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Diagrammatic Formulae for Conjugate Systems and Transport Maps in the q-Gaussians, and Free Moment Measures

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by

Nicholas James Worthington Boschert

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ABSTRACT OF THE DISSERTATION

Diagrammatic Formulae for Conjugate Systems and Transport Maps in the q-Gaussians, and Free Moment Measures

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Doctor of Philosophy in Mathematics
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We investigate the problem of finding explicitly a transport map from the q-Gaussians to the free group factors. To whit, we introduce a new graphical structure called semi-knots–diagrams which contain half the information of knots. With these we find semi-knot formulae for the conjugate system, and show that there is a formal semi-knot formula for the transport map. We further introduce free moment measures, which extend the classical notion, and show that a broad class of free Gibbs laws are free moment measures, in joint work with June Bahr.

This dissertation of Nicholas James Worthington Boschert is approved.

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This manuscript is dedicated to my friends, family, and partner, who have helped make my time in graduate school fulfilling, fun and just risky enough.

Contents

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Chapter 1

Introduction

1.1 von Neumann Algebras and Non-Commutative Probability

Initially studied by John von Neumann and Francis Murray, a von Neumann algebra is an algebra of operators on a Hilbert space which is closed in the strong operator topology, ($T_\alpha \to T$ if $||T_\alpha \xi - T\xi|| \to 0$ for all ξ). There are two remarkable properties of such algebras, first that they allow one to take $f(x)$ where f is any bounded Borel measurable function $\mathbb{C} \to \mathbb{C}$ and x is any normal operator (which makes them the natural setting for the spectral theorem). Second, when the algebra is abelian (i.e. multiplication is commutative), and equipped with an appropriate functional, it exactly describes a probability space, where functional is the expected value functional.

As such, von Neumann algebras can be considered to be the non-commutative analog of probability/measure theory. In particular, a non-commutative probability space is defined to be a pair (M, τ) where

1. M is a von Neumann algebra

2. τ is a (faithful, normal) trace on M, i.e. $\tau(xy) = \tau(yx)$, $\tau(x^*x) \geq 0$ with equality only

if $x = 0$, and τ is weak operator topology continuous.

We further call an element of such a space a non-commutative random variable.

The definition restricts itself to traces because there is an intimate relationship between the trace on a von Neumann algebra and the algebra itself. In particular, if assumed to be faithful and normal, a trace on the $*$-algebra generated by some elements $\{x_i\}$ determines the von Neumann algebra they generate. Moreover, if M is a factor, meaning it has no center except $\mathbb{C}$, then it will have a unique trace (depending on the details, this may be only defined on projections and may be allowed to take the value ∞). Non-commutative probability spaces must be built out of type I_n and II_1 factors, which are (so called finite) factors with minimal projections (i.e. $M_n(\mathbb{C})$), and without minimal projections, respectively. In the analogy to probability spaces these should be thought of as the atomic portion of the probability space and the diffuse portion of the probability space respectively.

Examples of II_1 factors require a bit more effort to describe, with the two most common classes of examples being the hyperfinite II_1 factor, R, and the group von Neumann algebra construction $L(\Gamma)$. Although R is of fundamental importance in the theory of von Neumann algebras, it is not vital to this work, so we will not describe it here. On the other hand, $L(\Gamma)$, also due to Murray and von Neumann, will play an important role in this work, so we provide a rapid summary:

Given a discrete, countable group Γ, the von Neumann algebra $L(\Gamma)$ is the closure of the group algebra $\mathbb{C}[\Gamma]$ where we consider the group algebra as operators on $l^2(\Gamma)$. In particular, we denote the group algebra elements by u_g and the natural basis of $l^2(\Gamma)$ by δ_h and have that

$$u_g \delta_h = \delta_{gh}$$

So far, this only constructs the algebra $L(\Gamma)$, but it is worth noting here the result that $L(\Gamma)$ is a factor exactly when Γ has the infinite conjugacy class property, i.e. all conjugacy classes have infinitely many elements. Of particular importance to us are the groups $\mathbb{F}_n$, the free groups on n generators. These can be easily seen to have the i.c.c. property, and so $L(\mathbb{F}_n)$ are all (type II_1) factors. Curiously, it is not yet known whether $L(\mathbb{F}_n) \simeq L(\mathbb{F}_k)$ when $n \neq k$. Indeed, this question, called the "free group factor isomorphism problem" is often considered the biggest open question in von Neumann algebras.

$L(\Gamma)$ is automatically equipped with a trace, namely

$$\tau(x) = \langle \delta_e, x\delta_e \rangle$$

(note that any group element could be chosen in place of the identity with the same result).

1.2 Free Probability

In trying to solve the free group factor isomorphism problem, Dan Voiculescu created the field of free probability. This field can be motivated by several important observations

1. Non-commutative probability spaces are a natural language for discussing many properties of random matrices. In particular, empirical measures (and most measurements that are invariant under unitary conjugation) can be obtained by exclusively applying the trace to the algebra generated by a random matrix.

2. When the random matrices are large, independent, and have probability distributions that are invariant under unitary conjugation, the traces of the algebra they generate approximately satisfy a relation called free independence, described shortly.

3. This free independence relationship allows for a novel product of von Neumann algebras called the free product $M_1 * M_2$. This satisfies the important property that $L(\Gamma * \beta) = L(\Gamma) * L(\beta)$ when Γ and β are countable discrete groups.

This notion of free independence is given by the relation

$$\tau(a_1 b_1 a_2 ... b_n) = 0$$

whenever $a_i \in M_1$, $b_j \in M_2$, and $\tau(a_i) = \tau(b_j) = 0$. This should be compared to the relation of classical independence $\tau(ab) = \tau(a)\tau(b)$. Note that free independence naturally implies that $[a, b] \neq 0$ as long as neither a nor b is a scalar. Indeed, since the free product on von Neumann algebras respects that of groups, we can see that it encodes a lack of any algebraic relations between random variables from different algebras.

These observations also open up random matrix theory as a tool for studying the free group factor isomorphism problem, since $L(\mathbb{Z}) \simeq L^\infty[0,1]$, $\mathbb{F}_n = \mathbb{Z}*...*\mathbb{Z}$, and since e.g. large Gaussian unitary ensembles approximately generate $L^\infty(\mathbb{Z})$. In particular, this produces a description of a family of non-commutative random variables s_i which generate $L(\mathbb{F}_n)$, which we describe in the next subsection.

1.3 Free Semicircular variables and the q-Gaussians

We begin by making precise the statement that a large GUE approximately generates $L^\infty[0,1]$. Let x_N be the random variable that is the N by N GUE with unit variance. This has a trace, namely $\mathbb{E}(\frac{1}{N}tr(\cdot))$. When N is large,

$$\mathbb{E}(\tfrac{1}{N}tr(x_N^m)) \to \frac{1}{2\pi}\int_{-2}^{2} s^m\sqrt{4-s^2}ds$$

where the measure on the right is known as the Wigner semicircle law. This can be proven for example using Wick's theorem and a technique called the genus expansion [BIPZ78], the upshot of which is the modification of wick's theorem: when s is a random variable distributed according to the unit variance semi-circle law,

$$\tau(s^n) = \sum_{\pi \text{ a non-crossing pairing of } \{1,...,n\}} 1$$

which is distinct from Wick's theorem only in that the pairings must be non-crossing (i.e. let $\pi(i)$ be the index to which i is paired. Then $i < j < \pi(i) < \pi(j)$ is forbidden). Indeed, the multivariable Wick's theorem (for independent centered Gaussians) generalizes to a formula for the law for *freely* independent semi-circular variables.

$$\tau(s_{i_1}...s_{i_n}) = \sum_{\pi \text{ n.c. pairing}} \prod_j \delta_{i_j i_{\pi(j)}}$$

This formula can be further generalized to the q-Gaussians, $q \in [-1, 1]$, which interpolate between ordinary (Bosonic) Gaussians on $\mathbb{R}^n$ at $q = 1$, free semi-circular variables at $q = 0$ and boolean (Fermionic) variables at $q = -1$ [FB70]. The revised formula is

$$\tau(x_{i_1}...x_{i_n}) = \sum_{\pi \text{ pairing}} q^{\#\text{crossings}(\pi)} \prod_j \delta_{i_j i_{\pi_j}}$$

It is worth noting that the q-Gaussians can be built using a Fock space construction. In this construction, we define a basis for our Hilbert space beginning with Ω, which we call the vacuum, and then by considering inductively

$$e_w = l_{w_1}...l_{w_{|w|}}\Omega$$

where w is taken to be a word in $\{1, ..., n\}$, and l_i is intended to be the i-th creation operator, and which is desired to satisfy the commutation relation

$$l_i^* l_j = q l_j l_i^* + \delta_{ij}$$

which we can see generalizes the canonical commutation relations ($q = 1$) and canonical anticommutation relations ($q = -1$) often used to define Bosons and Fermions. This relation is enough, with the conditions $l^*\Omega = 0$ and $\langle \Omega, e_w \rangle = 0$ whenever $|w| \neq 0$, to define the q-Gaussian Fock space with the inner product

$$\langle e_w, e_v \rangle = \langle \Omega, l_{w_{|w|}}^* ... l_{w_1}^* l_{v_1} ... l_{v_{|v|}} \Omega \rangle = \sum_{\sigma \in S_{|w|}} \delta_{\sigma(w) v} q^{\#\text{inversions}(\sigma)}$$

whenever $|w| = |v|$, and 0 otherwise. The q-Gaussians are then the non-commutative random variables $X_i = l_i + l_i^*$.

For all interior q the q-Gaussians are a genuine non-commutative probability space (while for $q = 1$ they are commutative, and for $q = -1$ they are anti-commutative). We will refer to this algebra as $q\mathcal{G}_n$, where n is the number of generators. It has been shown that

1. [GS12] When $|q|$ is small and the number of indices is finite, the $q\mathcal{G}_n$ are isomorphic to $L(\mathbb{F}_n)$.

