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On the Additivity of Equivariant Operads

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by

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ABSTRACT OF THE DISSERTATION

On the Additivity of Equivariant Operads

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In this dissertation, we investigate equivariant generalisations of Dunn Additivity. We first build equivariant operads called little star operads, which encompass little cube and little disk operads and prove they provide models of $\mathbb{E}_V$-operads. We then show general conditions for when additivity holds for these operads. In particular, we prove that an equivariant additivity theorem holds for simplex-shaped operads. We then consider another operad construction aimed to model more general $\mathbb{N}$-operads. We show that while they provide an approximation for $\mathbb{N}_\infty$-operads, they seem to fail a corresponding additivity theorem.

The dissertation of Benjamin Jack Szczesny is approved.

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For my parents.

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Introduction

The Boardman-Vogt tensor product of operads is a mysterious and often frustrating construction in operadic theory. Yet, as it encodes interchanging algebraic structures, it plays a significant role in homotopical algebra. This thesis was born out of a desire to comprehend the tensor in an equivariant context better.

The tensor is best described by its algebras. Suppose P and Q are operads, and X is an object with the structure of a P-algebra and a Q-algebra. If the two structures interchange, then X is an algebra of the operad $P \otimes Q$. Rephrased differently, X is a $P \otimes Q$-algebra if it is a P-algebra in the category of Q-algebras and vice versa. This construction appears in many guises; for instance, given an abelian group M that has the structure of a left S-module and a right R-module. The interchange condition here is nothing more than

$$s \cdot (m \cdot r) = (s \cdot m) \cdot r \text{ for all } m \in M, s \in S, r \in R.$$

This then lets us equivalently describe M as a left $S \otimes R^{\mathrm{op}}$-module. The tensor of operads is a vast generalisation of this.

One of the most important kinds of operads in homotopical algebra is the $\mathbb{E}_k$-operads. A $\mathbb{E}_1$-operad encodes a homotopy associative monoidal structure, while the rest encode an increasing amount of coherence data ending in $\mathbb{E}_\infty$ which are, at least homotopically, encodes commutative monoids. These kinds of operads are intimately linked with the operadic tensor. A result of Dunn [Dun88] gives us that $\mathbb{E}_k$-operads are additive:

$$\mathbb{E}_m \otimes \mathbb{E}_n \simeq \mathbb{E}_{m+n}.$$

Dunn specifically proved this for the little k-cubes operad $\mathcal{C}_k$, a specific model for $\mathbb{E}_k$-operads. A concrete example of this equivalence in action is for loop spaces. A classic result of May [May72] tells us that a group-like algebra X over the k-little cubes operad $\mathcal{C}_n$ in spaces is equivalent to a n-fold loop space $X \simeq \mathrm{Hom}(S^n, Y) = \Omega^n Y$. Dunn additivity then appears as the property that if Y is also a grouplike $\mathbb{E}_m$-algebra such that its algebra structure interchanges with the one on X, then we get that

$$X \simeq \mathrm{Hom}(S^n, Y) \simeq \mathrm{Hom}(S^n, \mathrm{Hom}(S^m, Z)) \cong \mathrm{Hom}(S^{n+m}, Z)$$

and so X is a $n+m$-fold loop space.

The main aim of this dissertation is to generalise Dunn additivity to an equivariant context. What this means is slightly ambiguous. Just as in much of equivariant homotopy, we can generalise to varying levels of "equivariant genuine-ness". The "naive" generalisation would be to prove that $\mathcal{C}_m \otimes \mathcal{C}_n \simeq \mathcal{C}_{m+n}$ where these operads are now in equivariant spaces. However, there is almost nothing to prove in this situation as the original proof goes through without much issue.

Another level up the equivariance scale would be to prove additivity for operads that encode equivariant loop spaces. An equivariant loop space $\Omega^V Y$ has loops coming from a representation sphere S^V. The equivariant little disks operad $\mathcal{D}(V)$ is the natural generalisation of $\mathcal{C}_k$ to encode this structure. Our aim is to then prove that for G-

representations V and W, there is an equivalence

$$\mathscr{D}(V) \otimes \mathscr{D}(W) \simeq \mathscr{D}(V \oplus W).$$

We essentially - but not quite - prove this. To explain what we mean by this, we must bring up a problem with the Boardman-Vogt tensor that we have glossed over. *The tensor is not homotopical.* This is why we say the tensor is mysterious and frustrating at the beginning of this introduction. What we do is construct operads that we call the equivariant little simplex operads $\mathcal{D}_\Delta(V)$, prove these are weakly equivalent to $\mathscr{D}(V)$ and prove that

$$\mathcal{D}_\Delta(V) \otimes \mathcal{D}_\Delta(W) \simeq \mathcal{D}_\Delta(V \oplus W).$$

This doesn't imply the additivity of the little disks, but it does give us an explicit case of an additivity of $\mathbb{E}_V$-operads.

There is one more level in the equivariance ladder. Blumberg-Hill in [BH15] define and study a class of equivariant operads called $\mathbb{N}_\infty$-operads which are a further generalisation of $\mathbb{E}_\infty$-operads that supersedes that of $\mathbb{E}_U$-operads. Here U is a G-universe, a type of countably infinite dimensional G-representation. What Blumberg-Hill show is that these types of operads are only a single type in a larger family of operads that span from the most incomplete, the $\mathbb{E}_\infty$-operads, to the most complete, the so-called $G - \mathbb{E}_\infty$-operads which do encode equivariant commutative monoids in a homotopically coherent fashion. The $\mathbb{N}_\infty$-operads sit inside a finite lattice of families, and Blumberg-Hill conjectured that the tensor corresponds to the lattice join. A derived version of this conjecture was proven by Rubin [Rub21c].

We have specialised to the ∞-case here because there is no theory of general $\mathbb{N}$-operads. In the last chapter of this thesis, we present some work in progress where we build operads that attempt to model general $\mathbb{N}$-operads. We will call these twisted little cube operads $\mathcal{C}_n^{\mathcal{F}}$, and we show that they can model both $\mathbb{N}_\infty$-operads and also "finite truncations" of them. We then end by considering how an additivity proof may work for these operads, and identify an obstruction for this model. We conjecture that this obstruction is also what stops $\mathcal{C}_n^{\mathcal{F}}$ from serving as true models for general $\mathbb{N}$-operads.

Overview

Let us explain the structure of this dissertation in more detail and provide a guide for the reader.

Chapter 1: Equivariant Operads In this chapter, we review the basic definitions and properties of equivariant operads. This chapter mostly sets the notation and conventions we will use throughout this thesis, and a reader familiar with the subject is more than welcome to skim it. One section that is not standard is on "operads indexed by other categories"; our treatment of this notion is non-standard, and it is not something that we will explore in any real depth in this dissertation. The purpose of this section is to define a set of technical conditions that imply some of our constructs give operads.

Chapter 2: Little star operads While Dunn additivity holds for little cubes $\mathcal{C}_n$ non-equivariantly, it is unknown if the same is true for non-equivariant little disks. This is problematic since little cubes don't work over representations. Moreover, while the two kinds of operads are equivalent, this equivalence in the literature is exhibited via a zigzag of equivalences. This chapter defines little star operads $\mathcal{S}^M$, which uniformly treat little cubes, disks and other star shapes over a G-representation. We then prove they are all weakly equivalent over the same representation V.

Chapter 3: The Boardman-Vogt tensor product In this chapter, we present a self-contained account of the construction of the tensor product. We have done this as most contemporary accounts in the literature tend only to offer the generators and relations of the tensor, but gloss over the nuts and bolts of it. This can make understanding how the tensor functions difficult for the novice. We hope that this chapter will be helpful for those new to the tensor.

Chapter 4: Equivariant Dunn additivity We finally prove an equivariant version of Dunn additivity. This chapter looks more generally at the conditions needed to prove additivity theorems. We will show that the only gap for verifying additivity for all little star operads $\mathcal{S}^M$ is in showing the induced maps from the tensor is injective. We then establish general criteria for the injectivity of tensor maps and show the equivariant little simplex operads $\mathcal{D}_\Delta$ satisfy this criterion, giving us the desired additivity result.

Chapter 5: Operads that encode norm maps In this section, we present a work in progress to find general models for $\mathbb{N}$-operads. We will restrict ourselves to an Abelian group G, and start this section with a quick overview on $\mathbb{N}$-operads. We then move on to constructing twisted little cube operads $\mathcal{C}_n^{\mathcal{F}}$ for each transfer system $\mathcal{F}$. We show that these encode the correct norm maps and, in the ∞-case give a model for $\mathbb{N}_\infty$-operads. We end with a conjecture on why this model fails additivity and how we may fix it.

Conventions

Throughout this thesis, G universally means a finite group. Top is the category of compactly generated weak Hausdorff spaces, and Top^G is the category of G-spaces with equivariant maps as morphisms. A G-representation V is always a countable sum of finite orthogonal G-representations.

Chapter 1

Equivariant Operads

1.1 Operads

Operads were first defined by May [May72] in his work on iterated loop spaces. They can be viewed as special cases of the earlier notion of PROPS due to Adams and MacLane [Mac63] and extended by Boardman and Vogt [BV68].

The core idea of an operad is to encode spaces of "n-ary" composition maps. This is useful because (among many other reasons) each operad has an associated category of algebras. Morphisms between operads, then let us compare different kinds of algebra structures. Moreover, the use of topological operads allows us to keep track of homotopical information of these algebras.

We will only recall the main definitions here and set our notation. There are many references available for more information on operads. Some recent examples include [Fre17] and [Man20]. We will denote by V a bicomplete closed symmetric monoidal category with monoidal product $\otimes$ and identity $\mathbb{1}$. A V-operad is usually defined as a sequence $\{P(n)\}_{n\in\mathbb{N}}$ of objects in V with defined composition maps

$$\gamma_{n,k_1,\ldots,k_n} : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(\sum k_i)$$

that satisfy certain associativity, unitality, and equivariance conditions. However, we prefer a "coordinate-free" approach to simplify notation dramatically. We will write the category finite sets as $\mathcal{F}in$ and the subcategory of finite sets and bijections by $\mathcal{B}ij$. For each $n \in \mathbb{N}$ we will write $\underline{n} = \{1, \ldots, n\}$ with $\underline{0} = \emptyset$.

1.1.1 Definition. A (left) V-symmetric collection is a functor $F : \mathcal{B}ij \to V$. A morphism of V-collections is a natural transformation, and we denote the category of V-collections by $\mathrm{Coll}(V)$.

Given a function of finite sets $f : K \to J$ and $j \in J$, we will usually denote the fiber of f at j by $K_j := f^{-1}(j)$. Given a composition

$$L \xrightarrow{g} K \xrightarrow{f} J$$

we write g_j for the pullback

$$\begin{tikzcd} L_j \arrow[r] \arrow[d, "g_j"'] \arrow[dr, phantom, "\lrcorner", very near start] & L \arrow[d, "g"] \\ K_j \arrow[r, hook] & K. \end{tikzcd}$$

1.1.2 Definition. A V-operad is a V-symmetric collection $P : \mathcal{B}ij \to V$ such that for every map $\alpha : K \to J$ in $\mathcal{F}in$ there is a composition map

$$\gamma_P(\alpha) : P(J) \otimes \bigotimes_{j \in J} P(K_j) \to P(K)$$

and for each singleton S, there exists maps $\mathrm{id}_P(S) : \mathbb{1} \to P(S)$ such that the following hold:

(1) *Composition is associative.* Given any morphisms

$$L \xrightarrow{g} K \xrightarrow{f} J$$

in $\mathcal{F}in$, the following diagram commutes.

$$\begin{tikzcd}
\left(P(J) \otimes \bigotimes_{j\in J} P(K_j)\right) \otimes \bigotimes_{k\in K} P(L_k) \arrow[r, "\gamma_P(f)\otimes \mathrm{id}"] \arrow[d, "\cong"'] & P(K) \otimes \bigotimes_{k\in K} P(L_k) \arrow[dr, "\gamma_P(g)"] & \\
& & P(L) \\
P(J) \otimes \bigotimes_{j\in J} \left(P(K_j) \otimes \bigotimes_{k\in K_j} P(L_k)\right) \arrow[r, "\mathrm{id}\otimes \bigotimes_{j\in J}\gamma_P(g_j)"'] & P(J) \otimes \bigotimes_{j\in J} P(L_j) \arrow[ur, "\gamma_P(fg)"'] &
\end{tikzcd}$$

(2) *Composition is natural.* For every commuting sequence of maps in $\mathcal{F}in$

$$\begin{tikzcd}
I \arrow[r, "p"] \arrow[d, "f"'] & K \arrow[d, "g"] \\
J \arrow[r, "q"] & L
\end{tikzcd}$$

where p, q are bijections. The following diagram in V commutes

$$\begin{tikzcd}
P(J) \otimes \bigotimes_{j\in J} P(I_j) \arrow[r, "P(q)\otimes\bigotimes_{l\in L} P(p_l)"] \arrow[d, "\gamma_P(f)"'] & P(L) \otimes \bigotimes_{l\in L} P(K_l) \arrow[d, "\gamma_P(g)"] \\
P(I) \arrow[r, "P(p)"] & P(K)
\end{tikzcd}$$

(3) *The map* id_P *is a unit.*

(a) For any finite set J, the following commutes in V.

$$\begin{tikzcd}
P(J) \otimes \bigotimes_{i\in I} \mathbb{1} \arrow[rr, "P(\mathrm{id}_J)\otimes\bigotimes_{j\in J}\mathrm{id}_P(J_j)"] \arrow[dr, "\cong"'] & & P(J) \otimes \bigotimes_{j\in J} P(J_j) \arrow[dl, "\gamma_P(\mathrm{id}_J)"] \\
& P(J) &
\end{tikzcd}$$

Here J_j is the fiber with respect to the identity map on J.

(b) for any map of finite set $f : T \to S$ where S is a singleton set, the following commutes.

$$\begin{tikzcd} \mathbb{1} \otimes P(T) \arrow[r, "\mathrm{id}_{P(S)} \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & P(S) \otimes P(T) \arrow[d, "\gamma_P(f)"] \\ & P(T) \end{tikzcd}$$

1.1.3 Remark. This definition for operads is equivalent to the usual definition. See, for instance, [Bat08] for details. Be aware that our operads have symmetric action on the left instead of the more common right-sided notion. These are equivalent but do change some of the diagrams. We do this to follow the convention in [BH15]. The advantage of this convention is that we deal primarily with operads in Top^G, which puts both the $\mathcal{B}ij$ and group actions on the left.

1.1.4 Definition. A morphism of operads $f : P \to Q$ is a morphism of the underlying symmetric collections such that for any morphism of finite sets $\alpha : J \to K$ and singleton set S, the following diagrams in V commute.

$$\begin{tikzcd} P(K) \otimes \bigotimes_{k\in K} P(J_k) \arrow[r, "f_K \otimes \bigotimes_{k\in K} f_{J_k}"] \arrow[d, "\gamma_P(f)"'] & Q(K) \otimes \bigotimes_{k\in K} Q(J_k) \arrow[d, "\gamma_Q(f)"] \\ P(J) \arrow[r, "f_J"] & Q(J) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r, "\mathrm{id}_{P(S)}"] \arrow[dr, "\mathrm{id}_{Q(S)}"'] & P(S) \arrow[d, "f_S"] \\ & Q(S) \end{tikzcd}$$

We denote the category of operads in V by $\mathrm{Oper}(V)$.

We will occasionally use another characterisation for operads due to Markl [Mar96]. Recall that for finite sets J, K and an element $j \in J$ that we have the partial disjoint union

$$J \sqcup_j K = (J \backslash \{j\}) \cup K.$$

It may be helpful to think of this construction as taking K as a fibre over j of some map and then collapsing. The connection with operads comes from the following.

1.1.5 Definition. A *partial composition product* on a V-collection P is a collection of maps

$$\circ_{j,J,K} : P(J) \otimes P(K) \to P(J \sqcup_j K)$$

for each $j \in J$ and K such that for any morphisms of finite set $f : J \to J'$, $g : K \to K'$ the following commutes.

$$\begin{tikzcd}
P(J) \otimes P(K) \arrow[r, "\circ_{j,J,K}"] \arrow[d, "(P(f),P(g))"'] & P(J \sqcup_j K) \arrow[d, "P(f \sqcup_j g)"] \\
P(J') \otimes P(K') \arrow[r, "\circ_{f(j),J',K'}"'] & P(X' \sqcup_{f(x)} Y')
\end{tikzcd} \tag{1.1}$$

(1) The partial composition on P *it is associative* if for all $j \in J, k \in K, L$ the following commutes

$$\begin{tikzcd}
(P(J) \otimes P(K)) \otimes P(L) \arrow[r, "\alpha_{P(J),P(K),P(L)}"] \arrow[d, "\circ_{j,J,K} \otimes \mathrm{id}"'] & P(J) \otimes (P(K) \otimes P(L)) \arrow[d, "\mathrm{id} \otimes \circ_{k,K,L}"] \\
P(J \sqcup_j K) \otimes P(L) \arrow[d, "\circ_{k,J \sqcup_j K,L}"'] & P(J) \otimes P(K \sqcup_k L) \arrow[d, "\circ_{j,J,K \sqcup_k L}"] \\
P((J \sqcup_j K) \sqcup_k L)) \arrow[r, "P(\mathrm{id})"'] & P(J \sqcup_j (K \sqcup_k L))
\end{tikzcd} \tag{1.2}$$

Here α is the associator of V.

(2) The partial composition on P *has a two-sided unit* if there exists maps $u_S : I \to P(S)$ for all singleton sets $S = \{s\}$ such that the following commutes for all $j \in J$.

$$\begin{tikzcd}
 & \mathbb{1} \otimes P(J) \arrow[dl, "u_S \times \mathrm{id}"'] & & P(J) \otimes \mathbb{1} \arrow[dr, "\mathrm{id} \times u_{\{j\}}"] & \\
P(S) \otimes P(J) \arrow[dr, "\circ_{s,S,J}"'] & & P(J) \arrow[ul, "\lambda_{P(J)}"'] \arrow[ur, "\rho_{P(J)}"] & & P(J) \otimes P(\{j\}) \arrow[dl, "\circ_{j,J,\{j\}}"] \\
 & P(S \sqcup_s J) \arrow[ur, "P(\mathrm{id})"'] & & P(J \sqcup_j \{j\}) \arrow[ul, "P(\mathrm{id})"] &
\end{tikzcd} \tag{1.3}$$

Here λ and ρ are the left and right unitors of V.

1.1.6 Definition. A pseudo operad P is a symmetric collection with an associative partial product. If the partial product also has a unit, then this is equivalent to the definition of an operad.

We will mostly work with operads in concrete categories V, and instead of the above diagrams, we will work with the point sets themselves. Given a map $f : K \to J$ in $\mathcal{F}in$, and elements $x \in P(J), y_j \in P(K_j)$, we write the composition as

$$x \circ (y_j)_{j \in J} \in P(K)$$

and the partial compositions as

$$x \circ_j y_j \in P(J \sqcup_j K_j).$$

The *arity* of an element $x \in P(J)$ is the set J which we also denote by $\mathrm{ar}(x)$.

1.2 Operads indexed on other categories

This section is a bit technical and is only needed to justify a few constructions that produce operads. As such, the reader is welcome to skip this chapter until they need the results of this section.

First, let us review the basic definitions and properties of opfibrations of categories. These ideas go back to Grothendieck [Gro71]. A recent reference that the reader may find helpful is [Vis07].

Given a functor of categories $\pi : \mathcal{F} \to \mathcal{C}$, an arrow $f : X \to Y$ of $\mathcal{F}$ is π-*cocartesian* if for all other arrows $g : X \to Z$ such that its image $\pi(g)$ factors through $\pi(f)$, this factorization lifts uniquely. That is, if we have h such that $\pi(g) = h \circ \pi(f)$ then there exists a unique h' such that $\pi(h') = h$ and $g = h' \circ f$.

$$\begin{tikzcd}
 & & Z \arrow[dd, squiggly] \\
X \arrow[r, "f"'] \arrow[urr, "g"] \arrow[dd, squiggly] & Y \arrow[ur, dashed, "\exists! h'"'] \arrow[dd, squiggly] & \\
 & & \pi(Z) \\
\pi(X) \arrow[r, "\pi(f)"'] \arrow[urr, "\pi(g)"] & \pi(Y) \arrow[ur, "h=\pi(h')"'] &
\end{tikzcd}$$

A functor $\pi : \mathcal{F} \to \mathcal{C}$ is an *opfibration* if for every arrow $f : x \to y$ in $\mathcal{C}$, and object $X \in \mathcal{F}$, there exists a π-cocartesian lift of f at X: $f' : X \to Y$ such that $\pi(f') = f$. The idea behind this definition is the following. For every object $x \in \mathcal{C}$, we can define the *fibre of $\mathcal{F}$ over x* as a category $\mathcal{F}(x)$ with objects and maps those that lift x and id_x respectively. We would then like to define a functor $\mathcal{C} \to \mathbf{Cat}$ via these fibres. The definition of an opfibration is almost enough to do this. Instead, the resulting construction is a pseudo-functor[1]. The issue stems from the fact that given two π-cocartesian lifts $f', \hat{f}$ of the same map $f : x \to y$ at the object X:

$$\begin{tikzcd}
 & & Z \arrow[dd, squiggly] \\
X \arrow[r, "f'"'] \arrow[urr, "\hat{f}"] \arrow[dd, squiggly] & Y \arrow[ur, "\cong"'] \arrow[dd, squiggly] & \\
 & & y \\
x \arrow[r, "f"'] \arrow[urr, "f"] & y \arrow[ur, "\mathrm{id}_y"'] &
\end{tikzcd}$$

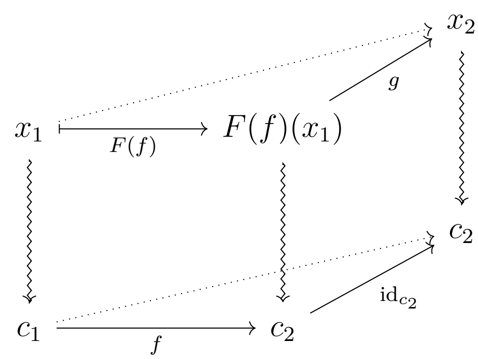
The identity id_y is only guaranteed to lift to an isomorphism. So there are choices in constructing a corresponding fibre functor. An opfibration $\pi : \mathcal{F} \to \mathcal{C}$ is *split* if the lifts

[1] A pseudo-functor is a functor where the relations defining functors only hold up to isomorphism (and obey lax coherent conditions).

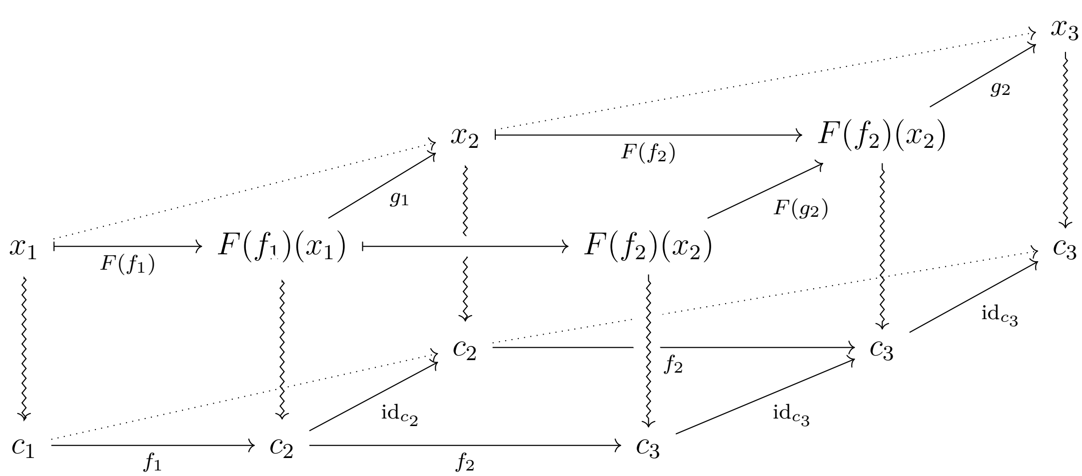
can be chosen such that the resulting fibre pseudo-functor is a functor.

We can also go in the opposite direction. Given a pseudo-functor $F : \mathcal{C} \to \mathbf{Cat}$, we can construct an opfibration via the *(covariant) Grothendieck construction* $\int_{\mathcal{C}} F \to \mathcal{C}$.

1.2.1 Definition. Given a pseudo-functor $F : \mathcal{C} \to \mathbf{Cat}$, the (covariant) Grothendieck construction $\int_{\mathcal{C}} F$ is a category with objects pairs (c, x) where $c \in \mathcal{C}$ and $x \in F(c)$, and morphisms are pairs $(f, g) : (c_1, x_1) \to (c_2, x_2)$ where $f : c_1 \to c_2$ and $g : F(f)(x_1) \to x_2$. It may be helpful to think of this definition in terms of the diagram



The dotted maps here don't signify anything besides showing the similarity to the above diagrams. Composition in $\int_{\mathcal{C}} F$ is given by $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ F(g_2))$. Again, we can see think of this in terms of the following diagram.



The projection $\int_{\mathcal{C}} F \to \mathcal{C}$ defines an opfibration. Hopefully, the above diagrams make this easy to see.

The Grothendieck construction gives us a correspondence between opfibrations and psuedofunctors, and for us, correspondence between functors $F : \mathcal{B}ij \to \mathbf{Cat}$ and split opfibrations $\int F \to \mathcal{B}ij$. If F is moreover a **Cat**-operad, then we can think of the operad structure as determining a "partial disjoint union" on the Grothendieck construction $\int F$ that lifts the one on $\mathcal{B}ij$. We then might want to consider "operads indexed on $\int F$".

To make this explicit, let us define the following.

1.2.2 Definition. Given a functor $\pi : \mathcal{C} \to \mathcal{B}ij$. A *(strict) partial monoidal product* on $\mathcal{C}$ is:

(1) for every object $X \in \underline{\mathcal{C}}(J), Y \in \underline{\mathcal{C}}(K)$ and $j \in J$, a choice of object $X \sqcup_j Y \in \underline{\mathcal{C}}(J \sqcup_j K)$;

(2) For every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in C and $j \in \pi(X)$, a choice of morphism

$$f \sqcup_j g : X \sqcup_j Y \to X' \sqcup_{\pi(f)(j)} Y'$$

where $f \sqcup_j g$ is an identity morphism if both f and g are;

(3) for every singleton set $S = \{s\}$, a distinguished object $\mathbb{1}_S \in \underline{\mathcal{C}}(S)$.

Such that the following hold:

(1) *functoriality.* For every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in C and $j \in \pi(X)$ the following commutes

$$\begin{tikzcd}
X \sqcup_j Y \arrow[r, "\mathrm{id}_X \sqcup_j g"] \arrow[d, "f \sqcup_j \mathrm{id}_Y"'] & X \sqcup_j Y' \arrow[d, "f \sqcup_j \mathrm{id}_{Y'}"] \\
X' \sqcup_{\pi(f)(j)} Y \arrow[r, "\mathrm{id}_{X'} \sqcup_{\pi(f)(j)} g"'] & X' \sqcup_{\pi(f)(j)} Y'
\end{tikzcd} \tag{1.4}$$

and the diagonal is equal to the morphism $f \sqcup_j g$.

