neglected compared to terms of order one. Of course, such an approximation becomes more valid as the magnitude of N becomes larger. However, an important criterion for the usefulness of such equations is that N be in a region below which $A_s(t,z)$ may be parametrized by s- and u-channel resonances. Thus in practice N is not taken to infinity , but is restricted to the intermediate region. In general such values of N are large enough to allow neglecting terms of order 1/N compared to one. However, in the case of our sum rule there exists another small quantity , namely $\mathrm{Im}(\alpha)$, and we must be carefull to ensure that the quantities we finally keep in our sum rule are of order one in both 1/N and $\mathrm{Im}(\alpha)$. In the rest of this section we discuss in detail this double expansion and the validity of certain approximations. The result of these discussions will be that in order to separate the quantities involved into terms of different order with respect to $\mathrm{Im}(\alpha)$ and 1/N , and then neglect the higher order terms, the quantity $\cot(2\mathrm{Im}\alpha \ln 2N)$ should not be very large. In other words, in certain terms of our expansion, $\mathrm{Im}(\alpha)$ and N will occur in the combination $2\mathrm{Im}\alpha \ln 2N$. We will require that the cotangent of this quantity should not be much greater than one. For N in the intermediate region, it turns out that this is only possible if Im(α) is not much smaller than .1 .

We well now engage in the detailed discussion of our approximations. We need to consider the integrals

$$\lim_{J\to\alpha^*}\int_N^\infty dz\, P_\alpha(z)\, Q_J(z) = \frac{(-\sin y + i\cos y)}{2(2\mathrm{Re}\,\alpha+1)\,\mathrm{Im}(\alpha)}\Big[1 + 2ic_1\,\mathrm{Im}(\alpha) + O(\mathrm{Im}\,\alpha)^2\Big] + O(1/N) \qquad (8)$$

$$\lim_{J\to\alpha^*}\int_1^N dz\, P_\alpha(z)\, Q_J(z) = \frac{\sin y + i(1-\cos y)}{2(2\mathrm{Re}\,\alpha+1)\,\mathrm{Im}(\alpha)}\Big[1 + O(\mathrm{Im}\,\alpha)\Big] + O(1/N) \qquad (9)$$

and their sum

$$\lim_{J\to\alpha^*}\int_1^\infty dz\, P_\alpha(z)\, Q_J(z) = \frac{i}{2(2\mathrm{Re}\,\alpha+1)\,\mathrm{Im}(\alpha)}$$

with

$$y = 2\,\mathrm{Im}(\alpha)\,\ln(2N) \quad \text{and,}$$

$$c_1 = 1/(2\mathrm{Re}\,\alpha+1) + \psi(\alpha+1/2) - \psi(\alpha+1)$$

where $\psi(\alpha)$ is the logarithmic derivative of the Γ-function.

The integral in Eq. (8) is proportional to the Regge integral of Eq. (7) . Since experimentally the Regge contribution approximates the average magnitude of the imaginary part of the amplitude for small values of z (or s) [1], the integral in Eq. (9) can give a good estimate of the size of the first integral in Eq. (7).

From Eq. (1) the explicit expression for $R_s(t,z)$ is

$$\begin{aligned} R_s^{\pm}(t,z) &= \pi(2\alpha+1)\,\beta(t)\,P_\alpha(z) && \text{for right signature,} \\ &= 0 && \text{for wrong signature.} \end{aligned} \tag{10}$$

Substituting this in Eq. (7) and with the aid of Eq.(8) we obtain ,

$$\frac{\mathrm{Re}\left[\beta(M^2)\right]}{\mathrm{Im}(\alpha)} \sin y \left[1 + \frac{\mathrm{Im}\left[\beta(M^2)\right]}{\mathrm{Re}\left[\beta(M^2)\right]} \cot y + 2c_2\,\mathrm{Im}(\alpha)\cot y + O(1/N) + O(\mathrm{Im}\,\alpha)^2 \right]$$

$$\mathrm{Re}\left[\frac{2}{\pi}\int_{z_0}^{N} dz\, A_s^{\pm}(t,z)\, Q_{\alpha^*}(z)\right] + \mathrm{Re}\left[\frac{2}{\pi}\int_{N}^{\infty} dz\, \bar{R}_s(t,z)\, Q_{\alpha^*}(z)\right] \tag{11}$$

where $c_2 = c_1 + 1/(2\mathrm{Re}\alpha+1)$.

The first approximation concerns the left hand side of Eq. (11) . Eq. (4) implies that the quantity $\mathrm{Im}(\beta)/\mathrm{Re}(\beta)$ is of order $\mathrm{Im}(\alpha)$. Therefore if cot(y) is not much greater than one, we can neglect the two terms $\mathrm{Im}(\beta)/\mathrm{Re}(\beta)$ cot y, and $2c_2\,\mathrm{Im}(\alpha)$ cot y inside the bracket. The left-hand side will then be equal to $\left[\mathrm{Re}(\beta) / \mathrm{Im}(\alpha) \right]$ sin y. When cot y is not very large (sin y is not small) this quantity is of order one, since from Eq. (3) $\mathrm{Re}(\beta) / \mathrm{Im}(\beta)$ is of order one.

