Title
A p, q-Analogue of the Generalized Derangement Numbers

Journal
Annals of Combinatorics, 13(1)

ISSN
0219-3094

Authors
Briggs, Karen S.
Remmel, Jeffrey B.

Publication Date
2009-07-01

DOI
10.1007/s00026-009-0010-4

Peer reviewed

Ann. Comb. 13 (2009) 1–25
DOI 10.1007/s00026-009-0010-4
Published online May 14, 2009

Annals of Combinatorics

A p, q-Analogue of the Generalized Derangement Numbers

Karen S. Briggs[1] and Jeffrey B. Remmel[2*]

[1]Department of Mathematics and Computer Science, North Georgia College & State University, Dahlonega, GA 30597, USA
kbriggs@ngcsu.edu

[2]Department of Mathematics, University of California, San Diego, La Jolla, CA 92093, USA
jremmel@ucsd.edu

Received August 29, 2006

AMS Subject Classification: 05A30; 05A19, 05A15, 05A05, 05E15

Abstract. In this paper, we study the numbers $D_{n,k}$ which are defined as the number of permutations σ of the symmetric group S_n such that σ has no cycles of length j for $j \leq k$. In the case $k = 1$, $D_{n,1}$ is simply the number of derangements of an n-element set. As such, we shall call the numbers $D_{n,k}$ generalized derangement numbers. Garsia and Remmel [4] defined some natural q-analogues of $D_{n,1}$, denoted by $D_{n,1}(q)$, which give rise to natural q-analogues of the two classical recursions of the number of derangements. The method of Garsia and Remmel can be easily extended to give natural p, q-analogues $D_{n,1}(p, q)$ which satisfy natural p, q-analogues of the two classical recursions for the number of derangements. In [4], Garsia and Remmel also suggested an approach to define q-analogues of the numbers $D_{n,k}$. In this paper, we show that their ideas can be extended to give a p, q-analogue of the generalized derangements numbers. Again there are two classical recursions for generalized derangement numbers. However, the p, q-analogues of the two classical recursions are not as straightforward when $k \geq 2$.

Keywords: permutations, derangements, p, q-analogues

1. Introduction

In this paper, we study the numbers $D_{n,k}$ which are defined as the number of permutations σ of the symmetric group S_n such that σ has no cycles of length j for $j \leq k$. In the case $k = 1$, $D_{n,1}$ is simply the number of derangements of an n-element set. As such, we shall call the numbers $D_{n,k}$ *generalized derangement numbers*. There are two classical recursions for the number of derangements. In particular, it is easy to see that $D_{1,1} = 0$ and $D_{2,1} = 1$, so that for $n \geq 2$,

$$D_{n+1,1} = nD_{n,1} + nD_{n-1,1}, \tag{1.1}$$

* Partially supported by NSF grant DMS 0400507.

and for $n \geq 1$,

$$D_{n+1,1} = (n+1)D_{n,1} + (-1)^{n+1}. \tag{1.2}$$

Garsia and Remmel [4] defined a natural q-analogue of $D_{n,1}$, denoted by $D_{n,1}(q)$, which satisfy two natural q-analogues of (1.1) and (1.2). That is, let $\mathcal{D}_{n,k}$ denote the set of permutations $\sigma \in S_n$ such that σ has no cycles of length j for $j \leq k$. Then given a $\sigma \in \mathcal{D}_{n,1}$, Garsia and Remmel arranged the cycles of σ so that the second smallest element in each cycle is on the right and the cycles are ordered from left to right by increasing second smallest elements. We shall refer to such an arrangement of cycles of $\sigma \in \mathcal{D}_{n,1}$ as the 1-*standard order* of σ. For example, $\sigma = (3,1,11,2)(10,4,5)(9,8,12,6,13,7)$ is in 1-standard order. Having written σ in 1-standard order, Garsia and Remmel then set $\overline{\sigma}$ to be the permutation in one line notation that results from the 1-standard order of σ by erasing the parentheses and commas. Thus in our case, $\overline{\sigma} = 3\ 1\ 11\ 2\ 10\ 4\ 5\ 9\ 8\ 12\ 6\ 13\ 7$.

Garsia and Remmel defined their q-analogue of the derangement numbers by setting

$$D_{n,1}(q) = \sum_{\sigma \in \mathcal{D}_{n,1}} q^{inv(\overline{\sigma})}, \tag{1.3}$$

where for any $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $inv(\sigma) = \left|\left\{1 \leq i < j \leq n \mid \sigma_i > \sigma_j\right\}\right|$ denotes the number of inversions of σ. With $D_{1,1}(q) = 0$ and $D_{2,1}(q) = 1$ following immediately from this definition, Garsia and Remmel then proved the following q-analogues of (1.1) and (1.2):

$$D_{n+1,1}(q) = q[n]_q D_{n,1}(q) + [n]_q D_{n-1,1}(q), \quad \text{for } n \geq 2 \tag{1.4}$$

and

$$D_{n+1,1}(q) = [n+1]_q D_{n,1}(q) + (-1)^{n+1}, \quad \text{for } n \geq 1, \tag{1.5}$$

where $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$ is the usual q-analogue of n.

In the same paper, Garsia and Remmel defined a second q-analogue of the derangement numbers by setting

$$\overline{D}_{n,1}(q) = \sum_{\sigma \in \mathcal{D}_{n,1}} q^{coinv(\overline{\sigma})}, \tag{1.6}$$

where for $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $coinv(\sigma)$ equals the number of pairs $1 \leq i < j \leq n$ such that $\sigma_i < \sigma_j$. Again, with $\overline{D}_{1,1}(q) = 0$ and $\overline{D}_{2,1}(q) = q$ following immediately from (1.6), Garsia and Remmel showed that for $n \geq 2$,

$$\overline{D}_{n+1,1}(q) = [n]_q \overline{D}_{n,1}(q) + q^n [n]_q \overline{D}_{n-1,1}(q), \tag{1.7}$$

and for $n \geq 1$,

$$\overline{D}_{n+1,1}(q) = [n+1]_q \overline{D}_{n,1}(q) + (-1)^{n+1} q^{\binom{n+1}{2}}. \tag{1.8}$$

Generalizing Garsia and Remmel's definition in (1.3), one can define an obvious p,q-analogue of the derangement numbers by setting

$$D_{n,1}(p,q) = \sum_{\sigma \in \mathcal{D}_{n,1}} q^{inv(\overline{\sigma})} p^{coinv(\overline{\sigma})}. \tag{1.9}$$

From this definition, it is easy to see that $D_{1,1}(p,q)=0$ and $D_{2,1}(p,q)=p$. Furthermore, with the usual p,q-analogue of n given by $[n]_{p,q}=p^{n-1}+qp^{n-2}+\cdots+q^{n-2}p+q^{n-1}=\frac{p^n-q^n}{p-q}$, we claim that the $D_{n,1}(p,q)$ given in (1.9) satisfy the following p,q-analogue of (1.1)

$$D_{n+1,1}(p,q)=q[n]_{p,q}D_{n,1}(p,q)+p^n[n]_{p,q}D_{n-1,1}(p,q), \tag{1.10}$$

for $n\geq 2$. In order to prove (1.10), we classify the elements of $\mathcal{D}_{n+1,1}$ according to whether $n+1$ lies in a cycle of length j for $j\geq 3$ or $n+1$ lies in a cycle of length 2. If $\sigma\in\mathcal{D}_{n+1,1}$ is such that $n+1$ lies in a j-cycle where $j\geq 3$, then we can remove $n+1$ from its cycle to get a permutation $\sigma'\in\mathcal{D}_{n,1}$. For example, if $\sigma=(3,1,11,2)(10,4,5)(9,8,12,6,13,7)\in\mathcal{D}_{13,1}$ is written in 1-standard order, then $\sigma'=(3,1,11,2)(10,4,5)(9,8,12,6,7)\in\mathcal{D}_{12,1}$. In such a situation, since $n+1$ is not the second smallest element in its cycle, the result of removing $n+1$ from the cycle structure of σ will leave a permutation σ' whose cycle structure is still in 1-standard order. Moreover, it is easy to see that for each such σ', there are n permutations $\tau\in\mathcal{D}_{n+1,1}$ which yield σ' upon removing $n+1$ from τ. That is, each such τ is the result of inserting $n+1$ directly in front of some element of σ' in its cycle structure.

It is easy to see that if τ_i is the result of inserting $n+1$ directly in front of the i-th element of σ' reading from left to right in the 1-standard order of σ', then τ_i is in 1-standard order and

$$inv(\overline{\tau_i})=n-i+1+inv\left(\overline{\sigma'}\right) \quad \text{and} \quad coinv\left(\overline{\tau_i}\right)=i-1+coinv\left(\overline{\sigma'}\right).$$

Therefore, it easily follows that

$$\begin{aligned}
&\sum_{\substack{\tau\in\mathcal{D}_{n+1,1}\\ n+1 \text{ is in a } j\text{-cycle for } j\geq 3}} q^{inv(\overline{\tau})}p^{coinv(\overline{\tau})}\\
&\quad=\left(q^n+q^{n-1}p+\cdots+q^2p^{n-2}+qp^{n-1}\right)\sum_{\sigma\in\mathcal{D}_{n,1}} q^{inv(\overline{\sigma})}p^{coinv(\overline{\sigma})}\\
&\quad=q[n]_{p,q}D_{n,1}(p,q).
\end{aligned}$$

On the other hand, if $\sigma\in\mathcal{D}_{n+1,1}$ is such that $n+1$ is in a 2-cycle, then there must be some $i\in\{1,\ldots,n\}$ such that $(i,n+1)$ is a 2-cycle in σ. Moreover, in the 1-standard order of σ, the last cycle of σ is precisely $(i,n+1)$ since $n+1$ is the second smallest element of the cycle $(i,n+1)$ and the cycles are ordered from left to right by increasing second smallest elements. Given such a σ, we can obtain an element of $\sigma''\in\mathcal{D}_{n-1,1}$ by removing the cycle $(i,n+1)$ and replacing the numbers $i+1,i+2,\ldots,n$ in the rest of cycle structure by $i,i+1,\ldots,n-1$ respectively. For example, if $\sigma=(3,1,6,11,2)(10,4,5)(9,12,7)(8,13)$, then $\sigma''=(3,1,6,10,2)(9,4,5)(8,11,7)$. Moreover, if we are given $\sigma''\in\mathcal{D}_{n-1,1}$, then there are n permutations $\tau\in\mathcal{D}_{n+1,1}$ such that $n+1$ is in a 2-cycle and $\tau''=\sigma''$ depending on which number $i\in\{1,\ldots,n\}$ is in the 2-cycle with $n+1$. Moreover, it is easy to see that if $(i,n+1)$ is a 2-cycle of τ and $\tau''=\sigma''$, then

$$q^{inv(\overline{\tau})}p^{coinv(\overline{\tau})}=q^{n-i}p^{i-1}p^nq^{inv\left(\overline{\sigma''}\right)}p^{coinv\left(\overline{\sigma''}\right)},$$

so that

$$\begin{aligned}
&\sum_{\substack{\tau\in\mathcal{D}_{n+1,1}\\ n+1 \text{ is in a 2-cycle}}} q^{inv(\overline{\tau})}p^{coinv(\overline{\tau})}\\
&\qquad = p^n\left(q^{n-1}+q^{n-2}p+\cdots+qp^{n-2}+p^{n-1}\right)\sum_{\sigma\in\mathcal{D}_{n-1,1}} q^{inv(\overline{\sigma})}p^{coinv(\overline{\sigma})}\\
&\qquad = p^n[n]_{p,q}D_{n-1,1}(p,q).
\end{aligned}$$

Thus (1.10) holds as claimed.