(2) *(strict) associativity.* For objects $X \in \underline{\mathcal{C}}(J), Y \in \underline{\mathcal{C}}(K), Z \in \underline{\mathcal{C}}(L)$ and $j \in J, k \in K$

then we have that

$$(X \sqcup_j Y) \sqcup_k Z = X \sqcup_j (Y \sqcup_k Z) \tag{1.5}$$

(3) *(strict) unitality.* For any singleton $S = \{s\}$ and finite set J with $j \in J$ we have that

$$\mathbb{1}_S \sqcup_s X = X = X \sqcup_j \mathbb{1}_{\{j\}}. \tag{1.6}$$

The following is straightforward, and we omit the proof. It follows from comparing Definitions 1.1.5 and 1.2.2, noting that the partial disjoint union on $\mathcal{B}ij$ is strict, and since our opfibration is split, we have that the composition of the fibre functors is strict.

1.2.3 Lemma. *Let $\pi : \mathcal{C} \to \mathcal{B}ij$ be a split opfibration. Then there is a bijective correspondence between strict partial monoidal products on $\mathcal{C}$ and the structure of a* **Cat***-operad on the fibre functor $\underline{\mathcal{C}} : \mathcal{B}ij \to$* **Cat***.*

1.2.4 Definition. Let $\pi : \mathcal{C} \to \mathcal{B}ij$ be a split opfibration with a strict partial monoidal product. A functor $F : \mathcal{C} \to V$ is a *$\mathcal{C}$-indexed operad* if for finite sets $j \in J, K$ and objects $X \in \underline{\mathcal{C}}(J)$, $Y \in \underline{\mathcal{C}}(K)$, there exists a morphism

$$\circ_{j,X,Y} : F(X) \otimes F(Y) \to F(X \sqcup_j Y)$$

and for each singleton S, there exists a morphism $u_S : \mathbb{1} \to F(\mathbb{1}_S)$ such that the analogues of the diagrams in Definition 1.1.5 hold.

1.2.5 Theorem. *Let $\pi : \mathcal{C} \to \mathcal{B}ij$ be a split opfibration with a strict partial monoidal product. For any $\mathcal{C}$-indexed operad $F : \mathcal{C} \to V$, the left Kan extension*

$$\mathrm{Lan}_\pi F : \mathcal{B}ij \to V$$

is an operad.

Proof. Since π is an opfibration, for $J \in \mathcal{F}in$, the left Kan extension $\operatorname{Lan}_\pi F(J)$ is given by the colimit over the fibre of J,

$$\operatorname{Lan}_\pi F(J) \cong \operatorname*{colim}_{\underline{\mathcal{C}}(J)} F.$$

The symmetric monoidal category V is closed, and so the tensor commutes with colimits. Hence, for $j \in J, K$ we have induced maps

$$\begin{aligned}\odot_{j,J,K} : \operatorname*{colim}_{X \in \underline{\mathcal{C}}(J)} F(X) \otimes \operatorname*{colim}_{Y \in \underline{\mathcal{C}}(K)} F(Y) &\xrightarrow{\cong} \operatorname*{colim}_{(X,Y) \in \underline{\mathcal{C}}(J) \times \underline{\mathcal{C}}(K)} F(X) \otimes F(Y) \\ &\xrightarrow{\overline{\circ}} \operatorname*{colim}_{Z \in \underline{\mathcal{C}}(J \sqcup_j K)} F(Z).\end{aligned}$$

The map $\overline{\circ}$ is the map induced by the partial composition of F. It is then straightforward to prove that $\odot_{j,J,K}$ defines an associative partial composition for $\operatorname{Lan}_\pi F$ using the universal property of colimits. We get two-sided units via the maps

$$\mathbb{1} \xrightarrow{u_F(S)} F(\mathbb{1}_S) \to \operatorname*{colim}_{\underline{\mathcal{C}}(S)} F,$$

that satisfies the required diagrams from similar arguments used previously. $\square$

After examining the proof of the previous theorem, one realises that it essentially says that colimits along the fibres "preserve the operadic structure". This leads to a mild generalisation, where we instead take weighted colimits. First, let us observe that if $\pi : \mathcal{C} \to \mathcal{B}ij$ has a strict partial monoidal product, then so does $(\pi)^{\mathrm{op}} : (\mathcal{C})^{\mathrm{op}} \to (\mathcal{B}ij)^{\mathrm{op}} \cong \mathcal{B}ij$. This follows by noticing that for every pair of morphisms $f : X \to X'$, $g : Y \to Y'$ in C and $j \in \pi(X')$, we simply define

$$f^* \sqcap_j g^* := (f \sqcup_{\pi(f)^{-1}(j)} g)^*.$$

This satisfies the conditions of Definition 1.2.2 (where the base is $(\mathcal{B}ij)^{\mathrm{op}}$ instead). Note that we are identifying $(\mathcal{B}ij)^{\mathrm{op}}$ with $\mathcal{B}ij$ through the inverse functor in this case. In particular, the $f^* : X \to Y$ in $(\mathcal{B}ij)^{\mathrm{op}}$ corresponds to the map $f^{-1} : X \to Y$ in $\mathcal{B}ij$. Pay attention here to the fact that *the variance of these maps is the same.*

1.2.6 Definition. Let $\pi : \mathcal{C} \to \mathcal{B}ij$ be a split opfibration with a strict partial monoidal product. A $\mathcal{C}$-indexed weight is a $(\mathcal{C})^{\mathrm{op}}$-indexed operad.

Suppose we have a split opfibration $\pi : \mathcal{C} \to \mathcal{B}ij$ such that $(\pi)^{\mathrm{op}}$ is also a split opfibration, a $\mathcal{C}$-indexed operad $F : \mathcal{C} \to V$, and a $\mathcal{C}$-indexed weight $G : (\mathcal{C})^{\mathrm{op}} \to V$. We can construct a functor

$$G \otimes_{\underline{\mathcal{C}}} F : \mathcal{B}ij \to V$$

as follows: First, as V is closed and bicomplete, we can do the canonical base-change and upgrade our functors F, and G into enriched V-functors. For $J \in \mathcal{B}ij$, we set

$$(G \otimes_{\underline{\mathcal{C}}} F)(J) := G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \star F\Big|_{\underline{\mathcal{C}}(J)}.$$

Note that this is just the unenriched functor tensor product $G|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \otimes_{\underline{\mathcal{C}}(J)} F|_{\underline{\mathcal{C}}(J)}$. We are viewing these as enriched functors to easily justify a future step. Given a map $\phi : J \to K$ in $\mathcal{B}ij$, we get from the opfibration structures that there are natural transformations

$$(\underline{\mathcal{C}})^{\mathrm{op}}((\phi^{-1})^*) : G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \Rightarrow G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(K)} \text{ and } \underline{\mathcal{C}}(\phi) : F\Big|_{\underline{\mathcal{C}}(J)} \Rightarrow F\Big|_{\underline{\mathcal{C}}(K)}.$$

These behave well with compositions and induce maps in the colimit.

$$(G \otimes_{\underline{\mathcal{C}}} F)(\phi) := G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \star F\Big|_{\underline{\mathcal{C}}(J)}.$$

Hence, we have constructed a functor $(G \otimes_{\underline{\mathcal{C}}} F)$.

We have phrased this construction in terms of V-weighted colimits to make use of the fact that, as V is closed, V-colimits commute with the tensor. In particular, for $j \in J, K$ we have the composition

$$\begin{aligned}\odot_j : \left(G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \star F\Big|_{\underline{\mathcal{C}}(J)}\right) \otimes \left(G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(K)} \star F\Big|_{\underline{\mathcal{C}}(K)}\right) \\ \xrightarrow{\cong} \left(G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J)} \otimes G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(K)}\right) \star \left(F\Big|_{\underline{\mathcal{C}}(J)} \otimes F\Big|_{\underline{\mathcal{C}}(K)}\right) \\ \xrightarrow{\sqcap_j \otimes \sqcup_j} \left(G\Big|_{(\underline{\mathcal{C}})^{\mathrm{op}}(J \sqcup_j K)} \star F\Big|_{\underline{\mathcal{C}}(J \sqcup_j K)}\right).\end{aligned}$$

and via the same reason as in Theorem 1.2.5, this induces an operad structure on $(G \otimes_{\underline{\mathcal{C}}} F)$. To summarise, we have that:

1.2.7 Theorem. *Suppose we have a split opfibration $\pi : \mathcal{C} \to \mathcal{B}ij$ such that $(\pi)^{op}$ is also a split opfibration, a $\mathcal{C}$-indexed operad $F : \mathcal{C} \to V$, and a $\mathcal{C}$-indexed weight $G : (\mathcal{C})^{op} \to V$. Then the induced functor $G \otimes_{\underline{\mathcal{C}}} F : \mathcal{B}ij \to V$ inherits an operadic structure from those of F and G.*

1.3 A category of trees

Operads and trees are intimately related. Trees are convenient combinatorial gadgets to keep track of repeated compositions of an operad, and something we will make use of throughout this dissertation. Roughly, by a tree, we mean a rooted tree – a graph with no loops and a distinguished root node. Since we need to be exact later, we will use the following model for trees. Other options include the broad posets [Wei17], and polynomial functors [Koc11]. We denote the category of finite pointed sets by $\mathcal{F}in_*$.

1.3.1 Definition. A (symmetric) rooted tree T is the data of a finite set of vertices $\mathrm{V}(T)$, a pointed finite set of edges $\mathrm{E}(T)$ where the distinguished point is the *root edge* which we denote by r, and also two functions:

(1) an injective function $o_T : \mathrm{V}(T) \longrightarrow \mathrm{E}(T)$ where we call $o_T(v)$ the output edge of vertex v,

(2) a function $i_T : \mathrm{E}(T)\backslash\{r\} \longrightarrow \mathrm{V}(T)$ where we call edge e an input edge of vertex v if $i_T(e) = v$.

From this data we have the *edge walking function*

$$\omega_T : \mathrm{E}(T)\backslash\{r\} \longrightarrow \mathrm{E}(T)$$

given by $\omega_T = o_T \circ i_T$. We require that for every non-root edge e, there is a finite k such that $\omega_T^k(e) = r$. The edge walking function lets us define a *descendant poset structure* on edges where for edges $e, f \in \mathrm{E}(T)$ we have $e \leq_d f$ if $\omega(f)^k = e$ for some k. We also have a descendant poset structure on vertices in the same manner.

1.3.2 Example. Let us give a quick illustration of this definition. Consider fig. 1.1. This is a tree with vertex set

$$\mathrm{V}(T) = \{x, y, w, u\}$$

and edge set

$$\mathrm{E}(T) = \{a, b, c, d, e, f\}$$

where f is the basepoint of $\mathrm{E}(T)$. The functions i_T and o_T are also illustrated. The descendant poset structure then corresponds to edges coming from lower edges.

Given a tree T, we will define two functors (viewing sets as discrete categories)

$$\begin{aligned} |-| : \mathrm{V}(T) &\to \mathcal{F}in \\ v &\mapsto i_T^{-1}(v) \end{aligned}$$

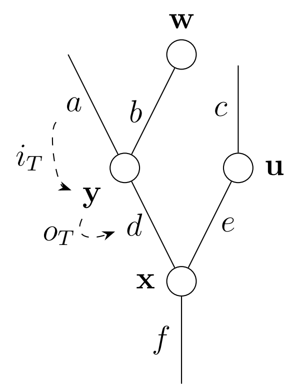


Figure 1.1: An example of a rooted tree

and,

$$
\begin{aligned}
[-] : \mathrm{V}(T) &\to \mathscr{F}in_* \\
v &\mapsto i_T^{-1}(v) \cup o_T(v)
\end{aligned}
$$

where the output edge $o_T(v)$ is the base point of $[v]$. We will call the edges $|v|$ the *input edges of* v and the edges $[v]$ the *adjacent edges of* v.

1.3.3 Convention. We will restrict ourselves to only trees where each vertex $v \in \mathrm{V}(T)$ is the set $|v|$. i.e., the vertices are the sets of its incoming edges. We want to do this so our trees are completely determined by $\mathrm{E}(T)$ and the relations between them (This is made explicit by broad posets). If we don't, we will not have a strict identity in some future constructions.

Some notation and terminology for trees that we will use are:

- A tree with no vertices and a single edge is called a trivial tree.
- The vertex v of a tree T such that $o_T(v)$ (if it exists) is the root edge is called the root vertex.
- A vertex with no input edge is called a *stump*.

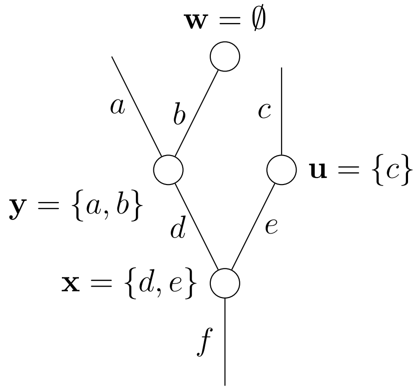


Figure 1.2: The tree of fig. 1.1 using Convention 1.3.3 on vertices

- A tree with a single vertex is called a *corolla*. Given a based finite set (X, x_0), we denote by t_X the corolla with edge set $E(t_X) = X$. Given a tree T and vertex $v \in \mathrm{V}(T)$, the corolla based at v is given by the corolla with edge set $[v]$. The corolla based at v is then $t_{[v]}$.

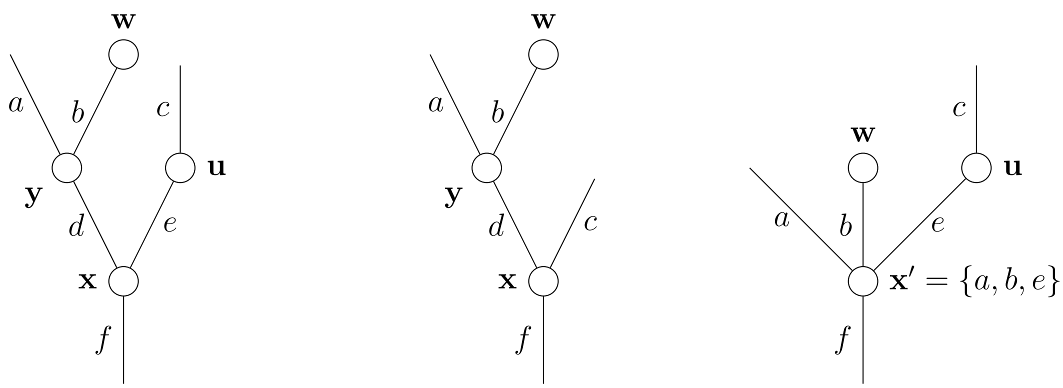
- Edges that aren't output edges of vertices are called input edges of the tree. We will denote the set of input edges of T by $|T|$.

- The set of internal edges (those edges neither an input edge nor a root edge) will be denoted by $\mathrm{E}^{\mathrm{inn}}(T)$.

- The *valence* or *degree* of a vertex is the cardinality of the input edges $|v|$. A vertex with one input edge is called an *unary* vertex

Given a tree T, we will use the following constructions generated from T.

(1) For a *unary* vertex $v \in \mathrm{V}(T)$, the tree $T \backslash v$ is given by removing the edge $o_T(v)$ (and the vertex v), and for $e \in |v|$, redefining the edge output to be $o_{T\backslash v}(i) = \omega_T(v)$. Geometrically, this removes the vertex v and reattaching the input edges to the vertex to which v was attached. We will not apply this to the root vertex. See fig. 1.3b.



(a) Tree T (b) Tree $T\backslash u$ (c) Tree T/d

Figure 1.3: Examples of tree constructions.

(2) For an internal edge $e \in \mathrm{E}^{\mathrm{inn}}(T)$, the tree T/e is given by removing the edge e and joining the vertices $a = o_T^{-1}(e)$ and $b = i_T(e)$ into a new vertex given by $|b| \sqcup_e [a]$. Geometrically, this corresponds to collapsing the edge e. See fig. 1.3c.

(3) Given two trees T and T', for an input edge $e \in |T|$ the *tree grafting along* e, written $T \circ_e T'$ is given by removing the root edge of T', taking the union of the edge and vertex sets and setting $o_{T \circ_e T'}(v) = e$ where v is the root vertex of T'.

1.3.4 Definition. We will define a category $\mathfrak{T}$ as follows: The objects will be rooted trees. The morphisms will be maps on the vertices generated by three kinds of maps:

(1) *inner face maps.* Given an internal edge $e \in \mathrm{E}^{\mathrm{inn}}(T)$, we have inner face maps

$$d_v : \mathrm{V}(T) \to \mathrm{V}(T/e)$$

these are identity on vertices except those on the collapsed edge which are mapped to the joined vertex in T/e.

(2) *degeneracies.* For a vertex $v \in \mathrm{V}(T)$ of degree 1, we have degenerate maps

$$s_v : \mathrm{V}(T\backslash v) \to \mathrm{V}(T).$$

This is the injective map on vertices that misses the vertex v of T.

(3) *tree isomorphisms.* Bijective functions $\mathrm{V}(T) \to \mathrm{V}(T')$ that commute with the edge walking functions ω.

We will write $\mathcal{T}_{\mathrm{iso}}$ for the subcategory generated by just the isomorphisms, and $\mathcal{T}_{\mathrm{inert}}$ for the subcategory generated by isomorphisms and degeneracies.

1.3.5 Remark. Readers familiar with the dendroidal category can identify this with a subcategory of the opposite dendroidal category. Another related interpretation is that this category is the category of "augmented dendroidal intervals".

We have the following, which is dual to the usual factorisation in the dendroidal category (see [MW07], also [HM22, Proposition 3.9] for a direct proof).

1.3.6 Lemma. *For a morphism $f : T \to T'$ in $\mathcal{T}$. There exists a factorisation*

$$f : T \xrightarrow{F} S \xrightarrow{I} S' \xrightarrow{D} T'$$

where F is an inner face map, I an isomorphism, and D a degeneracy map. Moreover, given any other factorization (F', I', D') there exists unique isomorphisms α and β such that the following commutes.

$$\begin{tikzcd}
 & S \arrow[r, "I"] \arrow[dd, "\alpha"] & S' \arrow[dr, "D"] \arrow[dd, "\beta"] & \\
T \arrow[ur, "F"] \arrow[dr, "F'"'] & & & T' \\
 & S'' \arrow[r, "I'"] & S'' \arrow[ur, "D'"'] &
\end{tikzcd}$$

The input function on trees $|T|$ defines a functor

$$\pi : \mathcal{T} \to \mathcal{B}ij.$$

This maps into $\mathcal{B}ij$ because the only morphism of our category that affects the inputs of the tree are the isomorphisms. In particular, we see that for face maps and degeneracy maps, we have that $\pi(d) = \pi(s) = \mathrm{id}_{|T|}$. Since each morphism ϕ in $\mathcal{B}ij$ has a lift starting (ending) at any object in the fibre of the source (target) of ϕ that is an isomorphism, we conclude that the functor π is a split bifibration. Moreover, an easy check shows us that tree grafting satisfies the conditions of a strict partial monoidal product from Definition 1.2.2. Hence we have the following:

1.3.7 Lemma. *The categories $\mathcal{T}_{\mathrm{iso}}$, $\mathcal{T}_{\mathrm{inert}}$, and $\mathcal{T}$ have the structure of* **Cat***-operads induced from tree grafting.*

1.4 Resolutions of operads

As mentioned in remark 1.3.5, we can consider the category $\mathcal{T}$ as a " category of augmented dendroidal intervals". Our purpose for introducing them is that they are natural categories to build simple operadic resolutions in a similar way the augmented simplex category leads to simplicial resolutions. This section will introduce some notation and give a simple example of this approach in action.

1.4.1 Definition. Given a G-symmetric collection $K : \mathcal{B}ij \to \mathrm{Top}^G$, we define a functor $\mathrm{R}_{\mathrm{iso}}(K) : \mathcal{T}_{\mathrm{iso}} \to \mathrm{Top}^G$ given on objects by

$$\mathrm{R}_{\mathrm{iso}}(K)(T) := \left\{ f \in \mathrm{Top}^G\left(\mathrm{V}(T), \coprod_{v \in \mathrm{V}(T)} K(|v|)\right) \,\middle|\, f(v) \in K(|v|) \right\}.$$

This is topologized as a subspace of $\prod_{v\in \mathrm{V}(T)} K(|v|)$. One can think of $f \in \mathrm{R}_{\mathrm{iso}}(K)(T)$ as a vertex labelling function that labels a vertex of valence J with an element from $K(J)$. The functor on an isomorphism $\phi : T \to T'$ is given by $\mathrm{R}_{\mathrm{iso}}(K)(\phi)(f) = f \circ \phi^{-1}$, the pullback of the inverse.

1.4.2 Definition. Given a *based* G-symmetric collection $K : \mathcal{B}ij \to \mathrm{Top}^G$, we can extend the previous functor to $\mathrm{R}_{\mathrm{inert}}(K) : \mathcal{T}_{\mathrm{inert}} \to \mathrm{Top}^G$ where on degeneracy maps $s_v : T\backslash v \to T$, we set

$$\mathrm{R}_{\mathrm{inert}}(K)(s_v)(f) = \begin{cases} f(w) & w \neq v \\ * & w = v. \end{cases}$$

That is, it's the map extending f by putting the base point on the new vertex v.

1.4.3 Definition. Given a *G-operad* $K : \mathcal{B}ij \to \mathrm{Top}^G$, we can extend the previous functor to $\mathrm{R}(K) : \mathcal{T} \to \mathrm{Top}^G$ where on internal face maps d_e where the edge is $e = (v, w)$ by

$$\mathrm{R}(K)(d_e)(f)(u) := \begin{cases} f(u) & u \neq v, w \\ f(w) \circ_e f(v) & u = \{v, w\}. \end{cases}$$

That is, we compose the vertices along the collapsing edge.

Since tree grafting doesn't affect the vertices, the following is straightforward.

1.4.4 Lemma. *Let $\mathcal{T}_* = \mathcal{T}, \mathcal{T}_{\mathrm{inert}}$, or $\mathcal{T}_{\mathrm{iso}}$. Tree grafting is a strict partial monoidal product on the tree category $\mathcal{T}_*$ and the functor $\mathrm{R}_*(K) : \mathcal{T}_* \to \mathrm{Top}^G$, from the correct form of Definitions 1.4.1 to 1.4.3, has a partial composition product given by taking the coproduct of functions.*

So $\mathrm{R}_*(K)$ are $\mathcal{T}_*$-indexed operads. From Theorem 1.2.5, the left Kan extensions of these then have induced operad structures. These operads are familiar. Let $\sigma : \mathcal{B}ij \to \mathcal{T}_*$ be the functor that maps $J \mapsto t_{[J \sqcup *]}$. Note that σ is cofinal for $\mathcal{T}$ and $\pi\sigma = \mathrm{id}$. Since the left Kan extension is given by the colimit over each fibre, we immediately have that $\mathrm{Lan}_\pi \mathrm{R}(P) \cong P$ is an operad.

1.4.5 Lemma. *The composite functor*

$$\mathcal{F} = \mathrm{Lan}_\pi \circ \mathrm{R} : \mathrm{Coll}^G \to \mathrm{Oper}^{\mathrm{G}}$$

is the free functor from G-collections to G-operads. Similarly,

$$\mathcal{F}_* = \mathrm{Lan}_\pi \circ \mathrm{R} : \mathrm{Coll}_*^G \to \mathrm{Oper}^{\mathrm{G}}$$

is the free functor from based G-collections to G-operads.

Proof. We will just prove this for $\mathcal{F}$. The other case follows similarly. Let $K \in \mathrm{Coll}^G$, and $P \in \mathrm{Oper}^{\mathrm{G}}$. We will denote the forgetful functor $\mathrm{Oper}^{\mathrm{G}} \to \mathrm{Coll}^G$ by U. Take any Coll^G-map $f : K \to UP$ and apply the functor $\mathrm{R}_{\mathrm{iso}}$ to get

$$\mathrm{R}_{\mathrm{iso}}(f) : \mathrm{R}_{\mathrm{iso}}(K) \Rightarrow \mathrm{R}_{\mathrm{iso}}(UP).$$

Note for later that $\mathrm{R}_{\mathrm{iso}}$ is faithful on Coll^G. The isomorphism $\mathrm{Lan}_\pi \mathrm{R}\, P \cong P$ corresponds to a map $\mathrm{R}(P) \to P \circ \pi$ which we precompose with the inclusion functor $i : \mathcal{T}_{\mathrm{iso}} \to \mathcal{T}$ to get

$$\mathrm{R}(P) \circ i \Rightarrow P \circ \pi \circ i.$$

Notice that $\mathrm{R}(P) \circ i = \mathrm{R}_{\mathrm{iso}}\, UP$ and so we get a composite

$$\mathrm{R}_{\mathrm{iso}}(K) \Rightarrow \mathrm{R}_{\mathrm{iso}}(UP) = \mathrm{R}(P) \circ i \Rightarrow P \circ \pi \circ i.$$

Via the left Kan extension adjunction, we get that this corresponds to a map

$$\mathrm{Lan}_{\pi_{\mathrm{iso}}} \mathrm{R}_{\mathrm{iso}}(K) \Rightarrow \mathrm{Lan}_{\pi_{\mathrm{iso}}} \mathrm{R}_{\mathrm{iso}}(UP) \overset{\alpha}{\Rightarrow} P$$

where $\pi_{\mathrm{iso}} = \pi \circ i$. Since $\mathrm{R}_{\mathrm{iso}}$ is faithful we see that (UP, α) is terminal in $\mathrm{Lan}_{\pi_{\mathrm{iso}}} \mathrm{R}_{\mathrm{iso}} / P$ and hence we conclude that $\mathrm{Lan}_{\pi_{\mathrm{iso}}} \mathrm{R}_{\mathrm{iso}}$ is left adjoint to U. $\square$

1.5 Homotopy of equivariant operads

We denote the category of topological G-spaces and equivariant maps Top^G. A G-operad is an operad in $\mathrm{Oper}\,\mathrm{Top}^G =: \mathrm{Oper}^{\mathrm{G}}$. A *reduced*[2] G-operad P is one in which $P(0) = P(\emptyset) = *$. We will denote the category of reduced operads by $\mathrm{Oper}_0^{\mathrm{G}}$. An important example of G-operads is the following.

We recall the following model structure on Top^G (The homotopical structure is due to Bredon [Bre67]).

1.5.1 Theorem. *For any family of subgroups $\mathcal{F}$ of G, there exists a cofibrantly generated model structure on Top^G, called the $\mathcal{F}$-model structure, where*

- *a map $f : X \to Y$ in Top^G is a fibration (weak equivalence) if and only if $f^H : X^H \to Y^H$ is a fibration (weak equivalence) in Top for all $H \in \mathcal{F}$,*
- *and the generating (trivial) cofibrations are given by*

$$I_{\mathcal{F}} = \left\{ G/H \times S^n \to D^{n+1} \;\middle|\; n \geq 0,\ H \in \mathcal{F} \right\}$$
$$J_{\mathcal{F}} = \left\{ G/H \times D^n \to D^n \;\middle|\; n \geq 0,\ H \in \mathcal{F} \right\}.$$

We will primarily work in the model structure associated with the complete family. Unless we specify otherwise, weak equivalence, fibrations, and cofibrations mean being taken with respect to the complete model structure.

