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Author
Grabiak, M.

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M. Grabiak

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Weakly Interacting Nielsen-Olesen Vortices.*

M. Grabiak
Lawrence Berkeley Laboratory
Nuclear Science Division
Berkeley CA 94720

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Abstract

In this article the interaction of parallel and antiparallel Nielsen-Olesen vortices are studied numerically on a lattice. Parallel vortices interact only very weakly for a large range of parameters whereas antiparallel vortices can annihilate each other from a nonzero distance by a kind of string flip mechanism.

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In a previous article [1] we have studied the Nielsen-Olesen model [2] as a phenomenological model for confinement which might give insight into some qualitative features of the strong interaction in heavy ion collisions. For example, the fact that independent string fragmentation models [3, 4] work so well might be attributed to the fact that flux tubes are found to be small and weakly interacting. The Nielsen-Olesen model is the relativistic analogue of a superconductor. As a model for QCD it is a dual model in the sense that the roles of the electric and the magnetic fields are exchanged. Instead of electric charges it confines magnetic monopoles. Two distant monopoles lead to the formation of a flux tube. As discussed in reference [1] the energy per unit length of such a flux tube can be identified with the QCD string tension, whereas the monopole charge can be related to the strong coupling constant α_s. For an intermediate case between type I and type II superconductor where the Higgs mass m_H is equal to the gauge boson mass m_A one finds that parallel flux tubes do not interact at all [5, 6, 1]. Furthermore, for this case one also finds that the mean square radius $<r^2>$ is proportional to the total flux [1]. This behaviour is very similar to flux tubes in the MIT-bag model, where the area is exactly proportional to the magnetic flux, with the magnetic field being constant and the energy being proportional to the total flux, independent of the shape of the flux tubes. In this article a numerical study on a two-dimensional lattice is presented for the interaction of flux tubes for $m_H \neq m_A$ as well as the interaction of antiparallel flux tubes. One finds that the interaction of parallel flux tubes changes only slowly for $m_H \neq m_A$, being only a few percent of the total energy for $m_H \approx m_A$. As I shall show below this effect can be related to the behaviour of flux tubes for $m_H = m_A$. For antiparallel flux tubes one finds that they can annihilate each other with some kind of flip mechanism.

In the Nielsen-Olesen [2] model the electromagnetic field is coupled to a relativistic charged scalar field ϕ, which is the analogue of the Cooper pair condensate field in a superconductor. One introduces a term of the form $(|\phi|^2 - \phi_V^2)^2$ which leads to spontaneous symmetry breaking, so that the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \left|(i\partial_\mu - qA_\mu)\phi\right|^2 - \frac{h}{4}\left(\phi_V^2 - |\phi|^2\right)^2 \quad . \tag{1}$$

In the vacuum state with $|\phi| = \phi_V =$ const. the term $-q^2|\phi|^2A^2$ becomes a mass term for the gauge bosons with $m_A = \sqrt{2}q\phi_V$, which is the inverse of the London penetration length. The Higgs mass is given by $m_H = (-1/2\, \partial^2\mathcal{L}/\partial|\phi|^2)^{1/2}\Big|_{\phi_V} = h^{1/2}\phi_V$. Defining the variables $\vec{\rho}$, $\vec{a}$, η, ϑ, and λ by

$$\begin{aligned} \vec{\rho} &= \sqrt{2}q\phi_V\vec{r} = m_A\vec{r} \quad , \quad & \vec{A} &= \sqrt{2}\phi_V\vec{a} \quad , \\ \phi &= \phi_V\eta\exp(i\vartheta) \quad , \quad & \lambda &= h/(2q^2) = m_H^2/m_A^2 \quad , \end{aligned} \tag{2}$$

the energy for a static configuration with no electric field reads

$$E = 2\phi_V^2 \int d^3r \left\{ \frac{1}{2}(\vec{\nabla}_\rho \times \vec{a})^2 + \frac{1}{2}(\vec{\nabla}_\rho\eta)^2 + \frac{1}{2}|\vec{\nabla}_\rho\vartheta - \vec{a}|^2\eta^2 + \frac{\lambda}{8}(1-\eta^2)^2 \right\} \quad . \tag{3}$$

$\lambda \ll 1$ means that the Higgs mass is much smaller than the gauge boson mass, i. e. the penetration length is much smaller than the coherence length. This corresponds to a type I superconductor. Vice versa, $\lambda \gg 1$ corresponds to a type II superconductor.

