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AN AVERAGING THEOREM FOR TIME-PERIODIC DEGREE ZERO HOMOGENEOUS DIFFERENTIAL EQUATIONS

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Abstract. This paper considers the stability of the differential equation $\dot{x} = \epsilon X(t, x, \epsilon)$, $x \in \mathbb{R}^n$, where $X(t, x, \epsilon)$ is a time-periodic, degree zero homogeneous vector field and $\epsilon > 0$ is a parameter. It is shown that asymptotic stability of the time-averaged equation implies asymptotic stability of the original system for ϵ sufficiently small.

Keywords: homogeneous, time-periodic, averaging, dilation, non-Lipschitz

1. Introduction

This note proves a stability result for time-periodic degree zero homogeneous differential equations. In particular, if the averaged equations are asymptotically stable then the original equations are asymptotically stable provided the vector field scaling factor is sufficiently small. A preliminary version of Theorem 3.1 appeared in [14]. Homogeneous differential equations play a prominent role in several aspects of nonlinear control theory. The work of Hermes [5, 6, 7] and Kawski [8, 9] developed the theory of homogeneous systems in the context of feedback control problems. The idea in these references is to utilize approximating homogeneous systems of the control system and then restrict the feedback functions to be homogeneous as well. Choosing the feedback to be homogeneous imparts desirable properties to the closed-loop system. For example, solutions will converge exponentially to the equilibrium point when the system is uniformly asymptotically stable and degree zero homogeneous. However, in this case, the closed-loop system is not Lipschitz continuous at the equilibrium point when the dilation is non-standard.

This property complicates the stability analysis of the system since the Jacobian linearization is not defined.

This averaging result was motivated by the need to analyze the stability of time-periodic homogeneous equations. In an effort to improve the closed-loop convergence rate of stabilizing control laws for driftless (or nonholonomic) systems, M'Closkey and Murray applied the notion of homogeneity to a class of driftless systems in [13]. However, stabilizing control laws for driftless systems that are continuous functions of the state are necessarily time-varying (see [2]) so ascertaining stability of the homogeneous closed-loop system usually involves the analysis of a time-varying non-Lipschitz differential equation. Application of this averaging theorem can eliminate the need for analyzing a time-varying system which considerably simplifies the analysis.

The averaging result presented here has directly facilitated the stability analysis of an under-actuated surface vessel and a spacecraft with two controls (see [18] and [19]). Additional background on physical systems with driftless models may be found in [11, 16, 17]. These systems include kinematic models of mobile robots, spacecraft attitude dynamics and underwater vehicles. Interested readers are directed to consult [15] and references therein for a comprehensive introduction to the application of homogeneous approximations and feedback to driftless systems.

2. Preliminaries

This section establishes some notation and definitions. A general reference for dilation groups is Goodman's book [3]. In the context of feedback stabilization, references [8, 9] should be consulted for some original definitions.

Definition 1. A *dilation* $\Delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is defined with respect to a fixed choice of coordinates $x = (x_1, x_2, \dots, x_n)$ on $\mathbb{R}^n$ by assigning n positive rationals $r = (r_1 = 1 \le r_2 \le \cdots \le r_n)$ and positive real parameter $\lambda > 0$ such that

$$\Delta_\lambda x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n), \quad \lambda > 0.$$

When the differential equations are homogeneous it is convenient to use a homogeneous norm as a measure of the state size.

Definition 2. A continuous map from $\mathbb{R}^n$ to $\mathbb{R}$, $x \mapsto \rho(x)$, is called a *homogeneous norm* with respect to the dilation Δ_λ when

1. $\rho(x) \geq 0, \quad \rho(x) = 0 \iff x = 0,$
2. $\rho(\Delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda > 0.$

For example, a homogeneous norm which is smooth on $\mathbb{R}^n \setminus \{0\}$ may always be defined as

$$\rho(x) = |x_1^{c/r_1} + x_2^{c/r_2} + \cdots + x_n^{c/r_n}|^{1/c},$$

where c is some positive integer evenly divisible by r_i, $i = 1, \ldots, n$.