Using the equivalence between $\mathcal{B}ij$-modules and Σ-modules, we have that

$$\mathrm{Coll}^G := \mathrm{Coll}(\mathrm{Top}^G) \simeq \prod_{n \geq 0} \mathrm{Top}^{G \times \Sigma_n}.$$

So given a sequence of subgroup families $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ where $\mathcal{F}$ is a subgroup family of $G \times \Sigma_n$,

[2]These are sometimes called unital operads.

we can put a model structure onto Coll^G via the product model structure. A result of Gutiérrez-White [GW18] tells us that we can transfer this model structure to $\mathrm{Oper}(\mathrm{Top}^G)$ via the free-forgetful adjunction

$$F : \mathrm{Coll}^G \rightleftarrows \mathrm{Oper}^{\mathrm{G}} : U.$$

1.5.2 Theorem. *For a sequence of subgroup families $\{\mathcal{F}_n\}$ as above, there exists a model structure on* $\mathrm{Oper}^{\mathrm{G}}$*, called the $\mathcal{F}$-model structure, where a morphism G-operads $f : P \to Q$ is (1) a fibration (weak equivalence) if the underlying morphism of symmetric collections $Uf : UP \to UQ$ is a fibration (weak equivalence) in the product model structure, (2) a cofibration if it has the left lifting property with respect to the acyclic fibrations.*

We will generally work in the complete version, just as in the case of Top^G.

We won't use the model structures too much in this thesis. Much of what we do will be in terms of explicit homotopies.

1.5.3 Definition. Given a map of G-symmetric collections A and B, a G-homotopy H of symmetric collections from A to B, written as

$$H : A \times I \to B$$

is the data of a collection of maps

$$\{H(J) : A(J) \times I \to B(J)\}_{J \in \mathcal{B}ij}$$

where each $H(J)$ is $G \times \mathrm{Aut}(J)$-equivariant.

1.5.4 Remark. Another way to think of Definition 1.5.3 is to observe that there is a faithful functor

$$\iota : \mathrm{Top}^G \hookrightarrow \mathrm{Coll}^G$$

where $\iota(X)$ is the symmetric collection given by

$$\iota(X)(J) = X$$

for all $J \in \mathcal{B}ij$. Here we take the trivial $\mathrm{Aut}(J)$-action on X. Then a G-homotopy of symmetric collections is a morphism of symmetric collections

$$H : A \times \iota(I) \to B.$$

1.5.5 Remark. Since the model structure on $\mathrm{Oper}^{\mathrm{G}}$ is the transferred model category from the product model structure on Coll^G. A morphism of operads $f : P \to Q$ is a weak equivalence if it has a weak left and right inverse in *symmetric collections* Coll^G.

1.5.6 Convention. We will often call symmetric collections just spaces and use common terminology of topology for symmetric collections. When we say this, we view symmetric collections as diagrams of spaces, or, phrased in the terminology of [Man+01], $\mathcal{B}ij$-spaces. Properties (definitions) that we say the space has means that every space in the diagram has that property (satisfies that definition).

A stronger form of homotopy of operads is the following.

1.5.7 Definition. We can put a monoid structure on the unit interval I by setting $ab = \max(a, b)$ for $a, b \in I$. Then $\iota(I)$ has the structure of an operad. If P and Q are operads, then a G-homotopy of operads is an operadic map

$$H : P \times \iota(I) \to Q$$

1.6 Little disks and $\mathbb{E}_V$-operads

The following is an operad of central importance to us.

1.6.1 Definition. Given a G-representation V and finite subrepresentation W, a little disk in W is an affine map $f : D(W) \to D(W)$ of the form $f(\vec{v}) = \alpha\vec{v} + \vec{b}$ where $\alpha \in \mathbb{R}_{>0}$ and $\vec{b} \in D(W)$. Here $D(W)$ means the unit disk in W. We define a G-operad $\mathcal{D}_W(V)$ where, for $J \in \mathcal{F}in$, the component $\mathcal{D}_W(V)(J)$ is given by tuples $(f_j)_{j\in J}$ of little W-disks in V such that

$$f_i(D(W)) \cap f_j(D(W)) = \emptyset$$

for all $i \neq j$. The operadic composition is induced by function composition. Explicitly, given $K \xrightarrow{\alpha} J$ in $\mathcal{F}in$, $g_j = (g_{j,k})_{k\in K_j} \in \mathcal{D}_W(V)(K_j)$, and $f = (f_j)_{j\in J} \in \mathcal{D}_W(V)(J)$ we have that

$$(f \circ (g_j)_{j\in J})_k := f_{\alpha(k)} \circ g_{j,k}$$

For finite subrepresentations $W \subseteq W'$ we have a morphism of operads $\mathcal{D}_W(V) \to \mathcal{D}_{W'}(V)$ induced by the map

$$f(\vec{w}) = \alpha\vec{w} + \vec{b} \mapsto f(\vec{w'}) = \alpha\vec{w'} + \vec{b}$$

on little disks. The G-operad of equivariant little V-disks is then given by

$$\mathcal{D}(V) = \operatorname*{colim}_{W\subseteq V} \mathcal{D}_W(V).$$

1.6.2 Definition. A G-operad is an $\mathbb{E}_V$-operad if there exists a zigzag of weak equivalences of G-operads to the little disk operad $\mathcal{D}(V)$.

1.7 The Boardman-Vogt tensor product

Given two operads P and Q, and an object X which has the structure of a P-algebra and Q-algebra.

$$\eta_P : P \to \mathrm{End}(X)$$
$$\eta_Q : Q \to \mathrm{End}(X)$$

We say these algebra structures *interchange* on X if for each $(K, L) \in \mathcal{B}ij \times \mathcal{B}ij$ the following commutes and is natural for each $\alpha \in P(K)$ and $\beta \in Q(L)$.

$$\begin{tikzcd}
 & \prod_{l \in L} \left(\prod_{k \in K} X\right) \arrow[r, "\prod_{l \in L} \eta_P(\alpha)"] & \prod_{l \in L} X \arrow[dr, "\eta_Q(\beta)"] & \\
\prod_{(k,l) \in K \times L} X \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & X \\
 & \prod_{k \in K} \left(\prod_{l \in L} X\right) \arrow[r, "\prod_{k \in K} \eta_Q(\beta)"'] & \prod_{k \in K} X \arrow[ur, "\eta_P(\alpha)"'] &
\end{tikzcd}$$

This diagram amounts to the X being a P-algebra in Q-algebras and vice-versa. Note that the isomorphisms on the left shuffle the factors around. An alternative version of this interchange condition is that the following commutes and is natural for each $K, L \in \mathcal{B}ij$.

$$
\begin{tikzcd}
 & P(K) \times \prod_{k \in K} Q(L) \arrow[r, "\eta_P \prod_{k \in K} \eta_Q"] & E(K) \times \prod_{k \in K} E(L) \arrow[ddr, "\gamma_E"] & \\
P(K) \times Q(L) \arrow[ur, "\mathrm{id} \times \Delta"] \arrow[dd, "\cong"] & & & \\
 & & & E(K \times L) \\
Q(L) \times P(K) \arrow[dr, "\mathrm{id} \times \Delta"] & & & \\
 & Q(L) \times \prod_{l \in L} P(K) \arrow[r, "\eta_Q \prod_{l \in L} \eta_P"'] & E(L) \times \prod_{l \in L} E(K) \arrow[uur, "\gamma_E"'] &
\end{tikzcd}
$$

where $E = \mathrm{End}(X)$ is the endomorphism operad of X. In general, this diagram works for any pair of operad maps $P \to Z, Q \to Z$.

1.7.1 Definition. Given operad maps $P \xrightarrow{f} Z$ and $P \xrightarrow{g} Z$, we say these interchange if they satisfy the interchange relations above. In this case we will call the pair (f, g) an interchanging pair. Given any morphism of operads $W \xrightarrow{h} Z$, then the composition induces a morphism of interchanging pairs $(f, g) \xrightarrow{h} (hf, hg)$ and we get a category of interchanging pairs.

The Boardman-Vogt tensor product is then defined by the following universal property.

1.7.2 Definition. The Boardman-Vogt tensor product is the initial object in the category of interchanging pairs. For operads P and Q, we will denote this by (i_P, i_Q) where

$$P \xrightarrow{i_P} P \otimes Q \xleftarrow{i_Q} Q.$$

Of course, we haven't justified that this exists. We will provide a proof of this in a future chapter.

Chapter 2

Little star operads

In this chapter, we build a generalisation to equivariant little disks that we call little star operads. The purpose of this generalisation is two-fold: (1) we want to build equivariant versions of other little shape operads that are better suited for additivity results that generalise those of Dunn [Dun88]; and (2) we want to show that these operads all encode the same operad up to homotopy. We do this in this chapter by first examining the relationship between embedding operads and the "ambient" operad, which doesn't enforce a non-overlapping condition. We also restrict the "flavour of embedding" which we call an affine type. This leads us to our definition of little star operads $\mathcal{S}^M(T)$, where M is the affine type, and T are star-shaped subsets of a G-representation V. We then start working on understanding how the affine type M and star-shape T change the homotopy type of the operad. To do this, we build intermediate little star operads $\mathcal{S}^M(S,T)$ for two different star shapes S and T. We will use these to show that *all our little star operads for the same representation are weakly equivalent.* This generalises the well-known fact that the little cubes and little disks are weakly equivalent in the non-equivariant setting.

Throughout this chapter, a G-representation V means a countable sum of finite dimensional orthogonal G-representation with an inner product. We will also use algebraic objects in Top^G, which we will call G-(algebraic object). e.g., a group object in Top^G

will be called G-group.

2.1 Embedding operads and affine types

First of all, let us recall what the embedding operad is non-equivariantly.

2.1.1 Definition. Given a topological space $U \subseteq \mathbb{R}^n$, the *embedding operad* $\mathrm{Emb}(U)$ has J-component given by J-tuples $(f_j)_{j\in J}$ where each f_j are embeddings $U \to U$ and

$$f_i(U) \cap f_j(U) = \emptyset$$

for all $i \neq j$. Composition is given by function composition, just as in the little disk operad.

The non-equivariant little cube and little disk operads are just special cases where we restrict the form the embeddings take and use specific subsets U. One thing to observe is that the embeddings of little cubes and disks are well-defined homeomorphisms on the entirety of $\mathbb{R}^n$. Little star operads will be embedding operads where we have chosen slightly more general kinds of embeddings and shapes than the little cube and disks.

As the name suggests, we want to embed the following type of shape.

2.1.2 Definition. Given a G-representation V, a G-star domain S in V is a G-invariant subset $S \subseteq V$ such that $0 \in S$ and for all points $x \in S$ and $t \in [0,1]$, we have that $tx \in S$. We also require that

(1) S is open, and

(2) it is non-degenerate: for all $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(1-\varepsilon)S + B(0;\delta) \subseteq S.$$

i.e., if we shrink S by any amount, then there is some small translation perturbation we can do that remains in the original shape S.

A *general or non-equivariant star domain* S is an $\{e\}$-star domain.

Besides the shape between the little disks and cubes, the type of embedding is slightly different. For little disks, these embeddings are translations and scaling. However, for cubes, the embeddings are rectilinear. i.e., maps of the form $\vec{v} \mapsto A\vec{v} + \vec{b}$ where A is diagonal with strictly positive diagonal elements. We want to consider both these kinds of embeddings and combinations of them. One viewpoint we take is that for a given G-representation V, we want to treat the indecomposable subrepresentations as "axis' of V". We don't want to scale differently in different directions in the same indecomposable. On the other hand, scaling differently for different indecomposables is fair game.

2.1.3 Definition. For a G-representation V, a *decomposition* is a collection of sub G-representations $\{V_i\}_{i\in I}$ such that $\bigoplus_{i\in I} V_i = V$. We will call it a *finite decomposition* if $|I|$ is finite. If every V_i is finite-dimensional, we will say it is a decomposition into finite subrepresentations.

The type of maps that build the equivariant little disks is the following type.

2.1.4 Definition. For a G-representation V, and finite dimensional subrepresentation $W \subseteq V$, we have a topological G-group given by

$$\Lambda_W(V) := \left\{ w \mapsto \alpha \operatorname{id}_W w + b \;\middle|\; \alpha \in \mathbb{R}_{>0}, b \in W \right\}$$

where the G-action is given by conjugation. This is topologized as a subspace $\mathbb{R}_{>0} \times W$. For each inclusion $W \subseteq W'$ there exists a continuous injective G-homomorphisms

$$\Lambda_W(V) \hookrightarrow \Lambda_{W'}(V)$$
$$(w \mapsto \alpha w + b) \mapsto (w' \mapsto \alpha w' + b).$$

We then define the G-group of *rigid dilations* on V by

$$\Lambda(V) := \operatorname*{colim}_{W \subseteq V} \Lambda_W(V).$$

The equivariant candidate for the types of embeddings we want is then the following.

2.1.5 Definition. Let V be a G-representation and $F = \{V_i\}_{i \in I}$ be a decomposition of V with each V_i finite, or I finite. For each inclusion of subsets $K \subseteq J \subseteq I$ where K, J are finite, we have a G-homomorphism

$$\prod_{k \in K} \Lambda(V_k) \hookrightarrow \prod_{j \in J} \Lambda(V_j)$$
$$(f_k)_{k \in K} \mapsto (\overline{f}_j)_{j \in J}$$

where

$$\overline{f}_j = \begin{cases} f_j & \text{if } j \in K \\ \mathrm{id}_{V_k} & \text{otherwise.} \end{cases}$$

The *affine type* on V generated by the decomposition F is given by

$$M(F) := \operatorname*{colim}_{K \subseteq I} \prod_{k \in K} \Lambda(V_k).$$

Note that we can identify $M(F)$ with the subspace

$$M(F) \cong \mathbb{R}_{>0}^{\widetilde{\oplus}|I|} \times V$$

where

$$\mathbb{R}_{>0}^{\widetilde{\oplus}|I|} = \{(\lambda_i)_{i \in I} \mid \lambda_i \neq 1 \text{ for finitely many } i\}.$$

i.e., $\mathbb{R}_{>0}^{\widetilde{\oplus}|I|} \cong \mathbb{R}^{\oplus|I|}$ under a homeomorphism induced by the exponential.

2.1.6 Definition. Given any $f \in \Lambda(V)$, we have that $f(D(V)) = \lambda D(V)$ for a scalar $\lambda \in \mathbb{R}_{>0}$. We will call this scalar λ the *dilation factor* of f and denote it by $\delta(f)$. In general, for $f \in M(F)$, restricting f onto any $V_i \in F$ has a dilation factor. Moreover, there are only finitely many possible values as we iterate over V_i. Hence, the minimal dilation factor exists, and we set in this case $\Delta(f) := \min_{i \in I} \delta(f|_{V_i})$.

2.1.7 Remark. It is also possible to extend affine types to include rotations. This gives us generalisations for operads like framed disks and the skew-cubes of Dwyer-Hess-Knudsen [DHK18]. Much of what we do extends to this context; however, the number of edge cases and other complications that arise increases exponentially. Since these cases aren't directly related to the overarching thesis of this dissertation, we have decided not to include these types to make our current presentation more transparent.

2.2 Little star operads of a single star domain

Before we move on to defining little star operads, there is one thing we would like to clarify. While embedding operads are geometric, they live in very much algebraically defined operads – a viewpoint we will find helpful throughout this dissertation. To explain this, let us review an important adjunction between monoids and operads ([Igu82], see also [FV15, Lemma 7.2]).

2.2.1 Definition. Given a topological G-monoid A, we can construct a reduced G-operad $\mathcal{O}(A)$ as follows. The J-ary components are given by

$$\mathcal{O}(A)(J) = \prod A^J,$$

and the composition is given by distributing the multiplication across the elements. That is, for a map of finite sets $\alpha : J \to K$, the composition is

$$\begin{aligned}\gamma(\alpha) : \mathcal{O}(A)(K) \times \prod_{k\in K} \mathcal{O}(A)(J_k) &\to \mathcal{O}(A)(J) \\ ((a_k)_{k\in K}, ((b_j)_{j\in J_k})_{k\in K}) &\mapsto (a_{\alpha(j)} b_j)_{j\in J}.\end{aligned}$$

2.2.2 Theorem. *The construction above is functorial and has a left adjoint given by taking the* 1*-ary components.*

$$\begin{tikzcd} \mathrm{Oper}_0^{\mathrm{G}} \arrow[r, bend left, "(\cdot)(1)"] \arrow[r, phantom, "\perp"] & \mathrm{Mon}(\mathrm{Top}^G) \arrow[l, bend left, "\mathcal{O}"] \end{tikzcd}$$

All our operads of interest then fall into operads of this type once we forget the non-overlapping condition of the embedding operads. For instance, for D the unit disk in V, if we write $\Lambda(V)(D)$ for the G-monoid of all maps $f \in \Lambda(V)$ such that $f(D) \subseteq D$. Then the little disk operad naturally sits inside the following.

$$\mathscr{D}(V) \subseteq \mathcal{O}(\Lambda(V)(D)) \subseteq \mathcal{O}(\Lambda(V))$$

Extending this idea, we define a coloured operad as a natural space for the little star operads to live in. First, the following is the generating G-monoid we want to construct it from.

2.2.3 Definition. Given an affine type M of V, and general star domains S, T, define a subspace of M given by

$$M(S,T) := \{f \in M \mid (g \cdot f)(S) \subseteq T \text{ for all } g \in G\}.$$

2.2.4 Remark. When S and T are G-star domains, then this simplifies and we have

$$M(S,T) = \{f \in M \mid f(S) \subseteq T\}.$$

2.2.5 Remark. It is natural to think of elements $f \in M(S,T)$ as restricted functions $f : S \to T$; indeed, we have chosen notation to reflect this. However, it is crucial to keep in mind that they are not. For instance, the group conjugation on f wouldn't make sense. We may occasionally refer to S as the domain and T as the codomain - but we never restrict our functions in this way.

2.2.6 Example. Let us illustrate what the space $M(S,T)$ can look like. Consider the nontrivial irreducible representation of $C_3 = \mathbb{Z}/3\mathbb{Z}\langle\tau\rangle$; the affine type M is the rigid dilations Λ, S is a triangle, and T is a square. We have two maps $f, g \in M$ as described in fig. 2.1 by their images. The map f is in $M(S,T)$ because its entire orbit is inside

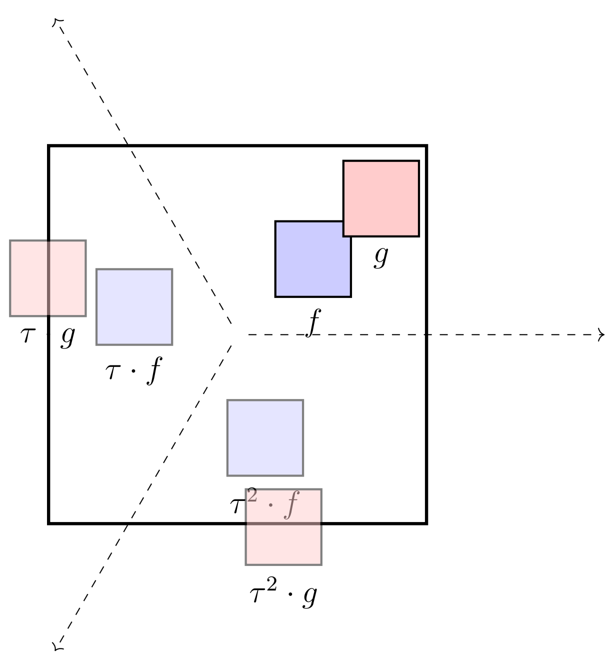


Figure 2.1: Example of $M(S,T)$ for C_3 representation.

the square. The map g, on the other hand, isn't because its orbit isn't contained in the square. Note that for any $g \in G$, $(g \cdot f)(S) \subseteq T$ is equivalent to $f(g^{-1}S) \subseteq g^{-1}T$, and so we can instead geometrically view the containment condition as not requiring the orbit of f being contained, but instead that the orbit of the domain shape is included in the orbit of the codomain shape.

2.2.7 Lemma. *The subspace $M(S,T)$ is G-invariant and closed in M. Moreover, if $T \subseteq S$, it is a G-semigroup, and if $S = T$, it is a G-monoid.*

Proof. The fact it is G-invariant is clear from the definition. The semigroup and monoid statements are clear. Let us now prove that this is a closed subspace of M. Observe that M is a metric space so we can test closure with sequences. Suppose we have a sequence $(f^k)_{k\in\mathbb{N}}$ in $M(S,T)$ that converges to $f \in M$. Suppose for contradiction that $f(S) \nsubseteq T$. Since f is a homeomorphism on V and S is open, we have that $\overline{f(S)} \nsubseteq \overline{T}$ and there exists a point $f(x) \in \overline{f(S)} \backslash \overline{T}$. Now, we have that $f^k(x) \to f(x)$, and we have that $f^k(x) \in T$, which is a contradiction. Hence $f(S) \subseteq T$. We can similarly show that $(g \cdot f)(S) \subseteq T$ and so $f \in M(S,T)$. □

2.2.8 Definition. Given an affine type M on G-representation V, the *coloured ambient star operad* $\mathcal{A}(M)$ is a coloured Top^G-operad which has as set of objects $\mathrm{ob}(\mathcal{A}(M))$ star domains of V. The morphisms are given by

$$\mathcal{A}(M)(\{S_1, \ldots, S_n\}, T) = M(S_1, T) \times M(S_2, T) \times \cdots M(S_n, T)$$

and composition is given by distributing across tuples (as in $\mathcal{O}(M)$).

2.2.9 Definition. We also have the *coloured star embedding operad* $\mathcal{E}(M)$ given as a suboperad of $\mathcal{A}(M)$ with the same objects and morphisms $\mathcal{E}(M)(\{S_1, \ldots, S_n\}, T)$ given by $(f_i)_{i=1}^n \in \mathcal{A}(M)(\{S_1, \ldots, S_n\}, T)$ such that for all $i \neq j, g \in G$,

$$(g \cdot f_i)(S_i) \cap (g \cdot f_j)(S_j) = \emptyset.$$

In general, we will write ${}^nS := \{\underbrace{S, \ldots, S}_{n \text{ times}}\}$ and the corresponding 1-coloured operads we will write slightly differently as

$$\mathcal{A}^M(S)(n) := \mathcal{A}(M)({}^nS, S)$$
$$\mathcal{E}^M(S)(n) := \mathcal{E}(M)({}^nS, S).$$

Let us record a lemma about their topology here.

2.2.10 Lemma. *For affine type M on V and star domains $S_1, \ldots, S_n, T$, the sets*

$$\mathcal{A}(M)(\{S_1, \ldots, S_n\}, T) \textit{ and } \mathcal{E}(M)(\{S_1, \ldots, S_n\}, T)$$

are closed in M^n. One consequence of this is that the induced subspace topologies are compactly generated.

Proof. That $\mathcal{A}(M)(\{S_1, \ldots, S_n\}, T)$ is closed in M^n follows from Lemma 2.2.7.

We will show that the sets

$$\overline{\mathcal{E}}(M)(\{S_1, \ldots, S_n\}, T) := \left\{ (f_1, \ldots, f_n) \in M^n \,\middle|\, (g \cdot f_i)(S_i) \cap (g \cdot f_j)(S_j) = \emptyset \text{ for all } i \neq j \right\}$$

are closed in M^n. This implies the rest of the lemma as

$$\mathcal{E}(M)(\{S_1, \ldots, S_n\}, T) = \overline{\mathcal{E}}(M)(\{S_1, \ldots, S_n\}, T) \cap \mathcal{A}(M)(\{S_1, \ldots, S_n\}, T).$$

Suppose we have a sequence $(f^k)_{k\in\mathbb{N}}$ where

$$f^k = (f_1^k, \dots, f_n^k) \in \overline{\mathcal{E}}(M)(\{S_1, \dots, S_n\}, T)$$

that converges to $f = (f_1, \dots, f_n)$ in M^n. Let S be an arbitrary general star domain. We must have that the minimum dilation factors converge

$$\Delta(f_i^k) \to \Delta(f_i).$$

Hence, for some $\epsilon > 0$ small enough, there exists $K > 0$ such that

$$B(0; \epsilon) \subseteq f_i^k(B(0,1)) \text{ for all } i = 1, \dots, n, \text{ and } k > K.$$

Suppose $x \in S$. Then for some $\alpha > 0$ we have that $x \in B(x; \alpha) \subseteq S$ since S open, and as f_i^k are affine maps, we get for all $k > K$,

$$\begin{aligned} B(f_i^k(x); \alpha\epsilon) &= B(f_i^k(0); \alpha\epsilon) + f_i^k(x) \\ &\subseteq f_i^k(B(0; \alpha)) + f_i^k(x) \\ &= f_i^k(B(0; \alpha) + x) \\ &= f_i^k(B(x; \alpha)). \end{aligned}$$

In summary, for any $x \in X$, and i, we can find $\alpha > 0$ with $B(x; \alpha) \subseteq S$ such that for all $k > K$ we have $B(f_i^k(x); \alpha\epsilon) \subseteq f_i^k(B(x, \alpha))$.

Now, suppore we have for some $i \neq j$ that there exists $y \in f_i(S_i) \cap f_j(S_j)$. Then for some $x_1 \in S_i$, and $x_2 \in S_j$ such that $f_i(x_1) = f_j(x_2) = y$. The previous argument tells us that we can find $\alpha > 0$ with $B(x_1; \alpha) \subseteq S_i$ and $B(x_2; \alpha) \subseteq S_j$ such that for all large enough k, we have that

$$B(f_i^k(x_1);\alpha\epsilon) \subseteq f_i^k(B(x_1;\alpha)),$$

and

$$B(f_j^k(x_2);\alpha\epsilon) \subseteq f_j^k(B(x_2;\alpha)).$$

However, $f_i^k(x_1), f_j^k(x_2) \to y$ and so eventually y is in both $B(f_i^k(x_1);\alpha\epsilon)$ and $B(f_j^k(x_2);\alpha\epsilon)$. This contradicts that $f_i^k(S_i) \cap f_j^k(S_j) = \emptyset$. Hence $f_i(S_i) \cap f_j(S_y) = \emptyset$, and we get similar statements for the conjugated elements. Therefore we are done. $\square$

Unsurprisingly, we define the following.

2.2.11 Definition. Given a G-representation V, affine type M on V, and general star domain S. The *little star operad* $\mathcal{S}^M(S)$ is given by

$$\mathcal{S}^M(S) := \mathcal{E}^M(S).$$

Some examples of little star operads are the following.

2.2.12 Definition. For a G-representation V and M an affine type, let $C(V)$ be the unit cube with respect to some orthonormal basis. The *equivariant little cube operad* is $\mathcal{C}^M(V) := \mathcal{S}^M(C(V))$.

2.2.13 Remark. The equivariant little cubes $\mathcal{C}^M(V)$ are very similar to the little skew cube operads of Dwyer-Hess-Knudson [DHK18].