In this model magnetic charges are confined, two distant magnetic monopoles lead to the formation of a magnetic flux tube. In the limit that the two monopoles are far apart one can assume translational symmetry along the flux tube. The two-dimensional configuration corresponding to the cross section through such a flux tube is called a vortex. A vortex is characterized by the fact that the phase of the Higgs field changes by a multiple of 2π going around the vortex[1]:

$$\oint_{\text{vortex}} d\vec{\rho}\cdot\vec{\nabla}\vartheta = 2\pi n_{\text{vortex}} \iff \vec{\nabla}\times(\vec{\nabla}\times\vartheta) = 2\pi n_{\text{vortex}}\delta^2(\vec{\rho}-\vec{\rho}_{\text{vortex}}) \quad . \tag{4}$$

For the energy to remain finite $\vec{a}$ has to approach $\vec{\nabla}\vartheta$ at large distances. Thus eq. (4) implies that the magnetic flux is quantized. There is also a problem at $\vec{\rho}_{\text{vortex}}$ where $\vec{\nabla}\vartheta$ has a singularity. However, in this case $\vec{a}$ cannot cancel $\vec{\nabla}\vartheta$ like for large distances because then the magnetic field would have a singularity. The only way to avoid an infinite contribution to the energy is to have a zero of the Higgs field at the position of the vortex:

$$\eta(\vec{\rho}_{\text{vortex}}) = 0 \quad . \tag{5}$$

[1]In the following we drop the index ρ from $\vec{\nabla}$ for convenience

The Lagrangian (1) has three free parameters ϕ_V, q, and h. Two of these parameters can be adjusted by fitting the charmonium potential $V(R) \approx Q^2/(4\pi R)+tR$ for the limiting cases of very short and very large distances [7]. ϕ_V is chosen so that the string tension of a flux tube, i. e. the energy per unit length of a vortex solution, is equal to t, and q is related to Q by Dirac's monopol quantization condition $Q = 2\pi N/q$, which implies $\alpha_s = 3\pi/(4q^2)$ for the choice $N = 1$. This leaves λ as a free parameter. As discussed in reference [1] it might be related to the QCD vacuum energy as obtained by QCD sumrules [8, 9]. Higher λ corresponds to higher vacuum energy leading to thinner strings.

Remarkably, for the case $\lambda = 1$, an intermediate case between type I and type II superconductors, one finds that parallel vortices have no interaction at all. This is so because the energy per unit length of a configuration with translational symmetry along the z-axes can be written as

$$\int d^2\rho\, \varepsilon/(2\phi_V^2) =$$
$$\pm\pi n + \int d^2 r \left\{ \left[\vec{\nabla}\eta \pm \vec{e}_z \times (\vec{\nabla}\vartheta - \vec{a})\eta\right]^2 + \left[\vec{e}_z \cdot (\vec{\nabla} \times \vec{a}) \mp \frac{1}{2}(1-\eta^2)\right]^2 \right\} \quad . \tag{6}$$

n is the sum of all n_{vortex}, corresponding to the total magnetic flux. Choosing the sign so that the first term on the rhs is positive, $\pi|n|$, one can show that for any multi-vortex configuration there exists a solution for which the remaining terms vanish [5, 6], independent of the positions of the vortices – defined by eqs. (4) and (5) – i. e. it is possible to fulfil the conditions[2]

$$\begin{aligned} \vec{\nabla}\eta &= \mp\vec{e}_z \times (\vec{\nabla}\vartheta - \vec{a})\eta \\ \vec{e}_z \cdot (\vec{\nabla} \times \vec{a}) &= \pm\frac{1}{2}(1-\eta^2) \quad . \end{aligned} \tag{7}$$

This means that the energy depends only on the total magnetic flux, but not on the positions of the vortices, i. e. the vortices do not interact at all.