Next, we claim that for $n\geq 1$, $D_{n,1}(p,q)$ satisfy the following p,q-analogue of (1.2):

$$D_{n+1,1}(p,q)=[n+1]_{p,q}D_{n,1}(p,q)+(-1)^{n+1}p^{\binom{n+1}{2}}. \tag{1.11}$$

We will prove (1.11) by induction on n. First, one can easily verify that (1.11) holds when $n=1$ and assuming that (1.11) holds for $n<r$, then

$$\begin{aligned}
D_{r+1,1}(p,q) &= q[r]_{p,q}D_{r,1}(p,q)+p^r[r]_{p,q}D_{r-1,1}(p,q)\\
&= q[r]_{p,q}D_{r,1}(p,q)+p^r\left(D_{r,1}(p,q)-(-1)^r p^{\binom{r}{2}}\right)\\
&= (p^r+q[r]_{p,q})D_{r,1}(p,q)+(-1)^{r+1}p^{\binom{r+1}{2}}\\
&= [r+1]_{p,q}D_{r,1}(p,q)+(-1)^{r+1}p^{\binom{r+1}{2}}.
\end{aligned}$$

For the generalized derangement numbers, $D_{n,k}$, we note that when $k\geq 1$, there are again two natural recursions. First, one can easily derive the recursion

$$D_{n+1,k}=nD_{n,k}+(n)\downarrow_k D_{n-k,k}, \tag{1.12}$$

where $(n)\downarrow_k = n(n-1)\cdots(n-k+1)$, by classifying the elements of $\mathcal{D}_{n+1,k}$ according to whether $n+1$ is in a j-cycle for $j>k+1$ or whether $n+1$ is in a $(k+1)$-cycle. The second recursion on $D_{n,k}$ can easily be obtained from the following application of the theory of exponential structures:

$$\sum_{n=0}^{\infty}\frac{D_{n,k}t^n}{n!}=e^{\sum_{m\geq k+1}\frac{(m-1)!t^m}{m!}}. \tag{1.13}$$

From (1.13), we obtain

$$\sum_{n=0}^{\infty}\frac{D_{n,k}t^n}{n!}=e^{\sum_{m\geq k+1}\frac{t^m}{m}}=e^{ln(1/(1-t))-\left(t+\frac{t^2}{2}+\cdots+\frac{t^k}{k}\right)}=\frac{e^{-t}e^{-\frac{t^2}{2}}\cdots e^{-\frac{t^k}{k}}}{1-t}. \tag{1.14}$$

Multiplying both sides of (1.14) by $1-t$ and expanding the exponential functions gives

$$(1-t)\sum_{n=0}^{\infty}\frac{D_{n,k}t^n}{n!}=e^{-t}e^{-\frac{t^2}{2}}\cdots e^{-\frac{t^k}{k}}=\prod_{i=1}^{k}\sum_{a_i\geq 0}\frac{(-1)^{a_i}t^{ia_i}}{i^{a_i}(a_i!)}. \tag{1.15}$$

Then, taking the coefficient of t^{n+1} on both sides of (1.15) and multiplying by $(n+1)!$ yields

$$D_{n+1,k} - (n+1)D_{n,k} = \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{a_1+\cdots+a_k} \frac{(n+1)!}{(1^{a_1}a_1!)\,(2^{a_2}a_2!)\cdots(k^{a_k}a_k!)}. \tag{1.16}$$

Solving for $D_{n+1,k}$ in (1.16) gives

$$\begin{aligned} D_{n+1,k} &= (n+1)D_{n,k} \\ &\quad + \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{a_1+\cdots+a_k} \binom{n+1}{a_1,\,2a_2,\dots,\,ka_k} \prod_{j=1}^{k} Fact(a_j,\,j), \end{aligned} \tag{1.17}$$

where

$$Fact(a_j,\,j) = \frac{(ja_j)!}{j^{a_j}\,(a_j!)} = \prod_{s=0}^{a_j-1} (sj+1)(sj+2)\cdots(sj+j-1). \tag{1.18}$$

The main goal of this paper is to define a p, q-analogue of the generalized derangement numbers $D_{n,k}$ which yields natural p, q-analogues of recursions (1.12) and (1.17). The outline of this paper is as follows. In Section 2, we follow a suggestion of Garsia and Remmel [4] and define a p, q-analogue of the generalized derangement numbers, which we denote by $D_{n,k}(p,\,q)$, so that the following p, q-analogue of (1.12) holds:

$$D_{n+1,k}(p,\,q) = q[n]_{p,q}D_{n,k}(p,\,q) + p^n[n]_{p,q}\downarrow_k D_{n-k,k}(p,\,q)$$

where $[n]_{p,q}\downarrow_k = [n]_{p,q}[n-1]_{p,q}\cdots[n-k+1]_{p,q}$. However, we shall see that a p, q-analogue of (1.17) is not as straightforward. In Section 3, we will consider the special case when $k=2$ in which (1.17) becomes

$$\begin{aligned} D_{n+1,2} &= (n+1)D_{n,2} + \sum_{\substack{a_1,a_2 \geq 0 \\ a_1+2a_2=n+1}} (-1)^{a_1+a_2} \binom{n+1}{a_1,\,2a_2} Fact(a_2,\,2) \\ &= (n+1)D_{n,2} + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j \binom{n+1}{2j} 1\cdot 3\cdots(2j-1). \end{aligned}$$

One might hope that the p, q-analogue of (1.17) would be of the form

$$\begin{aligned} D_{n+1,2}(p,\,q) &= q^{a_n}p^{b_n}[n+1]_{p,q}D_{n,2}(p,\,q) \\ &\quad + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j q^{c_{n,j}} p^{d_{n,j}} \end{aligned}$$

$$\cdot \begin{bmatrix} n+1 \\ 2j \end{bmatrix}_{p,q} [1]_{p,q} \cdot [3]_{p,q} \cdots [2j-1]_{p,q} \tag{1.19}$$

for appropriate choices of a_n, b_n, $c_{n,j}$, and $d_{n,j}$. However, we will show in Section 3 that this is not possible. Instead, we will show that the p,q-analogue of $\sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j \binom{n+1}{2j} 1 \cdot 3 \cdots (2j-1)$ arises by p,q-counting a certain set of words counted by $\sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j \binom{n+1}{2j} 1 \cdot 3 \cdots (2j-1)$. Finally, in Section 4, we shall show how the arguments of Section 2 can be generalized to give p, q-analogues of recursions (1.12) and (1.17) for an arbitrary integer $k \geq 1$.

2. p, q-Analogues of $D_{n,k}$

In this section, we describe our p, q-analogues of the generalized derangement numbers. For a fixed $k \geq 1$, arrange the cycles of each $\sigma \in \mathcal{D}_{n,k}$ so that

(1) each cycle of σ is arranged with the $(k+1)$-th smallest element on the right and
(2) the cycles are ordered from left to right by increasing $(k+1)$-th smallest elements.

We will refer to such an arrangement of σ as the *k-standard order* of σ. For example, suppose $k = 3$ and

$$\sigma = (1, 4, 14, 11)(2, 6, 5, 3, 15)(7, 13, 8, 12, 9, 10) \in \mathcal{D}_{15,3}.$$

Then, the 3-standard order of the σ is given by

$$(5, 3, 15, 2, 6)(7, 13, 8, 12, 9, 10)(11, 1, 4, 14).$$

Next, for each $\sigma \in \mathcal{D}_{n,k}$, we let $\sigma^{(k)}$ denote the permutation that results by starting with the k-standard order of σ and erasing the parentheses and commas. In our example,

$$\sigma^{(3)} = 5\ 3\ 15\ 2\ 6\ 7\ 13\ 8\ 12\ 9\ 10\ 11\ 1\ 4\ 14.$$

Definition 2.1. *For each natural number n and k, we define the p, q-analogue of $D_{n,k}$ by*

$$D_{n,k}(p, q) = \sum_{\sigma \in \mathcal{D}_{n,k}} q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)}. \tag{2.1}$$

Our first theorem shows that our definition of $D_{n,k}(p, q)$ satisfies a natural p, q-analogue of (1.12).

Theorem 2.2. *If $k \geq 1$, then*

(1) $D_{n,k}(p, q) = 0$, *for* $1 \leq n \leq k$,
(2) $D_{k+1,k}(p, q) = p^k [k]_{p,q}!$, *and*
(3) $D_{n+1,k}(p, q) = q[n]_{p,q} D_{n,k}(p, q) + p^n [n]_{p,q} \downarrow_k D_{n-k,k}(p, q)$,

where $[n]_{p,q} \downarrow_k = [n]_{p,q} [n-1]_{p,q} \cdots [n-k+1]_{p,q}$ *and* $[k]_{p,q}! = [k]_{p,q} \downarrow_k$.

Proof. (1) is trivial since $\mathcal{D}_{n,k}$ is empty for $n \leq k$.

For (2), note that the only elements of $\mathcal{D}_{k+1,k}$ are the permutations $\sigma \in S_{k+1}$ consisting of a single $(k+1)$-cycle. Under our k-standard ordering of cycles, the $(k+1)$-th smallest element in each cycle must be on the right. That is, for each $\sigma \in \mathcal{D}_{k+1,k}$, $\sigma^{(k)}$ has the form $\sigma^{(k)} = \sigma_1 \cdots \sigma_k \ (k+1)$, for some $\sigma_1 \cdots \sigma_k \in S_k$. Moreover, for each $\sigma \in \mathcal{D}_{k+1,k}$, it is easy to see that $q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)} = p^k q^{inv(\alpha)} p^{coinv(\alpha)}$ where $\alpha = \sigma_1 \cdots \sigma_k \in S_k$. Thus, (2) follows immediately from MacMahon's result that $\sum_{\sigma \in S_n} q^{inv(\sigma)} p^{coinv(\sigma)} = [n]_{p,q}!$.