2.2.14 Definition. Let V be a G-representation and suppose $S = \{\vec{s}_i\}_{i\in I}$ is an affine independent subset of V such that (1) its affine span aff-span(S) is the entirety of V, and (2) the interior of its convex hull conv S contains the origin. Then the S-simplex

$$\Delta^S = \operatorname{conv}(S) = \left\{ \sum_{i\in I} \lambda_i \vec{s}_i \;\middle|\; \sum_{i\in I} \lambda_i = 1 \text{ where } \lambda_i \geq 0, \text{ finitely many non-zero} \right\}$$

is a general star domain. We call the resulting little star operad the *equivariant little simplex operad.*

$$\mathcal{D}^{\mathrm{S}}_{\Delta}(V) := \mathcal{S}^{\Lambda(V)}((\Delta^S)^{\circ})$$

2.2.15 Definition. Given a G-representation V and a decomposition $F = \{V_i\}_{i\in I}$, we can choose affine independent subsets S_i for each subspace V_i, and get a corresponding simplex Δ^{S_i} as in Definition 2.2.14. The *equivariant little product simplex operad* for the decomposition F is given by

$$\mathcal{D}^{\mathrm{F}}_{\Delta}(V) := \mathcal{S}^{M(F)}\left((\times_{i\in I}\Delta^{S_i})^{\circ}\right).$$

2.3 Little star operads of pairs of star domains

We want to compare little star operads $\mathcal{S}^M(S)$ for different star domains S. Given another star domain T, the little star operads $\mathcal{S}^M(S)$ and $\mathcal{S}^M(T)$ are different subspaces of the ambient star operad $\mathcal{A}^M(V)$, or in a more refined sense, objects of the coloured embedding operad $\mathcal{E}(M)$. From this viewpoint, there is an almost obvious candidate to use as a go-between for $\mathcal{S}^M(S)$ and $\mathcal{S}^M(T)$. Consider G-star domains S, T such that $T \subseteq S$. Then for all n we have that

$$\mathcal{E}(M)\left({}^nS, T\right) \subseteq \mathcal{E}(M)\left({}^nS, S\right) = \mathcal{E}^M(S)(n).$$

You can think of this as reinterpreting functions into ones with a larger codomain - although we never actually restrict our functions (See remark 2.2.5). Similarly, we have that

$$\mathcal{E}(M)\left({}^nS, T\right) \subseteq \mathcal{E}(M)\left({}^nT, T\right) = \mathcal{E}^M(T)(n).$$

This can be viewed as a restriction on a smaller domain. Observe that we can view $\{\mathcal{E}(M)\,({}^{n}S,T)\}_{n}$ as a sub-pseudo operad of $\mathcal{E}^{M}(T)$ since it is closed under composition, but it doesn't contain the unit. Note that a similar statement holds for $\mathcal{A}(M)$.

2.3.1 Definition. Given an affine type M on V and G-star domains $T \subseteq S$ then we have the G-pseudo operads with n-arity components given by

$$\mathcal{E}^{M}(S,T)(n) := \mathcal{E}(M)\,({}^{n}S,T)$$
$$\mathcal{A}^{M}(S,T)(n) := \mathcal{A}(M)\,({}^{n}S,T)\,.$$

Ultimately we want to compare the embedding 1-operads $\mathcal{E}^{M}(S)$ for different S by using $\mathcal{E}^{M}(S,T)$. Writing ι for the inclusion $T \hookrightarrow S$, this situation above can be drawn as:

$$\begin{tikzcd}
 & \mathcal{E}^{M}(S,T) \arrow[dl, "\iota^{*}"'] \arrow[dr, "\iota_{*}"] & \\
\mathcal{S}^{M}(T) = \mathcal{E}^{M}(T) & & \mathcal{E}^{M}(S) = \mathcal{S}^{M}(S)
\end{tikzcd}$$

The problem is that this is a diagram in pseudo operads, not operads. We will build little star operads of pairs that are a correction to this problem.

2.3.2 Definition. For a G-representation V and affine type M generated by the decomposition $F = \{V_i\}$. Write $\Lambda_I^F(V)$ for the G-monoid

$$\Lambda_I^F(V) := \prod_{i\in I}\{\lambda_i \,\mathrm{id}_{V_i} \mid \lambda_i \in (0,1] \text{ and finitely many } \lambda_i \neq 1.\}$$

The main idea to introduce an identity to the pseudo operads $\mathcal{E}^{M}(S,T)$ is to observe that for any $f \in M(S,T)$ and $\alpha \in \Lambda_I^F(V)$ we have that

$$f \circ \alpha, \text{ and } \alpha \circ f \in M(S,T)$$

since $\alpha(S) \subseteq S$ and $\alpha(T) \subseteq T$. Given a general element $f = (f_i)_{i \in I} \in \mathcal{E}^M(S,T)(I)$ where $|I| > 1$. We have for any $i \neq j \in I$ that

$$\begin{aligned} f_i \circ \alpha(S) \cap f_j \circ \alpha(S) &\subseteq f_i(S) \cap f_j(S) \\ &= \emptyset \end{aligned}$$

and

$$\begin{aligned} \alpha \circ f_i(S) \cap \alpha \circ f_j(S) &= \alpha\left(f_i(S) \cap f_j(S)\right) \\ &= \emptyset \end{aligned}$$

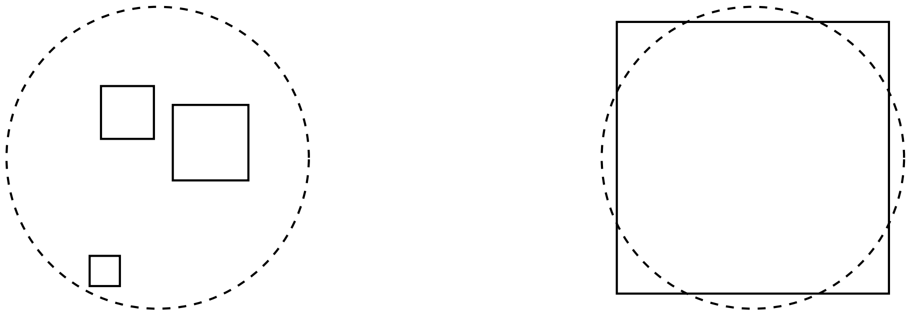
Hence, composition with elements of $\Lambda_I^F(V)$ doesn't change the non-intersecting condition. This means if we add elements from $\Lambda_I^F(V)$ to the unary component of $\mathcal{E}^M(S,T)$, this gives it an identity, but also, we don't need to change any other component to preserve its operad structure. We will use this idea to define little star operads. However, we will also give a variation required for the next chapter.

2.3.3 Definition. Let $M = M(F)$ be an affine type generated by a decomposition $F = \{V_i\}$ of V, S, T general star domains such that $T \subseteq S$. The *little star operad for the pair* S, T, $\mathcal{S}^M(S,T)$, is the operad

$$\mathcal{E}^M(S,T) \cup \Lambda_I^F(V)$$

If $S = T$, observe that $\Lambda_I(V) \subseteq \mathcal{E}^M(S,S)$ and we have $\mathcal{S}^M(S) = \mathcal{S}^M(S,S)$.

2.3.4 Example. Consider the case non-equivariant case when $V = \mathbb{R}^2$. Let $C = [-1,1] \times [-1,1]$ and D be the unit disk, then $D \subseteq C$ and so we have the operad $\mathcal{S}^M(C,D)$. For arity > 1, we can visualise elements as in fig. 2.2a, which are embeddings of cubes into the disk that don't overlap. For arity 1, we now allow our cube to be outside the disk as long as it is centred at the origin, see fig. 2.2b.



(a) An element of $\mathcal{S}^M(C, D)(3)$ (b) An element of $\mathcal{S}^M(C, D)(1)$

Figure 2.2: Elements of the operad $\mathcal{S}^M(C, D)$.

We also have the following variation, which has slightly better properties for the next chapter.

2.3.5 Definition. Let $M = M(F)$ be an affine type generated by a finite decomposition $F = \{V_i\}$ of V. For star domains $T \subseteq S$ in V, the *thick little star operad* is given by

$$\mathcal{S}_t^M(S, T) := \mathcal{E}^M(S, T) \cup (M(S, S) \cap M(T, T)) .$$

Here we again interpret the G-monoid $(M(S, S) \cap M(T, T))$ as a G-operad in degree 1. Note that we still have $\mathcal{S}_t^M(S) = \mathcal{S}^M(S)$. We will often write $\mathcal{S}_{(t)}^M(S, T)$ in statements to signify that they hold for both the normal and thick variants of little star operads.

Our small discussion at the beginning of this section justifies that $\mathcal{S}^M(S, T)$ is an operad. Let us now show that the fat variation is also an operad. We will also show that it satisfies a topological property we will need for the next chapter (and also the reason we call it "thick")

2.3.6 Lemma. *The thick little star operad $\mathcal{S}_t^M(S, T)$ is a G-operad. Moreover, for each $s \in \Lambda_I^F(V)$, and $\lambda \in (0, 1)$, there exists an open ball centered at λs of M, $B(\lambda s, \varepsilon) \subseteq M$, such that $B(\lambda s, \varepsilon) \subseteq M(S, S) \cap M(T, T)$.*

Proof. Write $\overline{M}$ for $M(S, S) \cap M(T, T)$. Given any $g \in \overline{M}$ and $f \in \mathcal{E}^M(S, T)$, we have that

$$(f \circ g)(S) \subseteq f(S) \subseteq T$$
$$(g \circ f)(S) \subseteq g(T) \subseteq T.$$

Hence, for any general element $(f_i)_{i\in I} \in \mathcal{E}^M(S,T)(I)$ we have for $g \in \overline{M}, (g_i)_{i\in I} \in \overline{M}^I$ that

$$g \circ (f_i)_{i\in I}, (f_i \circ g_i)_{i\in I} \in \mathcal{E}^M(S,T)(I).$$

So $\mathcal{S}_t^M(S,T)$ is an operad.

The second statement follows from the non-degenerate condition we impose on star domains. We require that the decomposition F is finite so that λs is a well-defined element of $\Lambda_I^F(V)$. Since $\lambda < 1$ we have that λs is shrinking in all directions. Using the non-degenerate condition of S and T, we know there exists a $\delta > 0$ such that

$$\lambda A + B(0;\delta) \subseteq A$$

where $A = S$, or T. Hence, for small enough $\varepsilon > 0$, there exists a ball $B(\lambda s;\varepsilon) \subseteq M = \mathbb{R}_{>0}^{\widetilde{\oplus} n} \times V$ such that

$$f(A) \subseteq A$$

for all $f \in B(\lambda s;\varepsilon)$. We are then finished the lemma. $\square$

2.3.7 Remark. The non-degenerate condition for our star domains is needed only for Lemma 2.3.6. If we remove that condition, then nothing changes for this chapter. In particular, the comparison theorems of the next chapter still work.

2.4 Comparison of little star operads

For this section, we will only work with the non-thick variant $\mathcal{S}^M(S,T)$ since this gives us the required comparison theorem without any extra complications. Although, the same statements hold with the thick variant.

To get a comparison between $\mathcal{S}^M(S)$ and $\mathcal{S}^M(T)$, we use that for $T \subseteq S$; we have the inclusions of operads.

$$\begin{tikzcd} & S^M(S,T) \arrow[dl, hook, "\iota^*"'] \arrow[dr, hook, "\iota_*"] & \\ S^M(T) & & S^M(S) \end{tikzcd}$$

Eventually, we want to show that $\mathcal{S}^M(S) \simeq \mathcal{S}^M(T)$. To do this, we can turn $\mathcal{S}^M(S,T)$ into $\mathcal{S}^M(T)$ by "shrinking the domain". Similarly, we can turn $\mathcal{S}^M(S,T)$ into $S^M(S)$ by "enlarging the codomain". We claim that such a transformation doesn't change the homotopy type. The following lemma makes this rigorous for scaling by fixed constants.

2.4.1 Lemma. *For $\lambda \leq 1$ and star domains S, T such that $T \subseteq \lambda S$. The inclusion by restriction*

$$\mathcal{S}^M(S,T) \hookrightarrow \mathcal{S}^M(\lambda S, T)$$

is a weak equivalence of G-operads. Similarly, for $\mu \geq 1$ and star domains S, T such that $\mu T \subseteq S$. The inclusion

$$\mathcal{S}^M(S,T) \hookrightarrow \mathcal{S}^M(S, \mu T)$$

is a weak equivalence of G-operads.

Proof. For $\alpha \in \Lambda_I^F(V)$, write $p^\alpha : I \to \Lambda_I^F(V)$ for the path

$$p^\alpha(t) := ((1-t)\,\mathrm{id}_V + t\alpha).$$

The homotopy of G-symmetric sequences $H : \mathcal{A}^M(V) \times I \to \mathcal{A}^M(V)$ given by

$$H\left((f_1, \ldots, f_n), t\right) := \left(f, \circ \rho^\lambda(t), \ldots, f_n \circ \rho^\lambda(t)\right)$$

restricts to homotopies

$$\begin{aligned} H_1 &:= H|_{\mathcal{S}^M(\lambda S, T)} : \mathcal{S}^M(\lambda S, T) \times I \to \mathcal{S}^M(\lambda S, T) \\ H_2 &:= H|_{\mathcal{S}^M(S, T)} : \mathcal{S}^M(S, T) \times I \to \mathcal{S}^M(S, T). \end{aligned}$$

Note that for any $0 < \lambda < 1$, then $\lambda\,\mathrm{id}_V \in \Lambda_I^F(V)$ and as $p^{\lambda\,\mathrm{id}_V}(1)(S) = \lambda S$, we have that for $\alpha = \lambda\,\mathrm{id}_V$ in the above homotopies that $H_1(\mathcal{S}^M(\lambda S, T), 1) \subseteq \mathcal{S}^M(S, T)$. We then deduce the inclusion $\mathcal{S}^M(S, T) \hookrightarrow \mathcal{S}^M(\lambda S, T)$ is a weak equivalence. The second statement follows similarly except with the homotopy $\widetilde{H} : \mathcal{A}^M(V) \times I \to \mathcal{A}^M(V)$ given by

$$\widetilde{H}\left((f_1, \ldots, f_n), t\right) := (p^{\mu^{-1}}(t) \circ f_1, \ldots, p^{\mu^{-1}}(t) \circ f_n).$$

$\square$

We can extend this lemma to get a more general statement.

2.4.2 Theorem. *Let S, T be G-star domains in V. For $0 < \lambda \leq 1$ and star domain S' with $T \subseteq S'$ and $\lambda S \subseteq S' \subseteq S$. The inclusion*

$$\mathcal{S}^M(S, T) \hookrightarrow \mathcal{S}^M(S', T)$$

is a weak equivalence of operads. Similarly, for $\mu \geq 1$ and star domain T' with $T' \subseteq S$ and $T \subseteq T' \subseteq \mu T$ then the inclusion

$$\mathcal{S}^M(S, T) \hookrightarrow \mathcal{S}^M(S, T')$$

is a weak equivalence of operads.

Proof. Note that we have

$$T \subseteq S' \subseteq S \subseteq \lambda^{-1}S' \subseteq \lambda^{-1}S$$

and so we have a sequence of inclusions

$$\mathcal{S}^M(\lambda^{-1}S, T) \hookrightarrow \mathcal{S}^M(\lambda^{-1}S', T) \hookrightarrow \mathcal{S}^M(S, T) \hookrightarrow \mathcal{S}^M(S', T)$$

by lemma 2.4.1 and two-out-of-six these are weak equivalences. The second statement follows similarly. □

This theorem is the claim we made at the beginning of this section. In particular, it shows that we get the following as a special case.

2.4.3 Corollary. *For star domains S, T where there exists constants $\alpha, \beta > 0$ such that $\alpha S \subseteq T \subseteq \beta S$, the following are equivalent*

$$\mathcal{S}^M(S) \simeq \mathcal{S}^M(T).$$

We will end this section with a comparison between different affine types.

2.4.4 Theorem. *Let F be a finite decomposition of V and T a general star domain. The inclusion*

$$\mathcal{S}^{\Lambda(V)}(T) \hookrightarrow \mathcal{S}^{M(F)}(T)$$

is a weak equivalence of G-operads.

Proof. This follows from simply shrinking the radius of each element of $\mathcal{S}^{M(F)}(T)$ so that all its dilation factors are equal to the minimum one. Since F is finite, this is always possible. □

Chapter 3

The Boardman-Vogt tensor product

This chapter is on the construction of the Boardman-Vogt tensor product. Our goal for this chapter is to (1) present a self-contained construction that is easily accessible, and (2) make the construction less mysterious for the reader. In particular, we want to make the relations and properties of the tensor more explicit - with the ultimate goal of (hopefully) making them more intuitive for the reader. We do this by making how the defining generators and relations interact more explicit with a notion we will call "trees in superposition". We will then show that the tensor product can be constructed as a left Kan extension.

3.1 Trees in superposition

3.1.1 Definition. A (2-colour) tree in superposition is the data $(T, c^T, \chi_W^T, \chi_B^T, \phi^T)$ where:

(1) T is a rooted tree,

(2) we have a vertex colouring function

$$c : V(T) \longrightarrow \{w, b, g\}.$$

Here, w is for white, b for black, and g for grey. We will say that a vertex is *pure* if it's coloured white or black. We will say a grey node is in *superposition*.

(3) Writing $S(T) := c^{-1}(g)$ for the set of vertices in superposition. Viewing this as a discrete category, we have *colour state functors*, or a *colour state assignment* for each vertex in superposition.

$$\chi_W : S(T) \longrightarrow \mathcal{F}in$$
$$\chi_B : S(T) \longrightarrow \mathcal{F}in$$

(4) Writing $\mathrm{In}(v) := |v|$, we have a natural isomorphism

$$\phi : \chi_W \times \chi_B \overset{\cong}{\Longrightarrow} \mathrm{In}\,|_{S(T)} : S(T) \to \mathcal{F}in$$

which we call the *state labelling transformation*.

3.1.2 Example. Let us illustrate the various parts of this definition. An example is shown in fig. 3.1. We have omitted the edge and vertex labellings and have only shown the colours and values of the colour state functor at the grey vertex. The idea behind the grey nodes is that they represent both a white node and a black node. The colour state functors χ_W, and χ_B are then the record of how the edges are related.

3.1.3 Definition. A morphism of trees in superposition $T \longrightarrow T'$ is the triple (f, α_W, α_B) where:

(1) f is a function of based sets $f : V(T)_+ \to V(T')_+$ such that

   (a) it preserves colours of pure vertices

   (b) it partially preserves vertices in superposition. That is, for $v \in S(T)$ we either have $f(v) \in S(T')$ or $f(v) = *$

   (c) it preserves the restricted descendant poset structure on $f^{-1}(V(T'))$

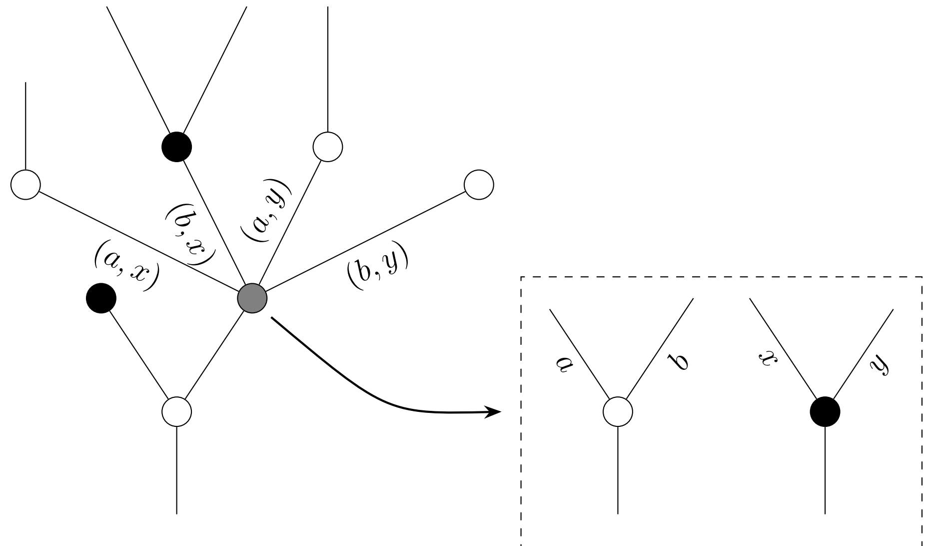


Figure 3.1: An example of a tree in superposition

(d) if $v \in f^{-1}(S(T'))$, the function f restricts to a well-defined function

$$|v| \longrightarrow |f(v)|.$$

This allows us to build a natural transformation

$$\overline{f} : |-| \Rightarrow |-| \circ f : f^{-1}(S(T')) \to \mathfrak{F}in$$

(2) α_W, α_B are natural transformations

$$\alpha_W : \chi_W^T \Rightarrow \chi_W^{T'} \circ f : f^{-1}(S(T')) \to \mathfrak{F}in$$
$$\alpha_B : \chi_B^T \Rightarrow \chi_B^{T'} \circ f : f^{-1}(S(T')) \to \mathfrak{F}in$$

such that the following commutes

$$\begin{tikzcd}
\chi_W^T \times \chi_B^T \arrow[r, Rightarrow, "\alpha_W \times \alpha_B"] \arrow[d, Rightarrow, "\phi^T"'] & \chi_W^{T'} \circ f \times \chi_B^{T'} \circ f \arrow[d, Rightarrow, "\phi^T \circ f"] \\
|-|_T \arrow[r, Rightarrow, "\overline{f}"'] & |-|_{T'}
\end{tikzcd}$$

The composition of two morphisms

$$T \xrightarrow{(f,\alpha_W,\alpha_B)} T' \xrightarrow{(f',\alpha'_W,\alpha'_B)} T''$$

is given by

$$T \xrightarrow{(f'f,(\alpha'_W f)\circ\alpha_W,(\alpha'_B f)\circ\alpha_B)} T''.$$

While we have defined morphisms generally, we won't need all possible morphisms. The following three are the expected generalisations from those of $\mathcal{T}$. For a tree T in superposition, we have

(1) *Pure inner face maps.* Given an edge e with adjacent vertices both pure and of the same colour, we can put the obvious superposition tree structure on T/e and get inner face maps $d_e : T \to T/e$.
(2) *Pure degeneracy maps.* Given a pure unary vertex v, the tree $T\backslash v$ has an obvious structure as a superposition tree by restricting the component functions. We then have degeneracy maps $s_v : T\backslash v$
(3) *Isomorphisms.*

We do not have face or degeneracy maps for grey vertices. The new morphisms for grey vertices are what we will call *blow-up maps*. Given finite sets A, B, we define the *white first interchange tree* $I_W(A, B)$ as a tree where every path from input to edge is precisely two nodes, and no stumps exist. The root vertex we colour white, and all other vertices we colour black. We use A for the input edges of the white vertex, and for each black vertex v, we use $\{o(v)\} \times B$ for its input edges. See fig. 3.2 for an example.

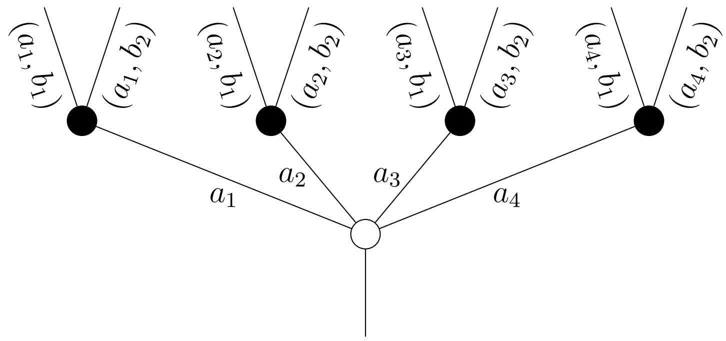


Figure 3.2: The tree $I_W(A, B)$ where $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2\}$.

We similarly have *black first interchange trees* $I_B(A, B)$ where instead the root node is black and other nodes white. The set B is the edges of the black node, and for white node u, it has edges given by $A \times \{o(u)\}$.

Given a tree in superposition T and grey vertex v, we have a tree in superposition $T \uparrow_v^W$ which is the tree T, but we replace v with the interior of $I_W(\chi_W(v), \chi_B(v))$. By interior, we mean the graph constructed by removing the input and root edges. We similarly have $T \uparrow_v^B$ where we use $I_B(\chi_W(v), \chi_B(v))$ instead. The obvious morphisms

$$b_v^W : T \longrightarrow T \uparrow_v^W$$

$$b_v^B : T \longrightarrow T \uparrow_v^B$$

are called the *white-first blow-up* and *black-first blow-up* respectively.

3.1.4 Definition. The category of trees in superposition $\mathcal{T}_{\text{sup}}$ has objects trees in superposition and morphisms generated by the following types:

(1) isomorphisms of superposition trees

(2) for a pure unary vertex v with all adjacent edges of the same colour, we take the degeneracy maps $s_v : T \backslash v \longrightarrow T$. Note we aren't assuming v has two adjacent

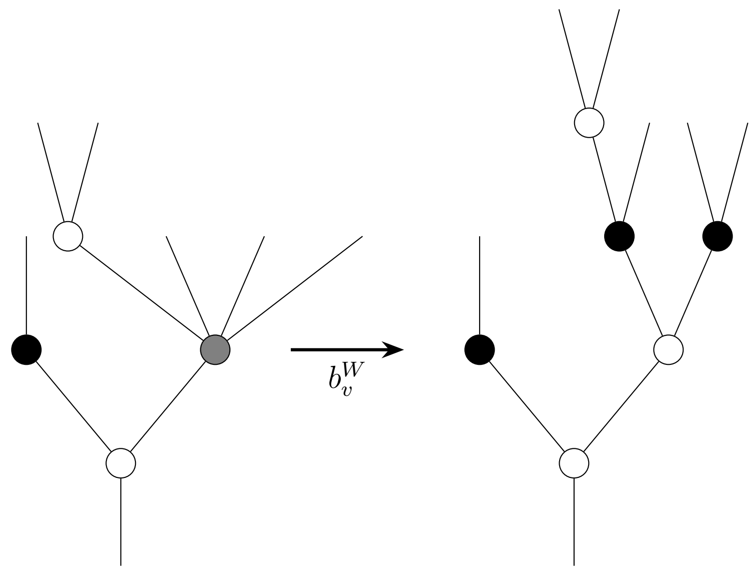


Figure 3.3: An example of a white-first blow-up map

vertices. i.e., the 1-corollas give degeneracy maps from the trivial tree | for both colours.

(3) for an edge e with pure adjacent vertices of the same colour, the inner face maps $d_v : T \to T/e$

(4) for grey nodes v, the black-first b_v^W and white-first blow-up maps b_v^B.

The full subcategory spanned by objects with no grey nodes will be denoted by $\mathcal{T}_{\text{bin}}$ and be called the binary tree category. The full subcategory of non-trivial binary trees will be denoted by $\mathcal{T}_{\text{nbin}}$.

Just as for $\mathcal{T}$, we have the functor $\pi : \mathcal{T}_{\text{sup}} \to \mathcal{B}ij$ that sends a tree T to its input set $|T|$. Also, grafting trees makes just as much sense for trees in superposition as it does for ordinary trees. Unsurprisingly, we have the following.

3.1.5 Theorem. *The functor* $\pi : \mathcal{T}_{\text{sup}} \to \mathcal{B}ij$ *is a strict opfibration, and tree grafting determines a* **Cat***-operad structure on the corresponding fibre functor.*

3.2 Reduced trees and components

3.2.1 Definition. For a tree in superposition, we will say it is *fully reduced* if it has vertices, none of which is in superposition and no edge that connects vertices of the same colour. More generally; for any tree in superposition, if it has any vertices, then it has a single vertex grey vertex and no edges that connect two vertices of the same colour, we will say it is *partially reduced.* A *reduced* tree is one that is either fully or partially reduced.

We then have the following subcategories of $\mathcal{T}_{\mathrm{sup}}$.

(1) The full subcategory of reduced trees which we denote by $\mathcal{T}^{\mathrm{red}}_{\mathrm{sup}}$

(2) The full subcategory of fully reduced trees $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$. The subscript is chosen because these are all the reduced trees in $\mathcal{T}_{\mathrm{bin}}$.