I studied the interaction of Nielsen-Olesen vortices in a two dimensional variational lattice calculation, the parameters to be varied being the Higgs

[2]These conditions are reminiscent of self-duality conditions in non-Abelian gauge theories.

fields on the lattice sites and the gauge fields along the links. I worked in the unitary gauge, i. e. I set the phase ϑ of the Higgs field to zero, making the replacement $\vec{a}-\vec{\nabla}\vartheta \rightarrow \vec{a}$. In other words, the phase change of the Higgs field is absorbed into the gauge potential[3]. Given the Higgs field on the lattice sites and the gauge potential on the links the fields were interpolated within each plaquettes as

$$\eta = \bar{\eta} + \eta_x x + \eta_y y + \eta_{xy} xy \qquad ,$$
$$a_x = \bar{a}_x + a_{xy} y \qquad , \qquad a_y = \bar{a}_y + a_{yx} x \qquad , \tag{8}$$

where the eight parameters on the right-hand sides are uniquely determined by the fields on the plaquette boundary. The above interpolation also ensures that neighbouring plaquettes can be smoothly joined together. For example, the Higgs field along a link changes linearly according to the fields on the adjacent sites, being the same for both plaquettes on each side of the link. Note that the sole purpose for including the term $\eta_{xy}xy$ is to be able to interpolate the Higgs field within the plaquette. It has nothing to do with a Taylor expansion, where it would only make sense to include it together with the other second order terms going like x^2 and y^2. The above parametrization also ensures that a_x is smooth along the y-direction and a_y along x, so that the magnetic field remains finite.

Given the above interpolation the energy was calculated exactly, summing up the contributions from each plaquette for various distances between the vortices, where the position of a vortex is defined as a zero of the Higgs field with the constraint (4), which reads $\oint_{\text{vortex}} d\vec{\rho} \cdot \vec{a} = 2\pi n$ in the unitary gauge. The energy was minimized with respect to the values of the Higgs fields on the plaquettes and the vector potential on the links, with the boundary condition $\vec{a} = 0$ and $\eta = 1$ along the boundary of the lattice. The calculation was actually performed only in one quadrand, the other three can be obtained by symmetry arguments. In terms of the penetration length the lattice sizes used ranged from 8×8 for two vortices on top of each other to to 8×16 for a mutual distance of 16. For each configuration I performed three calculations

[3]There is one technical complication in this gauge. (4) implies that $\vec{\nabla} \times \vec{a}$ has a delta-function singularity at the position of the vortex, analogous to the Dirac string of a magnetic monopole. This singularity must not be taken into account in the magnetic energy density $(\vec{\nabla} \times \vec{a})^2/2$.

with 8×8, 12×12, and 18×18 lattice points per unit square and extrapolated these results to lattice spacing zero. Comparing that result with the solution of the radially symmetric differential equations for distance zero and distance infinity showed that the lattice calculation reproduced the right energy with a precision better than 0.1 %.

First I calculated the vortex-vortex potential for two parallel for $\lambda = 1$. In agreement with the non-interaction theorem for $\lambda = 1$ the resulting values for the energy were $E/(2\phi_V) = 2\pi$ to a precision better than 0.1 %, independent of the distance between the vortices, as expected from (6) and (7). Fig. 1 shows the different contributions to the total energy. Due to (7) the magnetic energy is equal to the potential energy of the Higgs field and $(\vec{\nabla}\eta)^2/2$ is equal to the gauge invariant combination $(\vec{\nabla}\vartheta - \vec{a})^2\eta^2/2$. Fig. 1 shows that the contributions from the different terms change only by a few percent. For two separate vortices the magnetic energy and the potential energy both contribute about 23 % of the total energy and the other two terms 27 % each, whereas for two vortices on top of each other, i. e. one vortex with winding number 2, it is the other way round. This result can be understood in the following way. In a previous publication we showed that the mean square radius $<r^2>$ of the energy distribution is proportional to the winding number [1][4] :

$$<r^2> = n/(q^2\phi_V^2) \quad . \tag{9}$$

Qualitatively this means that the total area of the flux tubes does not change very much. This is very similar to the MIT-bag model, where the total area remains exactly the same, which implies that the electric flux and the volume energy remain constant. While the corresponding quantities in the Nielsen-Olesen model, the magnetic flux and the potential energy of the Higgs field, are no longer exactly constant they change only by a few percent.