2. [Cas23][BCKW23]When the number of generators is infinite then $q\mathcal{G}_\infty$ is non-isomorphic to $L(\mathbb{F}_\infty)$ for all $q \neq 0$.

The q-Gaussians are not yet fully understood, but they hint at several extensions of important probabilistic and physical notions. Firstly, we direct attention to the free Brownian motion of Biane and Speicher [BS98a], which gives a notion of stochastic dynamics in the free setting, replete with an Ito like lemma. We would like to be able to consider a notion of q-Brownian motion, although existing attempts at a definition have thus far failed to satisfy. Secondly, it is known [Dab14] that the q-Gaussians are free Gibbs laws, meaning they can be constructed as the stationary point of a free Brownian motion. This can afford an interpretation of the q-Gaussians as an interacting process in free statistical mechanics. The sufficiently dedicated scholar of the q-Gaussians might hope that they can be used to construct a broader class of quantum field theory that interpolates between Fermions and Bosons, and indeed, does so in such a way that nearby values of q can be interpreted as interacting q' quantum field theories.

1.4 Tracial Formulas and the Notation

Toward the end of this work we will be trying to follow in the footsteps of David Jekel et. al. with our notation. A fundamental object of study will be trace polynomials and their extensions. A trace polynomial in the (self adjoint) non-commutative random variables x_i is a formula which is a sum of terms of the form

$$tr(P_1)...tr(P_n)P_{n+1}$$

where each $P_i(x_1, ..., x_m)$ is a (non-commutative) polynomial. We can further extend these by including $f(y)$ where f is a function from $\mathbb{R} \to \mathbb{R}$ and y is a self adjoint variable, or has only scalar terms (terms where all polynomials are inside traces). With some technical details (see [JLS22]) these define the classes C and C_{tr} of tracial formulas, where $C_{tr} \subset C$ are the scalar valued formulas. We may also define separate class C_m of tracial formulas in $x \cup S$ where S is an infinite family of symbols s_i. This class is required to be m-multilinear in S (i.e. must be the sum of terms that each contain exactly m s_i, but in which each s_i appears at most once), and will be denoted as $f(x)[s_1, s_2, ...] = f(x)[s]$. Several derivative analogs have been introduced by Jekel and Voiculescu:

$$\partial : C_m \to C_{m+1}$$

defined by

$$\partial(ab)[s] = \partial(a)[s]b + a\partial(b)[s]$$

$$\partial(x_i)[s] = s_i$$

$$\partial(tr(a))[s] = tr\left(\partial(a)[s]\right)$$

Which then defines the gradient

$$\nabla : C_{tr} \to C$$

satisfying

$$tr(P^*\nabla f) = \partial(f)[P] \text{ for all } P(x) \text{ polynomials}$$

In addition to these, we will also need the cyclic gradient, $\mathcal{D}$ which is zero on C_{tr} and on polynomials is

$$\mathcal{D}P = \nabla tr(P)$$

We will make explicit note of the product on C_1^n that we will later make use of, namely

$$f[s] \cdot g[s] = f[g[s]]$$

Once we evaluate these formulas in $x_i \in M$ a particular von Neumann algebra generated by the x, we can identify C_1^n with $M \otimes M^{op}$, where *op* indicates the opposite algebra and n is the number of generators.

1.5 Classical Moment Measures

Fix a measure μ on $\mathbb{R}^n$; following [CEK13], we say that μ is a *moment measure* with *potential* u when u is a convex function satisfying $\mu = (\nabla u)_{\#}\rho$ and ρ is the Gibbs measure $\frac{1}{Z}e^{-u}dx$. We also say μ is the moment measure of u. Cordero-Erausquin and Klartag in [CEK13] show that a finite Borel measure μ is a moment measure with some convex essentially continuous potential u if and only if μ has barycenter zero (in particular, a finite first moment) and is not supported in a lower dimensional hyperplane. This result is proven variationally, although we will rely more directly on another variational approach taken in [San15a] which is more closely related to optimal transport. In Section 2 we describe a functional in terms of μ considered in [San15a] whose optimizer is $\rho = e^{-u}dx$, as well as this functional's analog in free probability.

Voiculescu introduced free probability theory in [Voi86]. He later introduced the notion of free entropy in a series of papers [Voi93, Voi94, Voi96, Voi99, Voi98]; see also [Voi02] for a summary. The setting for free probability is that of non-commutative (nc) probability spaces—pairs (M, τ), where M is a $*$-algebra (often a C^* or W^* algebra) and τ is a state, a functional which is both positive ($\tau(x^*x) \geq 0$) and satisfies $\tau(1) = 1$. In this paper we will further assume our state τ is a trace, i.e., $\tau(ab) = \tau(ba)$. The analogy to classical probability spaces $(\Omega, \mathcal{F}, P)$ is made by interpreting M as the space of $\mathcal{F}$-measurable essentially bounded functions on Ω, and τ as the expectation on this space with respect to P.

Consistent with this analogy, a nc random variable is an element of M. Similarly, a vector valued nc random variable is an n-tuple $(X_1, ..., X_n)$ of elements of M. Note that when M is a C^* or W^* algebra this can be slightly more restrictive than the classical notion, since we assume that these random variables have bounded norm, corresponding classically to an almost surely bounded random variable. The linear map sending non-commutative polynomials P to $\tau(P(X_1, \ldots, X_n))$ is the *law* of these random variables.

It is thus natural to ask if moment measures have an analog in free probability. This is especially of interest to us because moment measures μ are in a sense parametrized by their potentials u. Of course there is a natural way of doing this in $\mathbb{R}^n$, considering the density with respect to the Lebesgue measure. However, in the free case, the notion of density is ill-defined.

There is an analog of Gibbs measures $\frac{1}{Z}e^{-u}dx$ to free probability: free Gibbs laws (see [BS98b, Voi02]). Where Gibbs laws minimize

$$\mathcal{E}(\mu) + \int u d\mu,$$

with $\mathcal{E}$ is the classical entropy ($\mathcal{E}(f\,dx) = \int f \log f\,dx$), free Gibbs laws minimize

$$-\chi(\tau) + \tau(U),$$

where χ is free entropy, first defined by Voiculescu (see the survey paper [Voi02] for more information). Here U, which is assumed to be self-adjoint, is the potential for the free Gibbs law τ.

Definition 1.5.1 ([Voi02, Gui06, GMS06])**.** The free Gibbs law τ_U associated to the potential

U is the minimizer of $-\chi(\tau) + \tau(U)$ if it exists.

There are two cases when such laws are known to exist. The first is when U is a n.c. power series which is a small perturbation of quadratic (see [GMS06]).

The second is in the single variable case when U is bounded below, satisfies a growth condition, and satisfies a locally Hölder continuity-like condition (see [dPS95, Remark 3]) where we also get uniqueness. In this latter case, the free entropy is the negative of log energy, $\iint \log|s-t| d\mu(s) d\mu(t)$, (see [Voi02]). The above optimization implies (and by [GMS06], for U which are small perturbations of quadratic the above, is equivalent to) the integration by parts formula or Schwinger-Dyson (type) equation:

$$\tau(P \cdot \mathcal{D}U) = \tau \otimes \tau \otimes Tr(JP)$$

where $U \in \mathbb{C}\langle X_1, ..., X_n \rangle$ is the potential of the law which is assumed to be self-adjoint, and P is an arbitrary n-tuple of nc polynomials in X. Letting $M = W^*(X_1, ..., X_n)$, we have that Voiculescu's cyclic gradient $\mathcal{D} = (\mathcal{D}_{x_1}, ..., \mathcal{D}_{x_n})$, the difference quotient derivative $\partial = (\partial_{x_1}, \ldots, \partial_{x_n})$, and the (difference quotient) Jacobian J are linear maps on the following spaces

$$\begin{aligned} \mathcal{D}_{x_i} &: M \to M \\ \partial_{x_i} &: M \to M \otimes M^{op} \\ J &: M^n \to M_{n \times n} (M \otimes M^{op}), \end{aligned}$$

defined by

$$
\begin{aligned}
\mathcal{D}_{x_i}(x_{i_1}\cdots x_{i_n}) &= \sum_{j=1}^{n} \delta_{i,i_j} x_{i_{j+1}}\cdots x_{i_n} x_{i_1}\cdots x_{i_{j-1}} \\
\partial_{x_i}(x_{i_1}\cdots x_{i_n}) &= \sum_{j=1}^{n} \delta_{i,i_j} x_{i_1}\cdots x_{i_{j-1}} \otimes x_{i_{j+1}}\cdots x_{i_n}. \\
(JP)_{ij} &= \partial_{x_j} P_i
\end{aligned}
$$

The above Schwinger-Dyson equation is the nc analog of

$$\mathbb{E}(f\cdot\nabla U) = \mathbb{E}(Tr(\mathrm{Jac}f))$$

which holds for log concave Gibbs laws $\frac{1}{Z}e^{-U}\,dx$, where Jac is the classical Jacobian. These free Gibbs laws are known to exist in the multi-variable case when U is a small perturbation of the semi-circle potential ([GMS06]). In the single variable case, this can be relaxed to ordinary convexity along with growth conditions: $U(x)$ must go to infinity as $|x|$ does (and thus must grow at least as $|x|$).

We then define free moment laws as follows

Definition 1.5.2. The law τ of the nc random variables $X_1,\dots,X_n$ is a free moment law if there exists a self-adjoint nc power series U such that the free Gibbs law τ_U is well defined and is the law of nc random variables $Y_1,\dots,Y_n$ such that

$$(X_1,\dots,X_n) = (\mathcal{D}U)(Y_1,\dots,Y_n)$$

In the single variable case, laws have corresponding measures, and so we will discuss free moment measures instead of free moment laws.

