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UNIVERSITY OF CALIFORNIA SAN DIEGO

**Invariant Theory of Preprojective Algebras**

A dissertation submitted in partial satisfaction of the  
requirements for the degree  
Doctor of Philosophy

in

Mathematics

by

Stephan Weispfenning

Committee in charge:

Professor Daniel S. Rogalski, Chair  
Professor Kenneth A. Intriligator  
Professor Alireza Salehi Golsefidy  
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2018

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The dissertation of Stephan Weispfenning is approved,  
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Chair

University of California San Diego

2018

## DEDICATION

To my wife Angelika.

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Chapters 1, 2, 4 and 5 contain material from "Properties of the fixed ring of a preprojective algebra", Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of this paper.

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ABSTRACT OF THE DISSERTATION

**Invariant Theory of Preprojective Algebras**

by

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Doctor of Philosophy in Mathematics

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Studying invariant theory of commutative polynomial rings has motivated many developments in commutative algebra and algebraic geometry. For a finite group acting on a polynomial ring, the remarkable Chevalley-Shephard-Todd Theorem proves that the fixed subring is isomorphic to a polynomial ring if and only if the group is generated by pseudo-reflections. Another interesting question is to find properties of a fixed ring for a group action satisfying certain attributes. In recent years, progress was made in work of

Jing, Jørgensen, Kirkman, Kuzmanovich, Walton, Zhang, and others to extend the theory to regular algebras which are a noncommutative generalization of polynomial rings.

Naturally, the question arises if the theory generalizes further to non-connected noncommutative algebras. Our objects of study will be preprojective algebras which are certain factor algebras of path algebras corresponding to extended Dynkin diagrams of type  $A$ ,  $D$  or  $E$ . This dissertation answers the question what conditions need to be satisfied by the fixed ring in order to make a rich theory possible. Moreover, we give a sufficient condition on the finite group acting on a preprojective algebra to guarantee that the fixed ring has finite injective dimension and satisfies a generalized Gorenstein condition. Part of this result is the construction of a homological determinant of a non-connected algebra which turns out to be particularly nice for the examined preprojective algebras.

# Chapter 1

## Introduction

Throughout this dissertation, fix an algebraically closed field  $k$  of characteristic zero. All vector spaces and algebras are taken over  $k$ . Every algebra and every module is graded and locally finite.

### 1.1 Overview

Commutative invariant theory mainly studies (finite) linear group actions on polynomial rings, their fixed points, and related objects. Starting to develop in the mid-19th century, by now crucial connections to commutative algebra, Galois theory, representation theory, algebraic topology, combinatorics and other fields have been well-established and been used to solve open questions. Its undeniable beauty encouraged many famous mathematicians such as Boole, Caley, Chevalley, Coxeter, Dickson, Gauß, Gordan, Hilbert, Klein, Molien, Nagata, Noether, Salmon, Serre, Shephard, Sylvester, Todd, and many

more to contribute to this field which put commutative invariant theory at the forefront of mathematics for the second half of the 19th century. At that time, a central question was if there is always a finite set of fundamental invariants for arbitrary groups. While Gordan and Hilbert proved this statement for  $\mathrm{SL}_2(\mathbb{C})$  and  $\mathrm{GL}_n(\mathbb{C})$ , respectively, acting on a polynomial ring over  $\mathbb{C}$ , Nagata gave a counterexample where a finite product  $G \cong \mathbb{C}^+ \times \dots \times \mathbb{C}^+$  acts on a polynomial ring such that the fixed ring is not finitely generated. An important high point was reached when Hilbert proved the central finiteness problem of invariant theory which is based on Hilbert's Basissatz and says that under some assumptions the invariant ring is finitely generated as a  $k$ -algebra. Since this approach did not help to construct a system of generators, Hilbert published a second paper including Hilbert's Nullstellensatz, Noether's Normalization Lemma and the Hilbert-Mumford-Criterion to give a more constructive advance. After this immense progress, the subject became more quiet. However, it never died completely and was brought back to the main stage just a few decades later.

In 1954 G. C. Shephard and J. A. Todd proved one of the most powerful theorems in commutative invariant theory. Shortly after, C. Chevalley provided a uniform proof in 1955. Now known as the Chevalley-Shephard-Todd Theorem, it says that for a finite group  $G$  acting faithfully on a polynomial ring  $k[x_1, \dots, x_n]$  its fixed ring  $k[x_1, \dots, x_n]^G$  is isomorphic to  $k[x_1, \dots, x_n]$  if and only if  $G$  is generated by pseudo-reflections. An element of  $G$  is a pseudo-reflection if its restriction to the degree 1 piece of  $k[x_1, \dots, x_n]$  fixes a hyperplane. In 1987 M. Artin and W. Schelter introduced a new class of noncom-

mutative graded algebras which can be understood as noncommutative generalizations of polynomial rings: an Artin-Schelter regular algebra is defined to be a finitely generated connected  $\mathbb{N}$ -graded ring  $S = k \oplus \bigoplus_{j>0} S_j$  satisfying the following three conditions:

(AS 1)  $S$  has finite (graded) global dimension  $d$ .

(AS 2)  $S$  has finite Gelfand-Kirillov dimension.

(AS 3)  $\underline{\text{Ext}}_S^m(k, S) \cong \underline{\text{Ext}}_{S^{\text{op}}}^m(k, S) \cong \delta_{md} k[\ell]$  for some  $\ell$ .

Chan, Jørgensen, Kirkman, Kuzmanovich, Walton, Zhang and many others generalized invariant theory to finite group actions ([KKZ08, KKZ10, JZ00]) and finite Hopf algebra actions ([KKZ09, CKWZ16]) on Artin-Schelter regular algebras. There has been extensive success in finding analogues of the Chevalley-Shephard-Todd Theorem and other very important theorems from commutative invariant theory to this new setting. For instance, also a version of Watanabe's Theorem, Stanley's Theorem and many more have been found. One condition all these theorems have in common is to require the algebra to be *connected*. An  $\mathbb{N}$ -graded algebra  $S$  is called connected if its degree zero piece  $S_0$  is isomorphic to  $k$ .

Kirkman, Kuzmanovich and Zhang succeeded in proving a more general version of the Chevalley-Shephard-Todd Theorem as follows. After expanding the polynomial ring to an Artin-Schelter regular algebra  $S$ , their first question was to find the suitable generalization of pseudo-reflections. While the definition from the commutative world to fix a hyperplane of the degree 1 piece fails terribly in this new setting (see [Kir16, Example

1.4]), the characterization using the trace of  $g$  works well. More precisely, the trace of  $g$ , which is the following formal Laurent series:

$$\mathrm{Tr}_S(g, t) = \sum_{j \in \mathbb{N}} \mathrm{tr}(g|_{S_j}) t^j$$

has the special form  $e_g(t)^{-1}$  for some polynomial  $e_g(t) \in k[t]$  by [JZ97, Theorem 2.3]. Moreover, for a nontrivial automorphism  $g$  acting on an Artin-Schelter regular domain  $S$ , the multiplicity of 1 as a root of  $e_g(t)$  is always strictly smaller than  $\mathrm{GKdim}(S)$  ([KKZ08, Proposition 1.8]). The term quasi-reflection was introduced in [KKZ08, Definition 2.2] to describe graded automorphisms  $g$  for which 1 is a root of the polynomial  $e_g(t)$  of multiplicity  $\mathrm{GKdim}(S) - 1$ . This definition coincides with the definition of pseudo-reflections for polynomial rings. Later, the foremost generalization of the Chevalley-Shephard-Todd Theorem was proved by Kirkman, Kuzmanovich and Zhang in [KKZ10]:

**Theorem 1.1.1** ([KKZ10, Theorem 5.5 and Theorem 6.3]). *Let  $S$  be a noetherian and Koszul Artin-Schelter regular domain and  $G$  a finite subgroup of  $\mathrm{Aut}_{\mathrm{gr}}(S)$ . Suppose either  $G$  is abelian or  $S$  is a skew polynomial ring. Then the fixed subring  $S^G$  is Artin-Schelter regular if and only if  $G$  is generated by quasi-reflections.*

Additionally, the proof of this beautiful result uses the observation that if the fixed ring  $S^G$  has finite global dimension it is already Artin-Schelter regular (see [KKZ08, Lemma 1.10(c)]). At the same time, this is equivalent to  $S$  being a free module over the fixed ring.

On their way, Kirkman, Kuzmanovich and Zhang also show that quasi-reflections

of finite order do not exist for every Artin-Schelter regular algebra:

**Theorem 1.1.2** ([KKZ08, Proposition 6.4]). *Let  $S$  be a noetherian Artin-Schelter regular algebra of global dimension 3 that is generated by two elements in degree 1. Then  $S$  has no quasi-reflection of finite order, and hence no regular fixed subrings  $S^G$  for  $G$  a finite group.*

Naturally, one question to ask is how often do quasi-reflections exist. The proof technique of Theorem 1.1.2 encouraged the work done in Chapter 3. While Artin-Schelter regular algebras get too complicated very quickly, some progress has been made for global dimension 4. If  $S$  is an Artin-Schelter regular domain of global dimension 4, generated by three elements and properly  $\mathbb{Z}^{\times 2}$ -graded, the situation is understood in Theorem 3.2.8. Then either  $S$  does not have a quasi-reflection of finite order or  $S = B[x; \sigma]$  where  $B$  is an Artin-Schelter regular algebra of global dimension 3 generated by two elements, and a quasi-reflection  $g$  fixes  $B$  pointwise and sends  $g(x) = \lambda x$  for some nontrivial root of unity  $\lambda$ . In other words,  $g$  must be a classical reflection.

Unfortunately, the techniques used above do not work for larger dimensions. Even for global dimension 4 but generated by two elements in degree 1 we cannot present a similar result. We have to settle with five equations in Proposition 3.3.5 instead, for which we believe no solutions exist. If this is true, also for this class of algebras no quasi-reflections exist.

Another approach to generalize invariant theory to noncommutative algebras was

made by Peter Jørgensen and James J. Zhang in their paper *Gourmet's Guide to Gorensteinness* [JZ00]. More precisely, Jørgensen and Zhang prove the following remarkable theorem:

**Theorem 1.1.3** ([JZ00, Theorem 3.3]). *Suppose that  $S$  is noetherian and Artin-Schelter Gorenstein (i.e.  $S$  is connected, has finite injective dimension over itself and satisfies (AS 3)), and that  $G$  is a finite subgroup of  $\text{Aut}_{\text{gr}}(S)$ . If  $\text{hdet}(g) = 1$  for each  $g \in G$ , then the fixed ring  $S^G$  is Artin-Schelter Gorenstein.*

This theorem requires one to understand the construction of the homological determinant  $\text{hdet}(-)$ . It involves the local cohomology functors

$$H_{\mathfrak{m}}^i(M) = \varinjlim \text{Ext}_S^i(S/S_{\geq m}, M)$$

for  $S$  an  $\mathbb{N}$ -graded connected locally finite  $k$ -algebra, a right  $S$ -module  $M$  and  $\mathfrak{m} = S_{\geq 1}$ . In one of their first steps the authors showed that an Artin-Schelter Gorenstein algebra  $S$  with injective dimension  $d$  satisfies  $H_{\mathfrak{m}}^i(S) = \delta_{id} S^*[\ell]$ . For a graded automorphism  $g$  of  $S$ , this motivated to consider  $H_{\mathfrak{m}}^d(g) : S^*[\ell] \rightarrow S^*[\ell]$ . A small technical analysis reveals that this map equals a scalar multiple of the map  $(g^{-1})^* : S^* \rightarrow S^*$ ,  $\varphi \mapsto \varphi \circ g^{-1}$ . This scalar will be called the inverse of the homological determinant of  $g$ .

With this large variety of results generalized to the noncommutative setting, this dissertation attempts to find some suitable versions for the non-connected noncommutative world. One of the main motivations for this dissertation was to find a version of

the Chevalley-Shephard-Todd Theorem for some class of non-connected noncommutative algebras. However, this turned out to be substantially harder than hoped.

A particularly nice class of non-connected noncommutative algebras are the so-called preprojective algebras. Their construction begins with a quiver  $Q$  which for us is a directed graph with finitely many vertices  $Q_0 = \{e_i \mid 1 \leq i \leq n\}$  and finitely many arrows  $Q_1$ . The double  $\bar{Q}$  of  $Q$  is the quiver obtained by keeping the vertex set and adding a new arrow  $\alpha^*$  from  $e_j$  to  $e_i$  for each arrow  $\alpha$  from  $e_i$  to  $e_j$ . The preprojective algebra  $\Pi_Q(k)$  is then defined as

$$A = \Pi_Q(k) = k\bar{Q} / \left( \sum_{\alpha \in Q_1} \alpha\alpha^* - \sum_{\alpha \in Q_1} \alpha^*\alpha \right)$$

where  $k\bar{Q}$  denotes the path algebra of  $\bar{Q}$ .

Primarily, we focus on preprojective algebras for the extended Dynkin diagrams  $\widetilde{A}_{n-1}$ ,  $\widetilde{D}_{n-1}$  and  $\widetilde{E}_m$  for  $m = 6, 7, 8$ . Preprojective algebras form a nice class of non-connected noncommutative algebras with natural grading, global dimension and Gelfand-Kirillov dimension equal to 2 and nice homological properties such as the  $\chi$ -condition. In particular, this is a class of twisted Calabi-Yau algebras. We define twisted Calabi-Yau algebras in Definition 2.3.7 which form a non-connected generalization of Artin-Schelter regular algebras. In fact, Reyes, Rogalski and Zhang showed in [RRZ14, Lemma 1.2] that every connected twisted Calabi-Yau algebra is indeed Artin-Schelter regular. While much of this dissertation is done for preprojective algebras, the main theories developed in Chapters 5 and 6 apply to noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebras with degree zero piece  $B_0 \cong k^n$ .

The first steps towards finding a Chevalley-Shephard-Todd Theorem consist of finding the right definition of a quasi-reflection and the right condition to require the fixed ring to have. As mentioned earlier, in [KKZ10]’s version a key argument is that finite global dimension already implies the important homological properties to make the fixed ring Artin-Schelter regular. Unfortunately, the situation with non-connected algebras is significantly more difficult. To obtain a fixed ring of a twisted Calabi-Yau algebra  $B$  with  $B_0 \cong k^n$  which is again twisted Calabi-Yau it is not enough to require only finite global dimension. We moreover need to ask for global dimension equal to  $\text{gl.dim}(B)$ . Additionally, if the fixed ring is twisted Calabi-Yau the strongest statement possible is that  $B$  is projective as a  $B^G$ -module rather than free. The following is the main result of Chapter 5, which answers one of the two main questions before attempting to find a Chevalley-Shephard-Todd Theorem for preprojective algebras.

**Theorem 5.2.5.** *Assume  $B$  is a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra of global dimension  $d \geq 1$  with  $B_0 \cong k^n$ . Suppose  $\text{pdim}(S_i) = \text{pdim}(S_j)$  and  $\text{pdim}(S_i^*) = \text{pdim}(S_j^*)$  for all  $i, j = 1, \dots, n$  where  $S_i = e_i B^G / (e_i B^G)_{\geq 1}$  and  $S_i^* = B^G e_i / (B^G e_i)_{\geq 1}$  denote the simple modules. Then the following are equivalent:*

- (1)  $\text{gl.dim}(B^G) = \text{gl.dim}(B)$ .
- (2)  $B^G$  is twisted Calabi-Yau.
- (3)  $B$  is a projective  $B^G$ -module.

In particular, for  $Q = \widetilde{A_{n-1}}$  and  $A = \Pi_Q(k)$  the conditions on the projective

dimensions of the simple modules  $\text{pdim}(S_i) = \text{pdim}(S_j)$  and  $\text{pdim}(S_i^*) = \text{pdim}(S_j^*)$  are satisfied for the fixed ring whenever  $A^G$  is twisted Calabi-Yau as worked out in Section 5.3. This means, for  $A$  the conditions (1) - (3) are always equivalent.

The other big question for extending the theory to a preprojective algebra  $A$  is to find the correct definition for a quasi-reflection. Being used to define quasi-reflections before, one of the crucial tools of invariant theory is a trace function as defined above. For non-connected algebras and automorphisms which fix the idempotents  $e_i$ , it is possible to encode more information using a vector trace in which each entry equals the trace of  $g|_{e_i A}$  for  $i = 1, \dots, n$ . We find concrete formulas to calculate this vector trace of a graded automorphism which fixes the idempotents in Sections 4.1, 6.1 and 6.2. These formulas for the vector trace enable us to find a concrete description of a matrix homological determinant which we will introduce in Section 6.3.

In order to understand invariant theory of preprojective algebras better, it was crucial to work out several examples. Three of the most interesting examples are presented in Section 4.2 all of which use  $Q = \widetilde{A_2}$ . More precisely, the first example gives a fixed ring such that  $A$  is free over  $A^G$  as it happens for reflection groups in the commutative case. The second example discusses a fixed ring of global dimension 2 where  $A$  is only projective but not free over its fixed ring. Lastly, Example 4.2.3 works out a fixed ring with global dimension 3 unlike the preprojective algebra which has global dimension 2. As described before the last two examples display situations impossible for connected algebras which have to be evaluated.

It becomes apparent that the usual trace function is no longer of the form  $p(t)^{-1}$  for some polynomial  $p(t)$ . Instead, it is possible to get rational functions which make a straightforward generalization of quasi-reflection by using the multiplicity of 1 as a root of the denominator not well-defined. Simply using the multiplicity of the pole 1 of the trace function does not solve the problem either as Example 7.2.4 presents a graded automorphism whose trace has 1 as a pole of order  $1 = \text{GKdim}(A) - 1$  but  $A^{\langle g \rangle}$  has infinite global dimension. Additionally, it is possible to have a nonidentity automorphism whose trace has 1 as a pole of multiplicity  $\text{GKdim}(A) = 2$ . How to define quasi-reflections remains an unsolved problem.

In Chapter 6 we follow the construction from [JZ00] for certain non-connected algebras satisfying analogous properties. In particular, the theory applies to preprojective algebras. Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ , the simple right  $B$ -modules by  $S_j = e_j B / (e_j B)_{\geq 1}$  and the simple left  $B$ -modules by  $S_j^* = B e_j / (B e_j)_{\geq 1}$ , respectively. Then  $B$  is called generalized Gorenstein if it satisfies

- (1)  $\text{injdim}_B(B) = N < \infty$  and
- (2)  $\underline{\text{Ext}}_B^i(S_j, B) \cong \delta_{Ni} S_{\sigma(j)}^*$  for some  $\sigma \in \text{Sym}(n)$ . (generalized Gorenstein condition)

Fix such a  $B$ . Additionally, we use the definition for local cohomology

$$H_{\mathfrak{r}}^i(M) = \varinjlim \underline{\text{Ext}}_B^i(B(B/B_{\geq r})_B, M_B)$$

for a right  $B$ -module  $M$  and  $\mathfrak{r} = B_{\geq 1}$ . Luckily, this construction and several related results found in [AZ94] work for non-connected generalized Gorenstein algebras. Also, it turns out that the same statement as in [JZ00, Lemma 2.1] for connected Artin-Schelter Gorenstein algebras is satisfied: we obtain  $H_{\mathfrak{r}}^i(B) = \delta_{iN} B^*[\ell]$  as ungraded modules where  $B^*$  is the Matlis dual of  $B$  and  $N = \text{injdim}(B)$  in Lemma 6.3.5.

On the other hand, the definition of a homological determinant is more challenging. For a graded automorphism  $g$  fixing the idempotents we also observe that  $H_{\mathfrak{r}}^N(g)$  can be related to  $(g^{-1})^*$ . However, here the connection requires  $n$  scalars, one for each idempotent  $e_i$ . Naturally, we put these scalars into a diagonal matrix denoted by  $\text{Hdet}(g)^{-1}$ . From here, several technical lemmas later we can prove the following theorem:

**Theorem 6.3.9.** *Let  $B$  be a noetherian  $\mathbb{N}$ -graded generalized Gorenstein algebra with degree zero piece  $B_0 \cong k^n$ . Suppose  $B$  has finite global dimension  $d$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Suppose  $G$  is a finite subgroup of graded automorphisms of  $B$  such that every  $g \in G$  fixes the primitive idempotents. Assume that every  $g \in G$  satisfies  $\text{Hdet}(g) = I_n$ . Then  $B^G$  is generalized Gorenstein.*

Section 6.4 focuses on the homological determinant for preprojective algebras  $A$ . In the connected case, for algebras of global dimension 2, the homological determinant of  $g$  is the same as the scalar the defining relation is multiplied by the action of  $g$  (see [MS16, Theorem 3.3]). This inspires the suspicion that also all diagonal entries of  $\text{Hdet}(g)$  equal the scalar the defining relation of a preprojective algebra is multiplied by the action of  $g$ . Essentially, it will be enough to show that  $A$  is a finitely generated module over an

Artin-Schelter regular algebra and some calculations to prove the following result:

**Corollary 6.4.6.** *Let  $Q$  be the quiver corresponding to an extended Dynkin graph of type  $\widetilde{A}_{n-1}$ ,  $\widetilde{D}_{n-1}$  or  $\widetilde{E}_m$  for  $m = 6, 7, 8$  and  $A = \Pi_Q(k)$ . Let  $g$  be a graded automorphism of  $A$  which fixes the idempotents. Use the usual convention  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Then  $\text{Hdet}(g) = c_1 t_1 I_n$  ( $n = m + 1$  for  $\widetilde{E}_m$ ). In particular, it makes sense to define  $\text{hdet}(g) = c_1 t_1$ .*

In Chapter 7 we work with  $A = \Pi_Q(k)$  for  $Q = \widetilde{A}_{n-1}$ . An attempt to define a quasi-reflection is finally given in Definition 7.2.1. We call a graded automorphism  $g$  of  $A$  a *quasi-reflection* if  $g$  is not the identity and  $\text{Tr}_{e_j A}(g|_{e_j A}, t)$  has 1 as a pole of order 1 for at least one of  $j = 1, \dots, n$ . The preparatory work done in Section 7.1 is necessary to reveal how the different traces  $\text{Tr}_{e_j A}(g|_{e_j A}, t)$  behave. In particular, Proposition 7.2.2 proves that a quasi-reflection always has a nontrivial homological determinant which then combined with Section 7.1 says that for a quasi-reflection  $g$  no  $\text{Tr}_{e_j A}(g|_{e_j A}, t)$  can have 1 as a pole of multiplicity 2. The following theorem is inspired by [KKZ08, Theorem 2.4].

**Theorem 7.2.3.** *Let  $Q = \widetilde{A}_{n-1}$  and  $A = \Pi_Q(k)$ . Assume  $g$  is a graded automorphism of  $A$  which fixes the idempotents  $e_1, \dots, e_n$ . Use the usual convention  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Assume that every nontrivial element  $g \in G$  satisfies the following property: if  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  then  $\text{hdet}(g) = c_1 t_1 \neq 1$ . If  $\text{gl. dim}(A^G) = 2$ , then  $G$  contains a quasi-reflection.*

The main result of Section 7.3 demonstrates that in order to obtain a fixed ring

which is twisted Calabi-Yau it must be generated by  $3n$  elements. This encouraged the final chapter which works through the smallest example  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$ . Worth pointing out is that for this small algebra we can define an automorphism  $g$  to be a quasi-reflection by simply checking the multiplicity of 1 as a pole of  $\text{Tr}(g, t)$ . Finally, Proposition 8.2.1 presents all possible fixed rings with 9 generators and if some of the twisted Calabi-Yau cases arise from a cyclic group action, we show that the group must be generated by quasi-reflections.

## 1.2 Structure of this dissertation

In this last section of the introduction we present a concise overview over the chapters of this dissertation. All necessary definitions and background on graded rings, invariant theory of Artin-Schelter regular algebras and preprojective algebras is given in Chapter 2. Chapter 3 describes the existence of quasi-reflections of Artin-Schelter regular algebras of small global dimensions which is independent from the rest of this dissertation. We start studying preprojective algebras of type  $\widetilde{A}_{n-1}$  in Chapter 4. In particular, formulas for the vector trace of certain graded automorphisms are developed in section 4.1 followed up by interesting examples.

Chapter 5 contains one of the main theorems which proves that three properties of a fixed ring of a twisted Calabi-Yau algebra are equivalent. A different approach to invariant theory is worked out in Chapter 6. Necessary conditions for a fixed ring of a

generalized Gorenstein algebra to be again generalized Gorenstein are given in Section 6.3. Also, formulas for the vector trace for the cases  $\widetilde{D}_{n-1}$  and  $\widetilde{E}_m$  for  $m = 6, 7, 8$  are explored in Sections 6.1 and 6.2. Concluding Chapter 6, the inevitable tool, the homological determinant, is examined for preprojective algebras in 6.4 which reveals its nice form  $c_1 t_1 I_n$ . Hereafter, Chapter 7 discusses a possible definition of a quasi-reflection. After tedious calculations, it turns out that such a group element exists whenever the preprojective algebra is projective over its fixed ring. At last, a discussion of invariant theory of the preprojective algebra with respect to  $Q = \widetilde{A}_2$  is given in Chapter 8.

This chapter contains material from “Properties of the fixed ring of a preprojective algebra”, Weispfenning, Stephan, to appear and “Generalized Gorensteinness and a homological determinant for preprojective algebras”, Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of both papers.

# Chapter 2

## Background

This chapter consists of three parts. First, the general conventions and notation regarding general graded rings will be introduced including the concepts of dimension for noncommutative algebras which are of interest to this project. The remaining two sections deal with the two general classes of algebras this dissertation concerns about. Naturally, the background on Artin-Schelter regular algebras is given first. This entails the invariant theoretic prospective such as introducing the trace function of a graded automorphism, the Hilbert series of a graded algebra as well as remarkable results including a generalization of the Chevalley-Shephard-Todd Theorem. Lastly, the most detailed part introduces the main object of this dissertation. In addition to defining preprojective algebras in Definition 2.3.3, several technical results are given peaking in Proposition 2.3.9 which lists important properties of preprojective algebras. The most visible change from Artin-Schelter regular algebras to preprojective algebras consists in loosening that the algebra is connected, i.e.

the zero degree piece equals the field. This causes many technical changes with respect to trace functions and Hilbert series. The necessary modifications are given at the end of this chapter together with two general results. These say for our situation that given a finite group acting, the ring is always finitely generated over its fixed ring and the fixed ring satisfies the so-called  $\chi$ -condition.

## 2.1 General definitions

For a commutative monoid  $\Gamma$ , a  $k$ -algebra  $R$  is called  $\Gamma$ -graded provided  $R$  has a vector space decomposition  $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$  such that  $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$  for every  $\gamma, \delta \in \Gamma$ . Every algebra in this dissertation is either  $\mathbb{N}$ -,  $\mathbb{Z}$ - or  $\mathbb{Z}^{\times 2}$ -graded. Unless said otherwise, every algebra is assumed to be locally finite, i.e. each graded piece is finite dimensional over  $k$ . Lastly, an  $\mathbb{N}$ -graded  $k$ -algebra  $R$  is called connected if  $R_0 = k$ .

Let  $R = \bigoplus_{j \in \mathbb{N}} R_j$  be a graded algebra over  $k$  and  $g$  be an automorphism of  $R$ .

- We call  $g$  a *graded automorphism* if  $g(R_i) = R_i$  for all graded pieces  $R_i$  of  $R$ . The *group of all graded automorphisms of  $R$*  is denoted by  $\text{Aut}_{\text{gr}}(R)$ .
- A right  $R$ -module  $M$  is called *graded* if  $M = \bigoplus_{j \in \mathbb{Z}} M_j$  and all  $r \in R_n$  and  $m \in M_\ell$  satisfy  $mr \in M_{n+\ell}$  for all  $n, \ell \in \mathbb{Z}$ .
- A graded right  $R$ -module  $M$  is called *left bounded* if there exists an integer  $N$  such that  $M_j = 0$  for all  $j \leq N$ . Analogously,  $M$  is called *right bounded* if there exists

an integer  $N$  such that  $M_j = 0$  for all  $j \geq N$ . The module  $M$  is called *bounded* if it is both left bounded and right bounded.

- For a graded  $R$ -module  $M = \bigoplus_{j \in \mathbb{Z}} M_j$ , the *shifted module*  $M[\ell] = \bigoplus_{j \in \mathbb{Z}} (M[\ell])_j$  is defined by  $(M[\ell])_j = M_{\ell+j}$ .
- For graded  $R$ -modules  $M = \bigoplus_{j \in \mathbb{Z}} M_j$  and  $N = \bigoplus_{i \in \mathbb{Z}} N_i$ , a linear map  $\varphi$  satisfying  $\varphi(M_j) \subseteq N_{j+\ell}$  is called *homogeneous of degree  $\ell$* . Denote the vector space of all right  $R$ -module homomorphisms from  $M$  to  $N$  which are homogeneous of degree  $\ell$  by  $\text{Hom}_R^\ell(M, N)$ .
- For graded  $R$ -modules  $M$  and  $N$ , the elements of  $\text{Hom}_R^0(M, N)$  are called *graded* and the set of all graded automorphisms of  $M$  is denoted by  $\text{Aut}_{\text{gr}}(M)$ .
- For graded  $R$ -modules  $M$  and  $N$ , the vector space  $\text{Hom}_R(M, N)$  denotes the usual  $\text{Hom}$  in the category  $R\text{-Mod}$  of all right  $R$ -modules. We have:

$$\underline{\text{Hom}}_R(M, N) = \bigoplus_{\ell=-\infty}^{\infty} \text{Hom}_R^\ell(M, N) \subseteq \text{Hom}_R(M, N).$$

It is well-known that if  $M$  is also finitely generated, then  $\underline{\text{Hom}}(M, N) = \text{Hom}(M, N)$  which is the case for all appearances of  $\underline{\text{Hom}}$  in this dissertation.

- Denote the right derived functors of  $\underline{\text{Hom}}$  by  $\underline{\text{Ext}}_R^i = R^i \underline{\text{Hom}}_R$ .
- For a right  $R$ -module  $M$ , the  *$g$ -twisted module*  $M^g$  is defined as  $M^g = M$  with changed action  $m * r = mg(r)$  for  $m \in M$ ,  $r \in R$ .

- Let  $M$  and  $N$  be graded right  $R$ -modules. A  $k$ -linear map  $\varphi : M \rightarrow N$  is called *g-linear* if  $\varphi(mr) = \varphi(m)g(r)$  for all  $m \in M$  and  $r \in R$ . As noted in [JZ97], a graded map  $\varphi : M \rightarrow N$  is *g-linear* if and only if it is a graded  $R$ -homomorphism from  $M$  to  $N^g$ .

For Chapter 3 we need the following definition taken from [GW04, CHAPTER 2].

**Definition 2.1.1.** Let  $R$  be a ring,  $\alpha$  a ring endomorphism of  $R$ , and  $\delta$  an  $\alpha$ -derivation of  $R$ . This means  $\delta : R \rightarrow R$  is an additive map satisfying  $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$  for all  $r, s \in R$ . The ring  $S = R[x; \alpha, \delta]$  is called an *Ore extension of  $R$*  provided

- (a)  $S$  is a ring, containing  $R$  as a subring,
- (b)  $x$  is an element of  $S$ ,
- (c)  $S$  is a free left  $R$ -module with basis  $\{1, x, x^2, \dots\}$  and
- (d)  $xr = \alpha(r)x + \delta(r)$  for all  $r \in R$ .

In this dissertation we will mostly see Ore extensions with  $\delta = 0$  where we simply omit  $\delta$  and write  $S = R[x; \alpha]$  instead.

**Definition 2.1.2.** For  $R$  as before, let  $M$  be a graded locally finite right  $R$ -module. Further, let  $g$  be a graded endomorphism of  $M$ . Then:

- The *Hilbert series*  $H_M(t)$  of  $M$  is the formal Laurent series

$$H_M(t) = \sum_{j \in \mathbb{Z}} \dim_k(M_j) t^j.$$

- As in [JZ97], the *trace*  $\mathrm{Tr}_M(g, t)$  of  $g$  is the formal Laurent series

$$\mathrm{Tr}_M(g, t) = \sum_{j \in \mathbb{Z}} \mathrm{tr}_k(g|_{M_j}) t^j.$$

Many different notions of dimension have been introduced for noncommutative rings. For this work, the most crucial definitions are as follows. Let  $R$  be a ring and  $M$  be an  $R$ -module.

- The *projective dimension* of  $M$ , denoted  $\mathrm{pdim}_R(M)$ , is defined to be the minimal nonnegative number  $d$  such that there exists a projective resolution of  $M$  of length  $d$ . If no such  $d$  exists then  $\mathrm{pdim}_R(M) = \infty$ .
- The *(right) global dimension* of  $R$ , denoted  $\mathrm{gl.dim}(R)$ , is defined to be the supremum of the projective dimensions of all (right)  $R$ -modules. For a finitely generated connected  $\mathbb{N}$ -graded algebra  $S$  it is well-known that  $\mathrm{gl.dim}(S) = \mathrm{pdim}_S(k)$  where  $k = S/S_{\geq 1}$  is the trivial module (see [BRS<sup>+</sup>16, Proposition 1.30]). A similar result is true for a not necessarily connected noetherian  $\mathbb{N}$ -graded algebra  $B$  with degree zero piece  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$  and call the simple right modules  $S_j = e_j B / (e_j B)_{\geq 1}$ . Then [RR18, Proposition 3.19 (3)] says that  $\mathrm{gl.dim}(B) = \max\{\mathrm{pdim}(S_j) \mid j = 1, \dots, n\}$ .
- The *Gelfand-Kirillov dimension* of  $R$ , denoted  $\mathrm{GKdim}(R)$ , is defined by

$$\mathrm{GKdim}(R) = \limsup_{\ell \rightarrow \infty} \log_\ell \dim_k V^\ell$$

where  $V$  is a finite dimensional  $k$ -subspace of  $R$  which contains 1 and generates  $R$  as an algebra. As shown in [KL00] the definition does not depend on the choice of  $V$ .

- The *injective dimension* of  $M$ , denoted  $\text{injdim}_R(M)$ , is defined to be the minimal nonnegative number  $d$  such that there exists an injective resolution of  $M$  of length  $d$ . If no such  $d$  exists then  $\text{injdim}_R(M) = \infty$ .

For a finite group  $G$  of automorphism of a ring  $R$ , the *fixed ring* is denoted by  $R^G = \{r \in R \mid g(r) = r, \text{ for all } g \in G\}$ .

## 2.2 Preliminaries to invariant theory of AS-regular algebras

A particularly nice class of noncommutative  $\mathbb{N}$ -graded algebras was introduced by M. Artin and W. Schelter in 1987.

**Definition 2.2.1.** A finitely generated connected  $\mathbb{N}$ -graded ring  $S = k \oplus \bigoplus_{j \geq 1} S_j$  satisfying the following three conditions:

(AS 1)  $S$  has finite (graded) global dimension  $d$ ,

(AS 2)  $S$  has finite Gelfand-Kirillov dimension, and

(AS 3)  $\underline{\text{Ext}}_S^m(k, S) \cong \underline{\text{Ext}}_{S^{\text{op}}}^m(k, S) \cong \delta_{md} k[\ell]$  for some  $\ell$

is called *Artin-Schelter regular*. Sometimes, we abbreviate Artin-Schelter regular by *AS-regular* or just *regular*.

Fix an Artin-Schelter regular algebra  $S$  together with a graded automorphism  $g \in \text{Aut}_{\text{gr}}(S)$ . One of the first milestones to develop invariant theory for Artin-Schelter regular algebras was to consider the trace function of graded automorphisms. First, Jing and Zhang proved in [JZ97, Theorem 2.3] that the trace of  $g$  has the form

$$\text{Tr}_S(g, t) = \frac{1}{e_g(t)}$$

for some polynomial  $e_g(t)$  called the *Euler polynomial of  $g$* . Further, they showed that if  $g$  has finite order all the roots of  $e_g(t)$  are roots of unity. Over ten years later, Kirkman, Kuzmanovich and Zhang proved in [KKZ08, Lemma 1.7 and Proposition 1.8] that the multiplicity of 1 as a root of  $e_g(t)$  must be less than or equal to  $\text{GKdim}(S)$  with equality if and only if  $g$  is the identity. This motivated their definition of quasi-reflection as follows.

**Definition 2.2.2.** Let  $S$  be an Artin-Schelter regular algebra with GK-dimension equal to  $m$ . A graded automorphism  $g$  of  $S$  is called a *quasi-reflection* if the trace has the form

$$\text{Tr}_S(g, t) = \frac{1}{(1-t)^{m-1}q(t)}$$

such that  $q(1) \neq 0$ .

In the same paper, the authors prove the following remarkable classification of all quasi-reflections for quantum polynomial rings, Artin-Schelter regular algebras whose Hilbert series equals  $H_S(t) = \frac{1}{(1-t)^m}$ .

**Theorem 2.2.3** ([KKZ08, Theorem 3.1]). *Let  $S$  be a quantum polynomial ring and let  $g$  be a quasi-reflection. Then  $g$  is one of the following:*

- *$g|_{S_1}$  is a classical reflection, i.e. fixes a hyperplane. In this case, we call  $g$  a classical reflection.*
- *There is a basis of  $S_1$ , say  $\{b_1, b_2, \dots, b_r\}$ , such that  $g(b_1) = ib_1$ ,  $g(b_2) = -ib_2$  and  $g(b_j) = b_j$  for all  $j \geq 3$ . In this case, we call  $g$  a mystic reflection.*

The three papers [JZ97], [KKZ08] and [KKZ10] have been the main motivation for the work in this dissertation. In particular, the following remarkable theorems were inspiring.

**Theorem 2.2.4** ([KKZ08, Theorem 2.4]). *Let  $S$  be noetherian and Artin-Schelter regular. Let  $G$  be a finite subgroup of  $\text{Aut}_{\text{gr}}(S)$ . If  $S^G$  has finite global dimension, then  $G$  contains a quasi-reflection.*

**Proposition 2.2.5** ([KKZ08, Proposition 6.4]). *Let  $S$  be a noetherian Artin-Schelter regular algebra of global dimension 3 that is generated by two elements in degree 1. Then  $S$  has no quasi-reflection of finite order, and hence no regular fixed subrings  $S^G$  for  $G$  a finite group.*

Kirkman, Kuzmanovich and Zhang's work builds up to the following main theorem which is a generalization of the famous Chevalley-Shephard-Todd Theorem.

**Theorem 2.2.6** ([KKZ10, Theorems 5.5 and 6.3]). *Let  $S$  be a quantum polynomial ring, i.e. Artin-Schelter regular with  $H_S(t) = (1 - t)^{-m}$ , and  $G$  a finite subgroup of  $\text{Aut}_{\text{gr}}(S)$ .*

*Suppose either  $G$  is abelian or  $S$  is a skew polynomial ring. The fixed subring  $S^G$  is Artin-Schelter regular if and only if  $G$  is generated by quasi-reflections.*

The work in Section 3 was mostly encouraged by the first two statements. There, the main results state that  $S$  does not have any quasi-reflections of finite order if

- $S$  is isomorphic to the Jordan plane  $k\langle x, y \rangle / (yx - xy - x^2)$  or
- $S$  is an Artin-Schelter regular domain of global dimension 4 which is generated by three elements, properly  $\mathbb{Z}^{\times 2}$ -graded and not a normal extension of global dimension 4.

Moreover, the situation where  $S$  is an Artin-Schelter regular domain of global dimension 4 which is generated by three elements and properly  $\mathbb{Z}^{\times 2}$ -graded is understood in Theorem 3.2.8. The analysis done afterwards with Artin-Schelter regular algebras of global dimension 4 generated by two elements in degree 1 shows that this approach gets complicated too quickly. Based on five equations which need to be satisfied in order to obtain a quasi-reflection of finite order in this case, we conjecture that again no quasi-reflection of finite order exists.

## 2.3 Preliminaries to invariant theory of preprojective algebras

Finally, the foundation for our approach of invariant theory for a class of non-connected graded algebras will be formulated. In order to construct preprojective algebras, we need to exploit quivers which are directed graphs. Here, all quivers have finitely many vertices and finitely many arrows. In detail, taking the double of an extended Dynkin quiver (see Euclidean graphs in [EE07, Chapter VII.2]), forming the path algebra and factoring out a predetermined relation finishes the construction of a preprojective algebra we will be interested in. As a convention, the symbol  $A$  is reserved to be the preprojective algebra  $\Pi_Q(k)$  of an extended Dynkin quiver  $Q$  as defined in Definition 2.3.3. After this significant definition, the  $\chi$ -condition (Definition 2.3.5) and the definition of a twisted Calabi-Yau algebra (Definition 2.3.7) are introduced. Before Proposition 2.3.9 collects essential properties of preprojective algebras we state an important result giving equivalent conditions for an algebra to be twisted Calabi-Yau. As seen in the previous section, important tools for any sort of invariant theory are the construction of a Hilbert series and a trace function which we will extend to a vector and matrix notation starting with Definition 2.3.13. One other theorem worthwhile pointing out is a generalization of Molien's Theorem 2.3.16 which connects the (matrix) Hilbert series of the fixed ring to the (matrix) traces of the finite group acting. In particular, it is very useful to calculate examples of fixed rings. The very end of the chapter cites results saying that in a very

general setting an algebra is finitely generated as a module over the fixed ring under a finite group action and that the homological  $\chi$ -condition is satisfied by the fixed rings we care about.

**Definition 2.3.1.** Following [ASS06, Chapter II.1], a *quiver*  $Q = (Q_0, Q_1, s, t)$  is a quadruple consisting of vertices  $Q_0 = \{e_i \mid i \in I\}$ , arrows  $Q_1$  and two maps  $s, t : Q_1 \rightarrow Q_0$  called the *source*  $s$  and *target*  $t$ . For every arrow  $\alpha \in Q_1$  the maps associate the corresponding source  $s(\alpha)$  and target vertex  $t(\alpha)$  in  $Q_0$ . Usually, the quiver is just denoted by  $Q$ .

Further using the notation of [ASS06], a *path of length*  $\ell \geq 1$  with source  $a$  and target  $b$  is a sequence

$$(a \mid \alpha_1, \alpha_2, \dots, \alpha_\ell \mid b),$$

where  $\alpha_k \in Q_1$  for all  $1 \leq k \leq \ell$  as well as  $s(\alpha_1) = a$ ,  $t(\alpha_k) = s(\alpha_{k+1})$  for each  $1 \leq k < \ell$  and  $t(\alpha_\ell) = b$ . Additionally, each vertex  $e_i \in Q_0$  has a path of length  $\ell = 0$  associated to it, called the *trivial path* at  $e_i$  also denoted by  $e_i = (e_i \mid \mid e_i)$ .

The path algebra  $kQ$  is defined in the usual way with its natural grading

$$kQ = kQ_0 \oplus kQ_1 \oplus kQ_2 \oplus \dots \oplus kQ_\ell \oplus \dots$$

where  $kQ_\ell$  denotes the  $k$ -subspace of  $kQ$  generated by the set  $Q_\ell$  of all paths of length  $\ell$ .

**Definition 2.3.2.** For a quiver  $Q$ , its *double*  $\bar{Q}$  is obtained by keeping the vertex set  $Q_0$  and adding a new arrow  $\alpha^*$  from  $e_j$  to  $e_i$  for each arrow  $\alpha \in Q_1$  from  $e_i$  to  $e_j$ .

**Definition 2.3.3.** Using [EE07, Definition 3.1.1.], we can define the preprojective algebra associated to a quiver  $Q$ . Let  $Q$  be a quiver and  $\bar{Q}$  its double. The *preprojective algebra* is defined as

$$A = \Pi_Q(k) = k\bar{Q} / \left( \sum_{\alpha \in Q_1} \alpha \alpha^* - \sum_{\alpha \in Q_1} \alpha^* \alpha \right).$$

Inspired by [BRS<sup>+</sup>16, Lemma 1.5.5], this following standard lemma becomes handy later when we compute  $\underline{\text{Ext}}^j((e_i A)/(e_i A)_{\geq 1}, A)$ .

**Lemma 2.3.4.** *Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with degree zero piece  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Also, denote the indecomposable projective right  $B$ -modules by  $P_i = e_i B$ .*

(a) *For  $s_j \in \{1, \dots, n\}$  for  $j = 1, \dots, N$ , there exists a left  $B$ -module isomorphism*

$$\text{Hom}_B \left( \bigoplus_{j=1}^N P_{s_j}, B \right) \cong \bigoplus_{j=1}^N B e_{s_j}.$$

(b) *Given a graded right  $B$ -module homomorphisms  $\Gamma : P = \bigoplus_{i=1}^N P_{s_i} \rightarrow Q = \bigoplus_{j=1}^M P_{t_j}$ , represented by left multiplication by a matrix  $T$ , i.e.  $\Gamma(\beta) = T \cdot \beta$ , whose entries  $T_{ji}$  lie in  $e_{t_j} B e_{s_i}$ . Applying the functor  $\text{Hom}(-, B)$  yields a left module homomorphism  $\Gamma^* : \text{Hom}(Q, B) \rightarrow \text{Hom}(P, B)$ . By (a),*

$$\Gamma^* : \bigoplus_{j=1}^M B e_{t_j} \rightarrow \bigoplus_{i=1}^N B e_{s_i}$$

*and  $\Gamma^*$  is given by right multiplication by  $T$ .*

*Proof.* We start with (a). Since  $\text{Hom}_B(-, B)$  is linear with respect to finite sums, it is enough to show that  $\text{Hom}_B(P_\ell, B) \cong B e_\ell$  as left modules. Every  $\phi \in \text{Hom}_B(P_\ell, B)$  is

uniquely determined by  $\phi(e_\ell)$  since  $\phi(\beta) = \phi(e_\ell\beta) = \phi(e_\ell)\beta$  for all  $\beta \in e_\ell B$ . Moreover,  $\phi(e_\ell) = \phi(e_\ell)e_\ell$  and so  $\phi(e_\ell) \in Be_\ell$ . On the other hand, every choice of  $\psi(e_\ell) \in Be_\ell$  gives  $\psi \in \text{Hom}(P_\ell, B)$ .

For part (b), let  $\Gamma : P = \bigoplus_{i=1}^N P_{s_i} \xrightarrow{T} Q = \bigoplus_{j=1}^M P_{t_j}$  and  $\mathcal{B}_P = \{e_{s_1}, \dots, e_{s_N}\}$  be the standard set of generators for  $P$  while  $\mathcal{B}_Q = \{e_{t_1}, \dots, e_{t_M}\}$  generates  $Q$  as  $B$ -modules. Then, the matrix  $T$  comes from

$$\Gamma(e_{s_v}) = \sum_{\ell=1}^M e_{t_\ell} t_{\ell v}.$$

Let  $\mathcal{B}_P^* = \{e_{s_1}^*, \dots, e_{s_N}^*\}$  and  $\mathcal{B}_Q^* = \{e_{t_1}^*, \dots, e_{t_M}^*\}$  be the dual bases where  $e_a^*(e_b) = \delta_{ab}1_B$  in  $\text{Hom}(P, B)$  and  $\text{Hom}(Q, B)$ , respectively, for all feasible  $a, b$ . Then by part (a), we obtain

$$\Gamma^* : \bigoplus_{j=1}^M Be_{t_j} \longrightarrow \bigoplus_{i=1}^N Be_{s_i}$$

and  $\mathcal{B}_Q^*$  and  $\mathcal{B}_P^*$  are the corresponding standard bases. Let  $T^*$  be the matrix which by right-multiplication is associated to  $\Gamma^*$  with respect to the chosen bases. Then, by definition,  $T^*$  satisfies

$$\Gamma^*(e_{t_u}^*) = \sum_{i=1}^N t_{ui}^* e_{s_i}^* \quad \text{and} \quad (\Gamma^*(e_{t_u}^*)) (e_{s_v}) = \left( \sum_{i=1}^N t_{ui}^* e_{s_i}^* \right) (e_{s_v}) = t_{uv}^*.$$

If we go back to how  $\Gamma^*$  was defined, we find that

$$(\Gamma^*(e_{t_u}^*)) (e_{s_v}) = (e_{t_u}^* \circ \Gamma) (e_{s_v}) = (e_{t_u}^*) \left( \sum_{\ell=1}^M e_{t_\ell} t_{\ell v} \right) = \sum_{\ell=1}^M e_{t_u}^*(e_{t_\ell}) t_{\ell v} = t_{uv}.$$

and combined this says  $t_{uv}^* = t_{uv}$  which finishes the proof.  $\square$

Weak homological properties introduced by Artin and Zhang are the  $\chi^\circ$ - and  $\chi$ -condition for a noetherian  $\mathbb{N}$ -graded algebra from [AZ94, Definitions 3.2 and 3.7].

**Definition 2.3.5.** Let  $R$  be a noetherian  $\mathbb{N}$ -graded algebra and  $M$  be a right  $R$ -module.

- We say  $\chi_i^\circ(M)$  holds if  $\underline{\text{Ext}}_R^j(R_0, M)$  is bounded for all  $j \leq i$ . If  $\chi_i^\circ(M)$  holds for every finitely generated  $R$ -module  $M$  we say that  $\chi_i^\circ$  holds for  $R$ . If  $\chi_i^\circ$  holds for every  $i$ , we say  $\chi^\circ$  holds for  $R$ .
- We say  $\chi_i(M)$  holds if for all  $\ell$  and all  $j \leq i$  there is an integer  $m_0$  such that  $\underline{\text{Ext}}^j(A/A_{\geq m}, M)_{\geq \ell}$  is a finitely generated  $R$ -module, when  $m \geq m_0$ . If  $\chi_i(M)$  holds for every finitely generated  $R$ -module  $M$ , we say  $\chi_i$  holds for  $R$ . If  $\chi_i$  holds for every  $i$  we say that  $\chi$  holds for  $R$ .

By [AZ94, Proposition 3.11 (2)], a locally finite noetherian  $\mathbb{N}$ -graded algebra satisfies  $\chi_i^\circ$  if and only if  $\chi_i$  holds. Thankfully, preprojective algebras and their fixed rings satisfy these equivalent conditions as pointed out in the following lemma and Corollary 2.3.15.

**Lemma 2.3.6.** *Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . If  $B$  has finite global dimension  $d$  and  $\underline{\text{Ext}}_B^i(S_j, B) \cong \delta_{di} S_{\sigma(j)}^*$  for some  $\sigma \in \text{Sym}(n)$  and  $S_j = e_j B / (e_j B)_{\geq 1}$ ,  $S_j^* = B e_j / (B e_j)_{\geq 1}$  for all  $j$ , then  $B$  satisfies the  $\chi$ -condition.*

*Proof.* Since  $B$  is locally finite, [AZ94, Proposition 3.11 (2)] says condition  $\chi_m^\circ$  is equivalent to  $\chi_m$ . Let  $M$  be a finitely generated graded  $B$ -module. It suffices to show that

$\underline{\text{Ext}}^i(B_0, M)$  is bounded for all  $i$ . We proceed by induction on  $\text{pdim}(M)$ . If  $\text{pdim}(M) = 0$ , then  $M$  is projective and therefore  $M$  is isomorphic to a finite sum of shifted copies of  $P_\ell = e_\ell B$  for  $\ell = 1, \dots, n$ . Using  $B_0 \cong \bigoplus S_j$  and the hypothesis,  $\underline{\text{Ext}}^i(B_0, B)$  is bounded. Therefore, its summands  $\underline{\text{Ext}}^i(B_0, e_\ell B)$  are bounded for  $\ell = 1, \dots, n$ . Being a sum of shifted copies of the  $e_\ell B$ 's, also for  $M$  is true that  $\underline{\text{Ext}}^i(B_0, M)$  is bounded. If  $\text{pdim}(M) > 0$ , there is a short exact sequence

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

where  $P$  is projective and  $P \longrightarrow M$  is a minimal surjection. This means  $P \longrightarrow M$  is the start of a minimal projective resolution of  $M$  and it follows that  $\text{pdim}(N) = \text{pdim}(M) - 1$ . By induction,  $\underline{\text{Ext}}^i(B_0, N)$  and  $\underline{\text{Ext}}^i(B_0, P)$  are bounded for all  $i$ . Consider the long exact sequence

$$\begin{aligned} \dots &\longrightarrow \underline{\text{Ext}}^i(B_0, N) \longrightarrow \underline{\text{Ext}}^i(B_0, P) \longrightarrow \underline{\text{Ext}}^i(B_0, M) \\ &\longrightarrow \underline{\text{Ext}}^{i+1}(B_0, N) \longrightarrow \underline{\text{Ext}}^{i+1}(B_0, P) \longrightarrow \underline{\text{Ext}}^{i+1}(B_0, M) \longrightarrow \dots \\ &\longrightarrow \underline{\text{Ext}}^d(B_0, N) \longrightarrow \underline{\text{Ext}}^d(B_0, P) \longrightarrow \underline{\text{Ext}}^d(B_0, M) \longrightarrow 0, \end{aligned}$$

where  $d = \text{gl. dim}(B) < \infty$ . This shows that  $\underline{\text{Ext}}^i(B_0, M)$  is always squeezed between two bounded  $\underline{\text{Ext}}$ 's and therefore,  $\underline{\text{Ext}}^i(B_0, M)$  is also bounded for all  $i$ .  $\square$

Finally, let us introduce the term Calabi-Yau.

**Definition 2.3.7.** [RR18, Definition 1.2] Let  $B$  be a  $k$ -algebra and call  $B^e = B \otimes_k B^{\text{op}}$ .

We say that  $B$  is *twisted Calabi-Yau of dimension  $d$*  if

- (CY 1)  $B$  has a resolution of finite length by finitely generated projective  $(B, B)$ -bimodules and

(CY 2) there exists an invertible  $k$ -central  $(B, B)$ -bimodule  $U$  such that

$$\mathrm{Ext}_{B^e}^i(B, B^e) \cong \begin{cases} 0, & i \neq d \\ U, & i = d \end{cases}$$

as  $(B, B)$ -bimodules, where each  $\mathrm{Ext}_{B^e}^i(B, B^e)$  is considered as a right  $B^e$ -module via the “inner” right  $B^e$ -structure of  $B^e$ .

In order to complete the main Proposition 2.3.9 which collects important properties of preprojective algebras, we need one more tool. The next result from [RR18] will be used to show that the fixed ring is Calabi-Yau by only proving the generalized Gorenstein condition.

**Proposition 2.3.8.** *[RR18, Theorem 5.15] Let  $B$  be an  $\mathbb{N}$ -graded algebra with  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ , the simple right  $B$ -modules by  $S_j = e_j B / (e_j B)_{\geq 1}$  and the simple left  $B$ -modules by  $S_j^* = B e_j / (B e_j)_{\geq 1}$ , respectively. Then the following are equivalent:*

(1)  *$B$  is twisted Calabi-Yau of dimension  $d$ .*

(2)  *$\mathrm{gl. dim}(B) = d$  and  $B$  satisfies  $\underline{\mathrm{Ext}}_B^i(S_j, B) \cong \delta_{di} S_{\sigma(j)}^*[\ell_j]$  for some  $\sigma \in \mathrm{Sym}(n)$  and some  $\ell_j \in \mathbb{N}$  for all  $j$ . This is called the generalized Gorenstein condition.*

*Proof.* Notice that  $S = B/J(B)$  is separable. According to [RR18, Theorem 5.15]  $B$  is twisted Calabi-Yau if and only if it satisfies [RR18, Theorem 5.2(a)], finite global dimension and  $\underline{\mathrm{Ext}}_B^i(S_j, B) = \delta_{di} S_{\sigma(j)}^*$  for some  $\sigma \in S_n$ .  $\square$

**Proposition 2.3.9.** *Here are some facts about a preprojective algebra  $A = \Pi_Q(k)$  of an extended Dynkin quiver  $Q$  with vertex set  $\{e_1, \dots, e_n\}$ .*

(a) *A has global dimension 2 and satisfies the generalized Gorenstein condition*

$$\underline{\text{Ext}}_A^i(S_j, A) \cong \delta_{di} S_{\sigma(j)}^*[\ell_j]$$

*for the simple right modules  $S_j = e_j A / (e_j A)_{\geq 1}$  for all  $j = 1, \dots, n$  and  $\sigma = \text{id}$ .*

(b) *A is twisted Calabi-Yau of dimension  $d = 2$ .*

(c) *Lemma 2.3.6. A satisfies the  $\chi$ -condition from Definition 2.3.5.*

(d) *[DR81, Theorem].  $\text{GKdim}(\Pi_Q(k)) = 2$ . (The typo was noted in [BGL87, Proposition 6.10]).*

(e) *[ASS06, Corollary 5.17]. The trivial paths  $\{e_1, \dots, e_n\}$  associated to the vertices form the set of primitive and pairwise orthogonal idempotents. Then, the modules  $P_i = e_i A$  for  $i = 1, \dots, n$  are the only indecomposable projective modules and thus every finitely generated projective graded module is a sum of finitely many possibly shifted copies of the  $P_i$ 's.*

(f) *[BGL87, Theorem 6.5]. A is a noetherian prime polynomial identity ring.*

*Proof.* By Proposition 2.3.8 it follows that showing (a) also proves (b). In other words, we need to show that  $\text{gl. dim}(A) = d$  is finite and that  $\underline{\text{Ext}}^i(S_j, A) \cong \delta_{di} S_j^* = \delta_{di} A e_j / (A e_j)_{\geq 1}$  for all simple modules  $S_j = e_j A / (e_j A)_{\geq 1}$  for  $j = 1, \dots, n$ . This requires minimal projective resolutions of the  $S_j$ 's.

Consider  $Q = \widetilde{A_{n-1}}$  first - for basic information on  $A = \Pi_Q(k)$  look at Chapter 4. The proof of [Boc08, Theorem 3.2] gives the start of the minimal projective resolution  $(P_j)_\bullet$  of  $S_j$  which is also where it stops:

$$0 \longrightarrow e_j A \xrightarrow{\delta_2 = \begin{pmatrix} \alpha_j^* \\ -\alpha_{j-1} \end{pmatrix}} e_{j+1} A \oplus e_{j-1} A \xrightarrow{\delta_1 = \begin{pmatrix} \alpha_j & \alpha_{j-1}^* \end{pmatrix}} e_j A \xrightarrow{\delta_0} S_j \longrightarrow 0,$$

where  $j = 1, \dots, n$  and we understand all indices modulo  $n$  if necessary. This shows that  $\text{gl.dim}(A) = 2 = d$  by [RR18, Proposition 3.19 (3)]. Applying  $\underline{\text{Hom}}(-, A)$  to  $(P_j)_\bullet$ , Lemma 2.3.4 yields

$$0 \longrightarrow Ae_j \xrightarrow{\delta_1^* = \begin{pmatrix} \alpha_j & \alpha_{j-1}^* \end{pmatrix}} Ae_{j+1} \oplus Ae_{j-1} \xrightarrow{\delta_2^* = \begin{pmatrix} \alpha_j^* \\ -\alpha_{j-1} \end{pmatrix}} Ae_j \longrightarrow 0.$$

Again from the proof of [Boc08, Theorem 3.2] it follows that this is the start of a minimal projective resolution of  $S_j^*$ . Later in Section 4.1 we discuss how a Gröbner basis of  $A$  looks like. It will show that the reduced form of a word ending in  $\alpha_{j-1}^*$  will also end in  $\alpha_{j-1}^*$  and this implies that  $\delta_1^*$  is injective. These two observations yield  $\underline{\text{Ext}}^2(S_j, A) = S_j^*$  and  $\underline{\text{Ext}}^i(S_j, A) = 0$  for  $i \neq 2$  which shows (a). Proposition 2.3.8 implies (b).