The magnitude of the second integral on the right of Eq. (11) can be estimated by the contribution of the next leading singularity in the J-plane. If α_2 and β_2 are the position and the residue of the singularity, then this contribution is of order $\beta_2 / N^{(\alpha - \alpha_2)}$. As in the case of finite-energy sum rules, for large enough N this term is much smaller than the left hand side. However, in order to completely justify neglecting this term, we must show that the first integral on the right is also of order one. As mentioned before, the magnitude of the latter quantity can be estimated by considering the integral of Eq. (9). As N is increased from its minimum value, this integral starts from zero and grows in magnitude. For some value of N its magnitude actually becomes comparable to that of the integral in Eq. (8). This condition is realized when cot y $\lesssim$ 1.

We can thus see that cot y is a measure of the accuracy of our approximations, and the condition cot y ≈ 1 gives the desired relation between N and $\mathrm{Im}(\alpha)$. If $\mathrm{Im}(\alpha)$ is very small, cot y ≈ 1 demands an extremely large N. However, for the experimental values of $\mathrm{Im}(\alpha) \approx 0.1$, this condition is satisfied when $N \approx 10$, which corresponds to intermediate energies when t is in the resonance region. The next oscillation of cot y occurs for a very large value of N and does not concern us here.

Including the above approximations in Eq. (11) we write our sum rule in the final form of Eq. (12), correct to leading order in 1/N and Im(α) :

$$\sin(2\,\mathrm{Im}\,\alpha\,\ln 2N) = 2\,\rho(M^2)\,\mathrm{Re}\left[\frac{1}{\pi}\int_{z_0}^{N} dz\, A_s^{\pm}(M^2,z)\, Q_{\alpha^*}(z)\right] \qquad (12)$$

The successful calculations with the finite-energy sum rules suggest that for $z < N$, $A_s^{\pm}(t,z)$ may be approximated by s- and u-channel resonances, even though t is outside the ellipse of convergence of the partial-wave expansion in the s and u channels. We will parametrize $A_s^{\pm}(M^2,z)$ by a sum of resonances and apply Eq. (12) to the calculation of some parameters in the $\pi\pi \rightarrow N\bar{N}$ and $\pi\pi \rightarrow \pi\pi$ processes. We will use δ-functions for the s-and u-channel resonances for the sake of simplicity. When compared to the Breit-Wigner formula, this approximation causes only a 1% error for Im(α)$<$0.2 . However, it should be emphasized that by using δ-functions in the right hand side of Eq. (12) we will not contradict the previous statement that we do not take the limit Im(α)$\rightarrow$0. As is clear from the discussion of this section, the condition Im(α)$\neq$ 0 is crucial to justifying a finite value of N in Eq. (12) . The correct parametrization of $A_s(t,z)$ for t above its threshold is one of the major unsolved problems in the application of finite-energy sum rules to the bootstrap problem. (Some discussion of this question can be found in Ref. 1) . In this paper, we will take the success of some calculations with finite-energy sum rules in which $A_s(t,z)$ is parametrized by a sum of zero-width resonances as an indication that such a parametrization may be reasonable. This problem which is probably the source of the greatest uncertainties in the numerical results certainly deserves a separate and thorough investigation.

III. CALCULATION OF ρ RESIDUES IN $\pi\pi \rightarrow N\bar{N}$

The derivation of sum rules like Eq. (12) for inelastic amplitudes with spin contains no essential complications. The procedure, which is similar to that for the spinless elastic case, is : (A) substitute the fixed-t dispersion relation into the partial wave formula (i.e., derive the Froissart-Gribov formula); (B) continue to $J = \alpha^*$ after introducing the leading Regge pole; (C) use unitarity to evaluate the amplitude at $J = \alpha^*$;(D) analyze the resulting equation as in section II. There is only one modification of the basic formula, Eq.(7) . If an inelastic amplitude is being considered, the left hand side of Eq. (7) will be $-1/2i\rho$ times a ratio of Regge residues. If 1 and 2 label two communicating channels and t lies between the thresholds of these channels (1 corresponds to the lowest threshold) , we can write the unitarity equation as

$$a_{12}(J,t) - a_{12}^*(J^*,t) = 2i\rho \; a_{11}^*(J^*,t) \; a_{12}(J,t) \qquad (13)$$

Again, if there is a pole at $J = \alpha$, Eq. (13) implies,

$$-1 \,/\, 2i\rho \left[\beta_{11}^*(t) \,/\, \beta_{12}^*(t) \right] = a_{12}(\alpha^*,t) \qquad (14)$$

However, in our approximation the imaginary part of the ratio, $\beta_{11}^*(t)/\ \beta_{12}^*(t)$ is proportional to the phase of the residues, which is of order Im(α) . Consequently , it can be neglected compared to terms of order one.

Let us now apply this procedure to the spin-flip and non-spin-flip amplitudes in the $\pi\pi \rightarrow N\bar{N}$ channel and calculate the ρ residues. The t channel is $\pi\pi \longrightarrow N\bar{N}$, the usual invariant amplitudes are A and B, and someuseful kinematical factors are

$$2p = (t - 4M^2)^{1/2} ,$$
$$2q = (t - 4\mu^2)^{1/2},$$
$$\Delta = (t - 2\mu^2 - 2M^2),$$
$$z = \frac{s + \Delta}{2pq}$$
$$x = |z| ,$$

and

$$\nu = 2M \frac{s + \Delta}{4M^2 - t} \qquad (15)$$

The nucleon and pion masses are denoted by M and μ , respectively. The value $t = m_\rho^{\;2}$ is below the $N\bar{N}$ threshold, so that p and z are both pure imaginary $(z = ix)$.