For (3), we classify the elements of $\mathcal{D}_{n+1,k}$ according to whether $n+1$ lies in a cycle of length j for $j \geq k+2$ or $n+1$ lies in a cycle of length $k+1$. If $\sigma \in \mathcal{D}_{n+1,k}$ is such that $n+1$ lies in a j-cycle where $j \geq k+2$, then we can remove $n+1$ from σ to get a permutation $\sigma' \in \mathcal{D}_{n,k}$.

We note that in such a situation, $n+1$ is not the $(k+1)$-th smallest element in its cycle. So, removing $n+1$ from the cycle structure of σ will leave a permutation $\sigma' \in \mathcal{D}_{n,k}$ whose cycle structure is still in $(k+1)$-standard order. As in the $k=1$ case, it is easy to see that each σ' arises from n different $\tau_i \in \mathcal{D}_{n+1,k}$ by removing $n+1$ from τ_i. In particular, for $1 \leq i \leq n$, τ_i is the permutation obtained from σ' by inserting $n+1$ directly in front of the i-th element of σ' reading from left to right in the k-standard order of σ'. We note that for each $1 \leq i \leq n$, τ_i will still be in k-standard order and

$$inv\left(\tau_i^{(k)}\right) = n-i+1+inv\left(\left(\sigma'\right)^{(k)}\right) \quad \text{and} \quad coinv\left(\tau_i^{(k)}\right) = i-1+coinv\left(\left(\sigma'\right)^{(k)}\right).$$

It easily follows that

$$\begin{aligned}
&\sum_{\substack{\tau \in \mathcal{D}_{n+1,k} \\ n+1 \text{ is in a } j\text{-cycle for } j \geq k+2}} q^{inv\left(\tau^{(k)}\right)} p^{coinv\left(\tau^{(k)}\right)} \\
&\qquad = \left(q^n + q^{n-1}p + \cdots + q^2 p^{n-2} + q p^{n-1}\right) \sum_{\sigma \in \mathcal{D}_{n,k}} q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)} \\
&\qquad = q[n]_{p,q} D_{n,k}(p,q).
\end{aligned}$$

On the other hand, if $\sigma \in \mathcal{D}_{n+1,k}$ is such that $n+1$ is in a $(k+1)$-cycle, then there must be some sequence $i_1, \ldots, i_k$ of elements in $\{1, \ldots, n\}$ such that $(i_1, \ldots, i_k, n+1)$ is a $(k+1)$-cycle in σ. Moreover, in the $(k+1)$-standard order of σ, the last cycle of σ is precisely $(i_1, \ldots, i_k, n+1)$ since $n+1$ is the $(k+1)$-th smallest element of its cycle and we are ordering the cycles from left to right by increasing $(k+1)$-th smallest elements. Given such a σ, we can obtain an element $\sigma'' \in \mathcal{D}_{n-k,k}$ by removing the cycle $(i_1, \ldots, i_k, n+1)$ and replacing the remaining elements $\{j_1 < \cdots < j_{n-k}\} = \{1, \ldots, n\} - \{i_1, \ldots, i_k\}$ respectively by $1, \ldots, n-k$ in the rest of the cycle structure. For example, if $\sigma = (3,1,2,15,6)(10,4,5,11)(9,12,7,16)(14,8,13,17)$ is an element of $\mathcal{D}_{17,3}$, whose cycles are written in 3-standard order, then $\sigma'' = (3,1,2,12,6)$ $(9,4,5,10)(8,11,7,13)$. Moreover, for each $\sigma'' \in \mathcal{D}_{n-k,k}$, there are $n\downarrow_k$ permutations $\tau \in \mathcal{D}_{n+1,k}$ such that $n+1$ is in a $(k+1)$-cycle and $\tau'' = \sigma''$ depending on which sequences of numbers $i_1,\ldots,i_k \in \{1,\ldots,n\}$ are in the $(k+1)$-cycle $(i_1,\ldots,i_k,n+1)$ with $n+1$.

Now suppose that we are given $\sigma \in \mathcal{D}_{n-k,k}$ whose cycles are written in k-standard order. We would like to compute

$$S = \sum_{\tau} q^{inv\left(\tau^{(k)}\right)} p^{coinv\left(\tau^{(k)}\right)}, \tag{2.2}$$

where the sum runs over all $\tau \in \mathcal{D}_{n+1,k}$ such that $n+1$ is in a $(k+1)$-cycle and $\tau'' = \sigma$. Suppose that τ contains the cycle $(i_1,\ldots,i_k,n+1)$. Then since τ'' equals σ, we can see that by p,q-enumerating the elements in the cycles preceding $(i_1,\ldots,i_k,n+1)$ by inversions and coinversions we obtain a factor of $q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)}$ to S. Moreover, $n+1$ contributes no inversions and n coinversions to $\tau^{(k)}$ and thus contributes a factor of p^n to S. The remaining contribution to S comes from our choices of $i_1,\ldots,i_k$ from $\{1,\ldots,n\}$. Note that i_k contributes $n-i_k$ inversions and i_k-1 coinversions to the elements preceding i_k in $\tau^{(k)}$, and hence, the choice of i_k gives a contribution of $q^{n-1}+pq^{n-2}+\cdots+qp^{n-2}+p^{n-1} = [n]_{p,q}$ to S. Fixing our choice of i_k, we can repeat the same argument to show that if we count the number of inversions and coinversions with all the elements preceding i_{k-1} in $\tau^{(k)}$ over all choices of i_{k-1}, then we get a contribution $q^{n-2}+pq^{n-3}+\cdots+qp^{n-3}+p^{n-2} = [n-1]_{p,q}$ to S. Continuing in this way, we see that

$$S = q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)} p^n [n]_{p,q}[n-1]_{p,q}\cdots[n-k+1]_{p,q}. \tag{2.3}$$

Therefore, it follows that

$$\begin{aligned}\sum_{\substack{\tau\in\mathcal{D}_{n+1,k}\\ n+1 \text{ is in a } k+1\text{-cycle}}} q^{inv\left(\tau^{(k)}\right)} p^{coinv\left(\tau^{(k)}\right)} &= p^n [n]_{p,q}\downarrow_k \sum_{\sigma\in\mathcal{D}_{n-k,k}} q^{inv\left(\sigma^{(k)}\right)} p^{coinv\left(\sigma^{(k)}\right)} \\ &= p^n [n]_{p,q}\downarrow_k D_{n-k,k}(p,q).\end{aligned}$$

Thus, $D_{n+1,k}(p,q) = q[n]_{p,q}D_{n,k}(p,q) + p^n[n]_{p,q}\downarrow_k D_{n-k,k}(p,q)$ as claimed. ∎

3. A Second Recursion for $D_{n,2}(p,q)$

Our p,q-analogue of the recursion in (1.17) will be given in the next section. In this section, we would like to motivate that recursion by considering the case when $k=2$. From (1.17), it follows that the generalized derangement numbers $D_{n,2}$ satisfy the following recursion for $n \geq 1$,

$$D_{n+1,2} = (n+1)D_{n,2} + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j \binom{n+1}{2j} Fact(j,2), \tag{3.1}$$

where $Fact(0,2) = 1$ and $Fact(n,2) = 1\cdot 3\cdots(2n-1)$ if $n \geq 1$. One might hope that our $D_{n,2}(p,q)$'s would satisfy a recursion like

$$D_{n+1,2}(p,q) = q^{a_n} p^{b_n} [n+1]_{p,q} D_{n,2}(p,q)$$

$$+(-1)^{n+1}\sum_{j=0}^{\lfloor (n+1)/2\rfloor}(-1)^j q^{c_{n,j}}p^{d_{n,j}}\begin{bmatrix} n+1 \\ 2j\end{bmatrix}_{p,q} Fact(j,2)_{p,q}, \quad (3.2)$$

for some a_n, b_n, $c_{n,j}$, and $d_{n,j}$ where $Fact(0,2)_{p,q}=1$ and $Fact(n,2)_{p,q}=[1]_{p,q}\cdot[3]_{p,q}\cdots[2n-1]_{p,q}$ if $n\geq 1$. However, we can show that even in the case $p=1$, this is not possible. That is, suppose we define

$$D_{n,k}(q)=D_{n,k}(1,q)=\sum_{\sigma\in\mathcal{D}_{n,k}} q^{inv\left(\sigma^{(k)}\right)}.$$

Then, we will show that there are no a_n and $c_{n,j}$ for which

$$D_{n+1,2}(q)=q^{a_n}[n+1]_q D_{n,2}(q)+(-1)^{n+1}\sum_{j=0}^{\lfloor (n+1)/2\rfloor}(-1)^j q^{c_{n,j}}\begin{bmatrix} n+1 \\ 2j\end{bmatrix}_{q} Fact(j,2)_{q}, \quad (3.3)$$

where $Fact(0,2)_q=1$ and $Fact(n,2)_q=[1]_q\cdot[3]_q\cdots[2n-1]_q$ if $n\geq 1$. One can easily see that $D_{1,2}=D_{2,2}=0$ since $\mathcal{D}_{1,2}=\mathcal{D}_{2,2}=\emptyset$. Furthermore, there are 2 elements of $\mathcal{D}_{3,2}$, $(1,2,3)$ and $(2,1,3)$, written in 2-standard order. As such, $D_{3,2}(q)=q^{inv(1\,2\,3)}+q^{inv(2\,1\,3)}=1+q=[2]_q$. We can compute $D_{4,2}(q)$ and $D_{5,2}(q)$ using the following recursion for $D_{n,k}(q)$ obtained from Theorem 2.2 by setting $p=1$:

$$D_{n+1,k}(q)=q[n]_q D_{n,k}(q)+([n]_q)\downarrow_k D_{n-k,k}(q).$$

That is,

$$D_{4,2}(q)=q[3]_q D_{3,2}(q)+[3]_q[2]_q D_{1,2}=q[3]_q[2]_q$$

and

$$D_{5,2}(q)=q[4]_q D_{4,2}(q)+[4]_q[3]_q D_{2,2}=q^2[4]_q[3]_q[2]_q.$$

In the case $n+1=5$, (3.3) becomes

$$D_{5,2}(q)=q^{a_4}[5]_q D_{4,2}(q)+(-1)^5\left(q^{c_{4,0}}-q^{c_{4,1}}\begin{bmatrix}5\\2\end{bmatrix}_q+q^{c_{4,2}}\begin{bmatrix}5\\4\end{bmatrix}_q[3]_q\right). \quad (3.4)$$

Since $\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right]_q=[5]_q\left(1+q^2\right)$, $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]_q=[5]_q$, $D_{4,2}(q)=q[3]_q[2]_q=\left(q+2q^2+2q^3+q^4\right)$, and $D_{5,2}(q)=q^2[4]_q[3]_q[2]_q=\left(q^2+3q^3+5q^4+6q^5+5q^6+3q^7+q^8\right)$, we can rewrite (3.4) as

$$\begin{aligned} &q^{c_{4,0}}+q^2+3q^3+5q^4+6q^5+5q^6+3q^7+q^8 \\ &=[5]_q\left(q^{a_4}\left(q+2q^2+2q^3+q^4\right)+q^{c_{4,1}}\left(1+q^2\right)-q^{c_{4,2}}\left(1+q+q^2\right)\right). \quad (3.5)\end{aligned}$$

By setting $E=\left(q^2+3q^3+5q^4+6q^5+5q^6+3q^7+q^8\right)$ and $F=\left(q^{a_4}\left(q+2q^2+2q^3+q^4\right)+q^{c_{4,1}}\left(1+q^2\right)-q^{c_{4,2}}\left(1+q+q^2\right)\right)$, we can rewrite (3.5) as $q^{c_{4,0}}+E=[5]_qF$.