The next case is  $Q = \widetilde{D_{n-1}}$  and so define  $A = \Pi_Q(k)$ . Looking at the minimal projective resolutions for the simple modules  $S_i = e_i A / (e_i A)_{\geq 1}$  provides again  $\text{gl.dim}(A) = 2 = d$ . We need to compute the  $\underline{\text{Ext}}$ -groups. For  $i = 4, \dots, n-3$ , the resolutions and observations are identical to the case  $\widetilde{A_{n-1}}$ . As before, the first three

steps of the minimal projective resolutions come from the proof of [Boc08, Theorem 3.2] and all of them stop there. The two easier cases are for  $i = 1, 2$  and  $\ell = n - 1, n$ . They are

$$0 \longrightarrow e_i A \xrightarrow{\delta_2 = (\alpha_i^*) \cdot} e_3 A \xrightarrow{\delta_1 = (\alpha_i) \cdot} e_i A \xrightarrow{\delta_0} S_i \longrightarrow 0, \quad \text{and}$$

$$0 \longrightarrow e_\ell A \xrightarrow{\delta_2 = (\alpha_\ell) \cdot} e_{n-2} A \xrightarrow{\delta_1 = (\alpha_\ell^*) \cdot} e_\ell A \xrightarrow{\delta_0} S_\ell \longrightarrow 0$$

which after applying  $\underline{\text{Hom}}(-, A)$  to the minimal projective resolutions become

$$0 \longrightarrow Ae_i \xrightarrow{\delta_1^* = \cdot(\alpha_i)} Ae_3 \xrightarrow{\delta_2^* = \cdot(\alpha_i^*)} Ae_i \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow Ae_\ell \xrightarrow{\delta_1^* = \cdot(\alpha_\ell^*)} Ae_{n-2} \xrightarrow{\delta_2^* = \cdot(\alpha_\ell)} Ae_\ell \longrightarrow 0.$$

As in the case  $\widetilde{A_{n-1}}$  we know that applying  $\underline{\text{Hom}}(-, A)$  provides minimal projective resolutions of  $S_i^*$  for  $i = 1, 2$  and  $S_\ell^*$  for  $\ell = n - 1, n$ , respectively. The same reasoning as before shows that the  $\delta_1^*$ 's are injective, the claim  $\underline{\text{Ext}}^j(S_i, A) \cong \delta_{jd} S_i^*$  for  $i = 1, 2, n - 1, n$  follows. Being analogues of each other, it is enough to only do the case  $i = 1, 2$ . One more time, a Gröbner basis argument based on the relations guarantees that the map  $\delta_1^*$  is injective and therefore we have  $\underline{\text{Ext}}^0(S_i, A) = 0$ . The concatenation of the maps  $\delta_2^* \circ \delta_1^*(\beta) = \beta \alpha_i \alpha_i^* = 0$  and thus  $\text{Im}(\delta_1^*) \subseteq \ker(\delta_2^*)$ . In order to prove the other inclusion, let  $\beta = \sum_{j=1}^N c_j \beta_j \in \ker(\delta_2^*)$  for paths  $\beta_j \in Ae_3$ . This implies

$$\delta_2^*(\beta) = \sum_{j=1}^N c_j \beta_j \alpha_i^* = 0.$$

From the relations we know that  $\beta_j = \beta'_j \alpha_i$  and therefore  $\beta = \beta' \alpha_i \in \text{Im}(\delta_1^*)$ . The equality  $\text{Im}(\delta_1^*) = \ker(\delta_2^*)$  gives  $\underline{\text{Ext}}^1(S_i, A) = 0$ . At the last step, every element in  $(Ae_i)_{\geq 1}$  is a linear combination of paths that all end in  $\alpha_i^*$  and therefore  $\text{Im}(\delta_2^*) = (Ae_i)_{\geq 1}$ . This finishes the case  $i = 1, 2$  with  $\underline{\text{Ext}}^2(S_i, A) \cong Ae_i / (Ae_i)_{\geq 1} \cong S_i^*$ .

The two remaining resolutions are a little more complicated:

$$0 \longrightarrow e_3 A \xrightarrow{\delta_2 = \begin{pmatrix} \alpha_3^* \\ -\alpha_2 \\ -\alpha_1 \end{pmatrix}} e_4 A \oplus e_2 A \oplus e_1 A \xrightarrow{\delta_1 = \begin{pmatrix} \alpha_3 & \alpha_2^* & \alpha_1^* \end{pmatrix}} e_3 A \xrightarrow{\delta_0} S_3 \longrightarrow 0$$

$$0 \longrightarrow e_{n-2} A \xrightarrow{\delta_2 = \begin{pmatrix} \alpha_{n-1}^* \\ \alpha_n^* \\ -\alpha_{n-3} \end{pmatrix}} e_{n-1} A \oplus e_n A \oplus e_{n-3} A \xrightarrow{\delta_1 = \begin{pmatrix} \alpha_{n-1} & \alpha_n & \alpha_{n-3}^* \end{pmatrix}} e_{n-2} A \xrightarrow{\delta_0} S_{n-2} \longrightarrow 0$$

By symmetry, it is enough to deal with the case  $i = 3$ . Applying  $\underline{\text{Hom}}(-, A)$  to the minimal projective resolution and applying Lemma 2.3.4 yields

$$0 \longrightarrow Ae_3 \xrightarrow{\delta_1^* = \begin{pmatrix} \alpha_3 & \alpha_2^* & \alpha_1^* \end{pmatrix}} Ae_4 \oplus Ae_2 \oplus Ae_1 \xrightarrow{\delta_2^* = \begin{pmatrix} \alpha_3^* \\ -\alpha_2 \\ -\alpha_1 \end{pmatrix}} Ae_3 \longrightarrow 0.$$

Identically to the cases before,  $\delta_1^*$  is injective and the proof of [Boc08, Theorem 3.2] says this is the minimal projective resolution of  $S_3^*$ . Therefore,  $\underline{\text{Ext}}^j(S_3, A) = \delta_{jd} S_3^*$ .

Finally, the cases  $\widetilde{E}_m$  for  $m = 6, 7, 8$  are left where the underlying quivers are as described in Figure 2.3.1. To produce the preprojective algebras, one has to consider the

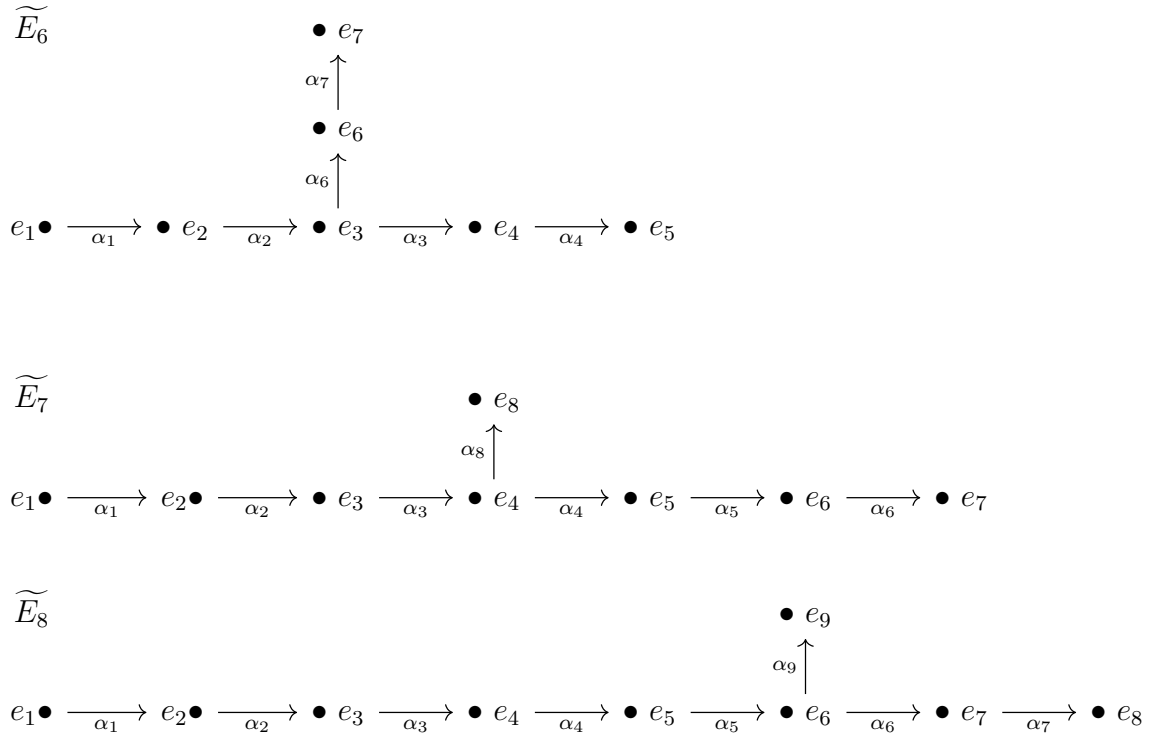


**Figure 2.3.1:** Dynkin diagrams of  $\widetilde{E}_m$  for  $m = 6, 7, 8$ .

doubles, construct the path algebras and factor out the following relations:

$$\begin{aligned}
\widetilde{E}_6 : \quad & \alpha_1 \alpha_1^* = \alpha_4^* \alpha_4 = \alpha_7^* \alpha_7 = 0, \\
& \alpha_3 \alpha_3^* + \alpha_6 \alpha_6^* = \alpha_2^* \alpha_2, \quad \text{and} \\
& \alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 4, 7. \\
\widetilde{E}_7 : \quad & \alpha_1 \alpha_1^* = \alpha_6^* \alpha_6 = \alpha_8^* \alpha_8 = 0, \\
& \alpha_4 \alpha_4^* + \alpha_8 \alpha_8^* = \alpha_3^* \alpha_3, \quad \text{and} \\
& \alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 3, 5, 6.
\end{aligned}$$

$$\begin{aligned}
\widetilde{E}_8 : \quad & \alpha_1 \alpha_1^* = \alpha_7^* \alpha_7 = \alpha_9^* \alpha_9 = 0, \\
& \alpha_6 \alpha_6^* + \alpha_9 \alpha_9^* = \alpha_5^* \alpha_5, \quad \text{and} \\
& \alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 3, 4, 5, 7.
\end{aligned}$$

All these relations are of the same form as in the case  $\widetilde{D}_{n-1}$  and the minimal projective resolutions as well as the arguments using the proof of [Boc08, Theorem 3.2] are identical. It follows that  $\text{gl. dim}(A) = 2 = d$  and  $\underline{\text{Ext}}^j(S_i, A) \cong \delta_{jd} S_i^*$  for  $A = \Pi_Q(k)$  for  $Q = \widetilde{E}_6, \widetilde{E}_7$  and  $\widetilde{E}_8$ . This finishes the proof of part (a) and part (b) can be deduced from Proposition 2.3.8.  $\square$

Most of the main theory developed in Chapters 5 and 6 is worked out for the more general class of algebras which are generalized Gorenstein. Based on [JZ00, Definition 0.2], the following definition arises naturally.

**Definition 2.3.10.** A noetherian  $N$ -graded algebra  $B$  with degree zero piece  $B_0 \cong k^n$  and primitive orthogonal idempotents  $\{e_1, \dots, e_n\}$  is called *generalized Gorenstein* if it satisfies:

- (1)  $\text{injdim}_B(B) = N < \infty$  and
- (2)  $\underline{\text{Ext}}_B^i(S_j, B) \cong \delta_{Ni} S_{\sigma(j)}^*$  for some  $\sigma \in \text{Sym}(n)$  and the simple right  $B$ -modules  $S_i = e_i B / (e_i B)_{\geq 1}$  and simple left  $B$ -modules  $S_i^* = B e_i / (B e_i)_{\geq 1}$ , respectively.  
(*generalized Gorenstein condition*)

Even weaker is the term AS-Cohen-Macaulay taken from [JZ00, Definition 0.1].

**Definition 2.3.11.** An algebra  $R$  is called *AS-Cohen-Macaulay* if there exists an integer  $N$  such that for  $i \neq N$  it follows that  $H_{\mathfrak{n}}^i(R) = H_{\mathfrak{n}^{\text{op}}}^i(R) = 0$  where  $\mathfrak{n} = R_{\geq 1}$  and the local cohomology  $H_{\mathfrak{n}}^i(R) = \varinjlim \text{Ext}^i({}_R(R/R_{\geq n}), R_R)$ .

As mentioned before, for the non-connected preprojective algebras it very often is not enough to only consider the Hilbert series and trace as defined in Definition 2.1.2. Instead, for automorphisms fixing the idempotents, one has to focus on traces with respect to the different indecomposable projective modules as introduced in Definition 2.3.13.

**Definition 2.3.12.** Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Let  $g \in \text{Aut}_{\text{gr}}(B)$  be a graded automorphism of  $B$  such that  $g(e_i) = e_i$  for every  $i = 1, \dots, n$ . Notice that  $e_i B$  is a projective right  $B$ -module and for each  $e_i$ , there is an induced map

$$g_i : e_i B \longrightarrow e_i B, \quad e_i \beta \mapsto e_i g(\beta),$$

denoted by  $g_i$ . Further,  $g_i = g|_{e_i B}$  is a  $g$ -linear right  $B$ -module isomorphism from  $e_i B$  to  $e_i B$ .

**Definition 2.3.13.** Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . The *matrix Hilbert series*  $\underline{H}_{\mathbf{M}_B}(t)$  or  $\underline{H}_{\mathbf{M}}(t)$  is the matrix with entries

$$[\underline{H}_{\mathbf{M}_B}(t)]_{ij} = [\underline{H}_{\mathbf{M}}(t)]_{ij} = \sum_{s=0}^{\infty} (\dim_k e_i B_s e_j) t^s.$$

For  $g \in \text{Aut}_{\text{gr}}(B)$  with  $g(e_i) = e_i$ , the *matrix trace*  $\underline{\text{Tr}}_{\mathbf{M}_B}(g, t)$  or  $\underline{\text{Tr}}_{\mathbf{M}}(g, t)$  (*vector traces*  $\overline{\text{Tr}}_B(g, t)$  and  $\overline{\text{Tr}}_B(g, t)$  or  $\underline{\text{Tr}}(g, t)$  and  $\overline{\text{Tr}}(g, t)$ , respectively) is the matrix (vector) with

entries

$$\begin{aligned} [\underline{\text{Tr}}_{\mathbf{M}_B}(g, t)]_{ij} &= [\underline{\text{Tr}}_{\mathbf{M}}(g, t)]_{ij} = \sum_{s=0}^{\infty} \text{tr}(g|_{e_i B_s e_j}) t^s, \\ [\underline{\text{Tr}}_B(g, t)]_j &= [\underline{\text{Tr}}(g, t)]_j = \sum_{s=0}^{\infty} \text{tr}(g|_{e_j B_s}) t^s, \quad \text{and} \\ [\overline{\text{Tr}}_B(g, t)]_j &= [\overline{\text{Tr}}(g, t)]_j = \sum_{s=0}^{\infty} \text{tr}(g|_{B_s e_j}) t^s. \end{aligned}$$

The following lemma sums up Molien's Theorem [JZ97, Lemma 5.2] and some of its applications. The first part, for instance, will be used to connect local cohomology of the fixed ring to the preprojective algebra. The second part guarantees that all our algebras are noetherian. Molien's Theorem itself will be helpful to calculate the examples in Section 4.2 and to prove Theorem 7.2.3.

**Lemma 2.3.14.** *Let  $R$  be an  $\mathbb{N}$ -graded ring and  $G$  be a finite subgroup of  $\text{Aut}_{\text{gr}}(R)$ . The fixed ring equals  $R^G = \{x \in R \mid g(x) = x, \text{ for all } g \in G\}$ . Then:*

(a) [JZ97, Lemma 5.1].  $R \cong R^G \oplus C$  as  $(R^G, R^G)$ -bimodules, where  $C$  is the kernel of

$$\pi_G : x \mapsto \frac{1}{|G|} \sum_{g \in G} g(x).$$

(b) [Mon80, Corollary 1.12]. If  $R$  is right noetherian, then  $R^G$  is right noetherian.

(c) Molien's Theorem, [JZ97, Lemma 5.2]. The Hilbert series of the fixed ring equals:

$$H_{R^G}(t) = \frac{1}{|G|} \sum_{g \in G} \text{Tr}_R(g, t).$$

With this lemma there is an alternative way to show that  $A$  and  $A^G$  satisfy the  $\chi$ -condition as follows.

**Corollary 2.3.15.** *Let  $G$  be a finite group acting on  $A$  which fixes the vertices. Then  $A^G$  satisfies the  $\chi$ -condition.*

*Proof.* By Proposition 2.3.9(f) we have  $A$  is noetherian and so by Lemma 2.3.14(b) also  $A^G$  is noetherian. Proposition 2.3.9(g) makes both  $A$  and  $A^G$  into PI algebras and so [AZ94, Proposition 5.1] says  $A^G$  satisfies the  $\chi$ -condition.  $\square$

The aforementioned Molien's Theorem has a straightforward generalization to the language of matrix Hilbert series and matrix traces:

**Theorem 2.3.16.** *Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra with degree zero piece  $B_0 \cong k^n$ . Denote the orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Let  $G$  be a finite subgroup of  $\text{Aut}_{\text{gr}}(B)$  in which every  $g \in G$  satisfies  $g(e_i) = e_i$  for all  $i$ . Then:*

$$\begin{aligned} \underline{H}_{\mathbf{M}_{B^G}}(t) &= \frac{1}{|G|} \sum_{g \in G} \underline{\text{Tr}}_{\mathbf{M}_B}(g, t), \\ H_{e_j B^G}(t) &= \frac{1}{|G|} \sum_{g \in G} [\underline{\text{Tr}}_B(g, t)]_j. \end{aligned}$$

We conclude this introduction by stating two important results. First, under weak conditions the ring is always finitely generated as a module over its fixed ring and second, the fixed ring satisfies the  $\chi$ -condition if the ring satisfies this property.

**Proposition 2.3.17.** ([Mon80, Corollary 5.9]) *Let  $R$  be a right (left) noetherian ring,  $G$  be a finite subgroup of  $\text{Aut}(R)$  such that  $|G|^{-1} \in R$ . Then  $R$  is a finitely generated right (left)  $R^G$ -module.*

**Proposition 2.3.18.** ([AZ94, Remark after Proposition 8.7]) *Let  $R$  be a noetherian graded ring and  $G$  be a finite subgroup of  $\text{Aut}_{\text{gr}}(R)$ . If  $R$  satisfies the  $\chi$ -condition then so does  $R^G$ .*

*Convention 2.3.19.* To sum up let's state the most important conventions. Throughout let  $k$  be an algebraically closed field of characteristic 0. Unless said otherwise, every algebra is noetherian,  $\mathbb{N}$ -graded with degree zero piece isomorphic to  $k^n$  and locally finite. Our convention is to use right modules. Every automorphism obeys the grading and all groups  $G$  and automorphisms  $g$  have finite order. Moreover, every  $g \in G$  fixes the idempotents. Unless said otherwise,  $A = \Pi_Q(k)$  for an extended Dynkin quiver  $Q$  with vertices  $\{e_1, \dots, e_n\}$ . On  $A$ , all automorphisms act by extending scalar multiplication on the arrows of degree 1. Unless said otherwise,  $B$  denotes a noetherian  $\mathbb{N}$ -graded generalized Gorenstein algebra with  $B_0 \cong k^n$  such that  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition. To simplify notation, the primitive orthogonal idempotents of  $B$  are also denoted by  $\{e_1, \dots, e_n\}$ . The modules  $S_i$  and  $S_i^*$  denote the simple right and left modules of the algebra we focus on. Call  $\mathfrak{m} = A_{\geq 1}$ ,  $\mathfrak{r} = B_{\geq 1}$  and  $\mathfrak{s} = (B^G)_{\geq 1}$ . The fixed ring is denoted by

$$B^G = \{b \in B \mid g(b) = b, \text{ for all } g \in G\}.$$

This chapter contains material from “Properties of the fixed ring of a preprojective algebra”, Weispfenning, Stephan, to appear and “Generalized Gorensteinness and a homological determinant for preprojective algebras”, Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of both papers.

# Chapter 3

## Quasi-reflections of AS-regular algebras of small dimensions

### 3.1 Quasi-reflections of AS-regular algebras of global dimension 2

It is well-known (see [Rog14, Theorem 2.18]) that an Artin-Schelter regular algebra of global dimension 2 is isomorphic to either

- (i) the Jordan plane  $k\langle x, y \rangle / (yx - xy - x^2)$  or
- (ii) the quantum plane  $k\langle x, y \rangle / (yx - qxy)$  for some  $q \in k^*$ .

In this first warm-up section the occurrence of quasi-reflections of finite order of Artin-Schelter regular algebras of global dimension 2 is discussed. It turns out that the Jordan

plane does not have any quasi-reflections of finite order. In light of Theorem 2.2.4 this implies that the Jordan plane has no fixed subring under a finite group action which has finite global dimension. For the quantum plane the situation is different. Since it is a quantum polynomial ring the two possible types of quasi-reflections are mystic reflections and classical reflections by Theorem 2.2.3. While any quantum plane has classical reflections, it turns out that mystic reflections arise precisely for  $q = -1$ .

**Proposition 3.1.1.** *If  $S = k\langle x, y \rangle / (yx - xy - x^2)$ , then  $S$  does not have any quasi-reflections of finite order.*

*Proof.* Let  $g \in \text{Aut}_{\text{gr}}(S)$  and define  $g(x) = a_{11}x + a_{21}y$  and  $g(y) = a_{12}x + a_{22}y$ . Then  $g$  applied to the relation  $yx - xy - x^2$  gives

$$\begin{aligned}
& (a_{12}x + a_{22}y)(a_{11}x + a_{21}y) - (a_{11}x + a_{21}y)(a_{12}x + a_{22}y) - (a_{11}x + a_{21}y)(a_{11}x + a_{21}y) \\
&= (a_{12}a_{21} - a_{11}a_{22} - a_{11}a_{21})xy + (a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{21})yx - a_{11}^2x^2 - a_{21}^2y^2 \\
&= (a_{12}a_{21} - a_{11}a_{22} - a_{11}a_{21} + a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{21})xy \\
&\quad + (a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{21} - a_{11}^2)x^2 - a_{21}^2y^2 \\
&= -2a_{11}a_{21}xy + (a_{11}a_{22} - a_{12}a_{21} - a_{11}a_{21} - a_{11}^2)x^2 - a_{21}^2y^2.
\end{aligned}$$

Since  $x^2, xy, y^2$  are linearly independent,  $a_{21} = 0$  and  $a_{11}(a_{22} - a_{11}) = 0$  by looking at the coefficients of  $y^2$  and  $x^2$ . This simplifies the determinant of  $g|_{S_1}$  to  $a_{11}a_{22} \neq 0$  and therefore in order to achieve  $a_{11}(a_{22} - a_{11}) = 0$  we must have  $a_{11} = a_{22} \neq 0$ . As a side

result we notice that

$$\mathrm{Aut}_{\mathrm{gr}}(S) = \left\{ \begin{pmatrix} \lambda & a_{12} \\ & \lambda \end{pmatrix} \middle| \lambda \in k^*, a_{12} \in k \right\}.$$

If  $g$  has finite order this forces  $a_{12} = 0$  as one can see from the following formula for the powers and using that the characteristic of  $k$  is 0:

$$\begin{pmatrix} \lambda & a_{12} \\ & \lambda \end{pmatrix}^m = \begin{pmatrix} \lambda^m & m\lambda^{m-1}a_{12} \\ & \lambda^m \end{pmatrix}.$$

Let  $g \in \mathrm{Aut}_{\mathrm{gr}}(S)$  be a quasi-reflection of  $S$  of finite order. From above we get that  $g(x) = \lambda x$  and  $g(y) = \lambda y$ . We use [KKZ08, Lemma 1.6 (d)] to obtain

$$\mathrm{Tr}(g, t) = \frac{1}{(1-t)(1-\zeta t)} = 1 + (1+\zeta)t + \dots$$

and comparing traces of  $g|_{S_1}$  gives  $\lambda + \lambda = 1 + \zeta$ . Clearly,  $\lambda$  must be a root of unity and [KKZ08, Lemma 1.6 (e)] says the same about  $\zeta$ . Therefore, [KKZ08, Lemma (3.2)] says that if  $2\lambda - \zeta = 1$  has a solution then one of  $\{\lambda, -\zeta\}$  must be equal to 1. It is left to consider the cases where  $\zeta = -1$  or  $\lambda = 1$ : If  $\lambda = 1$  then  $g$  is the identity which is excluded from the definition of a quasi-reflection. If  $\zeta = -1$  then  $\lambda = 0$  and  $g$  is not an automorphism.  $\square$

As a consequence, we know the following for fixed rings of a finite group acting on a Jordan plane.

**Corollary 3.1.2.** *Let  $S = k\langle x, y \rangle / (yx - xy - x^2)$ . Then:*

(a) *For every finite group  $G \leq \mathrm{Aut}_{\mathrm{gr}}(S)$  the fixed ring  $S^G$  has infinite global dimension.*

(b) For every finite group  $G \leq \text{Aut}_{\text{gr}}(S)$  the fixed ring  $S^G$  is not isomorphic to  $S$ .

*Proof.* Every finite subgroup  $G$  of  $\text{Aut}_{\text{gr}}(S)$  does not contain any quasi-reflections by Proposition 3.1.1. Therefore, [KKZ08, Theorem 2.4] implies part (a). If  $S^G$  is isomorphic to  $S$ , then  $S^G$  has finite global dimension which is impossible by (a). This proves (b).  $\square$

For the remainder of this section, we will understand the quasi-reflections of finite order of the quantum plane. First, we find its graded automorphism group. From there it is an easy calculation to check that if  $q \neq -1$  only classical reflections occur. At last, we conclude for  $q = -1$  that both types of quasi-reflections come up.

**Proposition 3.1.3.** *If  $S = k\langle x, y \rangle / (yx - qxy)$  with  $q^2 \neq 1$  then  $\text{Aut}_{\text{gr}}(S) = k^* \times k^*$  acting separately on  $x$  and  $y$ .*

*Proof.* Let  $g \in \text{Aut}_{\text{gr}}(S)$  and define  $g(x) = a_{11}x + a_{21}y$  and  $g(y) = a_{12}x + a_{22}y$ . Applying  $g$  to the relation  $yx - qxy$  gives

$$\begin{aligned} & (a_{12}x + a_{22}y)(a_{11}x + a_{21}y) - q(a_{11}x + a_{21}y)(a_{12}x + a_{22}y) \\ &= a_{11}a_{12}(1 - q)x^2 + a_{12}a_{21}xy - qa_{12}a_{21}yx + a_{21}a_{22}(1 - q)y^2 \\ &= a_{11}a_{12}(1 - q)x^2 + a_{12}a_{21}(1 - q^2)xy + a_{21}a_{22}(1 - q)y^2. \end{aligned}$$

As  $x^2, xy$  and  $y^2$  are linearly independent and  $(1 - q) \neq 0$  as well as  $(1 - q^2) \neq 0$ , we get  $a_{12}a_{21} = 0$  from the coefficient of  $xy$ . This implies  $a_{11} \neq 0 \neq a_{22}$  and therefore from the coefficients of  $x^2$  and  $y^2$  we get that both  $a_{12} = a_{21} = 0$ . Clearly, acting separately on  $x$  and  $y$  by a nonzero scalar gives an automorphism and this finishes the proof.  $\square$

**Corollary 3.1.4.** *Let  $S = k\langle x, y \rangle / (yx - qxy)$  with  $q^2 \neq 1$ . If  $g \in \text{Aut}_{\text{gr}}(S)$  is a quasi-reflection of finite order then  $g$  is a classical reflection. In other words,  $S$  does not have any mystic reflections.*

*Proof.* From Proposition 3.1.3 we obtain  $g(x) = a_{11}x$  and  $g(y) = a_{22}y$  where  $a_{11}$  and  $a_{22}$  are roots of unity. As before in Proposition 3.1.1, we use the trace function to obtain the equation  $a_{11} + a_{22} = 1 + \zeta$  for some root of unity  $\zeta$ . Then [KKZ08, Lemma (3.2)] gives us that this equation does not have a solution if all of  $\{a_{11}, a_{22}, -\zeta\}$  are not equal to 1. If  $\zeta = -1$  then  $a_{11} = -a_{22}$  and so computing the trace of the action of  $g$  on  $S_2$  in two ways gives

$$a_{11}^2 + a_{11}a_{22} + a_{22}^2 = 1 + \zeta + \zeta^2 = 1$$

and so  $a_{22}^2 = 1$  is saying  $a_{22} = \pm 1$ . In either case, one of the  $a_{ii}$  is 1 and so  $g$  is a classical reflection. If without loss of generality  $a_{11} = 1$  then  $a_{22} = \zeta$  and  $g$  is a classical reflection. □

**Proposition 3.1.5.** *Let  $S = k\langle x, y \rangle / (yx + xy)$ . Then  $S$  does have classical reflections and mystic reflections.*

*Proof.* From the proof of Proposition 3.1.3 we obtain  $2a_{11}a_{12} = 0 = 2a_{21}a_{22}$  using the coefficients of  $x^2$  and  $y^2$  together with  $q = -1$ . This implies that either  $a_{12} = a_{21} = 0$  or  $a_{11} = a_{22} = 0$  as the determinant of  $g|_{S_1}$  has to be nonzero. Now assume that  $g$  is a quasi-reflection of finite order.

If  $a_{12} = a_{21} = 0$  we can again compute traces and find that  $a_{11} + a_{22} = 1 + \zeta$  as

well as  $a_{11}^2 + a_{11}a_{22} + a_{22}^2 = 1 + \zeta + \zeta^2$ . The same analysis as in Corollary 3.1.4 gives that only classical reflections are possible. Clearly, any map  $g(x) = x$  and  $g(y) = \zeta y$  for some root of unity  $\zeta$  forms a graded isomorphism of  $S$ . A simple calculation verifies that

$$\mathrm{Tr}(g, t) = 1 + (1 + \zeta)t + (1 + \zeta + \zeta^2)t^2 + \dots = \frac{1}{(1 - t)(1 - \zeta t)}$$

and so  $g$  is a classical reflection. Analogously, one obtains a classical reflection by fixing  $y$  and scalar multiplying  $x$  by a root of unity.

If  $a_{11} = a_{22} = 0$  computing the trace yields  $0 = 1 + \zeta$  and thus  $\zeta = -1$ . Computing the trace of the action of  $g$  on  $S_2$  gives  $-a_{12}a_{21} = 1 + \zeta + \zeta^2 = 1$  and so  $a_{21} = -1/a_{12}$ . Thus the characteristic polynomial of  $g|_{S_1}$  is  $t^2 + 1$  and so the eigenvalues are  $i$  and  $-i$ . In other words,  $g$  is a mystic reflection. An example for this case to happen can be found in [Kir16, Example 1.8. (3)]. □

## 3.2 Quasi-reflections of AS-regular algebras of global dimension 4 generated by three elements

In [KKZ08, Proposition 6.4] it was shown that Artin-Schelter regular algebras of global dimension 3 generated by two elements in degree 1 do not admit any quasi-reflections of finite order. Therefore, we focus on Artin-Schelter regular algebras of global dimension 4 generated by three elements next. In particular, we investigate domains with a  $\mathbb{Z}^{\times 2}$ -grading classified in [RZ12, Theorem (0.1)].

Recall, that an element  $x$  of a ring  $R$  is called *normal* if  $xR = Rx$ . Additionally,  $x$  is called *regular* if is a non-zero-divisor of  $R$ . An Artin-Schelter regular domain  $S$  of global dimension  $d$  is called a normal extension if  $S/(x)$  for some normal regular element  $x$  is Artin-Schelter regular of global dimension  $d - 1$ . Thanks to the classification, we have to only deal with normal extensions and eight additional families. Our analysis will show that no quasi-reflections exist for the eight families. Further, a normal extension can only have a quasi-reflection in a very specific case.

**Definition 3.2.1.** We call an AS-regular algebra and domain  $S$  with global dimension  $d$  which also has a normal regular element  $x$  of degree 1 such that  $S/(x)$  is AS-regular of global dimension  $d - 1$  a *normal extension of global dimension  $d$* .

**Lemma 3.2.2.** *Let  $S$  be AS-regular of global dimension 4 which is generated by three elements in degree 1. Let  $x \in S_1$  be a normal regular element and  $g \in \text{Aut}_{\text{gr}}(S)$ . Then  $g(x) = \lambda x$  for some nonzero  $\lambda \in k$ .*

*Proof.* Using [LPWZ07, Proposition (1.4)], we know that

$$H_S(t) = \frac{1}{(1-t)^3(1-t^2)} = 1 + 3t + 7t^2 + 13t^3 + \text{higher terms}.$$

Assume that  $g(x) = y \in S_1$  is linearly independent from  $x$  and let  $\{x, y, z\}$  be a basis of  $S_1$ . As both  $x$  and  $y$  are regular normal elements, a subset of  $\{x^2, xy, xz, y^2, yz, z^2\}$  forms a basis for  $S_2$ . This contradicts  $\dim S_2 = 7$ .  $\square$

Now that we understand how a graded automorphism  $g$  acts on a normal regular element of degree 1 we need to connect the trace function of  $g$  to the trace function of the induced automorphism on the factor ring.

**Lemma 3.2.3.** *Let  $S$  be a normal extension of global dimension  $d$ , let  $x$  be a normal regular element as in 3.2.1 and denote  $S/(x)$  by  $B$ . Let  $g$  be a graded automorphism of  $S$  such that  $g(x) = \lambda x$  for some nonzero  $\lambda \in k$ . The induced graded automorphism on  $B$  is denoted by  $\tilde{g}$ . Then it is true that*

$$\text{Tr}_B(\tilde{g}, t) = (1 - \lambda t) \text{Tr}_S(g, t).$$

*Proof.* Using the normality of  $x$  we understand  $S_\ell$  as  $\sum_{k=0}^{\ell} x^k B_{\ell-k}$ . Then

$$\begin{aligned} \text{Tr}_S(g, t) &= \sum_{\ell=0}^{\infty} \text{tr}(g|_{S_\ell}) t^\ell = \sum_{\ell=0}^{\infty} \left( \sum_{k=0}^{\ell} \lambda^k \text{tr}(\tilde{g}|_{B_{\ell-k}}) \right) t^\ell \\ &= \left( \sum_{k=0}^{\infty} (\lambda t)^k \right) \left( \sum_{\ell=0}^{\infty} \text{tr}(\tilde{g}|_{B_\ell}) t^\ell \right) = \frac{\text{Tr}_B(\tilde{g}, t)}{1 - \lambda t} \end{aligned}$$

finishes the proof.  $\square$

The following proposition describes how a quasi-reflection  $g$  can occur if our algebra is a normal extension. More precisely, the algebra must then be an Ore extension  $B[x; \sigma]$  where  $g$  fixes  $B$  pointwise.

**Proposition 3.2.4.** *Let  $S$  be a normal extension of global dimension 4 which is generated by three elements in degree 1 and let  $x \in S_1$  be a normal regular element as in 3.2.1. Let  $g$  be a quasi-reflection of finite order. Then the induced automorphism  $\tilde{g}$  on  $B := S/(x)$  is the identity and  $S \cong B[x; \sigma]$ .*

*Proof.* Recall that  $g$  is not the identity. By definition,  $B = S/(x)$  is AS-regular of global dimension 3 generated by two elements. By [Rog14, Lemma (2.22)] its Hilbert series is

$$H_B(t) = \frac{1}{(1-t)^2(1-t^2)}.$$

Let  $g$  be a quasi-reflection of finite order. By [KKZ08, Definition (2.2) and Lemma 1.6 (d) and (e)] we have

$$\mathrm{Tr}_S(g, t) = \frac{1}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)}$$

where  $\zeta_1$  and  $\zeta_2$  are roots of unity. Additionally, Lemma 3.2.2 implies  $g(x) = \lambda x$  for some root of unity  $\lambda$ . So  $g$  induces a graded automorphism  $\tilde{g}$  on  $B$  whose trace computes as

$$\mathrm{Tr}_B(\tilde{g}, t) = (1 - \lambda t) \mathrm{Tr}_S(g, t) = \frac{1 - \lambda t}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)}$$

using Lemma 3.2.3. By [JZ97, Theorem (2.3) 4.],  $\mathrm{Tr}_B(\tilde{g}, t)$  is of the form  $1/p(t)$ . Therefore, we have the two cases  $\lambda = 1$  and  $\lambda = \zeta_2$ .

Suppose  $\lambda = 1$ . This implies

$$\mathrm{Tr}_B(\tilde{g}, t) = \frac{1}{(1-t)^2(1-\zeta_1 t)(1-\zeta_2 t)}$$

and so  $\tilde{g}$  is a quasi-reflection of  $B$ . This is a contradiction to [KKZ08, Proposition (6.4)].

This means for the remainder of this proof we suppose  $\lambda = \zeta_2$ . This implies

$$\mathrm{Tr}_B(\tilde{g}, t) = \frac{1}{(1-t)^3(1-\zeta_1 t)}.$$

We notice that  $\mathrm{GKdim}(B) = 3$  equals the multiplicity of 1 of the Euler polynomial of  $\tilde{g}$ . Therefore, [KKZ08, Proposition (1.8)] provides that  $\tilde{g} = \mathrm{id}$  which gives the first part of the proposition. This implies that  $\mathrm{Tr}_B(\tilde{g}, t) = H_B(t)$  saying  $\zeta_1 = -1$ . Thus, we obtain

$$\begin{aligned} \mathrm{Tr}_S(g, t) &= \frac{1}{(1-t)^3(1+t)(1-\zeta_2 t)} = \frac{1}{(1-t)^3(1+t)(1-\lambda t)} \\ &= 1 + (2 + \lambda)t + (4 + 2\lambda + \lambda^2)t^2 + \text{higher terms.} \end{aligned}$$

Due to  $\tilde{g}$  being the identity, we know that we can choose a basis  $\{x, y, z\}$  of  $S_1$  such that  $g(x) = \lambda x$ ,  $g(y) = y + c_y x$  and  $g(z) = z + c_z x$ . This observation fixes the eigenvalues of  $g$  and we can replace  $y$  and  $z$  to obtain a basis  $\{x, y, z\}$  of  $S_1$  such that  $g(x) = \lambda x$ ,  $g(y) = y$  and  $g(z) = z$ .

By [LPWZ07, Proposition 1.4] we know that  $S$  is minimally presented with two relations of degree 2 and two relations of degree 3. Normality of  $x$  together with the form of  $g$  forces the two quadratic relations to be

$$r_1 : \quad xy = a_{11}yx + a_{12}zx, \quad \text{and} \tag{3.1}$$

$$r_2 : \quad xz = a_{21}yx + a_{22}zx. \tag{3.2}$$

From the Hilbert series we get  $\dim(S_3) = 13$  and  $\text{tr}(g|_{S_3}) = \lambda^3 + 2\lambda^2 + 4\lambda + 6$ . Using the normality of  $x$  we know the following fifteen elements span  $S_3$ :

$$x^3, x^2y, x^2z, xy^2, xz^2, xyz, xzy, y^3, y^2z, yz^2, yzy, z^3, z^2y, zy^2, zyz.$$

As we know how  $g$  acts on the basis and we already know the trace, the cubic relations of  $S$  are limited to the latter eight elements including only  $y$  and  $z$  and so the two relations are identical to the relations determining an AS-regular algebra of global dimension 3 generated by two elements. Thus, the subalgebra  $C$  of  $S$  generated by  $y$  and  $z$  is isomorphic to  $B$ .

Due to  $x$  being normal, we can always write  $f = a_0 + a_1x + \dots + a_mx^m$  for some  $a_i \in C$  whenever  $f \in S$ . Define  $d(f) = \max\{i \mid a_i \neq 0\}$ . We also know from the normality of  $x$  that  $\dim_k(S_\ell) = \sum_{j=0}^{\ell} \dim_k(C_j)$ . Suppose the expression of  $f$  is not unique, then this equality would become ' $<$ ' which gives a contradiction.

Clearly,  $d(f) \geq 0$  and  $d(f - g) \leq \max\{d(f), d(g)\}$ . Additionally, as  $S$  is a domain and  $x$  is normal,  $d(fg) = d(f) + d(g)$  for all  $f, g \in S$ . Moreover, normality allows us to write  $xa = a^\sigma x$  for any  $a \in C$  and some automorphism  $\sigma$ .

Now [Coh89, Chapter 12.2, Theorem 1] applies. To sum up,  $S = C[x; \sigma, \delta]$  and the form of  $r_1$  and  $r_2$  implies  $\sigma(y) = a_{11}y + a_{12}z$  and  $\sigma(z) = a_{21}y + a_{22}z$  as well as  $\delta \equiv 0$ . Finally,  $C[x; \sigma] \cong B[x; \sigma]$  finishes the proof.  $\square$

It is easy to see that the situation described in Proposition 3.2.4 really occurs.

**Example 3.2.5.** Let  $B$  be any AS-regular algebra of global dimension 3 generated by

2 elements. Then for  $B[x]$ , the automorphism  $g$  defined by  $g(b) = b$  for all  $b \in B$  and  $g(x) = \zeta_m x$  where  $\zeta_m$  is a nontrivial  $m$ -th root of 1 defines a quasi-reflection of  $B[x]$ .

The other possibility for a  $\mathbb{Z}^{\times 2}$ -graded Artin-Schelter regular domain generated by three elements in degree 1 is to be one of eight 1 or 2-parameter families. In order to show that no quasi-reflections exist in this case it is necessary to examine the trace function. The following lemma shows that for a quasi-reflection of finite order only five different types of trace functions occur.

**Lemma 3.2.6.** *Let  $S$  be AS-regular of global dimension 4 which is generated by three elements in degree 1. Let  $g$  be a quasi-reflection of  $S$  of finite order whose trace is*

$$\mathrm{Tr}_S(g, t) = \frac{1}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)},$$

where  $\zeta_1, \zeta_2$  are roots of unity not equal to 1 ([KKZ08, Definition 2.2 and Lemma 1.6 (d) and (e)]). Further, let  $\{x_1, x_2, x_3\}$  be a basis consisting of eigenvectors of  $g$ , say  $g(x_i) = \lambda_i x_i$  for  $i = 1, 2, 3$ . Then up to interchanging  $\zeta_1$  and  $\zeta_2$  and relabeling the  $x_i$ , the only possibilities are:

- (i)  $\zeta_1 = -1 = \zeta_2$  and  $\lambda_3 = 1$  and  $\lambda_1 = -\lambda_2$  is a root of unity not equal to  $\pm 1$ .
- (ii)  $\zeta_1 = -1 = \zeta_2$  and  $\lambda_3 = 1 = \lambda_2$  and  $\lambda_1 = -1$ .
- (iii)  $\zeta_2 = -1$  and two of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1$  are  $\xi_6 := \exp(2\pi i/6)$  and the other two are  $\bar{\xi}$ .
- (iv)  $\zeta_2 = -1$  and  $\lambda_2 = \lambda_3 = 1$  and  $\lambda_1 = \zeta_1 \neq \pm 1$ .

(v)  $\lambda_3 = 1$  and two of  $\lambda_1, \lambda_2, -\zeta_1, -\zeta_2$  are  $\xi_6 := \exp(2\pi i/6)$  and the other two are  $\bar{\xi}$ .

*Proof.* Before we start, we recall that the identity is by definition not a quasi-reflection.

We compute the trace of  $g|_{S_1}$  in two ways. First, we use our basis  $\{x_1, x_2, x_3\}$  and second, we use  $\text{Tr}_S(g, t)$  yielding

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 + \zeta_1 + \zeta_2 \Leftrightarrow \lambda_1 + \lambda_2 + \lambda_3 - \zeta_1 - \zeta_2 = 3. \quad (3.3)$$

We consider cases how many of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1.

- Assume that precisely zero of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1. Then Equation (3.3) is of the form

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 3$$

with all  $\alpha$ 's roots of unity not equal to 1 which by [KKZ08, Lemma (3.2)] does not have a solution.

- Assume that precisely one of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equals 1. First, suppose  $\zeta_2 = -1$ . This implies by [KKZ08, Lemma (3.2)] that two of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1$  are  $\xi_6$  and the other two are  $\bar{\xi}_6$  which is case (iii). Second, suppose  $\lambda_1 = 1$ . Here, two of  $\lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  are  $\xi_6$  and the other two are  $\bar{\xi}_6$  which is case (v).
- Assume that precisely two of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1. Then Equation (3.3) reduces to

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

with all  $\alpha$ 's roots of unity not equal to 1 which by [KKZ08, Lemma (3.2)] does not have a solution.

- Assume that precisely three of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1. First, suppose both  $\zeta$ 's are  $-1$  and  $\lambda_3 = 1$ . Then, we are in case (i). Second, suppose one of the  $\zeta$ 's, say  $\zeta_2$ , equals  $-1$ . This means without loss of generality  $\lambda_2 = \lambda_3 = 1$  and we are in case (iv). Finally, if all three  $\lambda_i = 1$  then  $g$  is the identity and we get a contradiction.
- Assume that precisely four of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1. If all three  $\lambda$ 's equal 1 then  $g$  must be the identity, a contradiction. If all but  $\lambda_1$  are 1, then  $\lambda_1 = -1$  and we are in case (ii).
- Assume that precisely five of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  equal 1. This immediately contradicts Equation (3.3).

This finishes the proof of the lemma. □

We are ready to prove that if  $S$  is no normal extension it does not have any quasi-reflections of finite order. In [RZ12, Theorem 0.1] graded automorphisms are described in detail. This information and the technique to calculate traces in two different ways rule out any quasi-reflections of finite order:

**Proposition 3.2.7.** *Let  $S$  be an AS-regular domain of global dimension 4 which is generated by three elements and properly  $\mathbb{Z}^{\times 2}$ -graded. Suppose  $S$  is not a normal extension of global dimension 4. Then  $S$  does not have a quasi-reflection of finite order.*

*Proof.* We know by [RZ12, Theorem 0.1] that  $S$  must be isomorphic to one of the eight 1 or 2-parameter families  $\mathcal{A}(b, q)$ ,  $\mathcal{B}(b)$ ,  $\mathcal{C}(b)$ ,  $\mathcal{D}(b)$ ,  $\mathcal{E}(b, \gamma_1)$ ,  $\mathcal{F}(b, \gamma_2)$ ,  $\mathcal{G}(b, \gamma_3)$  or  $\mathcal{H}(b)$  with  $\gamma_1 = \pm i$ ,  $\gamma_2 = \text{primitive } \sqrt[3]{1}$  and  $\gamma_3 = \frac{1 \pm i}{2}$ .

Suppose  $g \in \text{Aut}_{\text{gr}}(S)$  is a quasi-reflection of finite order. By [RZ12, Lemma (5.1)], we understand that either

(a)  $g$  is given by  $x_1 \mapsto \lambda x_1$ ,  $x_2 \mapsto \lambda x_2$ ,  $x_3 \mapsto \mu x_3$  for some nonzero  $\lambda, \mu \in k$  or

(b)  $g$  is given by  $x_1 \mapsto \lambda x_2$ ,  $x_2 \mapsto \rho x_1$ ,  $x_3 \mapsto \mu x_3$

where  $\{x_1, x_2, x_3\}$  is the basis given in the description of  $\mathcal{A}(b, q)$  -  $\mathcal{H}(b)$ . Additionally, case (b) does not happen for  $\mathcal{H}(b)$ .

We deal with each case separately. From the relations in [RZ12, Chapter 3] we obtain that  $\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3, x_3x_1\}$  is a basis of  $S_2$ . Also notice that we use the notation  $g(x_i) = \lambda_i x_i$  as in Lemma 3.2.6 as well as the description in (a).

Let's start with case (a). Suppose  $g$  is of type (a) with trace

$$\begin{aligned} \text{Tr}_S(g, t) &= \frac{1}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)} \\ &= 1 + (3 + \zeta_1 + \zeta_2)t + (6 + 3\zeta_1 + 3\zeta_2 + \zeta_1\zeta_2 + \zeta_1^2 + \zeta_2^2)t^2 + \text{higher terms.} \end{aligned}$$

Here, Lemma 3.2.6 applies and so we can do the cases (i) - (v).

(i) Case (i) cannot happen as all of  $\lambda_1, \lambda_2, \lambda_3$  are different. Type (a) requires  $x_1$  and  $x_2$  to be eigenvectors of the same eigenvalue.

- (ii) If  $\zeta_1 = \zeta_2 = -1$  and precisely two of  $\lambda_1, \lambda_2, \lambda_3$  are 1 then  $\lambda = 1$  and  $\mu = -1$ .

Comparing the traces of the action on  $S_2$  computed in two ways leads to

$$6 + 3\zeta_1 + 3\zeta_2 + \zeta_1\zeta_2 + \zeta_1^2 + \zeta_2^2 = 3 \neq 1.$$

This gives a contradiction.

- (iii) As two of  $\lambda_1, \lambda_2, \lambda_3, -\zeta_1, -\zeta_2$  must be  $\xi_6$  and  $\bar{\xi}_6$ , respectively, we see that  $\lambda = \xi_6$  and  $\mu = \bar{\xi}_6$  or  $\xi_6$  and  $\bar{\xi}_6$  flipped. Without loss of generality, we can pick  $\zeta_2 = -1$  and therefore  $-\zeta_1 = \mu = \bar{\xi}_6$ . As before, we can compute the traces in two different ways:

$$6 + 3\zeta_1 + 3\zeta_2 + \zeta_1\zeta_2 + \zeta_1^2 + \zeta_2^2 = 4 + \bar{\xi}_6^2 - 2\bar{\xi}_6 \neq 3 + 3\xi_6^2 + \bar{\xi}_6^2.$$

This gives a contradiction and the other case leads to the conjugates of the same equations.

- (iv) This case forces  $\zeta_2 = -1$ ,  $\lambda = 1$  and  $\mu = \zeta_1$ . The same strategy as before gives

$$6 + 3\zeta_1 + 3\zeta_2 + \zeta_1\zeta_2 + \zeta_1^2 + \zeta_2^2 = 4 + \mu^2 + 2\mu = 3 + 3\mu + \mu^2$$

and this results in  $\mu = 1$ . This is a contradiction as  $g$  is not the identity.

- (v) This case forces  $\mu = 1$  and either  $\lambda = \xi_6$  and  $\zeta_1 = \zeta_2 = -\bar{\xi}_6$  or  $\xi_6$  and  $\bar{\xi}_6$  flipped.

Computing the trace of  $g|_{S_2}$  gives

$$6 + 3\zeta_1 + 3\zeta_2 + \zeta_1\zeta_2 + \zeta_1^2 + \zeta_2^2 = 6 - 6\bar{\xi}_6 + 3\bar{\xi}_6^2 = 3\xi_6^2 + 3\xi_6 + 1$$

which leads to a contradiction. The other case gives the conjugate of the same equation.

Suppose  $g$  is of type (b). Here, we use the basis  $\{x_1, x_2, x_3\}$  as in (b) which does not consist of eigenvectors. The trace of  $g$  restricted to  $S_1$  computed in two ways gives

$$\mu = 3 + \zeta_1 + \zeta_2.$$

As  $\mu$ ,  $\zeta_1$  and  $\zeta_2$  are roots of unity, this leads to  $\mu = 1$  and  $\zeta_1 = \zeta_2 = -1$ . Therefore, we obtain

$$\text{Tr}_S(g, t) = \frac{1}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)} = 1 + t + 3t^2 + \text{higher terms}.$$

First, we can rule out the case  $\mathcal{F}$ . In this case we have  $x_2 x_1 = \gamma_2^2 x_1 x_2$  with  $\gamma_2$  a primitive third root of 1. Applying  $g$  to this equation reveals

$$\lambda \rho x_1 x_2 = \gamma_2^2 \lambda \rho x_2 x_1 = \gamma_2 \lambda \rho x_1 x_2.$$

Since  $\lambda$  and  $\rho$  are nonzero this is impossible.

We observe that applying  $g$  to the first relation of  $\mathcal{A}(b, q)$ ,  $x_2 x_1 = q^{-1} x_1 x_2$ , yields

$$\lambda \rho x_1 x_2 = q^{-1} \lambda \rho x_2 x_1 = q^{-2} \lambda \rho x_1 x_2$$

and as  $q \neq 1$  we get  $q = -1$ . Using this, we know that the first two relations of the remaining algebras  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$  and  $\mathcal{G}$  are always

$$\begin{aligned} r_1 : x_2 x_1 &= -x_1 x_2, \quad \text{and} \\ r_2 : x_3 x_2 &= -\frac{1}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3, \quad (b \neq 0). \end{aligned}$$

We compute the trace of  $g$  restricted to  $S_2$  using the basis of  $S_2$  from before. Written in order, we must have

$$\text{tr}(g)|_{S_2} = 0 + 0 + 1 - \lambda \rho + 0 + 0 - \frac{\lambda}{b} = 3. \tag{3.4}$$

Additionally,  $g$  needs to preserve  $r_2$ :

$$\begin{aligned}
r_2 : \quad & \rho x_3 x_1 = -\frac{\lambda}{b} x_3 x_2 + \lambda x_2 x_3 + b \rho x_1 x_3 \\
\Leftrightarrow \quad & \rho x_3 x_1 = -\frac{\lambda}{b} \left( -\frac{1}{b} x_3 x_1 + x_1 x_3 + b x_2 x_3 \right) + \lambda x_2 x_3 + b \rho x_1 x_3 \\
\Leftrightarrow \quad & \rho x_3 x_1 = \frac{\lambda}{b^2} x_3 x_1 - \frac{\lambda}{b} x_1 x_3 + b \rho x_1 x_3 \\
\Leftrightarrow \quad & \left( \rho - \frac{\lambda}{b^2} \right) x_3 x_1 = \left( b \rho - \frac{\lambda}{b} \right) x_1 x_3.
\end{aligned}$$

As  $x_3 x_1$  and  $x_1 x_3$  are linearly independent we yield  $\rho b^2 = \lambda$ . Going back to Equation (3.4), we can compute

$$-\lambda \rho - \frac{\lambda}{b} = 2 \quad \Leftrightarrow \quad -\lambda \rho - (\pm) \sqrt{\lambda \rho} = 2.$$

The order of  $g$  is finite and therefore  $\lambda \rho$  is a root of unity. Therefore, this forces  $\lambda \rho = -1$  and  $\pm \sqrt{\lambda \rho} = -1$  which gives a contradiction.  $\square$

Combined, we have proved the following theorem.

**Theorem 3.2.8.** *Let  $S$  be an AS-regular domain of global dimension 4 which is generated by three elements and properly  $\mathbb{Z}^{\times 2}$ -graded. Then either  $S$  does not have a quasi-reflection of finite order or  $S = B[x; \sigma]$  where  $B$  is an AS-regular algebra of global dimension 3 generated by two elements, and a quasi-reflection  $g$  of finite order fixes  $B$  pointwise and sends  $g(x) = \lambda x$  for some nontrivial root of unity  $\lambda$ .*

*Proof.* By [RZ12, Theorem (0.1)], either  $S$  is a normal extension or  $S$  is one of the eight 1 or 2-parameter families  $\mathcal{A}(b, q)$ ,  $\mathcal{B}(b)$ ,  $\mathcal{C}(b)$ ,  $\mathcal{D}(b)$ ,  $\mathcal{E}(b, \gamma_1)$ ,  $\mathcal{F}(b, \gamma_2)$ ,  $\mathcal{G}(b, \gamma_3)$  or  $\mathcal{H}(b)$  with

$\gamma_1 = \pm i$ ,  $\gamma_2 = \text{primitive } \sqrt[3]{1}$  and  $\gamma_3 = \frac{1 \pm i}{2}$ . Therefore, this follows from Propositions 3.2.4 and 3.2.7. □

### 3.3 Quasi-reflections of AS-regular algebras of global dimension 4 generated by two elements

Next, let  $S$  be AS-regular of global dimension 4, generated by two elements in degree 1. Then, using [LPWZ07, Proposition (1.4)], we know that

$$H_S(t) = \frac{1}{(1-t)^2(1-t^2)(1-t^3)} = \frac{1}{(1-t)^4 p(t)}$$

with one cubic relation and one relation of degree four. This section attempts the same analysis as the previous two for  $\mathbb{Z}^{\times 2}$ -graded Artin-Schelter regular algebras of global dimension 4 generated by two elements in degree 1. In particular, the first proposition states the classification of these algebras done by Lu, Palmieri, Wu and Zhang. Based on only having to deal with four families, the first step is to show that every graded automorphism acts diagonally on the generic generators. While this can be shown for most families almost instantaneously with a calculation, the case  $D(-2, -1)$  has to be treated separately. Eventually we yield that all the generators are eigenvectors for a finite group of graded automorphisms and so we can try the analysis as before. Unfortunately, the tools at hand are not enough to prove that there do not exist any quasi-reflections of finite order. Instead, we end with five equations involving only roots of unity. Our conjecture is that these equations do not have a solution which would imply no quasi-reflections of finite order exist.

**Proposition 3.3.1.** *Let  $S$  be a noetherian AS-regular  $\mathbb{Z}^{\times 2}$ -graded algebra of global dimension 4 generated by two elements  $x, y$  with degree  $(1, 0)$  and  $(0, 1)$ , respectively. Then*

by [LPWZ07, Theorems A and B],  $S$  is isomorphic to one of the following algebras:

(a)  $A(p) = k\langle x, y \rangle / (xy^2 - p^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$ , where  $0 \neq p \in k$ .

(b)  $B(p) = k\langle x, y \rangle / (xy^2 + ip^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$ , where  $i^2 = -1$  and  $0 \neq p \in k$ .

(c)  $C(p) = k\langle x, y \rangle / (xy^2 + pyxy + p^2y^2x, x^3y + jp^3yx^3)$ , where  $j^2 - j + 1 = 0$  and  $0 \neq p \in k$ .

(d)  $D(v, p) = k\langle x, y \rangle / (xy^2 + vyxy + p^2y^2x, x^3y + (v + p)x^2yx + (p^2 + pv)xyx^2 + p^3yx^3)$ ,  
where  $v, p \in k$  and  $p \neq 0$ .

If  $g \in \text{Aut}_{\text{gr}}(S)$ , then  $g$  acts diagonally on  $x$  and  $y$  unless  $S = D(-2, -1)$  in which case only  $y$  needs to be an eigenvector.

*Proof.* By [LPWZ07, Lemmas 7.10 and 7.13] the statement is true for  $B(p)$  and  $C(p)$ .

Let  $S = A(p)$  and  $g(x) = a_{11}x + a_{21}y$  and  $g(y) = a_{12}x + a_{22}y$  for some  $g \in \text{Aut}_{\text{gr}}(S)$ .

Applying  $g$  to the cubic relation gives

$$\begin{aligned}
0 &= a_{11}a_{12}^2x^3 + a_{11}a_{12}a_{22}x^2y + a_{11}a_{22}a_{12}xyx + a_{11}a_{22}^2xy^2 \\
&\quad + a_{21}a_{12}^2yx^2 + a_{21}a_{12}a_{22}yxy + a_{21}a_{22}a_{12}y^2x + a_{21}a_{22}^2y^3 \\
&\quad - p^2 [a_{11}a_{12}^2x^3 + a_{12}^2a_{21}x^2y + a_{12}a_{22}a_{11}xyx + a_{12}a_{22}a_{21}xy^2] \\
&\quad - p^2 [a_{22}a_{12}a_{11}yx^2 + a_{22}a_{12}a_{21}yxy + a_{22}^2a_{11}y^2x + a_{22}^2a_{21}y^3] \\
&= a_{11}a_{12}^2(1 - p^2)x^3 + (a_{11}a_{12}a_{22} - p^2a_{12}^2a_{21})x^2y + a_{11}a_{12}a_{22}(1 - p^2)xyx \\
&\quad + a_{12}a_{21}a_{22}(1/p^2 - p^2)xy^2 + (a_{12}^2a_{21} - p^2a_{11}a_{12}a_{22})yx^2 + a_{12}a_{21}a_{22}(1 - p^2)yxy \\
&\quad + a_{21}a_{22}^2(1 - p^2)y^3.
\end{aligned}$$

At the end we gave the expression in terms of the basis  $\{x^3, xy^2, x^2y, xyx, yx^2, yxy, y^3\}$  of  $S_3$  suggested by the cubic relation  $y^2x = 1/p^2 xy^2$ . Therefore, the coefficients of  $xyx$ ,  $yx^2$  and  $y^3$  must be zero:

$$0 = a_{11}a_{12}a_{22} - p^2a_{11}a_{12}a_{22}, \quad (3.5)$$

$$0 = a_{12}^2a_{21} - p^2a_{11}a_{12}a_{22}, \quad \text{and} \quad (3.6)$$

$$0 = a_{21}a_{22}^2 - p^2a_{21}a_{22}^2. \quad (3.7)$$

Equations (3.5) and (3.6) imply  $a_{12}^2a_{21} = a_{11}a_{12}a_{22}$  and so either  $a_{12} = 0$  or  $a_{21}a_{12} = a_{11}a_{22}$ . The second option is impossible as the determinant of  $g|_{S_1}$  is nonzero. Therefore, we have  $a_{12} = 0$ . The determinant now implies  $a_{11}a_{22} \neq 0$  and so using Equation (3.7) we either have  $p^2 = 1$  or  $a_{21} = 0$ .

We show that also in case  $p^2 = 1$  the scalar  $a_{21}$  must be zero. Before we proceed, we need to understand the relations of degree 4. They arise from the given relation of degree 4 and four relations come up from the cubic relation by multiplying  $x$  and  $y$  from either side. This gives the following five relations of degree 4:

$$0 = x^3y + px^2yx + p^2xyx^2 + p^3yx^3,$$

$$0 = x^2y^2 - p^2xy^2x,$$

$$0 = yxy^2 - p^2y^3x,$$

$$0 = xy^2x - p^2y^2x^2, \quad \text{and}$$

$$0 = xy^3 - p^2y^2xy.$$

These relations suggest  $\{x^4, x^3y, x^2yx, x^2y^2, xyx^2, xyxy, xy^3, yx^2y, yxyx, yxy^2, y^4\}$  as a basis of  $S_4$ . We notice that  $xyxy$  does not occur in any relation. Hence, we are ready to apply  $g$  to our relation of degree four to obtain

$$\begin{aligned} & a_{11}^3 a_{22} x^3 y + a_{11}^2 a_{21} a_{22} (x^2 y^2 + xyxy + yx^2 y) + a_{11} a_{21}^2 a_{22} (xy^3 + yxy^2 + y^2 xy) + a_{21}^3 a_{22} y^4 \\ & + p [a_{11}^3 a_{22} x^2 yx + a_{11}^2 a_{21} a_{22} (x^2 y^2 + xy^2 x + yxyx) + a_{11} a_{21}^2 a_{22} (xy^3 + yxy^2 + y^3 x) + a_{21}^3 a_{22} y^4] \\ & + p^2 [a_{11}^3 a_{22} xyx^2 + a_{11}^2 a_{21} a_{22} (xyxy + xy^2 x + y^2 x^2) + a_{11} a_{21}^2 a_{22} (xy^3 + y^2 xy + y^3 x) + a_{21}^3 a_{22} y^4] \\ & + p^3 [a_{11}^3 a_{22} yx^3 + a_{11}^2 a_{21} a_{22} (yx^2 y + yxyx + y^2 x^2) + a_{11} a_{21}^2 a_{22} (yxy^2 + y^2 xy + y^3 x) + a_{21}^3 a_{22} y^4] \end{aligned}$$

which must be equal to 0. Since  $xyxy$  cannot appear through a relation, we can read off its scalar which must be zero:

$$a_{11}^2 a_{21} a_{22} + p^2 a_{11}^2 a_{21} a_{22} = 0.$$

Recalling that  $a_{11}$  and  $a_{22}$  are nonzero as well as  $p^2 = 1$  yields  $a_{21} = 0$  as desired.