Figure 2 shows how the interaction changes with λ. As expected the vortices attract each other for $\lambda < 1$ corresponding to a type I superconductor and repel each other for type II superconductors with $\lambda > 1$. However, as one can see from fig. 2 the interaction energy is only a few percent of the total energy, i. e. the vortices interact only weakly. The reason for that is the following. According to (3) the derivative of the energy with respect to

[4]Beware the misprint in formula (15). Its right-hand side should be proportional to n, not to n^2

λ is

$$\frac{1}{2\phi_V^2}\frac{dE}{d\lambda} = \frac{1}{8}\int d^2\rho\,(1-\eta^2)^2 \quad . \tag{10}$$

The wave function also changes with λ, but to first order this does not change E for an energy minimum. The quantity (10) itself is rather large, being about 1/4 of the total energy as can be seen from fig. 1. However, the change of the interaction energy with λ is given by the difference of this quantity for $d = 0$ and $d = \infty$, and as fig. 1 shows the potential energy of the Higgs field changes only by a few percent. Consequently, the interaction energy changes only slowly with λ. Fig. 2 also shows that the interaction range is smaller for greater λ, in accordance with the fact that greater values of λ correspond to a higher vacuum energy leading to thinner strings.

While the above considerations show that *parallel* vortices interact weakly, there can still be a considerable interaction for an arbitrary orientation of two vortices. This is quite obvious for two antiparallel flux tubes. For two vortices far apart the energy is twice as large as the energy of a single vortex, whereas for distance zero they will annihilate each other so that the total energy becomes zero, since for total flux zero the configuration with minimal energy is just the vacuum configuration. Actually, the following discussion shows that two antiparallel vortices can even annihilate each other if they are a finite distance apart. The position of a vortex was defined as a zero of the Higgs field with the phase changing by a multiple of 2π when going around this point. This phase change is a topological invariant, i. e. it does not change when the path around the vortex is deformed. For two antiparallel vortices one gets two different winding numbers, corresponding to the phase change around each of the vortices. Since both winding numbers are topological invariants it seems that the vortices cannot annihilate each other. However, this is only true if one assumes that the Higgs field is zero only at the positions of the vortices. On the other hand, the constraint that the Higgs field is zero at two given points with an opposite phase change around each vortex can also be fulfilled in the following way: Along the connecting line between these points there is a thin, but finite region where the Higgs field is zero, and all the phase change occurs when passing through that region (see fig. 3), with the magnetic field being equal to zero. On the other hand, the phase has no meaning at all if the Higgs field vanishes. There is no contribution of $\vec{\nabla}\vartheta$ to the kinetic energy, and just setting ϑ to zero (or anything else) does not change the configuration. On the other hand, if there is no phase change there is no longer a reason for the Higgs field to be zero between the vortices. The Higgs field can be smoothly deformed so that it takes on its vacuum value everywhere. Thus the configuration depicted in fig. 3 serves as an intermediate configuration which makes it possible to deform a state with two antiparallel vortices into the vacuum state. The trick is that the topology of the configuration changes if the Higgs field becomes zero along one line, being equivalent to a plane with one hole instead of a plane with two holes as for two isolated vortices. In principle this mechanism works for any given distance between the two vortices. However, for very large separations the energy of the intermediate configuration grows linearly with their distance, so that they can annihilate each other only by tunneling through this configuration. In the limit that the connecting region with zero Higgs field becomes infinitely thin the energy of this configuration goes like

$$E/(2\phi_V^2) \longrightarrow \frac{2}{3}\sqrt{\lambda}\, d + 1.13 \tag{11}$$

for large d, where d is the distance between the vortices in terms of the penetration length is one. The energy per unit length can easily be determined analytically, whereas the constant term was found numerically with a lattice calculation. The intermediate configuration becomes favourable for

$$\frac{2}{3}\sqrt{\lambda}\, d + 1.13 \quad \approx \quad 2\pi \quad , \tag{12}$$

which is about $5{<}r^2{>}^{1/2}$ for $\lambda = 1$. At that point tunnelling is no longer necessary.