Now suppose $c_{4,0}\geq 11$. Then we must have that $E\big|_{q^j}=([5]_qF)\big|_{q^j}$ for $j\leq 10$ where for any q-series $f(q)=\sum_{n\geq 0}f_nq^n$, we let $f|_{q^j}=f_j$. By setting $E=\sum_{i\geq 0}E_iq^i$ and $F=\sum_{i\geq 0}F_iq^i$, it must follow that

$$E_0=F_0=0,$$

$$E_1 = F_0 + F_1 = 0,$$
$$E_2 = F_0 + F_1 + F_2 = 1,$$
$$E_3 = F_0 + F_1 + F_2 + F_3 = 3,$$
$$E_4 = F_0 + F_1 + F_2 + F_3 + F_4 = 5,$$
$$E_5 = F_1 + F_2 + F_3 + F_4 + F_5 = 6,$$
$$E_6 = F_2 + F_3 + F_4 + F_5 + F_6 = 5,$$
$$E_7 = F_3 + F_4 + F_5 + F_6 + F_7 = 3,$$
$$E_8 = F_4 + F_5 + F_6 + F_7 + F_8 = 1,$$
$$E_9 = F_5 + F_6 + F_7 + F_8 + F_9 = 0,$$
$$E_{10} = F_6 + F_7 + F_8 + F_9 + F_{10} = 0.$$

The unique solution to this system of equations is given by $F_0 = 0, F_1 = 0, F_2 = 1, F_3 = 2, F_4 = 2, F_5 = 1, F_6 = -1, F_7 = -1, F_8 = 0, F_9 = 1, F_{10} = 1$.

It is easy to see that there is no choice a_4, $c_{4,1}$, and $c_{4,2}$ which satisfy

$$\begin{aligned} &q^2 + 2q^3 + 2q^4 + q^5 - q^6 - q^7 + q^9 + q^{10} \\ &= \left(q^{a_4}\left(q + 2q^2 + 2q^3 + q^4\right) + q^{c_{4,1}}\left(1 + q^2\right) - q^{c_{4,2}}\left(1 + q + q^2\right)\right). \end{aligned}$$

As such, we must conclude that $c_{4,0} \leq 10$. However, one can use any computer algebra package to see that if we set $G_i = q^i + E$ for $0 \leq i \leq 10$, then G_i is not divisible by $[5]_q$. Thus, (3.5) has no solution.

Since we've shown that our obvious guess in (3.2) does not even work when $p = 1$, we must wonder *"What is a p,q-analogue of recursion (3.1)?"* To answer this question, we first develop a p,q-analogue of $\binom{n+1}{2j} Fact(j,2)$. To this end, let's consider the set $\mathcal{R}\left(0^{n+1-2j}, 1^2, \ldots, j^2\right)$ of all rearrangements of $(n+1-2j)$ 0's, two 1's, two 2's,..., two j's. Given $r = r_1 \cdots r_{n+1} \in \mathcal{R}\left(0^{n+1-2j}, 1^2, \ldots, j^2\right)$, we let $Fin(r) = \{i\colon r_i > 0 \ \&\ r_i \notin \{r_{i+1} \cdots r_{n+1}\}\}$. That is, $Fin(r)$ is the set of indices of the last occurrences of $1, \ldots, j$ in r. For example, if $n = 10$, $j = 3$, and

$$r = 0\,1\,3\,0\,2\,0\,0\,3\,0\,1\,2,$$

then $Fin(r) = \{8, 10, 11\}$. Now if $r = r_1 \cdots r_{n+1} \in \mathcal{R}\left(0^{n+1-2j}, 1^2, \ldots, j^2\right)$ and $Fin(r) = \left\{i_1 < \cdots < i_j\right\}$, then we let $fin(r) = r_{i_1} r_{i_2} \cdots r_{i_j}$ and note that $fin(r)$ will always be a permutation in S_j. In our example, $fin(r) = 3\,1\,2$.

Next, we define

$$\mathcal{T}_{n+1,j} = \left\{r \in \mathcal{R}\left(0^{n+1-2j}, 1^2, \ldots, j^2\right) : fin(r) = 1\,2 \cdots j\right\}. \tag{3.6}$$

Our next result will show that the cardinality of $\mathcal{T}_{n+1,j}$ is $\binom{n+1}{2j} Fact(j,2)$.

Theorem 3.1. *For $n \geq 1$ and $j \in \{0, \ldots, \lfloor (n+1)/2 \rfloor\}$,*

$$T_{n+1,j} = \left|\mathcal{T}_{n+1,j}\right| = \binom{n+1}{2j} Fact(j,2). \tag{3.7}$$

Proof. First, we note that (3.7) follows when $j = 0$ since $\mathcal{T}_{n+1,0} = \mathcal{R}\left(0^{n+1}\right)$ and $\binom{n+1}{0} Fact(0,2) = 1$. Similarly if $j = 1$, then $\mathcal{T}_{n+1,1} = \mathcal{R}\left(0^{n-1}, 1^2\right)$. In this case, we can construct an element $r = r_1 \cdots r_{n+1} \in \mathcal{T}_{n+1,1}$ in $\binom{n+1}{2}$ ways by choosing the positions of the two 1's in r. Therefore, $T_{n+1,1} = \binom{n+1}{2} = \binom{n+1}{2} Fact(1,2)$.

Now suppose that $j \geq 2$. We can construct an element $r = r_1 \cdots r_{n+1} \in \mathcal{T}_{n+1,j}$ as follows. First choose the positions $1 \leq p_1 < p_2 < \cdots < p_{2j} \leq n+1$ of the nonzero elements of r in $\binom{n+1}{2j}$ ways. Since $fin(r) = 12\cdots j$, it must follow that $r_{p_{2j}} = j$, leaving $2j-1$ positions for the leftmost occurrence of j in r. Having fixed the position of the leftmost occurrence of j in r, we know that the rightmost unassigned value in r must be $j-1$ leaving $2j-3$ positions for the leftmost occurrence of $j-1$ in r. Continuing in this way, we see that

$$T_{n+1,j} = \binom{n+1}{2j}(2j-1)(2j-3)\cdots 3 \cdot 1 = \binom{n+1}{2j} Fact(j,2),$$

as desired. ∎

Given (3.7), we can now rewrite the recursion in (3.1) as

$$D_{n+1,2} = (n+1)D_{n,2} + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n+1,j}. \tag{3.8}$$

In order to obtain a q- and p, q-analogue of (3.8), we must define a q-analogue and p, q-analogue of $T_{n+1,j}$. To this end, let's define for each $r \in \mathcal{T}_{n+1,j}$ and $1 \leq i \leq j$,

$$(i,i)_r = r_{j+1} \cdots r_{k-1},$$

where $j < k$ and $r_j = r_k = i$. We then let $(i,i)_{r,<i}$ be the number of elements r_a in the sequence $(i,i)_r$ such that $r_a < i$. That is, $(i,i)_r$ is the interval between the two occurrences of i in r and $(i,i)_{r,<i}$ is the number of elements of r which are less than i and fall between the two occurrences of i in r. Finally, we define for each $r \in \mathcal{T}_{n+1,j}$,

$$\theta(r) = \sum_{i=1}^{j} (i,i)_{r,<i}. \tag{3.9}$$

This given, we define our q-analogue of $T_{n+1,j}$ for $0 \leq j \leq \lfloor (n+1)/2 \rfloor$ to be

$$T_{n+1,j}(q) = \sum_{r \in \mathcal{T}_{n+1,j}} q^{\theta(r)},$$

and our p, q-analogue of $T_{n+1,j}$ for $0 \leq j \leq \lfloor (n+1)/2 \rfloor$ to be

$$T_{n+1,j}(p,q) = p^{\binom{n+1}{2}} T_{n+1,j}(q/p).$$

By making the convention that $T_{n,j} = T_{n,j}(q) = T_{n,j}(p,q) = 0$ if $j < 0$, we find that the $T_{n+1,j}(q)$'s and $T_{n+1,j}(p,q)$'s satisfy simple recursions given in the following theorem.

Theorem 3.2. *For all $n \geq 2$ and $0 \leq j \leq \lfloor (n+1)/2 \rfloor$,*

$$T_{n+1,j}(q) = T_{n,j}(q) + [n]_q T_{n-1,j-1}(q) \tag{3.10}$$

and

$$T_{n+1,j}(p,q) = p^n T_{n,j}(p,q) + p^n [n]_{p,q} T_{n-1,j-1}(p,q). \tag{3.11}$$

Proof. Note that when $j=0$, $\mathcal{T}_{n+1,0} = \left\{0^{n+1}\right\}$ for all $n \geq 0$. So by our definition, $T_{n+1,0}(q) = 1$ for all $n \geq 0$ and, hence,

$$T_{n+1,0}(p,q) = p^{\binom{n+1}{2}} T_{n+1,0}(q/p) = p^{\binom{n+1}{2}} = p^n T_{n,0}(p,q) + [0]_{p,q} T_{n-1,-1}(p,q)$$

as desired.

Now assume that $j \geq 1$. In this case, we will only prove (3.10) as (3.11) is a simple consequence of (3.10). To prove (3.10), we observe that if $r = r_1 \cdots r_{n+1} \in \mathcal{T}_{n+1,j}$, then either $r_{n+1} = 0$ or $r_{n+1} = j$. Clearly, when $r_{n+1} = 0$, $\theta(r_1 \cdots r_{n+1}) = \theta(r_1 \cdots r_n)$ which implies that $\sum_{r \in \mathcal{T}_{n+1,j}, r_{n+1}=0} q^{\theta(r)} = T_{n,j}(q)$. On the other hand, if $r_{n+1} = j$, let $T_{n+1,j,s,n+1}$ be the set of all $r = r_1 \cdots r_{n+1} \in \mathcal{T}_{n+1,j}$ such that $r_s = r_{n+1} = j$. Clearly if we remove r_s and r_{n+1} from such an r, we will get an element $r' \in \mathcal{T}_{n-1,j-1}$ for which

$$\theta(r) = n - s + \theta\left(r'\right),$$

since all the elements in the interval $(j,j)_r$ are less than j. It easily follows that

$$\begin{aligned}
\sum_{r \in \mathcal{T}_{n+1,j}, r_{n+1}=j} q^{\theta(r)} &= \sum_{s=1}^{n} \sum_{r \in \mathcal{T}_{n+1,j,s,n+1}} q^{\theta(r)} \\
&= \sum_{s=1}^{n} q^{n-s} \sum_{r' \in \mathcal{T}_{n-1,j-1}} q^{\theta\left(r'\right)} \\
&= [n]_q T_{n-1,j-1}(q).
\end{aligned}$$

Thus, (3.10) holds. ∎

Our next result shows that our $D_{n,2}(p,q)$'s satisfy a p,q-analogue of (3.8). That is,

Theorem 3.3. *For all $n \geq 1$,*

$$D_{n+1,2}(p,q) = [n+1]_{p,q} D_{n,2}(p,q) + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n+1,j}(p,q). \tag{3.12}$$

Proof. First, we establish (3.12) when $n=1$ and $n=2$. Clearly $D_{2,2}(p,q) = D_{1,2}(p,q) = 0$ since $\mathcal{D}_{2,2} = \mathcal{D}_{1,2} = \emptyset$. The two elements of $\mathcal{D}_{3,2}$ in 2-standard order are $(1,2,3)$ and $(2,1,3)$. Therefore, it follows that $D_{3,2}(p,q) = p^3 + p^2 q = p^2 [2]_{p,q}$.