Next, let  $S = D(v, p)$  and  $g \in \text{Aut}_{\text{gr}}(S)$  such that  $g$  acts by  $g(x) = a_{11}x + a_{21}y$  and  $g(y) = a_{12}x + a_{22}y$ . Recall the cubic relation

$$0 = xy^2 + v yxy + p^2 y^2 x \quad \Leftrightarrow \quad y^2 x = -1/p^2 xy^2 - v/p^2 yxy$$

of  $D(v, p)$ . This gives the same basis  $\{x^3, x^2y, yxy, xy^2, yx^2, yxy, y^3\}$  for  $S_3$  as in case  $A(p)$ . Similarly, this happens for degree 4. The relations of degree 4 come from the original simplified relation of degree four as well as the four simplified relations arising

from multiplying the relation of degree 3 by  $x$  and  $y$  from either side:

$$0 = x^3y + (v + p)x^2yx + (p^2 + pv)xyx^2 + p^3yx^3, \quad (3.8)$$

$$0 = x^2y^2 - pxyxy + p^2xy^2x \Leftrightarrow xy^2x = 1/p xxyxy - 1/p^2 x^2y^2, \quad (3.9)$$

$$0 = yxy^2 - py^2xy + p^2y^3x \Leftrightarrow y^3x = 1/p y^2xy - 1/p^2 yxy^2, \quad (3.10)$$

$$0 = xy^3 - pyxy^2 + p^2y^2xy \Leftrightarrow y^2xy = 1/p yxy^2 - 1/p^2 xy^3, \quad \text{and} \quad (3.11)$$

$$0 = xy^2x - pyxyx + p^2y^2x^2 \Leftrightarrow y^2x^2 = 1/p yxyx - 1/p^3 xxyxy + 1/p^4 x^2y^2. \quad (3.12)$$

Therefore, we choose  $\{x^4, x^3y, x^2yx, x^2y^2, xyx^2, xxyxy, xy^3, yx^2y, yxyx, yxy^2, y^4\}$  as basis of

$S_4$ . Now, applying  $g$  to the cubic relation gives

$$\begin{aligned} 0 = & a_{11}a_{12}^2x^3 + a_{11}a_{12}a_{22}x^2y + a_{11}a_{12}a_{22}xyx + a_{11}a_{22}^2xy^2 \\ & + a_{21}a_{12}^2yx^2 + a_{21}a_{12}a_{22}yxy + a_{21}a_{12}a_{22}y^2x + a_{21}a_{22}^2y^3 \\ & + v \cdot [a_{11}a_{12}^2x^3 + a_{11}a_{12}a_{22}x^2y + a_{12}^2a_{21}xyx + a_{12}a_{21}a_{22}xy^2] \\ & + v \cdot [a_{11}a_{12}a_{22}yx^2 + a_{11}a_{22}^2yxy + a_{12}a_{21}a_{22}y^2x + a_{21}a_{22}^2y^3] \\ & + p^2 \cdot [a_{11}a_{12}^2x^3 + a_{12}^2a_{21}x^2y + a_{11}a_{12}a_{22}xyx + a_{12}a_{21}a_{22}xy^2] \\ & + p^2 \cdot [a_{11}a_{12}a_{22}yx^2 + a_{12}a_{21}a_{22}yxy + a_{11}a_{22}^2y^2x + a_{21}a_{22}^2y^3]. \end{aligned} \quad (3.13)$$

Since the cubic relation has degree  $(1, 2)$ , only terms of this degree can appear when we write the sum as linear combination of basis elements. Therefore, we can restrict Equation

(3.13) to the degrees  $(3, 0)$ ,  $(0, 3)$  and  $(2, 1)$  to yield the following equations:

$$0 = (1 + v + p^2)a_{11}a_{12}^2, \quad (3.14)$$

$$0 = (1 + v + p^2)a_{21}a_{22}^2, \quad (3.15)$$

$$0 = a_{11}a_{12}a_{22} + va_{11}a_{12}a_{22} + p^2a_{12}^2a_{21}, \quad (3.16)$$

$$0 = a_{11}a_{12}a_{22} + va_{12}^2a_{21} + p^2a_{11}a_{12}a_{22}, \quad \text{and} \quad (3.17)$$

$$0 = a_{12}^2a_{21} + va_{11}a_{12}a_{22} + p^2a_{11}a_{12}a_{22}. \quad (3.18)$$

Combining Equations (3.17) and (3.18) gives

$$a_{11}a_{12}a_{22} + va_{12}^2a_{21} = a_{12}^2a_{21} + va_{11}a_{12}a_{22}$$

and so either  $a_{12} = 0$  or else  $v = 1$  as  $\det(g|_{S_1}) \neq 0$ .

For now suppose  $a_{12} \neq 0$  and therefore  $v = 1$ . Rewriting and combining Equations (3.16) and (3.17) yields

$$a_{11}a_{12}a_{22} + p^2a_{12}^2a_{21} = a_{12}^2a_{21} + p^2a_{11}a_{12}a_{22}. \quad (3.19)$$

As before  $a_{12}$  and  $\det(g|_{S_1})$  being nonzero implies  $p^2 = 1$ .

To sum up, we are confronted with two cases:

(a)  $a_{12} \neq 0$  and  $v = p^2 = 1$  or

(b)  $a_{12} = 0$ .

We consider each case individually.

First, let's look at (a). As  $a_{12} \neq 0$  and  $v = p^2 = 1$ , Equation (3.14) simplifies to  $3a_{11}a_{12}^2 = 0$  and so  $a_{11} = 0$ . The determinant again implies that  $a_{21} \neq 0$  and Equation (3.15) gives  $a_{22} = 0$ . To get a contradiction we analyze  $g$  applied to the cubic relation again:

$$a_{12}^2 a_{21} (yx^2 + vxyx + p^2 x^2 y) = 0.$$

But from our basis of  $S_3$  we know that  $yx^2$ ,  $xyx$ ,  $x^2y$  are linearly independent and so  $a_{12}^2 a_{21} = 0$  which gives a contradiction as  $g$  is an automorphism.

Case (b) is slightly more complicated. As  $a_{12} = 0$ , we use that the determinant  $\det(g|_{S_1}) = a_{11}a_{22} \neq 0$ . Now, Equation (3.15) simplifies to  $(1 + v + p^2)a_{21} = 0$  and so either  $a_{21} = 0$  and  $g$  acts diagonally on  $x$  and  $y$  or  $a_{21} \neq 0$  and  $(1 + v + p^2) = 0$ .

To finish the proof, we need to rule out the case  $a_{12} = 0$ ,  $(1 + v + p^2) = 0$  and  $a_{21} \neq 0$  and we do this by applying  $g$  to the relation of degree four. At first, we will only look at the coefficient of  $yx^2y$  in  $g$  applied to the original relation of degree 4. The five relations of degree 4 listed above in Equations (3.8) to (3.12) do not include  $yx^2y$ . Therefore, no relation makes a term  $yx^2y$  appear. As a consequence, the coefficient of  $yx^2y$  in  $g$  applied to the original relation of degree 4 equals  $(1 + p^3)a_{11}^2 a_{21}$ . It follows that  $(1 + p^3) = 0$ . Now, we consider two cases.

Let  $p \neq -1$ . As  $p^3 + 1 = 0$ , the minimal polynomial of  $p$  implies that  $p^2 - p + 1 = 0$ . Combined with  $1 + v + p^2 = 0$  we yield  $p = -v$ . This simplifies  $g$  applied to the relation

of degree four immensely to

$$\begin{aligned}
0 = & a_{11}^2 a_{21} a_{22} x^2 y^2 + a_{11}^2 a_{21} a_{22} x y x y + a_{11} a_{21}^2 a_{22} x y^3 + a_{11}^2 a_{21} a_{22} y x^2 y \\
& + a_{11} a_{21}^2 a_{22} y x y^2 + a_{11} a_{21}^2 a_{22} y^2 x y + a_{21}^3 a_{22} y^4 \\
& + p^3 \cdot (a_{11}^2 a_{21} a_{22} y x^2 y + a_{11}^2 a_{21} a_{22} y x y x + a_{11} a_{21}^2 a_{22} y x y^2) \\
& + p^3 \cdot (a_{11}^2 a_{21} a_{22} y^2 x^2 + a_{11} a_{21}^2 a_{22} y^2 x y + a_{11} a_{21}^2 a_{22} y^3 x + a_{21}^3 a_{22} y^4).
\end{aligned} \tag{3.20}$$

Recall that we are in case  $D(v, p)$ . As we fixed our basis of  $S_4$ , we use the five relations of degree four listed above to reduce Equation (3.20). Using the Equations (3.8) to (3.12) to write Equation (3.20) in the chosen basis provides the scalar of  $x^2 y^2$  to be  $a_{11}^2 a_{21} a_{22} \left(1 + \frac{1}{p}\right)$ . Since  $a_{11}^2 a_{21} a_{22}$  is nonzero, this gives  $p + 1 = 0$  which is impossible in our case.

On the other hand, let  $p = -1$ . From  $1 + v + p^2 = 0$  we get  $v = -2$ . In this case all the coefficients of  $g(r_3)$  and  $g(r_4)$  are zero and so this is the only exception.  $\square$

In order to get the same result for  $D(-2, -1)$  we have to take a small detour. First, we show that if  $x$  and  $y$  in  $D(-2, -1)$  are replaced by the eigenvectors of a graded automorphism the same presentation works. Afterwards, showing that every finite subgroup of graded automorphisms is abelian it is just one more step to gaining that every finite group of graded automorphisms is simultaneously diagonalizable.

**Proposition 3.3.2.** *For*

$$S = D(-2, -1) = k\langle x, y \rangle / (xy^2 - 2yxy + y^2x, x^3y - 3x^2yx + 3xyx^2 - yx^3)$$

the automorphism group  $\text{Aut}_{\text{gr}}(S)$  is isomorphic to  $\begin{pmatrix} k^* & \\ & k \end{pmatrix}$ . For every  $g \in \text{Aut}_{\text{gr}}(S)$  which has finite order,  $g|_{S_1}$  is diagonalizable with eigenvectors  $\tilde{x}, y$  such that

$$S = k\langle \tilde{x}, y \rangle / (\tilde{x}y^2 - 2y\tilde{x}y + y^2\tilde{x}, \tilde{x}^3y - 3\tilde{x}^2y\tilde{x} + 3\tilde{x}y\tilde{x}^2 - y\tilde{x}^3).$$

*Proof.* Consider the following graded Lie algebra  $\mathfrak{g} = kx + ky + kw + kv$  with grading defined by  $\deg(x) = \deg(y) = 1$ ,  $\deg(w) = 2$  and  $\deg(v) = 3$ . With the structural constants given by

$$[x, y] = w, \quad [x, w] = v, \quad \text{and} \quad [x, v] = [y, w] = 0,$$

it is easy to check that the Jacobi identity is satisfied. As a consequence, we observe that  $[y, v] = [v, w] = 0$ . Further, in the universal enveloping algebra

$$-[y, w] = [y, (yx - xy)] = y^2x - 2yxy + xy^2, \quad \text{and}$$

$$[x, v] = [x, [x, [x, y]]] = [x, x(xy - yx) - (xy - yx)x] = x^3y - 3x^2yx + 3xyx^2 - yx^3$$

form the two relations of  $D(-2, -1)$ . Since  $x$  and  $y$  generated the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  and all other relations are forced by the Jacobi identity, we proved that

$$\mathcal{U}(\mathfrak{g}) \cong D(-2, -1). \quad \text{Moreover, any transformation } g = \begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix} \text{ with } a_{11}a_{22} \neq 0$$

extends via  $g(w) = a_{11}a_{22}w$  and  $g(v) = a_{11}^2a_{22}v$  and forms an automorphism of  $\mathfrak{g}$ :

$$[a_{22}y, a_{11}a_{22}w] = a_{11}a_{22}^2[y, w] = 0,$$

$$[a_{11}x + a_{21}y, a_{11}^2a_{22}v] = a_{11}^3a_{22}[x, v] + a_{11}^2a_{21}a_{22}[y, v] = 0.$$

It is well-known that an automorphism of  $\mathfrak{g}$  lifts to an automorphism of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g}) = D(-2, -1)$ . In Proposition 3.3.1 we showed that any automorphism of  $D(-2, -1)$  can be described by its restriction to  $S_1$  by an invertible lower triangular matrix. Combined, we reached the claimed description of  $\text{Aut}_{\text{gr}}(S)$ .

In regards of the second claim consider  $g = \begin{pmatrix} a_{11} & \\ a_{21} & a_{22} \end{pmatrix}$  with  $a_{11}a_{22} \neq 0$  of finite order. If  $a_{21} = 0$  then  $g$  is diagonalized with eigenvectors  $x$  and  $y$  and so the claim follows trivially. Otherwise, as used before in the proof of Proposition 3.1.1, since  $g$  has finite order we know  $a_{11} \neq a_{22}$  and  $\{\tilde{x} = (a_{11} - a_{22})x + a_{21}y, y\}$  forms a basis of  $S_1$  consisting of eigenvectors of  $g$ . Consider the matrix

$$h = \begin{pmatrix} a_{11} - a_{22} & \\ a_{21} & 1 \end{pmatrix}$$

which sends  $x$  to  $\tilde{x}$  and  $y$  to  $y$ . From above we know that  $h$  extends to an automorphism of  $S$ . Recall the relations of  $S$  from its presentation

$$R_1 : xy^2 - 2yxy + y^2x = 0, \quad \text{and}$$

$$R_2 : x^3y - 3x^2yx + 3xyx^2 - yx^3 = 0.$$

Then clearly  $h(R_i) = 0$  and since  $h$  is an automorphism the relations

$$R'_1 : \tilde{x}y^2 - 2y\tilde{x}y + y^2\tilde{x} = 0, \quad \text{and}$$

$$R'_2 : \tilde{x}^3y - 3\tilde{x}^2y\tilde{x} + 3\tilde{x}y\tilde{x}^2 - y\tilde{x}^3 = 0.$$

follow as claimed. Therefore, the new basis  $\{\tilde{x}, y\}$  satisfies the same relations as  $\{x, y\}$  and

there cannot be any additional relations as the Hilbert series  $H_S(t)$  already determines the dimensions of each graded piece.  $\square$

**Lemma 3.3.3.** *Let  $G \leq \text{Aut}_{\text{gr}}(D(-2, -1))$  be a finite subgroup. Then  $G$  is abelian.*

*Proof.* Recall that the algebra  $D(-2, -1)$  is over a field  $k$  of characteristic 0. By Proposition 3.3.2,  $G \leq \begin{pmatrix} k^* & \\ & k \end{pmatrix} =: T$ . Further, let  $N := \begin{pmatrix} 1 & \\ k & 1 \end{pmatrix} \leq \begin{pmatrix} k^* & \\ & k^* \end{pmatrix}$  and notice that  $N \cong (k, +)$ . As  $(k, +)$  does not have any finite subgroups, the intersection  $G \cap N$  must be trivial. The first isomorphism theorem now implies

$$G \cong G/(G \cap N) \cong GN/N \hookrightarrow T/N \cong \begin{pmatrix} k^* & \\ & k^* \end{pmatrix}$$

and so  $G$  is abelian.  $\square$

**Lemma 3.3.4.** *Let  $G \leq \text{Aut}_{\text{gr}}(D(-2, -1))$  be a finite subgroup. Then without loss of generality,  $G$  acts diagonally on the generators  $x$  and  $y$  of  $D(-2, -1)$ .*

*Proof.* From the previous Lemma 3.3.3 we get that  $G$  is abelian and Proposition 3.3.1 says that all the elements of  $G$  have  $y$  as an eigenvector. In other words,  $ky$  is a  $G$ -invariant subspace of the vector space  $kx + ky$ . Maschke's Theorem gives a  $G$ -invariant complement of  $ky$  in  $kx + ky$  which must be of the form  $k\tilde{x}$  for  $\tilde{x} = (ax + by)$  for some  $a, b \in k$  where  $a$  is nonzero. This provides a basis  $\{\tilde{x}, y\}$  of  $S_1$  which consists of eigenvectors with respect to the action of  $G$ . Thanks to Proposition 3.3.2, the change of the first basis vector does not affect the relations of  $D(-2, -1)$  and this finishes the proof.  $\square$

Finally, we can again calculate the trace function of a graded automorphism using the two different approaches. In the end, the following proposition is the best we can do.

**Proposition 3.3.5.** *Let  $S$  be an AS-regular  $\mathbb{Z}^{\times 2}$ -graded algebra of global dimension 4 generated by two elements in degree  $(1, 0)$  and  $(0, 1)$ . Further, let  $g \in \text{Aut}_{\text{gr}}(S)$  be a quasi-reflection of finite order whose trace is*

$$\text{Tr}(g, t) = \frac{1}{(1-t)^3(1-\zeta_1 t)(1-\zeta_2 t)(1-\zeta_3 t)(1-\zeta_4 t)}$$

by [KKZ08, Definition 2.2 and Lemma 1.6 (d) and (e)]. Let  $\{x, y\}$  be a basis of  $S_1$  such that  $g(x) = \lambda_1 x$  and  $g(y) = \lambda_2 y$ . Then the following equations must be satisfied:

- $(\lambda_1 + \lambda_2) = 3 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4,$
- $(\lambda_1 + \lambda_2)^2 = 3 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2,$
- $(\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2^2 = 3 + \zeta_1^3 + \zeta_2^3 + \zeta_3^3 + \zeta_4^3,$
- $(\lambda_1 + \lambda_2)^4 - 4(\lambda_1^3\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2^3) = 3 + \zeta_1^4 + \zeta_2^4 + \zeta_3^4 + \zeta_4^4,$  and
- $15 + \lambda_1^3\lambda_2 + 3\lambda_1\lambda_2^2 - 10(\lambda_1 + \lambda_2) = \zeta_1\zeta_2\zeta_3\zeta_4.$

*Proof.* First, we recall that  $\zeta_i$  and  $\lambda_j$  for  $i = 1, 2, 3, 4$  and  $j = 1, 2$  are roots of unity. The traces can be computed in two different ways. Using  $\{x, y\}$  as basis for  $S_1$  and using the formula for  $\text{Tr}(g, t)$  gives the first equation:

$$\lambda_1 + \lambda_2 = 3 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4. \tag{3.21}$$

Similarly, the basis  $\{x^2, xy, yx, y^2\}$  of  $S_2$  can be used to find that  $\text{tr}(g|_{S_2})$  equals  $\lambda_1^2 + 2\lambda_1\lambda_2 + \lambda_2^2 = (\lambda_1 + \lambda_2)^2$ . Using  $\text{Tr}(g, t)$  we obtain

$$\begin{aligned}\text{tr}(g|_{S_2}) &= (\lambda_1 + \lambda_2)^2 \\ &= 6 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 + 3\zeta_1 + 3\zeta_2 + 3\zeta_3 + 3\zeta_4 \\ &\quad + \zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4.\end{aligned}\tag{3.22}$$

By squaring Equation (3.21) and subtracting (3.22) we get

$$0 = 3 + 3\zeta_1 + 3\zeta_2 + 3\zeta_3 + 3\zeta_4 + \zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4\tag{3.23}$$

which simplifies Equation (3.22) into the second equation

$$(\lambda_1 + \lambda_2)^2 = 3 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2.\tag{3.24}$$

During this calculation one notices that due to symmetry whenever a term  $\zeta_1$  occurs, also  $\zeta_i$  for all  $i$  occur with the same scalar. The same works for  $\zeta_i\zeta_j$ ,  $\zeta_i^2\zeta_j$ ,  $\zeta_i^2$  and so on for  $i \neq j$ . All possible combinations appear with the same multiplicity. This allows the following simplified notation. For instance, a coefficient  $c$  in  $\zeta_i\zeta_j$  means that the equation contains  $c \cdot (\zeta_1\zeta_2 + \zeta_1\zeta_3 + \zeta_1\zeta_4 + \zeta_2\zeta_3 + \zeta_2\zeta_4 + \zeta_3\zeta_4)$ , so all combinations with  $i \neq j$ . Similarly, for all the other products and powers. Next, we take the third power of Equation (3.21) and yield

$$(\lambda_1 + \lambda_2)^3 = 27 + 27\zeta_i + 9\zeta_i^2 + 18\zeta_i\zeta_j + \zeta_i^3 + 3\zeta_i^2\zeta_j + 6\zeta_i\zeta_j\zeta_k\tag{3.25}$$

as well as multiplying the Equation (3.24) with (3.21):

$$(\lambda_1 + \lambda_2)^3 = 9 + 3\zeta_i + 3\zeta_i^2 + \zeta_i^3 + \zeta_i^2\zeta_j.\tag{3.26}$$

As we know that  $\{x, y\}$  are the generators (Propositions 3.3.1 and 3.3.2), the cubic relation given suggests the following basis  $\{x^3, x^2y, xyx, xy^2, yx^2, yxy, y^3\}$  of  $S_3$ . Next, we compute  $\text{tr}(g|_{S_3})$  directly with this basis as well as  $\text{Tr}(g, t)$ :

$$\begin{aligned}\text{tr}(g|_{S_3}) &= \lambda_1^3 + 3\lambda_1^2\lambda_2 + 2\lambda_1\lambda_2^2 + \lambda_2^3 = (\lambda_1 + \lambda_2)^3 - \lambda_1\lambda_2^2 \\ &= 10 + 6\zeta_i + 3\zeta_i^2 + 3\zeta_i\zeta_j + \zeta_i^3 + \zeta_i^2\zeta_j + \zeta_i\zeta_j\zeta_k.\end{aligned}\tag{3.27}$$

Now we compute (3.25) -  $3 \times$  (3.27) and (3.26) - (3.27) and gain

$$3\lambda_1\lambda_2^2 - 2(\lambda_1 + \lambda_2)^3 = -3 + 9\zeta_i + 9\zeta_i\zeta_j - 2\zeta_i^3 + 3\zeta_i\zeta_j\zeta_k\tag{3.28}$$

and

$$\lambda_1\lambda_2^2 = -1 - 3\zeta_i - 3\zeta_i\zeta_j - \zeta_i\zeta_j\zeta_k.\tag{3.29}$$

Finally, negating one half of (3.28) +  $3 \times$  (3.29) gives the wanted third equation

$$(\lambda_1 + \lambda_2)^3 - 3\lambda_1\lambda_2^2 = 3 + \zeta_1^3 + \zeta_2^3 + \zeta_3^3 + \zeta_4^3.\tag{3.30}$$

In order to find the fourth equation we again compute  $\text{tr}(g|_{S_4})$  and the coefficient of  $t^4$  in  $\text{Tr}(g, t)$  and obtain

$$\begin{aligned}\text{tr}(g|_{S_4}) &= (\lambda_1 + \lambda_2)^4 - \lambda_1^3\lambda_2 - 2\lambda_1^2\lambda_2^2 - 2\lambda_1\lambda_2^3 \\ &= 15 + 10\zeta_i + 6\zeta_i^2 + 6\zeta_i\zeta_j + 3\zeta_i^3 + 3\zeta_i^2\zeta_j + 3\zeta_i\zeta_j\zeta_k + \zeta_i^4 + \zeta_i^3\zeta_j + \zeta_i^2\zeta_j^2 \\ &\quad + \zeta_i^2\zeta_j\zeta_k + \zeta_1\zeta_2\zeta_3\zeta_4.\end{aligned}\tag{3.31}$$

A more compact way to write one side of an equation and execute calculations can be done using a table as follows. Each column stands for a sum, power or product of the

right hand side of an equation while a scalar in each row means this term occurs with the multiplicity written in particular cell. For example, recall the  $\zeta$  part of (3.21), (3.24) and (3.30):

**Table 3.3.1:** Summary of formulas (3.21), (3.24) and (3.30).

|             | (3.21) | (3.24) | (3.30) |
|-------------|--------|--------|--------|
| 1           | 3      | 3      | 3      |
| $\zeta_i$   | 1      |        |        |
| $\zeta_i^2$ |        | 1      |        |
| $\zeta_i^3$ |        |        | 1      |

This allows us to do the calculations in Table 3.3.2 where  $*$  stands for the term from the previous column. Doing the same steps with the other side of the equations and dividing by 6 gives

$$(\lambda_1 + \lambda_2)^4 - 4(\lambda_1^3\lambda_2 + \lambda_1^2\lambda_2^2 + \lambda_1\lambda_2^3) = 3 + \zeta_1^4 + \zeta_2^4 + \zeta_3^4 + \zeta_4^4. \quad (3.32)$$

The following similar calculation, i.e. combining Equations (3.21), (3.24), and (3.30) - (3.32), provides the fifth equation:

$$\begin{aligned} & 2(3.31) - (3.24)(3.21)^2 - (3.32) - 3(3.21)^2 + 3(3.24) + 16(3.21) \\ & - ((3.21)^3 - 9(3.21)^2 - 3(3.21)(3.24) + 2(3.30) + 36(3.21) + 9(3.24)). \end{aligned}$$

**Table 3.3.2:** Summary of calculations.

|                                   | $24 \cdot (3.31)$ | $* - (3.21)^4$ | $* - 6(3.24)(3.21)^2$ | $* - 3(3.24)^2$ | $* - 8(3.21)(3.30)$ |
|-----------------------------------|-------------------|----------------|-----------------------|-----------------|---------------------|
| 1                                 | 360               | 279            | 117                   | 90              | 18                  |
| $\zeta_i$                         | 240               | 132            | 24                    | 24              |                     |
| $\zeta_i^2$                       | 144               | 90             | 18                    |                 |                     |
| $\zeta_i \zeta_j$                 | 144               | 36             |                       |                 |                     |
| $\zeta_i^3$                       | 72                | 60             | 24                    | 24              |                     |
| $\zeta_i^2 \zeta_j$               | 72                | 36             |                       |                 |                     |
| $\zeta_i \zeta_j \zeta_k$         | 72                |                |                       |                 |                     |
| $\zeta_i^4$                       | 24                | 23             | 17                    | 14              | 6                   |
| $\zeta_i^3 \zeta_j$               | 24                | 20             | 8                     | 8               |                     |
| $\zeta_i^2 \zeta_j^2$             | 24                | 18             | 6                     |                 |                     |
| $\zeta_i^2 \zeta_j \zeta_k$       | 24                | 12             |                       |                 |                     |
| $\zeta_1 \zeta_2 \zeta_3 \zeta_4$ | 24                |                |                       |                 |                     |

□

**Conjecture 3.3.6.** *We believe that there is no solution to the five equations in Proposition 3.3.5 consisting of roots of unity so that there are no quasi-reflections of finite order in this case.*

# Chapter 4

## Assessment of invariant theory on preprojective algebras of type $\widetilde{A}_{n-1}$

### 4.1 Basic properties

Let  $n \geq 3$  and let  $Q$  be the following quiver coming from the extended Dynkin diagram  $\widetilde{A}_{n-1}$ :

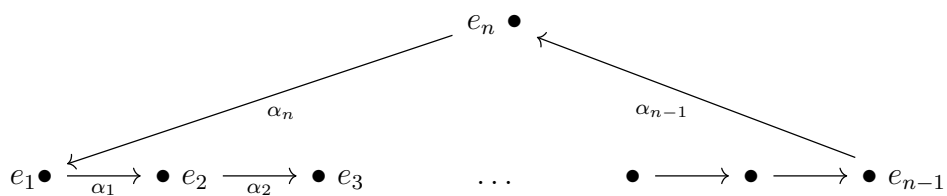


**Figure 4.1.1:** Dynkin diagram of  $\widetilde{A}_{n-1}$ .

In this chapter, let  $A = \Pi_Q(k)$  be the preprojective algebra with respect to  $Q$ . We

are going to present two formulas in Proposition 4.1.4 and Proposition 4.1.5 to compute  $\text{Tr}_A(g, t)$  for any graded automorphism  $g$  of  $A$  which fixes the primitive idempotents. In this situation, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  and reduce the indices modulo  $n$  if needed. Given an automorphism  $g$  of finite order, Proposition 4.1.4 also shows that  $\text{Tr}_A(g, t)$  can be written as a rational function where the zeros of the denominator are only roots of unity with 1 of multiplicity at most 2. Moreover, the graded automorphism group  $\text{Aut}_{\text{gr}}(A)$  is described as a semi-direct product in Proposition 4.1.6.

*Definition & Remark 4.1.1.* In Definition 2.3.3, the preprojective algebra was defined as the path algebra modulo the ideal  $I$  generated by the one relation

$$\sum_{\alpha \in Q_1} \alpha \alpha^* - \sum_{\alpha \in Q_1} \alpha^* \alpha. \quad (4.1)$$

By successively multiplying the trivial paths  $e_i$  for  $i = 1, \dots, n$  to Equation (4.1), relations of the form  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$  for all  $i = 1, \dots, n$  are obtained. Starting now, we think of  $I$  being generated by the relations of the form  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$ . As a consequence, we can always rearrange a path in  $A$  to first have the nonstar arrows occur followed by the star arrows. The absence of overlap or inclusion ambiguities with respect to the natural ordering  $\alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha_1^* < \dots < \alpha_n^*$  together with the Diamond Lemma [BRS<sup>+</sup>16, Theorem 1.15] says that the images of the reduced words form a  $k$ -basis. However, the leading terms of the relations are of the form  $\alpha_i^* \alpha_i$  for all  $i$  and so the paths of length  $j$  starting at  $e_\ell$  first having the nonstar arrows occur followed by the star

arrows form a basis of  $(e_\ell A)_j$ . This implies that  $\dim((e_\ell A)_j) = j + 1$  and therefore

$$H_{e_\ell A}(t) = 1 + 2t + 3t^2 + 4t^3 + \dots = \sum_{j=0}^{\infty} (j+1)t^j = \frac{1}{(1-t)^2}.$$

Since  $A = \bigoplus_{\ell=1}^n e_\ell A$ , this implies that  $H_A(t) = \frac{n}{(1-t)^2}$ .

Moreover, the relations  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$  force  $c_i t_i = c_{i-1} t_{i-1}$  for all  $i = 1, \dots, n$ . In other words,  $c_1 t_1 = c_i t_i$  for all  $i$  is necessary to obtain an automorphism  $g$ . On the other hand, for every choice of nonzero scalars  $c_1, \dots, c_n$ ,  $t_1, \dots, t_n$  with  $c_1 t_1 = c_i t_i$  one defines a graded automorphism of  $A$ .

Every element of  $A$  can be represented as a  $k$ -linear combination of paths in the double of  $Q$ . In order to simplify notation, we identify an element of  $A$  with any representative in  $k\bar{Q}$ . We call an element of the form  $\prod_j \alpha_{i_j} \prod_v \alpha_{u_v}^*$  a *simple path*. Due to the relations, every path has a unique representation as a simple path. In addition, every simple path in  $A$  is uniquely determined by its starting vertex, its length and the number of star arrows occurring. Further, every element of  $A$  is a unique linear combination of simple paths.

In order to simplify the language the following definitions are introduced:

**Definition 4.1.2.** We call a path *purely nonstar*, if it consists of only nonstar arrows and *purely star* if it consists of only star arrows. We call a path *pure* if it is either purely nonstar or purely star. We call a path *mixed* if it is not pure.

**Definition 4.1.3.** The *type*  $\text{type}(\beta) = (s, t)$  of a path  $\beta \in A$  is defined by  $s = \#$  of nonstar arrows in  $\beta$  and  $t = \#$  of star arrows in  $\beta$ . Thanks to the relations in  $A$  this

definition is well-defined. Two paths  $\beta_1$  and  $\beta_2$  are *of the same type* if  $\text{type}(\beta_1) = \text{type}(\beta_2)$ .

Now, we are ready to prove the main proposition of this chapter which gives valuable information about the vector trace and the trace of a graded automorphism of  $A$ .

**Proposition 4.1.4.** *Let  $A = \Pi_Q(k)$  and let  $g$  be a graded automorphism of  $A$  with  $g(e_i) = e_i$  for all  $i = 1, \dots, n$  of not necessarily finite order. Therefore, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Also, recall  $g_i = g|_{e_i A}$  and fix the convention that an empty product equals 1. Then we have the following:*

(a) *The vector  $(1 - t_1 \cdots t_n t^n) \cdot \underline{\text{Tr}}_A(g, t)$  equals*

$$\begin{pmatrix} 1 & -c_1 t & & & & \\ & 1 & -c_2 t & & & \\ & & 1 & -c_3 t & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & 1 & -c_{n-1} t \\ -c_n t & & & & & & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{k=0}^{n-1} \left( \prod_{j=n-k+1}^n t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+2}^n t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+3}^n t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+4}^n t_j \right) t^k \\ \vdots \\ \sum_{k=0}^{n-1} \left( \prod_{j=2n-k}^{2n-1} t_j \right) t^k \end{pmatrix}.$$

(b) *The trace  $\text{Tr}_A(g, t)$  can be written as fraction  $\frac{p(t)}{q(t)}$  of the two polynomials*

$$p(t) = \sum_{v=0}^{2n-2} \left( \sum_{\substack{v=k+s \\ 0 \leq k, s \leq n-1}} \sum_{\ell=1}^n \prod_{j=n-k+\ell}^{n+\ell-1} t_j \prod_{r=n-s+\ell}^{n+\ell-1} c_r \right) t^v, \quad \text{and}$$

$$q(t) = (1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n).$$

Notice that  $p(t)$  and  $q(t)$  may not be coprime.

(c) Similarly,  $\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{q_i(t)}$  with  $q_i(t) = (1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)$ .

(d) If  $g$  has finite order, all roots of  $q(t)$  ( $q_i(t)$ , respectively) are roots of unity. Additionally, the order of 1 as a pole of  $\text{Tr}_A(g, t)$  ( $\text{Tr}_{e_i A}(g_i, t)$ , respectively) can be at most two.

*Proof.* First, recall  $g_i : e_i A \rightarrow e_i A$ ,  $g_i(e_i \alpha) = g(\alpha)$  from Definition 2.3.12. Since the primitive idempotents are fixed, this is a well-defined vector space automorphism of  $e_i A$ . Understanding  $\alpha_i$  as a path in degree 1 from  $e_i$  to  $e_{i+1}$  (indices are taken modulo  $n$ ), the quotient vector space  $B_i = e_i A / \alpha_i e_{i+1} A$  arises naturally. Further using that  $g_i$  acts diagonally on  $\alpha_i$ , the induced maps

$$\tilde{g}_i : B_i \rightarrow B_i, \tilde{g}_i(\beta + \alpha_i e_{i+1} A) = g_i(\beta) + \alpha_i e_{i+1} A \in B_i$$

make sense. As noticed in the remark from the beginning of this section, the Diamond Lemma provides a basis of  $e_{i+1} A$  consisting of simple paths. This form of the basis and the relations guarantee that the map  $\gamma \mapsto \alpha_i \gamma$  for  $\gamma \in e_{i+1} A$  is injective. Therefore, due to the trace only seeing the vector space structure, we obtain

$$\text{Tr}_{B_i}(\tilde{g}_i, t) = \text{Tr}_{e_i A}(g_i, t) - c_i t \text{Tr}_{e_{i+1} A}(g_{i+1}, t). \quad (4.2)$$

To proceed,  $\text{Tr}_{B_i}(\tilde{g}_i, t)$  has to be understood better. Every path in  $e_i A$  is a linear combination of simple paths starting at  $e_i$ . Due to the relations in  $A$ , a simple path is uniquely determined by its starting vertex and its type. Further, every simple path

$\beta \in e_i A$  which has at least one nonstar arrow can be rewritten as  $\beta = \alpha_i \beta_1 \in \alpha_i e_{i+1} A$ .

Therefore, the only nonzero elements of  $B_i$  are idempotents and products of star arrows.

This provides the following  $k$ -basis for  $B_\ell$ :

$$\begin{aligned} & \overline{e_\ell}, \overline{\alpha_{\ell-1}^*}, \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^*}, \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \alpha_{\ell-3}^*}, \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \alpha_{\ell-3}^* \alpha_{\ell-4}^*}, \dots, \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \cdots \alpha_1^* \alpha_n^* \alpha_{n-1}^* \cdots \alpha_{\ell+1}^*} \\ & \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \cdots \alpha_1^* \alpha_n^* \alpha_{n-1}^* \cdots \alpha_{\ell+1}^* \alpha_\ell^*}, \overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \cdots \alpha_1^* \alpha_n^* \alpha_{n-1}^* \cdots \alpha_{\ell+1}^* \alpha_\ell^* \alpha_{\ell-1}^*}, \dots \end{aligned}$$

On this basis,  $\tilde{g}_\ell$  acts via scalar multiplication by 1,  $t_{\ell-1}$ ,  $t_{\ell-1}t_{\ell-2}$ ,  $t_{\ell-1}t_{\ell-2}t_{\ell-3}$ , and so on. Notice that  $\tilde{g}_\ell$  applied to  $\overline{\alpha_{\ell-1}^* \alpha_{\ell-2}^* \cdots \alpha_1^* \alpha_n^* \alpha_{n-1}^* \cdots \alpha_{\ell+1}^* \alpha_\ell^* \alpha_{\ell-1}^*}$  causes a scalar multiplication by  $t_1 \cdots t_n t_{\ell-1}$ . This repetition continues and so the trace computes as

$$\begin{aligned} \text{Tr}_{B_\ell}(\tilde{g}_\ell, t) &= \frac{1 + t_{\ell-1}t + t_{\ell-1}t_{\ell-2}t^2 + t_{\ell-1}t_{\ell-2}t_{\ell-3}t^3 + \dots + t_{\ell-1}t_{\ell-2} \cdots t_1 t_n t_{n-1} \cdots t_{\ell+1} t^{n-1}}{(1 - t_1 \cdots t_n t^n)} \\ &= \frac{\sum_{k=0}^{n-1} \left( \prod_{j=n-k+\ell}^{n+\ell-1} t_j \right) t^k}{(1 - t_1 \cdots t_n t^n)}. \end{aligned} \tag{4.3}$$

Now, we are ready to apply matrix notation to the formulas from equation (4.2):

$$\begin{pmatrix} \text{Tr}_{B_1}(\tilde{g}_1, t) \\ \text{Tr}_{B_2}(\tilde{g}_2, t) \\ \text{Tr}_{B_3}(\tilde{g}_3, t) \\ \text{Tr}_{B_4}(\tilde{g}_4, t) \\ \vdots \\ \text{Tr}_{B_n}(\tilde{g}_n, t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -c_1 t & & & \\ & 1 & -c_2 t & & \\ & & 1 & -c_3 t & \\ & & & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & 1 & -c_{n-1} t \\ -c_n t & & & & & & 1 \end{pmatrix}}_{=: C^{-1}} \cdot \begin{pmatrix} \text{Tr}_{e_1 A}(g_1, t) \\ \text{Tr}_{e_2 A}(g_2, t) \\ \text{Tr}_{e_3 A}(g_3, t) \\ \text{Tr}_{e_4 A}(g_4, t) \\ \vdots \\ \text{Tr}_{e_n A}(g_n, t) \end{pmatrix}. \quad (4.4)$$

Combining the equations (4.3) and (4.4) finishes the proof of part (a).

For part (b), Cramer's rule helps inverting the matrix  $C^{-1}$ . It says that the entries of  $C = (c_{ij})_{i,j}$  equal  $\left( \frac{1}{\det(C^{-1})} \text{adj}(C^{-1}) \right)_{i,j}$  where  $\text{adj}(-)$  denotes the adjugate matrix. Since  $C^{-1}$  is very sparse,  $\det(C^{-1})$  can be easily calculated by expanding along the first row. After the first step, one faces a constant term 1 plus  $(-1)^n c_1 t \cdot (-c_n t)$  multiplied by the determinant of the lower triangular matrix with diagonal entries  $(-c_1 t, \dots, -c_{n-1} t)$ . This gives  $(1 - c_1 \cdots c_n t^n)$ . Now, the  $(i, j)$ -th entry of  $\text{adj}(C^{-1})$  equals  $(-1)^{i+j}$  times the determinant of the minor of  $C^{-1}$  obtained by deleting the  $j$ -th row and the  $i$ -th column. The diagonal entries of  $\text{adj}(C^{-1})$  are easily seen to equal 1 since deleting an  $i$ -th row and column always isolates a diagonal entry 1 to be the only entry in a particular column. Every step produces a new column with just one entry on the diagonal equal to 1. As for  $i < j$ , expanding along the columns bigger than  $j$  and the rows smaller than  $i$  which all

only have one diagonal entry equal to 1 reduces the problem to a lower triangular matrix whose diagonal entries are  $(-c_1 t, \dots, -c_{j-1} t)$ . A similar analysis for  $i > j$  finally gives

$$\begin{aligned} (1 - c_1 \cdots c_n t^n) \cdot c_{ii} &= 1, \\ (1 - c_1 \cdots c_n t^n) \cdot c_{ij} &= \prod_{k=i}^{j-1} (c_k t), \quad i < j, \quad \text{and} \\ (1 - c_1 \cdots c_n t^n) \cdot c_{ij} &= \prod_{k=i}^{n+j-1} (c_k t), \quad i > j. \end{aligned}$$

This allows us to multiply  $C$  from the left on both sides of Equation (4.4). Being interested in  $\text{Tr}_A(g, t)$  means to multiply  $C$  from the left to the vector  $(\text{Tr}_{B_k}(\tilde{g}_k, t))_{k=1, \dots, n}$  and sum up the entries of the resulting vector. However, the same result can be obtained by first summing up each column of  $C$  to obtain a vector  $\text{col}(C)$  and then computing the dot product of  $\text{col}(C)$  and  $(\text{Tr}_{B_k}(\tilde{g}_k, t))_{k=1, \dots, n}$ . The  $j$ -th entry of  $\text{col}(C)$  equals:

$$(1 - c_1 \cdots c_n t^n) \cdot \text{col}(C)_j = \left( 1 + \sum_{i=1}^{j-1} \prod_{k=i}^{j-1} (c_k t) + \sum_{i=j+1}^n \prod_{k=i}^{n+j-1} (c_k t) \right) = \sum_{s=0}^{n-1} \left( \prod_{r=n-s+j}^{n+j-1} c_r \right) t^s. \quad (4.5)$$

Finally, putting together the Equations (4.3) and (4.5), the dot product equals

$$\begin{aligned} \text{Tr}(g, t) &= \frac{1}{(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)} \cdot \sum_{\ell=1}^n \left[ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+\ell}^{n+\ell-1} t_j \right) t^k \cdot \sum_{s=0}^{n-1} \left( \prod_{r=n-s+\ell}^{n+\ell-1} c_r \right) t^s \right] \\ &= \frac{1}{(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)} \cdot \sum_{v=0}^{2n-2} \left[ \sum_{\ell=1}^n \sum_{\substack{v=k+s \\ 0 \leq k, s \leq n-1}} \prod_{j=n-k+\ell}^{n+\ell-1} t_j \prod_{r=n-s+\ell}^{n+\ell-1} c_r \right] t^v \\ &= \frac{1}{(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)} \cdot \sum_{v=0}^{2n-2} \left[ \sum_{\substack{v=k+s \\ 0 \leq k, s \leq n-1}} \sum_{\ell=1}^n \prod_{j=n-k+\ell}^{n+\ell-1} t_j \prod_{r=n-s+\ell}^{n+\ell-1} c_r \right] t^v. \end{aligned}$$

Part (c) deals with  $\text{Tr}_{e_i A}(g_i, t)$  which equals the  $i$ -th entry of the vector  $\underline{\text{Tr}}_A(g, t)$ . By the proof of part (a) using Cramer's rule, we know that  $\text{Tr}_{e_i A}(g_i, t)$  must be a rational function with denominator of the form  $q(t) = (1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)$ . This is part (c). Regarding part (d), notice that all of  $c_1, \dots, c_n, t_1, \dots, t_n$  need to be roots of unity such that  $g$  has finite order. Therefore, the roots of  $1 - c_1 \cdots c_n t^n$  are roots of unity. Lastly, the maximal order of 1 as a root of  $q(t)$  is two since all roots of  $(1 - c_1 \cdots c_n t^n)$  are distinct.  $\square$

We can now prove a proposition which generalizes the ideas of [JZ97, Theorem 2.3]. It gives a different way to calculate the vector trace. In particular, using the following clear formula together with software like [WR, Mathematica, Version 10.4] provides the best way to effectively calculate traces for the examples in the next section.

**Proposition 4.1.5.** *Let  $A = \Pi_Q(k)$  and let  $g$  be a graded automorphism of  $A$  with  $g(e_i) = e_i$  for all  $i = 1, \dots, n$  of not necessarily finite order. Therefore, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Then the vector  $\underline{\text{Tr}}_A(g, t)$  equals*

$$\underline{\text{Tr}}_A(g, t) = \begin{pmatrix} 1 + c_1 t_1 t^2 & -c_1 t & & & -t_n t \\ & -t_1 t & 1 + c_1 t_1 t^2 & -c_2 t & \\ & & -t_2 t & 1 + c_1 t_1 t^2 & -c_3 t \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & -c_{n-1} t \\ -c_n t & & & & -t_{n-1} t & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

and for  $\overline{\text{Tr}}_A(g, t)$  a similar formula arises as

$$\overline{\text{Tr}}_A(g, t) = \begin{pmatrix} 1 + c_1 t_1 t^2 & -t_1 t & & & -c_n t \\ & -c_1 t & 1 + c_1 t_1 t^2 & -t_2 t & \\ & & -c_2 t & 1 + c_1 t_1 t^2 & -t_3 t \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots & -t_{n-1} t \\ -t_n t & & & & & -c_{n-1} t & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

*Proof.* Notice that  $g$  preserves the indecomposable projective modules  $P_i = e_i A$  as well as the graded submodules  $e_i A_{\geq 1}$ . The minimal projective resolution of  $S_i = e_i A / e_i A_{\geq 1}$  is

$$0 \longrightarrow e_i A \xrightarrow{\delta_2 = \begin{pmatrix} \alpha_i^* \\ -\alpha_{i-1} \end{pmatrix}} e_{i+1} A \oplus e_{i-1} A \xrightarrow{\delta_1 = \begin{pmatrix} \alpha_i & \alpha_{i-1}^* \end{pmatrix}} e_i A \xrightarrow{\delta_0} S_i \longrightarrow 0,$$

where  $i = 1, \dots, n$  and we understand all indices mod  $n$ . Denote  $g_i = g|_{e_i A}$  and let  $\tilde{g}_i$  be the induced map of  $g_i$  on  $S_i$ . Then  $\tilde{g}_i$  is a  $g$ -linear map from  $S_i$  to  $S_i$  or an  $A$ -linear map from  $S_i$  to  $S_i^g$ . Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_2 = e_i A & \xrightarrow{\delta_2} & Q_1 = e_{i+1} A \oplus e_{i-1} A & \xrightarrow{\delta_1} & Q_0 = e_i A & \xrightarrow{\delta_0} & S_i & \longrightarrow & 0 \\ & & \downarrow \phi_2^{(i)} & & \downarrow \phi_1^{(i)} & & \downarrow \phi_0^{(i)} = g_i & & \downarrow \tilde{g}_i = \text{id} & & \\ 0 & \longrightarrow & Q_2^g = e_i A^g & \xrightarrow{\delta_2} & Q_1^g = (e_{i+1} A \oplus e_{i-1} A)^g & \xrightarrow{\delta_1} & Q_0^g = (e_i A)^g & \xrightarrow{\delta_0} & S_i^g & \longrightarrow & 0. \end{array}$$

Thanks to the universal property of projective resolutions, [Jac80, Theorem 6.3], there exist graded  $A$ -linear maps  $\phi_j^{(i)} : Q_j \rightarrow Q_j^g$  such that  $\delta_2 \phi_2^{(i)} = \phi_1^{(i)} \delta_2$ ,  $\delta_1 \phi_1^{(i)} = \phi_0^{(i)} \delta_1$  and  $\delta_0 \phi_0^{(i)} = \tilde{g}_i \delta_0$ .

We compute the maps  $\phi_j^{(i)}$  explicitly. It is easy to see that  $\phi_0^{(i)} = g_i$  as  $\delta_0$  is the classical projection map. Let's look at the map

$$\phi_1^{(i)} : e_{i+1}A \oplus e_{i-1}A \rightarrow e_{i+1}A \oplus e_{i-1}A.$$

Since  $\phi_1^{(i)}$  is  $g$ -linear it is uniquely defined by knowing  $\phi_1^{(i)} \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) = \begin{pmatrix} de_{i+1} \\ se_{i-1} \end{pmatrix}$  and

$$\phi_1^{(i)} \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) = \begin{pmatrix} xe_{i+1} \\ ye_{i-1} \end{pmatrix} \text{ for some scalars } d, s, x, y \in k.$$

Since the square  $\phi_0^{(i)}\delta_1 = \delta_1\phi_1^{(i)}$  commutes, computing

$$\begin{aligned} \delta_1\phi_1^{(i)} \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) &= \delta_1 \left( \begin{pmatrix} de_{i+1} \\ se_{i-1} \end{pmatrix} \right) = d\alpha_i + s\alpha_{i-1}^*, \\ \phi_0^{(i)}\delta_1 \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) &= g_i\delta_1 \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) = g_i(\alpha_i) = c_i\alpha_i \end{aligned}$$

and

$$\begin{aligned} \delta_1\phi_1^{(i)} \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) &= \delta_1 \left( \begin{pmatrix} xe_{i+1} \\ ye_{i-1} \end{pmatrix} \right) = x\alpha_i + y\alpha_{i-1}^*, \\ \phi_0^{(i)}\delta_1 \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) &= g_i\delta_1 \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) = g_i(\alpha_{i-1}^*) = t_{i-1}\alpha_{i-1}^* \end{aligned}$$

gives  $s = x = 0$  as well as  $d = c_i$  and  $y = t_{i-1}$ . In order to find  $\phi_2^{(i)}$  we need to calculate

$$\phi_1^{(i)} \left( \begin{pmatrix} \alpha_i^* \\ -\alpha_{i-1} \end{pmatrix} \right) = \phi_1^{(i)} \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) g(\alpha_i^*) + \phi_1^{(i)} \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) g(-\alpha_{i-1}) = \begin{pmatrix} c_i t_i \alpha_i^* \\ -t_{i-1} c_{i-1} \alpha_{i-1} \end{pmatrix}. \quad (4.6)$$

As before,  $\phi_2^{(i)} : e_i A \rightarrow e_i A$  is uniquely defined by  $\phi_2^{(i)}(e_i) = z e_i$  for some  $z \in k$ .

Again, computing gives

$$\delta_2 \phi_2^{(i)}(e_i) = \delta_2(z e_i) = \begin{pmatrix} z \alpha_i^* \\ -z \alpha_{i-1} \end{pmatrix}, \quad \text{and}$$

$$\phi_1^{(i)} \delta_2(e_i) = \phi_1^{(i)} \left( \begin{pmatrix} \alpha_i^* \\ -\alpha_{i-1} \end{pmatrix} \right) \stackrel{(4.6)}{=} \begin{pmatrix} c_i t_i \alpha_i^* \\ -t_{i-1} c_{i-1} \alpha_{i-1} \end{pmatrix}.$$

As noticed in 4.1.1, the relations  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$  of  $\Pi_Q(k)$  force  $c_i t_i = c_{i-1} t_{i-1}$ . It follows that  $z = c_i t_i = c_1 t_1$ .

We now understand the  $\phi_j^{(i)}$  as  $g$ -linear maps from  $Q_j$  to  $Q_j$ . Since  $\phi_0^{(i)} = g_i$ , we get that  $\text{Tr}_{Q_0}(\phi_0^{(i)}, t) = \text{Tr}_{e_i A}(g_i, t)$ . If  $\phi_2^{(i)}$  is applied to  $\beta \in e_i A$  the  $g$ -linearity forces  $\phi_2^{(i)}(\beta) = \phi_2^{(i)}(e_i) g(\beta) = c_1 t_1 g(\beta)$  and therefore  $\text{Tr}_{Q_2}(\phi_2^{(i)}, t) = c_1 t_1 t^2 \text{Tr}_{e_i A}(g, t)$ . Notice that the  $t^2$  comes from the shift in the grading of  $Q_2$ . In the same way, using bases for each component individually, we obtain  $\text{Tr}_{Q_1}(\phi_1^{(i)}, t) = c_i t \text{Tr}_{e_{i+1} A}(g, t) + t_{i-1} t \text{Tr}_{e_{i-1} A}(g, t)$ .

Now, [JZ97, Lemma 2.1] applies and provides  $\sum_{j=0}^2 (-1)^j \text{Tr}_{Q_j}(\phi_j^{(i)}, t) = \text{Tr}_{S_i}(\tilde{g}_i, t)$

which translates to

$$\text{Tr}_{S_i}(\tilde{g}_i, t) = \text{Tr}_{e_i A}(g, t) - [c_i t \text{Tr}_{e_{i+1} A}(g, t) + t_{i-1} t \text{Tr}_{e_{i-1} A}(g, t)] + c_1 t_1 t^2 \text{Tr}_{e_i A}(g, t).$$

Finally, we can write the  $n$  equations using matrix notation as

$$\begin{pmatrix} \text{Tr}_{S_1}(\tilde{g}_1, t) \\ \text{Tr}_{S_2}(\tilde{g}_2, t) \\ \vdots \\ \text{Tr}_{S_n}(\tilde{g}_n, t) \end{pmatrix} = \begin{pmatrix} 1 + c_1 t_1 t^2 & -c_1 t & & -t_n t \\ -t_1 t & 1 + c_1 t_1 t^2 & -c_2 t & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & -c_{n-1} t \\ -c_n t & & & -t_{n-1} t & 1 + c_1 t_1 t^2 \end{pmatrix} \cdot \begin{pmatrix} \text{Tr}_{e_1 A}(g, t) \\ \text{Tr}_{e_2 A}(g, t) \\ \vdots \\ \text{Tr}_{e_n A}(g, t) \end{pmatrix}.$$

From  $g(e_i) = e_i$  we know that  $\tilde{g}_i = \text{id}$ . Therefore,  $\text{Tr}_{S_i}(\tilde{g}_i, t) = 1$  and bringing the matrix to the other side finishes the proof for  $\underline{\text{Tr}}_A(g, t)$ . The case  $\overline{\text{Tr}}_A(g, t)$  follows from analogous calculations.  $\square$

This closing proposition is independent from the rest of the section. While we will only be working with automorphisms which fix the vertices, it is nice to know that this subgroup of  $\text{Aut}_{\text{gr}}(A)$  is of finite index.

**Proposition 4.1.6.** *The graded automorphism group of  $A = \Pi_Q(k)$  for  $Q = \widetilde{A_{n-1}}$  is isomorphic to  $D_{2n} \ltimes N$ , where  $N = \{g \in \text{Aut}_{\text{gr}}(A) \mid g(e_i) = e_i, \text{ for } i = 1, \dots, n\}$  and  $D_{2n}$  denotes the dihedral group of order  $2n$ .*

*Proof.* We understand  $D_{2n}$  as the group of symmetries of the regular  $n$ -gon whose vertices are denoted by  $e_1, \dots, e_n$ . Let  $g \in \text{Aut}_{\text{gr}}(A)$ . Since  $g$  maps a pair of adjacent vertices to a pair of adjacent vertices, there exists an element  $d^{-1} \in D_{2n}$  such that  $g(e_i) = d^{-1}(e_i)$  for all  $i = 1, \dots, n$ . It follows that  $(d \circ g)(e_i) = e_i$  for all  $i = 1, \dots, n$ . Hence,  $d \circ g \in N$ . Further, for  $s \in D_{2n}$  and  $h \in N$  it is true that  $(s \circ h \circ s^{-1})(e_j) = (s \circ h)(e_i) = s(e_i) = e_j$

whenever  $s(e_i) = e_j$ . This shows that  $s \circ h \circ s^{-1} \in N$  which proves  $N$  is a normal subgroup of  $\text{Aut}_{\text{gr}}(A)$ .  $\square$

## 4.2 Examples

This section provides examples of automorphisms and fixed rings of  $A = \Pi_Q(k)$  for  $Q = \widetilde{A}_2$  where  $A_2$  stands for the quiver presented in Figure 4.2.1.

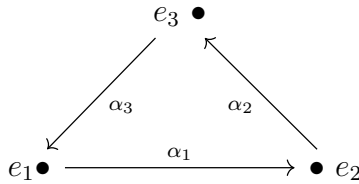


**Figure 4.2.1:** Dynkin diagram of  $\widetilde{A}_2$ .

We use the notation as before,  $g \in \text{Aut}_{\text{gr}}(A)$  maps:

$$\begin{aligned} \alpha_1 &\mapsto c_1 \alpha_1, & \alpha_1^* &\mapsto t_1 \alpha_1^*, \\ \alpha_2 &\mapsto c_2 \alpha_2, & \alpha_2^* &\mapsto t_2 \alpha_2^*, \\ \alpha_3 &\mapsto c_3 \alpha_3, & \alpha_3^* &\mapsto t_3 \alpha_3^*. \end{aligned}$$

Remark 4.1.1 proves  $H_A(t) = \frac{3}{(1-t)^2}$ . We always give an automorphism  $g$  and call  $G = \langle g \rangle$ .

Earlier in Section 4.1 was noted that the relations in  $A$  are  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$ . Recall that this forces  $c_1 t_1 = c_2 t_2 = c_3 t_3$  and whenever  $c_1 t_1 = c_2 t_2 = c_3 t_3$  for nonzero  $c_i$  and  $t_i$ , we obtain an automorphism of  $A$  by construction.

Proposition 4.1.4(b) provides the following formula of the trace of  $g$  as

$$\begin{aligned} \text{Tr}_A(g, t) = & \frac{3 + (c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t + (c_1c_2 + c_1c_3 + c_2c_3 + t_1t_2 + t_1t_3 + t_2t_3 + 3c_1t_1)t^2}{(1 - c_1c_2c_3t^3)(1 - t_1t_2t_3t^3)} \\ & + \frac{c_1t_1(c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t^3 + 3(c_1t_1)^2t^4}{(1 - c_1c_2c_3t^3)(1 - t_1t_2t_3t^3)}. \end{aligned}$$

Example 4.2.1 gives a fixed ring of global dimension 2 coming from an automorphism whose trace has 1 as a pole of order 1. A situation that does not happen in the case of Artin-Schelter regular algebras manifests in Example 4.2.2. Proposition 2.3.17 shows that  $A$  is a finitely generated  $A^G$ -module and Lemma 2.3.14(a) contributes  $A \cong A^G \oplus C$ . Thus, for AS-regular algebras, [KKZ08, Lemma 1.10 (c)] would imply that the fixed ring having finite global dimension implies it is regular. Then [KKZ08, Lemma 1.11 (d)] says  $\text{gl. dim}(A) = \text{gl. dim}(A^G)$  and  $A$  is free as an  $A^G$ -module. However, in Example 4.2.2,  $\text{gl. dim}(A^G) = 2$  and  $A$  is projective but not free as an  $A^G$ -module. Another situation impossible for AS-regular algebras to occur is revealed in Example 4.2.3 where the fixed ring has finite global dimension different from the global dimension of the preprojective algebra  $\Pi_Q(k)$ .

**Example 4.2.1.** Consider the following example where  $A$  is free over the fixed ring  $A^G$ .

We choose  $g$  with  $c_1 = 1$ ,  $c_2 = 1$ ,  $c_3 = 1$  and  $t_1 = -1$ ,  $t_2 = -1$ ,  $t_3 = -1$ . Thanks to the formula in the beginning of the section, the trace of  $g$  equals  $\text{Tr}_A(g, t) = \frac{3}{1 - t^2}$ .