(3) The full subcategory of partially reduced trees $\mathcal{T}^{\mathrm{pred}}_{\mathrm{sup}}$.

3.2.2 Remark. Our phrasing about having vertices in our definition is on purpose. Perhaps counterintuitively, we will consider the trivial tree to be partially reduced and not fully reduced. We partially do this so the following has a nicer form. However, we also find it more natural to think about the degeneracy maps from the trivial tree to secretly be types of blow-up maps. Not only because they come in two different varieties for each colour but because they also correspond to a sort of interchange between the identities of two different operads in the coproduct and tensor.

First, observe that the categories $\mathcal{T}^{\mathrm{pred}}_{\mathrm{sup}}$ and $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$ are groupoids, as well as their fibre variants $\mathcal{T}^{\mathrm{pred}}_{\mathrm{sup}}(J)$ and $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}(J)$ for some J in $\mathcal{B}ij$. For a non-trivial binary tree $T \in \mathcal{T}_{\mathrm{nbin}}$, we can view this as being constructed by grafting white and black trees. To make this more precise, let us define an equivalence on the vertices of T where $v \sim w$ if there exists a walk on T that only passes through vertices of the same colour (including the vertices v and w). Each equivalence class then gives us a connected subgraph of vertices of the same colour, which is maximal under subsets of connected vertices of the same colour. This allows us to define the following.

3.2.3 Definition. Given a non-trivial binary tree T, and vertex v, denote the maximal rooted subtree that has as vertex set the equivalence class of v by $C(v)$. We will call this the colour tree component of T corresponding to v.

These components are exactly the trees that form T under graft composition. We can construct a reduced tree from the data of its component trees $C(v)$ by grafting corollas with the same input edges in the same order that the components are grafted.

3.2.4 Definition. We will call the isomorphism class in $\mathcal{T}_{\mathrm{nbin}}$ of this constructed reduced tree the *total colour signature* of the non-trivial binary tree T, which we denote by $\xi(T)$. The isomorphism class of the reduced tree in $\mathcal{T}_{\mathrm{nbin}}(J)$ we will call the *relative colour signature* and denote this by $\xi_J(T)$.

An important observation about these signatures is that no morphism in $\mathcal{T}_{\mathrm{nbin}}$ ($\mathcal{T}_{\mathrm{nbin}}(J)$) changes a tree's total (relative) signature and, as we will see, they index the connected components of the category.

Given a tree in $\mathcal{T}_{\mathrm{nbin}}$, if we compose all possible inner face maps, we are left with an element T' of $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$. We will denote this morphism by $r_T : T \to T'$ and denote T' by rT, which we will call the *reduction of* T. Since this is reduced, this is inside the isomorphism class of the total and relative signature of T. Given any other tree with the same total (relative) signature, their reductions must be isomorphic in $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$ ($\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}(X)$), and so live in the same connected component of $\mathcal{T}_{\mathrm{nbin}}$ ($\mathcal{T}_{\mathrm{nbin}}(X)$). To summarise, we have the following.

3.2.5 Lemma. *There is a bijective correspondence between the connected components of $\mathcal{T}_{\mathrm{nbin}}$ ($\mathcal{T}_{\mathrm{nbin}}(X)$) and the set of isomorphism classes of $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$ in the form of a tree's* *total (relative) signature. Moreover, the subcategory $\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}$ ($\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}(X)$) is cofinal in $\mathcal{T}_{\mathrm{nbin}}$ ($\mathcal{T}_{\mathrm{nbin}}(X)$).*

This is a good time to prove a generalisation of Lemma 1.3.6.

3.2.6 Theorem. *Any morphism f in $\mathcal{T}_{\mathrm{sup}}$ can be factored into a product of compositions of the form*

$$f = D \circ I \circ F \circ B$$

where

- *B is the composition of blow-up maps,*
- *F is the composition of inner face maps,*
- *I is the composition of isomorphisms, and*
- *D is the composition of degeneracy maps.*

Moreover, given a different factorisation $f = D' \circ I' \circ F' \circ B'$ of the same form. Then there exist unique isomorphisms that make the following commute.

$$\begin{tikzcd}
 & U \arrow[r, "F"] \arrow[dd, "\cong"] & V \arrow[r, "I"] \arrow[dd, "\cong"] & W \arrow[dr, "D"] \arrow[dd, "\cong"] & \\
T \arrow[ur, "B"] \arrow[dr, "B'"'] & & & & T' \\
 & U' \arrow[r, "F'"'] & V' \arrow[r, "I'"'] & W' \arrow[ur, "D'"'] &
\end{tikzcd}$$

Proof. Observe that for a blow-up map b_v and any inner face map, degeneracy or isomorphism g, we have that $b_v g = g b_v$. Here we mean the commutativity of g to the left of blow-up maps. In the reverse direction, we need to be careful as it can't involve any of the vertices in the image of the blow-up. Hence we can factor f as $f = h \circ B$ where h is the composite of face, isomorphisms and degeneracy maps. The result then follows from Lemma 1.3.6. □

Now that we have this factorisation, we can more easily explain an extension to Lemma 3.2.5. Let us build an equivalence relation on $\mathfrak{T}^{\mathrm{red}}_{\mathrm{bin}}$: for two trees T, T' in $\mathfrak{T}^{\mathrm{red}}_{\mathrm{bin}}$, set $T \sim T'$ if they are spanned by a tree with a grey node (or the trivial tree), or they are isomorphic. That is, if they aren't isomorphic, then there exists a tree S which has a grey node or none and morphisms so that $T \leftarrow S \rightarrow T'$. We then consider the generated equivalence relation on $\mathfrak{T}^{\mathrm{red}}_{\mathrm{bin}}$.

We claim that it is sufficient to only require the spanning trees S to be partially reduced to generate the same equivalence relationship. To see this, by Theorem 3.2.6, we can assume that the spanning morphisms are of the form

$$\begin{tikzcd}
 & & S \arrow[dl, "r\circ b"'] \arrow[dr, "r'\circ b'"] & & \\
T & \widehat{T} \arrow[l, "\cong"'] & & \widehat{T'} \arrow[r, "\cong"] & T'
\end{tikzcd}$$

where b, b' are composites of blow-ups and r, r' are reduction maps. Since we are including isomorphisms in the generating relations, we can assume that $\widehat{T} = T$ and $\widehat{T'} = T'$. Moreover, since T, T' are binary trees, this implies that the blow-ups b, b' blow-up the same vertices. Namely, all the grey vertices of S. Since we can commute blow-ups past each other, b and b' differ only by whether the blow-ups are white or black on each grey vertex. Writing $v_1, v_2, \ldots, v_n$ for the grey nodes of S and b_v and b'_v for the corresponding blow-ups on that node from the morphisms b and b' respectively. We get the following commutative diagram.

$$\begin{tikzcd}[column sep=small]
 & & & & S \arrow[dlll, "\hat{b}_{v_1} b_{v_2} b_{v_3} \ldots b_{v_n}"'] \arrow[dl, "b'_{v_1} \hat{b}_{v_2} b_{v_3} \ldots b_{v_n}" description] \arrow[dr, "b'_{v_1} b'_{v_2} \hat{b}_{v_3} \ldots b_{v_n}" description] \arrow[drrr, "b'_{v_1} b'_{v_2} b'_{v_3} \ldots \hat{b}_{v_n}"] & & & & \\
 & S_1 \arrow[dl, "b_{v_1}"] \arrow[dr, "b'_{v_1}"'] & & S_2 \arrow[dl, "b_{v_2}"] \arrow[dr, "b'_{v_2}"'] & & S_3 \arrow[dl, "b_{v_3}"] \arrow[dr] & & S_n \arrow[dl] \arrow[dr, "b'_{v_n}"] & \\
\overline{T} \arrow[d, "r"] & & \overline{T}_1 \arrow[d, "r_1"] & & \overline{T}_2 \arrow[d, "r_2"] & & \cdots & & \overline{T}_n \arrow[d, "r'"] \\
T & & r\overline{T}_1 & & r\overline{T}_2 & & & & T'
\end{tikzcd}$$

Here $\hat{b}_v$ means that map is omitted in the composition. This justifies our claim since each S_i has exactly one grey node. We then have the following. Note, the above argument works just as well in the fibres $\mathfrak{T}_{\text{sup}}(X)$.

3.2.7 Lemma. *The connected components of* $\mathfrak{T}_{\text{sup}}$ *(*$\mathfrak{T}_{\text{sup}}(J)$*)are in bijection with equivalence classes on objects of* $\mathfrak{T}_{\text{bin}}^{\text{red}}$ *(*$\mathfrak{T}_{\text{bin}}^{\text{red}}(J)$*) under the equivalence generated by isomorphisms and spans by partially reduced trees. Moreover, we have that the subcategory* $\mathfrak{T}_{\text{sup}}^{\text{pred}}$ *(*$\mathfrak{T}_{\text{sup}}^{\text{pred}}(X)$*) is cofinal in* $\mathfrak{T}_{\text{sup}}$ *(*$\mathfrak{T}_{\text{sup}}(J)$*).*

3.3 Resolutions of the tensor

3.3.1 Definition. Given G-operads P, Q define a functor $\mathrm{R}_{\text{sup}}(P, Q) : \mathfrak{T}_{\text{sup}} \to \mathrm{Top}^G$ where on objects we define

$$\mathrm{R}_{\text{sup}}(P, Q)(T) := \prod_{v \in V(T)} [P, Q](v)$$

where

$$[P,Q](v) = \begin{cases} P(\mathrm{in}(v)) & \text{if } v \text{ is white,} \\ Q(\mathrm{in}(v)) & \text{if } v \text{ is black,} \\ P(\chi_W(v)) \times Q(\chi_B(v)) & \text{if } v \text{ is grey.} \end{cases}$$

The functor on isomorphisms is just the induced isomorphism on the product factors. The functor on the inner face and degeneracy maps is composition and inserting the identity from the corresponding operad, respectively. The functor on blow-up maps is the diagonal maps from the interchange diagram. We will denote the restricted functor on $\mathcal{T}_{\mathrm{bin}}$ by $\mathrm{R}_2(P,Q)$.

We have a couple of important functors relating our tree categories $\mathcal{T}$, $\mathcal{T}_{\mathrm{sup}}$ and $\mathcal{T}_{\mathrm{bin}}$. First, we have the *coloured inclusions*

$$i_W, i_B : \mathcal{T} \to \mathcal{T}_{\mathrm{sup}}$$

where i_W is mapping a rooted tree to the tree with all white vertices. i_B is similar except for black. We also have a forgetful functor $U : \mathcal{T}_{\mathrm{sup}} \to \mathcal{T}$ where we forget all data except the tree. Observe that the coloured inclusions form sections of the forgetful functor.

$$\mathrm{id}_{\mathcal{T}} : \mathcal{T} \xrightarrow{i_W, i_B} \mathcal{T}_{\mathrm{sup}} \xrightarrow{U} \mathcal{T}.$$

3.3.2 Theorem. *For G-operads P, Q we have that*

$$P \coprod Q \cong \mathrm{Lan}_\pi \mathrm{R}_2(P,Q) \text{ and } P \otimes Q \cong \mathrm{Lan}_\pi \mathrm{R}_{\mathrm{sup}}(P,Q).$$

Proof. From Theorem 1.2.5, we know these left Kan extensions are operads, so we just need to show that they satisfy the required universal properties. Suppose we have interchanging operad maps $P \xrightarrow{f} Z$ and $Q \xrightarrow{g} Z$. This determines a natural transformation

$$(f, g) : \mathrm{R}_{\mathrm{sup}}(P, Q) \Rightarrow \mathrm{R}(Z) \circ U$$

which is given on components $\mathrm{R}_{\mathrm{sup}}(P, Q)(T) \Rightarrow (\mathrm{R}(Z) \circ U)(T)$ by applying f and g to the decorations on the trees. That these are operadic maps means that this is natural with respect to the isomorphism, inner face and degeneracy maps. It is also well-behaved on the blow-up maps (the image of the maps b^W, b^B are equal) because the morphisms interchange.

Observe that the natural transformation $\mathrm{R}(f)$ factors as the following

$$\mathrm{R}(P) \overset{\cong}{\Longrightarrow} \mathrm{R}_{\mathrm{sup}}(P, Q) \circ i_W \overset{(f,g)i_W}{\Longrightarrow} \mathrm{R}(Z) \circ U \circ i_W = \mathrm{R}(Z)$$

via adjunction, this corresponds to a natural transformation

$$\mathrm{Lan}_{i_W} \mathrm{R}(P) \Longrightarrow \mathrm{R}_{\mathrm{sup}}(P, Q) \Longrightarrow R(Z) \circ U$$

which in turn corresponds to a natural transformation

$$\mathrm{R}(P) = \mathrm{Lan}_U \mathrm{Lan}_{i_W} \mathrm{R}(P) \Longrightarrow \mathrm{Lan}_U \mathrm{R}_{\mathrm{sup}}(P, Q) \Longrightarrow R(Z).$$

Since we have applied the left kan adjunction along an isomorphism, we know this map factors $\mathrm{R}(f)$. We get a similar expression for g and Q. Taking the left Kan extension of these diagrams along $p : \mathcal{T} \to \mathcal{B}ij$ then gives us the requisite maps for the universal property. Note that the induced map $\mathrm{Lan}_\pi \mathrm{R}_{\mathrm{sup}}(P, Q) \to Z$ is unique since we can view $\mathcal{B}ij$ as a cofinal subcategory of $\mathcal{T}$ and so, in this case, $Z = \mathrm{Lan}_p \mathrm{R}(Z) = \mathrm{R}(Z) \circ i$ where $i : \mathcal{B}ij \to \mathcal{T}$ is the inclusion. Hence we can backtrack any other possible map through our adjunctions. However, it is not hard to see that the map (f, g) above is forced if we require the correct restrictions. Hence $P \otimes Q \cong \mathrm{Lan}_\pi \mathrm{R}_{\mathrm{sup}}(P, Q)$. The case of the coproduct follows similarly. $\square$

Let us write $\mathrm{sk}(\mathcal{C})$ for the skeleton of a category $\mathcal{C}$, and b^W and b^B for the degeneracy maps from the trivial tree which insert a white and black node respectively. Then from 3.2.7 we get the following.

3.3.3 Corollary. *Given G-operads P, Q, for all $X \in \mathcal{B}ij$ the tensor component $(P \otimes Q)(X)$ is isomorphic to the coequalizer of*

$$\coprod_{S \in \mathrm{sk}\left(\mathcal{T}^{\mathrm{pred}}_{\mathrm{sup}}(X)\right)} \mathrm{R}_{\mathrm{sup}}(P, Q)(S) \underset{R(rb^B)}{\overset{R(rb^W)}{\rightrightarrows}} \coprod_{T \in \mathrm{sk}\left(\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}(X)\right)} \mathrm{R}_{\mathrm{sup}}(P, Q)(T)$$

Here, the maps rb^W mean to apply the corresponding white-first blow-up map on the component T and then take the reduction (if it's not already reduced). Similarly for rb^B.

3.3.4 Remark. Note that $\mathrm{R}_{\mathrm{sup}}(P, Q)|_{\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}} = \mathrm{R}_2(P, Q)|_{\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}}$ and so

$$\coprod_{T \in \mathrm{sk}\left(\mathcal{T}^{\mathrm{red}}_{\mathrm{bin}}(X)\right)} \mathrm{R}_{\mathrm{sup}}(P, Q)(T) \cong P \amalg Q.$$

i.e., the above shows us that the tensor is given by adding relations to the coproduct. So this gives us back the classical method the tensor is constructed.

3.4 Representatives of elements in the tensor

3.4.1 Definition. Given operads P, Q, and element $x \in P \otimes Q$, by a representative of x we mean an element $x' \in P \amalg Q$ whose equivalence class determined by Corollary 3.3.3 is x. We will use the notation $[x']$ to be this class. i.e., $x = [x']$. Also, given $a \in P(n)$, and $b \in Q(m)$, we will use the notation $a \otimes b := [a \circ b^n] = [b \circ a^m]$.

There are several quick simplifications we can make when dealing with representatives. The first is to realise that the interchange diagrams with one part given by a unary vertex allow us to commute unary elements up the tree. As we also compose matching elements in the coproduct, this tells us that every element $x \in P \otimes Q$ has a representative with no unary elements except for possibly at the top of the tree.

Moreover, if we add the extra condition of P and Q being reduced, then the interchange diagram works for any vertex ending in stumps. This effectively allows us to replace such vertices with a stump, which then immediately composes with its ancestor as they are of the same colour. Using this, along with the degeneracy maps, allows us to find a representative without any stumps for any element of $x \in P \otimes Q$. We can formalise this situation in our current framework as follows.

3.4.2 Definition. We will say a tree in superposition T is halfway reduced if it is fully reduced, the trivial tree, or it is partially reduced with its grey node having one of its colour states being empty or a singleton. Denote the full subcategory of halfway trees by $\mathcal{T}_{\mathrm{half}}$.

It is straightforward to see that this is a strict opfibration over $\mathcal{B}ij$. So in particular, we can form an operad by left Kan extension that lies halfway between the coproduct and tensor.

3.4.3 Definition. Given reduced operads P and Q, the *reduced coproduct* of P and Q is

$$\mathcal{F}_{\mathrm{red}}(P, Q) := \mathrm{Lan}_p \, \mathrm{R}_{\mathrm{sup}}(P, Q)$$

where $p : \mathcal{T}_{\mathrm{half}} \to \mathcal{B}ij$.

From the universal properties of left Kan extensions, we see that the quotient map

$P \coprod Q \to P \otimes Q$ factors as

$$P \coprod Q \to \mathcal{F}_{\mathrm{red}}(P, Q) \to P \otimes Q.$$

The advantage to this is that, unlike the coproduct, $\mathcal{F}_{\mathrm{red}}(P, Q)(n)$ is given by a *finite* disjoint union of products of the components of P and Q. i.e., there are only finitely many representative trees with n-many inputs.

3.4.4 Definition. Given reduced operads P, Q, and element $x \in P \otimes Q$, by a *proper representative of* x we mean an element $x' \in \mathcal{F}_{\mathrm{red}}(P, Q)$ with image x. We will use the notation $[x']$ to be this class. i.e., $x = [x']$.

Let us end this section by using this in an example.

3.4.5 Example. When S is a singleton set, it is well known ([FV15]) that if P and Q are reduced, then $(P \otimes Q)(S) \cong (P \times Q)(S)$ as monoids. This can also be seen from the map

$$\mathcal{F}_{\mathrm{red}}(P, Q) \to P \otimes Q.$$

By construction, this is surjective. The component $\mathcal{F}_{\mathrm{red}}(P, Q)(S)$ is easily seen to be isomorphic to $P(S) \times Q(S)$ as monoids; however, on singletons $\mathcal{F}_{\mathrm{red}}(P, Q)(S)$ and $(P \otimes Q)(S)$ are given by the same relations, and so are isomorphic.

Chapter 4

Equivariant Dunn additivity

In this chapter, we will investigate Dunn additivity. We aim to get as far as possible in proving additivity for the little star operads. i.e., there is a zigzag of weak equivalences between

$$\mathcal{S}^M(S,T) \otimes \mathcal{S}^{M'}(S',T') \rightsquigarrow \mathcal{S}^{M\times M'}(S\times S', T\times T').$$

Unfortunately, we do not get a complete version of this statement. However, we do show that the little equivariant simplex operads are additive.

Our approach is in the same vein as the original proof by Dunn [Dun88] and Brinkmeier [Bri00]. Our original proof used many of the ideas of Brinkmeier; however, recently, a paper by Barata and Moerdijk [BM22] greatly simplified many of the arguments involved in the classical additivity result. As a result, the proof we present here makes use of Barata and Moerdijk's approach. In particular, we generalise their proof of the injectivity of the tensor map to isolate a set of key conditions that imply injectivity. We also use their observation that it is easier to justify a map is an embedding by showing specific maps are proper rather than looking at the compact extensions of operads.

Let us now give an overview of our approach. First of all, unless our star domains and codomains are equal, we *do not* get a direct map of the form

$$\mathcal{S}^M(S,T) \otimes \mathcal{S}^{M'}(S',T') \to \mathcal{S}^{M\times M'}(S\times S', T\times T').$$

Instead, given little star operads $\mathcal{S}^M(S,T)$, $\mathcal{S}^{M'}(S',T')$, there are embeddings of operads

$$\mathcal{S}^M(S,T) \hookrightarrow \mathcal{S}^M(T) \hookrightarrow \mathcal{S}^{M\times M'}(T\times T')$$
$$(f_i)_{i\in I} \mapsto (f_i \times \mathrm{id}_{T'})_{i\in I}$$

and,

$$\mathcal{S}^{M'}(S',T') \hookrightarrow \mathcal{S}^{M'}(T') \hookrightarrow \mathcal{S}^{M\times M'}(T\times T')$$
$$(f_i)_{i\in I} \mapsto (\mathrm{id}_T \times f_i)_{i\in I}.$$

These interchange, and so we get an induced morphism of G-operads

$$\iota : \mathcal{S}^M(S,T) \otimes \mathcal{S}^{M'}(S',T') \to \mathcal{S}^{M\times M'}(T\times T'). \tag{4.1}$$

Our goal is to prove that this map is a weak equivalence. Recall from Theorem 2.4.2 that there is a weak equivalence

$$\mathcal{S}^{M\times M'}(T\times T') \simeq \mathcal{S}^{M\times M'}(S\times S', T\times T')$$

and so proving that the induced tensor (4.1) will prove that there exists the desired zigzag of weak equivalences.

4.0.1 Remark. There is a similar map to (4.1) using the inclusion maps $\mathcal{S}^M(S,T) \hookrightarrow \mathcal{S}^M(S)$ given by codomain extension instead. We can also take combinations of the two if we so desire.

The steps of showing that map (4.1) is a weak equivalence are to show that:

(1) ι has closed image,

(2) the image of ι is a deformation retract of $\mathcal{S}^{M\times M'}(T\times T')$,

(3) ι is injective.

We have ordered these by least to most difficult, and this is the order we shall tackle them in this chapter. We also require correspondingly more for each to hold. In particular, we shall see that:

(1) Tensors of all of our little star operads give closed images,

(2) as long as our little star operad is of the thick variant, the image is a deformation retract of the codomain, and

(3) injectivity is extremely hard to prove.

The fact that injectivity is so hard to establish can surprise those readers new to the tensor product. Especially since, as we will see, the ambient star operads satisfy an additivity theorem on the nose. Injectivity is so challenging to establish that we can only justify that the induced tensor maps are injective for a small class of shapes. Surprisingly, the little cubes for non-trivial representations do not fall into this class.

Throughout this chapter, let V, V' be G-representations, M, M' affine types on V, V' respectively, S, T be bounded G-star domains on V with $T\subseteq S$, and S', T' be bounded G-star domains on V' with $T'\subseteq S'$.

4.1 Algebraic additivity

Before we move on to proving the additivity of the little star operads. Let us first explain why there are isomorphisms of G-operads

$$\mathcal{A}^M(S,T)\otimes\mathcal{A}^{M'}(S',T')\cong\mathcal{A}^{M\times M'}(S\times S',T\times T').$$

This isomorphism isn't needed for what follows; however, we believe it adds useful context to the results we are trying to prove. None of this section is original and well-known to the experts (cf. axial operads of [FV15]).

Suppose we have G-monoids A and B. We have the usual projection morphisms

$$\mathrm{pr}_A : A \times B \to A \text{ and } \mathrm{pr}_B : A \times B \to B$$

which induce morphisms of G-operads

$$\mathcal{O}(\mathrm{pr}_A) : \mathcal{O}(A \times B) \to \mathcal{O}(A) \text{ and } \mathcal{O}(\mathrm{pr}_B) : \mathcal{O}(A \times B) \to \mathcal{O}(B).$$

Similarly, the inclusions $i_A : A \to A \times B, i_A : B \to A \times B$ given by $a \mapsto (a, \mathrm{id}), b \mapsto (\mathrm{id}, b)$ induce morphisms

$$\mathcal{O}(i_A) : \mathcal{O}(A) \to \mathcal{O}(A \times B) \text{ and } \mathcal{O}(i_B) : \mathcal{O}(B) \to \mathcal{O}(A \times B).$$

Together, this induces an isomorphism of G-operads

$$\mathcal{O}(\mathrm{pr}_A) \times \mathcal{O}(\mathrm{pr}_B) : \mathcal{O}(A \times B) \xrightarrow{\cong} \mathcal{O}(A) \times \mathcal{O}(B).$$

Perhaps less obviously, there is also an isomorphism

$$\mathcal{O}(A \times B) \cong \mathcal{O}(A) \otimes \mathcal{O}(B).$$

This can be seen by observing that $\mathcal{O}(A \times B)$ has the universal property of the tensor. In particular, suppose we have a G-operad P and interchanging morphisms

$$\mathcal{O}(A) \xrightarrow{f} P, \quad \mathcal{O}(B) \xrightarrow{g} P.$$

We then define a morphism $F : \mathcal{O}(A \times B) \longrightarrow P$ by $F = \mathcal{O}(U(f) \times U(g))$. Note that these morphisms interchange, which means that the images of $U(f)$ and $U(g)$ commute with each other, so we get a well-defined map of monoids

$$U(f) \times U(g) : A \times B \longrightarrow U(P).$$

The morphism F is unique since, via the adjunction, it must correspond to the map $U(f) \times U(g)$. To summarise, we get the following result which is an equivariant version of a well-known result.

4.1.1 Lemma (Remark 7.5 of [FV15]). *Given topological G-monoids A, B, the following are isomorphic as reduced G-operads*

$$\mathcal{O}(A) \times \mathcal{O}(B) \cong \mathcal{O}(A \times B) \cong \mathcal{O}(A) \otimes \mathcal{O}(B).$$

Since the ambient star operads are of this form, we immediately get the following.

4.1.2 Lemma. *There is an isomorphism of G-operads*

$$\mathcal{A}^M(S,T) \otimes \mathcal{A}^{M'}(S',T') \cong \mathcal{A}^{M \times M'}(S \times S', T \times T').$$

4.2 Proper composition maps in operads

The first step in proving the additivity theorem is to show that the image of the induced map is closed. We want this to show that the induced map is an honest embedding once we establish injectivity. We will show more than this and show that the map is closed. We do this by establishing the map is a proper map. By a proper map $f : X \to Y$, we mean that the preimage of all compact subsets are compact. References for proper maps are scattered, and there are multiple variations in definitions. We include direct proofs for some elementary facts about proper maps in the appendix for the reader's convenience. The main things to recall about proper maps are the following.

(1) Given a proper map $f : X \to Y$, then the function $f|_{f^{-1}(U)}$ restricted to any preimage $U \subseteq Y$ is also proper.

(2) If X has a finite closed cover $\{X_i\}$ such that $f|_{X_i} : X_i \to Y$ are all proper, then $f : X \to Y$ is proper. In the other direction, if $f : X \to Y$ is proper and $F \subseteq X$ is closed, then the restriction $f|_F$ is proper.

(3) Products of proper maps are proper.

(4) Compositions of proper maps are proper. Moreover, if $f \circ g$ is proper, then g is proper, and if g is surjective, then f is proper.

4.2.1 Definition. We will say an operad P is proper if all of its composition maps are proper. A morphism of operads $f : P \to Q$ is proper if all of its component maps $f(X) : P(X) \to Q(X)$ are proper.