To prove (3.12) when $n=1$, we must show that

$$D_{2,2}(p,q) = [2]_{p,q} D_{1,2}(p,q) + (-1)^2 \sum_{j=0}^{1} (-1)^j T_{2,j}(p,q)$$

$$= T_{2,0}(p,q) - T_{2,1}(p,q). \tag{3.13}$$

However, $\mathcal{T}_{2,0} = \{00\}$ and $\mathcal{T}_{2,1} = \{11\}$ so that $T_{2,0}(p,q) = T_{2,1}(p,q) = p^{\binom{2}{2}} = p$. Thus (3.13) holds. Likewise, to prove (3.12) when $n = 2$, we must show that

$$\begin{aligned} D_{3,2}(p,q) &= [3]_{p,q} D_{2,2}(p,q) + (-1)^3 \sum_{j=0}^{1} (-1)^j T_{3,j}(p,q) \\ &= T_{3,1}(p,q) - T_{3,0}(p,q). \end{aligned} \tag{3.14}$$

But, $\mathcal{T}_{3,0} = \{000\}$ and $\mathcal{T}_{3,1} = \{110, 101, 011\}$, so that $T_{3,0}(p,q) = p^{\binom{3}{2}} = p^3$ and $T_{3,1}(p,q) = p^{\binom{3}{2}}(1 + (q/p) + 1) = 2p^3 + qp^2$. Thus (3.14) holds.

Now assume that $n \geq 3$ and (3.12) holds for all $m \leq n$. Using (3.11) and the fact that $T_{m,-1}(p,q) = 0$ for all $m \geq 1$, it follows that for $n \geq 3$,

$$\begin{aligned} &[n+1]_{p,q} D_{n,2}(p,q) + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n+1,j}(p,q) \\ &= q[n]_{p,q} D_{n,2}(p,q) + p^n D_{n,2}(p,q) \\ &\quad + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} p^n (-1)^j \big(T_{n,j}(p,q) + [n]_{p,q} T_{n-1,j-1}(p,q)\big) \\ &= q[n]_{p,q} D_{n,2}(p,q) + p^n \left(D_{n,2}(p,q) - (-1)^n \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n,j}(p,q) \right) \\ &\quad + (-1)^{n+1} p^n [n]_{p,q} \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n-1,j-1}(p,q). \end{aligned} \tag{3.15}$$

By induction, we can assume that

$$D_{n,2}(p,q) - (-1)^n \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n,j}(p,q) = [n]_{p,q} D_{n-1,2}(p,q). \tag{3.16}$$

Using (3.16) in (3.15), we see that

$$\begin{aligned} &[n+1]_{p,q} D_{n,2}(p,q) + (-1)^{n+1} \sum_{j=0}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n+1,j}(p,q) \\ &= q[n]_{p,q} D_{n,2}(p,q) + p^n [n]_{p,q} D_{n-1,2}(p,q) \\ &\quad + (-1)^{n+1} p^n [n]_{p,q} \sum_{j=1}^{\lfloor (n+1)/2 \rfloor} (-1)^j T_{n-1,j-1}(p,q)) \\ &= q[n]_{p,q} D_{n,2}(p,q) \end{aligned}$$

$$+ p^n[n]_{p,q}\left(D_{n-1,2}(p,q) - (-1)^{n-1}\sum_{j=0}^{\lfloor (n-1)/2\rfloor}(-1)^j T_{n-1,j}(p,q)\right). \tag{3.17}$$

Again by induction, we can assume that

$$D_{n-1,2}(p,q) - (-1)^{n-1}\sum_{j=0}^{\lfloor (n-1)/2\rfloor}(-1)^j T_{n-1,j}(p,q) = [n-1]_{p,q}D_{n-2,2}(p,q). \tag{3.18}$$

Thus using (3.18) in (3.17), we find

$$\begin{aligned}
&[n+1]_{p,q}D_{n,2}(p,q) + (-1)^{n+1}\sum_{j=0}^{\lfloor (n+1)/2\rfloor}(-1)^j T_{n+1,j}(p,q)\\
&= q[n]_{p,q}D_{n,2}(p,q) + p^n[n]_{p,q}[n-1]_{p,q}D_{n-2,2}(p,q)\\
&= D_{n+1,2}(p,q),
\end{aligned}$$

where for the last step we have used the fact that Theorem 2.2 holds. ∎

4. The General Case

Suppose that $a_1 + 2a_2 + \cdots + ka_k = n$ and set

$$\mathcal{A}_{n,a_1,a_2,\ldots,a_k} = \{(i,j,a) \colon 1 \le i \le k,\ 1 \le j \le a_i,\ 1 \le a \le i\}.$$

Let $\mathcal{P}_{n,a_1,a_2,\ldots,a_k}$ denote the set of all permutations of the letters in the set $\mathcal{A}_{n,a_1,a_2,\ldots,a_k}$. Then, for each $i \ge 2$ and $r = r_1 \cdots r_n \in \mathcal{P}_{n,a_1,a_2,\ldots,a_k}$, we define

$$Fin(r,i) = \left\{ s \;\middle|\; \begin{array}{l} r_s = (i,j,a) \text{ for some } 1 \le j \le a_i \text{ and } 1 \le a \le i, \text{ and} \\ \text{no letter of the form } (i,j,-) \text{ occurs in } r_{s+1}\cdots r_n \end{array} \right\}. \tag{4.1}$$

If $Fin(r,i) = \{s_1 < s_2 < \cdots < s_{a_i}\}$, then set

$$fin(r,i) = r_{s_1} r_{s_2} \cdots r_{s_{a_i}}. \tag{4.2}$$

Definition 4.1. *Let $\mathcal{T}_{n,a_1,a_2,\ldots,a_k}$ denote the set of all $r \in \mathcal{P}_{n,a_1,a_2,\ldots,a_k}$ such that*

(1) $(1,1,1), (1,2,1), \ldots, (1,a_1,1)$ *is a subsequence of r,*
(2) *for all $2 \le i \le r$, $fin(r,i) = (i,1,i)(i,2,i)\cdots(i,a_i,i)$,*

and set $T_{n,a_1,a_2,\ldots,a_k} = \left|\mathcal{T}_{n,a_1,a_2,\ldots,a_k}\right|$.

Lemma 4.2. *For any natural numbers n and k,*

$$T_{n,a_1,a_2,\ldots,a_k} = \binom{n}{a_1, 2a_2, \ldots, ka_k}\prod_{j=2}^{k} Fact\,(a_j, j).$$

Proof. To construct a word in $\mathcal{T}_{n,a_1,a_2,\ldots,a_k}$, first note that we can select the $a_1, 2a_2, \ldots, ka_k$ positions of the letters of the form $(1,-,-), (2,-,-), \ldots, (k,-,-)$ in $\binom{n}{a_1, 2a_2, \ldots, ka_k}$ ways. Then, by condition (1) in Definition 4.1, there is only one way to place the letters $(1,1,1), (1,2,1), \ldots, (1,a_1,1)$ in the selected positions. Moreover, for each $2 \le j \le k$, condition (2) in Definition 4.1 forces the letter (j, a_j, j) to be placed in the last position among those selected for letters of the form $(j,-,-)$. Once (j, a_j, j) has been placed, there remain $ja_j - 1$ positions to place $(j, a_j, j-1)$, $ja_j - 2$ positions to place $(j, a_j, j-2), \ldots, j(a_j-1)+1$ positions to place $(j, a_j, 1)$. Next, among the remaining $j(a_j-1)$ positions, (j, a_j-1, j) must be placed in the last available position. Once (j, a_j-1, j) has been placed, there remain $j(a_j-1)-1$ positions to place $(j, a_j-1, j-1)$, $j(a_j-1)-2$ positions to place $(j, a_j-1, j-2), \ldots, j(a_j-2)+1$ positions to place $(j, a_j-1, 1)$. Continuing in this way, we find that there are a total of $(ja_j-1)\cdots(j(a_j-1)+1)(j(a_j-1)-1)\cdots(j(a_j-2)+1)(j(a_j-2)-1)\cdots(j+1)(j-1)\cdots 1 = Fact(a_j, j)$ ways to place the letters of the form $(j,-,-)$ in the ja_j selected positions, for each $2 \le j \le k$. ∎

We are now in a position to define a statistic Θ on words in the set $\mathcal{T}_{n,a_1,a_2,\ldots,a_k}$ from which we obtain a q- and p,q-analogue of the value $T_{n,a_1,a_2,\ldots,a_k}$.

Definition 4.3. *Let $r \in \mathcal{T}_{n,a_1,a_2,\ldots,a_k}$ and $i \ge 2$. Then for each $1 \le j \le a_i$, define $\Theta_{i,j}(r)$ as follows*:

(1) *Consider the position s of (i, j, i) in r.*
(2) *Let $\Gamma_{i,j}(r)$ be the word that arises from $r_1 \cdots r_s$ by eliminating all letters (a, b, c) such that a letter of the form $(a, b, -)$ occurs in $r_{s+1} \cdots r_n$. (By our convention, we note that each of the letters $(i, j, 1), \ldots, (i, j, i)$ must occur in $\Gamma_{i,j}(r)$ and the last letter of $\Gamma_{i,j}(r)$ is (i, j, i).)*
(3) *For each $1 \le a \le i-1$, let $c_{>(i,j,a)}(\Gamma_{i,j}(r))$ be the number of letters that follow (i, j, a) in $\Gamma_{i,j}(r)$ that are not of the form (i, j, b) with $b > a$.*
(4) *Set*

$$\Theta_{i,j}(r) = \sum_{a=1}^{i-1} c_{>(i,j,a)}\left(\Gamma_{i,j}(r)\right).$$

We then define $\Theta(r) = \sum_{i=2}^{k}\sum_{j=1}^{a_i} \Theta_{i,j}(r)$ and let

$$T_{n,a_1,a_2,\ldots,a_k}(q) = \sum_{r \in \mathcal{T}_{n,a_1,a_2,\ldots,a_k}} q^{\Theta(r)}. \tag{4.3}$$

Finally, we define

$$T_{n,a_1,a_2,\ldots,a_k}(p,q) = p^{\binom{n}{2}} T_{n,a_1,a_2,\ldots,a_k}(q/p) = \sum_{r \in \mathcal{T}_{n,a_1,a_2,\ldots,a_k}} q^{\Theta(r)} p^{\binom{n}{2}-\Theta(r)}. \tag{4.4}$$

Given Definition 4.3, we now prove a q-analogue of Lemma 4.2.