The amplitudes $A' = A + \nu B$ and B are proportional to the helicity amplitudes in the t channel:

$$T_+(t,z) = A(t,z) + \nu B(t,z)$$
$$T_-(t,z) = \sqrt{t}\ \sin\theta_t\ \ B(t,z) \qquad (16)$$

where θ_t is the c.m. scattering angle, and + and — refer to the ++ and + − helicity amplitudes. The isotopic-spin projections are

$$A'^{(-)} = \frac{1}{3}\left(A'^{I=\frac{1}{2}} - A'^{I=\frac{3}{2}} \right), \tag{17}$$

and a similar relation for B. The (-) amplitudes are pure $I=1$ in the t channel. The partial-wave unitarity relation for $T_{\pm}(t,z)$ with only the two-pion intermediate states are

$$T_{\pm}(J,t) - T_{\pm}^{*}(J^{*},t) = 2i\rho \; T_{\pm}(J,t) \; T_{\pi}^{*}(J^{*},t) \tag{18}$$

where $T_{\pi}(J,t)$ is the amplitude for $\pi\pi \rightarrow \pi\pi$.

We assume the ρ trajectory gives the asymptotic behaviour of $A'^{(-)}$ and $B^{(-)}$ as $z \longrightarrow \infty$. The Regge terms are[4]

$$A'_R(t,z) = -\pi(2\alpha+1)\,\frac{P_{\alpha}(-z) - P_{\alpha}(z)}{2\sin\pi\alpha}\,\frac{16\pi M}{4M^2 - t}\left(\frac{pq}{M^2}\right)^{\alpha} r_{+}(t) \tag{19}$$

$$B_R(t,z) = \pi\,\frac{P'_{\alpha}(-z) + P'_{\alpha}(z)}{2\sin\pi\alpha}\,\frac{8\pi}{M^2}\left(\frac{pq}{M^2}\right)^{\alpha-1} r_{-}(t) \tag{20}$$

where P'_{α} is the derivative of the Legendre polynomial. The normalization of the residues is identical to the normalization of the $r_{+}(t)$ and $r_{-}(t)$ defined by Desai in Ref 4. We can continue the Froissart-Gribov formula to $t = m_{\rho}^2$ and $J = \alpha^{*}$ and use unitarity to evaluate the amplitude at this point. We will find:

$$- \frac{1}{2i\rho} \frac{r_+^*(m_\rho^2)}{\beta_\pi^*(m_\rho^2)} = - \frac{r_+(m_\rho^2)}{2i\,\mathrm{Im}\,\alpha} (2i\bar{N})^{2i\,\mathrm{Im}\,\alpha}$$

$$+ \frac{4M^2 - m_\rho^2}{8\pi^2 M} \left(\frac{M^2}{i\bar{p}q}\right)^{\alpha} \int_{x_0}^{\bar{N}} (-i)dx \, A_s'(m_\rho^2, -ix) \, Q_{\alpha^*}(-ix) \tag{21}$$

and

$$- \frac{1}{2i\rho} \frac{r_-^*(m_\rho^2)}{\beta_\pi^*(m_\rho^2)} = - \frac{r_-(m_\rho^2)}{2i\,\mathrm{Im}\,\alpha} (2i\bar{N})^{2i\,\mathrm{Im}\,\alpha}$$

$$+ \frac{M^2}{4\pi^2} \left(\frac{M^2}{i\bar{p}q}\right)^{\alpha-1} \int_{x_0}^{\bar{N}} (-i)dx \, B_s(m_\rho^2, -ix) \left[Q_{\alpha^*-1}(-ix) - Q_{\alpha^*+1}(-ix) \right] \tag{22}$$

where the bars denote absolute values of quantities that become complex for $t < 4M^2$: $N = -i\bar{N}$, $p = i\bar{p}$, and $z = -ix$. The contribution of the other J-plane singularities (the $\bar{R}$ terms of Eq. (7)) and other terms of order $(1/N)$ and $\mathrm{Im}(\alpha)$ have been dropped from the right hand side of Eqs. (21) and (22) . To leading order in $\mathrm{Im}(\alpha)$ and $(1/N)$ then, the real part of these equations lead to the following sum rules for $r_+(m_\rho^2)$ and $r_-(m_\rho^2)$:

$$r_+(m_\rho^2) = - \lambda^{-1} \frac{(4M^2 - m_\rho^2)}{8\pi^2 \bar{p}q} \int_{x_0}^{N} dx \, A_s'(m_\rho^2, -ix) \, Q_1(-ix) \tag{23}$$

and

$$r_-(m_\rho^2) = \lambda^{-1} \frac{M^2}{4\pi^2} \int_{x_0}^{\bar{N}} dx \; B_s(m_\rho^2, -ix) \; \left[-iQ_0(-ix) + iQ_2(-ix) \right] \tag{24}$$

where

$$\lambda = e^{\pi \, \mathrm{Im}\alpha} \sin\left(2 \, \mathrm{Im}\alpha \, \ln 2\bar{N}\right) / \left(2 \, \mathrm{Im}\alpha\right)$$

and for example,

$$-iQ_0(-ix) = \arctan(1/x) \; .$$

In order to test these sum rules, we saturate the discontinuity of the amplitude with the s- and u- channel πN resonances. We parametrize these resonances as δ-functions and include the nucleon and all the established πN resonances up to 2 GeV (N corresponds to 2 GeV) . The masses, widths, and inelasticities can be obtained from the most recently available experimental data.[5]