Our main result is to show that certain free Gibbs laws are in fact free moment laws.

We organize the paper as follows. In Section 2, we discuss the single variable case where we prove the most general existence result for free moment measures using a variational approach. We will also provide a few examples and contrast them with the classical case. In Section 3, we discuss the existence of free moment laws for a certain class of free Gibbs laws which are close to the semicircular law. We proceed in this case by a contraction mapping argument.

1.6 What are We Trying To Achieve

There are two largely unrelated goals in this work. The second, the contents of chapter 4, are to generalize the notion of moment measures to the free probabilistic setting. In particular, since many of the classical notions in probability–densities, cumulative distribution functions, characteristic functions, etc.–are not defined in the context of general von Neumann algebras, we must translate the notion into the language of laws (aka moment methods) and free probability. With this translation, we succeed in showing that many non-commutative laws that are sufficiently close to the free semi-circle law are free moment measures.

The other two chapters of the thesis are all in service of answering the question: are the q-Gaussians isomorphic to the free group factors? As mentioned earlier, we have some answers, thanks to [GS12] [BCKW23]. In particular, we know that for an infinite collection of indices, the q-Gaussians are not isomorphic to the infinite free group factor. Conversely, we know that for $|q|$ small enough (in a way dependent on the number of indices), they are isomorphic. In fact, this result constructs an isomorphism that is also an optimal transport by constructing a "convex" potential whose gradient is the transport map. This result in fact extends beyond just the q-Gaussians, to a broad class of free Gibbs laws.

It is the opinion of the author that a diagrammatic language is the natural language for discussing the q-Gaussians, and so we attempt to explicitly compute the transport map in this diagrammatic language. We make considerable progress, but stop shy of achieving our result. We manage to find diagrammatic formulae for several intermediate steps, and show that there is a formal (i.e. infinite, potentially non-convergent) diagrammatic formula for a copy of the free group factor inside the q-Gaussians.

There are several gaps between our results and our goals:

1. We certainly require that this formula converges in the q-Gaussians. This is the most important gap, and will be an immediate concern of the author.

2. We need a formula in the opposite direction, finding the q-Gaussians in the free group factors. This is important, but should not require new techniques, and should be approachable one the first gap is closed.

3. We need to know that these two formulae give transport maps that are inverses of one another, and hence isomorphisms.

4. We would like to verify that these are the same transport maps given in [GS12]. The most natural way to do this would be to either reconstruct the diagrammatic formulae using the contraction mapping argument therein, or showing that the diagrammatic formulae we have satisfy a version of the Monge-Ampere equation.

All of this is following the suspicion of the author that the q-Gaussians are isomorphic to the free group factors for all $q \in (-1, 1)$, when the number of indices is finite. We further suspect that this isomorphism is implemented diagrammatically, and that it will only fail in the infinite index case because the terms it contains will involve sums over all index assignments, which cannot converge when there are infinitely many indices.

Chapter 2

Diagrammatic q-Gaussian Computations and Semi-Knots

The core of this work is centered on methods for performing q-Gaussian computations via diagrams. We must first understand how the q-Gaussians can be represented by diagrams, and produce some relevant results. We will then have to explore a richer diagrammatic setting which largely extends the q-Gaussian algebras which will be the home of the remaining, and central, results of the section.

2.1 Diagrams and the Conjugate System

It is well known that there is an isomorphism between $L^2(q\mathcal{G}_n)$ and the Hilbert space on which it acts. This isomorphism can be implemented using the q-Chebyshev polynomials, T_w, with the particular identification (for the rest of this work w will be assumed to be a word in the alphabet $\{1, .., n\}$)

$$T_w(X)\Omega = l_{w_0}...l_{w_{|w|-1}}\Omega = e_w$$

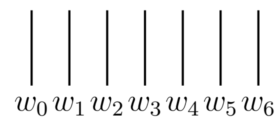
Since the q-Gaussian variables are defined to be $X_i = l_i + l_i^*$, We can find the q-Chebyshev polynomials satisfy the recursion relation

$$T_\emptyset = 1,\ T_i = X_i$$

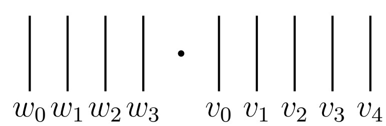
$$X_i T_w = T_{iw} + \sum_{j=0}^{|w|-1} \delta_{i w_j} q^j T_{W_{\hat{j}}}$$

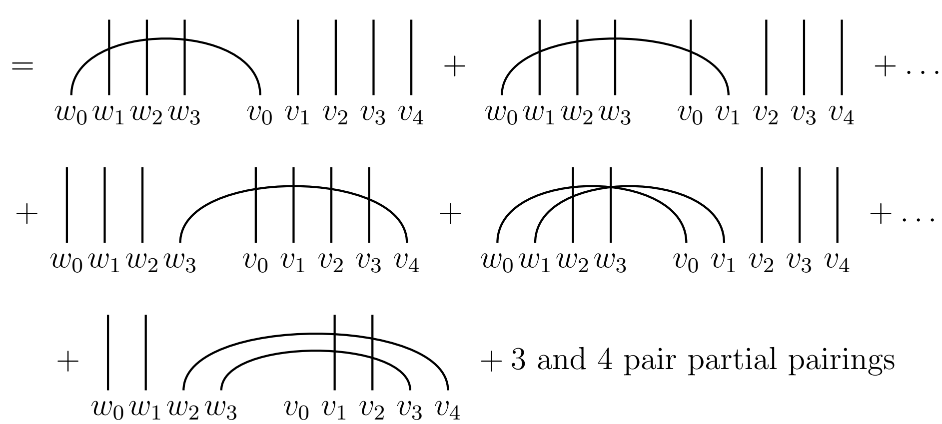
Where $w_{\hat{j}}$ is the word w with the j-th entry removed.

Appealing then to the q-Gaussian Wick formula, we can begin to understand the relevant diagrams. In particular, we can consider diagrams of the form



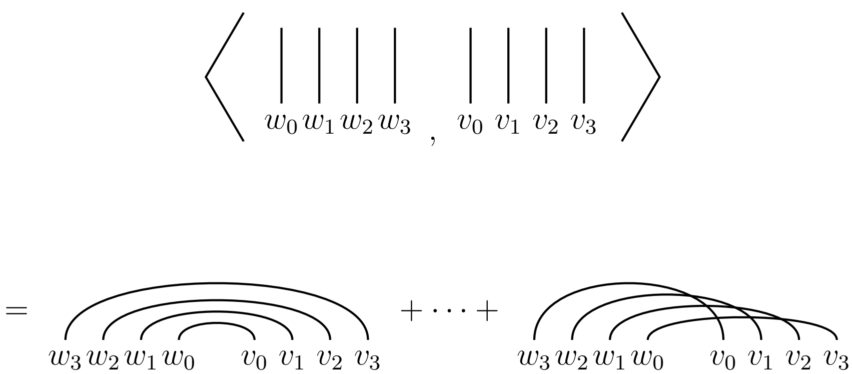
Which feature straight vertical lines indexed by w_0, ..., $w_{|w|-1}$ to represent T_w. Thus any element of $L^2(q\mathcal{G}_n)$ can be represented as a weighted sum of these diagrams. The product of two such diagrams is then quite simple:





Where a partial pairing is a diagram where some subset of w is paired with a subset of v with the same cardinality. This of course contains new diagrammatic elements not already apparent in the simplest diagrams. Two in particular are salient: lines that are not simply vertical represent pairs, and contribute a $\delta_{w_i v_{\pi(i)}}$, and crossings (between pairs or between a pair and a vertical line) which contribute a factor of q to the term. Next we can express the trace, which simply assigns 0 to any term with any unpaired vertical lines, and assigns the above coefficient $\delta_{w\pi(w)} q^{\#\text{cross}(\pi)}$ to fully paired diagrams. Finally among the basic algebraic operations, $*$ is simply implemented as flipping a diagram left-to-right.

Together, these produce a formula for the inner product of two diagrams (note that they must contain the same number of unpaired lines):



Which is simply the graphical version of the familiar formula [MS23]:

$$\langle T_w, T_v \rangle = \sum_{\sigma \in S_{|w|}} q^{\#\text{inv}(\sigma)} \delta_{v\sigma(w_{\text{rev}})}$$

where w_{rev} is the reversed word $w_{|w|}...w_1$.

Introducing now the dual basis f_w for the q-Gaussian Hilbert space which satisfies

$$\langle f_w, e_v \rangle = \delta_{wv}$$

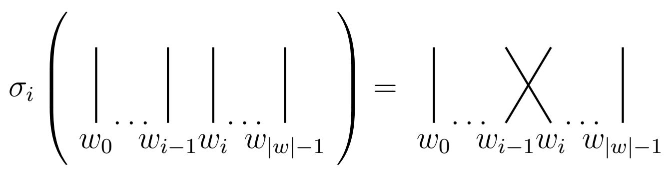
Then we see that

$$e_w = \sum_{\sigma \in S_{|w|}} q^{\#\text{inv}(\sigma)} f_{\sigma(w)} =: P_{|w|-1} f_w$$

This defines P_n which we set out to invert for the remainder of this subsection. We will summarize in words that P_n is the sum of all permutations on $n+1$ indices with the number of inversions incorporated as coefficients. To better understand this algebraically, we must now define the family of operators σ_i which we take to be generators of the braid groups (or, since we will only use positive powers of the generators, the braid semi-groups). They will satisfy $[\sigma_i, \sigma_j] = 0$ when $|i-j| \geq 2$, and always satisfy $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. On diagrams, we can implement these as

$$\sigma_i e_w = \begin{cases} 0 & |w| < i+1 \\ q e_{w_0 \ldots w_{i-2} w_i w_{i-1} w_{i+1} \ldots} & \text{else} \end{cases}$$

or diagrammatically:



in particular, each one creates a new crossing, and so creates a factor of q. We can express P_n in terms of these generators:

$$P_n = (1+\sigma_1)(1+\sigma_2+\sigma_2\sigma_1)\ldots(1+\sigma_n+\sigma_n\sigma_{n-1}+\ldots+\sigma_n\ldots\sigma_1)$$