Theorem 4.4. *For all natural numbers n and k,*

$$T_{n+1,a_1,a_2,\ldots,a_k}(q) = T_{n,a_1-1,a_2,\ldots,a_k}(q) + \sum_{i=2}^{k} [n]_q[n-1]_q \cdots [n-i+2]_q T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}(q). \tag{4.5}$$

Proof. We first note that the contribution to $T_{n+1,a_1,a_2,\ldots,a_k}(q)$ of all words in $\mathcal{T}_{n+1,a_1,a_2,\ldots,a_k}(q)$ which end in $(1, a_1, 1)$ is $T_{n,a_1-1,a_2,\ldots,a_k}(q)$.

Next note that for each $2 \le i \le k$, the term $[n]_q[n-1]_q \cdots [n-i+2]_q \times T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}(q)$ accounts for those words in $\mathcal{T}_{n+1,a_1,a_2,\ldots,a_k}$ which end in (i, a_i, i). To see this, let $\overline{r} \in \mathcal{T}_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}$. We can construct any $r \in \mathcal{T}_{n+1,a_1,a_2,\ldots,a_k}$ from $\overline{r}$ with $r_n = (i, a_i, i)$ by first placing (i, a_i, i) at the end of $\overline{r}$ and then inserting $(i, a_i, 1)$ at any of the $n-i+2$ positions before (i, a_i, i) labelled by $0, 1, \ldots, n-i+1$ as follows:

$$\underline{\ \ }_{0}\, \overline{r}_1 \,\underline{\ \ }_{1}\, \overline{r}_2 \,\underline{\ \ }_{2} \cdots \underline{\ \ }_{n-i}\, \overline{r}_{n-i+1} \,\underline{\ \ }_{n-i+1}\, (i, a_i, i).$$

For each $0 \le j \le n-i+1$, if $(i, a_i, 1)$ is inserted in the j-th position, then $c_{>(i,a_i,1)}(\Gamma_{i,a_i}(r)) = n-i+1-j$ which yields a factor of $[n-i+2]_q$ when summed over all j. Since $(i, a_i, 1)$ would contribute to $c_{>(i,a_i,2)}(\Gamma_{i,a_i}(r))$, it follows similarly that the insertion of $(i, a_i, 2)$ yields a factor of $[n-i+3]_q$. Continuing in this way, we find that the insertion of (i, a_i, a) yields a factor of $[n-i+1+a]_q$ for each $1 \le a \le i-1$, proving the claim. ∎

Corollary 4.5. *For all natural numbers n and k,*

$$T_{n+1,a_1,a_2,\ldots,a_k}(p,q) = p^n T_{n,a_1-1,a_2,\ldots,a_k}(p,q) + \sum_{i=2}^{k} p^n [n]_{p,q} \downarrow_{(i-1)} T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}(p,q). \tag{4.6}$$

Proof. This corollary follows immediately from (4.4) and (4.5). Namely,

$$\begin{aligned}
&T_{n+1,a_1,a_2,\ldots,a_k}(p,q) \\
&= p^{\binom{n+1}{2}} T_{n+1,a_1,a_2,\ldots,a_k}\left(\tfrac{q}{p}\right) \\
&= p^{\binom{n+1}{2}} \left(T_{n,a_1-1,a_2,\ldots,a_k}\left(\tfrac{q}{p}\right) + \sum_{i=2}^{k} [n]_{\frac{q}{p}} \downarrow_{(i-1)} T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}\left(\tfrac{q}{p}\right) \right) \\
&= p^n \left(p^{\binom{n}{2}} T_{n,a_1-1,a_2,\ldots,a_k}\left(\tfrac{q}{p}\right) \right) \\
&\quad + \sum_{i=2}^{k} p^n \prod_{j=0}^{i-2} \left(p^{n-j-1} [n-j]_{\frac{q}{p}} \right) \left(p^{\binom{n-i+1}{2}} T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}\left(\tfrac{q}{p}\right) \right)
\end{aligned}$$

$$= p^n T_{n,a_1-1,a_2,\ldots,a_k}(p,q) + \sum_{i=2}^{k} p^n [n]_{p,q}\downarrow_{(i-1)} T_{n-i+1,a_1,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_k}(p,q). \quad \blacksquare$$

We now give a p, q-analogue of (1.17).

Theorem 4.6. *For all natural numbers n and k,*

$$\begin{aligned} D_{n+1,k}(p,q) &= [n+1]_{p,q} D_{n,k}(p,q) \\ &\quad + (-1)^{n+1} \sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} T_{n+1,a_1,\ldots,a_k}(p,q). \end{aligned} \tag{4.7}$$

Proof. We will prove (4.7) by induction on k and then on n. First if $k = 1$, we note that for any $n \geq 1$,

$$\mathcal{T}_{n+1,n+1,0,\ldots,0} = \{(1,1,1)(1,2,1)\cdots(1,n+1,1)\},$$

so that $T_{n+1,n+1,0,\ldots,0}(p,q) = p^{coinv(1\,2\cdots n+1)} = p^{\binom{n+1}{2}}$. Therefore, by (1.11)

$$D_{n+1,1}(p,q) = [n+1]_{p,q} D_{n,1}(p,q) + (-1)^{n+1} T_{n+1,n+1,0,\ldots,0}(p,q).$$

Next, assume that (4.7) holds for some $k-1 \geq 1$ and all $n \in \mathbb{N}$. We will show that (4.7) holds for k by induction on n. We proceed by establishing k base cases. That is, we will show that (4.7) holds for each $1 \leq n \leq k$. Since $D_{n+1,k}(p,q) = 0$ for all $1 \leq n < k$ and $D_{k+1,k}(p,q) = p^k [k]_{p,q}!$, it suffices to prove that when $1 \leq n < k$,

$$\sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} T_{n+1,a_1,\ldots,a_k}(p,q) = 0, \tag{4.8}$$

and for $n = k$,

$$\sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=k+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} T_{k+1,a_1,\ldots,a_k}(p,q) = (-1)^{k+1} p^k [k]_{p,q}!. \tag{4.9}$$

To prove (4.8) for $1 \leq n < k$, we first apply the recursion in Corollary 4.5 to $T_{n+1,a_1,\ldots,a_k}$ to get

$$\begin{aligned} &\sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n+1,a_1,\ldots,a_k}(p,q) \\ &= p^n \sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} \\ &\quad \left(T_{n,a_1-1,a_2,\ldots,a_k}(p,q) + \sum_{i=2}^{k} [n]_{p,q}\downarrow_{(i-1)} T_{n-i+1,a_1,\ldots,a_i-1,\ldots,a_k}(p,q) \right) \end{aligned}$$

$$= p^n \left(\sum_{\substack{a_1 \geq 1, a_2, \ldots, a_k \geq 0 \\ a_1 + 2a_2 + \cdots + ka_k = n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n, a_1-1, a_2, \ldots, a_k}(p,q) \right.$$

$$\left. + \sum_{i=2}^{k} [n]_{p,q} \downarrow_{(i-1)} \sum_{\substack{a_i \geq 1, a_j \geq 0 \text{ for } j \neq i \\ a_1 + 2a_2 + \cdots + ka_k = n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-i+1, a_1, \ldots, a_i - 1, \ldots, a_k}(p,q) \right)$$

$$= p^n \left(\sum_{\substack{a_1, \ldots, a_k \geq 0 \\ a_1 + 2a_2 + \cdots + ka_k = n}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n, a_1, \ldots, a_k}(p,q) + \sum_{i=2}^{k} [n]_{p,q} \downarrow_{(i-1)} (-1)^{i-1} \times \right.$$

$$\left. \sum_{\substack{a_1, \ldots, a_k \geq 0 \\ a_1 + 2a_2 + \cdots + ka_k = n-i+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-i+1, a_1, \ldots, a_k}(p,q) \right). \tag{4.10}$$

Since $n < k$, it must be the case that $a_k = 0$ in both summations on the right hand side of (4.10). Furthermore, it follows by Definition 4.1 and (4.4) that

$$T_{n, a_1, \ldots, a_{k-1}, 0}(p,q) = T_{n, a_1, \ldots, a_{k-1}}(p,q) \tag{4.11}$$

for all natural numbers n and k. As such, the right hand side of (4.10) can be rewritten as

$$p^n \left(\sum_{\substack{a_1, \ldots, a_{k-1} \geq 0 \\ a_1 + \cdots + (k-1)a_{k-1} = n}} (-1)^{\sum_{j=2}^{k-1}(j-1)a_j} T_{n, a_1, \ldots, a_{k-1}}(p,q) + \sum_{i=2}^{k} (-1)^{i-1} [n]_{p,q} \downarrow_{(i-1)} \times \right.$$

$$\left. \sum_{\substack{a_1, \ldots, a_{k-1} \geq 0 \\ a_1 + \cdots + (k-1)a_{k-1} = n-i+1}} (-1)^{\sum_{j=2}^{k-1}(j-1)a_j} T_{n-i+1, a_1, \ldots, a_{k-1}}(p,q) \right), \tag{4.12}$$

which by induction equals

$$p^n \Bigg((-1)^n \left(D_{n,k-1}(p,q) - [n]_{p,q} D_{n-1,k-1}(p,q) \right) + \sum_{i=2}^{k} (-1)^{i-1} [n]_{p,q} \downarrow_{(i-1)} \times$$

$$(-1)^{n-i+1} \left(D_{n-i+1,k-1}(p,q) - [n-i+1]_{p,q} D_{n-i,k-1}(p,q) \right) \Bigg). \tag{4.13}$$

Since $1 \leq n < k$, Theorem 2.2 implies that $D_{n,k-1}(p,q) = D_{n-1,k-1}(p,q) = D_{n-i+1,k-1}(p,q) = D_{n-i,k-1}(p,q) = 0$ for each $i = 2, \ldots, k$. Therefore, the expression in (4.13) equals 0, proving (4.8).