The resonances and their contribution to the sum rule are listed in Table I. Also listed are the contributions of several prominent resonances above 2 GeV to show the size of these terms in the sum rule. (Of course they are not included in the final sum.) We set $m_\rho = 775$ MeV , and $\Gamma_\rho = 140$ MeV, and obtain

$$r_-(m_\rho^2) = 12.6 \; ,$$
$$r_+(m_\rho^2) = 2.45 \; .$$

These results are not very sensitive to the exact value of $\mathrm{Im}(\alpha)$ (or equivalently Γ_ρ) ; they change by only 10% if we set $\Gamma_\rho = 90$ MeV. As in other calculations of the ρ residues the ratio of r_- to r_+ is large, $r_- / r_+ = 5.1$. As discussed in Section II, in order for most of our approximations to be valid, cot y should not be much greater than one. We are forced by the experimental data to choose N to lie below 2 GeV. Then $\cot y = 1.7$, so that the intrinsic error of the sum rule is at least $\mathrm{Im}(\alpha) \cot y \approx .1 \cot y \approx 17\%$. The determination of $r_\pm (m_\rho^2)$ from this sum rule is about three times larger than the form factor calculations quoted by Desai (Desai quotes the numbers $r_+(m_\rho^2) = 0.87$ and $r_-(m_\rho^2) = 3.89$). However, the uncertainties involved in the models used in these calculations (for example, the resonance model in our calculation) are large enough that the exact numerical comparisons are not very meaningful.

Comparison with the finite-energy sum rules leads to much more interesting conclusions. We may use the same resonance model for $A_s(t,s)$ and $B_s(t,s)$ in both sum rules. If this model is a good model, we would be able to use sum rules of different physical content to derive restrictions on the free parameters of the model. The sum rule presented here makes some use of direct-channel unitarity, and therefore, it has a chance of differing with finite-energy sum rules. It is almost obvious from Table I that our sum rule is not equivalent to the positive-moment sum rules. Our rule emphasizes the nucleon and low-mass resonances, but the principal contribution to the lowest-moment finite-energy sum rule is the N(1688). The higher-moment sum rules emphasize the higher-mass resonances even more.

There is yet a better check on the nonequivalence of the two sum rules. If the sum rules had the <u>same</u> content, then a plot of $r_{\pm}(m_{\rho}^2)$ and m_{ρ} from our sum rule should be coincident with the curve from the finite-energy sum rule. (If the resonance model is not perfect, then the curves would only approximately duplicate each other.) If the sum rules have different content, the requirement of consistency within the model places bootstrap-like restrictions on other model parameters. In this case, the only free parameter is the ρ mass. Our sum rule is not dependent enough on the exact value of $\mathrm{Im}(\alpha)$ to calculate it; the finite-energy sum rule does not depend on $\mathrm{Im}(\alpha)$ at all. Since there are four sum rules and only three parameters $\left[r_{\pm}(m_{\rho}^2) \text{ and } m_{\rho}\right]$, we also have one internal check on the model.

The ρ residues are calculated from the finite-energy sumrules using equations like

$$r_{-}(m_{\rho}^2) = \frac{M^2}{2\pi^2 \bar{N}^2} \int_{x_o}^{\bar{N}} dx\, x\, B_s(m_{\rho}^2, -ix), \qquad (25)$$

and a similar relation for $r_+(m_\rho^2)$. We plot $r_-(m_\rho^2)$ in Fig. 1 from Eqs. (24) and (25). From this plot m_ρ is seen to be about 900 MeV. Similar results are obtained for $r_+(m_\rho^2)$ with $m_\rho = 1040$ MeV. This is consistent with the results of the $r_-(m_\rho^2)$ sum rules, because the $r_+(m_\rho^2)$ finite-energy sum rule is a rapidly varying function of m_ρ at $m_\rho \approx 900$ MeV and is very sensitive to the exact parameters (or existence) of the more massive resonances. Thus, there is enough internal consistency in the model so that we conclude that our sum rule has different content from the finite-energy sum rule, and may even be used with the finite-energy sum rules to obtain restrictions on the model parameters.

Our sum rule together with the finite-energy sum rule and the resonance model of A_s' and B_s yield $r_-(m_\rho^2) = 11$, $r_+(m_\rho^2) = 2.5$, and $m_\rho = 900$ MeV. It is difficult to attach errors to these numbers, but in view of the intrinsic error of 15% in our sum rule and an incomplete model for A_s' and B_s, an error of at least 40% is reasonable.