Definition 3. The *α-sphere* is defined as the set

$$S_\alpha = \{x \in \mathbb{R}^n | \rho(x) = \alpha\},$$

where ρ is a homogeneous norm corresponding to the dilation Δ_λ.

Definition 4. A *degree zero homogeneous* vector field $X(t, x)$ is invariant with respect to the dilation

$$(\Delta_\lambda)_* X(t, x) = X(t, \Delta_\lambda x) \quad \forall \lambda > 0.$$

Let $\phi(t, t_0, x_0)$ represent the solution of X through x_0 at time t_0. It is straight forward to verify that the degree zero property implies solutions scale to solutions under the dilation: $\Delta_\lambda \phi(t, t_0, x_0) = \phi(t, t_0, \Delta_\lambda x_0)$.

3. Averaging Results

Consider the differential equation

$$\dot{x} = \epsilon X(t, x, \epsilon), \tag{1}$$

where $x \in \mathbb{R}^n$, $\epsilon \geq 0$ is a real parameter, X is a continuous map from $\mathbb{R} \times \mathbb{R}^n \times [0, \infty)$ into $\mathbb{R}^n$, T-periodic with respect to t, $X(t, 0, \epsilon) \equiv 0$, and $X(\cdot, x, \cdot)$ is degree zero homogeneous with respect to the dilation Δ_λ. Time is rescaled so that the period is always 2π. The interval $[0, 2\pi)$ is denoted by I.

In the averaging theorem presented below, stability of the zero solution of equation (1) is inferred from stability of the zero solution of the *averaged* system,

$$\dot{x} = \epsilon X_0(x), \tag{2}$$

where

$$X_0(x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T X(t, x, 0) dt. \tag{3}$$

The vector field in (1) is 2π-periodic in t so the average in (3) is equivalent to

$$X_0(x) = \frac{1}{2\pi}\int_0^{2\pi} X(t,x,0)dt.$$

Note that X_0 is degree zero homogeneous with respect the Δ_λ. The main result is now stated.

Theorem 3.1. *Suppose $y = 0$ is an asymptotically stable fixed point of the associated averaged system $\dot{y} = \epsilon X_0(y)$. Then for $\epsilon > 0$ sufficiently small, the solution $x = 0$ is asymptotically stable in the time-periodic equations* (1)*.*

This result is already well known for C^1 vector fields where $x = 0$ is a hyperbolic fixed point. A brief review of the classical averaging results will be helpful in understanding the different approach taken in the proof of Theorem 3.1. Suppose X satisfies the conditions stated above. For any compact set Ω in $\mathbb{R}^n$ there exists an ϵ_0 and a function $u(t,y,\epsilon)$ such that the change of coordinates,

$$x = y + \epsilon u(t,y,\epsilon) \quad (t,y,\epsilon) \in \mathbb{R}\times\Omega\times[0,\epsilon_0), \tag{4}$$

applied to (1) yields the equation

$$\dot{y} = \epsilon X_0(y) + \epsilon F(t,y,\epsilon), \tag{5}$$

where X_0 is the averaged vector field (2). $F(t,y,\epsilon)$ is continuous for $(t,y,\epsilon) \in \mathbb{R}\times\Omega\times[0,\epsilon_0)$ and $F(t,y,0) = 0$. The function u possesses the following properties on $\mathbb{R}\times\Omega\times[0,\epsilon_0)$:

1. $u(t,y,\epsilon)$ is periodic with period 2π (same period as the vector field),
2. $u(t,y,\epsilon)$ has continuous derivatives with respect to t and derivatives of an arbitrary specified order with respect to y.
3. $\epsilon u(t,y,\epsilon)$ and $\epsilon\partial u(t,y,\epsilon)/\partial y$ approach 0 as $\epsilon \to 0$ uniformly with respect to $t \in \mathbb{R}$ and $y \in \Omega$.

The details of these constructions may be found in Hale [4] (Lemma V3.1, Lemma V3.2 and Lemma 5 of the appendix).