Fortunately, the tensor product behaves well with respect to properness. The following is essentially [BM22, Lemma 4].

4.2.2 Lemma. *For proper morphisms of reduced G-operads $\phi_i : P_i \to Q$ which interchange. If Q is a proper operad, then the induced morphism of reduced G-operads*

$$\phi : P_1 \otimes \cdots \otimes P_n \to Q$$

is proper.

Proof. Recall Definition 3.4.3. We have the following diagram of operads

$$\begin{tikzcd}
\mathcal{F}_{\mathrm{red}}(P_1, \ldots, P_n) \arrow[r, two heads, "q"] \arrow[dr, "\phi \circ q"'] & P_1 \otimes \cdots \otimes P_n \arrow[d, "\phi"] \\
& Q
\end{tikzcd}$$

The map $\phi \circ q$ is the map that applies the morphisms ϕ_i to each product in the components of $\mathcal{F}_{\text{red}}(P_1, \ldots, P_n)$ and then uses the composition maps in Q to get a single element of Q. This amount to taking a disjoint union of products of the maps ϕ_i and then composing with the composition maps of Q. Since $\mathcal{F}_{\text{red}}(P_1, \ldots, P_n)$ breaks up into finitely many components and so is a finite closed cover, this means the resulting morphism $\phi \circ q$ is proper. Hence we also conclude that ϕ is proper as q is surjective. $\square$

The following two lemmas follow easily from the elementary facts stated at the beginning of this section.

4.2.3 Lemma. *If we have G-operads $P \subseteq Q$ with P closed in Q, and the composition maps in Q proper. Then the composition maps in P are also proper.*

4.2.4 Lemma. *Let A be a monoid with proper multiplication maps. Then the operad $\mathcal{O}(A)$ has proper composition maps.*

We will now move on to proving the ambient and little star operads are proper, and the inclusion maps are proper.

4.2.5 Lemma. *The G-operads $\mathcal{A}^M(S)$, $\mathcal{S}^M(S)$, and $\mathcal{S}^M(S,T)$ are proper.*

Proof. From Lemma 2.2.10, $\mathcal{S}^M(S)$ and $\mathcal{S}^M(S,T)$ are closed in $\mathcal{A}^M(S)$. From Lemma 4.2.3, it is sufficient to prove the lemma for just $\mathcal{A}^M(S)$. Suppose M is an affine type generated by the decomposition $F = \{V_i\}$ and let $M_i := \{f|_{V_i} : V_i \to V_i \mid f \in M\}$. It follows we have an inclusion of subspaces

$$\mathcal{A}^M(S) \subseteq \prod_{i \in I} \mathcal{A}^{M_i}(S_{V_i})$$

where $S_{V_i} = S \cap V_i$. From Lemma 4.2.3 and that proper maps are closed under products, it is sufficient to prove $\mathcal{A}^M(S)$ has proper composition maps when M is the group of rigid dilations $M = \Lambda(V)$. We recall that for rigid dilations, we have that $\Lambda(V) = \mathbb{R}_{>0} \times V$, and so we have that

$$\mathcal{A}^{\Lambda(V)}(S)(\underline{1}) \subseteq (0,1] \times V.$$

Here, we interpret $(0,1] \times V$ as a monoid where we multiply in the first coordinate, and add in the second. The above inclusion is then a map of monoids. The monoid multiplication of $\mathbb{R}_{>0} \times V$ is proper, and as $\mathcal{A}^{\Lambda(V)}(S)(\underline{1})$ is closed in it, so too is the monoid multiplication of $\mathcal{A}^{\Lambda(V)}(S)(\underline{1})$. Since $\mathcal{A}^M(S) = \mathcal{O}\left(\mathcal{A}^M(S)(1)\right)$ the lemma now follows from Lemma 4.2.4. $\square$

We now want to show that the inclusion morphisms that generate the map from the tensor are proper.

4.2.6 Lemma. *The inclusion morphisms*

$$\iota : \mathcal{A}^M(S) \longrightarrow \mathcal{A}^{M \times M'}(S \times S')$$

are proper. Similarly, the little star operad variants also have proper inclusion morphisms.

Proof. The little star operad variants follow from the ambient case as they are closed subspaces. So we only need to prove the main statement. The inclusion map

$$\begin{aligned} \iota : \mathcal{A}^M(S) &\longrightarrow \mathcal{A}^{M \times M'}(S \times S') \\ (f_k)_{k \in K} &\mapsto (f_k \times \mathrm{id})_{k \in K} \end{aligned}$$

has a left inverse given by the projection map

$$\mathrm{pr}_1 : \mathcal{A}^{M \times M'}(S \times S') \to \mathcal{A}^{M_i}(S_i)$$

and this then implies that ι is proper. $\square$

We have now verified the conditions for Lemma 4.2.2 for the little star operads. Hence we have the following.

4.2.7 Theorem. *The induced morphism*

$$\iota : \mathcal{S}^M(S,T) \otimes \mathcal{S}^{M'}(S',T') \to \mathcal{S}^{M\times M'}(T \times T')$$

is proper. In particular, since our spaces are Hausdorff, this is a closed map.

4.3 Deformation retract onto the image

We will now prove that there exists a deformation retract onto the image of the induced tensor map (4.1). The idea is simple: we shrink the radius of our elements until they can be factorised. This is the same method used in the classical proofs of the additivity of the little cubes. However, our shapes are more general, so we must be more careful.

The main observation is that elements of a star operad lie in the induced image if they are "small enough". To explain this, and make it rigorous, let us introduce some notation.

4.3.1 Definition. Given an operad P, we will write $a \mid b$ for $a, b \in P$ if there exists $(t_i)_{i\in I} \in P^I$ where $\mathrm{ar}(a) = I$ and $|\mathrm{ar}(t_i)| > 0$ such that $b = a \circ (t_i)_{i\in I}$. In this case, we will say a is a left divisor of b and call the elements $(t_i)_{i\in I}$ the quotients of b by a. Given subspace $S \subseteq P$, $a \mid_S b$ signifies the quotients are in S.

4.3.2 Lemma. *Let $f = (f_i)_{i\in I}, g = (g_j)_{j\in J} \in \mathcal{S}^M_{(t)}(S,T)$. If $g \mid f$, then there exists a surjective set function $\alpha : I \to J$ such that $f_i(S) \subseteq g_j(S)$ for all i, j with $i \in \alpha^{-1}(j)$. As a partial converse, if there exists a surjective set function $\alpha : I \to J$ such that $f_i(S) \subseteq g_j(T)$ for all i, j with $i \in \alpha^{-1}(j)$ then $g \mid f$.*

Proof. If $g \mid f$ then by definition, then for each $i \in I$ there exists $h_i \in \mathcal{S}^M_{(t)}(S,T)(1)$ such that $f_i = g_j \circ h_i$ for some $j \in J$. This determines a function $\alpha : I \to J$ and we must have that

$$f_i(S) = (g_j \circ h_i)(S) \subseteq g_j(S).$$

Note here that the reason we can't claim $f_i(S) \subseteq g_i(T)$ and get an equivalence statement is because we don't, in general, have that $h_i \in \mathcal{S}^M_{(t)}(S,T)(1)$ are such that $h_i(S) \subseteq T$.

For the converse statement observe that for every $i \in \alpha^{-1}(j)$, since we have that M is a G-group, there exists $h_i \in M$ such that $f_i = g_j \circ h_i$. In M, we have that $g_j^{-1} \circ f_i = h_i$ and as, by assumption, we also have $(g_j^{-1} \circ f_i)(S) \subset T$ and so deduce $h_i(S) \subseteq T$. Moreover, for any other $i' \in \alpha^{-1}(j)$, and corresponding $h_{i'}$ we have that

$$\begin{aligned} g_j^{-1}\left(f_i(S) \cap f_{i'}(S)\right) &= (g_j^{-1} \circ f_i)(S) \cap (g_j^{-1} \circ f_{i'})(S)) \\ &= h_i(S) \cap h_{i'}(S) \end{aligned}$$

Hence, as $f_i(S) \cap f_{i'}(S) = \emptyset$, we get that $h_i(S) \cap h_{i'}(S) = \emptyset$. We then conclude for each $j \in J$ we get that $(h_i)_{i \in \alpha^{-1}(j)} \in \mathcal{S}^M_{(t)}(S,T)$ and so $g \mid f$. $\square$

4.3.3 Definition. Given elements $f = (f_i)_{i \in I} \in \mathcal{A}^M(T)(I)$, and $g = (g_j)_{j \in J} \in \mathcal{S}^M_{(t)}(S,T)(J)$. Let us say that f is *separated* by g if there exists a surjective set function $\alpha : I \to J$ such that $f_i(S) \subseteq g_j(T)$ for all i, j with $i \in \alpha^{-1}(j)$. We will say f is *completely separated* by g if it is separated and for all $i, i' \in \alpha^{-1}(j)$ we have that $f_i(0) = f_{i'}(0)$.

Figure 4.1 summarizes the relationship between elements $f \in \mathcal{A}^M(T)$ and $g \in \mathcal{S}^M_{(t)}(S,T)$.

Recall that we have projection maps

$$\begin{aligned} \mathrm{pr}_1 &: \mathcal{S}^{M \times M'}(T \times T') \to \mathcal{A}^M(T) \\ \mathrm{pr}_2 &: \mathcal{S}^{M \times M'}(T \times T') \to \mathcal{A}^{M'}(T'). \end{aligned}$$

$$\begin{tikzcd} \boxed{g \text{ completely separates } f} \arrow[r, Rightarrow] & \boxed{g \text{ separates } f} \arrow[r, Rightarrow] & \boxed{g \text{ divides } f} \arrow[l, Rightarrow, bend left, "\text{if } S = T"] \end{tikzcd}$$

Figure 4.1: Diagram of implications relating separability and division

Let $h_t = (1-t)\,\mathrm{id}_{V\times V'}$ for $t \in [0,1)$. This gives us a G-equivariant map

$$H : \mathcal{S}^{M\times M'}(T \times T') \times [0,1) \to \mathcal{S}^{M\times M'}(T \times T') \tag{4.2}$$

given by

$$((f_i)_{i\in I}, t) \mapsto (f_i \circ h_t)_{i\in I}.$$

Geometrically, as t increases, H shrinks the radial lengths of elements. Hence, it is not hard to see that for any fixed $f \in \mathcal{S}^{M\times M'}(T\times T')$, there exists a t large enough that both it's projections $\mathrm{pr}_1(f), \mathrm{pr}_2(f)$ are completely separated by elements of $\mathcal{S}^{M}_{(t)}(S,T), \mathcal{S}^{M'}_{(t)}(S',T')$ respectively. Such elements lie in the image of the tensor.

4.3.4 Lemma. *Let $f \in \mathcal{S}^{M\times M'}(T\times T')$ be such that $\mathrm{pr}_1(f), \mathrm{pr}_2(f)$ are completely separated by elements of $\mathcal{S}^{M}_{(t)}(S,T), \mathcal{S}^{M'}_{(t)}(S',T')$ respectively. Then f is in the image of map (4.1).*

Proof. Let $f \in \mathcal{S}^{M\times M'}(T\times T')(I)$. We will build an element $a \in \mathcal{S}^{M}_{(t)}(S,T) \otimes \mathcal{S}^{M'}_{(t)}(S',T')(I)$ such that $\iota(a) = f$. If $|I| = 1$ then as $\mathrm{pr}_1(f)$, and $\mathrm{pr}_2(f)$ is completely separated, they themselves must be elements of $\mathcal{S}^{M}(S,T)(I)$, and $\mathcal{S}^{M'}(S',T')(I)$ respectively. Hence, we can build a as

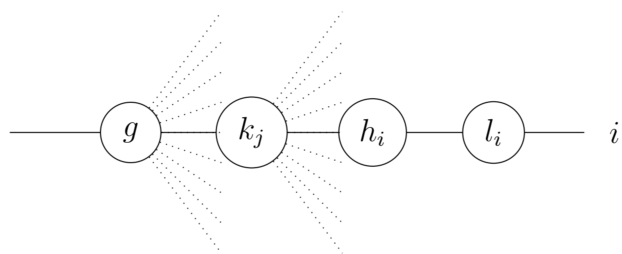
$$a = \mathrm{pr}_1(f) \circ \mathrm{pr}_2(f).$$

For $|I| > 1$, we must have at least one of the projections $\mathrm{pr}_1(f)$ or $\mathrm{pr}_2(f)$ to have components that don't share centres. Without loss of generality, suppose this is $\mathrm{pr}_1(f)$.

Then there exists $g = (g_j)_{j\in J} \in \mathcal{S}^M_{(t)}(S,T)(J)$ with $|J| > 1$ that separates $\mathrm{pr}_1(f)$. Let $\alpha : I \to J$ be the corresponding set function and as $\mathrm{pr}_1(f_i)(S) \subseteq g_j(T)$ for all $i \in \alpha^{-1}(j)$ there exists $h_i \in \mathcal{S}^M_{(t)}(S,T)(1)$ such that $\mathrm{pr}_1(f_i) = g_j \circ h_i$. This follows from the argument used in 4.3.2. Since $\mathrm{pr}_2(f)$ is completely separated by some element in $\mathcal{S}^{M'}_{(t)}(S',T')$, then so is each $\mathrm{pr}_2((f)_{i\in\alpha^{-1}(j)})$ for each $j \in J$ and let $k_j = (k_{j,i})_{i\in\alpha^{-1}(j)}$ be the corresponding separating element. Similarly to h_i, we have for each $i \in I$, an element $l_i \in \mathcal{S}^{M'}_{(t)}(S',T')(1)$ such that $\mathrm{pr}_2(f_i) = k_{j,i} \circ l_i$ where $i \in \alpha^{-1}(j)$. We can now construct the element a via the tree representative where each walk to the output i is decorated by

$$g \circ k_j \circ h_i \circ l_i.$$

i.e.,



□

4.3.5 Example. As an example of this factorisation, let us consider the simple case when $S = S' = T = T' = [0,1]$. We have the element $f \in \mathcal{S}(I \times I)$ given by

$$f = \Big([1/4, 1/2] \times [1/8, 1/3], [1/4, 1/2] \times [1/2, 3/4],$$
$$[2/3, 5/6] \times [1/8, 1/3], [2/3, 5/6] \times [1/2, 3/4], [2/3, 5/6] \times [7/8, 15/16]\Big)$$

We can picture this as in fig. 4.2. This is completely separated, and the corresponding factorisation is given by $\iota(a_1 \circ (a_2, a_3))$ where a_1, a_2, and a_3 are the elements in fig. 4.3.

Let us return to our discussion of the map H from (4.2), this lemma shows that each

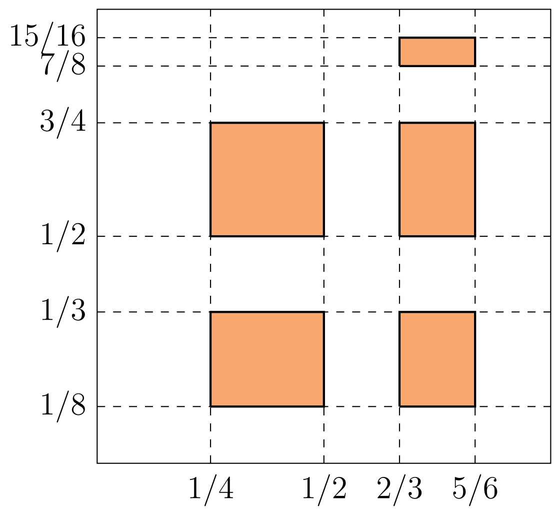


Figure 4.2: The element $f \in \mathcal{S}(I \times I)$

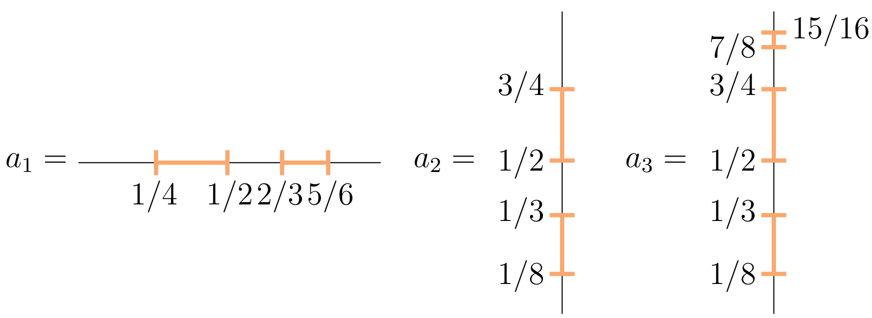


Figure 4.3: The factors a_1, a_2, and a_3.

f is eventually in the image of ι (4.1). We would then like to construct a deformation retract

$$\widetilde{H}(f,t) := H(f, s(f)t) \tag{4.3}$$

where

$$s(f) = \min\{t \in [0,1) \mid H(f,t) \in \mathrm{im}(\iota)\}.$$

The thick variant of little star operads is needed here to ensure that the map s is continuous.

4.3.6 Theorem. *The image of the induced map*

$$\iota : \mathcal{S}_t^M(S,T) \otimes \mathcal{S}_t^{M'}(S',T') \to \mathcal{S}^{M\times M'}(T\times T')$$

is weakly equivalent as G-operads to the codomain.

Proof. We must prove that (4.3) is a deformation retract. The image is closed from 4.2.7. Moreover, it is clear that if $H(f,t) \in \operatorname{im}(\iota)$ then $H(f,s) \in \operatorname{im}(\iota)$ for all $s \geq t$. It is then enough to show that for any $f \in \mathcal{S}^{M\times M'}(T\times T')(I)$ we have that

$$(\{f\} \times [0,1)) \cap H^{-1}(\partial \operatorname{im}(\iota)) = *.$$

See Lemma A.2.1 for details on why this is enough. Since $\operatorname{im}(\iota)$ is closed, the function

$$s(f) = \min\{t \in [0,1) \mid H(f,t) \in \operatorname{im}(\iota)\}$$

is well-defined. Suppose that $t_0 > s(f)$, and let $a \in \mathcal{S}_t^M(S,T) \otimes \mathcal{S}_t^{M'}(S',T')$ be such that $\iota(a) = H(f, s(f))$. Observe that it is sufficient to show some open neighbourhood of $H(f,t_0)$ is in $\operatorname{im}(\iota)$. Our plan to do this is the following. We have that

$$\iota(a \circ (\lambda \operatorname{id}_V \circ \lambda \operatorname{id}_{V'})_{i\in I}) = H(f,t_0)$$

for some $0 < \lambda < 1$. From Lemma 2.3.6, we can choose small enough open balls $B := B(\lambda \operatorname{id}_V; \varepsilon)$, and $B' := B(\lambda \operatorname{id}_V; \varepsilon)$ in M, and M' respectively such that $B \subseteq \mathcal{S}_t^M(T)(1)$, $B' \subseteq \mathcal{S}_t^{M'}(T')(1)$ and $(B\times B')(T\times T') \subseteq h_{s(f)}(T\times T')$. We then have that for each $i \in I$

$$\begin{aligned}(f_i \circ h_{t_0})(T\times T') &\subseteq (f_i \circ (B\times B'))(T\times T') \\ &\subseteq (f_i \circ h_{s(f)})(T\times T').\end{aligned}$$

Note that $(f_i \circ (B \times B'))$ is open in $M \times M'$ and so we have shown that

$$U := (f_i \circ (B \times B'))_{i \in I} = H(f, s_0) \circ (B \times B')$$

is an open neighborhood of $H(f, t_0)$ in $\mathcal{S}^{M \times M'}(T \times T')$. We now observe that

$$\iota(a \circ (B \circ B')_{i \in I}) = H(f, s(f)) \circ ((B \times B'))_{i \in I}$$

and so U is in the image of ι. Hence we have proven that a neighbourhood of $H(f, t_0)$ is in $\mathrm{im}(\iota)$ as required. □

4.4 Injectivity of tensor maps

We will now prove conditions for when the induced map map (4.1) is injective. Our method is a mild generalisation of the main idea in [BM22], which draws on ideas from the proofs of Dunn [Dun88] and Brinkmeier [Bri00].

The difficulty in proving injectivity for tensor maps stems from the fact that it requires understanding how elements factor in both the codomain and domain. We will adopt the following terminology to talk about these issues more succinctly.

4.4.1 Definition. A reduced G-operad P is *integral* if for all $x \in P(I)$ with $|I| > 0$, the corresponding composition maps

$$x \circ (-)_{i \in I} : \prod_{i \in I} P(J_i) \to P(\cup_{i \in I} J_i)$$

are injective for all J_i. This terminology was chosen as an analogy to integral domains.

We also want to extend our division terminology from the previous section.

4.4.2 Definition. Given $a, b \in P$, we call an element $c \in P$ such that $c|a$ and $c|b$ a *common (left) divisor*. If $|\operatorname{ar}(c)| > 1$ then we say it is a *non-trivial common divisor*. Similarly, for $a, b \in P$, we call an element $m \in P$ such that $a|m$ and $b|m$ a *common (left) multiple*.

4.4.3 Remark. Regarding tree representatives of the tensor, we can think of $a, b \in P \otimes Q$ having a common divisor as saying that they have representatives that share equivalent rooted subtrees.

4.4.4 Definition. An element of $a \in (P \otimes Q)(I)$ is *atomic* if $|\operatorname{ar}(I)| = 1$ or it has a tree representation by a corolla.

We will also need an efficient way to talk about the quotients of divisors. When $a|b$, we will write a choice of this division as

$$b = a \circ (q_i^{a|b})_{i \in \operatorname{ar}(a)}$$

and the corresponding surjective set function by $\alpha(a|b) : \operatorname{ar}(b) \to \operatorname{ar}(a)$.

Before specialising to little star operads, we want to develop a set of conditions sufficient for a map $P \otimes Q \to Z$ of operads to be injective. There are three different kinds of divisions we are interested in this respect.

4.4.5 Definition. For a tensor map of reduced operads $\phi : P \otimes Q \to Z$, with Z integral. For an element $x \in P \otimes Q$, define

$$\begin{aligned}
D_\phi(x) &:= \left\{ a \in P \otimes Q \;\middle|\; a|x \right\} \\
I_\phi(x) &:= \left\{ a \in P \otimes Q \;\middle|\; \phi(a) \underset{\operatorname{im}(\phi)}{|} \phi(x) \right\} \\
J_\phi(x) &:= \left\{ a \in P \otimes Q \;\middle|\; \phi(a)|\phi(x) \right\}.
\end{aligned}$$

Note that $D_\phi(x) \subseteq I_\phi(x) \subseteq J_\phi(x)$.

Given all this setup, we can now state the main theorem of this section.

4.4.6 Theorem. *Suppose we have a tensor map $\phi : P \otimes Q \to Z$ such that*

(1) Z is integral,

(2) $\phi(1)$ is injective,

(3) $I(x) = J(x)$ for all $x \in (P \otimes Q)(1)$,

(4) and for all $x \in P \otimes Q$ and all atomic elements $a \in D(x)$, and $b \in J(x)$, there exists a common multiple $m \in J(x)$ of a and b.

Then ϕ is injective.

We will use induction to prove this. To make this clearer, we will split this into several parts. Throughout the proof, I will be some finite set such that $|I| > 1$. We will then consider the following hypothesis

(H1) For all J with $|J| < |I|$, the map $\phi(J)$ is injective.

(H2) For all J with $|J| < |I|$, and $y \in (P \otimes Q)(J)$ we have that $I_\phi(y) = J_\phi(y)$.

(H3) For all $z \in (P \otimes Q)(K)$, and atomic elements $a, b \in P \otimes Q$ where $a \in D_\phi(z)$ and $b \in J_\phi(z)$. There exists a common multiple $m \in J_\phi(z)$ of a and b.

4.4.7 Lemma. *Assume that condition (H1) and condition (H2) hold. Suppose we have $x \in (P \otimes Q)(I)$ and $a \in D(x)$ with $|\mathrm{ar}(a)| > 1$. If $b \in J(x)$ such that $a|b$ then $b|x$.*

Proof. Writing our conditions out, we have that

$$x = a \circ (q_i^{a|x})_{i \in \mathrm{ar}(a)}, \tag{4.4}$$

$$b = a \circ (q_i^{a|b})_{i \in \mathrm{ar}(a)}, \tag{4.5}$$

and

$$\phi(x) = \phi(b) \circ (q_i^{\phi(b)|\phi(x)})_{i \in \mathrm{ar}(b)}. \tag{4.6}$$

Similar to the previous lemma, we get from eqs. (4.5) and (4.6) that

$$\begin{aligned}\phi(x) =&\phi(b) \circ (q_i^{\phi(b)|\phi(x)})_{i\in \mathrm{ar}(b)}\\ =&\phi\left(a \circ (q_i^{a|b})_{i\in\mathrm{ar}(a)}\right) \circ (q_i^{\phi(b)|\phi(x)})_{i\in\mathrm{ar}(b)}\\ =&\phi(a) \circ \left(\phi(q_i^{a|b}) \circ (q_j^{\phi(b)|\phi(x)})_{j\in\alpha(a|b)^{-1}(i)}\right)_{i\in\mathrm{ar}(a)}.\end{aligned}$$

Using eq. (4.4) and the integrality of Z we get that for each $i \in \mathrm{ar}(a)$

$$\phi(q_i^{a|x}) = \phi(q_i^{a|b}) \circ (q_j^{\phi(b)|\phi(x)})_{j\in\alpha(a|b)^{-1}(i)}.$$

i.e., $q_i^{a|b} \in J(q_i^{a|x})$. Since $|\mathrm{ar}(a)| > 1$ we deduce that $\mathrm{ar}(q_i^{a|x}) < |I|$, and from condition (H2), we get that there exist $\bar{q}_i$ for each $i \in \mathrm{ar}(b)$ such that

$$\phi(q_i^{a|x}) = \phi(q_i^{a|b}) \circ (\phi(\bar{q}_j))_{j\in\alpha(a|b)^{-1}(i)}.$$

From condition (H1) this implies that

$$q_i^{a|x} = q_i^{a|b} \circ (\bar{q}_j)_{j\in\alpha(a|b)^{-1}(i)}.$$

Using this, we then have that

$$\begin{aligned}b \circ (\bar{q}_i)_{i\in\mathrm{ar}(b)} &= \left(a \circ (q_i^{a|b})_{i\in\mathrm{ar}(a)}\right) \circ (\bar{q}_i)_{i\in\mathrm{ar}(b)}\\ &= a \circ (q_i^{a|x})_{i\in\mathrm{ar}(a)}\\ &= x.\end{aligned}$$

Hence $b|x$. $\square$

4.4.8 Lemma. *Assume (H1), (H2), and (H3) hold. Then for all $x \in (P \otimes Q)(I)$, we* *have that $D(x) = I(x) = J(x)$.*

Proof. Suppose $y \in J(x)$, and let a, b be the root element of a proper representative of x and y, respectively. We then have that $a \in D(x)$, and $b \in J(x)$, and so by (H3), there exists a common multiple $m \in J(x)$ of a and b. Since a is a root of a proper representative of an element of arity > 1, we must have that $|\mathrm{ar}(a)| > 1$ and so by Lemma 4.4.7 we have $m|x$. Since m is a common multiple of b, this implies $b|x$. If $\mathrm{ar}(y) > 1$ then as $y \in J(x)$, from Lemma 4.4.7 we get that $y|x$. If $\mathrm{ar}(y) = 1$, we can take $b = y$ at the beginning instead. □

Proof of Theorem 4.4.6. Because of 4.4.8, the missing step of a proof by induction is to show that if (H1) and (H2) hold ((H3) is one of the given conditions), then if $\phi(x) = \phi(y)$ where $x, y \in (P \otimes Q)(I)$ then $x = y$. Since $J(y) = J(x)$ we in fact have from 4.4.8 that $x|y$ and $y|x$. Using that $\phi(1)$ is injective, and Z is integral, we use the usual argument and see that $y = x$. □

4.4.9 Remark. There's nothing overly special about requiring the domain to be a tensor $P \otimes Q$ in Theorem 4.4.6. As long as we have some notion of atomic elements, the above theorem can prove any operadic map that meets the given conditions is injective.