Since  $c_1 = c_2 = c_3 = 1$ , all nonstar arrows are fixed. This implies that a mixed path is fixed if and only if its star part is fixed independently. Hence, it is enough to find the shortest fixed purely star paths. From  $t_1 = t_2 = t_3 = -1$  we know that these equal

$x = \alpha_3^* \alpha_2^*$ ,  $y = \alpha_1^* \alpha_3^*$  and  $z = \alpha_2^* \alpha_1^*$ . Using every path's expression as a simple path, we have found all generators of  $A^G$ . Since a path in  $A$  is uniquely determined by its length, its starting vertex and its number of nonstar arrows, the relations

$$x\alpha_2 = \alpha_1 y, \quad y\alpha_3 = \alpha_2 z \quad \text{and} \quad z\alpha_1 = \alpha_3 x$$

appear naturally. Hence, our initial guess is that the fixed ring is the path algebra

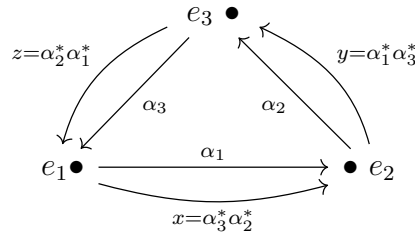


**Figure 4.2.2:** Path algebra underlying the fixed ring of Example 4.2.1.

with respect to the stated relations.

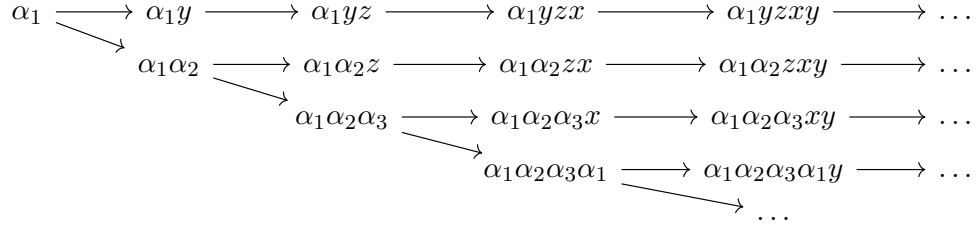
The following Gröbner basis calculation shows that we actually found a presentation of the fixed ring. We define the order  $\alpha_1 < \alpha_2 < \alpha_3 < x < y < z$ . Since there are no overlap or inclusion ambiguities, the paths not including the leading terms  $x\alpha_2$ ,  $y\alpha_3$  and  $z\alpha_1$  form a Gröbner basis by the Diamond Lemma. So we need to count them.

- Words starting with  $x$  can only look like

$$x \longrightarrow xy \longrightarrow xyz \longrightarrow xyzx \longrightarrow \dots$$

By symmetry, words starting with  $y$  or  $z$ , respectively, contribute the same to the Hilbert series of  $A^G$ . Each adds  $\frac{t^2}{(1-t^2)}$ .

- Words starting with  $\alpha_1$  can look like



where we see repetition starting at line 4. Combining the first, fourth, seventh and so on row gives  $\frac{t + t^3 + t^5}{(1 - t^6)(1 - t^3)}$ . Similarly, we get  $\frac{t^2 + t^4 + t^6}{(1 - t^6)(1 - t^3)}$  and  $\frac{t^3 + t^5 + t^7}{(1 - t^6)(1 - t^3)}$  from the second, fifth, eighth and so on line and third, sixth, ninth and so on line, respectively. This adds up to

$$\frac{t + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + t^7}{(1 - t^6)(1 - t^3)}.$$

The same has to be added for paths starting with  $\alpha_2$  and  $\alpha_3$ , respectively.

- Do not forget  $e_1, e_2$  and  $e_3$  which contribute 3.

In the end, we sum up

$$3 + \frac{3t^2}{(1 - t^2)} + \frac{3 \cdot (t + t^2 + 2t^3 + t^4 + 2t^5 + t^6 + t^7)}{(1 - t^6)(1 - t^3)} = \frac{3}{(1 - t)^2(1 + t)}.$$

We compare our result to the Hilbert series of  $A^G$ . According to Molien's Theorem, Lemma 2.3.14(c),  $H_{A^G}(t)$  equals  $\frac{3}{(1 - t)^2(1 + t)}$  which shows that we found a presentation for the fixed ring. It turns out that  $A \cong A^G \oplus A^G[-1]$  as right  $A^G$ -modules via the isomorphism  $\varphi : A^G \oplus A^G[-1] \longrightarrow A$ ,  $(\beta_1, \beta_2) \mapsto \beta_1 + \beta_2(\alpha_1^* + \alpha_2^* + \alpha_3^*)$ .

Linearity is clear. Also, it is easy to see that  $\varphi$  is surjective as every simple path with an even number of star arrows is fixed and thus in  $A^G$ , and every simple path with

an odd number of star arrows starting at  $e_i$  lives in  $A^G \alpha_i^*$ . Lastly,  $\varphi$  is injective since  $\varphi(\beta_1, \beta_2) = 0$  reduces to  $\beta_1 = \beta_2 = 0$  as follows. While  $\beta_1$  is a linear combination of simple paths with an even number of star arrows  $\beta_2(\alpha_1^* + \alpha_2^* + \alpha_3^*)$  can only be composed of simple paths with an odd number of star arrows. Therefore, there is no interaction between the two parts possible and the form of the basis of  $A$  and its relations do not allow  $\beta_2(\alpha_1^* + \alpha_2^* + \alpha_3^*)$  to be zero for a nonzero  $\beta_2$ .

In Example 4.2.1 everything is as for Artin-Schelter regular algebras. The trace of the examined graded automorphism has 1 as a pole of order  $1 = \text{GKdim}(A) - 1$ , the fixed ring  $A^G$  is twisted Calabi-Yau and  $A$  is a free module over  $A^G$ . The next examples will show that this does not always happen.

**Example 4.2.2.** This example provides a fixed ring of global dimension 2 where  $A$  is projective but not free over the fixed ring  $A^G$ . We choose  $g$  with  $c_1 = 1$ ,  $c_2 = -1$ ,  $c_3 = -1$  and  $t_1 = \zeta_3$ ,  $t_2 = -\zeta_3$ ,  $t_3 = -\zeta_3$  where  $\zeta_3 = \exp(2\pi i/3)$ . This means  $g$  acts as described in Figure 4.2.3.

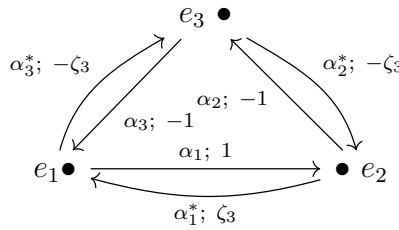


**Figure 4.2.3:** Dynkin diagram of  $\widetilde{A}_2$  including the group action of Example 4.2.2.

Clearly,  $g$  has order 6. Thanks to the formula in the beginning of the section and

Molien's Theorem, Lemma 2.3.14(c), the trace of the powers of  $g$  and the Hilbert series of  $A^G$  equal

$$\begin{aligned}\mathrm{Tr}_A(g, t) &= \frac{6 + (-1 - i\sqrt{3})t + (-4 + 4i\sqrt{3})t^2 + 2t^3 + (-3 - 3i\sqrt{3})t^4}{2(1 - t^3)^2}, \\ \mathrm{Tr}_A(g^2, t) &= \frac{6 + (3 - 3i\sqrt{3})t}{2(1 - t^3)}, \\ \mathrm{Tr}_A(g^3, t) &= \frac{3 - 5t + 3t^2}{(1 - t)(1 - t^3)}, \\ \mathrm{Tr}_A(g^4, t) &= \frac{6 + (3 + 3i\sqrt{3})t}{2(1 - t^3)}, \\ \mathrm{Tr}_A(g^5, t) &= \frac{6 + (-1 + i\sqrt{3})t + (-4 - 4i\sqrt{3})t^2 + 2t^3 + (-3 + 3i\sqrt{3})t^4}{2(1 - t^3)^2}, \\ \mathrm{Tr}_A(g^6, t) &= \mathrm{Tr}_A(\mathrm{id}_A, t) = \frac{3}{(1 - t)^2}, \quad \text{and} \\ \mathrm{H}_{A^G}(t) &= \frac{3 + t + t^2}{(1 - t^3)^2}.\end{aligned}$$

Notice that 1 is a pole of order 1 of  $\mathrm{Tr}_A(g, t)$ . In order to find generators for the fixed algebra  $A^G$ , where  $G = \langle g \rangle$ , we use the following strategy. Every minimal generator of  $A^G$  can have length at most four as every fixed path of length five or larger contains either three nonstar or three star arrows and therefore allows one to decompose the fixed element as concatenation of two individually fixed words. This allows us to find the generators

$$\alpha_1, \ x = \alpha_2\alpha_3, \ u_1 = \alpha_3^*\alpha_2^*\alpha_1^*, \ u_2 = \alpha_1^*\alpha_3^*\alpha_2^*, \ s = \alpha_3\alpha_1\alpha_2, \ t = \alpha_2^*\alpha_1^*\alpha_3^*$$

of the fixed ring.

Checking paths of the same type, we find at least the relations  $ts = st$ ,  $u_2x = xu_1$  and  $u_1\alpha_1 = \alpha_1u_2$ . With respect to the ordering  $\alpha_1 < x < u_1 < u_2 < s < t$  there do not

exist any overlap or inclusion ambiguities. Therefore, the reduced words form a basis by the Diamond Lemma. To sum up, we know the fixed ring must be isomorphic to a factor of the following path algebra  $B$  together with the stated relations:

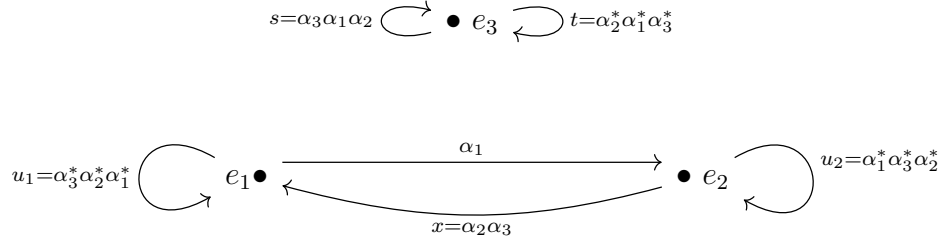
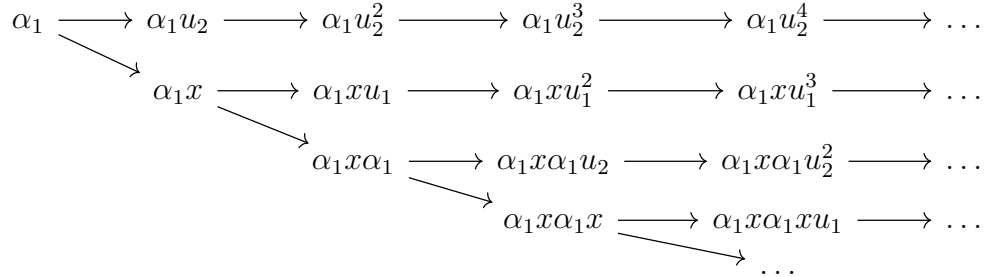


**Figure 4.2.4:** Path algebra underlying the fixed ring of Example 4.2.2.

To verify that this is the fixed ring, we count words not containing the leading terms  $ts$ ,  $u_2x$  and  $u_1\alpha_1$ .

- The part at  $e_3$  is isomorphic to a polynomial ring in two variables of degree 3. Its Hilbert series equals  $H_{e_3B}(t) = \frac{1}{(1-t^3)^2}$ .
- The relations imply that the loops  $u_i$  for  $i = 1, 2$  can only be followed up by the same loops again. Including the corresponding idempotents,  $\frac{1}{(1-t^3)}$  for each  $i$  is contributed to the Hilbert series.
- Lastly, we need to count paths starting with  $\alpha_1$  and  $x = \alpha_2\alpha_3$ . Let's start with  $\alpha_1$ .



Combine all even rows and all odd rows to see the repetition. The odd rows give  $\frac{t}{(1-t^3)^2}$  and the even rows give  $\frac{t^3}{(1-t^3)^2}$ . For paths starting with  $x$  a similar chain of possibilities arises:

$$\begin{array}{ccccccccccc}
x & \longrightarrow & xu_1 & \longrightarrow & xu_1^2 & \longrightarrow & xu_1^3 & \longrightarrow & xu_1^4 & \longrightarrow & \dots \\
& \searrow & & & & & & & & & \\
& & x\alpha_1 & \longrightarrow & x\alpha_1 u_2 & \longrightarrow & x\alpha_1 u_2^2 & \longrightarrow & x\alpha_1 u_2^3 & \longrightarrow & \dots \\
& & & \searrow & & & & & & & \\
& & & & x\alpha_1 x & \longrightarrow & x\alpha_1 x u_1 & \longrightarrow & x\alpha_1 x u_1^2 & \longrightarrow & \dots \\
& & & & & \searrow & & & & & \\
& & & & & & x\alpha_1 x \alpha_1 & \longrightarrow & x\alpha_1 x \alpha_1 u_2 & \longrightarrow & \dots \\
& & & & & & & & & & \searrow \\
& & & & & & & & & & \dots
\end{array}$$

Again, combining all the even rows and all the odd rows contributes a total of  $\frac{t^2 + t^3}{(1-t^3)^2}$  to the Hilbert series  $H_B(t)$ .

Putting the pieces together gives the following sum

$$H_B(t) = \frac{1}{(1-t^3)^2} + \frac{2}{(1-t^3)} + \frac{t+t^3}{(1-t^3)^2} + \frac{t^2+t^3}{(1-t^3)^2} = \frac{3+t+t^2}{(1-t^3)^2} = H_{A^{\langle g \rangle}}(t)$$

which shows that we have found a presentation for the fixed ring.

The previous calculation also helps to find  $H_{\underline{M}_{A^G}}(t)$ . Since the quiver underlying  $A^G$  has two connected components, only five entries in the  $(3 \times 3)$ -matrix are nonzero. The easiest part is  $H_{e_3 A^G e_3}(t) = \frac{1}{(1-t^3)^2}$  as noted above. Hence, the missing entries are  $H_{e_i A^G e_j}(t)$  for  $i, j = 1, 2$ . Regarding  $H_{e_1 A^G e_1}(t)$  (and analogously for  $H_{e_2 A^G e_2}(t)$ ) we find as basis  $e_1$ , powers of  $u_1$  and paths starting with  $\alpha_1$  and ending in either  $x$  or  $u_1$ . The first two combine to  $\frac{1}{1-t^3}$ . The latter part are exactly the even rows among the paths starting with  $\alpha_1$  which contribute  $\frac{t^3}{(1-t^3)^2}$ . This adds up to  $H_{e_1 A^G e_1} = \frac{1}{(1-t^3)^2}$ .

In the same way,  $H_{e_1 A^G e_2}$  has a basis consisting of all the elements starting with

$\alpha_1$  and ending in either  $u_2$  or  $\alpha_1$ . These elements are counted by the odd rows starting with  $\alpha_1$ . From above this means  $H_{e_1 A^G e_2} = \frac{t}{(1-t^3)^2}$ .

Analogously, a basis of  $e_2 A^G e_1$  consists of the elements of the odd rows starting with  $x$ . In other words,  $H_{e_2 A^G e_1} = \frac{t^2}{(1-t^3)^2}$ .

To sum up, we obtain

$$\underline{H}_{M_{A^G}}(t) = \frac{1}{(1-t^3)^2} \begin{pmatrix} 1 & t & \\ t^2 & 1 & \\ & & 1 \end{pmatrix}.$$

By [EE07, Proposition 3.2.1]

$$\underline{H}_{M_A}(t) = (I_3 - Ct + I_3 t^2)^{-1} = \frac{1}{(1-t^3)^2} \begin{pmatrix} 1+t^2+t^4 & t+t^2+t^3 & t+t^2+t^3 \\ t+t^2+t^3 & 1+t^2+t^4 & t+t^2+t^3 \\ t+t^2+t^3 & t+t^2+t^3 & 1+t^2+t^4 \end{pmatrix},$$

where  $C$  is the adjacency matrix of  $\bar{Q}$ . Inverting the matrix  $\underline{H}_{M_{A^G}}(t)$ , multiplying to  $\underline{H}_{M_A}(t)$  from the right and summing up finally gives

$$H_{e_1 A}(t) = (1+t^2) H_{e_1 A^G}(t) + t^2 H_{e_2 A^G}(t) + (t+t^2+t^3) H_{e_3 A^G}(t),$$

$$H_{e_2 A}(t) = (t+t^3) H_{e_1 A^G}(t) + H_{e_2 A^G}(t) + (t+t^2+t^3) H_{e_3 A^G}(t),$$

$$H_{e_3 A}(t) = (t+t^2) H_{e_1 A^G}(t) + t H_{e_2 A^G}(t) + (1+t^2+t^4) H_{e_3 A^G}(t)$$

or

$$\begin{aligned} H_A(t) &= (1+2t+2t^2+t^3) H_{e_1 A^G}(t) + (1+t+t^2) H_{e_2 A^G}(t) \\ &\quad + (1+2t+3t^2+2t^3+t^4) H_{e_3 A^G}(t) \end{aligned}$$

which shows that  $A$  is not free as an  $A^G$ -module. However, Proposition 5.1.5 will prove that  $A$  must be a projective  $A^G$ -module once we verify  $\text{gl. dim}(A^G) = 2$ .

With the notation  $P_i = e_i A^G$ , the proof of [Boc08, Theorem 3.2] states the beginning of the minimal projective resolution of a simple module of a factor of a path algebra. Applied to this example, thanks to computations we know that the minimal projective resolutions of  $S_i = P_i / (P_i)_{\geq 1}$  look like

$$\begin{aligned}
0 \longrightarrow P_2 &\xrightarrow{\begin{pmatrix} u_2 \\ -\alpha_1 \end{pmatrix}} P_2 \oplus P_1 \xrightarrow{(\alpha_1 \ u_1) \cdot} P_1 \longrightarrow S_1 \longrightarrow 0, \\
0 \longrightarrow P_1 &\xrightarrow{\begin{pmatrix} u_1 \\ -x \end{pmatrix}} P_1 \oplus P_2 \xrightarrow{(x \ u_2) \cdot} P_2 \longrightarrow S_2 \longrightarrow 0, \quad \text{and} \\
0 \longrightarrow P_3 &\xrightarrow{\begin{pmatrix} t \\ -s \end{pmatrix}} P_3 \oplus P_3 \xrightarrow{(s \ t) \cdot} P_3 \longrightarrow S_3 \longrightarrow 0.
\end{aligned}$$

Exactness at the 0-th and 1-st position comes for free thanks to [Boc08, Theorem 3.2].

At last, each map at the second position is injective because there are no relations of the form  $\alpha_1 \beta = 0$ ,  $x \beta = 0$  or  $s \beta = 0$  for appropriate nonzero  $\beta$ 's due to the relations and the chosen ordering. Therefore, [RR18, Proposition 3.19 (3)] provides  $\text{gl. dim}(A^G) = 2$ .

**Example 4.2.3.** In this example the fixed ring has global dimension 3. Define  $g$  to map

$$\begin{aligned}
\alpha_1 &\mapsto \zeta_3 \alpha_1, & \alpha_1^* &\mapsto \alpha_1^*, \\
\alpha_2 &\mapsto \alpha_2, & \alpha_2^* &\mapsto \zeta_3 \alpha_2^*, \\
\alpha_3 &\mapsto \zeta_3^2 \alpha_3, & \alpha_3^* &\mapsto \zeta_3^2 \alpha_3^*,
\end{aligned}$$

where  $\zeta_3 = \exp(2\pi i/3)$  and  $c_1 t_1 = \zeta_3$ . This means  $g$  acts as follows:

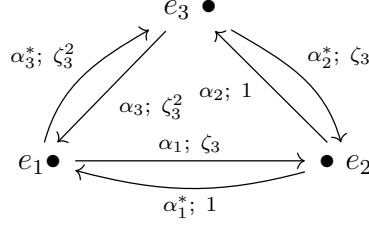


**Figure 4.2.5:** Dynkin diagram of  $\widetilde{A}_2$  including the group action of Example 4.2.3.

Thanks to the formula in the beginning of the section and Molien's Theorem, Lemma 2.3.14(c), the traces of  $g$  and  $g^2$  as well as the Hilbert series of  $A^G$  equal

$$\begin{aligned} \text{Tr}_A(g, t) &= \frac{3i(1-t^2)(-2i + (-i + \sqrt{3})t^2)}{2(1-t^3)^2}, \\ \text{Tr}_A(g^2, t) &= \frac{(-3i)(1-t^2)(2i + (i + \sqrt{3})t^2)}{2(1-t^3)^2}, \quad \text{and} \\ \text{H}_{A^G}(t) &= \frac{3 + 2t + 2t^2 + 2t^3}{(1-t^3)^2}. \end{aligned}$$

The same argument as in the previous example works, i.e. collecting all paths of length smaller than or equal to 4 which are fixed provides the generators of  $A^G$  for  $G = \langle g \rangle$ . Checking all words of length smaller than or equal to four provides a path algebra  $B$  as follows:

Notice that  $\alpha_1 \alpha_1^* \alpha_3^* = \alpha_3^* \alpha_2^* \alpha_2$ . We fix the ordering  $\alpha_2 < \alpha_1^* < x < y < u_1 < u_2$  and the following four relations become apparent:

$$u_1 y = y \alpha_2 x, \quad \alpha_1^* y \alpha_2 = \alpha_2 u_2, \quad \alpha_1^* u_1 = \alpha_2 x \alpha_1^*, \quad u_2 x = x \alpha_1^* y.$$

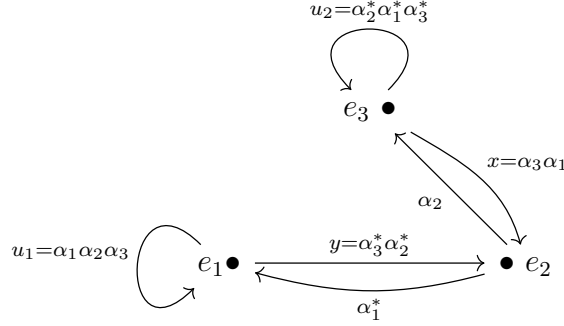


**Figure 4.2.6:** Path algebra underlying the fixed ring of Example 4.2.3.

The only overlap  $\alpha_1^* u_1 y$  resolves as

$$\alpha_1^*(y\alpha_2 x) - (\alpha_2 x \alpha_1^*)y = \alpha_2 u_2 x - \alpha_2 x \alpha_1^* y = \alpha_2 x \alpha_1^* y - \alpha_2 x \alpha_1^* y = 0.$$

This implies that a basis of  $B$  with the given relations consists of all words not containing any leading terms, i.e. no  $u_1 y$ ,  $\alpha_1^* y \alpha_2$ ,  $\alpha_1^* u_1$  and  $u_2 x$ . Now, we count words.

- Starting with  $u_1$ . Since  $u_1$  cannot come before  $y$ , the only possibilities are powers of  $u_1$ . Counting  $e_1$  here provides  $\frac{1}{1-t^3}$ .
- Starting with  $u_2$ . Similarly to  $u_1$  including  $e_3$ , we add  $\frac{1}{1-t^3}$  to the Hilbert series.
- To not forget  $e_2$ , we need to add 1.
- Starting with  $\alpha_1^*$ . Since  $\alpha_1^* u_1$  and  $\alpha_1^* y \alpha_2$  may not occur, the only sequence is

$$\alpha_1^* \longrightarrow \alpha_1^* y \longrightarrow \alpha_1^* y \alpha_1^* \longrightarrow \alpha_1^* y \alpha_1^* y \longrightarrow \dots$$

which contributes  $\frac{t+t^3}{1-t^3}$ .

- Starting with  $x$ . The following options occur

$$\begin{array}{ccccccc}
x & \longrightarrow & x\alpha_1^* & \longrightarrow & x\alpha_1^*y & \longrightarrow & x\alpha_1^*y\alpha_1^* & \longrightarrow & x\alpha_1^*y\alpha_1^*y & \longrightarrow & \dots \\
& \searrow & & & & & & & & & \\
& & x\alpha_2 & \longrightarrow & x\alpha_2u_2 & \longrightarrow & x\alpha_2u_2^2 & \longrightarrow & x\alpha_2u_2^3 & \longrightarrow & \dots \\
& & & \searrow & & & & & & & \\
& & & & x\alpha_2x & \longrightarrow & x\alpha_2x\alpha_1^* & \longrightarrow & x\alpha_2x\alpha_1^*y & \longrightarrow & \dots \\
& & & & & \searrow & & & & & \\
& & & & & & x\alpha_2x\alpha_2 & \longrightarrow & x\alpha_2x\alpha_2u_2 & \longrightarrow & \dots \\
& & & & & & & \searrow & & & \\
& & & & & & & & \dots & & 
\end{array}$$

Combining all the even rows as well as all odd rows, we count  $\frac{t^2 + t^3}{(1 - t^3)^2} + \frac{t^3}{(1 - t^3)^2}$ .

- Starting with  $y$ . This is very similar to terms beginning with  $x$ . The elements look like

$$\begin{array}{ccccccc}
y & \longrightarrow & y\alpha_1^* & \longrightarrow & y\alpha_1^*y & \longrightarrow & y\alpha_1^*y\alpha_1^* & \longrightarrow & y\alpha_1^*y\alpha_1^*y & \longrightarrow & \dots \\
& \searrow & & & & & & & & & \\
& & y\alpha_2 & \longrightarrow & y\alpha_2u_2 & \longrightarrow & y\alpha_2u_2^2 & \longrightarrow & y\alpha_2u_2^3 & \longrightarrow & \dots \\
& & & \searrow & & & & & & & \\
& & & & y\alpha_2x & \longrightarrow & y\alpha_2x\alpha_1^* & \longrightarrow & y\alpha_2x\alpha_1^*y & \longrightarrow & \dots \\
& & & & & \searrow & & & & & \\
& & & & & & y\alpha_2x\alpha_2 & \longrightarrow & y\alpha_2x\alpha_2u_2 & \longrightarrow & \dots \\
& & & & & & & \searrow & & & \\
& & & & & & & & \dots & & 
\end{array}$$

As for  $x$ , this adds  $\frac{t^2 + 2t^3}{(1 - t^3)^2}$  to the Hilbert series.

- Starting with  $\alpha_2$ . Consider

$$\begin{array}{ccccccc}
\alpha_2 & \longrightarrow & \alpha_2u_2 & \longrightarrow & \alpha_2u_2^2 & \longrightarrow & \alpha_2u_2^3 & \longrightarrow & \alpha_2u_2^4 & \longrightarrow & \dots \\
& \searrow & & & & & & & & & \\
& & \alpha_2x & \longrightarrow & \dots & & & & & & \\
& & & \searrow & & & & & & & \\
& & & & \dots & & & & & & 
\end{array}$$

Since no leading term of the relations starts with  $\alpha_2$  or  $\alpha_2x$ , everything but the first line contributes  $t$  multiplied to what words starting with  $x$  add to the Hilbert series.

The first row gives  $\frac{t}{1 - t^3}$ . Together, we get  $\frac{t}{1 - t^3} + \frac{t^3 + 2t^4}{(1 - t^3)^2}$ .

Finally, we can sum up and get

$$\begin{aligned} & \frac{1}{(1-t^3)} + \frac{1}{1-t^3} + 1 + \frac{t+t^3}{1-t^3} + \frac{2t^2+4t^3}{(1-t^3)^2} + \frac{t}{1-t^3} + \frac{t^3+2t^4}{(1-t^3)^2} \\ &= \frac{3+2t+2t^2+2t^3}{(1-t^3)^2} = H_{A^G}(t) \end{aligned}$$

which shows that we found a presentation for the fixed ring  $A^G$ .

In order to find the minimal projective resolutions we use the notation from before, i.e.  $S_i = e_i A / (e_i A)_{\geq 1} = e_i A^G / (e_i A^G)_{\geq 1}$  and  $P_i = e_i A^G$ . The proof of [Boc08, Theorem 3.2] again gives the beginnings of the easy minimal projective resolutions for  $S_1$  and  $S_3$ :

$$\begin{aligned} 0 \longrightarrow P_2 \xrightarrow{\begin{pmatrix} y \\ -\alpha_2 x \end{pmatrix}} P_1 \oplus P_2 &\xrightarrow{(u_1 \ y) \cdot} P_1 \longrightarrow S_1 \longrightarrow 0, \quad \text{and} \\ 0 \longrightarrow P_2 \xrightarrow{\begin{pmatrix} \alpha_1^* y \\ -x \end{pmatrix}} P_2 \oplus P_3 &\xrightarrow{(x \ u_2) \cdot} P_3 \longrightarrow S_3 \longrightarrow 0. \end{aligned}$$

As in the previous example, exactness at the 0-th and 1-st position is by construction and left multiplication by  $y$  ( $x$ , respectively) is injective due to the relations and the ordering.

This shows that we found the minimal projective resolutions of  $S_1$  and  $S_3$ . Let's look at  $S_2$ :

$$\dots \longrightarrow P_1 \oplus P_3 \xrightarrow{\begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix}} P_1 \oplus P_3 \xrightarrow{(\alpha_1^* \ \alpha_2) \cdot} P_2 \longrightarrow S_2 \longrightarrow 0.$$

Again we have exactness in the 0-th and 1-st spot by [Boc08, Theorem 3.2]. We need to

understand the kernel of the map

$$\Gamma : P_1 \oplus P_3 \longrightarrow P_1 \oplus P_3, \quad \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \mapsto \begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}.$$

Let  $\begin{pmatrix} f \\ g \end{pmatrix} \in \ker(\Gamma) \subseteq P_1 \oplus P_3$ . Due to the relations, we can write  $f = \sum_{j=0}^N c_j u_1^j + cyf_1$  and  $g = \sum_{\ell=0}^M d_\ell u_2^\ell + dxg_2$  for  $c, c_j, d, d_\ell \in k$  as well as  $M, N \in \mathbb{N}_0$  and  $f_1, g_2 \in P_2$ . Then we obtain

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \Gamma \left( \begin{pmatrix} f \\ g \end{pmatrix} \right) = \begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix} \cdot \begin{pmatrix} \sum_{j=0}^N c_j u_1^j + cyf_1 \\ \sum_{\ell=0}^M d_\ell u_2^\ell + dxg_2 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=0}^N c_j u_1^{j+1} + cu_1 y f_1 + \sum_{\ell=0}^M d_\ell y \alpha_2 u_2^\ell + dy \alpha_2 x g_2 \\ -\sum_{j=0}^N c_j x \alpha_1^* u_1^j - cx \alpha_1^* y f_1 - \sum_{\ell=0}^M d_\ell u_2^{\ell+1} - du_2 x g_2 \end{pmatrix}. \end{aligned}$$

In the first entry, only  $\sum_{j=0}^N c_j u_1^{j+1}$  consists of only nonstar arrows while the rest of terms always contain  $y$ , which has two star arrows. This implies  $c_j = 0$  for all  $j$ . Similarly,  $\sum_{\ell=0}^M d_\ell u_2^{\ell+1}$  are the only summands with only star arrows in the second entry and so  $d_\ell = 0$  for all  $\ell$ . Hence, we can assume that  $f = yf_1$  and  $g = xg_2$ .

It is easy to check that  $\begin{pmatrix} yf \\ -xf \end{pmatrix} \in \ker(\Gamma)$  for all  $f \in P_2$ :

$$\begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix} \cdot \begin{pmatrix} yf \\ -xf \end{pmatrix} = \begin{pmatrix} u_1 y f - y \alpha_2 x f \\ -x \alpha_1^* y f + u_2 x f \end{pmatrix} = \begin{pmatrix} (u_1 y - y \alpha_2 x) f \\ (-x \alpha_1^* y + u_2 x) f \end{pmatrix} = 0.$$

This implies that  $\begin{pmatrix} 0 \\ x(f_1 + g_2) \end{pmatrix} \in \ker(\Gamma)$  as we know that  $\begin{pmatrix} yf_1 \\ xg_2 \end{pmatrix}, \begin{pmatrix} y(-f_1) \\ xf_1 \end{pmatrix} \in \ker(\Gamma)$ .

Therefore, we can compute

$$\begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ x(f_1 + g_2) \end{pmatrix} = \begin{pmatrix} y\alpha_2 x(f_1 + g_2) \\ -u_2 x(f_1 + g_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first entry,  $y\alpha_2 x(f_1 + g_2) = 0$ , implies that  $g_2 = -f_1$ . This follows from  $y\alpha_2 x$  not appearing as or including a leading term of one of the relations. To sum up, we can write

$$\ker(\Gamma) = \left\{ \begin{pmatrix} y \\ -x \end{pmatrix} \cdot f_1 \mid f_1 \in P_2 \right\}$$

and the minimal projective resolution becomes

$$0 \longrightarrow P_2 \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} P_1 \oplus P_3 \xrightarrow{\begin{pmatrix} u_1 & y\alpha_2 \\ -x\alpha_1^* & -u_2 \end{pmatrix}} P_1 \oplus P_3 \xrightarrow{(\alpha_1^* \ \alpha_2)} P_2 \longrightarrow S_2 \longrightarrow 0,$$

which is injective at the last spot for the same reason as before: left multiplication by  $x$  is injective due to the relations and the order. Once again, [RR18, Proposition 3.19 (3)] says the global dimension of  $A^G$  equals the maximal projective dimension of the simple modules and thus  $\text{gl.dim}(A^G) = \text{pdim}(S_2) = 3$ .

The last piece of this chapter helps understanding graded automorphisms  $g$  of  $\Pi_Q(k)$  for  $Q = \widetilde{A}_2$  with 1 as a pole of order  $\text{GKdim}(\Pi_Q(k)) = 2$ . It was shown in [KKZ08, Proposition 1.8] that for regular domains, the denominator of the trace of a nonidentity automorphism cannot have 1 as a root of order equal to the GK-dimension

of the algebra. This result encouraged their definition of a quasi-reflection. Kirkman, Kuzmanovich and Zhang call a graded automorphism  $\sigma$  of an AS-regular domain  $S$  a quasi-reflection in [KKZ08, Definition 2.2] if it has 1 as a root of the denominator of its trace  $\text{Tr}_S(\sigma, t) = p_\sigma(t)^{-1}$  of multiplicity equal to  $\text{GKdim}(S) - 1$ . In contrast, for preprojective algebras many graded automorphisms have 1 as a pole of the trace of order  $2 = \text{GKdim}(\Pi_Q(k))$ . An easy example to see this occurs for  $Q = \widetilde{A}_2$  with  $c_1 = t_1 = 1$  and  $c_2 = c_3 = t_2 = t_3 = -1$ . Using the formula from the beginning of the section, we obtain  $\text{Tr}_A(g, t) = \frac{3 - 5t + 3t^2}{(1 - t)(1 - t^3)}$ .

A straight-forward approach to define quasi-reflections would be as follows. Instead of looking at 1 as a root of the denominator of the trace function, one could consider the order of 1 as a pole of  $\text{Tr}(g, t)$ . The natural definition would be to call an automorphism  $g$  a quasi-reflection if the order of 1 as a pole of  $\text{Tr}(g, t)$  equals  $\text{GKdim}(A) - 1$ . To get a feel for this idea one needs to understand the nontrivial graded automorphisms whose trace have 1 as a pole of order 2. For  $n = 3$ , the fixed rings  $A^{(g)}$  always have infinite global dimension which means we do not want to call these quasi-reflections.

**Proposition 4.2.4.** *Let  $g \neq \text{id}$  be a graded automorphism of  $A = \Pi_Q(k)$  for  $Q = \widetilde{A}_2$ . Denote  $\text{Tr}_A(g, t) = \frac{p(t)}{q(t)}$  and keep the formula given in the beginning of this section in mind. Then, the numerator  $p(t)$  factors for  $t = 1$  as follows:*

$$p(1) = (3 + c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3)(1 + (c_1t_1) + (c_1t_1)^2) \quad \text{if } c_1c_2c_3 = 1,$$

$$p(1) = (3 + t_1 + t_2 + t_3 + t_1t_2 + t_2t_3 + t_1t_3)(1 + (c_1t_1) + (c_1t_1)^2) \quad \text{if } t_1t_2t_3 = 1.$$

*If  $\text{Tr}_A(g, t)$  has 1 as a pole of order 2, then  $A^{(g)}$  has infinite global dimension.*

*Proof.* Recall the formula of the trace due to Proposition 4.1.4:

$$\begin{aligned}\mathrm{Tr}_A(g, t) &= \frac{3 + (c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t + (c_1c_2 + c_1c_3 + c_2c_3 + t_1t_2 + t_1t_3 + t_2t_3 + 3c_1t_1)t^2}{(1 - c_1c_2c_3t^3)(1 - t_1t_2t_3t^3)} \\ &\quad + \frac{c_1t_1(c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t^3 + 3(c_1t_1)^2t^4}{(1 - c_1c_2c_3t^3)(1 - t_1t_2t_3t^3)}.\end{aligned}$$

Let  $g$  be such that  $\mathrm{Tr}_A(g, t)$  has 1 as a pole of order 2. As noted before, the relations force  $c_1t_1 = c_2t_2 = c_3t_3$ . Also, in order to have 1 as a pole of order 2, we need  $c_1c_2c_3 = t_1t_2t_3 = 1$ . Hence,  $(c_1t_1)^3 = c_1c_2c_3 \cdot t_1t_2t_3 = 1$  and so  $c_1t_1$  must be a third root of unity. Consider the following calculation which shows how  $p(1)$  can be factored:

$$\begin{aligned}& [3 + (c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3)] \cdot [1 + c_1t_1 + (c_1t_1)^2] \\ &= 3 + (c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3) + 3c_1t_1 \\ &\quad + c_1t_1(c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3) \\ &\quad + 3(c_1t_1)^2 + (c_1t_1)^2(c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3) \\ &= 3 + (c_1 + c_2 + c_3 + (c_1t_1)(c_1c_2 + c_2c_3 + c_1c_3)) \\ &\quad + (c_1c_2 + c_2c_3 + c_1c_3 + (c_1t_1)^2(c_1 + c_2 + c_3) + 3c_1t_1) \\ &\quad + ((c_1t_1)(c_1 + c_2 + c_3) + (c_1t_1)^2(c_1c_2 + c_2c_3 + c_1c_3)) + 3(c_1t_1)^2 \\ &= 3 + (c_1 + c_2 + c_3 + t_3 + t_1 + t_2) + (c_1c_2 + c_2c_3 + c_1c_3 + t_2t_3 + t_1t_3 + t_1t_2 + 3c_1t_1) \\ &\quad + (c_1t_1)(c_1 + c_2 + c_3 + t_3 + t_1 + t_2) + 3(c_1t_1)^2.\end{aligned}$$

This means if

$$1 + c_1t_1 + (c_1t_1)^2 = 0 \quad \text{or}$$

$$3 + (c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3) = 0$$

we can have 1 as a pole of order at most 1. As  $(c_1 t_1)^3 = 1$ , the only possibility such that  $1 + c_1 t_1 + (c_1 t_1)^2$  is nonzero occurs if  $c_1 t_1 = 1$ . To sum up, our choices for  $c_i$  and  $t_i$  are

$$\begin{aligned} \alpha_1 &\mapsto c_1 \alpha_1, & \alpha_2 &\mapsto c_2 \alpha_2, & \alpha_3 &\mapsto \frac{1}{c_1 c_2} \alpha_3, \\ \alpha_1^* &\mapsto \frac{1}{c_1} \alpha_1^*, & \alpha_2^* &\mapsto \frac{1}{c_2} \alpha_2^*, & \alpha_3^* &\mapsto c_1 c_2 \alpha_3^*. \end{aligned}$$

Since  $c_1 c_2 c_3 = 1 = t_1 t_2 t_3$  the argument to find the generators of the fixed ring used in Example 4.2.2 applies: it is enough to look at fixed paths of degree at most four as every fixed path can be decomposed as a product of fixed paths of degree less than or equal to four. We look at the following two cases.

Either  $c_1 = 1, c_2 \neq 1$ , or  $c_2 = 1, c_1 \neq 1$  or  $c_1 \neq 1, c_2 \neq 1, c_1 c_2 = 1$ . The list of elements from Table 4.2.1 helps to understand the fixed ring.

**Table 4.2.1:** List of eigenvectors for all possible eigenvalues.

| eigenvalue       | eigenvector                                                                                                                                                                                                                                                                                                                                                                                              |
|------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| 1                | $e_1, e_2, e_3, \alpha_1 \alpha_1^*, \alpha_2 \alpha_2^*, \alpha_3 \alpha_3^*, \alpha_1 \alpha_2 \alpha_3, \alpha_3^* \alpha_2^* \alpha_1^*, \alpha_2 \alpha_3 \alpha_1, \alpha_1^* \alpha_3^* \alpha_2^*,$<br>$\alpha_3 \alpha_1 \alpha_2, \alpha_2^* \alpha_1^* \alpha_3^*, \alpha_1 \alpha_2 \alpha_2^* \alpha_1^*, \alpha_2 \alpha_3 \alpha_3^* \alpha_2^*, \alpha_3 \alpha_1 \alpha_1^* \alpha_3^*$ |
| $c_1$            | $\alpha_1, \alpha_3^* \alpha_2^*, \alpha_1 \alpha_2 \alpha_2^*, \alpha_1 \alpha_2 \alpha_3 \alpha_1, \alpha_1 \alpha_1^* \alpha_3^* \alpha_2^*$                                                                                                                                                                                                                                                          |
| $c_1^{-1}$       | $\alpha_1^*, \alpha_2 \alpha_3, \alpha_2 \alpha_2^* \alpha_1^*, \alpha_2 \alpha_3 \alpha_1 \alpha_1^*, \alpha_1^* \alpha_3^* \alpha_2^* \alpha_1^*$                                                                                                                                                                                                                                                      |
| $c_2$            | $\alpha_2, \alpha_1^* \alpha_3^*, \alpha_2 \alpha_3 \alpha_3^*, \alpha_2 \alpha_3 \alpha_1 \alpha_2, \alpha_2 \alpha_2^* \alpha_1^* \alpha_3^*$                                                                                                                                                                                                                                                          |
| $c_2^{-1}$       | $\alpha_2^*, \alpha_3 \alpha_1, \alpha_3 \alpha_3^* \alpha_2^*, \alpha_3 \alpha_1 \alpha_2 \alpha_2^*, \alpha_2^* \alpha_1^* \alpha_3^* \alpha_2^*$                                                                                                                                                                                                                                                      |
| $c_1 c_2$        | $\alpha_3^*, \alpha_1 \alpha_2, \alpha_1 \alpha_1^* \alpha_3^*, \alpha_1 \alpha_2 \alpha_3 \alpha_3^*, \alpha_3^* \alpha_2^* \alpha_1^* \alpha_3^*$                                                                                                                                                                                                                                                      |
| $(c_1 c_2)^{-1}$ | $\alpha_3, \alpha_2^* \alpha_1^*, \alpha_3 \alpha_1 \alpha_1^*, \alpha_3 \alpha_1 \alpha_2 \alpha_3, \alpha_3 \alpha_3^* \alpha_2^* \alpha_1^*$                                                                                                                                                                                                                                                          |

So, if precisely one of  $c_1$ ,  $c_2$ ,  $c_1c_2$  equals 1, we obtain the same diagram up to a rotation. For example in case  $c_1c_2 = 1$ , the path algebra underlying  $A^G$  is shown in Figure 4.2.7 with relations  $uw = vu$ ,  $uv = vu$ ,  $vw = wv$  and  $wv = u^3$  for the right connected component and relations  $xy = (\alpha_3^*\alpha_3)^2$ ,  $yx = (\alpha_3\alpha_3^*)^2$ ,  $x\alpha_3\alpha_3^* = \alpha_3^*\alpha_3x$ , and  $y\alpha_3^*\alpha_3 = \alpha_3\alpha_3^*y$  for the left connected component. Note that  $\alpha_1\alpha_1^* = \alpha_3^*\alpha_3$  and  $\alpha_2\alpha_3\alpha_3^*\alpha_2^* = u^2$ .

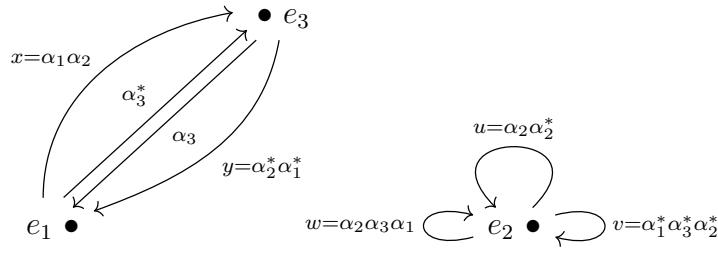


**Figure 4.2.7:** Path algebra underlying the fixed ring in case  $c_1c_2 = 1$ .

Consider the right component  $k[u, v, w]/(u^3 - vw)$  as the coordinate ring of the affine variety  $Y = Z(u^3 - vw)$ . The beginning of [Har77, Chapter 5] says that the point  $P = (0, 0, 0)$  is singular since the Jacobian matrix  $\begin{pmatrix} 3u^2 & -w & -v \end{pmatrix}$  is zero at  $P$ . Then, [Har77, Theorem 5.1] implies that the local ring  $\mathcal{O}_{P,Y}$  is not regular and as a localization this also makes the local ring  $k[u, v, w]/(u^3 - vw)$  not regular. By [Ser56, Théorème 3], a noetherian local ring which is not regular must have infinite global dimension.

In the second case, all three of  $c_1$ ,  $c_2$ ,  $c_1c_2$  are not equal to 1. Hence, the only

elements fixed of degree at most four are

$$e_1, e_2, e_3,$$

$$\alpha_1 \alpha_1^*, \alpha_2 \alpha_2^*, \alpha_3 \alpha_3^*,$$

$$\alpha_1 \alpha_2 \alpha_3, \alpha_2 \alpha_3 \alpha_1, \alpha_3 \alpha_1 \alpha_2, \alpha_3^* \alpha_2^* \alpha_1^*, \alpha_1^* \alpha_3^* \alpha_2^*, \alpha_2^* \alpha_1^* \alpha_3^*,$$

$$\alpha_1 \alpha_2 \alpha_2^* \alpha_1^*, \alpha_2 \alpha_3 \alpha_3^* \alpha_2^*, \alpha_3 \alpha_1 \alpha_1^* \alpha_3^*.$$

Therefore, the fixed ring is isomorphic to the path algebra from Figure 4.2.8 with relations

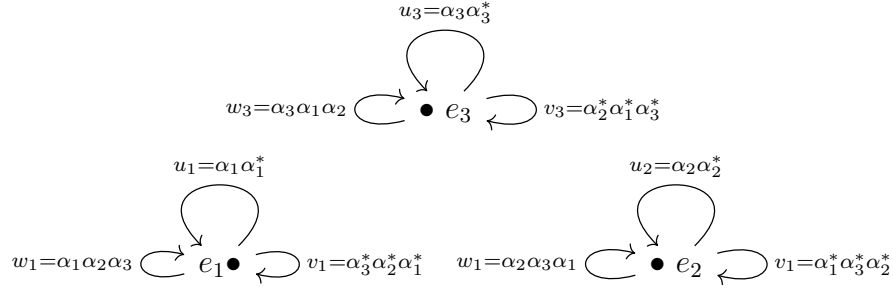


**Figure 4.2.8:** Path algebra underlying the fixed ring in case  $1 \notin \{c_1, c_2, c_1 c_2\}$ .

$u_i w_i = w_i u_i$ ,  $u_i v_i = v_i u_i$ ,  $v_i w_i = w_i v_i$  and  $w_i v_i = u_i^3$  for  $i = 1, 2, 3$ . The same reasoning as in part (a) proves that this algebra has infinite global dimension.  $\square$

### 4.3 Examples where $A^G$ is isomorphic to $A$

Again, fix  $Q = \widetilde{A_{n-1}}$  and let  $A = \Pi_Q(k)$ . We still consider graded automorphisms of  $A$  which fix the vertices. Recall the notation, for such  $g \in \text{Aut}_{\text{gr}}(A)$  we can define  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ .

In the introduction of [KKZ08], the authors discuss the remarkable rigidity theorem by Alev-Polo. Unlike for commutative algebras where every fixed ring under a group action of a reflection group is again isomorphic to the polynomial ring, noncommutative algebras are more rigid. Therefore, the question was raised in [KKZ08, Question 1.3] under what conditions on the ring  $R$  and the group  $G$  it follows that  $R$  is isomorphic to  $R^G$ . In this short section, we will provide some graded automorphism groups of  $A$  which satisfy that  $A^G$  is isomorphic to  $A$  as ungraded algebras.

**Proposition 4.3.1.** *Let  $g \in \text{Aut}_{\text{gr}}(A)$  with  $c_1 = c_2 = \dots = c_n$ . Since  $c_1 t_1 = c_i t_i$  for all  $i$ , this implies  $t_1 = t_2 = \dots = t_n$ . If  $\gcd(o(c_1), o(t_1)) = 1$  as well as  $o(c_1) \equiv o(t_1) \equiv 1 \pmod n$ , then  $A^{\langle g \rangle}$  is isomorphic to  $A$  as ungraded algebras.*

*Proof.* We need to find the generators of the fixed ring. By symmetry, it is enough to find the generators starting at  $e_1$ . Let's introduce the following notation:

$$o(c_1) = m_c = r_c \cdot n + 1, \quad \text{and}$$

$$o(t_1) = m_t = r_t \cdot n + 1.$$

Since  $g((\alpha_1 \cdots \alpha_n)^{r_c} \alpha_1) = c_1^{r_c \cdot n + 1} (\alpha_1 \cdots \alpha_n)^{r_c} \alpha_1 = (\alpha_1 \cdots \alpha_n)^{r_c} \alpha_1$  is fixed and no shorter purely nonstar path starting at  $e_1$  is fixed, we found a generator of  $A^{\langle g \rangle}$ , say  $\beta_1 = (\alpha_1 \cdots \alpha_n)^{r_c} \alpha_1$ . Analogously,  $\beta_n^* = (\alpha_n^* \cdots \alpha_1^*)^{r_t} \alpha_n^*$  and as always  $e_1$  are generators of  $A^{\langle g \rangle}$ . Corresponding definitions at the other vertices provide generators  $\beta_1, \dots, \beta_n$  and  $\beta_1^*, \dots, \beta_n^*$ .

It is left to show that these are all the generators of  $A^{\langle g \rangle}$ . Assume  $\gamma$  is a mixed

fixed nontrivial path of  $A$  written in its simple form, the nonstar arrows appearing first. Say,  $\text{type}(\gamma) = (a, b)$ . Without loss of generality we have  $(a, b) < (m_c, m_t)$  as otherwise we can decompose  $\gamma$ . Since  $g(\gamma) = \gamma$ , it follows that  $c_1^a t_1^b = 1$ , or  $c_1^a = t_1^{-b}$ . Taking the  $m_c$ -th power yields  $t_1^{-b \cdot m_c} = 1$ . Since  $\gcd(m_c, m_t) = 1$ , we must have that  $m_t$  divides  $b$ . This gives  $b = 0$ . By symmetry, also  $a = 0$  and this contradicts that  $\gamma$  is nontrivial. This means we found all generators of  $A^{\langle g \rangle}$ .

By construction, all  $\beta_i$  have the same degree in  $A$  and all  $\beta_i^*$  have the same degree in  $A$ . This implies that we must have the following relations in  $A^{\langle g \rangle}$ :

$$\beta_i \beta_i^* = \beta_{i-1}^* \beta_{i-1}$$

with indices taken modulo  $n$  if necessary. Consider the algebra  $B$  with generators  $e_i, \beta_i$  and  $\beta_i^*$  for  $i = 1, \dots, n$  and the relations  $\beta_i \beta_i^* = \beta_{i-1}^* \beta_{i-1}$  which is a quotient of the following path algebra:

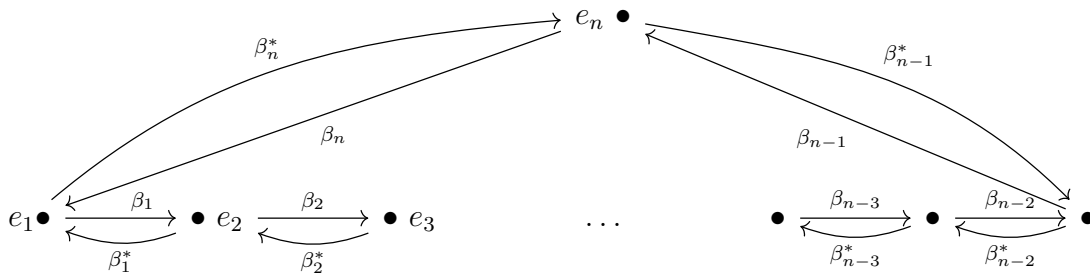


**Figure 4.3.1:** Path algebra underlying the fixed ring  $A^{\langle g \rangle}$  of Proposition 4.3.1.

These are the same relations as in  $A$ , so every path  $\gamma$  in  $B$  can be expressed by first writing the nonstar part and then the star part. Since this is the reduced form of  $\gamma$  in  $A$ , there cannot be any additional relations. This makes finding the Hilbert series of  $e_i B$  easy.  $H_{e_i B}(t) = \frac{1}{(1-t^{m_t})(1-t^{m_c})}$  and consequently  $H_B(t) = \frac{n}{(1-t^{m_t})(1-t^{m_c})}$ .

The only claim left to show is that  $A^{\langle g \rangle} \cong A$ . It is easy to see that

$$\varphi : A \rightarrow A^{\langle g \rangle}, \quad e_i \mapsto e_i, \quad \alpha_i \mapsto \beta_i, \quad \alpha_i^* \mapsto \beta_i^*$$

extends uniquely to a ring homomorphism. It is obvious that  $\varphi$  is bijective.  $\square$

Similarly, the fixed ring can interchange nonstar and star arrows and still be isomorphic to  $A$  as ungraded algebras which we state in the following corollary.

**Corollary 4.3.2.** *Let  $g \in \text{Aut}_{\text{gr}}(A)$  with  $c_1 = c_2 = \dots = c_n$ . Since  $c_1 t_1 = c_i t_i$  for all  $i$ , this implies  $t_1 = t_2 = \dots = t_n$ . If  $\gcd(o(c_1), o(t_1)) = 1$  as well as  $o(c_1) \equiv o(t_1) \equiv -1 \pmod{n}$ , then  $A^{\langle g \rangle}$  is isomorphic to  $A$  as ungraded algebras.*

*Proof.* This time, after noticing that the order of  $c_1$  and  $t_1$  is at least  $n-1$ , we use the adjusted notation:

$$o(c_1) = m_c = r_c \cdot n + n - 1, \quad \text{and}$$

$$o(t_1) = m_t = r_t \cdot n + n - 1.$$

As before, the generators starting at  $e_1$  give enough information to find all generators. Since  $\gcd(m_c, m_t) = 1$  again, the elements  $\beta_1 = (\alpha_1 \cdots \alpha_n)^{r_c} \alpha_1 \cdots \alpha_{n-2} \alpha_{n-1}$  and analogously  $\beta_1^* = (\alpha_n^* \alpha_{n-1}^* \cdots \alpha_1^*)^{r_t} \alpha_n^* \alpha_{n-1}^* \cdots \alpha_2^*$  as well as appropriate definitions for  $\beta_i$  and  $\beta_i^*$

for  $i = 2, \dots, n$  minimally generate the fixed ring. A similar analysis as in Proposition 4.3.1 using the types of these generators yields as fixed ring the following path algebra from Figure 4.3.2 subject to the relations  $\beta_i \beta_i^* = \beta_{i+1}^* \beta_{i+1}$ .

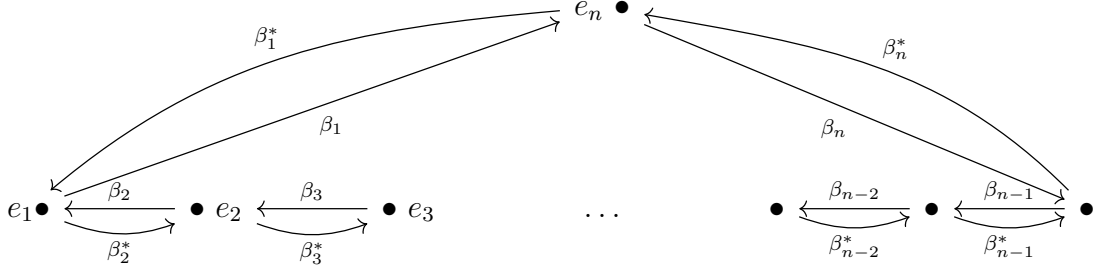


**Figure 4.3.2:** Path algebra underlying the fixed ring  $A^{(g)}$  of Corollary 4.3.2.

In conclusion, it is easy to see that

$$\varphi : A \rightarrow A^{(g)}, \quad e_i \mapsto e_i, \quad \alpha_i \mapsto \beta_{i+1}^*, \quad \alpha_i^* \mapsto \beta_{i+1}$$

extends uniquely to a ring homomorphism. It is obvious that  $\varphi$  is bijective.  $\square$

**Example 4.3.3.** Fix  $Q = \widetilde{A_2}$  and let  $A = \Pi_Q(k)$ . Let  $\zeta_7 = \exp(\frac{2\pi i}{7})$  and define  $g$  via

$$\alpha_i \mapsto i\alpha_i, \quad \alpha_i^* \mapsto \zeta_7 \alpha_i^*.$$

From Proposition 4.3.1 it follows that the fixed ring is isomorphic to the path algebra for the quiver from Figure 4.3.3 subject to the relations

$$x_1 y_1 = y_3 x_3, \quad x_2 y_2 = y_1 x_1, \quad x_3 y_3 = y_2 x_2,$$

which is isomorphic to  $A$  as ungraded algebras.

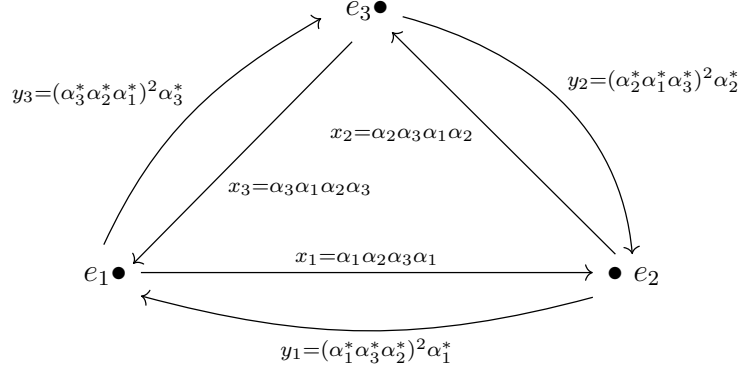


**Figure 4.3.3:** Path algebra underlying the fixed ring of Example 4.3.3.

This chapter contains material from “Properties of the fixed ring of a preprojective algebra”, Weispfenning, Stephan, to appear and “Generalized Gorensteinness and a homological determinant for preprojective algebras”, Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of both papers.

# Chapter 5

## Properties of the fixed ring of a preprojective algebra

### 5.1 Connecting three properties

In this section, let  $B$  be a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra of global dimension  $d \geq 1$  with degree zero piece  $B_0 \cong k^n$ . Lemma 2.3.6 combined with Proposition 2.3.8 implies that  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition. Denote the primitive orthogonal idempotents by  $e_i$ . This gives us all simple right  $B$ -modules  $S_i = e_i B / (e_i B)_{\geq 1}$  and all simple left  $B$ -modules  $S_i^* = B e_i / (B e_i)_{\geq 1}$ , respectively. Fix a finite subgroup  $G$  of the graded automorphism group of  $B$  such that every  $g \in G$  satisfies  $g(e_i) = e_i$ . Notice that this implies  $(B^G)_0 = B_0 \cong k^n$ .

We will present connections between  $B^G$  having finite global dimension together

with satisfying the generalized Gorenstein condition and  $\text{gl.dim}(B) = \text{gl.dim}(B^G)$  as well as  $B$  being a projective  $B^G$ -module. In particular, the only direction missing from proving that the mentioned three conditions are equivalent is to show that  $\text{gl.dim}(B^G) = \text{gl.dim}(B)$  implies  $B^G$  satisfies the generalized Gorenstein condition. We will be able to obtain this remaining piece under some stronger assumptions in the next section.

In particular, if  $Q$  is the quiver corresponding to an extended Dynkin graph of type  $\widetilde{A}_{n-1}$ ,  $\widetilde{D}_{n-1}$  or  $\widetilde{E}_m$  for  $m = 6, 7, 8$  then  $A = \Pi_Q(k)$  is an  $\mathbb{N}$ -graded Calabi-Yau algebra of global dimension 2 whose degree zero piece is isomorphic to  $k^n$  (where  $n = m + 1$  if  $Q = \widetilde{E}_m$ ). Moreover,  $A$  is noetherian and satisfies the  $\chi$ -condition by Proposition 2.3.9. Thus, for  $G$  a nontrivial finite subgroup of graded automorphisms of  $A$  satisfying that every  $g \in G$  fixes the primitive idempotents  $e_i$  for  $i = 1, \dots, n$  all results of this section apply.

Before we can start we need to introduce more notation. Most of the conditions are required in order to apply theorems from [AZ94]. Therefore, let's adopt the notation from [AZ94]:  $\text{qgr}(B)$  denotes the *quotient category*  $\text{gr}(B)/\text{tors}(B)$  where  $\text{gr}(B)$  is the category of all finitely generated graded right  $B$ -modules and  $\text{tors}(B)$  denotes the subcategory of finite dimensional modules. For every module  $M \in \text{gr}(B)$ , we denote its image in  $\text{qgr}(B)$  by  $\mathcal{M} = \pi(M)$ .