In the above mechanism two antiparallel strings annihilate each other by forming an infinitely long sheet of zero Higgs field between them. This is certainly a rather academical case since strings are never infinitely long and are never exactly parallel or antiparallel. Nevertheless, a similar effect could also take place between to finite pieces of strings, as indicated in fig. 4. There could be a finite region between the strings with zero Higgs field, as indicated by the hatched region. The interesting point is that this region may break up in a different way, corresponding to a string flip, as indicated in the third part of the figure.

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Figure Captions

Figure 1: The different contributions to the total energy of two vortices at distance d for $\lambda = 1$: $T_\eta = (\vec{\nabla}\eta)^2/2$, $V_{\phi,A} = (\vec{\nabla}\vartheta - \vec{a})^2\eta^2$, $E_{\text{magn}} = \vec{\nabla} \times \vec{a}$, and $U_\eta = (1-\eta^2)^2/8$

Figure 2: The vortex-vortex potential for $\lambda = 0.5$ and $\lambda = 2$.

Figure 3: An intermediate configuration during the annihilation of two antiparallel vortices, their positions being indicated by the crosses. The Higgs field is zero in the hatched region, and the arrows indicate that a phase change of 2π occurs when crossing the region between the two vortices.

Figure 4: A local annihilation of two string segments corresponds to a string flip.