IV. A MODEL OF THE ρ AND f_0 MESONS

We consider a model of the $\pi\pi \to \pi\pi$ amplitude in which only the ρ and f_0 resonances are included. The poles at $t = m_\rho^2$ and at $t = m_f^2$ lie on the leading Regge trajectories for isoopin zero and one, respectively. We may then write the sum rule, Eq. (12), at each of these values of t. If we also saturate the discontinuity of the cross-channel amplitudes with the ρ and f_0, we obtain a set of two equations,

$$\sin(\alpha_i' M_i \Gamma_i \ln 2N_i) = \sum_j 2X^{ij} \frac{m_j^2 \Gamma_j}{m_i q_i q_j} (2L_j+1) P_{L_j}(z_{ji}) Q_{L_i}(z_{ij}) \tag{26}$$

where i and j both correspond to the ρ or f_0. Other symbols are

$$4q_i^2 = m_i^2 - 4\mu^2 ,$$

$$z_{ij} = 1 + m_j^2/(2q_i^2) ,$$

$$N_i = 1 + s_i^2/(2q_i^2) . \tag{27}$$

Also, m_i, Γ_i, and L_i denote the mass, width, and spin, respectively of the ρ and f_0 ; X^{ij} is a 2-by2 submatrix of the isospin crossing matrix,

$$X = \begin{bmatrix} 1/3 & 1 \\ 1/3 & 1/2 \end{bmatrix} \tag{28}$$

and s_1 is a c.m. energy between 1250 MeV and 1600 MeV. We set $\alpha'_\rho = \alpha'_f = 1 \text{ (GeV)}^2$ in the following calculations.

Equation (26) is then two relations among the masses and widths of the ρ and f_0 mesons. The solutions of these equations can be studied numerically. The results are very encouraging, considering that only the grossest features of $\pi-\pi$ scattering have been included in this model. (For example, we have assumed that the imaginary part of the $I = 0$ amplitude is zero up to $t = m_f^2$.)

We search for solutions of Eq. (26) by scanning over values of the parameters $400 \text{ MeV} \leqslant m_\rho \leqslant 800 \text{ MeV}$, $800 \text{ MeV} \leqslant m_f \leqslant 1400 \text{ MeV}$, and $0.3 \leqslant (\Gamma_f/\Gamma_\rho) \leqslant 3$. Due to the factor sin y, the equation we are using is somewhat nonlinear. Although this nonlinearity serves to exclude some of the solutions, it cannot give an absolute scale for the widths. In one set of solutions we find that m_ρ increases from 500 to 700 MeV as (Γ_ρ/Γ_f) increases from 0.3 to 1.5 and as m_f decreases from 1200 to 900 MeV. To complete the bootstrap, we use the finite-energy sum rules which pick out the solution: $m_\rho = 540$ MeV; $m_f = 1150$ MeV; and $(\Gamma_f/\Gamma_\rho) = 3$.

Another disjoint set of solutions contain the physical masses of the ρ and the f meson but give a width of about 350 MeV for the f_0. We find that we can bring the solutions of Eq. (25) into agreement with experiment by either of two mechanisms. We can fit the experimental parameters by inserting a scalar meson. The scalar meson is broad with a mass of about 500 MeV. Also, simple models for the threshold bring the results into good accord with experiment.

V. THE BEHAVIOR OF THE SUM RULE NEAR THRESHOLD

In this section we consider the amplitude for the elastic scattering of spinless particles and study the behavior of the sum rule near the elastic threshold ($t \approx 4m^2$). Since we will be concerned with the $t \longrightarrow 4m^2$ limit of Regge trajectories and residues, we will work with the reduced quantities $b(J,E)$ and $\gamma(E)$ defined by,

$$a(J,E) \equiv E^J \, b(J,E)$$
$$\beta(E) \equiv E^J \, \gamma(E) \tag{29}$$

where,

$$E \equiv q^2 = t/4 - m^2 \tag{30}$$

The discontinuity equation around the elastic threshold can be written as,

$$b(J,E) - b(J,E_2) = 2i\rho \, (E/E_o)^J \, b(J,E) \, b(J,E_2) \tag{31}$$

In the following, we will not display the scale factor E_o explicitly, but whenever we speak of the small magnitude of E, it will be understood that the scale is set by the factor E_o. Eq. (31) can of course be continued to the entire E-plane. If $b(J,E)$ has a pole at $J = \alpha(t)$, then $b(J,E_2)$ will have a pole at $J = \alpha_2(t)$, and Eq. (31) will imply,

$$-1/(\, 2i\rho \, E^{\alpha_2} \,) = b(\alpha_2, E) \tag{32}$$

For the sake of simplicity, we will again restrict ourselves to positive E; our results will also hold for small negative values of E. Thus $E_2 = E^*$, and $\alpha_2(E) = \alpha^*(E)$. Following the same steps as in the deivation of section II, for any given positive value of E we obtain the sum rule,

$$-\frac{\sqrt{t}}{2E^{\mathrm{Re}\,\alpha+1/2}} \sin\left(\mathrm{Im}\,\alpha\,\ln E\right) = \mathrm{Re}\, I_1 + \mathrm{Re}\, I_2 + \mathrm{Re}\, I_3 \tag{33}$$

with,

$$I_1 = \frac{E^{-\alpha^*}}{\pi} \int_{S_0}^{S_1} A_s(t,s)\, Q_{\alpha^*}(1+s/2E)\, ds/2E \tag{34}$$

$$I_2 = \frac{E^{-\alpha^*}}{\pi} \lim_{J\to\alpha^*} \int_{S_1}^{\infty} R_s(t,s)\, Q_J(1+s/2E)\, ds/2E \tag{35}$$

$$\mathrm{Re}\, I_2 = -\frac{\mathrm{Re}\,\gamma(E)\,\sin y'}{2\,\mathrm{Im}\,\alpha}\left[1 + \frac{\mathrm{Im}\,\gamma(E)}{\mathrm{Re}\,\gamma(E)}\cot y' + 2c_2\,\mathrm{Im}(\alpha)\cot y' + O\left[\mathrm{Im}(\alpha)^2\right]\right] + O\left(\frac{1}{N}\right) \tag{36}$$

$$y' = 2\,\mathrm{Im}(\alpha)\,\ln s_1$$

and

$$I_3 = \frac{E^{-\alpha^*}}{\pi} \int_{S_1}^{\infty} \bar{R}_s(t,s)\, Q_{\alpha^*}(1+s/2E)\, ds/2E \tag{37}$$