Under the conditions that X is C^1 in x, F may be bounded by an arbitrarily small Lipschitz constant in y. In this case, if X_0 possesses a hyperbolic fixed point, y_0, then it is not difficult to show that X has a unique periodic solution in a neighborhood of y_0 with the same stability type as y_0.

Now if X is degree zero homogeneous but not necessarily C^1 these arguments must be modified. The transformation of variables given by (4) remains valid and the equations in the new coordinates still have the form given by (5). However converse Lyapunov functions are of no direct use in the stability analysis of (5) when X_0 has an asymptotically stable fixed point at the origin since F is not exclusively composed of degree one or higher order homogeneous terms.

Proof of Theorem 3.1. Without loss of generality, take Ω to be the unit ball in $\mathbb{R}^n$ and define the coordinate change according to (4). By hypothesis the solution $y = 0$ of (2) is asymptotically stable so there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with the following properties [10],

1. $V(y)$ is C^∞,
2. $V(0) = 0$, $V(y) > 0$ for all $y \neq 0$, and V is radially unbounded,
3. $\nabla V(y) \cdot X_0(y) < 0$ for all $y \neq 0$.

Define c_1 such that

$$D_{c_1} := \{y \in \mathbb{R}^n | V(y) \leq c_1\} \subset \Omega.$$

Compute c_2 such that

$$\sup_{\{y|V(y)=c_2\}} \rho(y) = \frac{1}{2} \inf_{\{y|V(y)=c_1\}} \rho(y), \tag{6}$$

and define

$$D_{c_2} := \{y | V(y) \leq c_2\}.$$

The Lyapunov function V is positive definite and continuous so c_1 and c_2 satisfying these conditions always exist. Denote the supremum on the left side of (6) as d_2 and the infimum on the right of (6) as d_1 (so $d_2 = \frac{1}{2} d_1$).

Evaluating V along solutions of the transformed vector field (5) yields

$$\frac{dV}{dt} = \epsilon \nabla V(y) \cdot X_0(y) + \epsilon \nabla V(y) \cdot F(t, y, \epsilon).$$

On the set $D_{c_1} \setminus D_{c_2}$ calculate

$$\beta := \sup_{y \in D_{c_1} \setminus D_{c_2}} \nabla V(y) \cdot X_0(y),$$

which is clearly less than zero because $D_{c_1} \setminus D_{c_2}$ is compact and excludes the origin. We also define $M(\epsilon)$ as

$$M(\epsilon) = \sup_{y \in D_{c_1}} \sup_{t \in I} \nabla V(y) \cdot F(t, y, \epsilon).$$

$M(\epsilon)$ is continuous because F is a continuous function. Furthermore, $M(0) = 0$ since $F(\cdot,\cdot,0) \equiv 0$. The coordinate change in (4) will not, in general, respect the dilation scaling. Hence the vector field $F(t,x,\epsilon)$ will not be homogeneous with respect to y. For example we may be forced to bound F with homogeneous functions of lower order than X_0 and asymptotic stability cannot be concluded with a standard Lyapunov analysis.

On $D_{c_1} \setminus D_{c_2}$ the time derivative of V is bounded by

$$\frac{dV}{dt} \leq \epsilon(\beta + M(\epsilon)).$$

Now choose $\epsilon_1 \in (0, \epsilon_0]$ such that $|M(\epsilon_1)| \leq -\frac{\beta}{2}$. The choice of ϵ_1 renders D_{c_1} and D_{c_2} invariant. Trajectories starting at points in D_{c_1} will reach D_{c_2} in a finite time no greater than

$$\tau = \frac{2(c_2 - c_1)}{\epsilon_1 \beta},$$

because $\dot{V} < \epsilon\beta/2$ on $D_{c_1} \setminus D_{c_2}$. Choosing any $\epsilon \in (0, \epsilon_1]$ does not change the invariance or attractive nature of the sets. The only modification in this case is τ. The functional relationship of τ is exactly the one given above with ϵ_1 replaced by the new ϵ. Nothing more can be stated about the stability of the system in the y-coordinates.