Hitherto, all of these statements and notations are versions of results already known. We will now need several new refined notations. As above P_n will represent the product of the above terms, while $P_{n,S}$, $S \subset \{1, ..., n\}$ will represent the sum of all those terms in P_n which can be written beginning with an element of S (and 1). For example,

$$P_{3,\{1,2\}} = (1+\sigma_2+\sigma_2\sigma_3)(1+\sigma_1+\sigma_1\sigma_2+\sigma_1\sigma_2\sigma_3)$$

Similarly, $M_{n,S}$ will be the sum of all terms which can only be written beginning with an element of S (and 1). For example, $\sigma_1\sigma_2\sigma_1$ is in $P_{n,\{1\}}$, but is not in $M_{n,\{1\}}$, since the braid group relations specify that this is equal to $\sigma_2\sigma_1\sigma_2$. Thus, the above factorization of P_n can be written as

$$P_1 M_{2,\{2\}} M_{3,\{3\}} ... M_{n,\{n\}}$$

More generally, we may also use $P_{S,T}$, $T \subset S$ to represent those words which are written using only elements of S and can begin with a letter of T, and $M_{S,T}$ for those which must begin with a letter of T. If $S = T$, then only one will be written.

With this we are equipped to state the main theorem of this subsection

Theorem 2.1.1.

$$P_n^{-1} = (1 + (-1)^{n+1} \sigma_1 \sigma_2 \sigma_1 ... \sigma_n \sigma_{n-1} ... \sigma_2 \sigma_1)^{-1} \left(\sum_i (-1)^{n-i} \sum_{|S|=i} P_{n,S}^{-1} \right)$$

and hence

$$f_w = P_{|w|-1}^{-1} e_w$$

has a diagrammatic expansion.

Following the proof, we will expound on the diagrammatic interpretation of this result. We prove the theorem with a series of claims:

Proposition 2.1.2. *The braid corresponding to a permutation can be written beginning with σ_i if and only if the $i-th$ and $(i-1)$-th strings cross.*

Proof. It is clear that any permutation can be drawn as a straight line-braid, wherein we place the starts of the strings at $(i, 0) \in \mathbb{R}^2$, and the ends at $(\sigma(i), 1)$, and connect these with straight lines, taking all crossings to be positive (say the line coming from the left crosses under that coming from the right). Clearly, if the braid can be written beginning with σ_i then the $i - 1th$ and ith strings must cross. On the other hand, suppose that they do cross in a given permutation. We can then homotope the default straight line braid so that the $i - 1th$ string and the ith string cross first: we simply move the starting points of these strings to $(i - 1 + t, 0)$ and $(i - t, 0)$ for a value of t close enough to $\frac{1}{2}$ that the corresponding lines must intersect at a smaller y than all other lines. Moreover, this is a genuine homotopy of braids, since we may embed this braid in $\mathbb{R}^3$ by first starting each string at $(i, 0, 0)$ then taking the *ith* string to $(i, 0, i)$, then connecting this to $(\sigma(i), 1, i)$ then again moving down to the $x - y$ plane. With this added elevation, any planar homotopies that restrict the strings to start along $y = 0$ and end at $y = 1$ cannot give rise to any intersections of the strings. □

Proposition 2.1.3. *A braid corresponds to a permutation if and only if the crossing numbers of all pairs of strings is at most one.*

Proof. Clearly the straight line braid representation of a permutation satisfies the above property (since any two straight line segments in the plane can intersect at most once). For the converse we again homotope the braid so the i-th string is at height i, which we do simply along the straight line homotopy that moves only the z coordinate. This homotopy does not create any intersections, since all crossings have the string from the left going under that from the right. Once each string is at a separate altitude we can then homotope the strings to be straight lines and we have a straight line braid for the corresponding permutation. □

Corollary 2.1.4. *A given braid (using only positive crossings) can be written with a squared generator iff there are two strings that cross at least twice in that braid.*

Proof. Only if is clear, since any squared generator means that two strings cross twice consecutively. If follows from the previous proposition; choose a pair of two strings i and j such that

1. i and j cross twice

2. no other pair of strings cross twice between two of the crossings of i and j.

We then split the braid into three parts: the portion until just after the first (qualifying) crossing of i and j, the portion after the second crossing of i and j, and the portion in between these two. Suppose that strings i and j are in the $(k-1)$-th and k-th position at the end of the first portion. We leave the beginning and end portions unchanged, and homotope the middle portion to begin with σ_k, which we may do because we know that this portion of the braid corresponds to a permutation (since all strings cross at most once in this portion) and by the previous proposition. Since the first portion must have ended by σ_k, this presentation of the braid must have a σ_k^2. □

Proposition 2.1.5. *If S, S' are not adjacent (i.e. $|i-j| \geq 2\ \forall i \in S,\ \forall j \in S'$) and are both intervals (i.e. $i, j \in S$, $i \leq k \leq j$ implies $k \in S$). Then*

$$P_n = P_S P_{S'} M_{n,(S \cup S')^c}$$

Proof. We have first that

$$P_n = P_S M_{n,S^c} = P_{S'} M_{n,S'^c}$$

by Proposition 1, since P_S is those permutations that affect only those strings in S, and M_{n,S^c} is all permutations of $\{0, ..., n\}$ which preserve the order of strings in S. Thus, an arbitrary permutation may be made by first rearranging only the strings in S to their appropriate reordering, and then permuting everything else without affecting the ordering of the strings in S. Similarly,

$$M_{n,S^c} = M_{S \cup S',S'} M_{n,(S \cup S')^c}$$

by the same reasoning restricted to those permutations that have no intersections among strings in S. Moreover, since S and S' are not adjacent, they commute with one another so $M_{S \cup S',S'}$ cannot contain any terms with a factor from S, so it is equal to $P_{S'}$. □

Moreover, since P_S and $P_{S'}$ commute when S and S' are not adjacent, we can see that P_S^{-1} is simply the product $P_{S_1}^{-1} P_{S_2}^{-1} ... P_{S_n}^{-1}$ where the S_i are the connected subsets (intervals) of S. The following is the main step in the theorem:

Lemma 2.1.6. *Suppose that we have constructed P_k^{-1} for all $k \leq n$. Then*

$$\left(\sum_{S\subsetneq\{1,\ldots,n+1\}}(-1)^{n-|S|}P_{n,S}^{-1}\right)P_{n+1}=(1+(-1)^n\sigma_1\sigma_2\ldots\sigma_{n+1}\sigma_1\ldots\sigma_n\sigma_1\ldots\sigma_1\sigma_2\sigma_1)$$

Where in particular the last braid represents the complete inversion permutation $\sigma(i)=n-i$.

Proof. Using the previous factorization, we can see that $P_{n+1}*P_S^{-1}=M_{n+1,S^c}$. So the above product becomes the sum over $S\neq\emptyset$ of $(-1)^{|S|+1}M_{n+1,S}$. If we consider π a permutation which can be written beginning with k different σ_i. This will then appear in

$$\sum_{i\geq 1}\binom{k}{i}\binom{n+1-k}{j-i}$$

different P_S with $|S|=j$. When k=0 (i.e. π is the identity permutation), we then get that it comes with a coefficient of

$$\sum_{i\geq 1}(-1)^{j+1}\binom{n+1}{j}=(1-1)^{n-1}+1=1$$

and the complete inversion, which is the only permutation that can be written with any starting letter, does not appear in any term except $M_{n+1,n+1}$, and so comes with a $(-1)^n$. For all other terms we have

$$\sum_{j=1}^{n+1}(-1)^{j+1}\sum_{i\geq 1}\binom{k}{i}\binom{n+1-k}{j-i}=0$$

Since the first binomial coefficient ensures that $i\leq k$ and the second binomial gives a factor of $(1-1)^{n+1-k}$ by summing first over j.

□

This affords us an inverse to P_{n+1}, namely

$$\left(\sum_{S\subsetneq\{1,\ldots,n+1\}}(-1)^{n-|S|}P_{n,S}^{-1}\right)(1+(-1)^n\sigma_1\sigma_2\sigma_1\ldots\sigma_1)^{-1}$$

For the sake of clarity, we first give the formulas for the first three inverses, then show that this converges in the q-Gaussians, and then give a graphical description.

$$P_1^{-1}=(1+\sigma_1)^{-1}$$

$$P_2^{-1}=\left((1+\sigma_1)^{-1}+(1+\sigma_2)^{-1}-1\right)(1-\sigma_1\sigma_2\sigma_1)^{-1}$$

$$P_3^{-1}=\Big[1-(1-\sigma_1)^{-1}-(1-\sigma_2)^{-1}-(1-\sigma_3)^{-1}+\left((1+\sigma_1)^{-1}+(1+\sigma_2)^{-1}-1\right)(1-\sigma_1\sigma_2\sigma_1)^{-1}$$
$$+(1+\sigma_1)^{-1}(1+\sigma_3)^{-1}+\left((1+\sigma_3)^{-1}+(1+\sigma_2)^{-1}-1\right)(1-\sigma_3\sigma_2\sigma_3)^{-1}\Big]$$
$$(1+\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1)$$

In understanding this expansion graphically, it will be useful to show that there is very little cancellation between these diagrams.

Lemma 2.1.7. *The above expansion converges in the q-Gaussians.*

Proof. Let π_n be the complete inversion permutation on n letters. By induction, this lemma amounts to showing that $(1\pm\pi_n)^{-1}$ exists in the q-Gaussians. In turn, this amounts to showing that the corresponding Neumann series converges. However, this is easy to see since

$$\sigma_i^2=q^2$$

so

$$\sum_n\sigma_i^n=(1+\sigma_i)\sum_n\sigma_i^{2n}$$

and the latter obviously converges. The same reasoning applies to π_n, since

$$\pi_n^2 = q^{n^2+n}$$

□

In describing the graphical interpretation of this expansion, it is useful to know that we change the formula only slightly to remove any cancellations among the diagrams.