4.5 Equivariant Dunn additivity

As a special case of Theorem 4.4.6, we can get an easily checkable criterion for the injectivity of the induced tensor map. First, let us define the following.

4.5.1 Definition. For the induced map

$$\iota : \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S') \to \mathcal{S}^{M \times M'}(S \times S'),$$

and an element $x \in \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S')(I)$ where $|I| > 1$. If for all elements $a, b \in \mathcal{S}^M(S)$

where a is a root element of a proper representation of x, and b is such that $b|\operatorname{pr}_1(\iota(x))$, there exists a common multiple $m \in \mathcal{S}^M(S)$ of a and b such that $m|\operatorname{pr}_1(\iota(x))$. We then say that x *has common refinements along* pr_1, and call m a common refinement of x and a. We similarly define common refinements along pr_2.

4.5.2 Lemma. *Given star domains S, S' in a G-representations V, V' with affine type M, M' respectively. For the induced tensor map*

$$\iota : \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S') \to \mathcal{S}^{M \times M'}(S \times S'),$$

if every $x \in \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S')$ has common refinements along both pr_1 *and* pr_2*, then ι is injective.*

Proof. We must verify the conditions of Theorem 4.4.6. That $\mathcal{S}^{M \times M'}(S \times S')$ is integral follows from the fact that it is constructed from the G-group $M \times M'$. The proof that the induced map is injective on unary elements was done in Example 3.4.5. That we have $J_\iota(x) = I_\iota(x)$ for all $|\operatorname{ar}(x)| = 1$ is straightforward. To see that common refinements exist along each projection, observe that for all $x \in \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S')$ and $a \in \mathcal{S}^M(S)$ we have that

$$a|\operatorname{pr}_1(\iota(x)) \iff \iota(a)|x.$$

This follows easily from the characterisation of divisibility in terms of separability (Lemma 4.3.2). Hence, if common refinements exist along each projection, this is exactly verifying the last condition on atomic elements in the tensor of the same colour. When they are of two different colours; i.e., suppose $a \in \mathcal{S}^M(S)$ is a root element of a representation of $x \in \mathcal{S}^M(S) \otimes \mathcal{S}^{M'}(S')$ and $b \in \mathcal{S}^{M'}(S')$ such that $\iota(b)|\iota(x)$. Then we get a common refinement by just taking $a \otimes b$ (See Definition 3.4.1). Therefore the conditions of Theorem 4.4.6 hold. $\square$

We will now show that the equivariant little simplex operads (Definition 2.2.14) satisfy this condition. The key lemma is the following.

4.5.3 Lemma. *Let V be a G-representation and S an affine independent subset of V as in Definition 2.2.14. Write $\Delta = \Delta^S$ for the corresponding simplex. Given maps $f, f' \in \Lambda(V)(\Delta, \Delta)$ such that $f(\Delta) \cap f'(\Delta) \neq \emptyset$. Then there exists a (unique) $g \in \Lambda(V)(\Delta, \Delta)$ such that*

$$g(\Delta) = f(\Delta) \cap f'(\Delta).$$

Proof. Since f is a homeomorphism, we can apply the inverse to the problem, and we see it is sufficient to show that for any $f \in \Lambda(V)$ such that $\Delta \cap f(\Delta) \neq \emptyset$ there exists a $g \in \Lambda(V)$ such that $g(\Delta) = \Delta \cap f(\Delta)$. For $i \in I$, denote by P_i the supporting affine hyperplane of the facet opposite the vertex $\vec{s}_i$.

$$P_i = \Big\{ \sum_{\substack{j \in I \setminus \{i\} \\ \text{finite}}} \lambda_j \vec{s}_j \;\Big|\; \sum_{\substack{j \in I \setminus \{i\} \\ \text{finite}}} \lambda_j = 1 \Big\}.$$

Let α_i be the affine form with $\ker \alpha_i = P_i$ and $\alpha_i(\vec{s}_i) = 1$ and define the following half-spaces

$$H_i := \Big\{ \vec{v} \in V \;\Big|\; \alpha_i(\vec{v}) \geq 0 \Big\}.$$

This is set up so that

$$\Delta = \bigcap_{i \in I} H_i$$

and this is a minimal intersection of half-spaces. Since the map f is affine, $f(H_i)$ are half-spaces. Moreover, since it is a rigid dilation, $f(P_i)$ is parallel to P_i. For each i, we either have that $\alpha_i(f(P_i)) \geq 0$ or $\alpha_i(f(P_i)) < 0$. Define the half-spaces K_i by

$$K_i := \begin{cases} H_i & \text{if } \alpha_i(f(P_i)) < 0 \\ f(H_i) & \text{if } \alpha_i(f(P_i)) \geq 0. \end{cases}$$

Observe that we have that $K_i = H_i \cap f(H_i)$ and so we have that

$$\Delta \cap f(\Delta) = \left(\bigcap_{i \in I} H_i\right) \cap \left(\bigcap_{i \in I} f(H_i)\right) = \bigcap_{i \in I} K_i.$$

So the intersection is a simplex where every facet is parallel to the corresponding facet of Δ and $f(\Delta)$. The vertices of the intersection determine an affine map g such that $g(\Delta) = \Delta \cap f(\Delta)$. Moreover, we pick this such that $g(H_i) = K_i$. We now need to show that $g \in \Lambda(V)$.

Let us first consider the case when V is finite-dimensional. Suppose that there is only a single $i \in I$ such that $K_i \neq H_i$. Then by Thales's theorem (otherwise known as the intercept theorem), we have that for every $j, j' \in I$ that $g(s_j \vec{s}_{j'}) = \lambda s_j \vec{s}_{j'}$ for the same constant $0 < \lambda \leq 1$. We then deduce that g is of the form $g(\vec{v}) = \lambda \vec{v} + \vec{c}$ in this case. For a general collection of the K_i, order these and apply the above, one hyperplane at a time. We deduce that g must still be of the same form. When V is infinite-dimensional, we just need to restrict to affine spans of finite subsets of S, where we see g is a rigid dilation on the restriction. But as we cover V by these, we get that $g \in \Lambda(V)$, and we are done. $\square$

We can also extend this to products.

4.5.4 Lemma. *For a G-representation V with decomposition $F = \{V_i\}_{i \in I}$ and affine type $M = M(F)$. Let Δ_i be simplices on V_i such as in Definition 2.2.14, and denote the product by $\Delta^F := \prod_{i \in I} \Delta_i$. For maps $f, f' \in M(\Delta^F, \Delta^F)$ such that $f(\Delta^F) \cap f'(\Delta^F) \neq \emptyset$,* *there exists a map $g \in M(\Delta^F, \Delta^F)$ such that*

$$g(\Delta^F) = f(\Delta^F) \cap f'(\Delta^F).$$

Proof. This lemma follows from the previous lemma by looking at the projections onto each subspace V_i. $\square$

We can now finally state an equivariant version of Dunn additivity

4.5.5 Theorem (Equivariant Dunn additivity)**.** *Let V and V' be G-representations with corresponding decompositions F and F'. The induced map*

$$\mathcal{D}_\Delta^{\mathrm{F}}(V) \otimes \mathcal{D}_\Delta^{\mathrm{F}'}(V') \to \mathcal{D}_\Delta^{\mathrm{F}\oplus\mathrm{F}'}(V \oplus V')$$

is a weak equivalence of G-operads.

Proof. All that is left to prove is that Lemma 4.5.2 holds. However, this is obvious after Lemma 4.5.4. $\square$

4.5.6 Remark. Interestingly, the equivariant little cubes of Definition 2.2.12 do not satisfy Lemma 4.5.2 unless we work with a trivial representation. This is because the dilation factor in the indecomposable must stay the same in that subspace. However, intersections of cubes can change their side proportions within the same subspace, so we don't have the required condition.

Chapter 5

Operads that encode norm maps

In this section, we will present some parts of a work-in-progress by the author on building general $\mathbb{N}$-operads. We start with reviewing the relevant ideas and literature about $\mathbb{N}_\infty$-operads before moving on to constructing twisted little cube operads $\mathcal{C}_n^{\mathcal{F}}$. These operads can be viewed as a model for operads between $\mathbb{E}_V$-operads and what should be called $\mathbb{N}$-operads. These operads encode norm maps determined by any choice of indexing system $\mathcal{F}$, and in the $n = \infty$ case, give us models for $\mathbb{N}_\infty$-operads. We then end this dissertation by discussing additivity for these operads.

We will change our notation for finite sets for this chapter and denote them instead by scripts $\mathscr{i}, \mathscr{j}, \mathscr{k}$, etc. *We will also assume that the group G is abelian throughout this chapter.* We do this to simplify the presented proofs. We expect our results to hold for general finite groups without much more work, and we will return to this in the future.

5.1 A conceptual overview of $\mathbb{N}$-operads

Conceptually, $\mathbb{E}$-operads encode algebras that have some portion of higher homotopy coherence data for homotopy commutative monoids. To expand on what we mean by this, let us take some $\mathbb{E}_k$-algebra X in Top and let us denote the structure map by

$\eta : \mathbb{E}_k \to \mathrm{End}(X)$. The basic structure of an algebra under an operad means that the operad parameterises "multiplication maps" on X. The key property of an $\mathbb{E}_k$-operad is that it will also point out homotopies between different multiplications, homotopies between homotopies and so on – at least up until k-cells.

Let us illustrate this idea in overbearing detail – we will find it useful shortly. Consider an element $f \in \mathbb{E}_k(n)$ where $k > 2$. Then $\eta(f)$ is a multiplication map

$$X \times X \times \cdots \times X \xrightarrow{\eta(f)} X,$$

but because this lives in a symmetric monoidal category, f determines more than one map. For instance, let $\sigma : X^n \to X^n$ be any permutation map, then the composition gives us a map

$$X \times X \times \cdots \times X \xrightarrow{\sigma} X \times X \times \cdots \times X \xrightarrow{\eta(f)} X.$$

These maps correspond to the Σ_n-orbit of f in $\mathbb{E}_k(n)$, which by definition must be all different unless f is identity. Another way to phrase this, is that f corresponds to a non-equivariant map $\bar{f} : * \to \mathbb{E}_k(n)$. Since $\mathbb{E}_k(n)$ is a Σ_n-space, the induction-restriction adjunction means this is equivalent to a Σ_n-map $\Sigma_n \times * \xrightarrow{\Sigma_n \times \bar{f}} \mathbb{E}_k(n)$ and the orbit of corresponding multiplication maps by the Σ_n-map

$$\Sigma_n \times * \xrightarrow{\Sigma_n \times \bar{f}} \mathbb{E}_k(n) \xrightarrow{\eta} \mathrm{Hom}(X^n, X).$$

For $k > 1$, then for $f, g \in \mathbb{E}_k(n)$, the data of $\mathbb{E}_k(n)$ also includes the data of a homotopy that connects these maps.

$$\begin{tikzcd}
\Sigma_n \times * \arrow[d, "\mathrm{id} \times i_0"'] \arrow[dr, "\Sigma_n \times f"] & & \\
\Sigma_n \times I \arrow[r, "\Sigma_n \times H"] & \mathbb{E}_k(n) \arrow[r, "\eta"] & \mathrm{Hom}(X^n, X) \\
\Sigma_n \times * \arrow[u, "\mathrm{id} \times i_1"] \arrow[ur, "\Sigma_n \times g"'] & &
\end{tikzcd}$$

We can keep doing this and adding higher homotopies, up to k-cells.

A $\mathbb{E}_\infty$-operad extends this and encodes the entire tower of homotopy coherence data. Homotopically speaking, we should expect no real difference between a $\mathbb{E}_\infty$-algebra and a commutative monoid. This is true and made rigorous in "rectification theorems" between operads. In particular, the homotopy categories of $\mathbb{E}_\infty$-algebras and commutative monoids are equivalent. Counterintuitively, this no longer works if we look at $\mathbb{E}_\infty$-algebras in an equivariant category.

To understand what is happening here, let us now take the $\mathbb{E}_k$-algebra X from above to be in Top^G. The $\mathbb{E}_k$ operad parameterises homotopy coherence data as in the non-equivariant case. Let us again consider a single element $f \in \mathbb{E}_k(n)$. This is determined by a G-map $\bar{f} : * \to \mathbb{E}_k(n)$, and inducing up to $G \times \Sigma_n$-spaces, this is a map $\Sigma_n \times * \xrightarrow{\Sigma_n \times f} \mathbb{E}_k(n)$ and the multiplication that f points out is given by the $G \times \Sigma_n$-map

$$\Sigma_n \times * \xrightarrow{\Sigma_n \times \bar{f}} \mathbb{E}_k(n) \xrightarrow{\eta} \mathrm{Hom}(X^n, X).$$

There is now a problem. The orbit of f can only be Σ_n/e or if it's the identity, Σ_n/Σ_n, and so f must map to points of $\mathrm{Hom}(X^n, X)$ with isotropy given by subgroups of Σ_n. However, $\mathrm{Hom}(X^n, X)$ is now a $G \times \Sigma_n$-space, and its isotropy groups can now involve the group G. We can think of these fixed points of $\mathrm{Hom}(X^n, X)$ as "twisted multiplications". This is because, after unpacking definitions, they correspond to multiplication maps

$f : X^n \longrightarrow X$ where for some homomorphism $\sigma : H \to \Sigma_n$ where $H < G$, we have for $h \in H$ that

$$hf(x_1, x_2, \ldots, x_n) = f(hx_{\sigma^{-1}(h)(1)}, hx_{\sigma^{-1}(h)(2)}, \ldots, hx_{\sigma^{-1}(h)(n)}).$$

So unlike in the non-equivariant case, there are multiplication maps that $\mathbb{E}_k$ cannot parameterise. This is the underlying reason for the failure of a rectification theorem of $\mathbb{E}_\infty$-operads in the equivariant setting. A commutative moniod in an equivariant category must automatically satisfy relations involving twisted multiplications. However, $\mathbb{E}_k$ cannot even parameterise these, let alone fill in the missing coherence data for the relations these maps must satisfy.

The aim of $\mathbb{N}$-operads is to extend $\mathbb{E}$-operads to the equivariant setting by being able to encode these twisted multiplication maps and associated coherence data. The study of these kinds of operads, and the term $\mathbb{N}$-operad, was initiated by Blumberg-Hill in [BH15] where they defined and studied $\mathbb{N}_\infty$-operads - the $\mathbb{E}_\infty$-case. Let us recall how these are defined.

5.1.1 Definition ([BH15], definition 1.1.)**.** A $\mathbb{N}_\infty$-operad P is a G-operad such that

(1) $P(0)$ is G-contractible,

(2) the $\mathrm{Aut}(j)$-action on $P(j)$ is free,

(3) $P(j)$ is a universal space for a family of subgroups $\mathcal{F}_j$ of $G \times \mathrm{Aut}(j)$ that includes the subgroups $H \times \{\mathrm{id}_j\}$.

The Σ_j-free condition implies that the groups in these families of subgroups are of the following form.

5.1.2 Definition. let j be a finite set. A G-graph subgroup Γ of arity j is a subgroup

$\Gamma \subseteq G \times \mathrm{Aut}(j)$ of the form

$$\Gamma = \{(h, \phi(h)) \mid h \in H\}$$

for some group homomorphism $\phi : H \to \mathrm{Aut}\, j$. We will call ϕ the underlying homomorphism of Γ and may use the notation ϕ_Γ for it.

5.1.3 Remark. G-graph subgroups Γ can also be characterized by the property that $\Gamma \cap \{e\} \times \mathrm{Aut}(J) = \{e\} \times \{\mathrm{id}_J\}$.

5.1.4 Definition. A *collection of G-graph subgroups* $\mathcal{F}$ is a collection of sets of G-graph subgroups $\mathcal{F} = \{\mathcal{F}_j\}_{j \in \mathcal{B}ij}$. Given a $\mathbb{N}_\infty$-operad P, we will denote the collection of G-graph subgroups in it's definition by $\mathcal{F}(P)$.

5.1.5 Remark. Unpacking these definitions so far, we have that for an $\mathbb{N}_\infty$-operad P that

$$P(j)^\Gamma \simeq \begin{cases} * & \text{if } \Gamma \in \mathcal{F}(P) \\ \emptyset & \text{if } \Gamma \notin \mathcal{F}(P). \end{cases}$$

This shows that an $\mathbb{N}_\infty$-operad is precisely the kind of G-operads that homotopically coherently encode some portion of the possible twisted multiplication maps.

The natural question is, then, what kinds of families of graph subgroups are admissible? i.e., what families $\mathcal{F}$ are $\mathcal{F}(P)$ for some $\mathbb{N}_\infty$-operad P. Blumberg-Hill gave a set of conditions such a family must satisfy and packaged this data in an *indexing system* [BH15, 3.3]. Moreover, there is a lattice of such indexing systems and a functor from $\mathrm{Ho}(\mathbb{N}_\infty)$, the homotopy category of $\mathbb{N}_\infty$-operads, to it. Blumberg-Hill conjectured that this functor was an equivalence. Several proofs ([GW18], [Rub21a], and [BP21]) have since confirmed this, each taking a different approach to the problem.

An alternative characterisation of indexing systems, due to both Rubin [Rub21b] and Balchin-Barnes-Roitzheim [BBR21], is the following

5.1.6 Definition. A *G-transfer system* is a subrelation $\dashrightarrow$ of the subgroup lattice $\mathbf{Sub}(G)$ such that the following conditions hold.

(1) *Identity.* $H \dashrightarrow H$ for all $H < G$,

(2) *Conjugation.* If $K \dashrightarrow H$ then ${}^gK \dashrightarrow {}^gH$ for all $g \in G$,

(3) *Restriction.* If $K \dashrightarrow H$ and $H' < H$, then $K \cap H' \dashrightarrow H'$.

(4) *Composition.* If $K \dashrightarrow H$ and $H \dashrightarrow J$ then $K \dashrightarrow J$.

The connection to the families $\mathcal{F}(P)$ is through the following. A G-graph subgroup $\Gamma < G \times \mathrm{Aut}(j)$ with underlying homomorphism $\phi_\Gamma : H \to \mathrm{Aut}(j)$ determines a H-action on j and so determines a H-set structure on j which we will denote by $\mathrm{set}(\Gamma)$. If a finite H-set T is such that for some $\Gamma \in \mathcal{F}(P)$ we have an isomorphism of H-sets $\mathrm{set}(\Gamma) \cong T$, then T is called an *admissible H-set* for the operad P. As a consequence of the definition of $\mathbb{N}_\infty$-operads, the set of all admissible sets is closed under both coproducts and subobjects. Hence, the admissible sets are completely determined by which finite H-sets of the form H/K are allowed. These are exactly the transfer maps in a transfer system.

We will end this section with an alternative characterisation for transfer systems which is a bit more categorical.

5.1.7 Definition. We will call an object of $\mathbf{Cat}^G$ a G-category. i.e., a category with an action by G via endofunctors. We will call a (commutative) monoid object in G-categories a (symmetric) G-monoidal category. i.e., a G-category with a (symmetric) monoidal category structure where the monoidal product is equivariant.

5.1.8 Lemma. *The subgroup lattice $\boldsymbol{Sub}(G)$ has the structure of a G-category. Moreover, it is a strict monoid object in the category $\mathbf{Cat}^G$ where subgroups' intersections give the monoidal product.*

Proof. The G-category structure comes from the lattice structure, which the G-action by conjugation preserves and acts as functors. It is straightforward that the intersection is a monoidal product on $\mathbf{Sub}(G)$ where G is the identity. It is strict since the set intersection is strict. The intersection is also equivariant under conjugation by G. $\square$

5.1.9 Theorem. *A transfer system $\dashrightarrow$ is equivalent to a wide G-monoidal subcategory of $\boldsymbol{Sub}(G)$.*

Proof. The identity, conjugation and composition conditions are just the conditions for a G-subcategory. We will denote this subcategory by $\overset{\dashrightarrow}{\mathcal{F}}$. We must justify that the restriction condition is equivalent to the monoidal product being closed on a subcategory. What this amounts to is showing that the restriction condition is equivalent to the intersection being a well-defined G-bifunctor:

$$- \cap - : \overset{\dashrightarrow}{\mathcal{F}} \times \overset{\dashrightarrow}{\mathcal{F}} \to \overset{\dashrightarrow}{\mathcal{F}}.$$

Let us use solid arrows $\to$ for arrows in $\mathbf{Sub}(G)$. The restriction condition diagrammatically is the following commutative square.

$$\begin{tikzcd}
H & K \arrow[l, dashed] \\
H' \arrow[u] & H' \cap K \arrow[l, dashed] \arrow[u]
\end{tikzcd}$$

Using this, we can build the following commutative diagram which implies the functor is well-defined on the subcategory $\overset{\dashrightarrow}{\mathcal{F}}$.

$$\begin{tikzcd}
 & H_1 & H_2 \arrow[l, dashed] & H_3 \arrow[l, dashed] \\
K_1 & K_1 \cap H_1 \arrow[l] \arrow[u] & K_1 \cap H_2 \arrow[l, dashed] \arrow[u] & K_1 \cap H_3 \arrow[l, dashed] \arrow[u] \\
K_2 \arrow[u, dashed] & K_2 \cap H_1 \arrow[l] \arrow[u, dashed] & K_2 \cap H_2 \arrow[l, dashed] \arrow[u, dashed] & K_2 \cap H_3 \arrow[l, dashed] \arrow[u, dashed] \\
K_3 \arrow[u, dashed] & K_3 \cap H_1 \arrow[l] \arrow[u, dashed] & K_3 \cap H_2 \arrow[l, dashed] \arrow[u, dashed] & K_3 \cap H_3 \arrow[l, dashed] \arrow[u, dashed]
\end{tikzcd}$$

Conversely, if we are given a wide G-monoidal subcategory $\mathcal{F}$ of $\mathbf{Sub}(G)$, then for any morphism $K \dashrightarrow H$ in $\mathcal{F}$ and subgroup $H' < H$, we get from functoriality of the product that

$$H' \cap (K \dashrightarrow H) = H' \cap K \dashrightarrow H'.$$

i.e., that restriction holds. $\square$

5.1.10 Remark. Note that the intersection of subgroups in $\mathbf{Sub}(G)$ is the categorical product. Because of this, another interpretation of a transfer system is an embedding that preserves products.

5.1.11 Convention. When we refer to an indexing system from now on, we will mean a wide G-monoidal subcategory of $\mathbf{Sub}(G)$.

The last thing we will recall in this section is that transfer systems themselves form a lattice of transfer systems.

5.1.12 Definition. Given two transfer systems $\mathcal{F}$ and $\mathcal{F}'$, the join $\mathcal{F} \vee \mathcal{F}'$ is the transfer systems generated by $\mathcal{F}$ and $\mathcal{F}'$. i.e., the smallest transfer system to contain both $\mathcal{F}$ and $\mathcal{F}'$.

5.2 Towards a model for $\mathbb{N}$-operads

In this section, we will build "twisted" little cube operads as a model for $\mathbb{N}$-operads. To do this, we will need to separate our operads from their underlying representations. In order to motivate the construction, let us first recall that the forgetful functor

$$(i^G)^* : \mathrm{Oper}^{\mathrm{G}} \to \mathrm{Oper}$$

has left and right adjoints, called induction and coinduction, respectively.

$$\begin{tikzcd} \mathrm{Oper} \arrow[rr, bend left, "(i_G)_*"] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, bend right, "(i_G)_!"'] \arrow[rr, phantom, shift right=2, "\perp"] & & \mathrm{Oper}^{\mathrm{G}} \arrow[ll, "i_G^*"] \end{tikzcd}$$

Here i^G is the map on the indexing categories $i^G : \mathcal{B}ij \to G \times \mathcal{B}ij$, which is compatible with the operadic structure. Induction and coinduction are given explicitly by

$$\mathrm{Ind}_e^G(P) = i_!^G(P) = \coprod_G P$$
$$\mathrm{Coind}_e^G(P) = i_*^G(P) = \prod_G P$$

Where this is the product and coproduct in $\mathrm{Oper}^{\mathrm{G}}$. The key idea for our constructions is to use the unit of the restriction-coinduction adjunction

$$\eta : \mathcal{D}(V) \to \mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$$
$$(f_j)_{j\in j} \mapsto (g \mapsto (g \cdot f_j)_{j \in j}).$$

The unit is injective, so there's a copy of $\mathcal{D}(V)$ living inside of $\mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$. *This copy no longer depends on the representation V.* Since every $\mathbb{E}_V$-operad lives inside operads of the form $\mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$, this raises the following question.

5.2.1 Question. *Are there suboperads of coinduced little something operads that can serve as models for* $\mathbb{N}$*-operads?*

We will provide a prototype of such a construction below. First, let us define a slight variation on coinduced operads that doesn't reference representations.

5.2.2 Definition. The *ambient twisted little n-cube operad* $\mathcal{A}_n$ has $\jmath$-component given by

$$\mathcal{A}_n(\jmath) = \mathrm{Hom}(G \times \mathrm{Aut}(\jmath), R(I^n, I^n))$$

where $R(I^n, I^n)$ is the space of rectilinear embeddings. For $\alpha : \mathscr{k} \to \jmath$, and $x \in \mathcal{A}_n(\jmath)$, $y_j \in \mathcal{A}_n(\mathscr{k}_j)$ we define the composition by

$$(x \circ (y_j))\,(g, k) := x(g, \alpha(k)) \circ y_{\alpha(k)}(g, k).$$

The G-action (actually, a $G \times \mathcal{B}ij$-action) comes from $((g, \sigma) \cdot x)(g', j) = x(g^{-1}g', \sigma^{-1}j)$.

In analogy with $\mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$, we think of an element $x \in \mathcal{A}_n$ as a $\jmath$-tuple of cubes given by $x(e, -)$. The other tuples $x(g, -)$ then describe the orbit of the element $x(e, -)$ under the G-action. We will sometimes call $x(-, j)$ the j-th cube of x. We will use the following terminology.

5.2.3 Definition. Given an element $x \in \mathcal{A}_n(\jmath)$ and a subset $\jmath' \subseteq \jmath$, the $\jmath'$-the cube of x is the restriction

$$x^{\jmath'} := x\Big|_{G \times \jmath'}.$$

For $j \in \jmath$, we will often use the shorthand $x^j(g) := x^{\{j\}}(g) = x(g, j)$.

Our aim is to find suboperads of $\mathcal{A}_n$ that have fixed points living in some prescribed family $\mathcal{F}$. Simply taking restrictions onto points with the correct stabilisers won't be enough as this won't ensure an operad structure. The key observation in preserving an operad structure is that *overlapping cubes introduce fixed points.* These fixed points come from possible compositions of these cubes. Because of this, identifying correct suboperads of $\mathcal{A}_n$ to serve as models for $\mathbb{N}$-operads boils down to correctly identifying how the cubes $x(g, j)$ of elements $x \in \mathcal{A}_n$ can overlap.