Also recall the definition of *cohomological dimension of*  $\text{proj}(B)$  from [AZ94, p.276] which uses  $H^i(\mathcal{M}) = H_B^i(\mathcal{M}) := \text{Ext}_{\text{qgr}(B)}^i(\mathcal{B}, \mathcal{M})$ :

$$\text{cd}(\text{proj } B) = \max\{i \mid H^i(\mathcal{M}) \neq 0 \text{ for some } \mathcal{M} \in \text{qgr}(B)\}.$$

Since  $\text{gl. dim}(B) = d$  is finite, [AZ94, Proposition 7.10(3)] says  $\text{cd}(\text{proj } B) \leq d - 1$  is finite as well. Then, we can use [AZ94, Proposition 7.10(1)] to get a different description of  $\text{cd}(\text{proj } B)$  involving  $\underline{H}^i(\mathcal{M}) := \bigoplus_{s=-\infty}^{\infty} H^i(\mathcal{M}[s])$ :

$$\text{cd}(\text{proj } B) = \max\{i \mid \underline{H}^i(\mathcal{B}) \neq 0\}.$$

This says, we only need to look at  $\mathcal{B}$  in  $\text{qgr}(B)$ . Even better, for  $i \geq 1$ , we have the equality  $\underline{H}^i(\mathcal{B}) = \lim_{m \rightarrow \infty} \text{Ext}_B^{i+1}(B/B_{\geq m}, B)$  thanks to [AZ94, Proposition 7.2(2)]. Moreover, the same proposition says  $\underline{H}^0(\mathcal{B})$  surjects onto  $\lim_{m \rightarrow \infty} \text{Ext}_B^1(B/B_{\geq m}, B)$ .

The following lemma connects the cohomological dimension to the global dimension of  $B$ .

**Lemma 5.1.1.**  $\text{cd}(\text{proj } B) = \text{gl. dim}(B) - 1$ .

*Proof.* We obtain  $\text{cd}(\text{proj } B) \leq \text{gl. dim}(B) - 1$  from [AZ94, Proposition 7.10(3)]. To achieve the equality  $\text{cd}(\text{proj } B) = \text{gl. dim}(B) - 1$ , we consider the following constructions and results to generalize the idea of [AZ94, Theorem 8.1(3)]. For  $d = \text{gl. dim}(B) \geq 1$  we are going to show that  $\lim_{m \rightarrow \infty} \text{Ext}_B^d(B/B_{\geq m}, B) \neq 0$ . By our assumptions  $B$  is twisted Calabi-Yau. This guarantees the existence of the  $k$ -central invertible bimodule  $U = \text{Ext}_{B^e}^d(B, B^e)$

as in Definition 2.3.7. From [RR18, Proposition 4.12] it follows that

$$\begin{aligned}
\lim_{m \rightarrow \infty} \operatorname{Ext}_B^d(B/B_{\geq m}, B) &\cong \lim_{m \rightarrow \infty} \operatorname{Tor}_{d-d}^B((B/B_{\geq m})^* \otimes_B U, B) \\
&\cong \lim_{m \rightarrow \infty} ((B/B_{\geq m})^* \otimes_B U) \otimes_B B \\
&\cong \lim_{m \rightarrow \infty} (B/B_{\geq m})^* \otimes_B U.
\end{aligned}$$

We can calculate the direct limit of  $(B/B_{\geq m})^*$ . Applying the left-exact contravariant functor  $\operatorname{Hom}(-, k)$  to the natural projections  $B/B_{\geq m+1} \rightarrow B/B_{\geq m} \rightarrow 0$ , we obtain exact maps

$$0 \longrightarrow \operatorname{Hom}_k(B/B_{\geq m}, k) \longrightarrow \operatorname{Hom}_k(B/B_{\geq m+1}, k).$$

Hence, the direct limit becomes the union of all  $\operatorname{Hom}_k(B/B_{\geq m}, k)$  which is nonzero. This does not change if we apply  $\otimes_B U$ .

Thus, for  $d = \operatorname{gl. dim}(B) \geq 2$ , we showed that  $\underline{H}^{d-1}(\mathcal{B}) = \lim_{m \rightarrow \infty} \operatorname{Ext}_B^d(B/B_{\geq m}, B) \neq 0$  using [AZ94, Proposition 7.2(2)]. For  $d = \operatorname{gl. dim}(B) = 1$ , we have the surjection of  $\underline{H}^0(\mathcal{B})$  onto  $\lim_{m \rightarrow \infty} \operatorname{Ext}_B^1(B/B_{\geq m}, B) \neq 0$ . Together, for  $\operatorname{gl. dim}(B) \geq 1$  we obtain  $\underline{H}^{d-1}(\mathcal{B}) \neq 0$ . Hence  $\operatorname{cd}(\operatorname{proj} B) = d - 1$ .  $\square$

**Proposition 5.1.2.** *If  $B^G$  is twisted Calabi-Yau, then  $\operatorname{gl. dim}(B^G) = \operatorname{gl. dim}(B)$ .*

*Proof.* The proof of [KKZ08, Lemma 1.11(d)] works in our case. We immediately gain that  $B^G$  is a noetherian  $\mathbb{N}$ -graded algebra (see Lemma 2.3.14(b)). Moreover,  $B^G$  satisfies the  $\chi$ -condition due to Proposition 2.3.18. These properties will be important to use results from [AZ94].

Now, Lemma 5.1.1 applies to both  $B$  and  $B^G$  saying  $\text{cd}(\text{proj } B) = \text{gl. dim}(B) - 1$  as well as  $\text{cd}(\text{proj } B^G) = \text{gl. dim}(B^G) - 1$ . Therefore, in order to finish the proof, it suffices to show that  $\text{cd}(\text{proj } B) = \text{cd}(\text{proj } B^G)$ .

Recall that  $B$  satisfies the  $\chi$ -condition and the natural inclusion of  $B^G \hookrightarrow B$  makes  $B$  a finitely generated  $B^G$ -module by Proposition 2.3.17. Hence, [AZ94, Corollary 8.4(1)] says  $\text{cd}(\text{proj } B) \leq \text{cd}(\text{proj } B^G)$ . Further, the natural projection

$$\pi : B \rightarrow B^G, \quad b \mapsto \frac{1}{|G|} \sum_{g \in G} g(b)$$

allows one to decompose  $B = B^G \oplus C$  as  $(B^G, B^G)$ -bimodules (see Lemma 2.3.14(a)).

The same calculation as in [KKZ08, Lemma 1.11(d)] for  $i = d - 1$  provides

$$\begin{aligned} \text{cd}(\text{proj } B) &= \max\{i \mid \underline{H}_B^i(\mathcal{B}) \neq 0\} \\ &= \max\{i \mid \underline{H}_B^i(\mathcal{B} \oplus \mathcal{C}) \neq 0\} \\ &= \max\{i \mid \underline{H}_{B^G}^i(\mathcal{B} \oplus \mathcal{C}) \neq 0\} \quad [\text{AZ94, Theorem 8.3(3)}] \\ &\geq \max\{i \mid \underline{H}_B^i(\mathcal{B}^G) \neq 0\} \\ &= \text{cd}(\text{proj } B^G). \end{aligned}$$

This shows that  $\text{gl. dim}(B^G) - 1 = \text{cd}(\text{proj } B^G) = \text{cd}(\text{proj } B) = \text{gl. dim}(B) - 1$ .  $\square$

In the case of a noetherian Artin-Schelter regular algebra  $S$  it cannot happen that  $S^G$  has finite global dimension but  $S^G$  is not AS-Gorenstein: Lemma 2.3.14 and Proposition 2.3.17 apply and so whenever the fixed ring has finite global dimension, the Gorenstein condition comes along with it due to [KKZ08, Lemma 1.10(c)]. However, for

preprojective algebras Example 4.2.3 shows that  $A^G$  having finite global dimension does not guarantee the generalized Gorenstein condition.

Next, an easy lemma gives that  $\text{gl.dim}(B^G)$  is always greater than or equal to  $\text{gl.dim}(B)$ . Afterwards, we prove that we obtain equality provided  $B$  is a projective  $B^G$ -module.

**Lemma 5.1.3.** *The fixed ring  $B^G$  satisfies  $\text{gl.dim}(B^G) \geq \text{gl.dim}(B)$ .*

*Proof.* If  $\text{gl.dim}(B^G) = \infty$  there is nothing to prove. So assume  $\text{gl.dim}(B^G)$  is finite. As in the proof of Proposition 5.1.2,  $B$  and  $B^G$  are right noetherian  $\mathbb{N}$ -graded algebras. Since both  $B$  and  $B^G$  have finite global dimension, [AZ94, Proposition 7.10(3)] gives  $\text{cd}(\text{proj } B) \leq \text{gl.dim}(B) - 1$  and  $\text{cd}(\text{proj } B^G) \leq \text{gl.dim}(B^G) - 1$ . Since  $B$  is twisted Calabi-Yau, Lemma 5.1.1 gives  $\text{cd}(\text{proj } B) = \text{gl.dim}(B) - 1$ . Since  $B$  satisfies the  $\chi$ -condition, [AZ94, Corollary 8.4(1)] applied to  $B^G \hookrightarrow B$  says  $\text{cd}(\text{proj } B) \leq \text{cd}(\text{proj } B^G)$ . Combining these results yields

$$\text{gl.dim}(B) - 1 = \text{cd}(\text{proj } B) \leq \text{cd}(\text{proj } B^G) \leq \text{gl.dim}(B^G) - 1$$

which gives  $\text{gl.dim}(B) \leq \text{gl.dim}(B^G)$  as claimed.  $\square$

**Proposition 5.1.4.** *If  $B$  is a projective  $B^G$ -module then  $\text{gl.dim}(B) = \text{gl.dim}(B^G)$ .*

*Proof.* Lemma 5.1.3 provides  $d = \text{gl.dim}(B) \leq \text{gl.dim}(B^G)$ . The other inequality follows from the fact that a minimal projective resolution of  $S_j = e_j B / e_j B_{\geq 1} = e_j B^G / (e_j B^G)_{\geq 1}$  as a  $B$ -module yields a projective resolution as a  $B^G$ -module. Hence, [RR18, Proposition 3.19 (3)] which says  $\text{gl.dim}(B^G) = \max\{\text{pdim}(S_j) \mid j = 1, \dots, n\}$  finishes the proof.  $\square$

After giving two necessary conditions for  $\text{gl. dim}(B) = \text{gl. dim}(B^G)$ , as a conclusion, we finish by showing that one of the conditions is actually equivalent.

**Proposition 5.1.5.**  *$B$  is a projective  $B^G$ -module if and only if  $\text{gl. dim}(B) = \text{gl. dim}(B^G)$ .*

*Proof.* One direction is Proposition 5.1.4. Assume that  $\text{gl. dim}(B) = \text{gl. dim}(B^G)$ . By [RR18, Proposition 3.19 (1)], we get  $\text{pdim}_{B^G}(B) = \sup\{q \in \mathbb{N} \mid \text{Tor}_q^{B^G}(B_0^G, B) \neq 0\}$  and so we need to prove that  $\text{Tor}_q^{B^G}(B_0^G, B) = 0$  for all  $q \geq 1$ . By [Wei94, Theorem 5.6.6] applied to  $B^G \hookrightarrow B$  with  $M = B_0^G$  and some left  $B^G$ -module  $N$  we obtain the spectral sequence

$$E_{pq}^2 = \text{Tor}_p^B(\text{Tor}_q^{B^G}(B_0^G, B), N) \Rightarrow \text{Tor}_{p+q}^{B^G}(B_0^G, N). \quad (5.1)$$

Thanks to the hypothesis, we can denote  $d = \text{gl. dim}(B) = \text{gl. dim}(B^G)$  as well as call  $e = \text{pdim}_{B^G}(B) = \max\{q \in \mathbb{N} \mid \text{Tor}_q^{B^G}(B_0^G, B) \neq 0\} \leq d$ . We need to show that  $e = 0$ . By using a minimal projective resolution of  $B$  as  $B^G$ -module of length  $e = \text{pdim}_{B^G}(B)$ , it follows that  $\text{Tor}_q^{B^G}(B_0^G, B) = 0$  for all  $q > e$ . Analogously,  $\text{Tor}_p^B(-, -) = 0$  for  $p > d$ . Therefore, the only nonzero terms of  $E_{pq}^2$  appear for  $0 \leq p \leq d$  and  $0 \leq q \leq e$  making  $E_{de}^2$  the top right corner of this rectangle. Since we know that  $E_{pq}^2 \Rightarrow \text{Tor}_{p+q}^{B^G}(B_0^G, N)$  we need to understand  $E_{pq}^r$  for  $r \geq 2$ . The data includes the differentials  $d_{pq}^r : E_{pq}^r \rightarrow E_{p-r, q+r-1}^r$  and thus for  $r \geq 2$  taking homology at  $E_{de}^r$  does not change  $E_{de}^r$  due to the involved maps being trivial. Therefore, the convergence in (5.1) yields

$$\text{Tor}_d^B(\underbrace{\text{Tor}_e^{B^G}(B_0^G, {}_{B^G}B_B)}_{=X_B}, N) \cong \text{Tor}_{d+e}^{B^G}(B_0^G, N). \quad (5.2)$$

First, we notice that  $X$  is finite dimensional over  $k$  since applying  $B_0^G \otimes -$  to a minimal finitely generated projective resolution  $P_\bullet \rightarrow 0$  of  $B$  only leaves finite dimensional pieces at each step. This does not change if  $X_B$  is calculated using a resolution of  $B_0^G$  instead. Next, we prove that  $\text{pdim}(X) = \text{gl.dim}(B)$ . Being a  $B$ -module gives  $\text{pdim}(X) \leq \text{gl.dim}(B)$  for free. For the converse, notice that every finite dimensional module over  $B$  has a composition series with factors which are shifted copies of simple modules. Therefore, by short exact sequences and the generalized Gorenstein condition which  $B$  satisfies by Proposition 2.3.8, we gain  $\text{Ext}_B^d(X, B) \neq 0$ . This is enough to guarantee  $\text{pdim}(X) = d$ .

From the characterization of  $\text{pdim}(X)$  and symmetry in the argument of [RR18, Proposition 3.19 (1)] it follows that

$$d = \text{gl.dim}(B) = \text{pdim}(X) = \max\{q \in \mathbb{N} \mid \text{Tor}_q^B(X, B_0) \neq 0\},$$

where  $B_0 = B/J(B)$ . Therefore by choosing  $N = B_0$ , the left hand side of Equation (5.2) is nonzero. The only way the right hand side can be nonzero is if  $d + e = d$  or  $e = 0$ .  $\square$

Proposition 5.1.5 shows that  $\text{gl.dim}(B^G) = \text{gl.dim}(B)$  if and only if  $B$  is projective as a  $B^G$ -module. This is true for the preprojective algebra  $A$ . In particular, it finally finishes the proof of Example 4.2.2 and makes sure that  $A$  is a projective  $A^G$ -module. However, it can happen that  $\text{gl.dim}(A^G)$  is finite but different from  $\text{gl.dim}(A)$  as described in Example 4.2.3. We have now proved that in Example 4.2.3  $A$  is not projective as an  $A^G$ -module. The situation for an Artin-Schelter regular algebra  $S$  with a finite group  $G$  acting on  $S$  is nicer. In [KKZ08, Lemma 1.10] and [KKZ08, Lemma 1.11] it was proved

that the fixed ring  $S^G$  has finite global dimension if and only if  $\text{gl. dim}(S) = \text{gl. dim}(S^G)$  if and only if the fixed ring is Artin-Schelter regular if and only if  $S$  is free over  $S^G$ .

## 5.2 The main theorem

Let  $B$  be a noetherian  $\mathbb{N}$ -graded algebra of finite global dimension  $d \geq 1$  with degree zero piece  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . This gives us all simple right  $B$ -modules  $S_i = e_i B / (e_i B)_{\geq 1}$  and all simple left  $B$ -modules  $S_i^* = B e_i / (B e_i)_{\geq 1}$ , respectively. Fix a finite subgroup  $G$  of the graded automorphism group of  $B$  such that every  $g \in G$  satisfies  $g(e_i) = e_i$ . Notice that this automatically implies  $(B^G)_0 = B_0 \cong k^n$ .

The purpose of this section is to find a converse of Proposition 5.1.2. In light of Example 4.2.3 it became apparent that in order to achieve the generalized Gorenstein condition, all simple modules need to have the same projective dimension over  $B^G$ . More precisely, this section starts with technical results leading to Theorem 5.2.3 which gives sufficient conditions for  $B$  to satisfy the generalized Gorenstein condition. This is enough to prove the main result: if additionally  $B$  is twisted Calabi-Yau and  $\text{pdim}_{B^G}(S_i) = \text{pdim}_{B^G}(S_j)$  as well as  $\text{pdim}_{B^G}(S_i^*) = \text{pdim}_{B^G}(S_j^*)$  for the simple modules  $S_i = e_i B^G / (e_i B^G)_{\geq 1} = e_i B / (e_i B)_{\geq 1}$  and  $S_i^* = B^G e_i / (B^G e_i)_{\geq 1} = B e_i / (B e_i)_{\geq 1}$  and all  $i, j$ , then the following are equivalent:

- (1)  $\text{gl. dim}(B^G) = \text{gl. dim}(B)$ .

(2)  $B^G$  is twisted Calabi-Yau.

(3)  $B$  is a projective  $B^G$ -module.

But first, let's see how the Ext-groups involving the simple modules determine Ext-groups for finite dimensional modules. The proof of this standard result using the long exact Ext sequence is omitted.

**Lemma 5.2.1.** *If  $F_B$  is a finite dimensional  $B$ -module and  $\text{Ext}_B^i(S_j, B) = 0$  for all  $j$  and all  $i \leq p$ , then  $\text{Ext}_B^i(F, B) = 0$  for  $i \leq p$ .*

**Proposition 5.2.2.** *Let  $B^{\text{op}}$  be the opposite ring of  $B$ . Denote the global dimension of  $B$  by  $d = \text{gl. dim}(B) < \infty$ . Further assume the equalities  $\text{pdim}_B(S_i) = \text{pdim}_B(S_j)$  and  $\text{pdim}_{B^{\text{op}}}(S_i^*) = \text{pdim}_{B^{\text{op}}}(S_j^*)$  for all  $i, j$  where  $S_i$  and  $S_i^*$  denote the simple right and left  $B$ -modules. If*

- $\text{Ext}_{B^{\text{op}}}^i(S_j^*, B) = 0 = \text{Ext}_B^i(S_j, B)$  for  $i \neq d$  and
- $\dim_k(\text{Ext}_{B^{\text{op}}}^d(S_j^*, B)) < \infty$  and  $\dim_k(\text{Ext}_B^d(S_j, B)) < \infty$  for all  $j$

*then  $\dim_k(\text{Ext}_B^d(S_j, B)) = 1$ .*

*Proof.* This uses ideas from [SZ97, Proposition 3.1]. Since  $B$  is noetherian, [Rot09, Theorem 8.27] implies  $\text{gl. dim}(B) = \text{gl. dim}(B^{\text{op}})$  and so [RR18, Proposition 3.19(3)] implies  $\text{pdim}(S_j) = \text{pdim}(S_1) = \text{pdim}(S_1^*) = \text{pdim}(S_j^*) = d$  for all  $j$ . Consider a minimal projective resolution of  $S_j$ :

$$0 \longrightarrow Q_d^{(j)} \longrightarrow \dots \longrightarrow Q_2^{(j)} \longrightarrow Q_1^{(j)} \longrightarrow e_j B \longrightarrow S_j \longrightarrow 0,$$

where each  $Q_i^{(j)}$  equals a sum of shifted copies of the indecomposable projectives  $e_m B[-\ell]$ .

If we apply  $\text{Hom}(-, B)$ , Lemma 2.3.4 applies and the result is

$$0 \longleftarrow (Q_d^{(j)})^* \longleftarrow \dots \longleftarrow (Q_2^{(j)})^* \longleftarrow (Q_1^{(j)})^* \longleftarrow Be_j \longleftarrow 0,$$

where  $(Q_i^{(j)})^*$  is the left  $B$ -module consisting of a sum of the corresponding indecomposable projectives  $Be_m[\ell]$ . Since  $\text{Ext}_B^i(S_j, B) = 0$  for  $i \neq d$ , the second resolution is exact everywhere but at the  $d$ -th position. Hence, it can be understood as a minimal projective resolution of  $F_j^* = \text{Ext}_B^d(S_j, B)$ . Notice, the  $*$  indicates that it is a left  $B$ -module. Since  $\text{Hom}(\text{Hom}(e_m B, B), B) = e_m B$ , it follows that  $\text{Ext}_{B^{\text{op}}}^d(F_j^*, B) = S_j$  and  $\text{Ext}_{B^{\text{op}}}^i(F_j^*, B) = 0$  for  $i \neq d$ . In particular,  $\dim_k(\text{Ext}_{B^{\text{op}}}^d(F_j^*, B)) = 1$ .

Define  $w_j = \dim_k(\text{Ext}_{B^{\text{op}}}^d(S_j^*, B)) < \infty$ . Notice, that  $\text{gl. dim}(B^{\text{op}}) = \text{pdim}(S_j^*) = d$ .

Therefore, we can take a minimal projective resolution  $\mathcal{P}_\bullet$  of  $S_j^*$  whose boundary maps can be described by matrices with entries of degree bigger than or equal to 1 due to being minimal. After applying  $\text{Hom}_{B^{\text{op}}}(-, B)$  to  $\mathcal{P}_\bullet$  we can use a symmetric version of Lemma 2.3.4. Now, at the  $d$ -th position we have

$$0 \longleftarrow \text{Hom}_{B^{\text{op}}}(\mathcal{P}_d, B) \xleftarrow{\delta_d^*} \dots$$

where  $\delta_d^*$  is a matrix with entries in  $B_{\geq 1}^{\text{op}}$ . Clearly, taking homology at the described spot leaves behind the lowest degree part of  $\text{Hom}_{B^{\text{op}}}(\mathcal{P}_d, B)$  which implies  $w_j > 0$ . Let  $M^*$  be a finite dimensional left  $B$ -module. We are going to show by induction on  $\dim_k(M^*)$  that

$$\dim_k(\text{Ext}_{B^{\text{op}}}^d(M^*, B)) = \sum_{s=1}^{\dim_k(M^*)} w_{m_s}, \quad \text{for some numbers } m_1, \dots, m_{\dim_k(M^*)} \in \{1, \dots, n\}.$$

If  $\dim_k(M^*) = 1$  we must have  $\text{Ext}_{B^{\text{op}}}^d(M^*, B) = \text{Ext}_{B^{\text{op}}}^d(S_{m_1}^*, B)$  and this implies that  $\dim_k(\text{Ext}_{B^{\text{op}}}^d(S_{m_1}^*, B)) = w_{m_1}$ . Otherwise, consider the short exact sequence

$$0 \longrightarrow K^* \longrightarrow M^* \longrightarrow S_m^* \longrightarrow 0$$

and notice that  $\dim_k(K^*) = \dim_k(M^*) - 1$ . Calculating the long exact Ext sequence gives

$$\text{Ext}_{B^{\text{op}}}^{d-1}(K^*, B) \longrightarrow \text{Ext}_{B^{\text{op}}}^d(S_m^*, B) \longrightarrow \text{Ext}_{B^{\text{op}}}^d(M^*, B) \longrightarrow \text{Ext}_{B^{\text{op}}}^d(K^*, B) \longrightarrow 0.$$

Due to  $\text{Ext}_{B^{\text{op}}}^r(S_j^*, B) = 0$  for all  $r \leq d-1$ , a symmetric version of Lemma 5.2.1 says  $\text{Ext}_{B^{\text{op}}}^{d-1}(K^*, B)$  vanishes. Hence, the last three terms form a short exact sequence and the induction hypothesis applies to give

$$\begin{aligned} \dim_k(\text{Ext}_{B^{\text{op}}}^d(M^*, B)) &= \dim_k(\text{Ext}_{B^{\text{op}}}^d(K^*, B)) + \dim_k(\text{Ext}_{B^{\text{op}}}^d(S_m^*, B)) \\ &= \sum_{s=1}^{\dim_k(M^*)-1} w_{m_s} + w_{m_{\dim_k(M^*)}} = \sum_{s=1}^{\dim_k(M^*)} w_{m_s}. \end{aligned}$$

To sum up, we apply this result to  $M^* = F_j^* = \text{Ext}_B^d(S_j, B)$  which is finite dimensional by our assumption. This yields

$$1 = \dim_k(\text{Ext}_{B^{\text{op}}}^d(F_j^*, B)) = \sum_{s=1}^{\dim_k(F_j^*)} w_{m_s}.$$

Here, a sum of positive integers equals 1 which can only happen with just one summand.

Therefore,  $\dim_k(\text{Ext}_B^d(S_j, B)) = \dim_k(F_j^*) = 1$ . □

The following theorem is a modification of [DW09, Theorem 3.6, (2)  $\Rightarrow$  (1)].

**Theorem 5.2.3.** *Let  $B^{\text{op}}$  be the opposite ring of  $B$ . Suppose that both  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition. Assume that  $\text{pdim}_B(S_i) = \text{pdim}_B(S_j)$  and  $\text{pdim}_{B^{\text{op}}}(S_i^*) = \text{pdim}_{B^{\text{op}}}(S_j^*)$*

for all  $i, j$  where  $S_i$  and  $S_i^*$  denote the simple right and left  $B$ -modules. Then  $B$  satisfies the generalized Gorenstein condition.

*Proof.* Since  $B$  is noetherian, [Rot09, Theorem 8.27] implies  $\text{gl.dim}(B) = \text{gl.dim}(B^{\text{op}})$  and thus  $\text{pdim}(S_j) = \text{pdim}(S_1) = \text{pdim}(S_1^*) = \text{pdim}(S_j^*) = d$  for all  $j$ . In particular, it follows that  $\text{gl.dim}(B^{\text{op}}) < \infty$  and the other assumptions imply  $b_1 = b_2 = \dots = b_n = b$  for

$$b_j = \text{pdim}_{B^{\text{op}}}(S_j^*) = \sup\{i \mid \text{Ext}_{B^{\text{op}}}^i(S_j^*, B) \neq 0\} < \infty.$$

Furthermore,  $b = \text{gl.dim}(B^{\text{op}}) = \text{gl.dim}(B) = d$ . We use the fourth quadrant spectral sequence from [Lev92, 2.2 Theorem]:

$$E_{p,-q}^2 = \text{Ext}_{B^{\text{op}}}^p(\text{Ext}_B^q(S_j, B), B) \quad \Rightarrow \quad \begin{cases} 0, & p \neq q \\ S_j, & p = q \end{cases}.$$

In particular,  $E_{p,-q}^\infty$  is nonzero precisely if  $p = q$ . Define

$$f_j = \inf\{i \mid \text{Ext}_B^i(S_j, B) \neq 0\} < \infty.$$

This implies all the nonzero entries of the spectral sequence can be found in the rectangle  $[0, d] \times [-d, -f_j]$ .

Additionally, the  $\chi$ -condition guarantees that  $X = \text{Ext}_B^{f_j}(S_j, B) \neq 0$  is finite dimensional. Every finite dimensional module over  $B^{\text{op}}$  has a composition series with factors which are shifted copies of simple modules. Therefore, by short exact sequences of the first component of  $\text{Ext}_{B^{\text{op}}}^i(-, B)$  and the definition of  $b = d$  we gain  $\text{Ext}_{B^{\text{op}}}^b(X, B) \neq 0$ . Combined, this implies  $E_{d,-f_j}^2 \neq 0$  which is the top right corner of the rectangle.

The data includes the differentials  $d_{p,q}^r$  of degree  $(r, 1 - r)$  as described in [Lev85, 1.1]. Thus, taking homology does not change  $E_{d,-f_j}^r$  for all  $r$  and so  $E_{d,-f_j}^2 \cong E_{d,-f_j}^\infty$  must be a nonzero entry. The spectral sequence shows now that  $d = f_j$  for all  $j$ . In other words, we proved  $d = f_1 = \dots = f_n =: f$ .

Finally, since  $d = f$  we get that  $\text{Ext}_B^i(S_j, B)$  is nonzero if and only if  $i = d$  and  $\text{Ext}_B^d(S_j, B)$  is finite dimensional due to the  $\chi$ -condition of  $B$ . Repeating the argument for  $B^{\text{op}}$  shows that  $\text{Ext}_{B^{\text{op}}}^i(S_j^*, B)$  is nonzero if and only if  $i = d$  and  $\text{Ext}_{B^{\text{op}}}^d(S_j^*, B)$  is finite dimensional. Proposition 5.2.2 gives in this case  $\text{Ext}_B^d(S_j, B) \cong S_{m_j}^*$  for some  $m_j \in \{1, \dots, n\}$  for all  $j$ . Since

$$S_j = \text{Ext}_{B^{\text{op}}}^d(\text{Ext}_B^d(S_j, B), B) = \text{Ext}_{B^{\text{op}}}^d(S_{m_j}^*, B),$$

we know that  $\{m_1, \dots, m_n\} = \{1, \dots, n\}$ , and therefore  $B$  satisfies the generalized Gorenstein condition.  $\square$

Since we want to apply the previous theorem to a fixed ring  $B^G$  of a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra  $B$  with degree zero piece  $B_0 \cong k^n$ , some of the assumptions are automatically satisfied. Therefore, we state the result for our purposes in the following corollary.

**Corollary 5.2.4.** *Assume  $B$  is a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra of global dimension  $d \geq 1$  with degree zero piece  $B_0 \cong k^n$ . Suppose  $\text{gl.dim}(B^G) = d$  is finite and that we have the equalities  $\text{pdim}(S_i) = \text{pdim}(S_j)$  as well as  $\text{pdim}(S_i^*) = \text{pdim}(S_j^*)$  for all  $i, j$  where  $S_i = e_i B^G / (e_i B^G)_{\geq 1}$  and  $S_i^* = B^G e_i / (B^G e_i)_{\geq 1}$ . Then  $B^G$  twisted Calabi-Yau.*

*Proof.* Notice that  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition by Lemma 2.3.6. By our convention of this section, every  $g \in G$  fixes the vertices. Therefore,  $(B^G)_0 = B_0 \cong k^n$ . Lemma 2.3.14(b) and Proposition 2.3.18 say that  $B^G$  is noetherian and that  $B^G$  and  $(B^G)^{\text{op}}$  satisfy the  $\chi$ -condition. The result follows from Theorem 5.2.3 applied to  $B^G$  combined with Proposition 2.3.8.  $\square$

We finish this section by putting together all the results from Sections 5.1 and 5.2. By additionally requiring that all simple modules have the same projective dimension over the fixed ring we reach our goal to prove that the following three conditions of the fixed ring are equivalent.

**Theorem 5.2.5.** *Assume  $B$  is a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra of global dimension  $d \geq 1$  with degree zero piece  $B_0 \cong k^n$ . Denote the pairwise orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Let  $G$  be a finite subgroup of  $\text{Aut}_{\text{gr}}(B)$  such that  $g(e_i) = e_i$  for all  $g \in G$  and all  $i = 1, \dots, n$ . Suppose  $\text{pdim}(S_i) = \text{pdim}(S_j)$  and  $\text{pdim}(S_i^*) = \text{pdim}(S_j^*)$  for the simple  $B^G$ -modules  $S_i = e_i B^G / (e_i B^G)_{\geq 1}$  and  $S_i^* = B^G e_i / (B^G e_i)_{\geq 1}$  and all indices  $i, j = 1, \dots, n$ . Then the following are equivalent:*

- (1)  $\text{gl. dim}(B^G) = \text{gl. dim}(B)$ .
- (2)  $B^G$  is twisted Calabi-Yau.
- (3)  $B$  is a projective  $B^G$ -module.

*Proof.* From Proposition 5.1.5 we know that (1) is equivalent to (3). Then the Propositions 2.3.8 and 5.1.2 show that (2) implies (1). The requirements of Corollary 5.2.4 are

satisfied and so (1) implies (2). □

### 5.3 The main theorem for $\widetilde{A_{n-1}}$

Concluding this chapter, we focus on the preprojective algebra  $A = \Pi_Q(k)$  with  $Q = \widetilde{A_{n-1}}$ . We continue to use the notation  $S_i = e_i A / (e_i A)_{\geq 1}$  and  $S_i^* = A e_i / (A e_i)_{\geq 1}$  for the simple right and left modules of both  $A$  and  $A^G$  where  $G$  is a finite subgroup of the graded automorphism group of  $A$  such that every  $g \in G$  satisfies  $g(e_i) = e_i$ . Another application of Theorem 5.2.3 together with some new terminology is Proposition 5.3.6. This is enough to show that  $A$  satisfies the required conditions for Theorem 5.2.5.

*Definition & Remark 5.3.1.* Let  $Q = \widetilde{A_{n-1}}$ . Define  $Q^G$  to be the quiver with weighted arrows which has the same vertices as  $Q$  and whose arrows are the paths in  $Q$  minimally generating  $A^G$ . Recall that  $G$  is finite.

Since  $G$  acts by scalar multiplication, a minimal linear combination of paths in  $A = \Pi_Q(k)$  for  $Q = \widetilde{A_{n-1}}$  is fixed if and only if each summand is fixed. Therefore, every generator of  $A^G$  is a path and it makes sense to define  $Q^G$ .

This section consists of proving that in case  $Q = \widetilde{A_{n-1}}$  the conditions of Corollary 5.2.4 are satisfied. More precisely, we are going to check that every simple  $A^G$ -module  $S_i = e_i A^G / (e_i A^G)_{\geq 1}$  and  $S_i^* = A^G e_i / (A^G e_i)_{\geq 1}$  has projective dimension greater than or equal to 2. For this, we need to understand the generators and relations of  $A^G$ . We start with the generators in the following lemma.

**Lemma 5.3.2.** *Let  $Q = \widetilde{A_{n-1}}$ . Every vertex  $e_i$  in  $Q^G$  must have a purely star arrow and a purely nonstar arrow starting at  $e_i$  and one of each ending at  $e_i$ . Further, there is a purely nonstar (purely star) path starting and ending at  $e_i$  which is fixed.*

*Proof.* We know  $G = \{g_1, \dots, g_\ell\}$  is a finite group. For  $g_j$ , we know  $g_j(\alpha_s) = c_s^{(j)}\alpha_s$  for  $s = 1, \dots, n$  where  $c_s^{(j)}$  is some root of unity. Therefore, for  $j = 1, \dots, \ell$  we have that  $c_1^{(j)}c_2^{(j)} \dots c_n^{(j)}$  is a root of unity whose order can be denoted by  $n_j$ . This gives that the purely nonstar path  $\beta_i = (\alpha_i\alpha_{i+1}\alpha_{i+2} \dots \alpha_n\alpha_1 \dots \alpha_{i-1})^{n_1 \dots n_\ell}$  starting and ending at the vertex  $e_i$  must be fixed. Now,  $\beta_i$  decomposes into indecomposable fixed paths in  $Q^G$ . The first one starts at  $e_i$  and the last one ends at  $e_i$ . Therefore, there exists a shortest fixed purely nonstar path starting at  $e_i$  as well as one ending at  $e_i$  for  $i = 1, \dots, n$ . Analogously, there are shortest fixed purely star paths starting at  $e_i$  and ending at  $e_i$ .  $\square$

**Corollary 5.3.3.** *For  $Q = \widetilde{A_{n-1}}$ , the fixed ring  $A^G$  has at least  $3n$  generators counting the primitive orthogonal idempotents.*

*Proof.* For every vertex  $e_i$ , there exist one purely nonstar and one purely star generator of  $A^G$  starting at  $e_i$  by Lemma 5.3.2 plus the idempotent  $e_i$  gives three generators at  $e_i$ .  $\square$

Since the generators of  $Q^G$  have been understood well enough we focus on the relations in  $A^G$ . It turns out that they have a nice form which enables us to prove the existence of a minimal relation starting and ending at each vertex.

**Proposition 5.3.4.** *Let  $Q = \widetilde{A_{n-1}}$ . Every relation in  $A^G$  can be written as a sum of homogeneous (with respect to the grading in  $A$ ) binomial relations in  $A^G$ .*

*Proof.* Let  $R = \beta_1 + b_2\beta_2 + \dots + b_m\beta_m$  be a relation in  $A^G$  where the  $\beta_i$  are nonzero paths in  $Q^G$  and  $b_2 \cdots b_m \neq 0$ . We show by induction on  $m$  that  $R$  is a sum of binomial relations. We can understand  $R$  as a relation in  $A$  and therefore we can assume that all  $\beta_i$  are homogeneous of the same degree in  $A$ , say  $\ell$ .

Suppose  $m = 1$ . This case cannot happen as a nontrivial path cannot equal zero.

If  $m = 2$ , then  $R$  is binomial itself,  $\beta_1 = -b_2\beta_2$ , so the claim follows.

Finally let  $m \geq 2$ . Since  $R = 0$  in  $A$ , we can write  $\beta_1 = -(b_2\beta_2 + \dots + b_{m+1}\beta_{m+1})$ . Define  $j$  and  $k$  such that  $\beta_1 \in e_j A e_k$ . Then all  $\beta_i \in e_j A e_k$  as otherwise multiplying  $R$  by  $e_j$  from the left and  $e_k$  from the right decreases  $m + 1$  and induction finishes the proof. Since every path in  $A$  is uniquely determined by its starting vector  $e_j$ , its length  $\ell$  and the number of nonstar arrows, we know a basis of  $(e_j A e_k)_\ell$  consists of one representative for each possible type  $(0, \ell), (1, \ell - 1), \dots, (\ell, 0)$ . Denote the type of  $\beta_i$  by  $(m_i, n_i)$  for all  $i$ . Suppose that  $(m_1, n_1) \neq (m_i, n_i)$  for all  $i = 2, \dots, m+1$ . Then  $\beta_1$  can be written as a linear combination of paths of different type which yields a basis of  $(e_j A e_k)_\ell$  of size  $\leq \ell$ . This is a contradiction as  $\dim((e_j A e_k)_\ell) = \ell + 1$ . Hence, we can assume  $(m_1, n_1) = (m_2, n_2)$ . This however gives  $\beta_1 = \beta_2$  and so we find the new relation  $R'$  as  $(b_2 - 1)\beta_2 + \dots + b_{m+1}\beta_{m+1}$  which by induction can be written as a sum of binomial relations. We notice that since  $\beta_1, \beta_2 \in A^G$ , the new relation holds in  $A^G$ .  $\square$

This proof also shows that the minimal relations of  $A^G$  are binomial of the form  $\beta_1 = \beta_2$  for two mixed paths  $\beta_1, \beta_2 \in Q^G$ . In particular,  $\beta_1$  and  $\beta_2$  must be of the same type in  $A$ . This forces  $\beta_1$  and  $\beta_2$  to be mixed paths.

**Lemma 5.3.5.** *Let  $Q = \widetilde{A_{n-1}}$ . Then for every vertex  $e_i$  in  $A^G$  there exists a minimal relation starting at  $e_i$  and a minimal relation ending at  $e_i$ .*

*Proof.* Fix an idempotent  $e_i \in A^G$  for some  $i$ . From Lemma 5.3.2 we obtain an indecomposable purely nonstar element  $x_i \in e_i A^G$  and an indecomposable purely nonstar element  $y_i \in A^G e_i$ . By construction,  $x_i$  and  $y_i$  are part of the set of minimal generators of  $A^G$ . For a vertex  $e_m$ , let  $\nu_m \in e_m A^G e_m$  be the shortest purely star closed path which is fixed under the action of  $G$ . Notice that  $\nu_m$  is not necessarily indecomposable in  $A^G$ . Similar to the proof of Lemma 5.3.2, it is easy to see that  $\nu_m$  is the first power of  $(\alpha_{m-1}^* \alpha_{m-2}^* \cdots \alpha_1^* \alpha_m^* \cdots \alpha_m^*)$  which is fixed. Since this construction is independent of  $m$ , we gain  $\deg(\nu_m) = \deg(\nu_\ell)$  for all  $m, \ell = 1, \dots, n$ .

This implies that  $\text{type}(x_i \nu_{t(x_i)}) = \text{type}(\nu_i x_i)$  and both these paths are from  $e_i$  to  $e_{t(x_i)}$ . Thus, they are equal and the following two relations arise:

$$x_i \nu_{t(x_i)} = \nu_i x_i \quad \text{and} \quad \nu_{s(y_i)} y_i = y_i \nu_i.$$

Since in the left relation the left hand side starts with a nonstar arrow while the right hand side starts with a star arrow, there must exist a minimal relation starting at  $e_i$ . Analogously, since the right relation ends with arrows of different type, there must exist a minimal relation ending at  $e_i$ .  $\square$

After this setup, we are ready to show that  $A$  satisfies the missing direction from Section 5.1. The key argument established by the previous statements is that the projective dimension of each simple module is at least 2 and therefore global dimension of  $A^G$

being 2 shows the necessary equality of all  $\text{pdim}(S_j)$  for  $j = 1, \dots, n$ .

**Proposition 5.3.6.** *Let  $Q = \widetilde{A_{n-1}}$ . If  $\text{gl. dim}(A^G) = \text{gl. dim}(A) = 2$ , then  $A^G$  satisfies the generalized Gorenstein condition  $\text{Ext}_{A^G}^i(S_j, A^G) = \delta_{di} S_{\sigma(j)}^*$  for some  $\sigma \in \text{Sym}(n)$ .*

*Proof.* Denote the indecomposable projectives  $e_j A^G$  by  $P_j$  for  $j = 1, \dots, n$ . We know the start of a minimal projective resolution of  $S_j$  and with  $\text{gl. dim}(A^G) = 2$  it follows that

$$0 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow P_j \longrightarrow S_j \longrightarrow 0,$$

where  $Q_1$  and  $Q_2$  are projective  $A^G$ -modules. As used before, the proof of [Boc08, Theorem 3.2] tells us that  $Q_1$  comes from the generators and  $Q_2$  is determined by the minimal relations. Lemma 5.3.2 shows that  $A^G$  has at least two minimal generators of positive degree starting at each vertex. Hence,  $Q_1 \neq 0$ . By Lemma 5.3.5 there exists at least one minimal relation starting at each vertex. This says the kernel of the map  $Q_1 \rightarrow P_j$  is nonzero or in other words,  $Q_2 \neq 0$ . Together, we get that  $\text{pdim}(S_j) = 2$  for all  $j$ . The same argument with generators and relations ending at  $e_j$  works for a minimal projective resolution of  $S_j^*$  and so  $\text{pdim}(S_j^*) = 2$  for all  $j$ . Corollary 5.2.4 entails that  $A^G$  satisfies the generalized Gorenstein condition.  $\square$

At last, we can combine the previous results to get the main theorem for  $Q = \widetilde{A_{n-1}}$ . We believe that the conditions are satisfied for all types  $\widetilde{A_{n-1}}$ ,  $\widetilde{D_{n-1}}$  or  $\widetilde{E_m}$  for  $m = 6, 7, 8$  rather than just  $\widetilde{A_{n-1}}$ . However, the relations of the fixed rings in the other two cases are much harder to understand. In particular, they are not binomial and it is unclear if every vertex has the needed generators and relations to get 2 as a lower bound for the projective

dimension of every simple module. Nevertheless, the following theorem motivates the decision to require  $\text{gl. dim}(A^G) = 2$  on the algebra side of a possible Chevalley-Shephard-Todd Theorem.

**Theorem 5.3.7.** *For type  $Q = \widetilde{A_{n-1}}$ ,  $A = \Pi_Q(k)$  and  $G$  a finite subgroup of  $\text{Aut}_{\text{gr}}(A)$  such that  $g(e_i) = e_i$  for all  $g \in G$  and all  $i = 1, \dots, n$ , the following conditions are equivalent:*

- (1)  $\text{gl. dim}(A^G) = \text{gl. dim}(A)$  (i.e.  $\text{gl. dim}(A^G) = 2$ ).
- (2)  $A^G$  is twisted Calabi-Yau.
- (3)  $A$  is a projective  $A^G$ -module.

*Proof.* Proposition 5.3.6 shows that (1) implies (2) using Proposition 2.3.8. The other implications hold due to 5.1.2 and 5.1.5. □

This chapter contains material from “Properties of the fixed ring of a preprojective algebra”, Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of this paper.

# Chapter 6

## Generalized Gorensteinness and a homological determinant

### 6.1 Summary of case $\widetilde{D}_{n-1}$

Let  $n \geq 5$  and let  $Q$  be the following quiver coming from the extended Dynkin diagram  $\widetilde{D}_{n-1}$ :

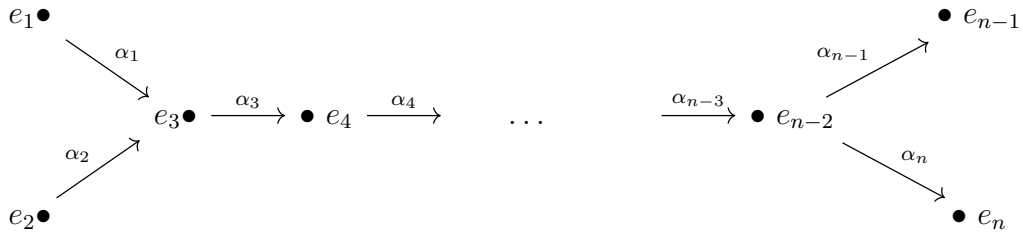


**Figure 6.1.1:** Dynkin diagram of  $\widetilde{D}_{n-1}$ .

As before, let  $A = \Pi_Q(k)$  be the preprojective algebra with respect to  $Q$ . We need

to give a description of  $\underline{\text{Tr}}(g, t)$  and  $\text{Tr}_{e_i A}(g_i, t)$  of a graded automorphism which fixes the primitive idempotents in this case. Before proving this formula, let's analyze the relations of  $A$  as they are crucial to find the minimal projective resolutions of the simple modules.

*Definition & Remark 6.1.1.* In Definition 2.3.3, the preprojective algebra was defined as the path algebra modulo the ideal  $I$  generated by the one relation

$$\sum_{\alpha \in Q_1} \alpha \alpha^* - \sum_{\alpha \in Q_1} \alpha^* \alpha. \quad (6.1)$$

By successively multiplying the trivial paths  $e_i$  for  $i = 1, \dots, n$  to Equation (6.1), for  $Q = \widetilde{D_{n-1}}$  relations of the form

$$\begin{aligned} \alpha_i \alpha_i^* &= 0, & i &= 1, 2 \\ \alpha_3 \alpha_3^* &= \alpha_1^* \alpha_1 + \alpha_2^* \alpha_2, \\ \alpha_i \alpha_i^* &= \alpha_{i-1}^* \alpha_{i-1}, & i &= 4, \dots, n-3, \\ \alpha_{n-3}^* \alpha_{n-3} &= \alpha_{n-1}^* \alpha_{n-1} + \alpha_n \alpha_n^* & \text{and} \\ \alpha_i^* \alpha_i &= 0, & i &= n-1, n \end{aligned} \quad (6.2)$$

are obtained. Starting now we think of  $I$  being generated by the relations in Equation (6.2). Moreover, notice that these relations force  $c_i t_i = c_1 t_1$  for all  $i = 1, \dots, n$  as it was the case for  $Q = \widetilde{A_{n-1}}$ . Additionally, whenever  $g$  satisfies  $c_1 t_1 = c_i t_i$  for all  $i$  we obtain a graded automorphism.

For the proof of the subsequent proposition, we follow [JZ97, Theorem 2.3] closely.

**Proposition 6.1.2.** *Let  $Q = \widetilde{D_{n-1}}$  and  $A = \Pi_Q(k)$ . Fix  $g$  to be a graded automorphism of  $A$  with  $g(e_i) = e_i$  for all  $i = 1, \dots, n$ . Therefore, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  for  $1 \leq i \leq n-3$  and  $i = n-1, n$ . Then,  $\underline{\text{Tr}}(g, t)$  equals*

$$\begin{pmatrix} 1 + c_1 t_1 t^2 & 0 & -c_1 t & & & \\ & 0 & 1 + c_1 t_1 t^2 & -c_2 t & & \\ & -t_1 t & -t_2 t & 1 + c_1 t_1 t^2 & -c_3 t & \\ & & & \ddots & \ddots & \ddots \\ & & & & -t_{n-3} t & 1 + c_1 t_1 t^2 & -c_{n-1} t & -c_n t \\ & & & & & -t_{n-1} t & 1 + c_1 t_1 t^2 & 0 \\ & & & & & & -t_n t & 0 & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

and similarly  $\overline{\text{Tr}}(g, t)$  equals

$$\begin{pmatrix} 1 + c_1 t_1 t^2 & 0 & -t_1 t & & & \\ & 0 & 1 + c_1 t_1 t^2 & -t_2 t & & \\ & -c_1 t & -c_2 t & 1 + c_1 t_1 t^2 & -t_3 t & \\ & & & \ddots & \ddots & \ddots \\ & & & & -c_{n-3} t & 1 + c_1 t_1 t^2 & -t_{n-1} t & -t_n t \\ & & & & & -c_{n-1} t & 1 + c_1 t_1 t^2 & 0 \\ & & & & & & -c_n t & 0 & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}.$$

*Proof.* Notice that  $g$  preserves the indecomposable projective modules  $P_i = e_i A$  as well as the graded submodules  $e_i A_{\geq 1}$ . Since  $\text{gl. dim}(A) = 2$ , the proof of [Boc08, Theorem 3.2] gives the minimal projective resolution of  $S_i = e_i A / e_i A_{\geq 1}$  for  $i = 4, \dots, n-3$ :

$$0 \longrightarrow e_i A \xrightarrow{\delta_2 = \begin{pmatrix} \alpha_i^* \\ -\alpha_{i-1} \end{pmatrix}} e_{i+1} A \oplus e_{i-1} A \xrightarrow{\delta_1 = \begin{pmatrix} \alpha_i & \alpha_{i-1}^* \end{pmatrix}} e_i A \xrightarrow{\delta_0} S_i \longrightarrow 0.$$

Denote  $g_i = g|_{e_i A}$  and let  $\tilde{g}_i$  be the induced map of  $g_i$  on  $S_i$ . Then  $\tilde{g}_i$  is a  $g$ -linear map from  $S_i$  to  $S_i$  or an  $A$ -linear map from  $S_i$  to  $S_i^g$ . This minimal projective resolution is identical to the ones coming up in Proposition 4.1.5 for case  $\widetilde{A_{n-1}}$ . In the same way, we consider the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q_2 = e_i A & \xrightarrow{\delta_2} & Q_1 = e_{i+1} A \oplus e_{i-1} A & \xrightarrow{\delta_1} & Q_0 = e_i A \xrightarrow{\delta_0} S_i \longrightarrow 0 \\ & & \downarrow \phi_2^{(i)} & & \downarrow \phi_1^{(i)} & & \downarrow \phi_0^{(i)} \quad \downarrow \tilde{g}_i \\ 0 & \longrightarrow & Q_2^g = e_i A^g & \xrightarrow{\delta_2} & Q_1^g = (e_{i+1} A \oplus e_{i-1} A)^g & \xrightarrow{\delta_1} & Q_0^g = (e_i A)^g \xrightarrow{\delta_0} S_i^g \longrightarrow 0. \end{array}$$

As before, we use the universal property of projective resolution, [Jac80, Theorem 6.3]

which provides graded  $A$ -linear maps  $\phi_j^{(i)} : Q_j \rightarrow Q_j^g$  such that all the squares commute:

$$\delta_2 \phi_2^{(i)} = \phi_1^{(i)} \delta_2, \quad \delta_1 \phi_1^{(i)} = \phi_0^{(i)} \delta_1 \quad \text{and} \quad \delta_0 \phi_0^{(i)} = \tilde{g}_i \delta_0.$$

Recall the relations of  $A$ :  $\alpha_1 \alpha_1^* = \alpha_2 \alpha_2^* = \alpha_{n-1}^* \alpha_{n-1} = \alpha_n^* \alpha_n = 0$ ,  $\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}$  for  $4 \leq i \leq n-3$  as well as  $\alpha_3 \alpha_3^* = \alpha_2^* \alpha_2 + \alpha_1^* \alpha_1$  and  $\alpha_n \alpha_n^* + \alpha_{n-1} \alpha_{n-1}^* = \alpha_{n-3}^* \alpha_{n-3}$ . These imply  $c_i t_i = c_1 t_1$  for all  $i$ .

$$\text{Calculations as in Proposition 4.1.5 give } \phi_0^{(i)} = g_i, \quad \phi_1^{(i)} \left( \begin{pmatrix} e_{i+1} \\ 0 \end{pmatrix} \right) = \begin{pmatrix} c_i e_{i+1} \\ 0 \end{pmatrix}$$

and  $\phi_1^{(i)} \left( \begin{pmatrix} 0 \\ e_{i-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ t_{i-1}e_{i-1} \end{pmatrix}$  as well as  $\phi_2^{(i)}(e_i) = c_1 t_1 e_i$ . Since the maps  $\phi_1^{(i)}$  and  $\phi_2^{(i)}$  are  $g$ -linear this defines the maps uniquely. Notice, that these minimal projective resolutions and their lifts only depend on the relation  $\alpha_i \alpha_i^* - \alpha_{i-1}^* \alpha_{i-1} = 0$  at  $e_i$  and the target vertices of  $\alpha_i$  and  $\alpha_{i-1}^*$ . Thus, for  $\widetilde{E}_m$ , the same conclusions - in particular the equations leading to the matrix formula - can be made again for the vertices at which the corresponding relation holds.

Using the same technique, we obtain the following minimal projective resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_2 = e_j A & \xrightarrow{\alpha_j^*} & Q_1 = e_3 A & \xrightarrow{\alpha_j} & Q_0 = e_j A & \xrightarrow{\delta_0} & S_j \longrightarrow 0 \\ & & \downarrow \phi_2^{(j)} & & \downarrow \phi_1^{(j)} & & \downarrow \phi_0^{(j)} & & \downarrow \tilde{g}_j \\ 0 & \longrightarrow & Q_2^g = (e_j A)^g & \xrightarrow{\alpha_j^*} & Q_1^g = (e_3 A)^g & \xrightarrow{\alpha_j} & Q_0^g = (e_j A)^g & \xrightarrow{\delta_0} & S_j^g \longrightarrow 0 \end{array}$$

for  $j = 1, 2$  with  $\phi_0^{(j)} = g_j$ ,  $\phi_1^{(j)}(e_3) = c_j e_3$  as well as  $\phi_2^{(j)}(e_j) = c_1 t_1 e_j$ . Similarly,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_2 = e_l A & \xrightarrow{\alpha_l} & Q_1 = e_{n-2} A & \xrightarrow{\alpha_l^*} & Q_0 = e_l A & \xrightarrow{\delta_0} & S_l \longrightarrow 0 \\ & & \downarrow \phi_2^{(l)} & & \downarrow \phi_1^{(l)} & & \downarrow \phi_0^{(l)} & & \downarrow \tilde{g}_l \\ 0 & \longrightarrow & Q_2^g = (e_l A)^g & \xrightarrow{\alpha_l} & Q_1^g = (e_{n-2} A)^g & \xrightarrow{\alpha_l^*} & Q_0^g = (e_l A)^g & \xrightarrow{\delta_0} & S_l^g \longrightarrow 0 \end{array}$$

for  $l = n-1, n$  with  $\phi_0^{(l)} = g_l$ ,  $\phi_1^{(l)}(e_{n-2}) = t_l e_{n-2}$  as well as  $\phi_2^{(l)}(e_l) = c_1 t_1 e_l$  are obtained. Once again, these cases only depend on  $\alpha_j \alpha_j^* = 0$  and  $t(\alpha_j)$  ( $\alpha_l^* \alpha_l = 0$  and  $t(\alpha_l^*)$ , respectively) which occur identically at the endpoints of the diagrams  $\widetilde{E}_m$ .

The two remaining resolutions are a little more complicated but can also be computed using that the squares commute and that all maps are  $g$ -linear from  $Q_r$  to  $Q_r$ . First

we have

$$\begin{array}{ccccccc}
& & \delta_2 = \begin{pmatrix} \alpha_3^* \\ -\alpha_2 \\ -\alpha_1 \end{pmatrix} & & \delta_1 = \begin{pmatrix} \alpha_3 \\ \alpha_2^* \\ \alpha_1^* \end{pmatrix}^T & & \\
0 & \longrightarrow & Q_2 = e_3 A & \xrightarrow{\quad} & Q_1 = e_4 A \oplus e_2 A \oplus e_1 A & \xrightarrow{\quad} & Q_0 = e_3 A \xrightarrow{\delta_0} S_3 \longrightarrow 0 \\
& & \downarrow \phi_2^{(3)} & & \downarrow \phi_1^{(3)} & & \downarrow \phi_0^{(3)} \quad \downarrow \tilde{g}_3 \\
0 & \longrightarrow & Q_2^g = (e_3 A)^g & \xrightarrow{\delta_2} & Q_1^g = (e_4 A \oplus e_2 A \oplus e_1 A)^g & \xrightarrow{\delta_1} & Q_0^g = (e_3 A)^g \xrightarrow{\delta_0} S_3^g \longrightarrow 0
\end{array}$$

with  $\phi_0^{(3)} = g_3$ ,  $\phi_2^{(3)}(e_3) = c_1 t_1 e_3$  and

$$\phi_1^{(3)} \left( \begin{pmatrix} e_4 \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} c_3 e_4 \\ 0 \\ 0 \end{pmatrix}, \quad \phi_1^{(3)} \left( \begin{pmatrix} 0 \\ e_2 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ t_2 e_2 \\ 0 \end{pmatrix}, \quad \phi_1^{(3)} \left( \begin{pmatrix} 0 \\ 0 \\ e_1 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ t_1 e_1 \end{pmatrix}$$

define the maps  $\phi_0^{(3)}$ ,  $\phi_1^{(3)}$  and  $\phi_2^{(3)}$  uniquely. Notice, that this case will not emerge for  $\widetilde{E}_m$ .

Finally, the last resolution computes in the same way yielding

$$\begin{array}{ccccccc}
& & \delta_2 = \begin{pmatrix} \alpha_{n-1}^* \\ \alpha_n^* \\ -\alpha_{n-3} \end{pmatrix} & & \delta_1 = \begin{pmatrix} \alpha_{n-1} \\ \alpha_n \\ \alpha_{n-3}^* \end{pmatrix}^T & & \\
0 & \longrightarrow & e_{n-2} A & \xrightarrow{\quad} & e_{n-1} A \oplus e_n A \oplus e_{n-3} A & \xrightarrow{\quad} & e_{n-2} A \xrightarrow{\delta_0} S_{n-2} \longrightarrow 0 \\
& & \downarrow \phi_2^{(n-2)} & & \downarrow \phi_1^{(n-2)} & & \downarrow \phi_0^{(n-2)} \quad \downarrow \tilde{g}_{n-2} \\
0 & \longrightarrow & (e_{n-2} A)^g & \xrightarrow{\delta_2} & (e_{n-1} A \oplus e_n A \oplus e_{n-3} A)^g & \xrightarrow{\delta_1} & (e_{n-2} A)^g \xrightarrow{\delta_0} S_{n-2}^g \longrightarrow 0
\end{array}$$

with  $\phi_0^{(n-2)} = g_{n-2}$ ,  $\phi_2^{(n-2)}(e_{n-2}) = c_1 t_1 e_{n-2}$  and

$$\begin{aligned} \phi_1^{(n-2)} \left( \begin{pmatrix} e_{n-1} \\ 0 \\ 0 \end{pmatrix} \right) &= \begin{pmatrix} c_{n-1} e_{n-1} \\ 0 \\ 0 \end{pmatrix}, & \phi_1^{(n-2)} \left( \begin{pmatrix} 0 \\ e_n \\ 0 \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ c_n e_n \\ 0 \end{pmatrix} \text{ and} \\ \phi_1^{(n-2)} \left( \begin{pmatrix} 0 \\ 0 \\ e_{n-3} \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ 0 \\ t_{n-3} e_{n-3} \end{pmatrix}. \end{aligned}$$

Due to the direction of the arrows we will choose for  $\widetilde{E}_m$  in the next section, this is the same calculation necessary for the unique vertex of  $\widetilde{E}_m$  which is connected to three vertices. However, in contrast, the neighboring vertices there have non-consecutive labels. For instance, for  $\widetilde{E}_6$  the vertex  $e_3$  is adjacent to  $e_2$ ,  $e_4$  and  $e_6$ . This affects which projective modules appear in the minimal projective resolution and this influences the position an entry occurs in the matrix formula at the end of the proof.

For  $s = 1, \dots, n$ , we now understand the  $\phi_r^{(s)}$  as  $g$ -linear maps from  $Q_r$  to  $Q_r$ . Since  $\phi_0^{(s)} = g_s$ , we get that  $\text{Tr}_{Q_0}(\phi_0^{(s)}, t) = \text{Tr}_{e_s A}(g, t)$ . Applying  $\phi_2^{(s)}$  to  $\beta \in e_s A$  gives  $\phi_2^{(s)}(\beta) = \phi_2^{(s)}(e_s)g(\beta) = c_1 t_1 g(\beta)$  and therefore  $\text{Tr}_{Q_2}(\phi_2^{(s)}, t) = c_1 t_1 t^2 \text{Tr}_{e_s A}(g, t)$ . Notice that the  $t^2$  comes from the shift in the grading of  $Q_2$  which is used so  $\delta_1$  and  $\delta_2$  are graded.