Lemma 2.1.8. *When considering $P_n^{-1} = A(1 + (-1)^n \pi_n)^{-1}$ we can remove all cancellation by removing the first term from all Neumann sums in A, and inverting the signs of all terms in A. For example,*

$$P_2^{-1} = \left(-\sigma_1(1+\sigma_1)^{-1} - \sigma_2(1+\sigma_2)^{-1} + 1\right)(1 - \sigma_1\sigma_2\sigma_1)^{-1}$$

Proof. We have need of an invariant for discerning when two diagrams in the sum might cancel. For this purpose we will use the matrix of (unsigned) intersection numbers of the strings in our braid. If two braids have different intersection matrices, then they must certainly be distinct braids (this follows from the fact that we are interested only in the braid semigroup with positive powers of the generators).

Let us consider more carefully the kinds of diagrams that appear in this expansion. In general, we begin by applying π_n some number of times, which adds one to all off diagonal elements of the intersection matrix. We then choose an $S \subset \{1, ..., n\}$, and apply the inverse for that subalphabet only. By induction, the inverse for this subalphabet then consists of first applying the complete inversion on each of its connected components some number of times, then choosing a further subalphabet. At each step, we are adding one to all elements of a block matrix that allows us to identify which subset we are doing the complete inversion on. Moreover, since later inversions must be on subsets of earlier inversions, we never have a situation where we can have two different blocks B_1 and B_2 where say B_2 is partly contained in B_1 and partly outside B_1 and both blocks are incremented. For example,

$$\begin{pmatrix} 0 & 2 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 2 & 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 5 \\ 1 & 1 & 1 & 5 & 0 \end{pmatrix}$$

is a valid intersection matrix that can occur in our inverse (obtained by inverting the whole braid once, then the first three strings once, and the last two strings 4 times) but

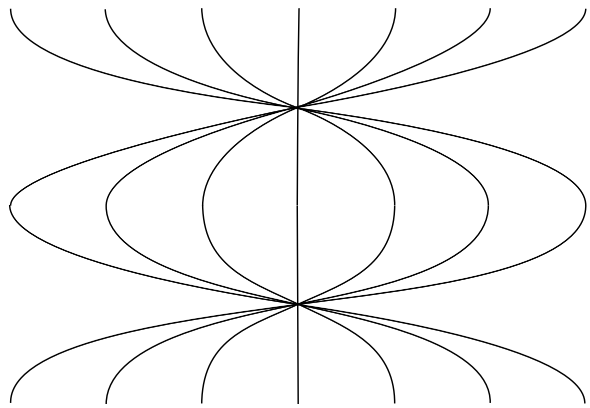
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Is not, since it would require first inverting the first three strings, and then inverting strings 2-4 (or vice versa) and neither set of strings is a subset of the other.

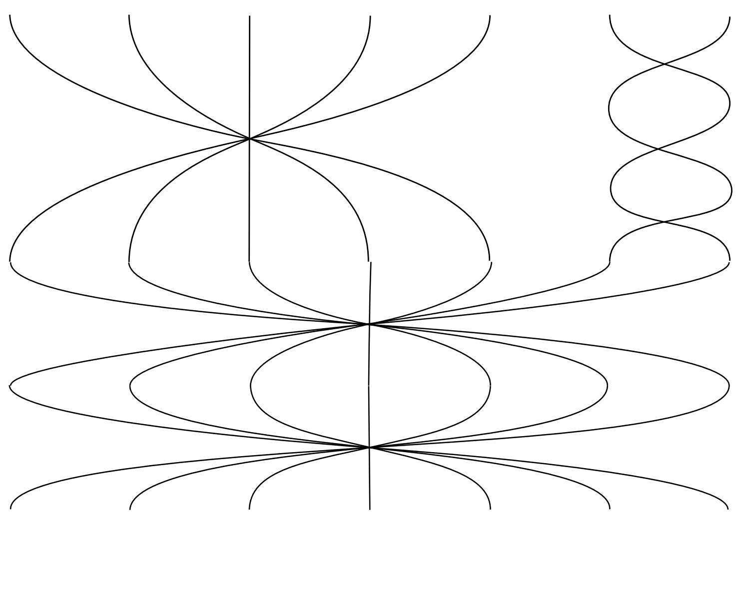
Considering the first example, we can see that we can read off the number of inversions and what subalphabet they applied to by identifying the blocks in the matrix. Indeed, the only problem is that we might choose a subalphabet S, then fail to make any inversions, and then choose a further subalphabet. Thus, as long as we *guarantee that we apply at least one inversion to each connected component of our choice of subalphabet* we can read off our choice of subalphabet, and number of complete inversions. This condition, that we apply the complete inversion to each connected component of the subalphabet, is exactly the condition that the Neumann sums begin at 1 instead of 0. □

Finally, we give a graphical description of this inverse. Our example will be on the braid with 7 strings.

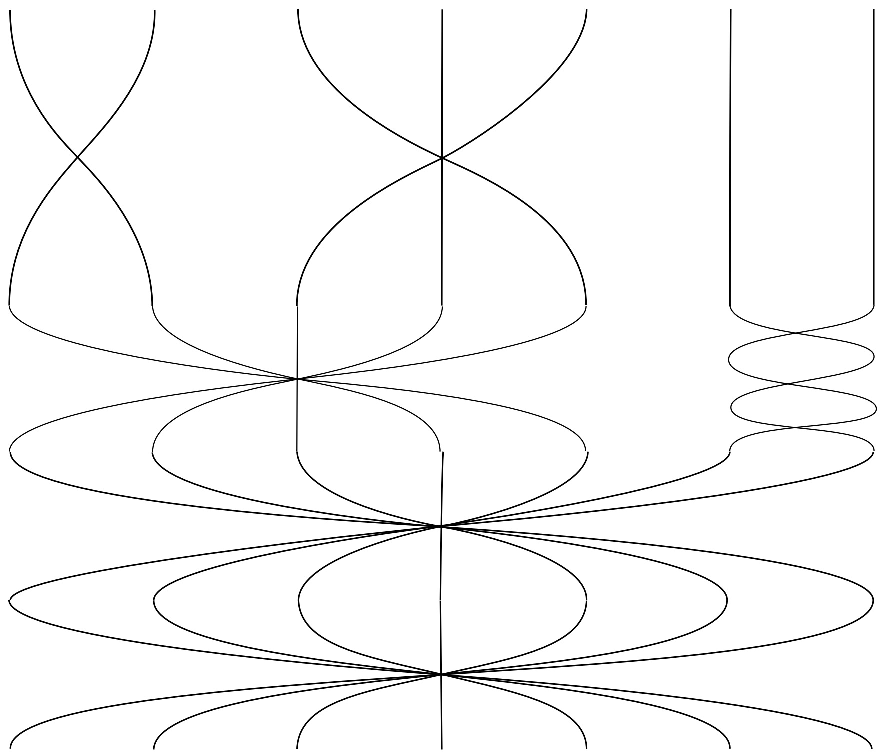
First, we begin with the complete inversion some number of times (this can be zero):



We then select two sub braids, which for our purposes will be the first five strings and the last two strings. We then apply the complete inversion to each of these substrings some number of times. In the example given below, the first five are inverted once, while the last two are inverted three times.

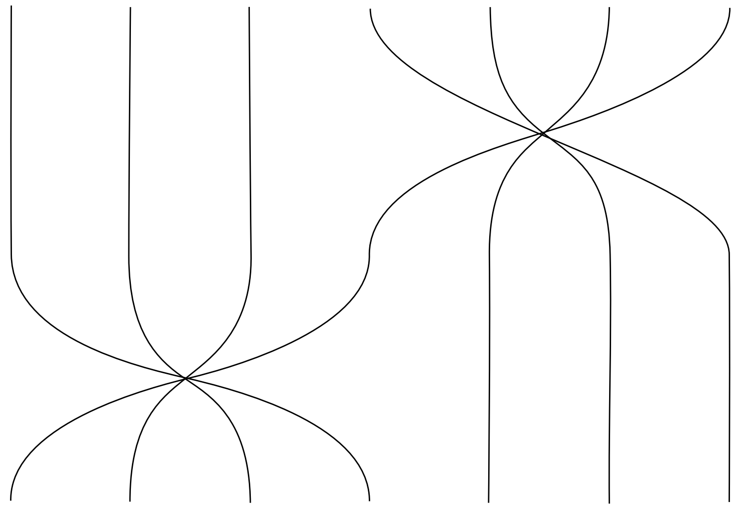


We then continue this process, selecting sub braids of each of the subbraids. We will select the first two strings, the second three strings and each of the last two strings as our subbraids. Applying the inversion to each relevant subbraid (only once this time) gives



Although there are further subbraids we could choose (namely in strings 3-5), we will stop this example here. We must then determine whether this diagram comes with a + or -. In the formula for the inverses we have a $(1+(-1)^{n+1}\pi_n)^{-1}$ which is taken to be a Neumann series. Thus, each inversion comes with a $(-1)^{\text{number of strings}}$. So here we get four factors of -1, two from the inversions of all seven strings, and one each from the five and three string inversions. Thus this diagram would come with a coefficient of 1.

Finally, we give an example of a diagram that cannot appear in this inverse.

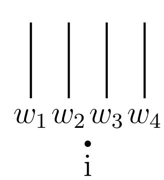


Which is disallowed because the second inversion is on a subbraid (strings 4-7) which is neither a subbraid of strings 1-4, nor of strings 5-7, which are the two subbraids the first inversion split the braid into.