The map $x = y + \epsilon u(t, y, \epsilon)$ is at least a C^1 diffeomorphism for $(t, y, \epsilon) \in I \times \Omega \times\ [0, \epsilon_0)$. As $\epsilon \to 0$ this map approaches the identity on Ω. Define the following sets in x-coordinates

$$D_1(\epsilon) := \bigcap_{t \in I} \{x = y + \epsilon u(t, y, \epsilon) | y \in D_{c_1}\}$$
$$D_2(\epsilon) := \bigcup_{t \in I} \{x = y + \epsilon u(t, y, \epsilon) | y \in D_{c_2}\}.$$

We may approximate D_{c_1} as closely as desired with D_1 since $\epsilon u(t, y, \epsilon) \to 0$ as $\epsilon \to 0$ uniformly with respect to t, $y \in \Omega$. Similarly, D_{c_2} may be approximated as closely as required with D_2 by suitable choice of ϵ. Now choose $\epsilon_2 \in (0, \epsilon_1]$ such that

1. $0 \in D_1(\epsilon_2)$ and $0 \in D_2(\epsilon_2)$,
2.

$$w_1 := \inf_{x \in \mathbb{R}^n \setminus D_1(\epsilon_2)} \rho(x) > w_2 := \sup_{x \in D_2(\epsilon_2)} \rho(x).$$

It is possible to find ϵ_2 satisfying these requirements since $0 \in D_{c_1}$ and $0 \in D_{c_2}$ and, at $\epsilon = 0$, $w_1 = d_1$ and $w_2 = d_2$.

Solutions of (1) starting at starting at points x_0 on the sphere S_{w_1} must reach the smaller homogeneous sphere S_{w_2} in time $\tau = 2(c_2 - c_1)/\epsilon_2\beta$ since x_0 maps to a point in D_{c_1} under (4) and all points y in the set D_{c_2} map into the homogeneous ball of radius w_2 under (4). This relationship may be summarized as

$$\rho(\phi(t, t_0, x_0)) \leq w_2 \quad \forall t_0 \in I,\, t \geq t_0 + \tau,\, \rho(x_0) = w_1. \tag{7}$$

To complete the proof, note that the vector field X may be written as a quotient system evolving on the homogeneous sphere and an auxiliary equation for ρ since X is invariant with respect to the one-parameter dilation group (see [1]). Let the components of X be defined as $a_i(t, x, \epsilon)$, $i = 1, \dots, n$. If we define $r = \rho(x)$ and $z \in S_1$ such that $z_i = x_i/\rho^{r_i}(x)$ then a straightforward but tedious computation yields

$$\dot{r}(t) = \underbrace{\left(\sum_{k=1}^{n} \frac{1}{r_k} y_k^{c/r_k - 1} a_k(t, z, \epsilon_2) \right)}_{=:g(t)} r(t), \tag{8}$$

and,

$$\dot{z}_i(t) = a_i(t, z, \epsilon_2) - z_i \sum_{k=1}^{n} \frac{r_i}{r_k} z_k^{c/r_k - 1} a_k(t, z, \epsilon_2), \quad i = 1, \dots, n.$$

Equation (7) implies that if $r(t_0) = w_1$ then $r(t) \leq w_2$ for all $t \geq t_0 + \tau$. Furthermore, using (8)

$$r(t) = \exp\left(\int_{t_0}^{t} g(s) ds \right) r(t_0),$$

so

$$\exp\left(\int_{t_0}^{t} g(s) ds \right) w_1 \leq w_2 \quad \forall t \geq t_0 + \tau.$$