It remains to understand  $\text{Tr}_{Q_1}(\phi_1^{(s)}, t)$ . First, consider  $s = 4, \dots, n-3$ . In the same way as before but using bases for each component individually, we immediately obtain the equation  $\text{Tr}_{Q_1}(\phi_1^{(s)}, t) = c_i t \text{Tr}_{e_{i+1} A}(g, t) + t_{i-1} t \text{Tr}_{e_{i-1} A}(g, t)$ . From the projective

resolutions above we read off the remaining four cases in Equation (6.3):

$$\begin{aligned}
\mathrm{Tr}_{Q_1}(\phi_1^{(j)}, t) &= c_j t \mathrm{Tr}_{e_3 A}(g, t), & \text{for } j = 1, 2, \\
\mathrm{Tr}_{Q_1}(\phi_1^{(l)}, t) &= t_l t \mathrm{Tr}_{e_{n-2} A}(g, t), & \text{for } l = n-1, n, \\
\mathrm{Tr}_{Q_1}(\phi_1^{(3)}, t) &= c_3 t \mathrm{Tr}_{e_4 A}(g, t) + t_2 t \mathrm{Tr}_{e_2 A}(g, t) + t_1 t \mathrm{Tr}_{e_1 A}(g, t), & \text{and} \\
\mathrm{Tr}_{Q_1}(\phi_1^{n-2}, t) &= c_{n-1} t \mathrm{Tr}_{e_{n-1} A}(g, t) + c_n t \mathrm{Tr}_{e_n A}(g, t) + t_{n-3} t \mathrm{Tr}_{e_{n-3} A}(g, t).
\end{aligned} \tag{6.3}$$

We noticed that the minimal projective resolutions of  $S_i$  together with their lifts only depend on the relation at  $e_i$  including the target vertices of the arrows starting at  $e_i$ .

Therefore, the stated equations for  $\mathrm{Tr}_{Q_*}(\phi_*^{(s)})$  can be found similarly for  $\widetilde{E}_m$ .

$$\text{Now, [JZ97, Lemma 2.1] applies and provides } \sum_{j=0}^2 (-1)^j \mathrm{Tr}_{Q_j}(\phi_j^{(i)}, t) = \mathrm{Tr}_{S_i}(\tilde{g}_i, t)$$

which translates to

$$\begin{aligned}
\mathrm{Tr}_{S_i}(\tilde{g}_i, t) &= \mathrm{Tr}_{e_i A}(g, t) - [c_i t \mathrm{Tr}_{e_{i+1} A}(g, t) + t_{i-1} t \mathrm{Tr}_{e_{i-1} A}(g, t)] + c_1 t_1 t^2 \mathrm{Tr}_{e_i A}(g, t), \\
& \quad i = 4, \dots, n-3,
\end{aligned}$$

$$\mathrm{Tr}_{S_j}(\tilde{g}_j, t) = \mathrm{Tr}_{e_j A}(g, t) - c_j t \mathrm{Tr}_{e_3 A}(g, t) + c_1 t_1 t^2 \mathrm{Tr}_{e_j A}(g, t), \quad j = 1, 2,$$

$$\mathrm{Tr}_{S_l}(\tilde{g}_l, t) = \mathrm{Tr}_{e_l A}(g, t) - t_l t \mathrm{Tr}_{e_{n-2} A}(g, t) + c_1 t_1 t^2 \mathrm{Tr}_{e_l A}(g, t), \quad l = n-1, n,$$

$$\begin{aligned}
\mathrm{Tr}_{S_3}(\tilde{g}_3, t) &= \mathrm{Tr}_{e_3 A}(g, t) - [c_3 t \mathrm{Tr}_{e_4 A}(g, t) + t_2 t \mathrm{Tr}_{e_2 A}(g, t) + t_1 t \mathrm{Tr}_{e_1 A}(g, t)] \\
& \quad + c_1 t_1 t^2 \mathrm{Tr}_{e_3 A}(g, t), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Tr}_{S_{n-3}}(\tilde{g}_{n-3}, t) &= \mathrm{Tr}_{e_{n-3} A}(g, t) - [c_{n-1} t \mathrm{Tr}_{e_{n-1} A}(g, t) + c_n t \mathrm{Tr}_{e_n A}(g, t) + t_{n-3} t \mathrm{Tr}_{e_{n-3} A}(g, t)] \\
& \quad + c_1 t_1 t^2 \mathrm{Tr}_{e_{n-3} A}(g, t).
\end{aligned}$$

Finally, we can write the  $n$  equations using matrix notation as

$$\begin{pmatrix} \text{Tr}_{S_1}(\tilde{g}_1, t) \\ \text{Tr}_{S_2}(\tilde{g}_2, t) \\ \vdots \\ \vdots \\ \vdots \\ \text{Tr}_{S_n}(\tilde{g}_n, t) \end{pmatrix} = \begin{pmatrix} 1 + c_1 t_1 t^2 & & & & \\ & 1 + c_1 t_1 t^2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \ddots \\ & & & & & 1 + c_1 t_1 t^2 \end{pmatrix} \cdot \begin{pmatrix} \text{Tr}_{e_1 A}(g, t) \\ \text{Tr}_{e_2 A}(g, t) \\ \vdots \\ \vdots \\ \vdots \\ \text{Tr}_{e_n A}(g, t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & c_1 t & & & \\ 0 & 0 & c_2 t & & & \\ t_1 t & t_2 t & 0 & c_3 t & & \\ & & \ddots & \ddots & \ddots & \\ & & & t_{n-3} t & 0 & c_{n-1} t & c_n t \\ & & & & t_{n-1} t & 0 & 0 \\ & & & & & t_n t & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{Tr}_{e_1 A}(g, t) \\ \text{Tr}_{e_2 A}(g, t) \\ \vdots \\ \vdots \\ \vdots \\ \text{Tr}_{e_n A}(g, t) \end{pmatrix}$$

On the left hand side,  $\text{Tr}_{S_i}(\tilde{g}_i, t) = 1$  as  $\tilde{g}_i$  sends  $e_i$  to  $e_i$ . Therefore, combining the matrices from the right and multiplying its inverse from the left finishes the proof for  $\underline{\text{Tr}}(g, t)$ . The case  $\overline{\text{Tr}}(g, t)$  follows from analogous calculations.  $\square$

## 6.2 Summary of case $\widetilde{E_m}$ for $m = 6, 7, 8$

Finally, the cases  $\widetilde{E_m}$  for  $m = 6, 7, 8$  remain. The construction of the preprojective algebra involved forming the path algebra of the double of the quiver  $Q$  and factoring out the unique relation

$$\sum_{\alpha \in Q_1} \alpha \alpha^* - \sum_{\alpha \in Q_1} \alpha^* \alpha.$$

As before, successfully multiplying idempotents from either side discloses the following relations instead.

$$\widetilde{E_6} : \quad \alpha_1 \alpha_1^* = \alpha_4^* \alpha_4 = \alpha_7^* \alpha_7 = 0,$$

$$\alpha_3 \alpha_3^* + \alpha_6 \alpha_6^* = \alpha_2^* \alpha_2, \quad \text{and}$$

$$\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 4, 7.$$

$$\widetilde{E_7} : \quad \alpha_1 \alpha_1^* = \alpha_6^* \alpha_6 = \alpha_8^* \alpha_8 = 0,$$

$$\alpha_4 \alpha_4^* + \alpha_8 \alpha_8^* = \alpha_3^* \alpha_3, \quad \text{and}$$

$$\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 3, 5, 6.$$

$$\widetilde{E_8} : \quad \alpha_1 \alpha_1^* = \alpha_7^* \alpha_7 = \alpha_9^* \alpha_9 = 0,$$

$$\alpha_6 \alpha_6^* + \alpha_9 \alpha_9^* = \alpha_5^* \alpha_5, \quad \text{and}$$

$$\alpha_i \alpha_i^* = \alpha_{i-1}^* \alpha_{i-1}, \quad i = 2, 3, 4, 5, 7.$$

As noticed for  $\widetilde{A_{n-1}}$  and  $\widetilde{D_{n-1}}$ , the relations force again  $c_1 t_1 = c_i t_i$  for all  $i$  as otherwise  $g$  is not an automorphism. Conversely, whenever  $g$  satisfies  $c_1 t_1 = c_i t_i$  for all  $i$  it is a graded automorphism of  $A$ . The relations are subject to the convention for the underlying quivers

$\widetilde{E}_m$  for  $m = 6, 7, 8$  as follows.

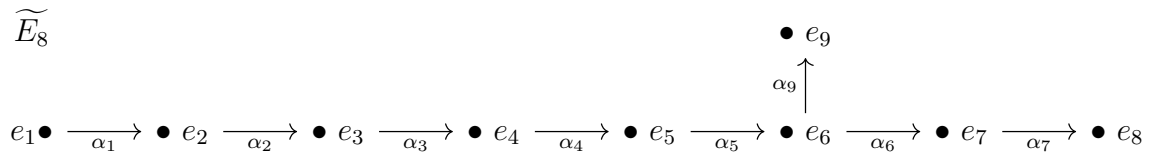
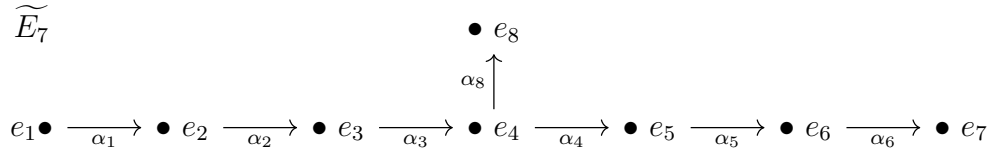
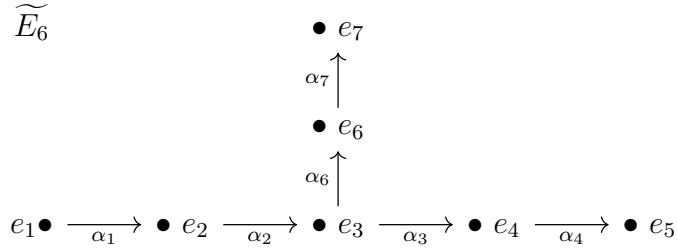


**Figure 6.2.1:** Dynkin diagram of  $\widetilde{E}_m$  for  $m = 6, 7, 8$ .

The goal of this section is the same as last section's for  $\widetilde{D}_{n-1}$ : we need to find a formula for  $\text{Tr}(g, t)$  for an automorphism  $g$  of  $A$  which fixes the vertices. Luckily, the same approach works. The following proposition uses the minimal projective resolutions of the simple modules together with the canonical lifts as done before. The required information manifests in the subsequent proposition.

**Proposition 6.2.1.** *Let  $Q = \widetilde{E}_m$  for  $m = 6, 7, 8$  and  $A = \Pi_Q(k)$ . Let  $g$  be a graded automorphism of  $A$  with  $g(e_i) = e_i$  for all  $i = 1, \dots, n$ . Hence, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  for all  $i$ . Then  $\underline{\text{Tr}}(g, t)$  equals*

$$\begin{pmatrix} 1 + c_1 t_1 t^2 & -c_1 t & & & & & & \\ & -t_1 t & 1 + c_1 t_1 t^2 & -c_2 t & & & & \\ & & -t_2 t & 1 + c_1 t_1 t^2 & -c_3 t & & -c_6 t & \\ & & & -t_3 t & 1 + c_1 t_1 t^2 & -c_4 t & & \\ & & & & -t_4 t & 1 + c_1 t_1 t^2 & & \\ & & & & -t_6 t & & 1 + c_1 t_1 t^2 & -c_7 t \\ & & & & & & -t_7 t & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

for  $\widetilde{E}_6$ ,

$$\begin{pmatrix} 1 + c_1 t_1 t^2 & -c_1 t & & & & & & \\ & -t_1 t & 1 + c_1 t_1 t^2 & -c_2 t & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -t_3 t & 1 + c_1 t_1 t^2 & -c_4 t & & -c_8 t \\ & & & & -t_4 t & 1 + c_1 t_1 t^2 & -c_5 t & \\ & & & & & -t_5 t & 1 + c_1 t_1 t^2 & -c_6 t \\ & & & & & & -t_6 t & 1 + c_1 t_1 t^2 \\ & & & & & & & -t_8 t & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

for  $\widetilde{E}_7$  and

$$\begin{pmatrix} 1 + c_1 t_1 t^2 & -c_1 t & & & & & \\ & -t_1 t & 1 + c_1 t_1 t^2 & -c_2 t & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -t_5 t & 1 + c_1 t_1 t^2 & -c_6 t & -c_9 t \\ & & & & -t_6 t & 1 + c_1 t_1 t^2 & -c_7 t \\ & & & & & -t_7 t & 1 + c_1 t_1 t^2 \\ & & & & & & -t_9 t & 1 + c_1 t_1 t^2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

for  $\widetilde{E}_8$ . Similar formulas arise for  $\overline{\text{Tr}}(g, t)$  where the  $c_i$ 's switch the role with the  $t_i$ 's.

*Proof.* This proof was prepared in the proof of Proposition 6.1.2. We need to find the minimal projective resolutions of the simple modules. They are immediate from knowing the generators, their target vertices and relations starting at  $e_i$  as done before. Then, the calculations for the connecting maps are the same as the corresponding ones in Proposition 6.1.2. For instance, for  $\widetilde{E}_6$ , the vertex  $e_1$  behaves like  $e_1$  in  $\widetilde{D}_{n-1}$ ,  $e_5$  and  $e_7$  behave like  $e_n$ , the formula for  $e_3$  can be extracted from  $e_{n-2}$  and at last the vertices  $e_2, e_4$  and  $e_6$  fall into the generic case for  $\widetilde{D}_{n-1}$ . Similar equations to (6.3) arise and written in matrix form together with remembering that  $\text{Tr}_{S_i}(\tilde{g}_i, t) = 1$  finishes the claim for  $\underline{\text{Tr}}(g, t)$ . The case  $\overline{\text{Tr}}(g, t)$  follows from analogous calculations.  $\square$

## 6.3 A sufficient criterion for generalized Gorenstein-ness

Let  $B$  be a noetherian  $\mathbb{N}$ -graded generalized Gorenstein algebra with degree zero piece  $B_0 \cong k^n$ . Suppose  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition from Definition 2.3.5. Denote the primitive orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . This gives us all simple right  $B$ -modules  $S_i = e_i B / (e_i B)_{\geq 1}$  and all simple left  $B$ -modules  $S_i^* = B e_i / (B e_i)_{\geq 1}$ , respectively. Every graded automorphism  $g \in \text{Aut}_{\text{gr}}(B)$  that we consider satisfies  $g(e_i) = e_i$  for all  $i = 1, \dots, n$ . Fix a finite subgroup  $G$  of  $\text{Aut}_{\text{gr}}(B)$  and some  $g \in \text{Aut}_{\text{gr}}(B)$ .

In this section, we develop the main theory of this chapter. In particular, we introduce the concept of a homological determinant  $\text{Hdet}(g)$  in matrix form for the graded automorphism  $g$ . The main Theorem 6.3.9 connects the homological determinant  $\text{Hdet}(g)$  for all  $g$  in  $G$  being trivial to the fixed ring being generalized Gorenstein. While stated in a more general setting, the results of this section apply to preprojective algebras.

The following is a generalization of the work of Peter Jørgensen and James J. Zhang in [JZ00]. Denote  $B_{\geq 1}$  by  $\mathfrak{r}$ . For a right  $B$ -module  $M$ , recall that

$$H_{\mathfrak{r}}^i(M) = \varinjlim \text{Ext}^i({}_B(B/B_{\geq r})_B, M_B)$$

has a right  $B$ -module structure since  $B/B_{\geq r}$  is a  $(B, B)$ -bimodule. The next definition is adapted from [AZ94]:

**Definition 6.3.1.** An element  $e$  of a  $B$ -module  $M$  is called *torsion* if there exists an  $i \in \mathbb{N}$  such that  $e B_{\geq i} = 0$ .

One of the key lemmas to prove the main theorem requires to understand the graded injective hulls of the simple modules  $S_j$ . As described in the following remark they have a nice form. In order to avoid confusion with the arrows  $\alpha^*$  of  $A$ , we denote elements of a dual space by  $(-)'$ .

*Remark 6.3.2.* The graded injective envelope of a simple right  $B$ -module  $S_j$  equals the graded Matlis dual  $P_j^* = \underline{\text{Hom}}_k(Be_j, k)$  with natural grading  $(P_j^*)_m = \underline{\text{Hom}}_k(B_{-m}e_j, k)$ .

*Proof.* Fix some homogeneous  $k$ -basis  $\mathcal{B}$  of  $Be_j$  which contains  $e_j$ . First, the graded  $B$ -module homomorphism  $\varphi : S_j \rightarrow P_j^*$ ,  $e_j \mapsto e'_j$  where  $e'_j$  denotes the dual map to  $e_j$ , sending  $e_j$  to 1 and all other elements of  $\mathcal{B}$  to 0, gives an inclusion of  $S_j$  into  $P_j^*$ .

Second, we need to show that  $P_j^*$  is the graded injective hull. Equivalently, it is enough to prove that  $P_j^*$  is an injective essential extension of  $S_j$ . Let  $N$  be a nonzero submodule of  $P_j^*$ . We denote the dual basis of  $P_j^*$  by  $\mathcal{B}' = \{\beta' \mid \beta \in \mathcal{B}\}$ . Let

$$\psi = \sum_{\beta \in \mathcal{B}} c_\beta \beta' : Be_j \rightarrow k$$

be a nonzero element from  $N$ . Since  $P_j^*$  lives entirely in nonpositive degrees, let  $\gamma'$  be of minimal degree such that  $c_\gamma$  is nonzero. We are going to show that  $\psi \cdot \gamma \in N \cap S_j \setminus \{0\}$ . For  $\beta'$  with  $\deg(\beta') > \deg(\gamma')$  in the expression for  $\psi$  we notice that

$$(\beta' \cdot \gamma)(\alpha) = \beta'(\gamma\alpha) = 0$$

since  $\gamma\alpha$  can never equal  $\beta$  as its degree is too big. Similarly, elements  $\beta' \in \mathcal{B}'$  with  $\deg(\beta') = \deg(\gamma')$  satisfy  $\beta' \cdot \gamma = e'_j$  if and only if  $\beta = \gamma$  and 0 otherwise. Combined, we

obtain

$$\psi \cdot \gamma = c_\gamma e'_j$$

which lives in  $N \cap S_j$ . Hence,  $P_j^*$  is an essential extension.

In regards to injectivity, recall that  $P_j$  is a projective locally finite right  $B$ -module.

Let  $M = \bigoplus_i M_i$  be a locally finite  $B$ -module. Then  $\underline{\text{Hom}}_k(\bigoplus M_i, k) = \bigoplus \underline{\text{Hom}}_k(M_{-i}, k)$  together with

$$\underline{\text{Hom}}_k(\underline{\text{Hom}}_k(M_i, k), k) \cong M_i$$

implies that taking the Matlis dual twice returns a module isomorphic to  $M$ . Therefore, the functor  $\underline{\text{Hom}}_k(-, k)$  forms an anti equivalence on the category of locally finite graded  $B$ -modules. In particular, the graded projective left  $B$ -module  $Be_j$  is sent to a graded injective right  $B$ -module  $P_j^*$ .  $\square$

This will now help proving the following lemma motivated by [JZ00, Lemma 2.1].

Notice that  $A$  is generalized Gorenstein by Proposition 2.3.9 and so this lemma applies to  $A$ .

**Lemma 6.3.3.** *Recall that  $B$  satisfies  $\underline{\text{Ext}}_B^i(S_j, B) \cong \delta_{Ni} S_{\sigma(j)}^*[\ell_j]$  for some  $\sigma \in \text{Sym}(n)$ .*

For  $\mathfrak{r} = B_{\geq 1}$  we have

$$H_{\mathfrak{r}}^i(B) \cong \begin{cases} 0 & \text{for } i \neq N, \\ \bigoplus_{b=1}^n P_b^*[\ell_b] & \text{for } i = N \end{cases}$$

where  $P_b^* = \underline{\text{Hom}}_k({}_B(Be_b)_k, k_k)$  and  $N = \text{injdim}(B)$ . In particular,  $B$  is AS-Cohen-Macaulay.

*Proof.* We calculate  $\underline{\text{Ext}}^i(B/B_{\geq r}, B)$  by resolving the second entry using a minimal graded injective resolution of length  $N$  of  $B$ , say

$$0 \longrightarrow B \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots \longrightarrow E^N \longrightarrow 0.$$

Fix some minimal  $j \leq N$  such that  $E^j$  has a nonzero torsion element  $e$ . We can assume that  $eB_{\geq i} = 0$  for some minimal  $i \in \mathbb{N}$ . By picking a nonzero element  $\tilde{e} \in eB_{\geq i-1}$  we have that  $\tilde{e}B_{\geq 1} = 0$ . This implies an embedding  $S_a[-\deg(\tilde{e})] \hookrightarrow E^j$  for some  $a \in \{1, \dots, n\}$  ( $\tilde{e} \cdot e_a \neq 0$  for some  $a$ ). Since  $S_a$  is a simple module and the resolution is minimal, this forces the composition  $S_a \rightarrow E^j \rightarrow E^{j+1}$  to be zero as otherwise the graded injective hull of  $S_a$  is a summand of both  $E^j$  and  $E^{j+1}$ . Thus, applying  $\underline{\text{Hom}}(S_a[-\deg(\tilde{e})], -)$  to the resolution and taking homology at the  $j$ -th spot, we obtain nonzero elements. In other words, using the minimality of  $j$ ,  $\underline{\text{Ext}}_B^j(S_a[-\deg(\tilde{e})], B) \neq 0$  and so  $j = N$  as well as  $E^0, \dots, E^{N-1}$  are torsionfree. This shows that  $\underline{\text{Hom}}(B/B_{\geq r}, E^i) = 0$  for  $i < N$  and the conclusion is  $H_{\mathfrak{r}}^i(B) = 0$  for  $i \neq N$ .

From the generalized Gorenstein condition it follows that  $\underline{\text{Ext}}_B^N(S_b, B) \cong S_{\sigma(b)}^*[\ell_b]$  for every  $b$  and some  $\sigma \in \text{Sym}(n)$ . As shown above,  $E^0, \dots, E^{N-1}$  are torsionfree. This means  $\underline{\text{Hom}}_B(S_b, E^N)$  is nonzero. More precisely, due to  $S_b$  being simple, we can find a right  $B$ -module embedding from  $S_b$  into  $E^N$ . Moreover, since  $E^N$  is an injective module, we obtain that  $S_b[\ell_b]$ 's graded injective envelope  $P_b^*[\ell_b]$  (see Remark 6.3.2) is a direct summand in  $E^N$ :

$$E^N \cong P_b^*[\ell_b] \oplus M.$$

To guarantee that  $\dim_k(\underline{\text{Ext}}_B^N(S_b, B)) = 1$ , there cannot be another copy of  $P_b^*$  in  $M$ .

Repeating this argument for  $b = 1, \dots, n$ , we obtain

$$E^N \cong \bigoplus_{b=1}^n P_b^*[\ell_b] \oplus I$$

where  $I$  is torsionfree. Taking  $H_{\mathfrak{r}}^N(-)$  provides

$$\begin{aligned} H_{\mathfrak{r}}^N(B) &= \varinjlim \underline{\text{Ext}}^N(B/B_{\geq r}, B) = \varinjlim \underline{\text{Hom}}_B(B/B_{\geq r}, E^N) \\ &= \varinjlim \underline{\text{Hom}}_B \left( B/B_{\geq r}, \bigoplus_{b=1}^n P_b^*[\ell_b] \oplus I \right) = \bigoplus_{b=1}^n P_b^*[\ell_b] \end{aligned}$$

where the last equality holds since the direct limit of  $\underline{\text{Hom}}_B(B/B_{\geq r}, M)$  always gives back the torsion part of  $M$  combined with  $P_b^*$  is torsion. This implies the statement.  $\square$

Lemma 6.3.3 shows that  $H_{\mathfrak{r}}^i(B) \cong B^*$  as ungraded modules. While the shifts of  $P_b^*[\ell_b]$  can be different for each  $b$ , this does not happen for preprojective algebras where each  $\ell_b = 2$  and therefore  $H_{\mathfrak{m}}^2(A) \cong A^*[2]$  as graded modules.

Our next goal will be to connect  $H_{\mathfrak{r}}^N(g) : H_{\mathfrak{r}}^N(B) \rightarrow H_{\mathfrak{r}}^N(B)$  to  $(g^{-1})^*$  for some  $g \in \text{Aut}_{\text{gr}}(B)$  with  $g(e_i) = e_i$  for all  $i = 1, \dots, n$ . Therefore, we need to understand  $(g^{-1})^*$  first. Moreover, it turns out that any  $g$ -linear map  $\Phi$  from  $B_B^* \rightarrow B_B^*$  is closely related to  $(g^{-1})^*$  via  $n$  scalars determined by  $\Phi(e'_i)$ . Notice that it is enough to look at  $B^*$  rather than  $\bigoplus_{b=1}^n P_b^*[\ell_b]$  since  $\Phi$  is  $g$ -linear and  $g$  fixes the idempotents. This guarantees that  $\Phi$  restricts to maps on each  $P_b^*[\ell_b]$  individually. For simplicity, we will therefore state the following remark and its subsequent lemma without worrying about shifts in the grading.

*Remark 6.3.4.* Recall that  $g \in \text{Aut}_{\text{gr}}(B)$  with  $g(e_i) = e_i$  for all  $i$ . Then  $g^{-1}$  becomes a  $g^{-1}$ -linear  $B$ -module homomorphism from  ${}_B B$  to  ${}_B B$ . Furthermore,

$$(g^{-1})^* : B_B^* \rightarrow B_B^*, \quad \varphi \mapsto \varphi \circ g^{-1}$$

is  $g$ -linear: let  $a, x \in B$  and  $\varphi \in B^* = \text{Hom}_k({}_B B_k, k_k)$ . Then

$$\begin{aligned} (g^{-1})^*(\varphi \cdot a)(x) &= (\varphi \cdot a)(g^{-1}(x)) = \varphi(ag^{-1}(x)) \\ &= \varphi(g^{-1}(g(a)x)) = (\varphi \circ g^{-1})(g(a)x) \\ &= ((\varphi \circ g^{-1}) \cdot g(a))(x) = (((g^{-1})^*(\varphi)) \cdot g(a))(x). \end{aligned}$$

Moreover, any  $g$ -linear  $B$ -module homomorphism  $\Phi : B_B^* \rightarrow B_B^*$  can be described as follows. In order to avoid double usage of  $(-)^*$ , we denote the dual basis by  $\mathcal{B}' = \{\beta' \mid \beta \in \mathcal{B}\}$  where  $\mathcal{B}$  stands for a fixed basis of  $B$ . In general we have

$$\Phi(e'_i) = \sum_{j=1}^n \lambda_{ij} e'_j, \quad \lambda_{ij} \in k.$$

However, since

$$\sum_{j=1}^n \lambda_{ij} e'_j = \Phi(e'_i) = \Phi(e'_i \cdot e_i) = \Phi(e'_i) \cdot g(e_i) = \sum_{j=1}^n \lambda_{ij} e'_j \cdot g(e_i) = \lambda_{ii} e'_i$$

we can define  $\lambda_i = \lambda_{ii}$  and know that  $\Phi(e'_i) = \lambda_i e'_i$ . We will show that these  $\lambda_i$  determine  $\Phi$  completely. Let  $\beta$  be a word in  $e_r B e_s$ . By definition it follows that  $\beta' \cdot \beta = e'_s$ . Now, the same trick as for  $e'_i$  calculates that

$$\lambda_s e'_s = \Phi(e'_s) = \Phi(\beta' \cdot \beta) = \Phi(\beta') \cdot g(\beta).$$

By checking on a basis, we conclude with  $\Phi(\beta') = \lambda_s(\beta' \circ g^{-1})$  for  $\beta \in B e_s$ .

In order to apply the previous remark it remains to show that  $H_{\mathfrak{r}}^N(g)$  is actually  $g$ -linear. The domain and codomain equal  $B_B^*$  thanks to Lemma 6.3.3.

**Lemma 6.3.5.** *The map  $H_{\mathfrak{r}}^N(g) : H_{\mathfrak{r}}^N(B) \rightarrow H_{\mathfrak{r}}^N(B)$ , where  $\mathfrak{r} = B_{\geq 1}$ , is a  $g$ -linear  $B$ -module homomorphism from  $B_B^*$  to  $B_B^*$ . Further, there exist scalars  $\lambda_i$  for  $i = 1, \dots, n$  such that  $H_{\mathfrak{r}}^N(g)(\beta') = \lambda_s(\beta' \circ g^{-1})$  for  $\beta \in Be_s$ .*

*Proof.* Thanks to Lemma 6.3.3, this statement makes sense. We adapt the ideas of [JZ00, Lemma 2.2] to our situation. The same proof, considering the  $B$ -linear map  $g : B \rightarrow (B)^g$  (the right  $B$ -module with action via  $g$ ) and applying  $H_{\mathfrak{r}}^N(-)$  gives the following  $B$ -linear map:

$$H_{\mathfrak{r}}^N(g) : H_{\mathfrak{r}}^N(B) \rightarrow H_{\mathfrak{r}}^N((B)^g) = (H_{\mathfrak{r}}^N(B))^g.$$

Now, Lemma 6.3.3 translates this  $g$ -linear map from  $H_{\mathfrak{r}}^N(B) \rightarrow H_{\mathfrak{r}}^N(B)$  into

$$H_{\mathfrak{r}}^N(g) : B_B^* \rightarrow B_B^*.$$

The previous Remark 6.3.4 gives us  $\lambda_i$  for  $i = 1, \dots, n$  such that  $H_{\mathfrak{r}}^N(g)(\beta') = \lambda_s(\beta' \circ g^{-1})$  for  $\beta \in Be_s$ . □

Finally, we can put the pieces together. Every graded automorphism  $g$  of  $B$  fixing the idempotents induces a  $g$ -linear map  $H_{\mathfrak{r}}^N(B) \rightarrow H_{\mathfrak{r}}^N(B)$ . Therefore, it comes along with  $n$  scalars related to  $H_{\mathfrak{r}}^N(g)(e'_i)$  which we will call  $\text{Hdet}(g)_i^{-1}$ . Putting these scalars into a diagonal matrix finishes the definition of the homological determinant:

**Definition 6.3.6.** Let  $g \in \text{Aut}_{\text{gr}}(B)$  as before. Define the *homological determinant*  $\text{Hdet}(g)$  to be the diagonal matrix with entries  $(\text{Hdet}(g))_i = \lambda_i^{-1}$  for the  $\lambda_i$ 's from Lemma 6.3.5.

From the definition it is unclear how to calculate the  $\lambda_i$ 's or whether they are related to each other. In Section 6.4 we will prove that for preprojective algebras with quivers of extended Dynkin type  $A$ ,  $D$  or  $E$  the homological determinant is a scalar matrix. However, this does not always need to be the case as one can see by looking at a direct product of two algebras.

While we do not need to worry about shifts in the definition of the homological determinant, we now need to be more careful again. This next lemma follows [JZ00, Lemma 3.1] closely. It lays the ground work for the main theorem 6.3.9. More precisely, satisfying the conditions from Lemma 6.3.3 that  $H_{\mathfrak{s}}^N(B^G) = \bigoplus_{b=1}^n (e_b B^G)^*[\ell_b]$  for some fixed ring  $B^G$  together with the weaker AS-Cohen-Macaulay property is enough to prove the generalized Gorenstein condition later. The only piece missing is to guarantee finite injective dimension which will be taken care of by requiring a balanced dualizing complex as done afterwards.

**Lemma 6.3.7.** *Let  $G$  be a finite group acting on  $B$  by fixing the idempotents  $e_i$  for  $i = 1, \dots, n$ . Denote the fixed ring  $B^G$  by  $C$  and call  $\mathfrak{r} = B_{\geq 1}$  and  $\mathfrak{s} = C_{\geq 1}$ .*

(1) *Let  $N = \text{injdim}_B(B)$ . Then for  $i \neq N$  it is true that  $H_{\mathfrak{s}}^i(C) = H_{\mathfrak{s}^{\text{op}}}^i(C) = 0$ . In other words,  $C$  is AS-Cohen-Macaulay.*

(2) *Assume every  $g \in G$  satisfies  $\text{Hdet}(g) = I_n$ . Then  $H_{\mathfrak{s}}^N(C) \cong \bigoplus_{b=1}^n (e_b C)^*[\ell_b]$  as right  $C$ -modules where  $(e_b C)^* = \underline{\text{Hom}}_k(Ce_b, k)$ .*

*Proof.* We start with (1). Recall that both  $B$  and  $C$  are noetherian by Lemma 2.3.14(b).

Also,  $B$  and  $B^{\text{op}}$  satisfy the  $\chi$ -condition. This allows us to use results from [AZ94]. In particular, [AZ94, Lemma 8.2] including its preparatory work holds in our case. Consequently, [YZ99, Lemma 4.15] (Note: In [YZ99], Lemma 4.15 is labeled 4.5 and appears between Theorem 4.14 and Theorem 4.16) is satisfied under our assumptions. This implies  $H_{\mathfrak{r}}^i = H_{\mathfrak{s}}^i$  and  $H_{\mathfrak{r}^{\text{op}}}^i = H_{\mathfrak{s}^{\text{op}}}^i$  for all  $i$ . Thanks to Lemma 2.3.14(a) we have  $B \cong C \oplus K$  as  $(C, C)$ -bimodules and so

$$H_{\mathfrak{r}}^i(B) = H_{\mathfrak{s}}^i(B) = H_{\mathfrak{s}}^i(C \oplus K) = H_{\mathfrak{s}}^i(C) \oplus H_{\mathfrak{s}}^i(K).$$

Since  $B$  is generalized Gorenstein, for  $i \neq N$ , the left hand side is zero and so as a summand also  $H_{\mathfrak{s}}^i(C) = 0$  for  $i \neq N$  due to Lemma 6.3.3. This makes  $C$  AS-Cohen-Macaulay. The same calculation works over the opposite rings which finishes the first part.

Now, we prove (2). Recall from Lemma 2.3.14(a), the projection  $\pi_G : B \rightarrow B^G = C$  via  $b \mapsto \frac{1}{|G|} \sum_{g \in G} g(b)$  is a splitting of the natural inclusion of  $C$  into  $B$ , so we have

$$0 \longrightarrow C \xrightleftharpoons[\pi_G]{\iota} B \longrightarrow B/C \longrightarrow 0.$$

Being an additive functor,  $H_{\mathfrak{s}}^N(\pi_G) : H_{\mathfrak{s}}^N(B) \rightarrow H_{\mathfrak{s}}^N(C)$  is the splitting of the inclusion of the right  $C$ -modules  $H_{\mathfrak{s}}^N(C)$  into  $H_{\mathfrak{s}}^N(B)$ :

$$0 \longrightarrow H_{\mathfrak{s}}^N(C) \xrightleftharpoons[H_{\mathfrak{s}}^N(\pi_G)]{\iota'} H_{\mathfrak{s}}^N(B) \longrightarrow H_{\mathfrak{s}}^N(B/C) \longrightarrow 0.$$

Then  $H_{\mathfrak{s}}^N(\pi_G) \circ \iota' = \text{id}_{H_{\mathfrak{s}}^N(C)}$  allows us to understand the module  $H_{\mathfrak{s}}^N(C)$  as the fixed points of  $H_{\mathfrak{s}}^N(\pi_G)$  via  $\iota'$ , i.e.

$$H_{\mathfrak{s}}^N(C) = \{z \in H_{\mathfrak{s}}^N(B) \mid H_{\mathfrak{s}}^N(\pi_G)z = z\}.$$

However, it is known that

$$H_s^N(\pi_G) = \frac{1}{|G|} \sum_{g \in G} H_s^N(g).$$

We are going to show the equality

$$H_s^N(C) = \{z \in H_s^N(B) \mid H_s^N(g)(z) = z \text{ for all } g \in G\}.$$

“ $\supseteq$ ” Pick  $z \in H_s^N(B)$  which is fixed under all  $H_s^N(g)$ . Clearly,  $z$  is also fixed under the weighted sum  $H_s^N(\pi_G)$  and thus  $z \in H_s^N(C)$ .

“ $\subseteq$ ” Conversely, let  $z \in H_s^N(C)$  which says  $H_s^N(\pi_G)(z) = z$ . The same calculation as in [JZ00, Lemma 3.1] provides

$$\begin{aligned} H_s^N(g)(z) &= H_s^N(g) (H_s^N(\pi_G)(z)) = H_s^N(g) \sum_{h \in G} \frac{1}{|G|} H_s^N(h)(z) \\ &= \sum_{h \in G} \frac{1}{|G|} H_s^N(gh)(z) = H_s^N(\pi_G)(z) = z. \end{aligned}$$

This shows the second inclusion.

Combined, together with the observation made in part (1) that  $H_{\mathfrak{r}}^N = H_s^N$ , it follows that

$$\begin{aligned} H_s^N(C) &= \{z \in H_{\mathfrak{r}}^N(B) \mid H_s^N(g)(z) = z \text{ for all } g \in G\} \\ &\stackrel{6.3.3}{=} \left\{ z \in \bigoplus_{b=1}^n P_b^*[\ell_b] \mid H_s^N(g)(z) = z \text{ for all } g \in G \right\}. \end{aligned} \tag{6.4}$$

The assumption  $\text{Hdet}(g) = I_n$  implies that  $H_s^N(g)(\beta') = (\beta' \circ g^{-1}) = (g^{-1})^*(\beta')$  for all  $g \in G$  and all  $\beta \in A$ . We apply this to Equation (6.4):

$$H_s^N(C) = \left\{ z \in \bigoplus_{b=1}^n P_b^*[\ell_b] \mid (g^{-1})^*(z) = z \text{ for all } g \in G \right\} \stackrel{(*)}{=} \bigoplus_{b=1}^n (e_b C)^*[\ell_b].$$

To see (\*), recall that any element in  $C$  is fixed under the action of  $G$ . Therefore, every  $\beta' \in (e_b C)^*$  satisfies  $(g^{-1})^*(\beta') = \beta' \circ g^{-1} = \beta'$ . The reverse argument is similar.  $\square$

**Lemma 6.3.8.** *Let  $C$  be a noetherian  $\mathbb{N}$ -graded AS-Cohen-Macaulay algebra with  $C_0 \cong k^n$  and denote its primitive pairwise orthogonal idempotents by  $\{e_1, \dots, e_n\}$ . Assume that  $C$  has a balanced dualizing complex. Write  $\mathfrak{n} = C_{\geq 1}$ . If  $C$  satisfies  $H_{\mathfrak{n}}^N(C) = \bigoplus_{b=1}^n (e_b C)^*[\ell_b]$  as right  $C$ -modules, then  $C$  is generalized Gorenstein.*

*Proof.* Abusing notation, let's denote the simple  $C$ -modules by  $S_i = e_i C / (e_i C)_{\geq 1}$  and  $S_i^* = C e_i / (C e_i)_{\geq 1}$ , respectively. Similarly to [JZ00, Lemma 3.2], the modified spectral sequence from [Jør99, Proposition 1.1] given by

$$E_2^{pq} = \underline{\mathrm{Ext}}_C^p(C_0, H_{\mathfrak{n}}^q(C)) \Rightarrow \underline{\mathrm{Ext}}_C^{p+q}(C_0, C)$$

collapses due to  $C$  being AS-Cohen-Macaulay. Moreover, as used before, all the  $(e_b C)^*$  are injective and so  $E^{pq}$  has only one nonzero entry at the  $p = 0$  and  $q = N$  position. Two conclusions are possible. First,  $\underline{\mathrm{Ext}}_C^i(C_0, C) = 0$  for  $i \neq N$  and second,

$$\underline{\mathrm{Ext}}_C^N(C_0, C) \cong \underline{\mathrm{Ext}}_C^0(C_0, H_{\mathfrak{n}}^N(C)) = \underline{\mathrm{Hom}}_C \left( C_0, \bigoplus_{b=1}^n (e_b C)^*[\ell_b] \right) = \bigoplus_{b=1}^n S_b^*[\ell_b]. \quad (6.5)$$

By a symmetric version of [RR18, Lemma 2.8] it follows that

$$\underline{\mathrm{Ext}}_C^N(e_i C_0, C) \cong \bigoplus_{b=1}^n S_b^*[\ell_b] e_i \neq \{0\}$$

for all  $i$  and together with Equation (6.5) this implies  $\dim_k(\underline{\mathrm{Ext}}_C^N(S_i, C)) = 1$ . In other words,  $C$  satisfies the generalized Gorenstein condition.

In conclusion, we need to show that  $\text{id}_C(C)$  is finite. For this, [vdB97, Theorem 6.3] which works for non-connected algebras and the fact that  $C$  has a balanced dualizing complex verifies both requirements of van den Bergh's Local Duality Theorem [vdB97, Theorem 5.1]. In particular, the  $\chi$ -condition implies that  $\text{Ext}^i(C_0, C_0)$  is finite dimensional for all  $i$  which is the non-connected equivalent of being Ext-finite necessary for [vdB97, Theorem 5.1]. Finally, [vdB97, Theorem 5.1(1)] together with the condition  $H_u^N(C) = C^*$  as ungraded modules says that  $C$  has finite injective dimension.  $\square$

At last, we can put together the previous lemmas. Fortunately, fixed rings have a balanced dualizing complex and so the following main theorem is the combination of 6.3.7 and 6.3.8.

**Theorem 6.3.9.** *Suppose  $B$  has finite global dimension  $d$ . Suppose  $G$  is a finite subgroup of graded automorphisms of  $B$  such that every  $g \in G$  fixes the primitive idempotents. Assume that every  $g \in G$  satisfies  $\text{Hdet}(g) = I_n$ . Then  $B^G$  is generalized Gorenstein.*

*Proof.* Call  $C = B^G$  and  $\mathfrak{s} = C_{\geq 1}$  as before. By Lemma 6.3.7 it follows that  $C$  is AS-Cohen-Macaulay and that  $H_{\mathfrak{s}}^N(C) \cong \bigoplus_{b=1}^n (e_b C)^*[\ell_b]$ . Thanks to Proposition 2.3.18 the algebra  $C$  satisfies the  $\chi$ -condition. Our assumptions and Lemma 2.3.14(b) guarantee that both  $B$  and  $C$  are noetherian.

We want to apply [vdB97, Theorem 6.3] to  $B$  which holds for non-connected algebras. The first condition is satisfied since  $B$  has finite global dimension. This instantly implies that  $\Gamma_m$  and  $\Gamma_{m^\circ}$  have finite cohomological dimension. Moreover, we require  $B$

and  $B^{\text{op}}$  to satisfy the  $\chi$ -condition. Hence,  $B$  has a balanced dualizing complex. Using the projection  $B \rightarrow B^G$  which is a finite homomorphism of graded  $k$ -algebras, [YZ99, Theorem 4.16] yields a balanced dualizing complex for  $B^G$ . With the requirements of Lemma 6.3.8 satisfied,  $C$  is generalized Gorenstein.  $\square$

Notice that  $B$  and  $B^{\text{op}}$  in Theorem 6.3.9 do not need to be assumed in advance to satisfy the  $\chi$ -condition. Since  $B$  has finite global dimension and satisfies the generalized Gorenstein condition, it must automatically satisfy  $\chi$  due to 2.3.6.

## 6.4 Homological determinants for preprojective algebras

Let  $Q = \widetilde{A_{n-1}}$ ,  $\widetilde{D_{n-1}}$  or  $\widetilde{E_m}$  for  $m = 6, 7, 8$  and  $A = \Pi_Q(k)$ . Further, let  $\mathfrak{m} = A_{\geq 1}$ .

While the construction of the homological determinant of a graded automorphism  $g$  of  $A$  in Section 6.3 is rather complicated, it turns out that for preprojective algebras in particular the homological determinant is quite simple. Thanks to the work in Sections 6.1, 6.2 and 4.1, we have a detailed description of the vector trace of  $g$ . Preprojective algebras satisfy many nice properties. For instance, they happen to be finite modules over polynomial rings in two variables. This is enough to extend the results from [JZ00, Section 4] to  $A$  which relate the trace of the projective modules of  $A$  to the trace of  $H_{\mathfrak{m}}^2(g|_{e_i A})$  which entails the  $i$ -th entry of the homological determinant. As a consequence, it turns out that the homological determinant equals the diagonal matrix whose entries

are all equal to  $c_1 t_1$  using the usual convention  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ .

**Proposition 6.4.1.** *The following reciprocity statement holds:*

$$\mathrm{Tr}_{Ae_i}((g|_{Ae_i})^{-1}, t^{-1}) = c_1 t_1 t^2 \mathrm{Tr}_{e_i A}(g|_{e_i A}, t).$$

*Proof.* We are going to use the formulas for  $\underline{\mathrm{Tr}}(g, t)$  and  $\overline{\mathrm{Tr}}(g, t)$  from the Propositions 4.1.5, 6.1.2 and 6.2.1. There, equations of the form

$$\underline{\mathrm{Tr}}(g, t) = M_1^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad \overline{\mathrm{Tr}}(g, t) = M_2^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

with explicit descriptions for  $M_1$  and  $M_2$  were found. Since  $g$  acts by scalar multiplication, we can obtain a formula for  $\overline{\mathrm{Tr}}(g^{-1}, t^{-1})$  by inverting each  $c_i, t_i$  and  $t$  that occurs in  $M_2$ . Denote the resulting matrix by  $\widetilde{M}_2$ . It is easy to see that  $c_1 t_1 t^2 \cdot \widetilde{M}_2$  equals  $M_1$  in each case: recall that  $c_1 t_1 = c_i t_i$  for all  $i$  and therefore multiplying by  $c_1 t_1 t^2$  turns  $t^{-1}$  into  $t$  and switches each  $c_i^{-1}$  into  $t_i$  and similarly  $t_i^{-1}$  into  $c_i$ . Therefore,  $c_1 t_1 t^2 \cdot \underline{\mathrm{Tr}}(g, t)$  equals

$$c_1 t_1 t^2 \cdot M_1^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = c_1 t_1 t^2 \cdot (c_1 t_1 t^2 \cdot \widetilde{M}_2)^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \widetilde{M}_2^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \overline{\mathrm{Tr}}(g^{-1}, t^{-1})$$

and looking at the  $i$ -th entry of the resulting vectors on each side reveals the claim.  $\square$

The next definition is taken from [JZ00, Definition 1.3].

**Definition 6.4.2.** Let  $R$  be an  $\mathbb{N}$ -graded locally finite  $k$ -algebra and call  $\mathfrak{r} = R_{\geq 1}$ . Let  $M$  be an  $R$ -module such that  $H_{\mathfrak{r}}^i(M) = 0$  for large  $i$  and each  $H_{\mathfrak{r}}^i(M)$  is locally finite and

right bounded. Let  $g \in \text{Aut}_{\text{gr}}(R)$  and  $f : M \rightarrow M$  be a  $g$ -linear homomorphism. We define:

$$\text{Br}_M^R(f, t) = \sum_i (-1)^i \text{Tr}_{H_{\mathfrak{r}}^i(M)}(H_{\mathfrak{r}}^i(f), t).$$

If  $M$  is left bounded and locally finite, we call  $f$  *rational* over  $k$  if the following two conditions are satisfied:

- (1)  $\text{Tr}_M(f, t)$  and  $\text{Br}_M^R(f, t)$  are rational functions over  $k$  (inside  $k((t))$  respectively  $k((t^{-1}))$ ) and
- (2)  $\text{Tr}_M(f, t) = \text{Br}_M^R(f, t)$  as rational functions over  $k$ .

If there is no confusion about the algebra  $R$ , we may omit the superscript.

Thanks to the nice reciprocity property, we are able to find a connection between  $\text{Br}_{e_i A}(g_i, t)$  and the corresponding trace function:

**Proposition 6.4.3.** *The projective  $A$ -module  $e_i A$  satisfies*

$$\text{Br}_{e_i A}^A(g, t) = \text{Hdet}(g)_i^{-1} c_1 t_1 \text{Tr}_{e_i A}(g, t)$$

*as rational functions over  $k$ .*

*Proof.* From a dual version of [RR18, Lemma 2.8] we know that

$$\begin{aligned} H_{\mathfrak{m}}^2(e_i A) &= \varinjlim \text{Ext}^2(A/A_{\geq m}, e_i A) = e_i \varinjlim \text{Ext}^2(A/A_{\geq m}, A) \\ &= e_i A^*[2] = e_i \underline{\text{Hom}}_k(A, k)[2] = \underline{\text{Hom}}_k(Ae_i, k)[2] = P_i^*[2]. \end{aligned}$$

Since  $H_{\mathfrak{m}}^i(-)$  is an additive functor and  $A = \bigoplus_i e_i A$ , Lemma 6.3.3 gives that  $H_{\mathfrak{m}}^i(e_i A)$  is nonzero only if  $i = 2$ . This allows the calculation as in [JZ00, Proposition 4.2]:

$$\begin{aligned}
\mathrm{Br}_{e_i A}(g, t) &= \mathrm{Tr}_{H_{\mathfrak{m}}^2(e_i A)}(H_{\mathfrak{m}}^2(g), t) \\
&= \mathrm{Tr}_{P_i^*[2]}(\mathrm{Hdet}(g)_i^{-1} (g^{-1})^*, t) \\
&= \mathrm{Hdet}(g)_i^{-1} t^{-2} \mathrm{Tr}_{P_i^*}((g^{-1})^*, t) \\
&= \mathrm{Hdet}(g)_i^{-1} t^{-2} \mathrm{Tr}_{Ae_i}(g^{-1}, t^{-1}) \\
&= \mathrm{Hdet}(g)_i^{-1} t^{-2} c_1 t_1 t^2 \mathrm{Tr}_{e_i A}(g, t) \\
&= \mathrm{Hdet}(g)_i^{-1} c_1 t_1 \mathrm{Tr}_{e_i A}(g, t)
\end{aligned}$$

where  $H_{\mathfrak{m}}^2(g|_{e_i A}) = \mathrm{Hdet}(g)_i^{-1} (g^{-1})^*$  comes from the construction of  $\mathrm{Hdet}$  in Lemma 6.3.5.

The second to the last equality was proved in Proposition 6.4.1.  $\square$

In contrast to Proposition 6.4.3, the induced maps  $g_i$  are rational over  $k$ . The following theorem is stated in a more general setting.

**Theorem 6.4.4.** *Let  $B$  be a noetherian  $\mathbb{N}$ -graded locally finite twisted Calabi-Yau algebra of global dimension  $d \geq 1$  with degree zero piece  $B_0 \cong k^n$ . Denote the primitive orthogonal idempotents by  $e_i$ . Further assume that there is a noetherian subring  $S$  of  $B$  which is Artin-Schelter regular of global dimension  $d$  such that  ${}_S B$  and  $B_S$  are finitely generated. Let  $g$  be a graded automorphism of  $B$  such that  $g(e_i) = e_i$  and  $g(S) = S$ . Let  $f : e_i B \rightarrow e_i B$  be a graded  $g$ -linear automorphism. Then  $\mathrm{Br}_{e_i B}(f, t) = \mathrm{Tr}_{e_i B}(f, t)$ . In other words,  $f$  is rational over  $k$ .*

*Proof.* Notice that  $B$  is automatically a generalized Gorenstein algebra by [RR18, Theorem 5.2] and [RR18, Theorem 5.15]. Fix  $\mathfrak{r} = B_{\geq 1}$  and  $\mathfrak{n} = S_{\geq 1}$ . The idea of this proof is extracted from [JZ00, Lemma 4.4]. Lemma 2.3.6 combined with Proposition 2.3.8 implies that  $B$  satisfies the  $\chi$ -condition. From [AZ94, Theorem 8.1(1)] we know that also  $S$  satisfies the  $\chi$ -condition. As done before in Lemma 6.3.7, [AZ94, Proposition 7.2(2)] and [AZ94, Theorem 8.3(3)] imply  $H_{\mathfrak{r}}^j = H_{\mathfrak{n}}^j$  for  $j \geq 1$ . As described in [JZ00, Lemma 4.4(2°)] this isomorphism is functorial and so  $f : e_i B \rightarrow e_i B$  induces the same maps on  $H_{\mathfrak{r}}^j(e_i B)$  and  $H_{\mathfrak{n}}^j(e_i B)$  for  $j \geq 1$ . Summarized, as only  $H^d(e_i B)$  is nonzero by Lemma 6.3.3 for  $N = d \geq 1$ , we obtain  $\mathrm{Br}_{e_i B}^B(f, t) = \mathrm{Br}_{e_i B}^S(f, t)$ . Since for the trace function only the vector space structure matters,  $\mathrm{Tr}_{e_i B}(f, t)$  does not depend on whether  $e_i B$  is understood as  $S$ - or  $B$ -module. Now,  $S$  is Artin-Schelter regular and [JZ00, Proposition 4.2] says that  $f$  is rational over  $k$ , i.e.

$$\mathrm{Br}_{e_i B}^S(f, t) = \mathrm{Tr}_{e_i B}(f, t)$$

as  $S$ -modules. Combining the two equations implies  $\mathrm{Br}_{e_i B}^B(f, t) = \mathrm{Tr}_{e_i B}(f, t)$ .  $\square$

*Remark 6.4.5.* For  $Q = \widetilde{A_{n-1}}$  it is possible to concretely find a polynomial ring  $S$  in two variables over which  $A = \Pi_Q(k)$  is finitely generated. Define  $\beta_i = \alpha_i \alpha_{i+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1}$  and  $\beta_i^* = \alpha_{i-1}^* \alpha_{i-2}^* \cdots \alpha_1^* \alpha_n^* \cdots \alpha_i^*$ . Further, call

$$\beta = \sum_{i=1}^n \beta_i, \quad \beta^* = \sum_{i=1}^n \beta_i^* \quad \text{and} \quad e = \sum_{i=1}^n e_i = 1.$$

We are going to prove that  $A$  is a finitely generated module over  $k[\beta, \beta^*]$  where

$k = k \cdot e$ . Let

$$\mathcal{B} = \{\gamma \text{ path in } \bar{Q} \mid \text{type}(\gamma) \leq (n-1, n-1)\}$$

and notice that  $\mathcal{B}$  is a finite set. Let  $x$  be the unique simple path in  $\bar{Q}$  from  $e_\ell$  to  $e_m$  of type  $(\lambda n + r, \lambda^* n + r^*)$  where  $0 < r, r^* < n$  and  $\lambda, \lambda^* \in \mathbb{N}_0$ . Then we have the following

$$\beta^\lambda (\beta^*)^{\lambda^*} \tilde{x} = \beta_\ell^\lambda (\beta_\ell^*)^{\lambda^*} \tilde{x} = x,$$

where  $\tilde{x}$  is the unique simple path from  $e_\ell$  to  $e_m$  of type  $(r, r^*)$ . This implies  $\tilde{x} \in \mathcal{B}$ . To sum up,  $x$  lies in the module generated by  $\mathcal{B}$  over  $k[\beta, \beta^*]$ . This shows that  $A$  is finitely generated over  $k[\beta, \beta^*]$  which is an Artin-Schelter regular algebra of global dimension 2 as used in Theorem 6.4.4.

The previous remark encourages to apply Theorem 6.4.4 to a general preprojective algebra  $A$ . Combined with Proposition 6.4.3, it immediately gives information on  $\text{Hdet}(g)_i$ . We conclude with a corollary stating that all entries of  $\text{Hdet}(g)$  equal  $c_1 t_1$ .

**Corollary 6.4.6.** *Let  $Q$  be the quiver corresponding to an extended Dynkin graph of type  $\widetilde{A_{n-1}}$ ,  $\widetilde{D_{n-1}}$  or  $\widetilde{E_m}$  for  $m = 6, 7, 8$  and  $A = \Pi_Q(k)$ . Let  $g$  be a graded automorphism of  $A$  which fixes the idempotents. Use the usual convention  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Then  $\text{Hdet}(g) = c_1 t_1 I_n$  ( $n = m + 1$  for  $\widetilde{E_m}$ ). In particular, it makes sense to define  $\text{hdet}(g) = c_1 t_1$ .*

*Proof.* First, we need to find an Artin-Schelter regular subalgebra  $S$  of  $A$  of global dimension 2 such that  $A$  is a finitely generated module over  $S$  in order to apply Theorem 6.4.4. From [BRS<sup>+</sup>16, Theorem 1.9.15(2)] we obtain that the center of  $A$  equals  $Z(A) = \mathcal{O}(\mathbb{A}^2)^\Gamma$

- a fixed ring of a polynomial ring in two variables which is a finitely generated noetherian algebra. Notice that  $g(Z(A)) = Z(A)$  since  $g$  is surjective. Consider  $F = Z(A)^{\langle g \rangle}$ , the fixed ring of  $Z(A)$  under the action of  $g$  restricted to  $Z(A)$ . Recall that  $A$  is a finitely generated noetherian  $k$ -algebra. Moreover, since  $A$  is a prime polynomial identity algebra  $A$  (see Proposition 2.3.9), [DF04, II.10.6 Corollary] says that  $A$  is a finitely generated module over  $Z(A)$ . Proposition 2.3.17 says that  $Z(A)$  is a finitely generated  $F$ -module. Combined, this gives that  $A$  is a finitely generated  $F$ -module. Then, the Artin-Tate Lemma [DF04, Theorem 11.4] applied to the chain

$$k \subseteq F = Z(A)^{\langle g \rangle} \subseteq A$$

gives that  $F$  is a finitely generated  $k$ -algebra. Now, [BGL87, 6.10 Proposition] says that the  $k$ -transcendence degree of the quotient field of the center of  $A$  equals 2 which passes on to  $F$  since  $Z(A)$  is a finitely generated  $F$ -module. Hence, the Noether Normalization Lemma gives a polynomial ring  $S = k[x_1, x_2] \subseteq F$ . Moreover, also due to the Noether Normalization Lemma we know that  $F$  is a finitely generated module over  $S$ . To sum up, we have the chain of algebras

$$S = k[x_1, x_2] \subseteq F \subseteq Z(A) \subseteq A$$

and we showed that each algebra is a finitely generated module over the previous one. This shows that  $A$  is a finitely generated  $S$ -module. Clearly,  $S$  is Artin-Schelter regular of global dimension 2. In addition to that,  $g(S) = S$  since  $S$  is a subalgebra of the fixed ring  $F = Z(A)^{\langle g \rangle}$ .

Note that  $A$  is a noetherian  $\mathbb{N}$ -graded twisted Calabi-Yau algebra of global dimension 2 by Proposition 2.3.9. This means we can apply Theorem 6.4.4 to  $A$ ,  $S$  and the map  $f = g_i : e_i B \rightarrow e_i B$  which proves that  $g_i$  is rational over  $k$ . Combined with Proposition 6.4.3 we get

$$\mathrm{Tr}_{e_i A}(g_i, t) = \mathrm{Br}_{e_i A}^A(g_i, t) = \mathrm{Hdet}(g)_i^{-1} c_1 t_1 \mathrm{Tr}_{e_i A}(g, t)$$

and thus  $\mathrm{Hdet}(g)_i = c_1 t_1$  using that  $g_i(e_i) = e_i$  which guarantees that  $\mathrm{Tr}_{e_i A}(g_i, t)$  is nonzero.  $\square$

This chapter contains material from “Generalized Gorensteinness and a homological determinant for preprojective algebras”, Weispfenning, Stephan, to appear. The dissertation author was the primary investigator and author of this paper.

# Chapter 7

## First steps towards a

## Chevalley-Shephard-Todd Theorem

### 7.1 Analyzing the trace for case $\widetilde{A}_{n-1}$

In this section, let  $n \geq 3$  and fix  $Q = \widetilde{A}_{n-1}$ . Unless said otherwise,  $A = \Pi_Q(k)$  and  $g$  is a nontrivial graded automorphism of  $A$  which fixes the primitive idempotents  $e_i$  for  $i = 1, \dots, n$ . Therefore, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  for  $i = 1, \dots, n$  as done before with indices taken modulo  $n$  if necessary. Consequently, the induced maps  $g_i : e_i A \rightarrow e_i A$  with  $g_i(\beta) = g(\beta)$  arise.

Since quasi-reflections for Artin-Schelter regular algebras have been defined using the multiplicity of 1 as a root of the denominator of the trace function, we are going to examine how  $\text{Tr}(g, t)$  and  $\text{Tr}_{e_i A}(g_i, t)$  behave. As mentioned before, instead of consider-

ing 1 as a root of the denominator we have to look at 1 as a pole of  $\text{Tr}(g, t)$ . One of the concerns to have 1 as a pole of order equal to  $2 = \text{GKdim}(A)$  can be solved in the first proposition. While this scenario never happens for Artin-Schelter regular domains ([KKZ08, Proposition 1.8]), a nonidentity automorphism of  $A$  with 1 as a pole of order 2 is presented before Proposition 4.2.4. This problem can be solved as long as the automorphism has nontrivial homological determinant. Moreover, in this case, 1 is a pole of order less than or equal to 1 of  $\text{Tr}(g, t)$  and also of  $\text{Tr}_{e_i A}(g_i, t)$  for all  $i = 1, \dots, n$ .