Next, to prove (4.9), we note that

$$\sum_{\substack{a_1,\ldots,a_k\geq 0\\ a_1+\cdots+ka_k=k+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} T_{k+1,a_1,\ldots,a_k}(p,q)$$

$$= (-1)^{(k-1)} T_{k+1,1,0,\ldots,0,1}(p,q)$$

$$+ \sum_{\substack{a_1,\ldots,a_{k-1}\geq 0, a_k=0\\ a_1+\cdots+(k-1)a_{k-1}=k+1}} (-1)^{a_2+2a_3+\cdots+(k-2)a_{k-1}} T_{k+1,a_1,\ldots,a_{k-1},0}(p,q), \tag{4.14}$$

which by (4.11) equals

$$(-1)^{(k-1)} T_{k+1,1,0,\ldots,0,1}(p,q)$$

$$+ \sum_{\substack{a_1,\ldots,a_{k-1}\geq 0\\ a_1+\cdots+(k-1)a_{k-1}=k+1}} (-1)^{a_2+2a_3+\cdots+(k-2)a_{k-1}} T_{k+1,a_1,\ldots,a_{k-1}}(p,q). \tag{4.15}$$

Clearly, $\mathcal{A}_{k+1,1,0,\ldots,0,1} = \{(1,1,1),(k,1,1),(k,1,2),\ldots,(k,1,k)\}$. So, for any $r \in \mathcal{T}_{k+1,1,0,\ldots,0,1}$, either $r_{k+1} = (1,1,1)$ and $r_k = (k,1,k)$ or $r_{k+1} = (k,1,k)$. In either case, we can build up a word by first placing the letter $(k,1,1)$, then placing the letter $(k,1,2)$, etc. Note that for any j, the placement of letters $(k,1,j+1),\ldots,(k,1,k-1)$ does not effect the statistic $c_{>(k,1,j))}\left(\Gamma_{k,j}(r)\right)$ so that we can easily compute the contribution to the placement of a letter $(k,1,j)$ to $T_{k+1,1,0,\ldots,0,1}(q)$. In the case where $r_{k+1} = (1,1,1)$ and $r_k = (k,1,k)$, we have only one choice where to put $(k,1,1)$ which contributes a factor of $1 = [1]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. Having placed $(k,1,1)$, we then have 2 choices where to place $(k,1,2)$ and it can easily be seen that the placement of $(k,1,2)$ contributes a factor of $1+q = [2]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. In general, having placed $(k,1,1),\ldots,(k,1,j-1)$, we then have j choices where to place $(k,1,j)$ and it can easily be seen that the placement of $(k,1,j)$ contributes a factor of $[j]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. It follows that the contribution of all words r where $r_{k+1} = (1,1,1)$ and $r_k = (k,1,k)$ is $[k-1]_q!$. Now if $r_{k+1} = (k,1,k)$, then we can start with the word $(1,1,1)\,(k,1,k)$ and then build up a word by first placing the letter $(k,1,1)$, then placing the letter $(k,1,2)$, etc. In this case, we have two choices where to put $(k,1,1)$ which contributes a factor of $1+q = [2]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. Having placed $(k,1,1)$, we then have 3 choices where to place $(k,1,2)$ and it can easily be seen that the placement of $(k,1,2)$ contributes a factor of $1+q+q^2 = [3]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. In general, having placed $(k,1,1),\ldots,(k,1,j-1)$, we then have $j+1$ choices where to place $(k,1,j)$ and it can easily be seen that the placement of $(k,1,j)$ contributes a factor of $[j+1]_q$ to $T_{k+1,1,0,\ldots,0,1}(q)$. It follows that the contribution of all words r where $r_{k+1} = (k,1,k)$ is $[k]_q!$. Thus $T_{k+1,1,0,\ldots,0,1}(q) = [k-1]_q! + [k]_q!$ and so

$$(-1)^{(k-1)} T_{k+1,1,0,\ldots,0,1}(p,q) = (-1)^{k-1}\left(p^{2k-1}[k-1]_{p,q}! + p^k[k]_{p,q}!\right). \tag{4.16}$$

Next, by induction (on k) and Theorem 2.2, it follows that

$$(-1)^{k+1} \sum_{\substack{a_1,\ldots,a_{k-1}\geq 0\\ a_1+\cdots+(k-1)a_{k-1}=k+1}} (-1)^{a_2+2a_3+\cdots+(k-2)a_{k-1}} T_{k+1,a_1,\ldots,a_{k-1}}(p,q)$$

$$
\begin{aligned}
&= D_{k+1,k-1}(p,q) - [k+1]_{p,q} D_{k,k-1}(p,q) \\
&= q[k]_{p,q} D_{k,k-1}(p,q) + p^k [k]_{p,q}\downarrow_{k-1} D_{1,k-1}(p,q) - [k+1]_{p,q} D_{k,k-1}(p,q) \\
&= (q[k]_{p,q} - [k+1]_{p,q}) D_{k,k-1}(p,q) + p^k [k]_{p,q}\downarrow_{k-1} D_{1,k-1}(p,q) \\
&= -p^{2k-1}[k-1]_{p,q}!. \qquad (4.17)
\end{aligned}
$$

Combining (4.15) and (4.17) proves (4.9).

Now assume that $n \geq k+1$ and (4.7) holds for all $m \leq n$. Using (4.6), the right hand side of (4.7) can be written as

$$
\begin{aligned}
& q[n]_{p,q} D_{n,k}(p,q) + p^n D_{n,k}(p,q) + (-1)^{n+1} \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k = n+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} \times \\
& \left(p^n T_{n,a_1-1,a_2,\dots,a_k}(p,q) + \sum_{i=2}^{k} p^n [n]_{p,q}\downarrow_{(i-1)} T_{n-i+1,a_1,\dots,a_i-1,\dots,a_k}(p,q) \right) \\
&= q[n]_{p,q} D_{n,k}(p,q) \\
&\quad + p^n \left(D_{n,k}(p,q) - (-1)^n \sum_{\substack{a_1 \geq 1, a_2,\dots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k = n+1}} (-1)^{\sum_{j=2}^k (j-1)a_j} T_{n,a_1-1,a_2,\dots,a_k}(p,q) \right) \\
&\quad - (-1)^n \sum_{i=2}^{k} p^n [n]_{p,q}\downarrow_{(i-1)} \\
&\quad \sum_{\substack{a_j \geq 0 \text{ for } j \neq i,\, a_i \geq 1 \\ a_1+2a_2+\cdots+ka_k = n+1}} (-1)^{\sum_{j=2}^k (j-1)a_j} T_{n-i+1,a_1,\dots,a_i-1,\dots,a_k}(p,q) \\
&= q[n]_{p,q} D_{n,k}(p,q) + p^n \Bigg(D_{n,k}(p,q) \\
&\quad - (-1)^n \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+2a_2+\cdots+ka_k = n}} (-1)^{\sum_{i=2}^k (i-1)a_i} T_{n,a_1,\dots,a_k}(p,q) \Bigg) \\
&\quad - (-1)^n \sum_{i=2}^{k} (-1)^{i-1} p^n [n]_{p,q}\downarrow_{(i-1)} \\
&\quad \sum_{\substack{a_1,\dots,a_k \geq 0 \\ a_1+\cdots+ka_k = n+1-i}} (-1)^{\sum_{j=2}^k (j-1)a_j} T_{n-i+1,a_1,\dots,a_k}(p,q). \qquad (4.18)
\end{aligned}
$$

By induction, we can assume that

$$D_{n,k}(p,q)-(-1)^n \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+2a_2+\cdots+ka_k=n}} (-1)^{\sum_{i=2}^{k}(i-1)a_i} T_{n,a_1,\dots,a_k}(p,q)=[n]_{p,q}D_{n-1,k}(p,q). \tag{4.19}$$

Therefore, using (4.19) in (4.18) yields

$$\begin{aligned}
&[n+1]_{p,q}D_{n,k}(p,q)+(-1)^{n+1} \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n+1,a_1,\dots,a_k}(p,q)\\
&= q[n]_{p,q}D_{n,k}(p,q)+p^n[n]_{p,q}D_{n-1,k}(p,q)\\
&\quad -(-1)^n\sum_{i=2}^{k}(-1)^{i-1}p^n[n]_{p,q}\downarrow_{(i-1)}\\
&\quad \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+ka_k=n+1-i}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-i+1,a_1,\dots,a_k}(p,q)\\
&= q[n]_{p,q}D_{n,k}(p,q)+p^n[n]_{p,q}\Bigg(D_{n-1,k}(p,q)\\
&\quad -(-1)^{n-1} \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+ka_k=n-1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-1,a_1,\dots,a_k}(p,q)\Bigg)\\
&\quad -(-1)^n\sum_{i=3}^{k}(-1)^{i-1}p^n[n]_{p,q}\downarrow_{(i-1)}\\
&\quad \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+ka_k=n+1-i}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-i+1,a_1,\dots,a_k}(p,q),
\end{aligned} \tag{4.20}$$

which again, by induction, equals

$$\begin{aligned}
&q[n]_{p,q}D_{n,k}(p,q)+p^n[n]_{p,q}[n-1]_{p,q}D_{n-2,k}(p,q)\\
&-(-1)^n\sum_{i=3}^{k}(-1)^{i-1}p^n[n]_{p,q}\downarrow_{(i-1)} \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+\cdots+ka_k=n+1-i}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n-i+1,a_1,\dots,a_k}(p,q).
\end{aligned}$$

Continuing in this way, we find that k total uses of induction (on n) yields

$$\begin{aligned}
&[n+1]_{p,q}D_{n,k}(p,q)+(-1)^{n+1} \sum_{\substack{a_1,\dots,a_k\geq 0\\ a_1+2a_2+\cdots+ka_k=n+1}} (-1)^{\sum_{j=2}^{k}(j-1)a_j} T_{n+1,a_1,\dots,a_k}(p,q)\\
&= q[n]_{p,q}D_{n,k}(p,q)+p^n[n]_{p,q}\downarrow_k D_{n-k,k}(p,q)
\end{aligned}$$

$$= D_{n+1,k}(p,q),$$

proving the theorem. ∎

To demonstrate the recursion given in Theorem 4.6, we now return to our example from Section 2. Namely, we will compute $D_{5,2}(p,q)$ given that $D_{4,2}(p,q) = qp^2[3]_{p,q}[2]_{p,q}$:

$$\begin{aligned} D_{5,2}(p,q) &= [5]_{p,q}D_{4,2}(p,q) + (-1)^5 \sum_{\substack{a_1,a_2\geq 0 \\ a_1+2a_2=5}} (-1)^{a_2} T_{5,a_1,a_2}(p,q) \\ &= [5]_{p,q}qp^2[3]_{p,q}[2]_{p,q} - (T_{5,5,0}(p,q) - T_{5,3,1}(p,q) + T_{5,1,2}(p,q)). \end{aligned} \tag{4.21}$$