With this, we can explicitly construct a purely diagrammatic formula for the conjugate system ξ_i.

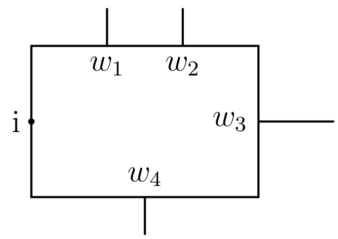
2.2 Diagrammatic Formula for the Conjugate System

In addition to the basic algebraic operations introduced before, we must now also consider Voiculescu's difference quotient derivative, ∂, diagrammatically. This will necessitate a slightly modified version of the diagrams. First, when considering vectors of elements, i.e. elements of $q\mathcal{G}_n^n$, we include a dot below the diagram whose index will be the vector index in which the diagram lives. For example,

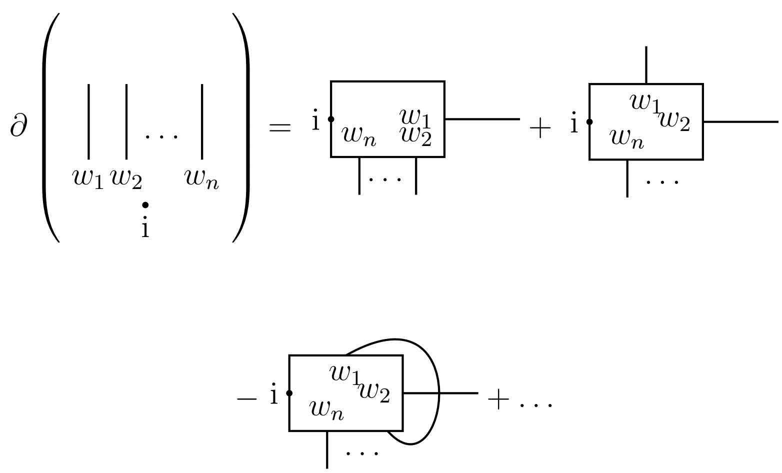


represents the vector which is zero in all entries except the i-th, where it is $T_{w_1 w_2 w_3 w_4}$.

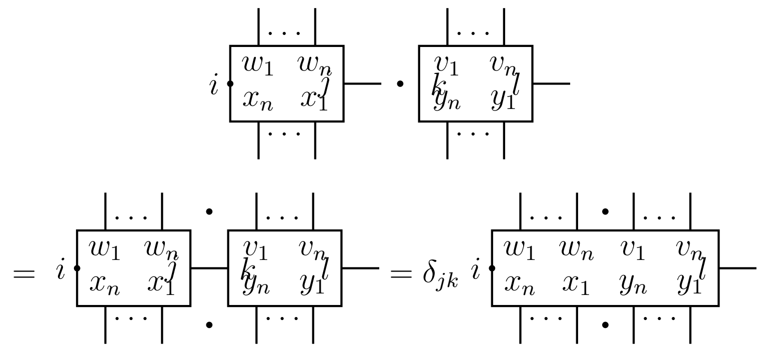
Second, those terms which are in the codomain of ∂, namely $M_n \otimes q\mathcal{G}_n \otimes q\mathcal{G}_n^{op}$ will be represented by a rectangle with several strings coming out of the top, right, and bottom sides, along with a dot on the left side. The dot and line coming out the right side represent the matrix indices of the diagram, while the lines that end facing upward represent the $q\mathcal{G}_n$ portion, while those that end facing down represent the $q\mathcal{G}_n^{op}$ portion. For example,



represents the matrix whose entries are all zero except the i, w_3-th entry, which is $T_{w_1 w_2} \otimes T_{w_4}$. With this we can now express ∂, Voiculescu's difference quotient derivative.

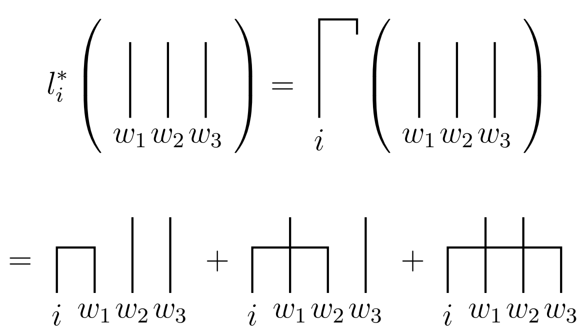


Which is perhaps most easily summarized in words. You proceed in two steps to compile a list of relevant diagrams. First, you place each unpaired line in turn on the right side of the box, moving all lines before it to the top side and all lines afterward to the bottom. Then for each such diagram we produce further diagrams by considering each partial pairing of the top with the bottom, where all pairing lines must take the right hand path around the the box. We then add these diagrams together, with the caveat that each diagram is multiplied by (-1) for each new crossing of the rightward line generated in this way. We include a proof for completeness, and to get accustomed to some diagrammatic calculations, although this result is known in [MS23] [DSS14]. Before the proof, however, it would be useful to see the product structure of these matrices graphically.



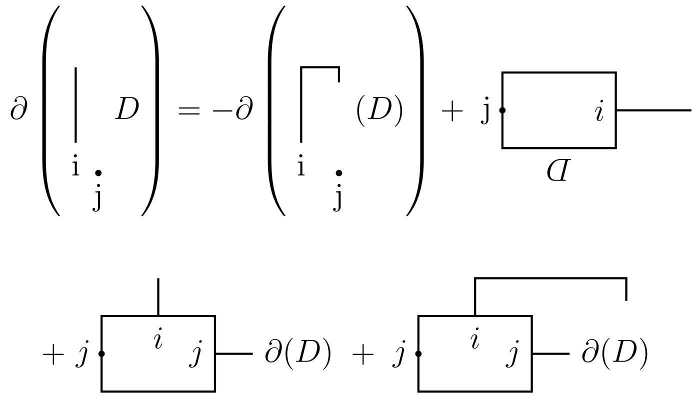
where the final dots indicate taking the product in $q\mathcal{G}_n$ alone, i.e. considering all partial pairings of the left with the right. In fact, we can extend this to the full algebra $M_n \otimes \mathcal{A} \otimes \mathcal{A}^{op} \supset M_n \otimes q\mathcal{G}_n \otimes q\mathcal{G}_n^{op}$ which includes the creation and annihilation operators. They will similarly act only on their side of the box, as will be relevant in the following proof.

Proof of Diagrammatic Formula for ∂. Graphically speaking, the left creation operators add a new (unpaired) line to the left. The left annihilation operators add a new line to the left, but then attempt to pair it with each unpaired line in turn. For example



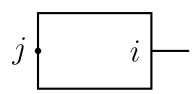
We can now use the fundamental recursion relation for ∂, which states that $\partial\left((x_i p(x))_j\right) =$

$(1 \otimes p(x))_{ij} + (x_i \otimes 1)\partial((p(x))_j)$. Since $x_i = l_i + l_i^*$, this can be rewritten graphically as



(2.1)

Where ᗡ indicates that the diagram D has been rotated 180 degrees and placed on the bottom of the box. This enables an induction on the number of unpaired strings. Certainly,

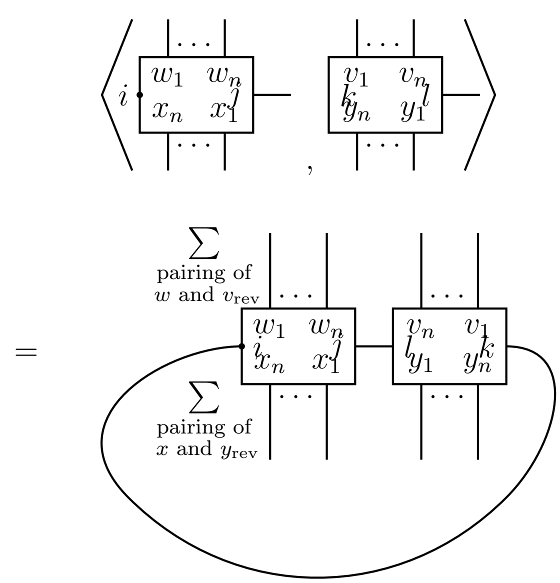
$\partial((x_i)_j) =$ 

and this can be extended to any diagram with a single unpaired string quite naturally; it amounts to bending the diagram around the base of the unpaired string and twisting it so the unpaired string goes to the right. Both of these are homotopies that do not change the number of crossings nor the indices of the ends of the paired strings, so they preserve the coefficient appropriately.

In general, the second and third terms in 2.1 produce all the terms where unpaired strings are extended to the right (the second term adds the diagram where i is chosen to go to the right, and the third term ensures that all the other such terms now have the requisite i string at the beginning. In fact, the third term accounts for all the terms which do not have the i chosen as the rightward string and do not have the i paired. Clearly then, any crossings of the rightward line in this term must be included in $\partial(D)$, and so the inductive hypothesis provides us with the correct sign. Among the terms we have claimed are in $\partial(|_i D)$, only those where i is paired remain. The claim is that i can only be paired with elements on the other side of the rightward line, that these pairings come with an extra factor of (-1), and that they must go around the right. The latter point is clear from the recursion–all the pairing goes to the right. The first point comes from a cancellation between terms one and four. Term 1 includes terms where i can pair with any index (except the rightward one), all with the extra (-1), but the fourth term contains only those terms with i is paired with another element on the top of the box. This cancels with all similar pairings in the first term, leaving only the claimed terms. In particular, we can then note inductively that since we get an extra factor of (-1), and an extra crossing of the rightward line, the power of (-1) must be equal to the number of new crossings of the rightward line, as claimed. □

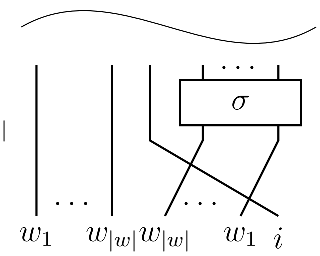
We now elucidate the diagrammatic expression for the trace in $M_n \otimes q\mathcal{G}_n \otimes q\mathcal{G}_n^{op}$, as before, it is zero for any diagram which has any unpaired lines on the top or bottom. However, we also connect the rightward line to the left edge, which, as with all pairings, produces a δ_{ij} if i and j are the indices of the left edge and rightward line respectively. We can also consider $*$, which, like before flips the diagram left-to-right.