With regard to defining quasi-reflections it would be nice to know if 1 as a pole of  $\text{Tr}_{e_i A}(g_i, t)$  behaves identically for all  $i$ . Unfortunately, this is only the case in two situations. If either

- (1) at most one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1 or
- (2) both  $c_1 \cdots c_n = t_1 \cdots t_n = 1$  and  $\text{hdet}(g) = 1$

the order of the pole 1 in  $\text{Tr}_{e_i A}(g_i, t)$  is the same for all  $i$ . In the remaining case, however, Example 7.1.8 presents an automorphism  $g$  where  $\text{Tr}_{e_1 A}(g_1, t)$  does not have 1 as a pole while  $\text{Tr}_{e_2 A}(g_2, t)$  and  $\text{Tr}(g, t)$  both have 1 as a pole of order 1.

**Proposition 7.1.1.** *If  $\text{hdet}(g) = c_1 t_1 \neq 1$  then  $\text{Tr}_{e_i A}(g_i, t)$  cannot have 1 as a pole of order greater than or equal to  $\text{GKdim}(A) = 2$  for all  $i = 1, \dots, n$ .*

*Proof.* Using Proposition 4.1.4(a) as well as the formula for the inverse of the matrix, we

can read off a formula for  $(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n) \text{Tr}_{e_i A}(g_i, t) = p_i(t)$  as

$$\left( \prod_{s=i}^n c_s t \prod_{s=i}^{n+1} c_s t \cdots \prod_{s=i}^{n+i-2} c_s t \underbrace{1}_i c_i t \cdots \prod_{s=i}^{n-1} c_s t \right) \cdot \begin{pmatrix} \sum_{k=0}^{n-1} \left( \prod_{j=n-k+1}^n t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+2}^{n+1} t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+3}^{n+2} t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+4}^{n+3} t_j \right) t^k \\ \vdots \\ \sum_{k=0}^{n-1} \left( \prod_{j=2n-k}^{2n-1} t_j \right) t^k \end{pmatrix}.$$

From knowing the denominator of  $\text{Tr}_{e_i A}(g_i, t)$  it is obvious that 1 cannot be a pole of order bigger than 2. Hence, the goal is to show that  $\text{Tr}_{e_i A}(g_i, t)$  cannot have 1 as a pole of order 2. The only way to obtain order 2 is if  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  and  $p_i(t)$  does not have 1 as a root.

Hence, assume that  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  as well as  $\text{hdet}(g) = c_1 t_1 \neq 1$ . We are going to prove that then  $p_i(t)$  also must have 1 as a root. More precisely, we will show that

$$p_i(1) = (1 + c_i + c_i c_{i+1} + \cdots + c_i \cdots c_n + c_i \cdots c_n c_1 + \cdots + c_i \cdots c_n c_1 \cdots c_{i-2}) \cdot (1 + c_1 t_1 + (c_1 t_1)^2 + \cdots + (c_1 t_1)^{n-1}). \quad (7.1)$$

From  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  we obtain  $(c_1 t_1)^n = c_1 \cdots c_n \cdot t_1 \cdots t_n = 1$  which then together with  $c_1 t_1 \neq 1$  says that  $c_1 t_1$  is a nontrivial  $n$ -th root of 1 and thus the second factor vanishes.

Let's look at the  $j$ -th summand for  $t = 1$  in the formula for  $p_i(1)$  individually by

separating into three cases,  $1 \leq j < i$ ,  $j = i$  and  $i < j \leq n$ . We use  $c_{n+\ell} = c_\ell$ ,  $t_{n+\ell} = t_\ell$  and  $c_1 \cdots c_n = 1$  in the form of  $c_i \cdots c_{n+i-1} = 1$  as well as  $c_j t_j = c_1 t_1$ .

- $j < i$ . The  $j$ -th summand computes as follows:

$$\begin{aligned}
& \left( \prod_{s=i}^{n+j-1} c_s \right) \cdot \sum_{k=0}^{n-1} \prod_{\ell=n+j-k}^{n+j-1} t_\ell \\
&= c_i \cdots c_{n+j-1} \cdot (1 + t_{j-1} + t_{j-1}t_{j-2} + \dots + t_{j-1} \cdots t_1 t_n \cdots t_{j+1}) \\
&= c_i \cdots c_{n+j-1} + (c_1 t_1) c_i \cdots c_{n+j-2} + \dots + (c_1 t_1)^{n+j-i} \\
&\quad + (c_1 t_1)^{n+j-i+1} c_i \cdots c_{n+i-2} + \dots + (c_1 t_1)^s c_i \cdots c_{n+j-s-1} \\
&\quad + \dots + (c_1 t_1)^{n-1} c_i c_{i+1} \cdots c_{n+j} \\
&= \sum_{s=0}^{n-1} (c_1 t_1)^s c_i \cdots c_{2n+j-s-1}.
\end{aligned}$$

- $j = i$ . The  $i$ -th summand computes as follows ( $\widehat{c_k}$  means omitting  $c_k$ ):

$$\begin{aligned}
& 1 \cdot \sum_{k=0}^{n-1} \prod_{\ell=n+i-k}^{n+i-1} t_\ell = 1 + t_{i-1} + t_{i-1}t_{i-2} + \dots + t_{i-1}t_{i-2} \cdots t_1 t_n \cdots t_{i+1} \\
&= 1 + (c_1 t_1) c_1 \cdots \widehat{c_{i-1}} \cdots c_n + \dots + (c_1 t_1)^s c_i \cdots c_{n+i-s-1} \\
&\quad + \dots + (c_1 t_1)^{n-1} c_i \\
&= \sum_{s=0}^{n-1} (c_1 t_1)^s c_i \cdots c_{n+i-s-1} = \sum_{s=0}^{n-1} (c_1 t_1)^s c_i \cdots c_{2n+i-s-1}.
\end{aligned}$$

- $j > i$ . This case is very similar to the case  $j < i$ . The  $j$ -th summand equals:

$$\left( \prod_{s=i}^{j-1} c_s \right) \cdot \sum_{k=0}^{n-1} \prod_{\ell=n+j-k}^{n+j-1} t_\ell = \sum_{s=0}^{n-1} (c_1 t_1)^s c_i \cdots c_{n+j-s-1} = \sum_{s=0}^{n-1} (c_1 t_1)^s c_i \cdots c_{2n+j-s-1}.$$

Finally, we sum up all  $j$ 's and obtain for fixed  $1 \leq s \leq n-1$ :

$$\begin{aligned} (c_1 t_1)^s & \left( c_i \cdots c_{2n+i-s-1} + \sum_{j=1}^{i-1} c_i \cdots c_{2n+j-s-1} + \sum_{j=i+1}^n c_i \cdots c_{2n+j-s-1} \right) \\ & = (c_1 t_1)^s \sum_{j=1}^n c_i \cdots c_{2n+j-s-1}. \end{aligned}$$

Here,  $(c_1 t_1)^s$  is multiplied by a sum of  $n$  terms each having an additional factor  $c_\ell$  in increasing order compared to the previous one. This means that

$$\begin{aligned} p_i(1) & = (1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2}) \\ & \quad \cdot (1 + (c_1 t_1) + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1}) \end{aligned}$$

which is equal to zero as the second factor vanishes due to  $c_1 t_1$  being a nontrivial  $n$ -th root of 1. Notice that no calculation used  $t_1 \cdots t_n = 1$ .  $\square$

**Corollary 7.1.2.** *Assume that  $\text{hdet}(g) = c_1 t_1 \neq 1$ . Then  $\text{Tr}(g, t)$  cannot have 1 as a pole of order greater than or equal to  $\text{GKdim}(A) = 2$ .*

*Proof.* From 7.1.1 we have that  $\text{Tr}_{e_i A}(g_i, t)$  cannot have 1 as a pole of order greater than or equal to  $2 = \text{GKdim}(A)$  for all  $i = 1, \dots, n$ . Since

$$\text{Tr}(g, t) = \sum_{j=1}^n \text{Tr}_{e_j A}(g_j, t),$$

also  $\text{Tr}(g, t)$  cannot have 1 as a pole of order greater than or equal to  $\text{GKdim}(A)$ .  $\square$

*Remark 7.1.3.* The corresponding equation for  $\text{Tr}(g, t) = \frac{p(t)}{q(t)}$  to  $\text{Tr}_{e_i A}(g_i, t)$  in Equation (7.1) equals

$$p(1) = \left( \sum_{k=0}^{n-1} \sum_{j=1}^n \prod_{s=j}^{j+k-1} c_s \right) \cdot (1 + (c_1 t_1) + (c_1 t_1)^2 + (c_1 t_1)^3 + \dots + (c_1 t_1)^{n-1}).$$

In the world of an Artin-Schelter regular domain  $R$ , [KKZ08, Proposition 1.8] proves that a nontrivial automorphism can never have 1 as a root of the denominator of its trace of multiplicity equal to  $\text{GKdim}(R)$ . Recall from Chapter 4, this can happen if  $A$  is the preprojective algebra corresponding to the extended Dynkin quiver  $\widetilde{A_{n-1}}$ . There, the example  $Q = \widetilde{A_2}$  with  $c_1 = t_1 = 1$  and  $c_2 = c_3 = t_2 = t_3 = -1$  was given. Using the formula from Proposition 4.2.4, we obtain  $\text{Tr}(g, t) = \frac{3 - 5t + 3t^2}{(1 - t)(1 - t^3)}$ .

In the remainder of this section, we analyze how the pole 1 in  $\text{Tr}_{e_i A}(g_i, t)$  behaves for different  $i$ . First, we manage to find a factorization of  $p_i(1)$  in case that at least one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1. This helps understanding how cancellation can happen. Therefore, this is enough to further examine the cases when at most one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1. The difficult part consists in understanding the derivative of the numerator  $p_i(t)$  of  $\text{Tr}_{e_i A}(g_i, t)$  which forms the end of this section.

**Lemma 7.1.4.** *Denote  $\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)}$  (see Proposition 4.1.4). Assume that at least one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1. Then  $p_i(1) = 0$  for some  $i$  if and only if  $p_j(1) = 0$  for  $j = 1, \dots, n$ . Further,*

$$p_i(1) = (1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_1 + \dots + c_i \cdots c_n c_1 \cdots c_{i-2}) \\ \cdot (1 + c_1 t_1 + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1}), \quad \text{if } c_1 \cdots c_n = 1, \quad \text{and}$$

$$p_i(1) = (1 + t_i + t_i t_{i-1} + \dots + t_{i-1} \cdots t_1 t_n \cdots t_{i+1}) \\ \cdot (1 + c_1 t_1 + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1}), \quad \text{if } t_1 \cdots t_n = 1.$$

*Proof.* There is only one direction to prove. Assume that  $p_i(1) = 0$  for some  $i$ . First, we

assume that  $c_1 \cdots c_n = 1$ . As shown in the proof of Proposition 7.1.1,  $c_1 \cdots c_n = 1$  implies that

$$p_i(1) = (1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2}) \\ \cdot (1 + (c_1 t_1) + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1}).$$

If the second factor  $(1 + (c_1 t_1) + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1})$  is zero,  $p_j(1) = 0$  for all  $j = 1, \dots, n$ .

Otherwise,  $p_i(1) = 0$  means

$$(1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2}) = 0.$$

But this is zero if and only if

$$(1 + c_{i+1} + c_{i+1} c_{i+2} + \dots + c_{i+1} \cdots c_n + c_{i+1} \cdots c_n c_{n+1} + \dots + c_{i+1} \cdots c_{n+i-1}) \\ = \frac{1}{c_i} (1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2}) = 0$$

using  $\frac{1}{c_i} = c_{i+1} \cdots c_{n+i-1}$ . Hence, from  $p_i(1) = 0$  it follows that  $p_{i+1}(1) = 0$  and by induction we get  $p_j(1) = 0$  for all  $j = 1, \dots, n$ . By symmetry, the result is true if  $t_1 \cdots t_n = 0$  instead.  $\square$

This is enough to prove that all  $\text{Tr}_{e_i A}(g_i, t)$  behave identically as long as not both  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equal 1.

**Corollary 7.1.5.** *Assume that at most one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1. Then, the order of 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  is the same for all  $j = 1, \dots, n$ .*

*Proof.* We use the notation of Proposition 4.1.4:

$$\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{(1 - c_1 \cdots c_n t^n)(1 - t_1 \cdots t_n t^n)}.$$

If  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  are both not equal to 1, all  $\text{Tr}_{e_j A}(g_j, t)$  do not have 1 as a pole. Otherwise, if precisely one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1, Lemma 7.1.4 says that either all  $p_j(t)$  have 1 as a root or none. In the first case, there is cancellation and the order of 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  equals zero for all  $j = 1, \dots, n$ . In the second case, there is no cancellation and 1 is a pole of  $\text{Tr}_{e_j A}(g_j, t)$  of order 1 for all  $j = 1, \dots, n$ .  $\square$

Next, we have to confront the case  $c_1 \cdots c_n = t_1 \cdots t_n = 1$ . The first approach also requires  $\text{hdet}(g) = 1$  but manages to describe this case completely.

**Proposition 7.1.6.** *Assume that  $c_1 \cdots c_n = t_1 \cdots t_n = 1$ . If  $\text{hdet}(g) = 1$ , then the order of 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  is the same for all  $j = 1, \dots, n$ . In particular, either 1 is a pole of order 0 for all  $j$  or a pole of order 2 for all  $j$ .*

*Proof.* From Lemma 7.1.4 we know that  $p_i(1) = 0$  for some  $i$  if and only if  $p_j(1) = 0$  for all  $j = 1, \dots, n$ . This happens precisely if

$$p_i(1) = (1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2}) \\ \cdot (1 + (c_1 t_1) + (c_1 t_1)^2 + \dots + (c_1 t_1)^{n-1}) = 0.$$

Since  $\text{hdet}(g) = c_1 t_1 = 1$ , the second factor is nonzero. If

$$(1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_n + c_i \cdots c_n c_{n+1} + \dots + c_i \cdots c_{n+i-2})$$

is nonzero, then for all  $j$  we have that  $p_j(1) \neq 0$  and therefore there is no cancellation and 1 is a pole of order 2 of  $\text{Tr}_{e_j A}(g_j, t)$  for all  $j = 1, \dots, n$ .

On the other hand, assume that

$$(1 + c_s + c_s c_{s+1} + \dots + c_s \cdots c_n + c_s \cdots c_n c_{n+1} + \dots + c_s \cdots c_{n+s-2}) = 0$$

for some fixed  $s$ . We are going to show that this implies  $p'_i(1) = 0$  for all  $i = 1, \dots, n$ .

Recall the formula of  $p_i(t)$  from Proposition 7.1.1:

$$\left( \prod_{s=i}^n c_s t \quad \prod_{s=i}^{n+1} c_s t \quad \dots \quad \prod_{s=i}^{n+i-2} c_s t \quad \underbrace{1}_i \quad c_i t \quad \dots \quad \prod_{s=i}^{n-1} c_s t \right) \cdot \begin{pmatrix} \sum_{k=0}^{n-1} \left( \prod_{j=n-k+1}^n t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+2}^{n+1} t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+3}^{n+2} t_j \right) t^k \\ \sum_{k=0}^{n-1} \left( \prod_{j=n-k+4}^{n+3} t_j \right) t^k \\ \vdots \\ \sum_{k=0}^{n-1} \left( \prod_{j=2n-k}^{2n-1} t_j \right) t^k \end{pmatrix}.$$

We fix an integer  $1 \leq \ell \leq n$  and work out where  $c_i \cdots c_\ell$  occurs in  $p_i(t)$ . In order to simplify notation, denote  $h = \text{hdet}(g) = c_1 t_1$ . Also notice that due to the formula, the products of  $c_r$ 's use increasing indices while the  $t_q$ 's are written as a product with decreasing indices. It turns out that  $c_i \cdots c_\ell$  appears in the  $j$ -th summand of the dot product precisely once. Let's see how by splitting into several cases.

- $\ell < j < i$ :

$$(c_i \cdots c_n c_1 \cdots c_{j-1} t^{n+j-i}) (t_{j-1} \cdots t_{\ell+1} t^{j-\ell-1}) = c_i \cdots c_n c_1 \cdots c_\ell \cdot h^{j-\ell-1} \cdot t^{n+2j-i-\ell-1}.$$

- $j \leq \min\{\ell, i\}$ :

$$\begin{aligned} & (c_i \cdots c_n c_1 \cdots c_{j-1} t^{n+j-i}) (t_{j-1} \cdots t_1 t_n \cdots t_{\ell+1} t^{n-\ell+j-1}) \\ &= c_i \cdots c_n c_1 \cdots c_\ell \cdot h^{n+j-\ell-1} \cdot t^{2n+2j-\ell-i-1}. \end{aligned}$$

- $j = i \leq \ell$ :

$$t_{i-1} \cdots t_1 t_n \cdots t_{\ell+1} t^{n-\ell+i-1} = c_i \cdots c_\ell \cdot h^{n+i-\ell-1} \cdot t^{n+i-\ell-1}.$$

- $\ell < i = j$ :

$$t_{i-1} \cdots t_{\ell+1} t^{i-\ell-1} = c_i \cdots c_n c_1 \cdots c_\ell \cdot h^{i-\ell-1} \cdot t^{i-\ell-1}.$$

- $i < j \leq \ell$ :

$$(c_i \cdots c_{j-1} t^{j-i}) (t_{j-1} \cdots t_1 t_n \cdots t_{\ell+1} t^{n+j-\ell-1}) = c_i \cdots c_\ell \cdot h^{n+j-\ell-1} \cdot t^{n+2j-\ell-i-1}.$$

- $\max\{\ell, i\} < j$ :

$$(c_i \cdots c_{j-1} t^{j-i}) \cdot (t_{j-1} \cdots t_{\ell+1} t^{j-\ell-1}) = c_i \cdots c_\ell \cdot h^{j-\ell-1} \cdot t^{2j-\ell-i-1}.$$

To finish the proof, we need to sum up all the factors of  $c_i \cdots c_\ell$ , take the derivative with respect to  $t$  at  $t = 1$  and replace  $h = 1$ . For  $i \leq \ell$ , this leads to the following coefficient of  $c_1 \cdots c_i$ :

$$\begin{aligned} & \sum_{j=1}^{i-1} 2n + 2j - \ell - i - 1 + (n + i - \ell - 1) + \sum_{j=i+1}^{\ell} h + 2j - \ell - i - 1 + \sum_{j=\ell+1}^n 2j - \ell - i - 1 \\ &= (2n - \ell - i - 1)(i - 1) + (i - 1)i + (n + i - \ell - 1) + (\ell - i)(n + i - \ell - 1) \\ & \quad + (\ell - i)(\ell - i + 1) + (n - \ell)(\ell - i - 1) + (n - \ell)(n - \ell + 1) \\ &= n(n - 1). \end{aligned}$$

A similar calculation for  $i > \ell$  gives

$$\sum_{j=\ell+1}^{i-1} n + 2j - \ell - i - 1 + \sum_{j=1}^{\ell} 2n + 2j - \ell - i - 1 + \underbrace{i - \ell - 1}_{j=i} + \sum_{j=i+1}^n 2j - \ell - i - 1 = n(n-1).$$

Since the coefficient of  $c_i \cdots c_\ell$  does not depend on  $\ell$ , we get that

$$p'_i(1) = n(n-1)(1 + c_i + c_i c_{i+1} + \dots + c_i \cdots c_{n+i-2}) = 0.$$

Summing up, this says that whenever  $p_i(1) = 0$  then  $p_i(t)$  has 1 as a zero of multiplicity at least 2 and therefore  $\text{Tr}_{e_i A}(g_i, t)$  does not have 1 as a pole. Lemma 7.1.4 says this works for all  $\text{Tr}_{e_j A}(g_j, t)$ .  $\square$

Combining the previous results we obtain:

**Theorem 7.1.7.** *In the following cases, the order of 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  is the same for every  $j = 1, \dots, n$ :*

- (1) *At most one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1.*
- (2) *Both  $c_1 \cdots c_n = t_1 \cdots t_n = 1$  and  $\text{hdet}(g) = 1$ . In particular, the order of 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  is either 0 for all  $j$  or 2 for all  $j$ .*

*Proof.* The first case is done in Corollary 7.1.5, the second case is Proposition 7.1.6.  $\square$

Further, this theorem discloses that the requirement  $\text{hdet}(g) \neq 1$  is necessary to guarantee that 1 is not a pole of order 2 of  $\text{Tr}_{e_i A}(g_i, t)$ . At last, the upcoming example shows that we cannot hope for 1 to be a pole of the same order of  $\text{Tr}_{e_j A}(g_j, t)$  in general.

The remaining case  $c_1 \cdots c_n = t_1 \cdots t_n = 1$  but  $\text{hdet}(g) \neq 1$  causes problems as described in the following example.

**Example 7.1.8.** Let  $n = 5$  and define  $g \in \text{Aut}_{\text{gr}}(A)$  by

$$\begin{aligned} \alpha_1 &\mapsto h^4 \alpha_1, & \alpha_1^* &\mapsto h^2 \alpha_1^*, \\ \alpha_2 &\mapsto \alpha_2, & \alpha_2^* &\mapsto h \alpha_2^*, \\ \alpha_3 &\mapsto h^2 \alpha_3, & \alpha_3^* &\mapsto h^4 \alpha_3^*, \\ \alpha_4 &\mapsto \alpha_4, & \alpha_4^* &\mapsto h \alpha_4^*, \\ \alpha_5 &\mapsto h^4 \alpha_5, & \alpha_5^* &\mapsto h^2 \alpha_5^*, \end{aligned}$$

where  $h = \text{hdet}(g) = \exp\left(\frac{2\pi i}{5}\right)$ . Let  $\text{Tr}(g, t) = \frac{p(t)}{(1-t^5)^2}$  and  $\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{(1-t^5)^2}$ .

From Lemma 7.1.4 we know that  $p_i(1) = p(1) = 0$ .

Calculations give

$$\begin{aligned} p_1(t) &= 1 + (c_1 + t_5)t + (c_1 c_2 + t_4 t_5 + h)t^2 + (c_1 c_2 c_3 + t_3 t_4 t_5 + h(c_1 + t_5))t^3 \\ &\quad + (c_1 c_2 c_3 c_4 + t_2 t_3 t_4 t_5 + h(c_1 c_2 + t_4 t_5) + h^2)t^4 \\ &\quad + (h(c_1 c_2 c_3 + t_3 t_4 t_5) + h^2(c_1 + t_5))t^5 + (h^2(c_1 c_2 + t_4 t_5) + h^3)t^6 \\ &\quad + (h^3(c_1 + t_5))t^7 + h^4 t^8 \\ &= 1 + (h^2 + h^4)t + (h + h^3 + h^4)t^2 + (h + h^3 + 1 + h^2)t^3 + \cancel{(h + h^3 + 1 + h^4 + h^2)}t^4 \\ &\quad + (h^2 + h^3 + h + h^4)t^5 + (h + 1 + h^3)t^6 + (h^2 + 1)t^7 + h^4 t^8, \end{aligned}$$

$$\begin{aligned}
p'_1(1) &= (h^2 + h^4) + 2(h + h^3 + h^4) + 3(1 + h + h^2 + h^3) + 5(h + h^2 + h^3 + h^4) \\
&\quad + 6(1 + h + h^3) + 7(1 + h^2) + 8h^4 \\
&= 16 + 16h + 16h^2 + 16h^3 + 16h^4 = 0,
\end{aligned}$$

$$\begin{aligned}
p_2(t) &= 1 + (c_2 + t_1)t + (c_2c_3 + t_5t_1 + h)t^2 + (c_2c_3c_4 + t_4t_5t_1 + h(c_2 + t_1))t^3 \\
&\quad + (c_2c_3c_4c_5 + t_3t_4t_5t_1 + h(c_2c_3 + t_5t_1) + h^2)t^4 \\
&\quad + (h(c_2c_3c_4 + t_4t_5t_1) + h^2(c_2 + t_1))t^5 + (h^2(c_2c_3 + t_5t_1) + h^3)t^6 \\
&\quad + (h^3(c_2 + t_1))t^7 + h^4t^8 \\
&= 1 + (1 + h^2)t + (h^2 + h^4 + h)t^2 + (h^2 + 1 + h + h^3)t^3 + \cancel{(h + h^4 + h^3 + 1 + h^2)t^4} \\
&\quad + (h^3 + h + h^2 + h^4)t^5 + (h^4 + h + h^3)t^6 + (h^3 + 1)t^7 + h^4t^8,
\end{aligned}$$

and

$$\begin{aligned}
p'_2(1) &= (1 + h^2) + 2(h + h^2 + h^4) + 3(1 + h + h^2 + h^3) + 5(h + h^2 + h^3 + h^4) \\
&\quad + 6(h + h^3 + h^4) + 7(1 + h^3) + 8h^4 \\
&= 11 + 16h + 11h^2 + 21h^3 + 21h^4 \neq 0.
\end{aligned}$$

A similar calculation for  $p(t)$  gives

$$\begin{aligned}
p(t) = & 5 + \left( \sum_{i=1}^5 c_i + \sum_{i=1}^5 t_i \right) t + \left( \sum_{i=1}^5 c_i c_{i+1} + \sum_{i=1}^5 t_i t_{i-1} + 5h \right) t^2 \\
& + \left( \sum_{i=1}^5 c_i c_{i+1} c_{i+2} + \sum_{i=1}^5 t_i t_{i-1} t_{i-2} + h \left( \sum_{i=1}^5 c_i + t_i \right) \right) t^3 \\
& + \left( \sum_{i=1}^5 c_i c_{i+1} c_{i+2} c_{i+3} + \sum_{i=1}^5 t_i t_{i-1} t_{i-2} t_{i-3} + h \left( \sum_{i=1}^5 c_i c_{i+1} + t_i t_{i-1} \right) + 5h^2 \right) t^4 \\
& + \left( h \left( \sum_{i=1}^5 c_i c_{i+1} c_{i+2} + t_i t_{i-1} t_{i-2} \right) + h^2 \left( \sum_{i=1}^5 c_i + t_i \right) \right) t^5 \\
& + \left( h^2 \left( \sum_{i=1}^5 c_i c_{i+1} + t_i t_{i-1} \right) + 5h^3 \right) t^6 + h^3 \left( \sum_{i=1}^5 c_i + t_i \right) t^7 + 5h^4 t^8.
\end{aligned}$$

Replacing all scalars by powers of  $h$  yields

$$\begin{aligned}
p(t) = & 5 + (2 + 2h + 3h^2 + 3h^4)t + (2 + 5h + 2h^2 + 3h^3 + 3h^4)t^2 \\
& + (5 + 5h + 5h^2 + 5h^3)t^3 + \cancel{(5 + 5h + 5h^2 + 5h^3 + 5h^4)t^4} \\
& + (5h + 5h^2 + 5h^3 + 5h^4)t^5 + (3 + 3h + 2h^2 + 5h^3 + 2h^4)t^6 \\
& + (3 + 3h^2 + 3h^3 + 2h^4)t^7 + 5h^4 t^8.
\end{aligned}$$

At last, we take the derivative and plug in  $t = 1$  which discloses

$$p'(1) = 60 + 70h + 80h^2 + 90h^3 + 100h^4 \neq 0.$$

In conclusion,  $\text{Tr}_{e_1 A}(g_1, t)$  does not have 1 as a pole while  $\text{Tr}_{e_2 A}(g_2, t)$  and  $\text{Tr}(g, t)$  both have 1 as a pole of order 1.

## 7.2 A necessary condition for a quasi-reflection

In this section, let  $n \geq 3$  and fix  $Q = \widetilde{A_{n-1}}$ . Unless said otherwise,  $A = \Pi_Q(k)$  and  $g$  is a graded automorphism of  $A$  which fixes the primitive idempotents  $e_i$  for  $i = 1, \dots, n$ . Therefore, we can denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  for  $i = 1, \dots, n$  as done before with indices taken modulo  $n$  if necessary. Consequently, the induced maps  $g_i : e_i A \rightarrow e_i A$  with  $g_i(\beta) = g(\beta)$  arise. In the same way,  $G$  is a nontrivial, finite group of graded automorphisms of  $A$  each of which fixes the primitive idempotents.

We are ready to give a definition of a quasi-reflection for  $g$  acting on our preprojective algebra  $A$ . In order to get enough information it is necessary to not only include  $\text{Tr}(g, t)$  but also to take  $\text{Tr}_{e_i A}(g_i, t)$  into account for all  $i$ . Two indications why this is a good approach are given in this section. First, a quasi-reflection must have a nontrivial homological determinant. This implies that  $\text{Tr}_{e_i A}(g_i, t)$  and  $\text{Tr}(g, t)$  cannot have 1 as a pole of order greater than or equal to 2 due to Proposition 7.1.1. Second, Theorem 7.2.3 proves that whenever  $A^G$  satisfies the equivalent conditions from Theorem 5.3.7 then  $G$  must contain a quasi-reflection. This gives hope to find a theorem similar to:

$$\text{gl. dim}(A^G) = 2 \text{ implies } G \text{ is generated by quasi-reflections.}$$

Sadly, the reverse direction is clearly wrong as shown in Example 7.2.4. There, a quasi-reflection  $g$  of order 6 is given but  $A^{\langle g \rangle}$  has infinite global dimension.

**Definition 7.2.1.** We call  $g$  a *quasi-reflection* if  $g$  is not the identity and  $\text{Tr}_{e_j A}(g_j, t)$  has 1 as a pole of order 1 for at least one of  $j = 1, \dots, n$ .

**Proposition 7.2.2.** *If  $\text{hdet}(g) = 1$  then  $g$  is not a quasi-reflection.*

*Proof.* If  $c_1 \cdots c_n \neq 1$  and  $t_1 \cdots t_n \neq 1$  then the denominator of  $\text{Tr}_{e_j A}(g_j, t)$  does not have 1 as a pole for all  $j = 1, \dots, n$  and therefore  $g$  is not a quasi-reflection.

If precisely one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1, then  $(c_1 t_1)^n \neq 1$  and therefore  $\text{hdet}(g) = c_1 t_1 \neq 1$ . Hence, this case does not happen.

If  $c_1 \cdots c_n = t_1 \cdots t_n = 1$ , Proposition 7.1.6 applies saying that 1 is a pole of each  $\text{Tr}_{e_j A}(g_j, t)$  of order 0 or 2 and therefore  $g$  is not a quasi-reflection.  $\square$

The previous proposition says that every quasi-reflection  $g$  satisfies  $\text{hdet}(g) \neq 1$ . With Proposition 7.1.1 this implies that  $\text{Tr}_{e_i A}(g_i, t)$  cannot have 1 as a pole of order greater than or equal to 2 for any  $i$ . To sum up, using Theorem 7.1.7, a quasi-reflection  $g$  with  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  can happen if either:

- exactly one of  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  equals 1 in which case all traces  $\text{Tr}_{e_i A}(g_i, t)$  have 1 as a pole of order 1 or
- both  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  and  $\text{hdet}(g) = c_1 t_1 \neq 1$ . In this case, all traces  $\text{Tr}_{e_i A}(g_i, t)$  have 1 as a pole of order at most 1 but for some 1 may not be a pole.

In particular, it cannot happen that  $g$  is a quasi-reflection and at the same time 1 occurs as a pole of order 2 for  $\text{Tr}_{e_j A}(g_j, t)$  for some  $j$ . This solves one of the problems to define a quasi-reflection. The next theorem shows why this definition of a quasi-reflection might be a good one.

**Theorem 7.2.3.** *Assume that every nontrivial element  $g \in G$  satisfies the following property: if  $c_1 \cdots c_n = 1 = t_1 \cdots t_n$  then  $\text{hdet}(g) = c_1 t_1 \neq 1$ . If  $\text{gl.dim}(A^G) = 2$ , then  $G$  contains a quasi-reflection.*

*Proof.* We assume that  $G$  does not have a quasi-reflection. Thanks to Proposition 2.3.9(f) and Lemma 2.3.14(b), the fixed ring  $A^G$  is noetherian. Additionally, Proposition 2.3.17 provides that  $A$  is finitely generated over  $A^G$ . Therefore  $\text{GKdim}(A^G) = \text{GKdim}(A) = 2$  by [KL00, Lemma 3.1] applied to  $A^G \subseteq A$  and  $(A^G)^m \rightarrow A$ .

By Proposition 5.1.5,  $A$  is projective as an  $A^G$ -module and therefore it can be decomposed as a direct sum of simple projectives  $e_i A^G$  ([ASS06, Corollary 5.17]). This implies that there exist polynomials  $F_j$  with nonnegative coefficients satisfying

$$\frac{n}{(1-t)^2} = H_A(t) = \begin{pmatrix} F_1 & F_2 & \cdots & F_n \end{pmatrix} \cdot \begin{pmatrix} H_{e_1 A^G}(t) \\ H_{e_2 A^G}(t) \\ \vdots \\ H_{e_n A^G}(t) \end{pmatrix}. \quad (7.2)$$

With Molien's Theorem 2.3.16, this leads to

$$\frac{n}{(1-t)^2} = \sum_{j=1}^n F_j \cdot H_{e_j A^G}(t) = \sum_{g \in G} \sum_{j=1}^n \frac{1}{|G|} F_j \cdot \text{Tr}_{e_j A}(g_j, t).$$

Our assumptions allow us to apply Proposition 7.1.1 and therefore  $\text{Tr}_{e_j A}(g_j, t)$  does not have 1 as a pole of order 2 and further not having a quasi-reflection prohibits a nonidentity element from having 1 as a pole of  $\text{Tr}_{e_j A}(g_j, t)$  altogether.

If we expand the right hand side into a Laurent series centered at  $t = 1$  and use that the first two terms come exclusively from  $\text{Tr}_{e_j A}(\text{id}, t)$  we obtain

$$\frac{n}{(1-t)^2} = \frac{1}{|G|} \left( \frac{\sum_{j=1}^n F_j(1)}{(1-t)^2} + \frac{-\sum_{j=1}^n F'_j(1)}{(1-t)} + \text{higher degree terms} \right).$$

Comparing the  $(1-t)^{-1}$  term gives  $\sum_{j=1}^n F'_j(1) = 0$ . Since each  $F_j \in \mathbb{Z}_+[t]$ , also  $F'_j(1) \geq 0$  and so each individual  $F'_j(1) = 0$ . Further, this implies that  $F'_j \equiv 0$  for all  $j$ , i.e. each  $F_j$  is constant. Going back to comparing coefficients, we obtain

$$\left( \sum_{j=1}^n F_j \right) (1) = \sum_{j=1}^n F_j(1) = n \cdot |G|$$

and being constant says that also  $\sum_{j=1}^n F_j(0) = n \cdot |G|$ . But this is impossible as Equation (7.2) for  $t = 0$  becomes  $n = n|G| > n$ , a contradiction.  $\square$

We give an example which makes it obvious that defining a quasi-reflection by requiring 1 to be a pole of order  $\text{GKdim}(A) - 1$  of  $\text{Tr}(g, t)$  does not guarantee that the fixed ring has finite global dimension.

**Example 7.2.4.** Consider the following example where the fixed ring  $A^G$  has infinite global dimension despite the trace of  $g$  at  $e_3$  having 1 as a pole of order 1. We choose  $g$  with  $c_1 = 1$ ,  $c_2 = \zeta_3$ ,  $c_3 = \zeta_3^2$  and  $t_1 = -1$ ,  $t_2 = -\zeta_3^2$ ,  $t_3 = -\zeta_3$  where  $\zeta_3 = \exp(2\pi i/3)$ .

Computations provide

$$\mathrm{Tr}_{e_3A}(g_3, t) = \frac{1 - (1 + 2 \exp(2\pi i/6))t^2 + t^4}{(1 - t^3)(1 + t^3)},$$

$$\mathrm{Tr}(g, t) = \frac{3(1 - t^2 + t^4)}{(1 - t^3)(1 + t^3)},$$

$$H_{A^G}(t) = \frac{3 - 2t + t^2 + 2t^3 + t^4}{(1 - t)(1 - t^6)},$$

and

$$\underline{H}_{M_{A^G}}(t) = \frac{1}{(1 - t)(1 - t^6)} \begin{pmatrix} 1 - t + t^3 & t - t^3 + t^4 & 0 \\ t^2 & 1 - t + t^3 & 0 \\ 0 & 0 & 1 - t + t^3 \end{pmatrix}.$$

Since the numerator of  $\mathrm{Tr}_{e_3A}(g_3, 1)$  equals  $1 - (1 + 2 \exp(2\pi i/6)) + 1 = -i\sqrt{3}$ , the automorphism  $g$  is a quasi-reflection according to Definition 7.2.1.

We find a presentation of  $e_3A^Ge_3$  to be:

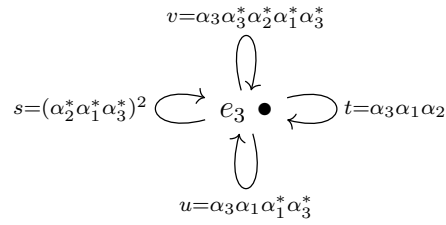


**Figure 7.2.1:** Path algebra underlying the fixed ring of  $e_3A^Ge_3$  in Example 7.2.4.

with ordering  $s < t < u < v$  and relations  $v^2 = su$ ,  $u^2 = vt$ ,  $uv = st$  in addition to the commutative property. It follows that  $e_3A^Ge_3$  is isomorphic to  $k[s, t, u, v]/I$  where

$I$  is the ideal generated by the relations ( $v^2 = su$ ,  $u^2 = vt$ ,  $uv = st$ ). The same reasoning as in Proposition 4.2.4 proves that  $A^G$  has infinite global dimension.

### 7.3 Describing the fixed ring of case $\widetilde{A_{n-1}}$

We use the well-tried notation as introduced in Chapter 4.1 for  $Q = \widetilde{A_{n-1}}$ . Let  $G$  be a finite group acting on  $A = \Pi_Q(k)$  by extending scalar multiplication on the arrows of degree 1 and let  $A^G$  be the fixed ring. As always, denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ .

The main result of this section says that if  $A^G$  has at least  $3n + 1$  generators then  $A^G$  cannot be twisted Calabi-Yau. Equivalently, combined with Corollary 5.3.3, we obtain: if  $A^G$  is twisted Calabi-Yau then  $A^G$  is generated by precisely  $3n$  elements. In order to prove this result we first need to give a detailed description of the relations of  $A^G$  and explain how the generators at different vertices correlate. Unfortunately, the key arguments of this section are limited to  $Q = \widetilde{A_{n-1}}$  since the minimal relations of  $\Pi_Q(k)$  for  $Q = \widetilde{D_{n-1}}$  and  $\widetilde{E_m}$  for  $m = 6, 7, 8$  are not binomial.

*Definition & Remark 7.3.1.* Define  $Q^G$  to be the quiver with weighted arrows which has the same vertices as  $Q$  and whose arrows are the paths in  $Q$  minimally generating  $A^G$ .

Since  $G$  acts by scalar multiplication, a minimal linear combination of paths in  $A$  is fixed if and only if each summand is fixed. Therefore, every generator of  $A^G$  is a path and it makes sense to define  $Q^G$ .

Recall from Proposition 5.3.4 that for  $Q = \widetilde{A_{n-1}}$  all relations of  $A^G$  can be written

as a sum of binomial relations in  $A^G$ . While in this case all relations are of this nice form, there is no similar result for  $Q = \widetilde{D_{n-1}}$  or  $Q = \widetilde{E_m}$  for  $m = 6, 7, 8$  possible. The crucial argument is that all minimal relations of  $A$  are binomial which we only have if  $Q = \widetilde{A_{n-1}}$ . Even more, the following two corollaries give an explicit description of all minimal relations of  $A^G$ .

**Corollary 7.3.2.** *Let  $\beta_1, \beta_2 \in e_i A^G \subseteq e_i A$  be nontrivial paths. Suppose  $\nu_1 \beta_1 = \nu_2 \beta_2$  is a relation in  $A^G$  for  $\nu_1, \nu_2 \in k$ . Then  $\nu_1 = \nu_2$ .*

*Proof.* As in the proof of Proposition 5.3.4,  $\beta_1$  and  $\beta_2$  have the same type. Thus  $\beta_1 = \beta_2$  and since  $\beta_1 \neq 0$  this proves the claim.  $\square$

**Corollary 7.3.3.** *For paths  $\beta_1, \beta_2 \in e_i A^G \subseteq e_i A$  we know that  $\beta_1 = \beta_2$  if and only if the type of  $\beta_1$  equals the type of  $\beta_2$ .*

*Proof.* Since  $\beta_1$  and  $\beta_2$  start at the same vertex  $e_i$ , have the same type and thus the same degree, the relations of  $A$  force  $\beta_1 = \beta_2$ . Conversely, if  $\beta_1 = \beta_2$ , then as in the proof of Proposition 5.3.4 we know that  $\beta_1$  and  $\beta_2$  must have the same type.  $\square$

Now that the minimal relations of  $A^G$  are understood, we need to take a look at the generators. First, we analyze how minimal fixed pure closed paths behave at different vertices. Then we compare them to minimal fixed pure generators of the same type. This section finishes with Proposition 7.3.5 which limits the generators of  $A^G$  to  $3n$  whenever  $A^G$  is twisted Calabi-Yau.

**Lemma 7.3.4.** *If  $x$  and  $y$  are minimal fixed purely nonstar (star, respectively) closed paths, then  $r = \deg(x) = \deg(y) = s$  in  $A$ . If  $z$  is a minimal fixed purely nonstar path (star, respectively), which is not a closed path, then  $\deg(z) < \deg(x)$ .*

*Proof.* Let  $x$  be a closed path at  $e_i$  and  $y$  be a closed path at  $e_j$ . By symmetry, it is enough to consider the case where  $x$ ,  $y$  and  $z$  are all purely nonstar. Therefore, we have  $x = (\alpha_i \alpha_{i+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{i-1})^m$  and it immediately follows that  $(\alpha_j \alpha_{j+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{j-1})^m$  is a closed path at  $e_j$  which must be fixed. This gives  $s \leq r$ . Symmetry again gives  $r \leq s$ . Assume  $z$  starts at  $e_k$ . Then  $\beta = (\alpha_k \alpha_{k+1} \cdots \alpha_n \alpha_1 \cdots \alpha_{k-1})^m$  is a fixed purely nonstar closed path starting at  $e_k$ . It is easy to see that  $\deg(z) \leq \deg(\beta)$  since  $z$  is the shortest purely nonstar path starting at  $e_k$ . However, since  $z \neq \beta$  and due to  $\deg(\beta) = \deg(x)$ , we conclude that  $\deg(z) < \deg(x)$ .  $\square$

For a  $k$ -algebra  $R$ , we call  $\mathcal{B}$  a *generating set* of  $R$  if every  $r \in R$  can be written as a  $k$ -linear combination of words from  $\mathcal{B}$ . The generating set  $\mathcal{B}$  is called *minimal* if any proper subset of  $\mathcal{B}$  is not a generating set of  $R$ . Notice that a generating set of a non-connected graded algebra  $B$  where  $B_0 \cong k^n$  with  $n \geq 2$  must contain elements to obtain the primitive pairwise orthogonal idempotents. The empty word equals 1.

Now we are ready to prove the final proposition of this chapter.

**Proposition 7.3.5.** *Assume that  $A^G$  has at least  $3n+1$  elements in a minimal generating set as an algebra. Then  $A^G$  cannot be twisted Calabi-Yau.*

*Proof.* Remember that being twisted Calabi-Yau implies having finite global dimension

and satisfying the generalized Gorenstein condition. Due to Corollary 5.3.3, we have at least  $3n$  generators of  $A^G$ . For each vertex  $e_i$ , denote the shortest fixed purely nonstar and the shortest fixed purely star path of  $A$  starting at  $e_i$  by  $x_i$  and  $y_i$ , respectively. Together with the idempotents  $e_i$ , this gives  $3n$  generators of  $A^G$  and thus there exists at least one additional generator, say  $z$ , which is a mixed path. Without loss of generality, we can say  $z$  starts at  $e_1$ .

We suppose for contradiction that  $A^G$  has finite global dimension and is generalized Gorenstein. By Proposition 5.1.2 we have that  $\text{gl.dim}(A^G) = 2$ . Therefore, there exists a unique minimal relation  $R_i : \beta_i = \delta_i$  starting at each vertex  $e_i$ . More precisely, both  $\beta_i$  and  $\delta_i$  are paths in  $Q^G$  each involving at least two minimal generators of  $A^G$ .

Suppose that  $x_1$  or  $y_1$  is a closed path. By symmetry, it is enough to consider just the case where  $x_1$  is a closed path. Due to  $G$  acting only diagonally on the arrows of degree 1, if  $x_1$  is fixed, the purely nonstar closed path starting at  $e_j$  of  $\deg_A(x_1)$  is fixed as well, denoted by  $\mu_j$  for  $j = 2, \dots, n$ . Due to Lemma 7.3.4 we notice that  $\mu_j$  is the shortest fixed purely nonstar closed path at  $e_j$ . Whenever the minimal fixed purely nonstar path  $x_j$  is a closed path, we obtain  $x_j = \mu_j$ . Otherwise,  $\mu_j$  decomposes as  $\mu_j = x_j \mu'_j$ . This gives rise to the following two relations:

$$P_1 : x_1 y_1 = y_1 \mu_{t(y_1)} \quad \text{and} \quad P_2 : x_1 z = z \mu_{t(z)},$$

where  $t(\cdot)$  denotes the target function which assigns every path to its ending vertex.

Consider the ideal of all relations

$$I = \{u_1 R_1 v_1 + \dots + u_n R_n v_n \mid u_i, v_i \in A^G \text{ for } i = 1, \dots, n\}.$$

Then  $x_1 y_1 - y_1 \mu_{t(y_1)} \in I$  and so  $x_1 y_1$  must occur as term in one of the  $u_\ell R_\ell v_\ell$ . As each summand of  $R_\ell$  is a product of at least two generators of  $A^G$ , we conclude that  $u_\ell = v_\ell = 1$  and  $\ell = 1$ . Moreover, Proposition 5.3.4 says that  $R_1$  is binomial and so we have

$$R_1 : x_1 y_1 = \delta_1.$$

Repeating the same argument for  $x_1 z - z \mu_{t(z)}$  results in  $R_1 : x_1 z = \delta'_1$ . Since  $x_1 y_1 \neq x_1 z$ , this is a contradiction.

Next, suppose that neither  $x_1$  nor  $y_1$  is a closed path while  $z$  is a closed path at  $e_1$ . Analogously to  $\mu_j$  being the shortest fixed purely nonstar closed path at  $e_j$ , define  $\nu_j$  to be the shortest fixed purely star closed path at  $e_j$ . The cycles  $\mu_j$  and  $\nu_j$  exist due to  $c_1 \cdots c_n$  and  $t_1 \cdots t_n$  having finite order for each  $g \in G$ . Then

$$P_1 : z \mu_1 = \mu_1 z \quad \text{and} \quad P_2 : z \nu_1 = \nu_1 z$$

must hold. Similarly to the first case, the word  $z \mu_1$  must appear in  $u_\ell R_\ell v_\ell$  for some  $\ell$ . If  $u_\ell \neq 1$ , then we can decompose  $u_\ell = z u'_\ell$  and  $R_\ell$  must contain a term which is purely nonstar. However, this cannot happen by the remark after Proposition 5.3.4. Hence,  $u_\ell = 1$  and  $R_\ell$  is a relation starting at  $e_1$  which implies that  $\ell = 1$ . In particular, we obtain the relation

$$R_1 : z \tilde{\mu}_1 = \delta_1,$$

where  $\tilde{\mu}_1$  is a purely nonstar path. Analogously for  $P_2$ , also  $R_1 : z\tilde{\nu}_1 = \delta'_1$  for some purely star path  $\tilde{\nu}_1$  must be satisfied. The usual argument about types yields  $z\tilde{\mu}_1 \neq z\tilde{\nu}_1$  which gives a contradiction.

In the last case, we assume that all three of  $x_1, y_1$  and  $z$  are not closed paths. As before, denote the shortest fixed purely nonstar cycle at  $e_j$  by  $\mu_j$ . Then

$$P_1 : y_1\mu_{t(y_1)} = \mu_1y_1 \quad \text{and} \quad P_2 : z\mu_{t(z)} = \mu_1z$$

arise by Corollary 7.3.3. The argument used in the second case works again for  $y_1\mu_{t(y_1)}$  and  $z\mu_{t(z)}$  which finishes the proof.  $\square$

# Chapter 8

## Case $\widetilde{A}_2$

### 8.1 An alternative definition of a quasi-reflection for

$$Q = \widetilde{A}_2$$

While Example 7.1.8 shows that the order of 1 as a pole of  $\mathrm{Tr}_{e_i A}(g_i, t)$  can be different for distinct  $i$  in general, we investigate the case  $n = 3$  separately. It turns out that here not only 1 is a pole of the same order for all  $\mathrm{Tr}_{e_i A}(g_i, t)$  and  $\mathrm{Tr}_A(g, t)$  but also all occurrences of 1 as a pole of order 1 can be described using Corollary 8.1.1. Moreover, this allows to define a quasi-reflection by requiring 1 to be a pole of order 1 of  $\mathrm{Tr}(g, t)$ .

In this section, let  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$ . As before, let  $g$  be a graded automorphism of  $A$  which fixes the primitive idempotents  $e_i$  for  $i = 1, 2, 3$ . Therefore, we can again denote  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$  for  $i = 1, 2, 3$  with indices taken modulo  $n$  if necessary. Consequently, the induced maps  $g_i : e_i A \rightarrow e_i A$  with  $g_i(\beta) = g(\beta)$  arise.

Now, let's recall some notation and formulas. Proposition 4.1.4 together with some small calculation shows that  $\text{Tr}(g, t) = \frac{p(t)}{q(t)}$  and  $\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{q_i(t)}$  where

$$\begin{aligned} q(t) &= q_i(t) = (1 - c_1 c_2 c_3 t^3)(1 - t_1 t_2 t_3 t^3), \\ p(t) &= 3 + \left( \sum_{i=1}^3 (c_i + t_i) \right) t + \left( 3c_1 t_1 + \sum_{i=1}^3 (c_i c_{i+1} + t_i t_{i+1}) \right) t^2 + c_1 t_1 \left( \sum_{i=1}^3 (c_i + t_i) \right) t^3 \\ &\quad + 3(c_1 t_1)^2 t^4, \end{aligned}$$

and

$$p_i(t) = 1 + (c_i + t_{i-1})t + (c_1 t_1 + c_i c_{i+1} + t_{i-1} t_{i-2})t^2 + c_1 t_1 (c_i + t_{i-1})t^3 + (c_1 t_1)^2 t^4$$

for all  $i = 1, 2, 3$ . Moreover, in Proposition 4.2.4 and Lemma 7.1.4 a factorization of  $p(1)$  and  $p_i(1)$ , respectively, was presented:

$$\begin{aligned} p(1) &= (3 + c_1 + c_2 + c_3 + c_1 c_2 + c_2 c_3 + c_1 c_3)(1 + (c_1 t_1) + (c_1 t_1)^2), & \text{if } c_1 c_2 c_3 = 1, \\ p(1) &= (3 + t_1 + t_2 + t_3 + t_1 t_2 + t_2 t_3 + t_1 t_3)(1 + (c_1 t_1) + (c_1 t_1)^2), & \text{if } t_1 t_2 t_3 = 1, \\ p_i(1) &= (1 + c_i + c_i c_{i+1}) \cdot (1 + (c_1 t_1) + (c_1 t_1)^2), & \text{if } c_1 c_2 c_3 = 1 \quad \text{and} \\ p_i(1) &= (1 + t_i + t_i t_{i-1}) \cdot (1 + (c_1 t_1) + (c_1 t_1)^2), & \text{if } t_1 t_2 t_3 = 1. \end{aligned} \tag{8.1}$$

This is enough to describe the situation  $c_1 c_2 c_3 = t_1 t_2 t_3 = 1$  entirely:

**Corollary 8.1.1.** *If  $c_1 c_2 c_3 = t_1 t_2 t_3 = 1$ , then the following cases occur:*

- (1)  $\text{hdet}(g) = c_1 t_1 = 1$  and  $c_1 + c_2 + c_3 + c_1 c_2 + c_2 c_3 + c_1 c_3 = -3$ . Then  $p(1) = p'(1) = 0$  and 1 is a pole of order 0 of  $\text{Tr}(g, t)$ .

(2)  $\text{hdet}(g) = c_1 t_1 = 1$  and  $c_1 + c_2 + c_3 + c_1 c_2 + c_2 c_3 + c_1 c_3 \neq -3$ . Then  $p(1) \neq 0$  and 1 is a pole of order 2 of  $\text{Tr}(g, t)$ .

(3)  $\text{hdet}(g) = c_1 t_1 = \zeta_3$  and  $c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = \zeta_3^2$ . Then  $p(1) = p'(1) = 0$  and 1 is a pole of order 0 of  $\text{Tr}(g, t)$ .

(4)  $\text{hdet}(g) = c_1 t_1 = \zeta_3$  and one of  $c_1, c_2, c_3, t_1, t_2, t_3$  does not equal  $\zeta_3^2$ . Then  $p(1) = 0$  and  $p'(1) \neq 0$  and 1 is a pole of order 1 of  $\text{Tr}(g, t)$ .

(5)  $\text{hdet}(g) = c_1 t_1 = \zeta_3^2$  and  $c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = \zeta_3$ . Then  $p(1) = p'(1) = 0$  and 1 is a pole of order 0 of  $\text{Tr}(g, t)$ .

(6)  $\text{hdet}(g) = c_1 t_1 = \zeta_3^2$  and one of  $c_1, c_2, c_3, t_1, t_2, t_3$  does not equal  $\zeta_3$ . Then  $p(1) = 0$  and  $p'(1) \neq 0$  and 1 is a pole of order 1 of  $\text{Tr}(g, t)$ .

*Proof.* The first two cases (1) and (2) follow from Propositions 4.2.4 and 7.1.6. Thanks to  $(c_1 t_1)^3 = (c_1 c_2 c_3)(t_1 t_2 t_3) = 1$ , we can assume that  $\text{hdet}(g) = c_1 t_1 = \zeta_3$  or  $\zeta_3^2$ . As recalled in the beginning of this section we know that

$$\begin{aligned} p(t) = & 3 + (c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t + (3c_1 t_1 + c_1 c_2 + c_1 c_3 + c_2 c_3)t^2 \\ & + (t_1 t_2 + t_1 t_3 + t_2 t_3)t^2 + c_1 t_1 (c_1 + c_2 + c_3 + t_1 + t_2 + t_3)t^3 + 3(c_1 t_1)^2 t^4 \end{aligned}$$

and due to  $c_1 t_1 = \zeta_3$  or  $\zeta_3^2$  the second factor in the factorization in Equation 8.1 guarantees  $p(1) = 0$ . Suppose  $p(t)$  factors as

$$p(t) = (1 - t)^2(a_0 + a_1 t + a_2 t^2) = a_0 + (a_1 - 2a_0)t + (a_2 - 2a_1 + a_0)t^2 + (a_1 - 2a_2)t^3 + a_2 t^4.$$

Comparing the coefficients of 1,  $t$  and  $t^4$  gives

$$a_0 = 3, \quad a_1 = 6 + c_1 + c_2 + c_3 + t_1 + t_2 + t_3, \quad \text{and} \quad a_2 = 3(c_1 t_1)^2.$$

To simplify notation, define  $x = c_1 + c_2 + c_3 + t_1 + t_2 + t_3$  and  $h = c_1 t_1$ . This leads to two equations for the coefficients of  $t^2$  and  $t^3$ :

$$-9 - 2x + 3h^2 = h^2 x + 3h, \quad \text{and}$$

$$6 + x - 6h^2 = hx.$$

Since  $h \neq 1$ , the second equation can be solved for  $x = \frac{6h^2 - 6}{1 - h} = -6(1 + h)$  and it is easy to check that the first equation is satisfied provided  $h = \zeta_3$  or  $\zeta_3^2$  and  $x = -6(1 + h)$ . Now, we see that  $x = -6(1 + h) = 6h^2$  which can be rewritten as  $hx = 6$ . Since  $hx$  is a sum of six roots of unity equaling 6, each summand must equal 1. In other words,  $hc_1 = 1$ , or  $c_1 = h^2$ . This shows cases (3) and (5). If one of  $c_1, c_2, c_3, t_1, t_2, t_3$  does not equal  $h^2$ , then  $p(t)$  cannot be factored as described. Since still  $p(1) = 0$ , it follows that  $\text{Tr}(g, t)$  has 1 as a pole of order 1 as claimed in the cases (4) and (6).  $\square$

This means we fully understand the case  $c_1 c_2 c_3 = t_1 t_2 t_3 = 1$ . Apart from the situations described in Corollary 8.1.1, we can only have 1 as a pole of positive order if precisely one of  $c_1 c_2 c_3$  and  $t_1 t_2 t_3$  equals 1 and at the same time  $p(1)$  or  $p_i(1)$ , respectively, do not vanish. This can easily be checked using the factorization of  $p(1)$  and  $p_i(1)$  from Equation 8.1. Next, we need to work to show that all traces  $\text{Tr}_{e_i A}(g_i, t)$  have 1 as a pole of the same order.

**Corollary 8.1.2.** *The trace  $\text{Tr}_{e_i A}(g_i, t) = \frac{p_i(t)}{(1 - c_1 c_2 c_3 t^3)(1 - t_1 t_2 t_3 t^3)}$  has 1 as a pole of the same order for all  $i = 1, 2, 3$ .*

*Proof.* If at most one of  $c_1 c_2 c_3$  and  $t_1 t_2 t_3$  equals 1, the claim is true by Corollary 7.1.5.

Let  $c_1 c_2 c_3 = 1 = t_1 t_2 t_3$ . If  $h = \text{hdet}(g) = c_1 t_1 = 1$ , then the result can be deduced from Proposition 7.1.6. Otherwise,  $h \in \{\zeta_3, \zeta_3^2\}$  where  $\zeta_3 = \exp\left(\frac{2\pi i}{3}\right)$ . The factorization in Equation 8.1 shows that  $p_i(1) = 0$  for all  $i = 1, 2, 3$ . Suppose  $p'_j(1) = 0$  for some  $j$ . Recall the formula of the numerator of  $\text{Tr}_{e_i A}(g_i, t)$  from the beginning of this section using  $h = c_1 t_1$ :

$$p_i(t) = 1 + (c_i + t_{i-1})t + (c_i c_{i+1} + t_{i-1} t_{i-2} + h)t^2 + h(c_i + t_{i-1})t^3 + h^2 t^4. \quad (8.2)$$

Since we assume  $p'_j(1) = 0$ , as in Corollary 8.1.1,  $p_j(t)$  must factor as

$$\begin{aligned} p_j(t) &= (1 - t)^2(a_0 + a_1 t + a_2 t^2) \\ &= a_0 + (a_1 - 2a_0)t + (a_2 - 2a_1 + a_0)t^2 + (a_1 - 2a_2)t^3 + a_2 t^4. \end{aligned}$$

Comparing the coefficients of  $1, t$  and  $t^4$  yields

$$a_0 = 1, \quad a_1 = 2 + c_j + t_{j-1}, \quad \text{and} \quad a_2 = h^2.$$

To simplify notation, denote  $x = c_j + t_{j-1}$  and find the following two equations as the coefficients of  $t^2$  and  $t^3$ :

$$a_2 - 2a_1 + a_0 = h^2 - 2x - 3 = c_j c_{j+1} + t_{j-1} t_{j-2} + h = h^2 x + h, \quad \text{and}$$

$$a_1 - 2a_2 = 2 + x - 2h^2 = hx.$$

The same calculation as in Corollary 8.1.1 works. Since  $h \neq 1$ , the second equation provides  $x = \frac{2(1-h^2)}{h-1} = -2(1+h) = 2h^2$ . It is easy to check that this choice satisfies the first equation. This implies  $hx = 2$  which is a sum of two roots of unity equaling 2 and so we get  $c_j = t_{j-1} = h^2$ . However, due to  $c_j t_j = h$  it follows that  $t_j = h^2$  and analogously it is true that  $c_{j-1} = h^2$ . Since  $c_j c_{j-1} c_{j-2} = 1$ , also  $c_{j-2} = h^2$ . To sum up, we know that  $c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = h^2$  and thus  $p'_i(1) = 0$  for all  $i = 1, 2, 3$ .  $\square$

While knowing  $p_i(1) = 0$  for all  $i$  immediately implies  $p(1) = 0$ , the reverse direction requires more finesse and the author gratefully acknowledges the help of Daniël Kroes who proved the following lemma.