Figure 1

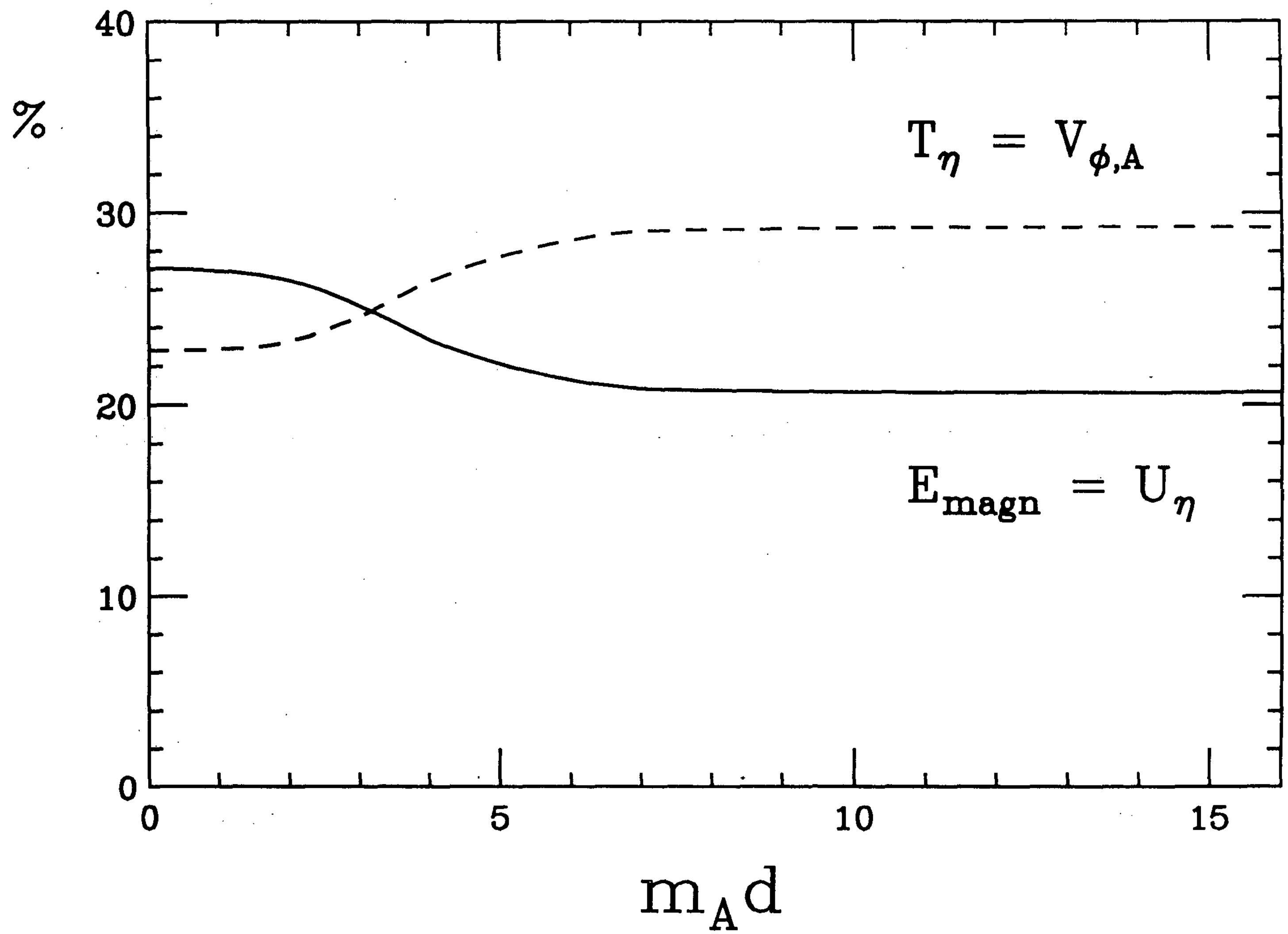


Figure 2

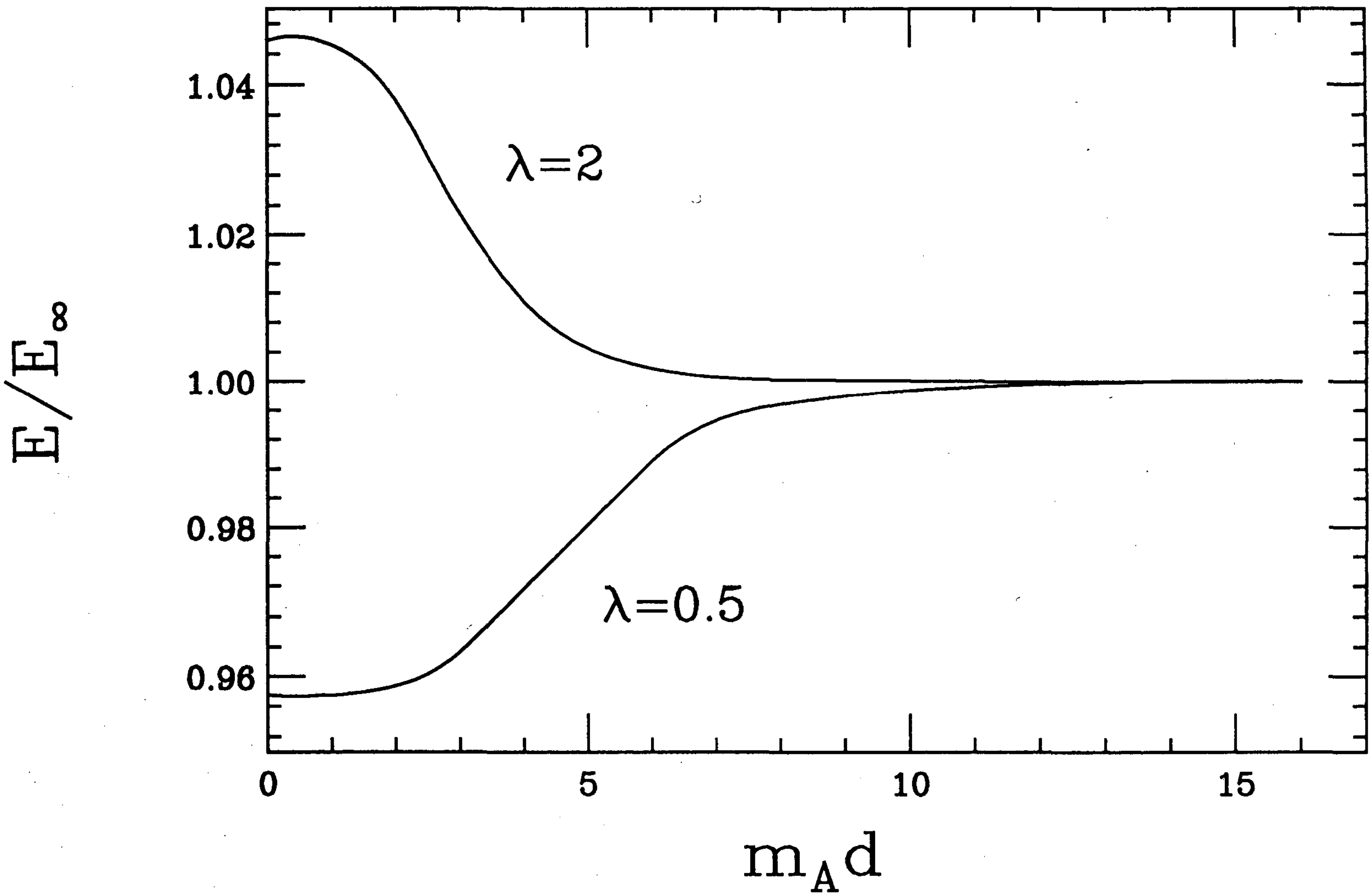
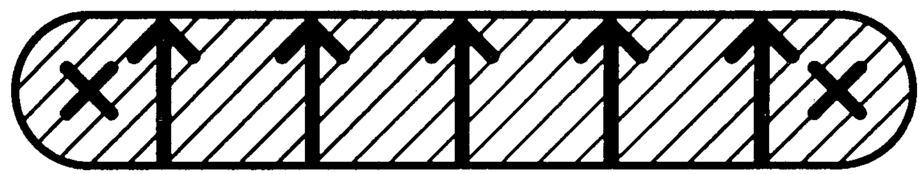