We can categorise how the cubes $x(g, j)$ overlap in two ways: as a partial overlap, where the intersection is nonempty, and as a complete overlap, where the two cubes are equal. Both of these types of overlaps are exhibited in the copy of $\mathcal{D}(V)$ in $\mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$. Given $(f_1, \ldots, f_n) \in \mathcal{D}(V)$, we can have that f_i and $g \cdot f_j$ partially overlap if $i \neq j$. If, on the other hand, $i = j$, then we must have f_i and $g \cdot f_i$ completely overlap or don't intersect at all. This apparent asymmetry is due to the underlying representation. However, if we forget about the origin of this condition, it still makes sense to enforce it. Elements in $\mathrm{Coind}_e^G(i_G^* \mathcal{D}(V))$, and also our ambient operad $\mathcal{A}_n$, are composed along the indices $j \in \jmath$. Forcing that the disks be equal or disjoint then preserves some overlaps after composition while completely removing the others. It, however, doesn't fundamentally change the remaining overlaps or introduce new ones. It seems reasonable that we will want to emulate this behaviour in our constructions.

5.2.4 Definition. Given an element $x \in \mathcal{A}_n(\jmath)$, we will say that the j-th cube x^j has *strict orbits of type* H, or H-orbits, if for $g_1, g_2 \in G$ we have that

(1) $x(g_1, j) = x(g_2, j)$ if $g_1^{-1} g_2 \in H$, and

(2) $x(g_1, j)(I^n) \cap x(g_2, j)(I^n) = \emptyset$ otherwise.

If every cube of x has strict orbits (for possibly different subgroups for each $j \in \jmath$), then we say it has strict orbits.

In order to simplify notation in the future, we will use the following convention.

5.2.5 Convention. We will often omit I^n when talking about statements of a maps image. i.e., we will write $x(g, j)(I^n)$ as $x(g, j)$.

The following lemma is straightforward but essential.

5.2.6 Lemma. *For a map of finite sets $\alpha : \mathscr{k} \to j$, and elements $x \in \mathcal{A}_n(j)$, and $y_j \in \mathcal{A}_n(\mathscr{k}_j)$ for each $j \in j$. If x, y_j all have strict orbits, then so does $x \circ (y_j)_{j \in j}$. In particular, if $x(-, j)$ has strict orbits of type H and $y_j(-, k)$ has orbits of type K, then $(x \circ (y_j)_{j\in j})(-, k)$ has strict orbits of type $K \cap H$.*

We want to restrict our suboperads only to have graph subgroups as isotropy subgroups – although, interestingly, we can build suboperads that don't obey this, as the following example shows.

5.2.7 Example. Consider the subcollection $\mathscr{P}$ of $\mathcal{A}_n$ given by

$$\mathscr{P}(j) = \left\{ x \in \mathcal{A}_n(j) \;\middle|\; \text{for all } g \in G \text{ we have } \cap_{j \in j}\, x(g, j) = \emptyset \right\}.$$

This forms a well-defined G-operad and as long as $\sigma \in \mathrm{Aut}(j)$ is not a full cycle, we have $\mathscr{P}^\sigma \neq \emptyset$.

The condition that will enforce graph subgroups as the only allowed isotropy groups is the following.

5.2.8 Definition. We will say an element $x \in \mathcal{A}_n(j)$ is *non-degenerate* if $x(g, j) \cap x(g, j') = \emptyset$ for all $g \in G, j \neq j' \in j$.

As for further restrictions, let us consider the elements in fixed points. First, since we can always compose with the zero element $* \in \mathcal{A}_n(0)$, we only need to focus on the basic orbit fixed points H/K as described by the transfer system. To better describe arrangements of cubes, we will extend the x^j notation as follows.

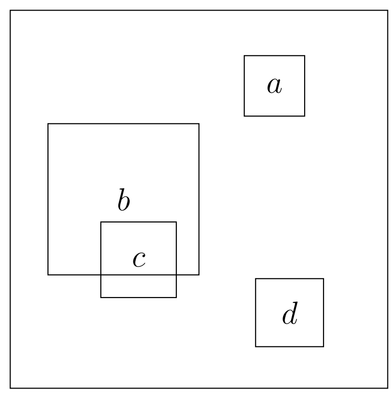
5.2.9 Definition. For a subset $S \subseteq G$, a *vertical S-slice* of $G \times j$ is a function $a : S \to j$. For an element $x \in \mathcal{A}_n(j)$, the a-cube x^a of x is the function $x^a(g) := x(g, a(g))$ for

$g \in S$. We can interpret the j-th cubes x^j as a special case of x^a where we treat j as the constant function $G \to j$ with output j.

5.2.10 Example. Let us consider the group $G = C_4 \times C_2$ and what a fixed point $x \in (\mathcal{A}_n)^\Gamma(2)$ where $\mathrm{set}(\Gamma) \cong C_2 \times C_2/C_2 \times \{e\}$ looks like. Here we will assume x has strict orbits. We can set out the cubes of x in an array as in fig. 5.1a. The array entries correspond to the cubes of x, which can be pictured as embeddings as in fig. 5.1b. Note that we can't have $a = c$ since by the strict orbits, this would force $b = d$, and then this would be a fixed point corresponding to $(C_4 \times C_2)/(C_4 \times \{e\})$.

	1	2
$(0,0)$	a	b
$(2,0)$	a	b
$(0,1)$	b	a
$(2,1)$	b	a
$(1,0)$	c	d
$(3,0)$	c	d
$(1,1)$	d	c
$(3,1)$	d	c

(a) cubes of $x \in (\mathcal{A}_n)^\Gamma(2)$ in an array.



(b) cubes of $x \in (\mathcal{A}_n)^\Gamma(2)$ as embeddings

Figure 5.1: An example of a fixed point of $\mathcal{A}_n$.

As we can see, cubes of fixed points follow specific patterns that get repeated. The following definitions are designed to encode such patterns and single out the combinations of overlaps that can generate them. Our plan for building our desired operad is to understand how these patterns behave under composition and only allow the correct type.

5.2.11 Definition. For an element $x \in \mathcal{A}_n(\jmath)$, $j \in \jmath$, and subgroups $K < H < G$. *H/K-twist of x based at j* is an a-cube x^a where a is a vertical gH-slice for some coset $gH \in G/H$ such that

(1) $j \in \operatorname{im}(a)$,

(2) we have $\bigcap_{h \in gH} x^a(h) \neq \emptyset$, and

(3) $a^{-1}(j)$ is a right K-coset.

5.2.12 Example. Considering again fig. 5.1, we can see that we have $(C_2 \times C_2)/(C_2 \times \{e\})$-twists (fig. 5.2a), but also $(\{e\} \times C_2)/(\{e\} \times \{e\})$-twists (fig. 5.2b). There are also "trivial twists" of type $(C_2 \times \{e\})/(C_2 \times \{e\})$ (fig. 5.2c), and $(\{e\} \times \{e\})/(\{e\} \times \{e\})$ (fig. 5.2d).

	1	2
$(0,0)$	a	b
$(2,0)$	a	b
$(0,1)$	b	a
$(2,1)$	b	a
$(1,0)$	c	d
$(3,0)$	c	d
$(1,1)$	d	c
$(3,1)$	d	c

(a)

	1	2
$(0,0)$	a	b
$(2,0)$	a	b
$(0,1)$	b	a
$(2,1)$	b	a
$(1,0)$	c	d
$(3,0)$	c	d
$(1,1)$	d	c
$(3,1)$	d	c

(b)

	1	2
$(0,0)$	a	b
$(2,0)$	a	b
$(0,1)$	b	a
$(2,1)$	b	a
$(1,0)$	c	d
$(3,0)$	c	d
$(1,1)$	d	c
$(3,1)$	d	c

(c)

	1	2
$(0,0)$	a	b
$(2,0)$	a	b
$(0,1)$	b	a
$(2,1)$	b	a
$(1,0)$	c	d
$(3,0)$	c	d
$(1,1)$	d	c
$(3,1)$	d	c

(d)

Figure 5.2: Examples of twists

5.2.13 Lemma. *Let $x \in \mathcal{A}_n(\jmath)$ be non-degenerate, $j \in \jmath$, and H, K subgroups of G. If x^a is an a-cube where a is a vertical gH-slice for some coset $gH \in G/H$ such that*

(1) $j \in \operatorname{im}(a)$, and

(2) we have $\bigcap_{h \in gH} x^a(h) \neq \emptyset$.

Then x^a is a $H/(H \cap K)$-twist based at j where K is the orbit type of x^j.

Proof. Consider the set $S = a^{-1}(j)$, which is nonempty as $j \in \mathrm{im}(a)$. We have $S \subseteq gH$ by definition and $S \subseteq \hat{g}K$ for some $\hat{g}$ by the orbit type of x^j. Hence $S \subseteq gH \cap \hat{g}K = k(H \cap K)$ for some $k \in G$. Suppose that there exists $h \in k(H \cap K) \setminus S$. Let $s \in S$ and then consider $x(h, a(h)) \cap x(s, j)$. Since $h \notin S$ we get $a(h) \neq j$, and since $h, s \in \hat{g}K$, we conclude that $x(h, a(h)) \cap x(s, j) = \emptyset$. However, this contradicts the second condition as this would imply $\bigcap_{h \in gH} x^a(h) = \emptyset$. □

5.2.14 Lemma. *Let $x \in \mathcal{A}_n(j)$ have strict orbits and be non-degenerate such that $\Gamma = \mathrm{Stab}(x)$ is nonempty and $\mathrm{set}(\Gamma) \cong H/K$. Then each j-th cube x^j has strict orbits of types K and every twist is of the form H'/K' where $K' = K \cap H'$ and $H' < H$.*

Proof. let $\phi : H \to \mathrm{Aut}(j)$ be the underlying homomorphism of Γ and suppose that some j-th cube x^j has orbit of type $K' > K$. Since x has isotropy Γ, we have that

$$x(g, i) = x(hg, \phi(h)(i)) \text{ for all } h \in H, g \in G, \text{ and } i \in j.$$

Since H/K is a transitive H-set, for all $i \in j$, there exists a $h \in H$ such that $\phi(h)(j) = i$. Hence, by the above, for all $k \in K'$, and $g \in G$ we get that

$$\begin{aligned} x(kg, j) &= x(kg, \phi(h^{-1})(i)) \\ &= x(hkg, i) \\ &= x(khg, i). \end{aligned}$$

Since x^j has strict orbit type K', we deduce that x^i must have strict orbit type K'. Moreover, since each x^i has the same orbit type and G is Abelian, this implies that we can extend ϕ to a map $K'H \to \mathrm{Aut}(J)$ which contradicts $\mathrm{set}(\Gamma) \cong H/K$ unless $K' = K$.

To show that each twist of the required form. Suppose we have a vertical gH'-slice

a. Since each x^j has orbit type K, and $|\jmath| = |H/K|$, we can have at most $|H|$ equal disks in a single vertical slice. Hence we must have that $H' < H$ and then the lemma follows from Lemma 5.2.13. $\square$

The following is the key lemma that describes how the H/K-twists behave under compositions.

5.2.15 Lemma. *Suppose we have a function of finite sets $\alpha : \mathscr{k} \to \jmath$, and non-degenerate elements $x \in \mathcal{A}_n(\jmath)$, and $y_j \in \mathcal{A}_n(\mathscr{k}_j)$ for $j \in \jmath$. Write $z = x \circ (y_j)_{j \in \jmath}$ for the composition and suppose that z^a is a H/K-twist based at $i \in \mathscr{k}$. Then for some subgroup $K < H' < H$, there exists a H/H'-twist x^b based at $\alpha(i)$, and a H'/K-twist y^c based at i.*

Proof. From our assumptions, we have a function of finite sets $a : gH \to \mathscr{k}$. Construct the function $b : gH \to \jmath$ by the composition $b = \alpha \circ a$. Since $z(g, i') \subseteq x(g, \alpha(i'))$ for all $g \in G$ and $i' \in \mathscr{k}$, it follows that x^b is such that $\cap_{h \in gH} x^b \neq \emptyset$. If K' is the orbit type of $x^{\alpha(i)}$ then Lemma 5.2.13 tells us that $b^{-1}(\alpha(i)) = g'H'$ for some $g'H' \in G/H'$ where $H' < H$. i.e., x^b is a H/H'-twist.

Now, consider the restriction $c = a|_{g'H'}$ and the cube $y^c_{\alpha(i)}$. This is such that $\cap_{p \in hH'} y^c_{\alpha(i)} \neq \emptyset$ and as z^a is a H/K-twist at i, we deduce that $y^c_{\alpha(i)}$ is a H'/K-twist at i. $\square$

5.2.16 Definition. Given an element $x \in \mathcal{A}_n(\jmath)$ and H/K-twist x^a. The *associated transfer map* is $K \dashrightarrow H$.

5.2.17 Corollary. *For a finite function $\mathscr{k} \to \jmath$, and elements $x \in \mathcal{A}_n(\jmath)$, and $y_j \in \mathcal{A}_n(\mathscr{k}_j)$ for $j \in \jmath$. Every associated transfer map to twists of $x \circ y_j$ lies in the same transfer system generated by associated transfers of twists of x and y_j.*

5.2.18 Definition. For a transfer system $\mathcal{F}$, the *$\mathcal{F}$-twisted little cube operad* $\mathcal{C}^{\mathcal{F}}_n$ is the suboperad of $\mathcal{A}_n$ where $x \in \mathcal{C}^{\mathcal{F}}_n$ if it

(1) has strict orbits,

(2) is non-degenerate, and

(3) if x^a is a H/K-twist, then $K \dashrightarrow H$ is in $\mathcal{F}$.

5.2.19 Theorem. *The operads $\mathcal{C}_n^{\mathcal{F}}$ are well-defined and $(\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma} \neq \emptyset$ if and only if* $\mathrm{set}(\Gamma)$ *is an admissible set of $\mathcal{F}$. Moreover, if $n = \infty$, then $(\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma} \simeq *$ if* $\mathrm{set}(\Gamma)$ *is an admissible set of $\mathcal{F}$.*

Proof. All that needs to be justified for $\mathcal{C}_n^{\mathcal{F}}$ being a well-defined operad is that it is closed under composition. This follows from Lemma 5.2.15. If a composite $x \circ (y_j)_j$ has a H/K-twist, then this must also be in $\mathcal{F}$ since we can use the properties of transfer systems to build it from those of x and y_j.

Suppose we have $\mathrm{set}(\Gamma) \cong \coprod H_i/K_i$. Then, if $x \in (\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma}$, from Lemma 5.2.14 we see that x must only have H/K-twists where for some i, $K = K_i$ and $H < H_i$. Hence, it follows that $\mathrm{set}(\Gamma)$ is an admissible set of $\mathcal{F}$ if and only if $(\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma} \neq \emptyset$.

As for the last claim, if $x \in (\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma}$, then x is a product of $G \times j$-many cubes, some of which are equal or otherwise completely disjoint. Whether they are equal or disjoint is fixed for all points in $(\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma}$ and so we see that

$$(\mathcal{C}_n^{\mathcal{F}}(j))^{\Gamma} \cong \mathcal{C}_n(j')$$

for some set j'. Hence when $n = \infty$, we get that this is contractible. □

5.3 Failure of additivity

This section will discuss how additivity fails for our model $\mathcal{C}_n^{\mathcal{F}}$, and, more generally, what we believe is the cause. Before considering twisted cubes, let us first consider the ambient case. Just as in the standard case, we have G-operad maps

$$\iota_1 : \mathcal{A}_n \to \mathcal{A}_{n+m}$$

$$x = (x(g, j))_{g \in G, j \in \jmath} \mapsto x \times \mathrm{id} := (x(g, j) \times \mathrm{id})_{g \in G, j \in \jmath},$$

and

$$\iota_2 : \mathcal{A}_m \to \mathcal{A}_{n+m}$$

$$y = (y(g, j))_{g \in G, j \in \jmath} \mapsto \mathrm{id} \times y := (\mathrm{id} \times y(g, j))_{g \in G, j \in \jmath}.$$

These maps also interchange. In order to see this, let $\jmath$ and $\Bbbk$ be finite sets, with $\alpha : \jmath \times \Bbbk \to \jmath$, and $\beta : \jmath \times \Bbbk \to \Bbbk$ the projection maps. Let $x \in \mathcal{A}_n(\jmath)$ and $y \in \mathcal{A}_n(\Bbbk)$, and write $\alpha^*(x \times \mathrm{id})(g, (j, k)) := (x \times \mathrm{id})(g, \alpha(j, k)) = (x \times \mathrm{id})(g, j)$. Similarly for $\beta^*(\mathrm{id} \times y)$. We then have that

$$\begin{aligned} ((x \times \mathrm{id}) \circ \beta^*(\mathrm{id} \times y))\,(g, (j, k)) &= (x \times \mathrm{id})(g, j) \circ (\mathrm{id} \times y)(g, \beta(j, k)) \\ &= (x \times \mathrm{id})(g, j) \circ (\mathrm{id} \times y)(g, k) \\ &= x(g, j) \times y(g, k), \end{aligned}$$

which by symmetry we get that

$$(\alpha^*(x \times \mathrm{id}) \circ (\mathrm{id} \times y))\,(g, (j, k)) = ((x \times \mathrm{id}) \circ \beta^*(\mathrm{id} \times y))\,(g, (j, k)).$$

This then implies that ι_1 and ι_2 interchange and so we have an induced map

$$\mathcal{A}_n \otimes \mathcal{A}_m \to \mathcal{A}_{n+m}.$$

In fact, the above calculation shows that this is an isomorphism of G-operads since any element $z \in \mathcal{A}_{n+m}$, we can write $z(g, j) = x(g, j) \times y(g, j)$ where $x \in \mathcal{A}_n$ and $y \in \mathcal{A}_m$.

5.3.1 Lemma. *The maps ι_1 and ι_2 induce an isomorphism of operads*

$$\mathcal{A}_n \otimes \mathcal{A}_m \to \mathcal{A}_{n+m}.$$

Let us now consider what happens to elements of $\mathcal{C}_n^{\mathcal{F}}$ and $\mathcal{C}_m^{\mathcal{F}'}$ under the maps ι_1 and ι_2. The following is straightforward.

5.3.2 Lemma. *Let $x \in \mathcal{C}_n^{\mathcal{F}}(j)$ and suppose $x \times \mathrm{id} \in \mathcal{A}_n(j)$. Then the j-th cube $(x \times \mathrm{id})^j$ has orbits of type K if and only if x^j does. Similarly, for the gH-slice $a : gH \to j$, $(x \times \mathrm{id})^a$ is a H/K-twist if and only if x^a is a H/K-twist.*

A consequence of this lemma and Lemma 5.2.15 is that elements in the image of $\mathcal{C}_n^{\mathcal{F}} \otimes \mathcal{C}_m^{\mathcal{F}'}$ must have twists with associated transfer maps in $\mathcal{F} \vee \mathcal{F}'$. Therefore, the induced map

$$\iota_1 \otimes \iota_2 : \mathcal{C}_n^{\mathcal{F}} \otimes \mathcal{C}_m^{\mathcal{F}'} \longrightarrow \mathcal{C}_{n+m}^{\mathcal{F} \vee \mathcal{F}'}$$

is well-defined.

The natural next question is whether this map is injective. We can use the theory we developed in the previous chapter to try and answer this. In particular, we can adapt Theorem 4.4.6 and the special case Lemma 4.5.2 for this situation.

For us to apply these lemmas, we need to understand how twists behave under projections. Observe that projections are well-behaved with respect to operadic composition. That is, given $x \in \mathcal{C}_n^{\mathcal{F}}(j)$ and $y \in \mathcal{C}_m^{\mathcal{F}'}(k)$ then using the projection $\mathrm{pr}_1 : \mathcal{A}_{n+m} \to \mathcal{A}_n$ we have that

$$\begin{aligned} \mathrm{pr}_1((x \times \mathrm{id}) \circ (\mathrm{id} \times y_j))(g,k) &= (\mathrm{pr}_1(x \times \mathrm{id}) \circ \mathrm{pr}_1(\mathrm{id} \times y_j))(g,k) \\ &(x \circ \mathrm{id})(g,k). \end{aligned}$$

So in the projection, we have just repeated columns depending on the arities of the

y_j. *This operation introduces new twisting.* In particular, suppose x^j has orbit type K where x is the element above. If we have enough copies of x^j in the composite, then for any subgroup $K' < K$, we can build K/K'-twists. Unfortunately, this is a considerable obstruction that we need to overcome – and likely to be impossible.

The primary cause seems to be that our constructions $\mathcal{C}_n^{\mathcal{F}}$ are still too big. We suspect this stems from the fact that there is one important feature of $\mathbb{E}_V$-operads that we have failed to capture. Each disk in $\mathcal{D}(V)$ has a *unique orbit*. This isn't true in $\mathcal{C}_n^{\mathcal{F}}$. Given any cube $x(e, -)$, almost every option for the orbits $x(g, -)$ is allowed. While in $\mathcal{D}(V)$, the representation forces what these will be. Perhaps even more crucially, if a disk contains another disk, then this containment is true for every disk in its orbit. We could avoid the above obstacle if we had such a property for $\mathcal{C}_n^{\mathcal{F}}$.

This also seems to be related to another problem of $\mathcal{C}_n^{\mathcal{F}}$ that we have glossed over until now. The operads $\mathcal{C}_n^{\mathcal{F}}$ don't appear to encode the correct algebras. A model for a "$\mathbb{N}_1$-operad" should be encoding a sort of "twisted homotopy associative monoid". However, the fixed points seem "too free" to correspond to the correct thing – precisely because the orbits aren't unique. Because of this, we conjecture the following.

5.3.3 Conjecture. *If we can define "canonical orbits" for little cubes $\mathcal{C}_n$ in the ambient operad $\mathcal{A}_n$, then we can define a model for $\mathbb{N}$-operads in which additivity will hold.*

Appendix A

Elementary topology results

A.1 Proper Maps

Here we will give proofs of the elementary facts stated at the beginning of section 4.2. Our presentation follows the set of notes by Schultz [Sch].

A.1.1 Lemma. *Given a proper map $f : X \to Y$, then the function $f|_{f^{-1}(U)}$ restricted to any preimage $U \subseteq Y$ is also proper.*

Proof. If $K \subseteq U$ is compact, then it is compact in Y, and as f is proper, $f^{-1}(K)$ is compact. Since $f|_U{}^{-1}(K) = f^{-1}(K)$ we are done. □

A.1.2 Lemma. *Let $f : X \to Y$ be a map. If X has a finite closed cover $\{X_i\}$ such that $f|_{X_i} : X_i \to Y$ are all proper, then $f : X \to Y$ is proper. In the other direction, if $f : X \to Y$ is proper and $F \subseteq X$ is closed, then the restriction $f|_F$ is proper.*

Proof. Given a compact subset $K \subseteq Y$, then for each i, $f|_{X_i}{}^{-1}(K)$ is compact. Since we have that

$$f^{-1}(K) = \bigcup_i (f\big|_{X_i})^{-1}(K)$$

and this is a finite union of compact subsets, we get that $f^{-1}(K)$ is compact. This proves the first statement. For the second statement, if f is proper, then $f^{-1}(K)$ is compact and since F is closed

$$f\big|_F^{-1}(K) = F \cap f^{-1}(K)$$

is compact and we are done. $\square$

A.1.3 Lemma. *Given a collection of proper maps $f_i : X_i \to Y_i$, $i \in I$. The product*

$$\prod_i f_i : \prod_i X_i \to \prod_i Y_i$$

is proper.

Proof. This follows from the fact that products of compact sets are compact and closed subsets of compact subsets are compact. $\square$

A.1.4 Lemma. *Compositions of proper maps are proper. Moreover, if $f \circ g$ is proper then g is proper, and if g is surjective, then f is proper.*

Proof. That the compositions of proper maps are proper follow from the definitions. If the composite $f \circ g$ is proper, then g is proper because the image of compact sets are compact. Suppose g is surjective and K in the codomain of f is compact. We have that $\widetilde{K} = (f \circ g)^{-1}(K)$ is compact, and and so $g(\widetilde{K})$ is compact. Since g is surjective, this means that $f^{-1}(K) = g(\widetilde{K})$ and so we are done. $\square$

A.2 Homotopy

A.2.1 Lemma. *Given a G-space X and a closed G-subspace $A \subseteq X$. If we have a G-map $H : X \times [0,1) \to X$ such that for all $x \in X$,*

(1) there exist $t \in [0,1)$ such that $H(x,t) \in A$,

(2) if $H(x,t) \in A$, then $H(x,s) \in A$ for all $s \geq t$,

(3) the set $(\{x\} \times [0,1)) \cap H^{-1}(\partial A)$ is a singleton.

Then the mapping $\phi : x \mapsto t_x$ where $t_x = \min\{t \in [0,1) \mid H(x,t) \in A\}$ is continuous and G-invariant. As a consequence, A is a strong deformation retract of X given by the deformation

$$\widetilde{H}(x,t) := H(x, t_x t).$$

Proof. Since A is closed, the map

$$\phi : X \to [0,1), \quad x \mapsto t_x$$

is well-defined, and as H is equivariant, it is G-invariant. So all we need to prove is that this map is continuous. We do this via the net characterization of continuity. Let $(x_i)_{i \in J}$ be a net in X and $x \in X$ such that $x_i \to x$ in X. We will consider ϕ to have codomain in $I = [0,1]$ which is compact. Hence, every net in I has a convergent subnet. This means to prove $\phi(x_i) \to \phi(x)$ in I it is sufficient to show that every convergent subnet of $\phi(x_i)$ must converge to $\phi(x)$. This follows since if we didn't have convergence, there exists a subnet $\phi(x_s)_{s \in S}$ which is eventually separated from $\phi(x)$, but this isn't possible if every subnet contains a convergent subnet that converges to $\phi(x)$.

Let $\left(\phi(x_{f(s)})\right)_{s \in S}$ be a subnet such that $\phi(x_{f(s)}) \to a \in I$. We want to show that $\phi(x) = a$. If the subset

$$C = \{s \in S \mid a = \phi(x_{f(s)})\}$$

is cofinal in S, then we must have that $a = \phi(x)$ since I is Hausdorff and limits are unique. Therefore, suppose C isn't cofinal. As we have that

$$H(x_{f(s)}, \phi(x_{f(s)})) \to H(x, a)$$

and each $H(x_{f(s)}, \phi(x_{f(s)})) \in A$, which is closed, we have that $0 \leq \phi(x) \leq a$ by the definition of ϕ. If $a = 0$, then $\phi(x) = a$ in this case. Therefore, suppose we have that

$a > 0$. Then for $\epsilon > 0$ consider the net $(y_s)_{s \in S}$ defined by

$$y_s := \min\Big(0, \phi(x_{f(s)}) - \epsilon|a - \phi(x_{f(s)})|\Big)$$

Since C isn't cofinal, the net (y_s) is such that eventually $y_s < \phi(x_{f(s)})$ and so $H(x_{f(s)}, y_s) \in X \backslash A$. Since $y_s \to a$ it follows that $(x, a) \in \{x\} \times [0, 1) \cap H^{-1}(\partial A)$. However, we also have that $(x, \phi(x)) \in \{x\} \times [0, 1) \cap H^{-1}(\partial A)$ and so as this set is a singleton we must have $a = \phi(x)$. $\square$

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