When E is not very close to zero, so that $\mathrm{Im}(\alpha)$ is not very small, a moderately large s_1 leads to a value of cot y' of order one. Since $\mathrm{Im}\,\gamma(E)/\,\mathrm{Re}\,\gamma(E)$ is of order $\mathrm{Im}(\alpha)$, we may neglect the terms inside the bracket compared to one and obtain the approximate sum rule we have used in the previous sections. However, as $E \rightarrow 0$ (as we approach the threshold) $\mathrm{Im}(\alpha)$ necessarily goes to zero and in order to keep cot y' near one, we would have to increase s_1 to extremely large values. On the other hand, if we wish to restrict ourselves to the intermediate values of s_1 and still use Eq. (33) to calculate the real part of the residue function, we would have to estimate the magnitude of the quantity $\mathrm{Im}\,\gamma(E)\,/\,\mathrm{Re}\,\gamma(E)$ for E near zero . For s_1 finite and $\mathrm{Im}(\alpha) \approx 0$, the sum rule becomes,

$$\frac{\sqrt{t}}{2E^{\mathrm{Re}\,\alpha + 1/2}}\,\mathrm{Im}(\alpha)\,\ln E = \mathrm{Re}\,\gamma(E)\,\left(\ln s_1 + c_2\right)$$

$$+ \frac{\mathrm{Im}\,\gamma(E)}{2\,\mathrm{Im}(\alpha)} + O\left[\mathrm{Im}(\alpha)^2\right] + O(1/N) - \mathrm{Re}\,I_1 - \mathrm{Re}\,I_3 \qquad (38)$$

In order to study the behavior of Eq.(38) as $E \rightarrow 0$, we need the behavior of $\alpha(E)$ and $\gamma(E)$ near threshold. This behavior is derived from the discontinuity equation for $b^{-1}(J,E)$ [6],

$$b^{-1}(J,E) - b^{-1}(J,E_2) = -2i\,(E)^{J+1/2} \,/\, \sqrt{t} \qquad (39)$$

The quantity $Y(J,E)$ defined as,

$$Y(J,E) = \sqrt{t}\,\cos(\pi J)\; b^{-1}(J,E) - (-E)^{J+1/2} \qquad (40)$$

can be easily shown to be free of the cut at the elastic threshold ($E=0$) . A Regge pole occurs when,

$$Y(J,E) + (-E)^{J+1/2} = 0 \qquad (41)$$

The residue of the pole, $\gamma(E)$, is $\sqrt{t}\,\cos\pi\alpha$ times the coefficient of $(J-\alpha)$ in the expansion of the left hand side of Eq. (41) around the point $J=\alpha$,

$$\gamma(E) = \frac{\sqrt{t}\,\cos\pi\alpha(E)}{\frac{\partial Y}{\partial J}\,(\alpha(E)\,,\,E) + \ln(-E)\,(-E)^{\alpha(E)+1/2}} \qquad (42)$$

Eq. (41) can also be expanded around the point ($\alpha(0)$, 0) to give the threshold behavior of $\alpha(E)$. Actually, only an expansion in the variable J is needed. It will be much simpler, although not absolutely necessary, for our purposes (we are only interested in the behavior of $\mathrm{Im}\,\alpha$) to expand around the point (α_0,E) defined in the following way: From Eq. (41) it is clear that $Y(\alpha(0)\,,\,0)=0$. If this zero is simple [a simple pole of $b(J,E)$] , then for small enough E , there exists a real point α_0 near to $\alpha(0)$ such that $Y(\alpha_0, E) = 0$. We will expand around this point and write,

$$Y(\alpha(E), E) = Y_J(\alpha(E) - \alpha_0) + Y_J'(\alpha(E) - \alpha_0)^2 + \cdots\cdots \tag{43}$$

Note that α_0 and the coefficients of the expansion are all real functions of E which go to their threshold value smoothly as $E \to 0$. (We can make them as close to their $E = 0$ value as we wish by making E small) . The function $\alpha(E)$ near $E = 0$ is thus the solution of the following equation:

$$\alpha(E) = \alpha_0 - Y_J^{-1} (-E)^{\alpha(E)+1/2} + Y_J^{-1} Y_J' (\alpha(E) - \alpha_0)^2 + \cdots\cdots\cdots \tag{44}$$