This bound implies that $r(t)$ converges to zero at an exponential rate for an arbitrary initial condition x_0 since we may express $r(t)$ as

$$\begin{aligned} r(t) = \exp &\left(\int_{t_0}^{t_0+\tau} g(s) ds \right) \exp\left(\int_{t_0+\tau}^{t_0+2\tau} g(s) ds \right) \cdots \\ &\cdots \exp\left(\int_{t_0+(m-1)\tau}^{t_0+m\tau} g(s) ds \right) \exp(\left(\int_{t_0+m\tau}^{t} g(s) ds \right) \rho(x_0) \\ \leq &\left(\frac{w_2}{w_1} \right)^m e^{\gamma\tau} \rho(x_0), \end{aligned}$$

for some $m \in [0, 1, 2, \dots]$ such that $t - m\tau \in [0, \tau)$ and

$$\gamma = \sup_{t \in I} \sup_{z \in S_1} \sum_{k=1}^{n} \frac{1}{r_k} z_k^{c/r_k - 1} a_k(t, z, \epsilon_2).$$

As $t \to \infty$ then $m \to \infty$ and so $r(t) \to 0$ since $w_2/w_1 < 1$. Thus the zero solution of (1) is asymptotically stable. □

4. Examples

For *illustrative purposes only*, we solve a small problem. The problem has no physical meaning. Readers are referred to [18] and [19] for actual applications.

Example 4.1. Consider the following ordinary differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{3}x_1 + \frac{x_2}{\rho(x)} \cos^2 t \\ -\frac{1}{2}\frac{x_1^2 x_2}{\rho^2(x)} + x_1^2 \cos t + x_2 \sin t \end{pmatrix} \quad \epsilon > 0, \tag{9}$$

where $\rho(x) = (x_1^4 + x_2^2)^{1/4}$. This system is homogeneous with respect the dilation $\Delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2)$, smooth on $\mathbb{R}^n \setminus \{0\}$, 2π-periodic with respect to t and not Lipschitz in any neighborhood of the origin. The averaged system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{3}x_1 + \frac{1}{2}\frac{x_2}{\rho(x)} \\ -\frac{1}{2}\frac{x_1^2 x_2}{\rho^2(x)} \end{pmatrix}. \tag{10}$$

A positive definite function, and its derivative along solutions of the averaged system, is

$$V = x_1^4 + x_2^2$$

$$\frac{dV}{dt} = -\epsilon x_1^2 \left(\frac{4}{3}x_1^2 - 2x_1 \frac{x_2}{\rho(x)} + \left(\frac{x_2}{\rho(x)} \right)^2 \right).$$

A simple calculation shows that the bracketed term in dV/dt is positive for all $x \in \mathbb{R}^2 \setminus \{0\}$. Thus dV/dt is negative semidefinite and the set where $dV/dt = 0$ is

$$W := \{x \in \mathbb{R}^2 | x_1 = 0\}.$$

However note that the averaged vector field (10) is transversal to W so the origin is asymptotically stable by LaSalle's invariance principle. Thus, by Theorem 3.1, we conclude that the original system of equations in (9) is asymptotically stable for ϵ sufficiently small.

Example 4.2. The second example demonstrates that the theorem is only a sufficient condition for asymptotic stability. Consider the degree zero homogeneous system

$$\begin{aligned} \dot{x}_1 &= \epsilon(-x_1 + \frac{x_3}{\rho(x)}\cos t) \\ \dot{x}_2 &= \epsilon(-x_2 + \frac{x_3^2}{\rho^3(x)}\sin t) \\ \dot{x}_3 &= \epsilon x_2(-x_1 + \frac{x_3}{\rho(x)}\cos t) \\ \rho(x) &= (x_1^4 + x_2^4 + x_3^2)^{1/4}. \end{aligned} \tag{11}$$

The dilation scalings are $r = (1, 1, 2)$. This example is the two-input, three dimensional "chained" system from [16] with homogeneous feedback. A detailed analysis proving asymptotic stability of (11) may be found in [12]. The averaged system is easily seen to be stable (but not asymptotically stable),

$$\begin{aligned} \dot{x}_1 &= -\epsilon x_1 \\ \dot{x}_2 &= -\epsilon x_2 \\ \dot{x}_3 &= -\epsilon x_1 x_2. \end{aligned}$$

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