**Lemma 8.1.3.** *Let  $1 + x + y + 3xy + x^2y + xy^2 + x^2y^2 = 0$  for  $x$  and  $y$  roots of unity. Then  $x = y = \exp(\frac{2\pi i}{3})$  or  $x = y = \exp(\frac{4\pi i}{3})$ .*

*Proof.* A calculation shows

$$1 + x + y + 3xy + x^2y + xy^2 + x^2y^2 = (1 + x + xy)(1 + y + xy) = 0.$$

If  $1 + x + xy = 0$  then  $1 + x = -xy$  which says that  $1 + x$  is a root of unity. Chose  $\theta$  such that  $x = \cos(\theta) + i \sin(\theta)$ . Then the norm of  $1 + x = 1 + \cos(\theta) + i \sin(\theta)$  computes as

$$||1 + x|| = \sqrt{(1 + \cos(\theta))^2 + \sin(\theta)^2} = \sqrt{2 + 2 \cos(\theta)} = \sqrt{4 \cos^2\left(\frac{\theta}{2}\right)} = \pm 2 \cos\left(\frac{\theta}{2}\right) = 1.$$

Hence,  $\cos\left(\frac{\theta}{2}\right) = \pm \frac{1}{2}$ , and we get  $\theta = \frac{2\pi i}{3}$  or  $\theta = \frac{4\pi i}{3}$ . This means that  $x = y$ .  $\square$

Finally, we are able to show that also in the remaining case  $\text{Tr}(g, t)$  and all  $\text{Tr}_{e_i A}(g_i, t)$  for  $i = 1, 2, 3$  have 1 as a pole of the same order.

**Proposition 8.1.4.** *For  $n = 3$ , the order of 1 as a pole is the same for  $\text{Tr}(g, t)$  and  $\text{Tr}_{e_i A}(g_i, t)$  for all  $i = 1, 2, 3$ .*

*Proof.* The denominator before cancellation is the same for  $\text{Tr}(g, t)$  and  $\text{Tr}_{e_i A}(g_i, t)$ . Hence, it is enough to find equality in the multiplicity of 1 as a root of  $p(t)$  and  $p_i(t)$  up to multiplicity 2. From Corollary 8.1.2 we have that all  $\text{Tr}_{e_i A}(g_i, t)$  behave identically. There are two main cases to consider.

First, let precisely one of  $c_1 c_2 c_3$  and  $t_1 t_2 t_3$  be equal to 1. By symmetry it is enough to only consider  $c_1 c_2 c_3 = 1$  and  $t_1 t_2 t_3 \neq 1$ . If  $p_j(1) = 0$  for some  $j$ , then  $p_i(1) = 0$  for  $i = 1, 2, 3$  and therefore  $p(1) = (p_1 + p_2 + p_3)(1) = 0$ . Hence, in all traces cancellation happens and both do not have 1 as a pole. Conversely, suppose  $p(1) = 0$ . We use the factorization of  $p(1)$  from the beginning of the section again:

$$p(1) = (3 + c_1 + c_2 + c_3 + c_1 c_2 + c_2 c_3 + c_1 c_3)(1 + (c_1 t_1) + (c_1 t_1)^2).$$

Since  $(c_1 t_1)^3 = t_1 t_2 t_3 \neq 1$ , the second factor is nonzero. Suppose the first factor is zero.

We multiply it through by  $c_1 c_2$  to obtain

$$1 + c_1 + c_2 + 3c_1 c_2 + c_1^2 c_2 + c_1 c_2^2 + c_1^2 c_2^2 = 0.$$

Lemma 8.1.3 provides  $c_1 = c_2 = \exp\left(\frac{2\pi i}{3}\right)$  or  $c_1 = c_2 = \exp\left(\frac{4\pi i}{3}\right)$  and Equation 8.1 forces  $p_i(1) = 0$  for all  $i = 1, 2, 3$ .

Next, let  $c_1 c_2 c_3 = 1 = t_1 t_2 t_3$ . First, assume that  $h = \text{hdet}(g) = c_1 t_1 = 1$ . If  $p_j(1) = 0$  for some  $j$ , then Proposition 7.1.6 shows that  $p_i(1) = p'_i(1) = 0$  for all  $i$  and therefore  $p(1) = p'(1) = 0$ . On the other hand with  $p(1) = 0$ , we are in case (1) of

Corollary 8.1.1 which says that  $p(1) = p'(1) = 0$  and  $3 + c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_1c_3 = 0$ .

The same argument as in case (1) of this proposition shows  $c_1 = c_2 = c_3 = \exp\left(\frac{2\pi i}{3}\right)$  or  $c_1 = c_2 = c_3 = \exp\left(\frac{4\pi i}{3}\right)$ . In this situation, plugging numbers into the formula again gives  $p_i(1) = 0$  and Proposition 7.1.6 adds  $p'_i(1) = 0$  for all  $i = 1, 2, 3$ . Combined, both  $\text{Tr}(g, t)$  and  $\text{Tr}_{e_i A}(g_i, t)$  do not have 1 as a pole.

Lastly, assume  $h = \text{hdet}(g) = c_1t_1 \neq 1$ . From Remark 7.1.3 and Lemma 7.1.4 it follows that  $p(1) = p_i(1) = 0$  for all  $i = 1, 2, 3$ . If  $p'_j(1) = 0$  for some  $j$ , Corollary 8.1.2 says  $p'_i(1) = 0$  for all  $i$  and therefore  $p'(1) = (p'_1 + p'_2 + p'_3)(1) = 0$ . Conversely, suppose  $p'(1) = 0$ . This situation is described in Corollary 8.1.1 cases (3) and (5). Focus on case (5),  $c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = \zeta_3$  and plug into the formulas for  $p_j(t)$  given in the beginning of this section to get

$$p_i(t) = 1 + 2\zeta_3 t + 3\zeta_3^2 t^2 + 2t^3 + \zeta_3 t^4,$$

$$p'_i(t) = 2\zeta_3 + 6\zeta_3^2 t + 6t^2 + 4\zeta_3 t^3, \quad \text{and}$$

$$p'_i(1) = 2\zeta_3 + 6\zeta_3^2 + 6 + 4\zeta_3 = 0$$

for all  $i = 1, 2, 3$  which finishes the proof.  $\square$

Finally, all cases are understood in regards of being a quasi-reflection as in Definition 7.2.1. Corollary 8.1.1 solves the case  $c_1c_2c_3 = t_1t_2t_3 = 1$ . If both  $c_1c_2c_3$  and  $t_1t_2t_3$  do not equal 1, no trace can have 1 as a pole altogether which makes a quasi-reflection impossible. The last situation is that precisely one of  $c_1c_2c_3$  and  $t_1t_2t_3$  equals 1 which by Proposition 8.1.4 means again that all traces  $\text{Tr}_{e_i A}(g_i, t)$  and  $\text{Tr}(g, t)$  have 1 as a pole of

the same order. To sum up, for  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$  an equivalent definition to 7.2.1 of a quasi-reflection was found:

**Definition 8.1.5.** Let  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$  and  $g$  defined as in the beginning of this section. Then  $g$  is called a *quasi-reflection* if 1 is a pole of order 1 of  $\text{Tr}(g, t)$ . In this case, this definition is equivalent to Definition 7.2.1.

For larger  $n$ , it remains unclear if this definition is equivalent to Definition 7.2.1. All we know is that it is possible to find an automorphism where the  $\text{Tr}_{e_i A}(g_i, t)$  do not have 1 as a pole of the same order (see Example 7.1.8). Unfortunately, the techniques used here are not good enough to deal with the more complicated formulas of the trace as  $n$  grows.

## 8.2 Twisted Calabi-Yau fixed rings of $A$ for $n = 2$

We continue to look at  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$ . Let  $G$  be a finite group acting on  $A$  by extending scalar multiplication on the arrows of degree 1 and denote the fixed ring by  $A^G$ .

Recall Corollary 5.3.3 which pointed out that  $A^G$  has at least 9 generators including the primitive pairwise orthogonal idempotents  $e_1, e_2$  and  $e_3$ . Further, Proposition 7.3.5 added that if  $A^G$  is twisted Calabi-Yau, then  $A^G$  must be generated by precisely 9 elements. This observation motivated the first proposition of this section. It is possible to characterize all possible quivers  $Q^G$  such that  $A^G$  is generated by 9 elements and can

be described as the path algebra of  $Q^G$  modulo some relations. Moreover, the quivers  $Q^G$  already tell us a lot about the minimal relations that arise in  $A^G$ . They are analyzed in Remark 8.2.2 and summarized in Remark 8.2.3. At last, Proposition 8.2.5 draws conclusions for half the cases when  $G$  is a cyclic group. In this case, we can say that  $G$  must be generated by quasi-reflections.

**Proposition 8.2.1.** *Assume that  $A^G$  can be generated by precisely 9 elements. Then  $A^G$  is isomorphic to a factor algebra of one of the path algebras from Figure 8.2.1 where  $\{i, j\} = \{1, 2\}$ .*

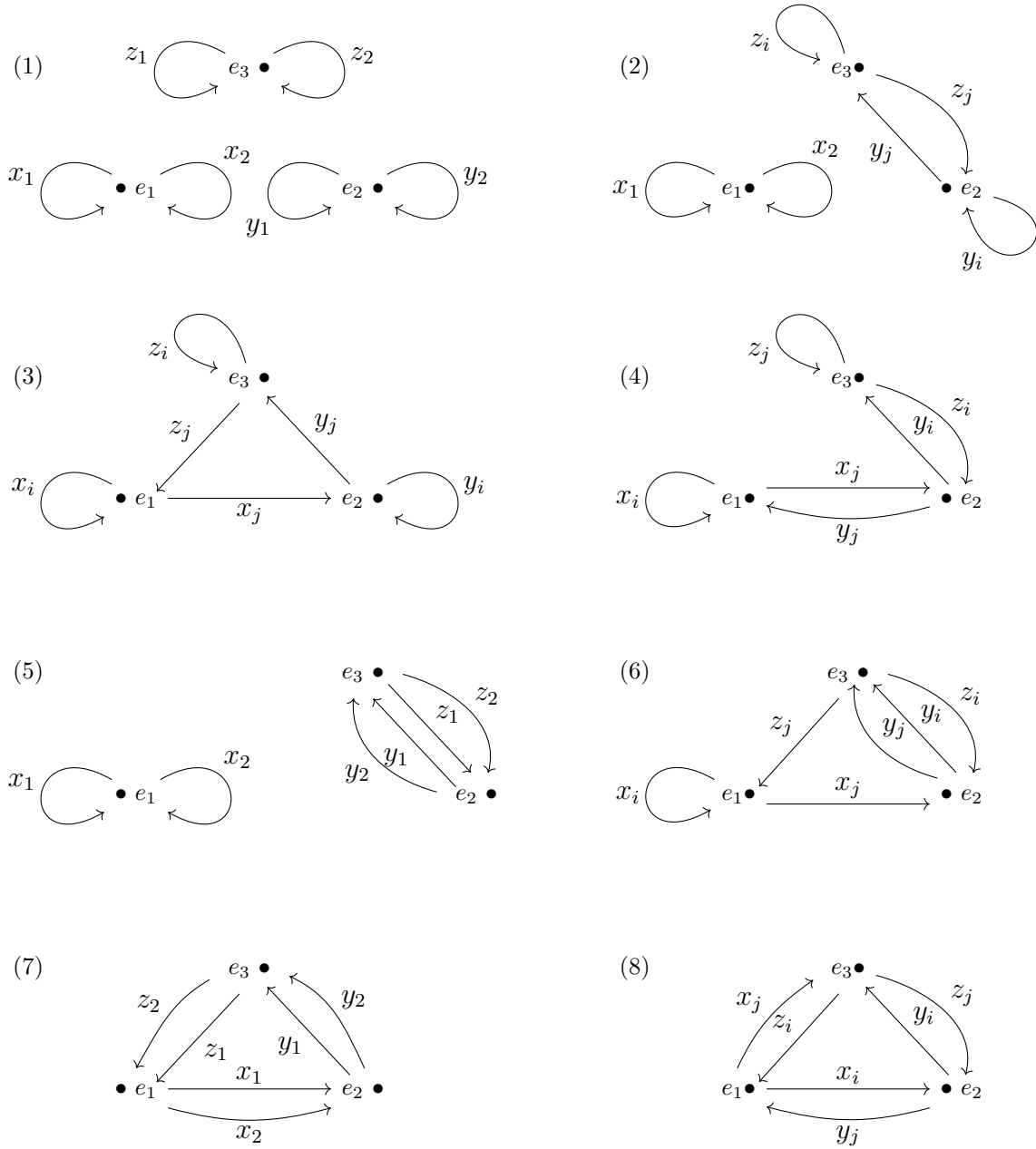


**Figure 8.2.1:** All possible path algebras underlying the fixed ring  $A^G$  for  $n = 3$  generated by precisely 9 elements.

*Proof.* Since the group  $G$  is finite, there exist shortest fixed pure paths starting at every

vertex by Lemma 5.3.2. In particular, call  $x_1, x_2$  the shortest such pure paths consisting of only nonstar arrows and only star arrows, respectively, which start at  $e_1$ . Similarly, call  $y_1, y_2$  and  $z_1, z_2$  the shortest pure paths starting at  $e_2$  and  $e_3$  that are fixed. Since these six elements cannot be decomposed, together with  $e_1, e_2$  and  $e_3$ , they must be the generators of  $A^G$ . Also by Lemma 5.3.2 it follows that  $\{t(x_1), t(y_1), t(z_1)\} = \{e_1, e_2, e_3\}$  and  $\{t(x_2), t(y_2), t(z_2)\} = \{e_1, e_2, e_3\}$  where  $t(\cdot)$  is the target function assigning each path to its ending vertex.

Consider the following two cases.

First, assume that at least one of  $\{x_1, x_2, y_1, y_2, z_1, z_2\}$  is a loop in  $Q^G$ . Without loss of generality, we can say  $x_i$  is a loop for  $i \in \{1, 2\}$ . By symmetry it is enough to only consider the case  $i = 1$ . Then  $y_1$  and  $z_1$  are either both loops or  $y_1$  ends at  $e_3$  and  $z_1$  ends at  $e_2$ . This gives two subgraphs of  $Q^G$ :

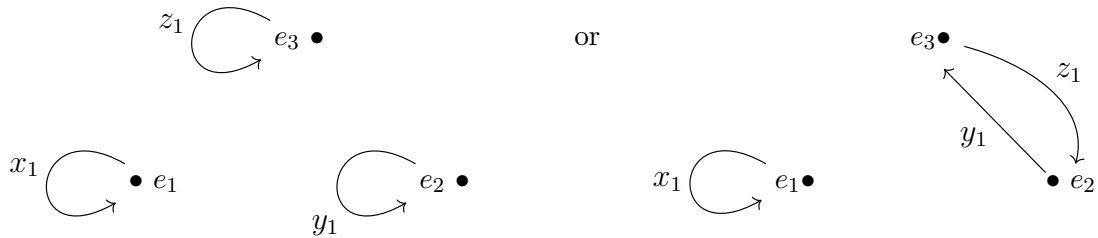


**Figure 8.2.2:** Possible subgraphs of  $Q^G$  including a loop.

Consider the left subgraph first. For  $x_2$  there are - up to symmetry - two options. Either  $x_2$  is a loop or  $x_2$  is not a loop. In the first case, we obtain  $Q^G$  to be the quiver (1)

or (2) depending on whether  $y_2$  is a loop or not. Otherwise, if  $x_2$  is not a loop, without loss of generality (reflection), we can assume it ends at  $e_2$  (this implies  $y_2$  cannot be a loop). Hence,  $y_2$  ends either at  $e_1$  which implies a rotation of case (2) since  $z_2$  must be a loop then. Alternatively,  $y_2$  ends at  $e_3$ , and therefore by Lemma 5.3.2 similarly  $z_2$  ends at  $e_1$  and we found case (3).

Otherwise, in the right subgraph,  $y_1$  ends at  $e_3$  as well as  $z_1$  ends at  $e_2$ . Again, there are two options for  $x_2$ . Either,  $x_2$  is a loop or it is not a loop. If  $x_2$  is a loop, we obtain either case (2) if  $y_2$  is a loop or case (5) if  $y_2$  is not a loop. Otherwise,  $x_2$  is not a loop and so without loss of generality ends at  $e_2$ . There are two options for  $y_2$ . Either, it ends at  $e_1$  or at  $e_3$  (as  $e_2$  is already taken by  $x_2$  this is not a possibility). If  $y_2$  ends at  $e_1$ , then  $z_2$  must be a loop and we get (4). If  $y_2$  ends at  $e_3$ , this forces  $z_2$  to end at  $e_1$  and case (6) arises.

In the other case, all of  $\{x_1, x_2, y_1, y_2, z_1, z_2\}$  are not loops. Since  $x_i$  is not a loop, we may assume it ends at  $e_2$ . This forces  $y_i$  - which is not a loop as we have no loops - to end at  $e_3$  as ending at  $e_1$  forces  $z_i$  to be a loop. Since an arrow of the same type as  $x_i$  needs to end at  $e_1$ , this forces  $z_i$  to be an arrow from  $e_3$  to  $e_1$ . Now, by symmetry, there are only two cases to consider. First, both  $x_1$  and  $x_2$  end at the same vertex, say  $e_2$  without loss of generality. The argument just given shows that we are in case (7). On the other hand, if  $x_1$  and  $x_2$  end at different vertices, say  $x_i$  ends at  $e_2$  and  $x_j$  ends at  $e_3$ , the above implies that we are in case (8).  $\square$

*Remark 8.2.2.* Let's see what we can say about the minimal relations of  $A^G$  in the eight

cases presented above. We keep the notation from Proposition 8.2.1, where  $x_1, y_1, z_1$  are the shortest fixed purely nonstar paths in  $Q$  starting at  $e_1, e_2, e_3$ . Analogously, let  $x_2, y_2, z_2$  be the shortest fixed purely star paths starting at  $e_1, e_2, e_3$ .

- (1) Since  $x_1$  and  $x_2$  are loops, we get  $x_1x_2 = x_2x_1$ . Further, since  $x_1$  is purely nonstar and  $x_2$  is purely star, there cannot be another minimal relation between them. Similarly, we obtain the total of three minimal relations  $x_1x_2 = x_2x_1$ , and  $y_1y_2 = y_2y_1$  as well as  $z_1z_2 = z_2z_1$ . In other words,  $A^G \cong k[x_1, x_2] \oplus k[y_1, y_2] \oplus k[z_1, z_2]$  which has global dimension 2 and satisfies the generalized Gorenstein condition.
- (2) As before,  $x_1x_2 = x_2x_1$ . Consider the other two vertices. Since an element is uniquely determined by its starting vertex, its length and the number of star arrows, the relation  $y_iy_j = y_jz_i$  appears automatically since  $\deg_A(y_i) = \deg_A(z_i)$  by Lemma 7.3.4. Notice that this relation cannot be further decomposed and therefore must be minimal. Similarly, we have the minimal relation  $z_iz_j = z_jy_i$  starting at  $e_3$ . We will show by induction on the degree that these are the only minimal relations starting at  $e_2$  and  $e_3$ . By Proposition 5.3.4 and Corollary 7.3.2, a general binomial relation at  $e_2$  looks like  $y_i^r y_j \beta = y_j \gamma$  for some appropriate  $\beta, \gamma \in A^G$ . Applying the relation  $y_iy_j = y_jz_i$  precisely  $r$ -times to the left side provides  $y_j z_i^r \beta = y_j \gamma$ . Dropping  $y_j$  yields a relation of lower degree which by induction must come from the two relations provided. Analogously, the claim is true for relations starting at  $e_3$ . We saw this example before in 4.2.2. It has global dimension 2, is generalized Gorenstein and  $A$  is projective as an  $A^G$ -module.

(3) Similarly to (2), at the vertex  $e_1$  arises the minimal relation  $x_i x_j = x_j y_i$  automatically thanks to Lemma 7.3.4 and the fact that a path is uniquely determined by its length, its starting vertex and the number of star arrows of  $A$ . Analogously, the minimal relations  $y_i y_j = y_j z_i$  and  $z_i z_j = z_j x_i$  must be satisfied starting at  $e_2$  and  $e_3$ , respectively. The same technique as before works. By Proposition 5.3.4 it is enough to consider binomial relations. Let  $x_i^r x_j \beta = x_j \gamma$  be a relation starting at  $e_1$ . Notice that  $x_i^r$  cannot be written differently and therefore there are no relations of the form  $x_i^r = \delta$ . Applying the relation  $x_i x_j = x_j y_i$  several times yields  $x_j y_i^r \beta = x_j \gamma$  and as before dropping  $x_j$  gives the relation  $y_i^r \beta = \gamma$  which starts at  $e_2$  and has lower degree. By induction it must come from the three stated relations. Hence, there is only one minimal relation starting at  $e_1$ . By symmetry, we obtain the same result at the vertices  $e_2$  and  $e_3$ . To sum up, we have precisely the three minimal relations

$$x_i x_j = x_j y_i, \quad y_i y_j = y_j z_i \quad \text{and} \quad z_i z_j = z_j x_i y_j.$$

As before, we denote the indecomposable projective modules by  $P_i = e_i A^G$ . This allows us to calculate the minimal projective resolution of  $S_1 = e_1 A^G / (e_1 A^G)_{\geq 1}$ :

$$0 \longrightarrow P_2 \xrightarrow{\begin{pmatrix} x_j \\ -y_i \end{pmatrix}} P_1 \oplus P_2 \xrightarrow{(x_i \quad x_j)^\bullet} P_1 \longrightarrow S_1 \longrightarrow 0.$$

By symmetry, this works for  $S_2$  and  $S_3$  in the same way and [RR18, Proposition 3.19 (3)] says that  $A^G$  has global dimension 2 and we checked that it satisfies the

generalized Gorenstein condition.

- (4) At  $e_1$  arises the relation  $x_i x_j = x_j y_i z_i$ . Since  $x_i$  and  $y_i z_i$  are minimal fixed closed paths, a similar argument as in Lemma 7.3.4 provides that  $\deg_A(x_i) = \deg_A(y_i z_i)$ . The technique from (2) and (3) shows that there cannot be a different minimal relation starting at  $e_1$ . Analogously, the same happens at  $e_3$ . Consider the following two relations at  $e_2$ :

$$y_i z_j = y_j x_j y_i \quad \text{and} \quad y_j x_i = y_i z_i y_j.$$

Since  $y_i z_j$  and  $y_j x_i$  do not contain a proper subword which can be written differently and  $y_i z_j \neq y_j x_i$ , these two relations must be minimal. We have seen this example in 4.2.3, it has global dimension 3 and is not generalized Gorenstein.

- (5) It is clear what happens at  $e_1$ . For  $e_2$  and  $e_3$ , notice that  $y_1 = (\alpha_2 \alpha_3 \alpha_1)^r \alpha_2$  for some  $r$  since it is a path from  $e_2$  to  $e_3$  and therefore  $\deg_A(y_1) \equiv 1 \pmod{3}$ . Analogously, we have  $\deg_A(z_1) \equiv 2 \pmod{3}$ . This means relations like  $z_1 y_2 = z_2 y_1$  or  $y_1 z_2 = y_2 z_1$  are not possible. Since every binomial relation consists of two mixed paths, there are no relations of degree 2 in  $A^G$  starting at  $e_2$  or  $e_3$ . Consider the following two relations both starting at  $e_2$ :

$$P_1 : y_1 z_1 y_2 = y_2 z_1 y_1 \quad \text{and} \quad P_2 : y_2 z_2 y_1 = y_1 z_2 y_2.$$

Since  $P_1$  and  $P_2$  cannot be built from a relation of degree 2 they must be minimal.

So there cannot be just one minimal relation starting at  $e_2$ .

- (6) Suppose  $A^G$  has global dimension 2 and is generalized Gorenstein. Therefore, there are unique minimal relations starting at each vertex. Then,  $e_1$  can be understood as in (3): the unique minimal relation  $R_1$  starting at  $e_1$  equals  $x_i x_j = x_j y_i z_i$  which ends at  $e_2$ . Consider the relations arising from having two paths starting at  $e_2$  of the same type:

$$P_1 : y_j z_i y_i = y_i z_i y_j \quad \text{and} \quad P_2 : y_j z_j x_j y_i = y_i z_j x_j y_j.$$

Every path in  $P_1$  and  $P_2$  must contain a subword which is a monomial in a minimal relation. We can choose the ordering  $y_i < y_j$  and thus obtain the leading terms  $y_j z_i y_i$  and  $y_j z_j x_j y_i$ . Reasoning as in case (5) says that there cannot be a minimal relation starting at  $e_2$  of degree 2 which makes  $P_1$  a minimal relation. Clearly,  $P_2$  cannot arise from  $P_1$  due to the degrees of the leading terms, i.e.  $\deg_A(z_i y_i) > \deg(y_i)$ . Therefore, there cannot be just one minimal relation starting at  $e_2$ .

- (7) The two relations starting at  $e_1$  arise:

$$P_1 : x_1 \cdot y_2 z_2 x_2 = x_2 y_2 z_2 \cdot x_1 \quad \text{and} \quad P_2 : x_2 \cdot y_1 z_1 x_1 = x_1 y_1 z_1 \cdot x_2.$$

As before, suppose  $\beta_1 = \beta_2$  is the unique minimal relation starting at  $e_1$ . Since the pure paths  $y_2 z_2 x_2$  and  $y_1 z_1 x_1$  and their subwords cannot be monomials in minimal relations, we know that  $\beta_i = x_1 \cdot y_2 z_2 x_2$  or  $\beta_i = x_1 \cdot y_2 z_2$  or  $\beta_i = x_1 \cdot y_2$  and  $\beta_j = x_2 \cdot y_1 z_1 x_1$  or  $\beta_j = x_2 \cdot y_1 z_1$  or  $\beta_j = x_2 \cdot y_1$ . Since  $t(\beta_1) = t(\beta_2)$ , the following

options are possible:

$$x_1 \cdot y_2 z_2 x_2 = x_2 \cdot y_1 z_1 x_1, \quad \text{impossible since } \deg_A(y_2 z_2 x_2) > \deg_A(x_2),$$

$$x_1 \cdot y_2 z_2 = x_2 \cdot y_1 z_1, \quad \text{impossible since } \deg_A(y_1 z_1) \neq \deg_A(x_1),$$

$$x_1 \cdot y_2 = x_2 \cdot y_1.$$

Let's give a little more details in the second case. Since  $x_1$ ,  $y_1$  and  $z_1$  are all purely nonstar paths from  $e_i$  to  $e_{i+1}$ , a similar analysis as in the beginning of case (5) gives

$$\deg_A(y_1) \equiv \deg_A(z_1) \equiv \deg_A(x_1) \equiv 1 \pmod{3}.$$

Therefore,  $\deg_A(y_1 z_1) \neq \deg_A(x_1)$ .

By symmetry, we can conclude that either  $A^G$  is not of global dimension 2 and generalized Gorenstein or the three minimal relations

$$R_1 : x_1 y_2 = x_2 y_1, \quad R_2 : y_1 z_2 = y_2 z_1, \quad \text{and} \quad R_3 : z_1 x_2 = z_2 x_1$$

must hold. It follows that  $\deg_A(x_i) = \deg_A(y_i) = \deg_A(z_i)$  in  $A$  for  $i = 1, 2$ . Notice that they imply  $P_1$  and  $P_2$ . The relations  $R_1$ ,  $R_2$  and  $R_3$  allow us to represent any given path by first putting all the purely nonstar paths and then the purely star paths. Let  $\beta_1 = \beta_2$  be a general nontrivial binomial relation. Since pure paths cannot be monomials in relations, both  $\beta_1$  and  $\beta_2$  must be mixed. Therefore, using  $R_1$ ,  $R_2$  and  $R_3$  we can put a purely nonstar path first in the expression for  $\beta_1$  and  $\beta_2$ . Since the sources  $s(\beta_1) = s(\beta_2)$ , the purely nonstar paths put first are identical.

This allows to shorten the degree of the relation and induction shows  $\beta_1 = \beta_2$  comes from the three relations  $R_1$ ,  $R_2$  and  $R_3$ .

- (8) Case (8) is very similar to (7). The same analysis concludes that either  $A^G$  is not both generalized Gorenstein and of global dimension 2 or the three minimal relations

$$R_1 : x_i y_j = x_j z_i, \quad R_2 : y_i z_j = y_j x_i, \quad \text{and} \quad R_3 : z_i x_j = z_j y_i$$

hold. In a generic nontrivial relation  $\beta_1 = \beta_2$  starting at  $e_1$ , we can rewrite  $\beta_1 = x_i \beta'_1$  as well as  $\beta_2 = x_i \beta'_2$  since both  $\beta_1$  and  $\beta_2$  must be mixed and the three relations  $R_1$ ,  $R_2$  and  $R_3$  allow us to put paths of type  $i$  first. Thus  $\beta_1 = \beta_2$  reduces to  $\beta'_1 = \beta'_2$  of lower degree and induction shows every relation starting at  $e_1$  comes from the three relations. Symmetry shows this for  $e_2$  and  $e_3$ .

*Remark 8.2.3.* To sum up, we understand the eight cases from Proposition 8.2.1 as follows:

- (1) Here,  $\text{gl. dim}(A^G) = 2$  and  $A^G$  is generalized Gorenstein. Further,  $A$  is free as an  $A^G$ -module.
- (2) Having precisely three minimal relations gives  $\text{gl. dim}(A^G) = 2$  and  $A^G$  is generalized Gorenstein.
- (3) Having precisely three minimal relations gives  $\text{gl. dim}(A^G) = 2$  and  $A^G$  is generalized Gorenstein.
- (4) Since there are two minimal relations starting at  $e_2$ , the fixed ring  $A^G$  cannot be generalized Gorenstein and of global dimension 2.

- (5) Occurring as a fixed ring, there are two minimal relations starting at  $e_2$ . Therefore, the fixed ring  $A^G$  cannot be generalized Gorenstein and of global dimension 2.
- (6) Having found two minimal relations starting at  $e_2$  implies that the fixed ring  $A^G$  cannot be generalized Gorenstein and of global dimension 2.
- (7) Either  $A^G$  is not both generalized Gorenstein and of global dimension 2 or the three relations  $x_1y_2 = x_2y_1$ ,  $y_1z_2 = y_2z_1$  and  $z_1x_2 = z_2x_1$  are the only minimal relations.
- (8) Either  $A^G$  is not both generalized Gorenstein and of global dimension 2 or the three relations  $x_iy_j = x_jz_i$ ,  $y_iz_j = y_jx_i$  and  $z_ix_j = z_jy_i$  are the only minimal relations.

As preparation to show that if a cyclic group  $G$  acts on  $A$  and produces a fixed ring of global dimension 2 which satisfies the generalized Gorenstein condition, it must be generated by quasi-reflections, we need the following lemma.

**Lemma 8.2.4.** *Let  $Q = \widetilde{A}_2$  and  $A = \Pi_Q(k)$ . Let  $g \in \text{Aut}_{\text{gr}}(A)$  have finite order and*

$$\begin{aligned} \alpha_1 &\mapsto c_1\alpha_1, & \alpha_1^* &\mapsto t_1\alpha_1^*, \\ \alpha_2 &\mapsto c_2\alpha_2, & \alpha_2^* &\mapsto t_2\alpha_2^*, \\ \alpha_3 &\mapsto c_3\alpha_3, & \alpha_3^* &\mapsto t_3\alpha_3^*. \end{aligned}$$

*Denote the order of  $c_1c_2c_3$  by  $k = 3^r\ell$  and the order of  $t_1t_2t_3$  by  $j = 3^sm$  where  $\ell$  and  $m$  satisfy  $\gcd(\ell, 3) = 1 = \gcd(m, 3)$ . Assume  $\text{hdet}(g)$  has order  $3kj$ .*

- *If  $c_1^k = c_2^k = c_3^k = \zeta_3$  or  $\zeta_3^2$  then there exists a fixed path under  $g$  with  $\ell j$  star arrows.*
- *If  $t_1^j = t_2^j = t_3^j = \zeta_3$  or  $\zeta_3^2$  then there is a fixed path under  $g$  with  $mk$  nonstar arrows.*

*Proof.* Let  $h = \text{hdet}(g) = c_1 t_1$ . Due to symmetry it is enough to find a fixed path with  $\ell j$  star arrows whenever  $c_1^k = c_2^k = c_3^k = \zeta_3 = \exp(\frac{2\pi i}{3})$ . For  $\zeta_k = \exp(\frac{2\pi i}{k})$ , we can write

$$c_1 = \zeta_k^{a_1} c, \quad c_2 = \zeta_k^{a_2} c, \quad \text{and} \quad c_3 = \zeta_k^{a_3} c$$

for some integers  $a_1, a_2, a_3 \in \mathbb{Z}$  and some root of unity  $c$ . Since

$$1 = (c_1 c_2 c_3)^k = c_1^k c_2^k c_3^k = c^{3k} = c^{3^{r+1}\ell} = (c^\ell)^{3^{r+1}}$$

we get that the order  $o(c^\ell)$  of  $c^\ell$  divides  $3^{r+1}$ . Suppose  $o(c^\ell) = 3^m$  for  $m < r + 1$ . Then  $c_1^k = (\zeta_k^{a_1} c)^k = c^k = c^{3^r \ell} = (c^\ell)^{3^r} = 1$  which contradicts  $c_1^k = \zeta_3$ . Thus  $o(c^\ell) = 3^{r+1}$ .

For some scalars  $s_1$  and  $s_2$  consider the element

$$\beta = (\alpha_1 \alpha_1^*)^{\ell j} \cdot (\alpha_1 \alpha_2 \alpha_3)^{s_1} \cdot \underbrace{\alpha_1 \alpha_2 \cdots \alpha_q}_{s_2 \cdot \ell\text{-many}}.$$

We will now try to find appropriate  $s_1$  and  $s_2$  such that  $g(\beta) = \beta$ . Since  $o(c_1 c_2 c_3) = k$ , acting on  $(\alpha_1 \alpha_2 \alpha_3)$  by  $g$  scales the path by a primitive  $k$ -th root of unity, say  $\zeta_k^{u_1}$ . Since  $o(h) = 3kj = 3^{r+1}\ell j$  we have that  $h^{\ell j}$  is a primitive  $3^{r+1}$ st root of unity. Then by construction we know that

$$g(\beta) = [h^{\ell j} \zeta_k^{s_1 \cdot u_1} \zeta_k^{\text{some power}} (c^\ell)^{s_2}] \beta.$$

By choosing  $s_2$  such that  $h^{\ell j} (c^\ell)^{s_2} = 1$  and then finding  $s_1$  to cancel out the  $k$ th roots of unity we obtain  $g(\beta) = \beta$ . □

**Proposition 8.2.5.** *Let  $G = \langle g \rangle$  with  $g(\alpha_i) = c_i \alpha_i$  and  $g(\alpha_i^*) = t_i \alpha_i^*$ . Let  $A^G$  be twisted Calabi-Yau such that  $Q^G$  is either of case (1), (2), (3), (7) or (8). Then  $G$  is generated*

by quasi-reflections. In other words, for  $Q = \widetilde{A}_2$  and a cyclic group  $G$ , the fixed ring  $A^G$  being twisted Calabi-Yau implies that  $G$  is generated by quasi-reflections.

*Proof.* To simplify notation call  $\zeta_m = \exp(\frac{2\pi i}{m})$ . Let  $h = \text{hdet}(g) = c_1 t_1$ . Denote the order of  $c_1 c_2 c_3$  by  $k$  and the order of  $t_1 t_2 t_3$  by  $j$ . Since the quivers  $Q^G$  from cases (1), (2), (7), (8) do not have any mixed generators we can make the following observation. If  $d = \gcd(k, j) > 1$ , we can construct a mixed closed path with  $\frac{3k}{d}$  nonstar arrows as follows:  $c_1 c_2 c_3$  is a primitive  $k$ -th root of 1, say  $\zeta_k^{p_k}$  and  $t_1 t_2 t_3$  is a primitive  $j$ -th root, say  $\zeta_j^{p_j}$ . Since  $d > 1$  we can find an  $r$  such that  $\zeta_k^{p_k k/d} \zeta_j^{p_j r} = 1$ . Therefore, by construction

$$\gamma = (\alpha_1 \alpha_2 \alpha_3)^{k/d} (\alpha_3^* \alpha_2^* \alpha_1^*)^r$$

is fixed with fewer than  $3k$  nonstar arrows. Since  $\gamma$  is a product of pure generators of  $A^G$  it follows that the nonstar part of  $\gamma$  must be fixed separately. In other words, we must have  $(c_1 c_2 c_3)^{k/d} = 1$  contradicting the definition of  $k$  to be the order of  $c_1 c_2 c_3$ . This means  $d = \gcd(k, j) = 1$ . This construction shows that every fixed closed path must consist of a multiple of  $3k$  nonstar arrows as well as a multiple of  $3j$  star paths. This also implies that  $o(h) = 3kj$  as

$$(\alpha_1 \alpha_1^*)^{o(h)}$$

must be a fixed closed path and therefore  $3k|o(h)$  and  $3j|o(h)$ . From  $d = 1$  it follows that  $kj|o(h)$  and since not both  $k$  and  $j$  can be multiples of 3 we know that  $o(h) = 3kj$ .

In the cases (7) and (8) whenever  $A^G$  is twisted Calabi-Yau, the relations imply that  $\deg(x_i) = \deg(y_i) = \deg(z_i)$ . Without loss of generality, in case (8) say type  $i$

stands for nonstar arrows. Hence,  $x_1 = (\alpha_1\alpha_2\alpha_3)^q\alpha_1$  and  $y_1 = (\alpha_2\alpha_3\alpha_1)^q\alpha_2$  for some  $q$ .

Then  $(c_1c_2c_3)^qc_1 = 1 = (c_1c_2c_3)^qc_2$  which implies  $c_1 = c_2$ . In the same way, we find that

$c_1 = c_2 = c_3$  and  $t_1 = t_2 = t_3$ . To sum up, we have

(7)  $c_1 = c_2 = c_3$  with  $o(c_1) = 3r + 1$  and  $t_1 = t_2 = t_3$  with  $o(t_1) = 3s + 2$  for some

nonnegative  $r, s \in \mathbb{Z}$  and

(8)  $c_1 = c_2 = c_3$  with  $o(c_1) = 3r + 1$  and  $t_1 = t_2 = t_3$  with  $o(t_1) = 3s + 1$  for some

nonnegative  $r, s \in \mathbb{Z}$ .

Since  $\gcd(3, 3r + 1) = 1$ , the order of  $c_1^3$  must be  $3r + 1$  as well. Therefore, we obtain

$k = 3r + 1$ . In the same way it follows that  $j = 3s + 1$  and  $j = 3s + 2$ , respectively.

Consider  $g^k$  with  $c_1^k = c_2^k = c_3^k = 1$ . A calculation using the formula from the beginning

of Section 8.1 simplifies the trace function to

$$\begin{aligned} \text{Tr}(g^k, t) &= \frac{3 + (3 + 3t_1^k)t + (3 + 3t_1^k + 3(t_1^k)^2)t^2 + (3t_1^k + 3(t_1^k)^2)t^3 + 3(t_1^k)^2t^4}{(1 - t^3)(1 - t_1^{3k}t^3)} \\ &= \frac{3}{(1 - t)(1 - t_1^k t)}. \end{aligned}$$

Since  $o(t_1) \neq k$ , this shows that  $g^k$  is a quasi-reflection. Analogously,  $g^j$  is a quasi-

reflection. Since  $(j, k) = 1$ , the group  $G$  can be generated by  $g^k$  and  $g^j$ .

In case (1), use the same notation as before. Denote the order of  $c_1c_2c_3$  by  $k$  and

the order of  $t_1t_2t_3$  by  $j$ . Recall that  $\gcd(k, j) = 1$ .

▷  $k = j = 1$ . Since  $\text{hdet}(g) \neq 1$  due to  $\alpha_1\alpha_1^*$  not being fixed, Corollary 8.1.1 implies

that  $g$  is not a quasi-reflection if and only if

$$c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = \exp\left(\frac{2\pi i}{3}\right) \quad \text{or} \quad \exp\left(\frac{4\pi i}{3}\right).$$

These two cases cannot happen since  $\alpha_1\alpha_2\alpha_2^*$  is not fixed. Hence,  $g$  is a quasi-reflection.

▷  $k = 1$  and  $j > 1$ . Then  $(c_1t_1)^3 = t_1t_2t_3 \neq 1$  and the trace of  $g$  computes as

$$\text{Tr}(g, t) = \frac{p(t)}{(1 - t^3)(1 - t_1t_2t_3t^3)}$$

with  $p(1) = (3 + c_1 + c_2 + c_3 + c_1c_2 + c_1c_3 + c_2c_3)(1 + c_1t_1 + (c_1t_1)^2)$  as we recalled in the beginning of Section 8.1. Either  $g$  is a quasi-reflection or  $p(1) = 0$ . This happens if and only if  $3 + c_1 + c_2 + c_3 + c_1c_2 + c_1c_3 + c_2c_3 = 0$ . Multiplying this equation by  $c_1c_2$  yields  $1 + c_1 + c_2 + 3c_1c_2 + c_1^2c_2 + c_1c_2^2 + c_1^2c_2^2 = 0$ . By Lemma 8.1.3 it follows that  $c_1 = c_2 = c_3 = \exp(\frac{2\pi i}{3})$  or  $c_1 = c_2 = c_3 = \exp(\frac{4\pi i}{3})$ . By symmetry it is enough to consider the case  $c_1 = c_2 = c_3 = \exp(\frac{2\pi i}{3})$ . As a consequence, we obtain  $t_1 = t_2 = t_3$  as well as  $t_1^3 \neq 1$ . Moreover, since in case (1) the purely nonstar generator at  $e_1$  is of the form  $(\alpha_3^*\alpha_2^*\alpha_1^*)^j$  we have that  $t_1^{3j} = 1$ . If  $t_1^j = 1$  then  $\underbrace{\alpha_3^*\alpha_2^*\alpha_1^* \cdots \alpha_i^*}_{j\text{-many}}$  is fixed contradicting that the shortest purely nonstar fixed path starting at  $e_1$  has  $3j$  star arrows. Therefore,  $t_1^j$  is a primitive third root of 1, say  $t_1^j = \exp(\frac{4\pi i}{3})$ . Therefore,  $\alpha_1 \underbrace{\alpha_1^*\alpha_3^*\alpha_2^* \cdots \alpha_i^*}_{j\text{-many}}$  is fixed, a contradiction as for mixed paths the nonstar path must be fixed separately but  $g(\alpha_1) = \exp(\frac{2\pi i}{3})\alpha_1$ .

▷  $k > 1$  and  $j = 1$ . This case is identical to  $k = 1$  and  $j > 1$ .

▷  $k > 1$  and  $j > 1$ . Consider the elements  $g^k$  and  $g^j$ . By symmetry it is enough to consider  $g^k$  whose trace is

$$\text{Tr}(g^k, t) = \frac{p(t)}{(1 - t^3)(1 - (t_1 t_2 t_3)^k t^3)}.$$

As before, either  $g^k$  is a quasi-reflection in which case we are done or  $p(1) = 0$ , where  $p(1) = (3 + c_1^k + c_2^k + c_3^k + (c_1 c_2)^k + (c_1 c_3)^k + (c_2 c_3)^k)(1 + (c_1 t_1)^k + (c_1 t_1)^{2k})$ . The second factor is nonzero since the order of  $c_1 t_1$  equals  $3kj$  and  $j > 1$ . Therefore, as in the case  $k = 1$  and  $j > 1$ , we know that  $p(1) = 0$  if and only if

$$3 + c_1^k + c_2^k + c_3^k + (c_1 c_2)^k + (c_1 c_3)^k + (c_2 c_3)^k = 0.$$

Once we multiply the equation by  $(c_1 c_2)^k$  we obtain

$$3c_1^k c_2^k + (c_1^k)^2 c_2^k + c_1^k (c_2^k)^2 + 1 + (c_1^k c_2^k)^2 + c_1^k + c_2^k = 0$$

and applying Lemma 8.1.3 says that  $c_1^k = c_2^k = c_3^k = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ . We introduce the notation from Lemma 8.2.4 that  $k = 3^r \ell$  with  $(\ell, 3) = 1$ . The statement of the lemma gives a fixed path with  $\ell j$  star arrows. This gives a contradiction since star arrows always occur in multiples of  $3j$  in fixed elements.

In case (2), use the same notation as before. Denote the order of  $c_1 c_2 c_3$  by  $k$  and the order of  $t_1 t_2 t_3$  by  $j$ . As before, if  $\gcd(k, j) > 1$ , we can construct a fixed closed path at  $e_1$  with less than  $3k$  nonstar arrows which gives a contradiction. Without loss of generality let's assume type  $i$  stands for nonstar arrows.

▷  $k = j = 1$ . This implies that  $t_1 t_2 t_3 = 1$  and so the minimal purely nonstar fixed path  $z_j$  starting at  $e_3$  must equal  $\alpha_2^*$  as it is a purely nonstar path from  $e_3$  to  $e_2$  with at most three arrows. In other words,  $t_2 = 1$  and  $(c_1 t_1)^3 = (c_1 c_2 c_3)(t_1 t_2 t_3) = 1$ . Notice that if type  $i$  stands for star arrows instead, we analogously get that  $c_2 = 1$ . Thus, we are in the situation of Corollary 8.1.1. From the beginning of the proof we know that  $o(\text{hdet}(g)) = 3kj = 3$ . Therefore  $\text{hdet}(g) \neq 1$  and together with  $t_2 = 1$ , the cases (4) and (6) force  $g$  to be a quasi-reflection.

▷  $k = 1$  and  $j > 1$ . As before, the trace of  $g$  looks like

$$\text{Tr}(g, t) = \frac{p(t)}{(1 - t^3)(1 - t_1 t_2 t_3 t^3)}.$$

Either  $g$  is a quasi-reflection or  $c_1 = c_2 = c_3 = \zeta_3$  or  $\zeta_3^2$  as worked out in case (1). However, this implies  $t_1 = t_2 = t_3$  which contradicts the shortest purely star fixed paths at  $e_1$  and  $e_2$  having different length. If type  $i$  stands for star arrows instead, we immediately get a contradiction from  $c_2 = \zeta_3$  or  $\zeta_3^2$  as  $y_j = \alpha_2$  must be fixed in this case.

▷  $k > 1$  and  $j = 1$ . The same analysis of the trace

$$\text{Tr}(g, t) = \frac{p(t)}{(1 - t^3)(1 - c_1 c_2 c_3 t^3)}.$$

gives that either  $g$  is a quasi-reflection or  $t_1 = t_2 = t_3$ . As before, this contradicts the shortest purely star fixed paths at  $e_1$  and  $e_2$  having different length. If type  $i$  stands for star arrows instead,  $t_1 = t_2 = t_3$  implies that  $c_1 = c_2 = c_3$  which again cannot happen due to  $y_j$  and  $z_j$  having different degree in  $A$ .

▷  $k > 1$  and  $j > 1$ . Consider the element  $g^k$  whose trace equals

$$\text{Tr}(g^k, t) = \frac{p(t)}{(1 - t^3)(1 - (t_1 t_2 t_3)^k t^3)}.$$

As before, either  $g^k$  is a quasi-reflection in which case we are done or  $p(1) = 0$ . Then, Lemma 8.1.3 applies again saying  $c_1^k = c_2^k = c_3^k = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ . We introduce the notation from Lemma 8.2.4 that  $k = 3^r \ell$  with  $(\ell, 3) = 1$ . The statement of the lemma gives a fixed path with  $\ell j$  star arrows starting at  $e_1$ . This gives a contradiction since star arrows always occur in multiples of  $3j$  in fixed elements starting at  $e_1$ . The same argument is valid for  $g^j$  and also works for rotated versions of case (2) and if type  $i$  stands for star arrows instead.

Case (3) is the most difficult case. We use the same notation as before. Denote the order of  $c_1 c_2 c_3$  by  $k$  and the order of  $t_1 t_2 t_3$  by  $j$ . Also recall that  $\gcd(k, j) = 1$  and for  $h = \text{hdet}(g)$  we know  $o(h) = 3kj$ . Without loss of generality let's assume type  $i$  stands for nonstar arrows.

▷  $k = j = 1$ . This situation is completely understood in Corollary 8.1.1 (3) - (6).

Since  $o(h) = 3$  we have that either  $g$  is a quasi-reflection or

$$c_1 = c_2 = c_3 = t_1 = t_2 = t_3 = h^2.$$

The second option is impossible since  $\alpha_1 \alpha_2 \alpha_2^*$  is not fixed by  $g$ .

▷  $k > 1$  and  $j = 1$ . Since  $o(t_1 t_2 t_3) = j = 1$  we know that at each vertex the shortest purely star path which is fixed has degree less than or equal to 3 in  $A$ . Since we are in

case (3), we know that  $x_j = (\alpha_3^* \alpha_2^* \alpha_1^*)^q \alpha_3^* \alpha_2^*$  for some  $0 \leq q \in \mathbb{Z}$  is the shortest fixed purely star path at  $e_1$ . This and symmetry for the other vertices yields  $x_j = \alpha_3^* \alpha_2^*$ ,  $y_j = \alpha_1^* \alpha_3^*$  and  $z_j = \alpha_2^* \alpha_1^*$ . In other words,  $t_2 t_3 = 1$ ,  $t_1 t_3 = 1$  and  $t_1 t_2 = 1$ . As a consequence,  $t_1 = t_2 = t_3$  and  $t_1^2 = 1$  since  $\alpha_3^* \alpha_2^*$  is fixed. Since  $1 = o(t_1 t_2 t_3) = o(t_1^3)$  we know that  $t_1 = t_2 = t_3 = 1$ . This is a contradiction since  $\alpha_1^*$  is not fixed.

▷  $k = 1$  and  $j > 1$ . Notice that  $h^3 = (c_1 t_1)^3 = t_1 t_2 t_3 \neq 1$  and consider

$$\text{Tr}(g, t) = \frac{p(t)}{(1 - t^3)(1 - t_1 t_2 t_3 t^3)}$$

with  $p(1) = (3 + c_1 + c_2 + c_3 + c_1 c_2 + c_1 c_3 + c_2 c_3)(1 + c_1 t_1 + (c_1 t_1)^2)$  as we used many times before and recalled in the beginning of Section 8.1. Either  $g$  is a quasi-reflection or the first factor of  $p(1)$  must be zero. In the second case, multiplying by  $c_1 c_2$  and applying Lemma 8.2.4 again gives  $c_1 = c_2 = c_3 = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ . This forces  $t_1 = t_2 = t_3$  and since  $x_j = (\alpha_3^* \alpha_2^* \alpha_1^*)^q \alpha_3^* \alpha_2^*$  for some  $0 \leq q \in \mathbb{Z}$  we find that  $o(t_1) = 3q + 2$ . We are going to show that  $g^3$  and  $g^{3q+2}$  are quasi-reflections and due to  $\gcd(3, 3q + 2) = 1$  they generate the group  $\langle g \rangle$ .

First, consider  $g^3$ . Recall that  $c_1 = c_2 = c_3$  is a third root of unity and therefore  $c_1 c_2 c_3 = 1$ . If  $(t_1 t_2 t_3)^3 = 1$  then  $t_1^9 = 1$  which contradicts  $o(t_1) = 3q + 2$ . Therefore, the trace of  $g^3$  equals

$$\text{Tr}(g^3, t) = \frac{p_3(t)}{(1 - t^3)(1 - (t_1 t_2 t_3)^3 t^3)}.$$

The same analysis as before says that either  $g^3$  is a quasi-reflection which is what

we want or  $c_1^3 = c_2^3 = c_3^2 = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$  (see Lemma 8.1.3). The second option is impossible since  $c_1^3 = 1$ .

For  $\text{Tr}(g^{3q+2}, t)$  we notice that  $(c_1 c_2 c_3)^{3q+2} = 1^{3q+2} = 1$  and  $t_1 = t_2 = t_3$  forces  $(t_1 t_2 t_3)^{3q+2} = (t_1^{3q+2})^3 = 1^3 = 1$ . However, for the homological determinant we know that  $\text{hdet}(g^{3q+2}) = (c_1 t_1)^{3q+2} = c_1^2 \neq 1$  since  $c_1$  is a nontrivial third root of 1. Hence, we know that

$$\text{Tr}(g^{3q+2}, t) = \frac{p_{3q+2}(t)}{(1-t^3)(1-t^3)}.$$

This situation is completely understood in Corollary 8.1.1. Since  $\text{hdet}(g^{3q+2}) \neq 1$  and  $o(t_1) = 3q+2 \neq 3$ , Corollary 8.1.1 (4) and (6) show that  $g^{3q+2}$  is a quasi-reflection. Combined, we showed that  $\langle g \rangle$  is generated by the two quasi-reflections  $g^3$  and  $g^{3q+2}$ .

▷  $k > 1$  and  $j > 1$ . We use the notation from before, and add  $k = 3^r \ell$  and  $j = 3^s m$  where  $m$  and  $\ell$  are coprime to 3. First, consider  $g^j$ . Its trace equals

$$\text{Tr}(g^j, t) = \frac{p_j(t)}{(1-t^3)(1-(c_1 c_2 c_3)^j t^3)}.$$

As done before, either  $g^j$  is a quasi-reflection or  $t_1^j = t_2^j = t_3^j = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ .

In the second case, Lemma 8.2.4 constructs a fixed path  $\beta$  with  $mk$  nonstar arrows. This is a contradiction since for all fixed paths the nonstar part must be fixed separately and therefore must be a closed path with a multiple of  $3k$  nonstar arrows.

Next, consider  $g^k$ . As for  $j$  above,

$$\text{Tr}(g^k, t) = \frac{p_k(t)}{(1-t^3)(1-(t_1 t_2 t_3)^k t^3)},$$

and either  $g^k$  is a quasi-reflection which shows that  $\langle g \rangle = \langle g^j, g^k \rangle$  is generated by quasi-reflections or  $c_1^k = c_2^k = c_3^k = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ . In order to finish the proof we need to take care of the second case. Assume that  $c_1^k = c_2^k = c_3^k = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$ . Consequently, we know that  $t_1^k = t_2^k = t_3^k$ . Lemma 8.2.4 hands to us a fixed path  $\beta$  with  $\ell j$  star arrows.

If  $\gcd(3, j) = 1$  then also  $\gcd(3k, j) = 1$  and we can look at  $g^{3k}$  whose trace equals

$$\text{Tr}(g^{3k}, t) = \frac{p_{3k}(t)}{(1 - t^3)(1 - (t_1 t_2 t_3)^{3k} t^3)}.$$

We know that  $(c_1 c_2 c_3)^{3k} = 1$  and  $\gcd(3k, j) = 1$  with  $j = o(t_1 t_2 t_3) > 1$  guarantees that  $(t_1 t_2 t_3)^{3k} \neq 1$ . Again, either  $g^{3k}$  is a quasi-reflection and  $\langle g \rangle = \langle g^j, g^{3k} \rangle$  is generated by quasi-reflections or  $c_1^{3k} = \exp(\frac{2\pi i}{3})$  or  $\exp(\frac{4\pi i}{3})$  by Lemma 8.2.4. This is a contradiction since  $c_1^{3k} = 1$ .

Finally, if  $\gcd(3, j) = 3$  we take a closer look at  $\beta$  which has degree  $\ell j$ . Since all generators of  $A^G$  are pure paths, the star part of  $\beta$  must be fixed separately. From  $3|\ell j$  we can conclude that the star part of  $\beta$  must be a closed path. This is a contradiction as fixed closed star paths must have multiples of  $3j$  star arrows but  $3j \nmid \ell j$  and so this case cannot happen.

By Remark 8.2.3  $A^G$  can only be twisted Calabi-Yau if  $Q^G$  is a quiver of type (1), (2), (3), (7) or (8). We showed that in each case  $G$  is generated by quasi-reflections. This finishes the proof of the proposition.  $\square$

# Chapter 9

## Future questions

Over the past decade, invariant theory of noncommutative algebras has seen extensive progress. Nevertheless, it is far from being as developed as commutative invariant theory. In particular, there are more than enough open questions just for connected noncommutative algebras. For instance, the following conjecture regarding a Chevalley-Shephard-Todd type theorem posed by Kirkman, Kuzmanovich and Zhang in [KKZ10] remains unsolved.

**Conjecture 9.0.1** ([KKZ10, Conjecture 1.2]). *Let  $S$  be a quantum polynomial ring which is a noetherian Artin-Schelter regular domain of global dimension  $d$  with Hilbert series  $H_S(t) = (1 - t)^{-d}$ . Suppose  $G$  is a finite subgroup of  $\text{Aut}_{\text{gr}}(S)$ . Then,  $B^G$  has finite global dimension if and only if  $G$  is generated by quasi-reflections.*

But even supposedly simpler projects for an Artin-Schelter regular algebra  $S$  continue to be unsolved:

**Question 9.0.2.** *What is a good definition for a quasi-reflection if  $S$  is not a domain?*

**Question 9.0.3.** *Let  $g$  be a graded automorphism of  $S$  with  $\mathrm{Tr}_S(g, t) = e_g(t)^{-1}$ . Can it happen that  $e_g(t)$  has 1 as a root of multiplicity  $\mathrm{GKdim}(S)$ ?*

Among the many open problems, one became apparent in Chapter 3. Clearly, it would be nice to first find out if Conjecture 3.3.6 turns out to be true. Spinning this question further, it would be interesting to see for which Artin-Schelter regular domains of higher dimension quasi-reflections can be found.

Looking more towards non-connected twisted Calabi-Yau algebras there are even more open problems. Since we have the nice dichotomy in regards of the fixed ring having finite global dimension equal to  $\mathrm{gl.dim}(S)$  or infinite global dimension for an Artin-Schelter regular domain  $S$ , Example 4.2.3 came as a surprise. As a consequence, the following question immediately arises.

**Question 9.0.4.** *Let  $B$  be a twisted Calabi-Yau algebra of global dimension  $d$ . Suppose a finite subgroup  $G$  of  $\mathrm{Aut}_{\mathrm{gr}}(B)$  acts on  $B$ . What are the possible global dimensions of  $B^G$ ?*

In light of the work done in Section 5.3, the following conjecture appears naturally:

**Conjecture 9.0.5.** *For  $Q = \widetilde{D_{n-1}}$  and  $Q = \widetilde{E_m}$  for  $m = 6, 7, 8$ , the preprojective algebra  $A = \Pi_Q(k)$  satisfies the equivalent conditions from Theorem 5.3.7.*

However, if proved, it automatically increases interest in finding new algebras for which Conjecture 9.0.5 holds.

After giving several specific open problems to think about, a simple question to ask is what happens if the automorphism group  $G$  acting on a preprojective algebra permutes the idempotents. Unfortunately, when considering this case one loses some of the tools used in invariant theory. To give one example, an automorphism  $g$  permuting the vertices can have a trace equal to zero. In particular, aside from permuting the vertices all information on how  $g$  acts on the arrows is lost. This remains to be a very hard problem. Hence, the following may be an optimistic vision to get a more general version of Theorem 6.3.9.

**Conjecture 9.0.6.** *Let  $B$  be a generalized Gorenstein algebra with finite global dimension. Suppose  $G$  is a finite subgroup of graded automorphisms of  $B$ . If every  $g \in G$  satisfies  $\text{Hdet}(g) = I_n$  then  $B^G$  is generalized Gorenstein.*

First, this requires to find an appropriate definition of a matrix homological determinant for an automorphism  $g$  not fixing the vertices. However, there is hope since  $H_s^d(g)$  entails the same permutation of the duals of the idempotents as  $g$ . This gives an intuitive way to obtain  $n$  scalars uniquely determining  $H_s^d(g)$ .

At last, there are many open problems motivated by Chapter 7. Since the work there is only done for  $Q = \widetilde{A_{n-1}}$ , the usual follow-up question how to deal with the cases  $Q = \widetilde{D_{n-1}}$  and  $Q = \widetilde{E_m}$  for  $m = 6, 7, 8$  comes up. Moreover, Section 7.2 leaves it open if the definition of a quasi-reflection given in Definition 7.2.1 turns out to be promising. Theorem 7.2.3 raises hope, but since it only works for  $A = \Pi_Q(k)$  in case  $Q = \widetilde{A_{n-1}}$  rather than for general twisted Calabi-Yau algebras, we cannot get an inductive argument to

work. Hence:

**Question 9.0.7.** *What is a definition for a quasi-reflection of a general twisted Calabi-Yau algebra  $B$  with  $B_0 \cong k^n$  such that whenever  $B^G$  is twisted Calabi-Yau, the group  $G$  must contain a quasi-reflection?*

It seems far out of reach to find a functioning definition since the trace functions turn out to be wild. Moreover, Example 7.2.4 presents a graded automorphism which we are tempted to call a quasi-reflection. But as worked out, the corresponding fixed ring has infinite global dimension.

Invariant theory of noncommutative algebras has advanced immensely. Not only is it a very interesting, beautiful and open subject, but it has given us many examples of Artin-Schelter regular algebras, Artin-Schelter Gorenstein algebras, non-connected algebras satisfying a generalized Gorenstein condition and more. Therefore, it has improved the understanding of these classes of algebras and we hope that the future brings an even more fruitful theory which will influence many other fields of mathematics.

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