In the following, we will use as our expansion parameter the quantity ζ defined as,

$$\zeta = Y_J^{-1} \cos\pi\alpha_0 (E)^{\alpha_0 + 1/2} \tag{45}$$

Therefore,

$$\alpha(E) = \alpha_0 + i\,\zeta\, e^{-i\pi\alpha_0} / \cos\pi\alpha_0 + O(\ln E\, \zeta^2)$$

$$\mathrm{Im}\,\alpha(E) = \zeta + O(\ln E\, \zeta^2) \tag{46}$$

Substituting the expansion for $\alpha(E)$ and $\frac{\partial Y}{\partial J}$ into Eq. (42), we get the following expansion of $\gamma(E)$ in terms of ζ ,

$$\gamma(E) = \sqrt{t}\, Y_J^{-1} \cos\pi\alpha_0 \left[1 + \frac{e^{-i\pi\alpha_0}}{\cos\pi\alpha_0} \zeta \left[\pi - i\pi\tan\pi\alpha_0 - 2i\, Y_J^{-1} Y_J' + i\, \ln E \right] + O(\ln E\, \zeta^2) \right] \tag{47}$$

Then, the quantity $\mathrm{Im}\,\gamma(E)/2\mathrm{Im}(\alpha)$ occuring in Eq. (38) is given by ,

$$\frac{\mathrm{Im}\,\gamma(E)}{2\mathrm{Im}(\alpha)} = -\sqrt{t}\, Y_J^{-1} \cos\pi\alpha_0 \left[\pi\tan\pi\alpha_0 + \pi Y_J^{-1} Y_J' - \frac{\ln E}{2} \right] + O(\zeta) \tag{48}$$

Substituting this in Eq. (38) and using the fact that $\sqrt{t}\, Y_J^{-1} \cos\pi\alpha_0 = \mathrm{Re}\,\gamma(E) + O(\zeta)$, we obtain,

$$0 = \mathrm{Re}\,\gamma(E) \left[\ln s_1 + c_2 - \pi\tan\pi\alpha_0 - \frac{\pi Y_J'\, \mathrm{Re}\,\gamma(E)}{\sqrt{t}\, \cos\pi\alpha_0} \right] - \mathrm{Re}I_1 - \mathrm{Re}I_3 + O(\zeta) + O(1/N) \tag{49}$$

Thus in the limit of very small $\mathrm{Im}(\alpha)$ but finite s_1 an unknown quantity , Y_J' , has appeared in our sum rule which no longer can be neglected. This quantity is actually closely related to the background term in b(J,E). For J near α we can write b(J,E) as,

$$b(J,E) = \frac{\gamma}{J-\alpha} \left[1 + h\, (J - \alpha) + \ldots \right] \tag{50}$$

Substituting this in the definition of Y(J,E) , Eq. (40) , and differentiating twice, we find,

$$\mathrm{Re}\,\gamma(E)\,Y_J' = -\pi\sqrt{t}\,\sin\pi\alpha_0 - \sqrt{t}\,\cos\pi\alpha_0\,h + O(\zeta) \quad (51)$$

Therefore, the sum rule has reduced to,

$$0 = \mathrm{Re}\,\gamma(E)\left[\ell n s_1 + c_2 + h\right] - \mathrm{Re}\,I_1 - \mathrm{Re}\,I_3 + O(\zeta) + O(1/N) \quad (52)$$

It is now possible to see explicitly from Eq. (52) that in this limit of $\mathrm{Im}(\alpha) \approx 0$ and s_1 finite, the sum rule has become an empty statement, since Eq. (52) is merely the definition of the background term h in terms of the Froissart-Gribov definition of the partial wave amplitude, and could have been immediately written down without any use of two body unitarity. In short, the resultsof this section indicate that at least for this sum rule, when the limit $\mathrm{Im}\alpha \to 0$ is taken, all the content of unitarity (and thus any non-linearity in the equation) is lost, and one is left with a trivial definition of a background term.

We will close this section with some rather vague remarks about the background. If we still wish to use Eq. (52) to calculate $\gamma(E)$, or if we wish to estimate the first correction to the approximate sum rule of the previous sections, we should calculate the quantity h. One possible model to use for this calculation is an N/D model. It is easy to see that if we write $b(J,E) = N(J,E)/D(J,E)$, and expand this as in Eq. (50) , the quantity h is given by,

$$h = N'(\alpha, E) \,/\, N(\alpha, E) \tag{53}$$

where $N'(\alpha, E)$ is the derivative of $N(J,E)$ with respect to J evaluated at $J = \alpha$. It seems plausible that even though an N/D model may not lead to the correct value of the position and the residue of the pole (which would involve the accurate calculation of both N and D) , a few iterations of the integral equation for N may give a good estimate of the background term h . In the case of the ρ residue in the $\pi\pi \rightarrow N\bar{N}$ reaction, the quantity Re I_1 for $E \approx 0$ is very large, because the nucleon pole gives a very large contribution. Thus if we neglect h and use the sum rule to calculate $r_{\pm}(E \approx 0)$ we obtain a result about ten times larger than what we expect from Regge fits for $t \leqslant 0$. This is because the left hand cut of $b(J,E)$ is very close to $E = 0$, a fact that is indicated by the large contribution of the nucleon pole. The crudest estimate of the function $N(J,E)$, but certainly not a numerically reliable one, is the sum of the Born terms, namely the quantity I_1 . In fact $(\partial I_1 / \partial J) / I_1$ has the correct sign and the correct order of magnitude to bring the value of $r_{\pm}$ closer to the expected value. These results are of course only qualitative , since an accurate calculation would involve a few iterations of the integral equation for N.

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FOOTNOTES AND REFERENCES

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Table I. Contribution of each resonance to our sum rules, Eqs. (23)and (24) , and to the finite-energy sum rule, Eq. (25) . The numerical values of the masses, widths, and inelasticities are found in Ref. 5 . A value of $g^2 = 14$ was used for the πN coupling.