Figure 3



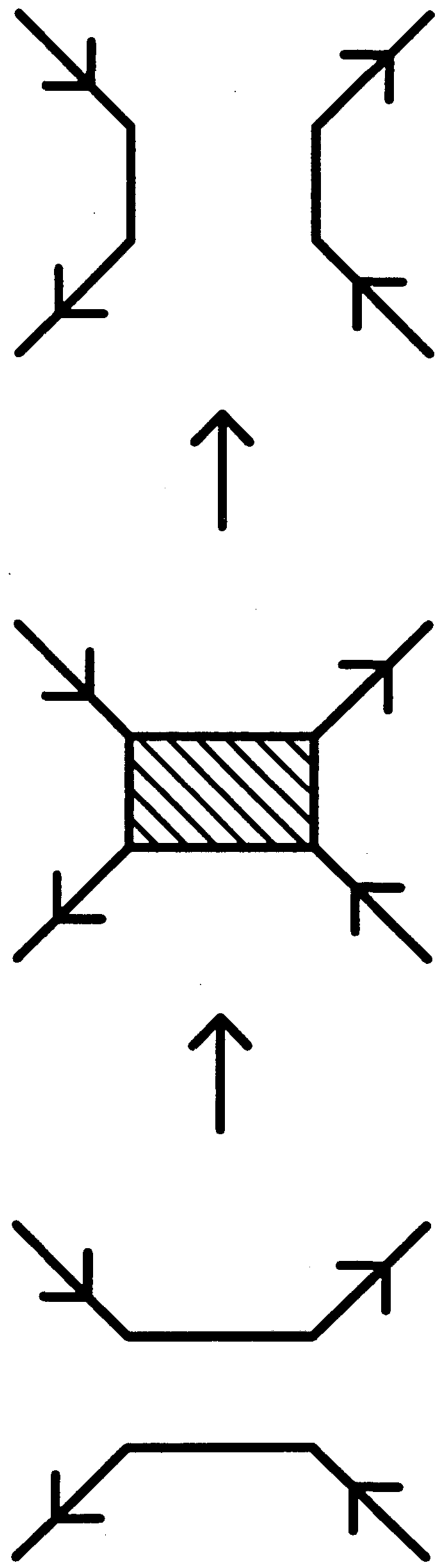


Figure 4