One can easily check that $\mathcal{T}_{5,5,0} = \{(1,1,1)(1,2,1)(1,3,1)(1,4,1)(1,5,1)\}$, so that $T_{5,5,0}(p,q) = p^{\binom{5}{2}} = p^{10}$. Next, we see that $\mathcal{T}_{5,3,1}$ consists of the words labelled in the following table by their p,q-weight.

p^{10}:	(2, 1, 1)(2, 1, 2)(1, 1, 1)(1, 2, 1)(1, 3, 1)
	(1, 1, 1)(2, 1, 1)(2, 1, 2)(1, 2, 1)(1, 3, 1)
	(1, 1, 1)(1, 2, 1)(2, 1, 1)(2, 1, 2)(1, 3, 1)
	(1, 1, 1)(1, 2, 1)(1, 3, 1)(2, 1, 1)(2, 1, 2)
p^9q:	(2, 1, 1)(1, 1, 1)(2, 1, 2)(1, 2, 1)(1, 3, 1)
	(1, 1, 1)(2, 1, 1)(1, 2, 1)(2, 1, 2)(1, 3, 1)
	(1, 1, 1)(1, 2, 1)(2, 1, 1)(1, 3, 1)(2, 1, 2)
p^8q^2:	(2, 1, 1)(1, 1, 1)(1, 2, 1)(2, 1, 2)(1, 3, 1)
	(1, 1, 1)(2, 1, 1)(1, 2, 1)(1, 3, 1)(2, 1, 2)
p^7q^3:	(2, 1, 1)(1, 1, 1)(1, 2, 1)(1, 3, 1)(2, 1, 2)

That is, $T_{5,3,1}(p,q) = p^7q^3 + 2p^8q^2 + 3p^9q + 4p^{10}$. Finally, $\mathcal{T}_{5,1,2}$ consists of the words labelled in the following table by their p,q-weight.

p^{10} :	(2, 1, 1)(2, 1, 2)(2, 2, 1)(2, 2, 2)(1, 1, 1)
	(2, 1, 1)(2, 1, 2)(1, 1, 1)(2, 2, 1)(2, 2, 2)
	(1, 1, 1)(2, 1, 1)(2, 1, 2)(2, 2, 1)(2, 2, 2)
p^9q :	(2, 1, 1)(2, 2, 1)(2, 1, 2)(2, 2, 2)(1, 1, 1)
	(2, 1, 1)(1, 1, 1)(2, 1, 2)(2, 2, 1)(2, 2, 2)
	(2, 1, 1)(2, 1, 2)(2, 2, 1)(1, 1, 1)(2, 2, 2)
	(1, 1, 1)(2, 1, 1)(2, 2, 1)(2, 1, 2)(2, 2, 2)
p^8q^2 :	(2, 2, 1)(2, 1, 1)(2, 1, 2)(2, 2, 2)(1, 1, 1)
	(2, 1, 1)(1, 1, 1)(2, 2, 1)(2, 1, 2)(2, 2, 2)
	(2, 1, 1)(2, 2, 1)(2, 1, 2)(1, 1, 1)(2, 2, 2)
	(1, 1, 1)(2, 2, 1)(2, 1, 1)(2, 1, 2)(2, 2, 2)
p^7q^3 :	(2, 1, 1)(2, 2, 1)(1, 1, 1)(2, 1, 2)(2, 2, 2)
	(2, 2, 1)(1, 1, 1)(2, 1, 1)(2, 1, 2)(2, 2, 2)
	(2, 2, 1)(2, 1, 1)(2, 1, 2)(1, 1, 1)(2, 2, 2)
p^6q^4 :	(2, 2, 1)(2, 1, 1)(1, 1, 1)(2, 1, 2)(2, 2, 2)

That is, $T_{5,1,2}(p,q) = p^6q^4 + 3p^7q^3 + 4p^8q^2 + 4p^9q + 3p^{10}$. Plugging these values into the right hand side of (4.21) gives

$$\begin{aligned} D_{5,2}(p,q) &= [5]_{p,q}qp^2[3]_{p,q}[2]_{p,q} - \big(p^{10} - \big(p^7q^3 + 2p^8q^2 + 3p^9q + 4p^{10}\big) \\ &\quad + p^6q^4 + 3p^7q^3 + 4p^8q^2 + 4p^9q + 3p^{10}\big) \\ &= p^2q^8 + 3p^3q^7 + 5p^4q^6 + 6p^5q^5 + 5p^6q^4 + 3p^7q^3 + p^8q^2 \\ &= p^2q^2[4]_{p,q}!. \end{aligned} \tag{4.22}$$

One can easily use Theorem 2.2 to verify that in fact $D_{5,2}(p,q) = p^2q^2[4]_{p,q}!$.

5. Conclusions and Perspectives

In the previous sections, we proved that our generalized p, q-derangement numbers, $D_{n,k}(p,q)$, defined in (2.1), satisfy the following p, q-analogues of (1.12) and (1.17):

$$D_{n+1,k}(p,q) = q[n]_{p,q}D_{n,k}(p,q) + p^n[n]_{p,q}\downarrow_k D_{n-k,k}(p,q),$$

where $D_{n,k}(p,q) = 0$ for $1 \le n \le k$, and

$$\begin{aligned} D_{n+1,k}(p,q) &= [n+1]_{p,q}D_{n,k}(p,q) \\ &\quad + (-1)^{n+1} \sum_{\substack{a_1,\dots,a_k \ge 0 \\ a_1+2a_2+\cdots+ka_k = n+1}} (-1)^{a_2+2a_3+\cdots+(k-1)a_k} T_{n+1,a_1,\dots,a_k}(p,q). \end{aligned}$$

While our definition of $D_{n,k}(p,q)$ was motivated by Garsia and Remmel's q-enumeration of derangements in S_n by the inversion statistic, we also note that q-derangements have also been defined in literature (see [3, 7]) using the major index statistic. In particular, Wachs defined a q-analogue of the derangements $\mathcal{D}_{n,1}$ in the symmetric group S_n by q-enumerating the derangements according to their major index. That is, Wachs defined

$$\tilde{D}_{n,1}(q) := \sum_{\sigma \in \mathcal{D}_{n,1}} q^{maj(\sigma)},$$

where for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $maj(\sigma) = \sum_{i:\, \sigma_i > \sigma_{i+1}} i$, and proved combinatorially that

$$\tilde{D}_{n,1}(q) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{[k]_q!}, \tag{5.1}$$

which is a natural q-analogue of the classical formula $D_{n,1} = n!\sum_{k=0}^{n}(-1)^k/k!$. It is not difficult to see that the $\tilde{D}_{n,1}(q)$ also satisfy the following recursion:

$$\tilde{D}_{n+1,1}(q) = [n+1]_q\tilde{D}_{n,1}(q) + (-1)^{n+1}q^{\binom{n+1}{2}}, \tag{5.2}$$

with initial conditions $\tilde{D}_{1,1}(q) = 1$ and $\tilde{D}_{2,1}(q) = q$. Comparing (5.2) with (1.8), we see that $\tilde{D}_{n,1}(p) = D_{n,1}(p,1)$. Moreover, it is easy to see that $D_{n,1}(p,q) = q^{\binom{n}{2}} D_{n,1}(p/q,1)$, so that

$$\begin{aligned} D_{n,1}(p,q) &= q^{\binom{n}{2}} D_{n,1}(p/q,1) \\ &= q^{\binom{n}{2}} \tilde{D}_{n,1}(p/q) \\ &= q^{\binom{n}{2}} \sum_{\sigma \in \mathcal{D}_{n,1}} (p/q)^{maj(\sigma)} \\ &= \sum_{\sigma \in \mathcal{D}_{n,1}} (p)^{maj(\sigma)} q^{comaj(\sigma)}, \end{aligned}$$

where for any permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, $comaj(\sigma) = \sum_{i:\, \sigma_i < \sigma_{i+1}} i$. Thus, we have two different combinatorial interpretations for $D_{n,1}(p,q)$ in this case.

It is natural to ask whether $\tilde{D}_{n,k}(q) := \sum_{\sigma \in \mathcal{D}_{n,k}} q^{maj(\sigma)}$ is also a specialization of $D_{n,k}(p,q)$. However, this is not the case. To see this, note there are six permutations in $\mathcal{D}_{4,2}$. The table below gives the statistics $maj(\sigma)$, $inv(\overline{\sigma})$, and $coinv(\overline{\sigma})$ for which it is easy to see that no specialization of $D_{4,2}(p,q)$ will yield $\tilde{D}_{4,2}(q)$.

σ in cyclic notation	σ in one line notation	$maj(\sigma)$	$inv(\overline{\sigma})$	$coinv(\overline{\sigma})$
$(1,2,3,4)$	2 3 4 1	3	0	6
$(1,3,2,4)$	3 4 2 1	5	1	5
$(2,1,3,4)$	3 1 4 2	4	1	5
$(2,3,1,4)$	4 3 1 2	3	2	4
$(3,1,2,4)$	2 4 1 3	2	2	4
$(3,2,1,4)$	4 1 2 3	1	3	3

We note that Chow [3] later extended Wachs' result by defining a q-analogue of the derangements d_n^B in the hyperoctahedral group B_n by q-enumerating the derangements according to their flag-major index (see [1, 2] for definition). That is, he defined

$$d_n^B(q) := \sum_{\sigma \in \mathcal{D}_n^B} q^{fmaj(\sigma)},$$

where $\mathcal{D}_n^B = \{\sigma \in B_n \colon \sigma(i) \neq i \text{ for all } i \in [n]\}$, and proved the following analogue of (5.1):

$$d_n^B(q) = [2]_q[4]_q \cdots [2n]_q \sum_{k=0}^{n} \frac{(-1)^k q^{2\binom{k}{2}}}{[2]_q[4]_q \cdots [2k]_q}. \tag{5.3}$$

There are many natural questions that arise from these developments, which we will pursue in subsequent work. In particular,

(1) How does Chow's model extend to $C_m \wr S_n$?
(2) Can we extend our model of generalized derangements to B_n?
(3) Can we extend our model of generalized derangements to $C_m \wr S_n$?

References

1. R.M. Adin, F. Brenti, and Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. **27** (2-3) (2001) 210–224.
2. R.M. Adin and Y. Roichman, The flag major index and group actions on polynomial rings, European J. Combin. **22** (4) (2001) 431–446.
3. C.-O. Chow, On derangement polynomials of type B, Sém. Lothar. Combin. **55** (2006) Article B55b.
4. A.M. Garsia and J.B. Remmel, A combinatorial interpretation of q-derangement and q-Laguerre numbers, European J. Combin. **1** (1980) 47–59.
5. P.A. MacMahon, Combinatory Analysis, Vol. I, II, Cambridge University Press, Cambridge, 1915, 1916.
6. J.B. Remmel, A note on a recursion for the number of derangements, European J. Combin. **4** (4) (1983) 371–374.
7. M.L. Wachs, On q-derangement numbers, Proc. Amer. Math Soc. **106** (1) (1989) 273–278.