Together, these produce a formula for the inner product of two elements of this algebra, namely



which we can simply reorient to determine $\partial^*(1 \otimes 1)$ in terms of f_w, and then simply combine this with the previous section to produce a diagrammatic formula for ∂^*.

$$\partial^*(I_n \otimes 1 \otimes 1) = \sum_{\text{words } w} \sum_{\sigma \in S_{|w|}} (-1)^{|w|}$$



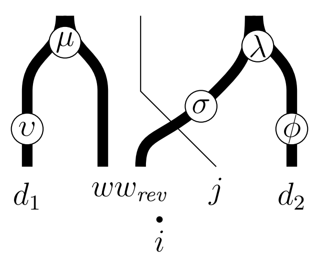
Where the tilde indicates that the given diagram is written in terms of f_w, and so must be followed by P^{-1} from the previous section.

We can extend this result to $\partial^*((d_1 \otimes d_2)_{ij})$, albeit with some notational adversity.

Theorem 2.2.1. *let d_1 and d_2 be two diagrams. We can find that*

$$\partial^*(d_1 \otimes d_2)_{ij} = \sum_{w} \sum_{\substack{\sigma \in S_{|w|} \\ \upsilon \in S_{|d_1|} \\ \phi \in S_{|d_2|}}} \sum_{\substack{\lambda \in \binom{|d_2|+|w|}{|w|} \\ \mu \in \binom{|d_1|+|w|}{|w|}}} (-1)^{|w|} P^{-1}(d_{w\sigma\upsilon\phi\lambda\mu})$$

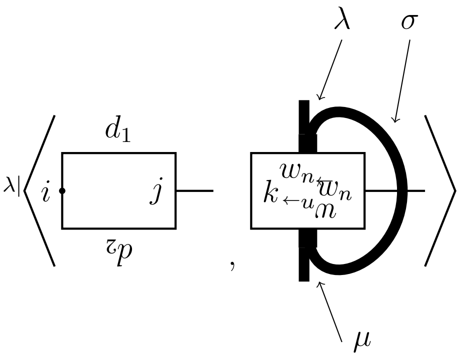
where $d_{w\sigma\upsilon\phi\lambda\mu}$ is the diagram



and σ, υ, and ϕ represent permutations of the corresponding braids, while μ and λ represent the choice of ways to intertwine two braids.

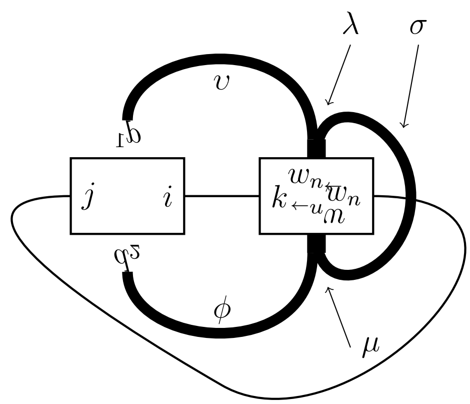
Proof. It is sufficient to consider the inner product of $(T_w)_k$ with $\partial^*(d_1 \otimes d_2)_{ij}$. This is of course $\langle (d_1 \otimes d_2)_{ij}, \partial(T_w)_k \rangle$, i.e.

$$\sum_n \sum_{\substack{\lambda\in 2^{n-1}\\ \mu\in 2^{|w|-n}\\ |\lambda|=|\mu|}} \sum_{\sigma\in S_{|\lambda|}} (-1)^{|\lambda|}$$

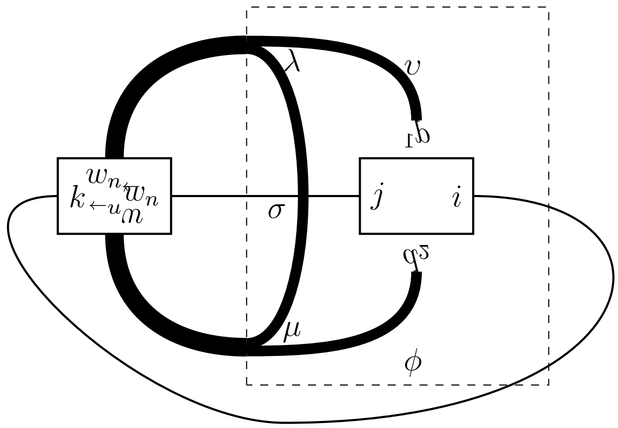


where λ and μ indicate choices of subset of $w_{n\leftarrow}$ (w ending just before n) and of $w_{n\rightarrow}$ (w beginning just after n). σ then indicates the fact that there may be crossings between the chosen subsets. Taken together, these three describe the partial pairings of $w_{n\leftarrow}$ with $w_{n\rightarrow}$. The inner product is then

$$\sum_n \sum_{\substack{\lambda\in 2^{n-1}\\ \mu\in 2^{|w|-n}\\ |\lambda|=|\mu|}} \sum_{\sigma\in S_{|\lambda|}} (-1)^{|\lambda|} \sum_{\upsilon\in S_{|d_1|}} \sum_{\phi\in S_{|d_2|}}$$



We will then modify this by either considering the inner product in the opposite order/flipping the diagram horizontally, and then performing a Möbius transform that moves the $i-k$ line to the $j-w_n$ line (all diagram inner products are real valued). We will then homotope all complexity toward the d_1 d_2 box, which will then then make clear our formula.



At this point we would like to simply take the contents of the dotted box to be the our new diagram. However, as of this subsection, we have not developed the machinery to process strings that do not originate from the x axis, so for now we content ourselves with cutting the line containing σ just below the $w_n - j$ line, turning it into the origin of a μ and a μ_{rev}. This end of these lines will be taken to be next to the j, and the μ_{rev} will immediately cross the j line. Immediately after that, σ will be applied to μ_{rev}. Then on either side of this we can place $d_{2,\text{rev}}$ and $d_{1,\text{rev}}$ respectively. To them we can then apply ϕ and υ respectively. The choices of μ and λ when considered in reverse amount to choosing how to intertwine the strings from d_i into those from μ or μ_{rev}, which we previously notated using binomial coefficients. Finally, we convert this into a diagram for a vector term by replacing the i side of the box by the $\cdot i$, unfolding the other three sides of the box to lie on the x-axis and, recalling that this is the term subject to $*$, flip it horizontally. $\square$

2.3 Semi-Knot Algebra and a Diagrammatic Formula for Free Semi-Circular Variables in $q\mathcal{G}_n$

The central idea of the previous proof amounts essentially to condensing all complexity in a diagram from $\langle T_w, \partial^*((d_1 \otimes d_2)_{ij})$ to one side of the diagram, so that it can be identified as a dual pairing of a complex diagram with a simple one. However, the need to cut the w

chord in the previous section suggests that there might be wisdom in extending our notion of diagrams beyond what we have previously considered. Although not elaborated on here, the current formulation also introduces some complexity when considering $\partial^*(AB)$ where A and B are both box diagrams which might have crossings of their rightward string. We therefore seek to formalize the kinds of manipulations we have been making heretofore.

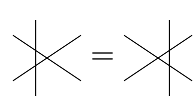
We will formalize our new family of more loose diagrams as *semi-knots* and *semi-links* and present the corresponding semi-link algebras, so called because they contain half the information of knots. When we trace the kinds of operations performed hitherto, we find that three axioms are clear:

Definition 2.3.1. A semi-knot (or semi-link) diagram is a knot diagram where all information about the handedness of crossings has been removed. That is to say, a semi-knot diagram is a map from $\bigsqcup_{i=1}^{n} S_1$ to S^2.

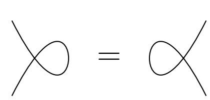
These diagrams can then be manipulated according to several axioms.

1. All the standard non-Reidemeister homotopies (i.e. post-composing with any isotopy of the plane) can be performed.

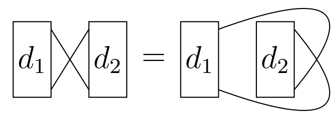
2. The third Reidemeister move (sliding) can be performed. (This was already used extensively in considering the braid semi-group.)



3. A weakened version of the first Reidemeister move (untwisting) can be performed:

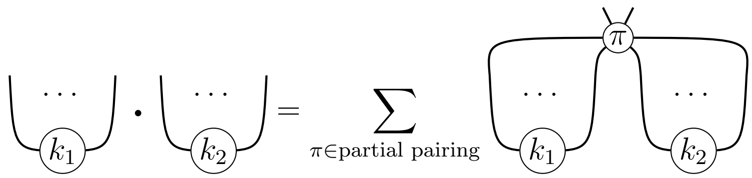


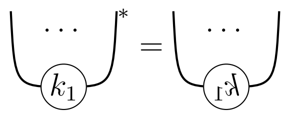
In fact, we extend this somewhat into: for any given crossing between a string and itself, if we can split the crossing either horizontally or vertically such that the resulting diagram has one more connected component than the initial diagram, then we can move these connected sub diagrams to either side of the crossing. i.e.