Resonance	$L_{2I,2J}$ Identification	Eqs. (23)and (24)		Finite-Energy Sum Rule	
		r_+	r_-	r_+	r_-
Nucleon		2.70	8.53	.05	.62
N(1470)	P_{11}	.09	.30	.09	.43
N(1518)	D_{13}	.21	.72	.28	1.27
N(1550)	S_{11}	.01	.003	.01	.01
N(1680)	D_{15}	.10	-.19	.29	-.55
N(1688)	F_{15}	.44	1.54	1.31	4.59
N(1710)	S_{11}	.02	.01	.08	.04
Δ(1236)	P_{33}	-.99	1.42	-.25	.76
Δ(1640)	S_{31}	-.01	-.003	-.02	-.01
Δ(1920)	F_{37}	-.12	.25	-.97	1.50
Total		2.45	12.6	.87	8.66
N(2190)	G_{17}	.11	.36	2.02	3.79
Δ(2420)	$H_{3,11}$	-.05	.11	-1.66	1.79

FIGURE CAPTION

Fig. 1. The ρ residue, $r_-(m_\rho^2)$, is plotted as a function of m_ρ. Curve (a) results from our sum rule, Eq. (24). The finite-energy sum rule, Eq. (25), produces curve (b). In both calculations we have used the resonances listed in Table I; the numerical values of the parameters are listed in Ref. 5.

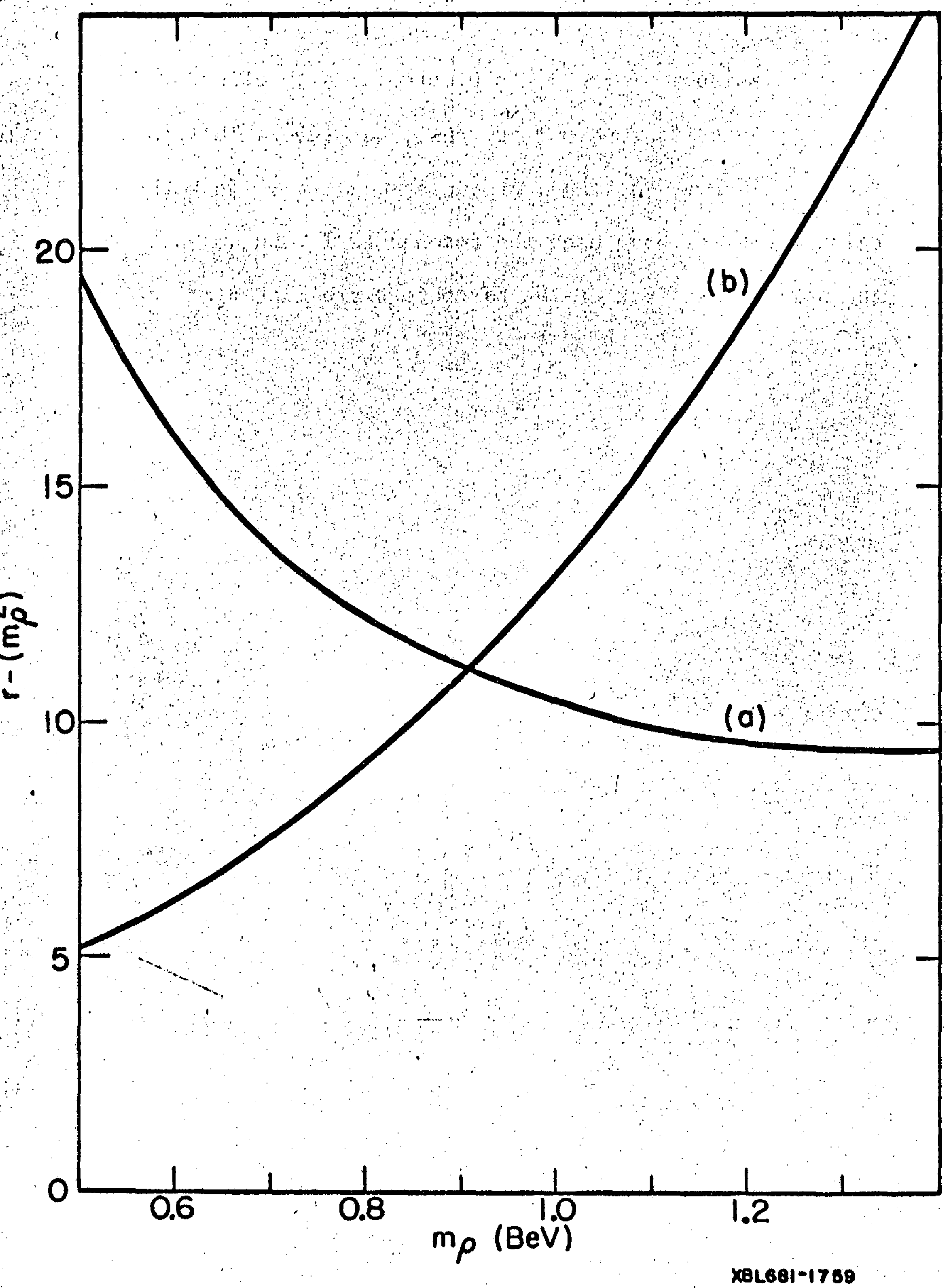

Fig. 1