Remark 2.3.2. *The final axiom in particular is not fixed by any considerations we will make in this paper. Indeed we have chosen it for two reasons: to maximize the strength of the available moves (i.e. to minimize the number of distinct semi-knots) and to try to preserve the intuition from knots to the greatest extent possible. However, by preserving as much of the structure of knots as possible, we reproduce one of the central difficulties of knot theory: deciding when two semi-knot diagrams are equivalent. It is the opinion of the author that this is most likely the morally correct set of axioms (in the sense that it should produce the most elegant final results by ignoring irrelevant complexity), but this complexity will have to be circumvented a few times by considering weak and fragile semi-knots. These replace the final axiom by only being able to flip loops, i.e. only considering the first diagram in axiom 3, or simply removing axiom 3 altogether. When relevant, we will refer to all these structures together as the family of extended semi-knots.*

To this family of semi-knots, we add the spiky semi-knots. I.e. those diagrams whose domain consists not only of copies of S_1 but also $[0, 1]$. For the semi-link algebra, we will insist that all the ends of open links (i.e. the spikes) terminate on a common horizontal line, which we take to be the x-axis in $\mathbb{R}^2$ and that the entire diagram be contained below this line. To construct an algebra of spiky semi-links $\mathcal{SL}$ from such spiky semi-links, we define a product structure, an involution and a trace exactly as with the ordinary diagrams at the beginning of this section. More explicitly, we take

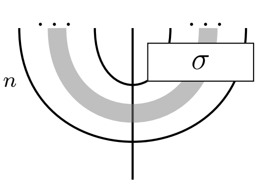




and of course, the trace of a spiky semi-link is 0 if there are any spikes, and q raised to the number of crossings otherwise. The algebra $\mathcal{SL}$ will then be the free algebra generated by these diagrams. We can then define a map from this algebra to $q\mathcal{G}_n$ quite simply: for a diagram, d, we produce first assign an index $\{1, ..., n\}$ to each string. The spikes then define a word w, and include a term of $q^{\#\text{cross}(d)}T_w$ in the result. We then add up the results across all possible assignments of indices. Note that this means that any closed links contribute a factor of n to the output. After one more slight modification, we will motivate this map by considering the formula for the conjugate system in this notation.

The slight modification will be how we can present diagrams in $\mathcal{SL}_v$, which is relevant when want to think of $q\mathcal{G}_n^n$. The only change is that we will now require that exactly one spike terminate below the diagram, and when mapping the diagram into $q\mathcal{G}_n^n$, the index of the string with the downward spike will determine which component of the vector is mapped to. With this we can express the formula for the conjugate system (note that though this is an infinite sum, we do not define any topology on $\mathcal{SL}$, instead treating it as the free vector space generated by the set of diagrams. To make sense of the infinite sums we must first take the maps into $q\mathcal{G}_n$.)

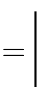
$$\xi = \sum_n \sum_{\sigma \in S_n} (-1)^n$$



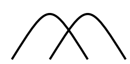
With this we are equipped to actually show the main result of this thesis. Which we state roughly here:

Theorem In $\mathcal{SL}_v$ there is a sum which will formally produce the a family of n free semi-circular elements from n q-Gaussian's. More precisely, there is an (infinite) diagrammatic formula f such that $tr(f_{w_1}...f_{w_{2n}})$ contains exactly the non-crossing pairings of w_1 through w_{2n}.

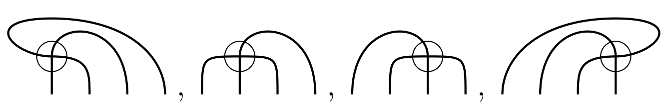
We will begin our argument beginning with the very simple diagram

$f =$ 

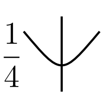
and consider what kinds of errors are produced, as well as how to remove them. When considering $\tau(x_i x_j)$ this produces no errors. $\tau(x_i x_j x_k x_l)$ however, produces the error



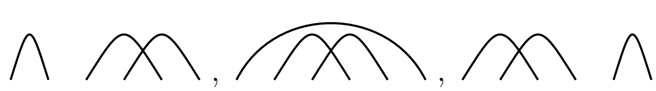
following our ethos of encapsulating all the complexity in one portion of the diagram and leaving the rest simple, we can homotope this to any of

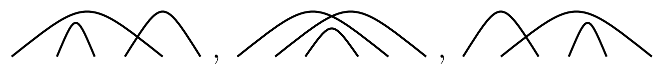


Which is to say, that by replacing each leg in the four base points one at a time with

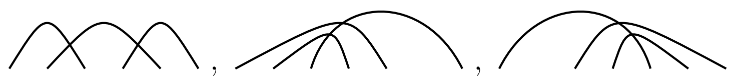
$-\frac{1}{4}$  (2.2)

we can cancel out all these errors. Other errors would of course be introduced, but for now we will ignore them. Instead, we will consider $\tau(x^6)$, which has several errors, including 6 terms with one crossing:

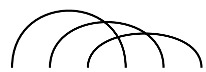




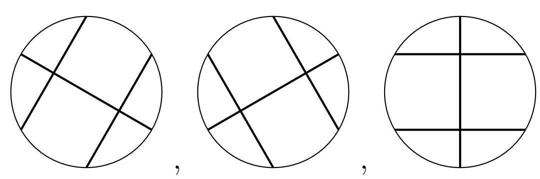
3 terms with 2 crossings:



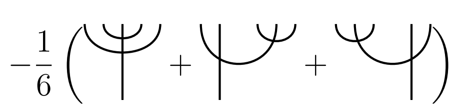
and one with 3 crossings:



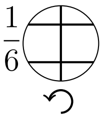
The one crossing terms can likewise be fixed by adding 2.2 to each of the stems that are involved in the intersection. For the two crossing terms, we will make an observation–all of these terms are rotations of each other. To make this more plain, we will rearrange our diagrams a bit. Instead of all spikes terminating at the bottom, we will declare that the spikes must terminate on outside edge of a disk (we will arbitrarily choose the first spike to be at the bottom). In this way, these three terms are now



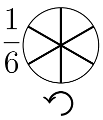
which is to say they are all the same. When we choose to retract the diagrams so all crossings are contained in one leg we will always get one of these three diagrams. As it happens, this will be true for all error terms (a symptom of traciality), so we will focus on these circular diagrams. This circular diagram has three distinct rotations, each of which has six legs. Thus when we want to cancel this error we can place each unique rotation at each of the six legs, producing 18 error correcting terms for 3 error terms, so we find that the correcting term must be



Or, using the notation ↺ to represent taking the sum of all unique rotations of a circular diagram,

$-\frac{1}{6}$ 

and indeed, this same notion suffices for the one three crossing error term, which can be corrected by

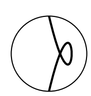
$-\frac{1}{6}$ 

These considerations already hint at several important ideas:

1. It is only connected spiky (or spoky if you will) semi-links that should appear in our formula, since different connected components of the error should be dealt with by adding different correction terms to the legs of each connected component.

2. We should only really be concerned with circular diagrams, since all the rotations of a given diagram can be produced by making different choices of which spike should be the bottom spike. Indeed, because of traciality all terms in the error (and hence in the correction) will appear with the same coefficient.

3. The rotational symmetry of the circular diagrams will be important for determining the coefficients of the error terms. It is here that the problem of determining equivalence of semi-knots will come into play. For example, the third axiom is needed to know that



has only one unique rotation.

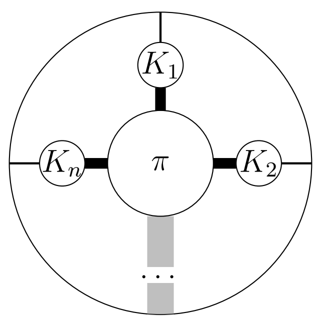
With the final complexity in mind, the proper statement of the theorem is

Theorem 2.3.3. *There is a formal infinite sum, f, in each of the extended families of semi-knots such that $\tau(f^n)$ formally contains only noncrossing pairings.*

Proof. Our strategy at this point is not explicit, but instead to show that there is a recursion relation that the corrections must satisfy, thereby glossing over the question of equivalence of diagrams. To begin with suppose we have that

$$f = \sum_{\text{Diagrams } K} a_K K$$

where $a_i \in \mathbb{C}$ and d_i are diagrams enumerated in some order. Then the relevant error diagrams that we can generate are any of the form

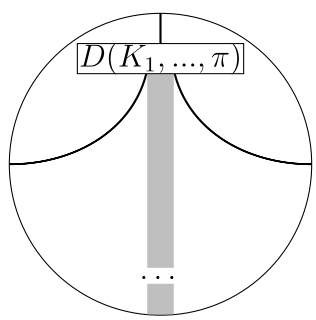


where π is some pairing, here meaning a spiky diagram consisting of only straight lines. However, we will only be concerned with connected error diagrams, so we will further insist that the resulting diagram be connected (note that this is not equivalent to π being connected since each K_i might send many lines to π, and even if these lines don't intersect in π, they may do so in the diagrams). This error term comes with a coefficient of $\prod_i a_{K_i}$. Of course, all rotations of this diagram will also appear as error terms with the same coefficient, so indeed we can consider our diagrams only up to rotation and assume that all of them have ↺. We can then cancel this error by contracting the entire interior of the circle onto one of the spikes. Referring to the above construction as $D(K_1, ..., K_n, \pi)$, we find the following recursion for f:

$$a_K = -\frac{1}{n} \sum_{\substack{K_1,\ldots,K_n,\pi \\ D(K_1,\ldots,K_n,\pi)=K \\ \text{not exactly one } K_i \text{ nontrivial}}} \prod_i a_{K_i}$$

$$a_{\bigcirc\mkern-13mu|} = 1$$

If this recursion is satisfied, then when we consider $\tau(f^n)$, we get all noncrossing pairings by default, and when we consider any diagram which has a crossing, we may consider each connected component in turn, and see that they are of the form $D(K_1, ..., K_m, \pi)$ for some choice of K_i and π. However, by the recursion relation, we then find that the coefficient of $D(K_1, ..., K_m, \pi)$ is 0, since all ways that it can be generated are included with the corresponding coefficients in the error term on the left hand side, but so also is the $-\frac{1}{n}$ of this coefficient included n times in the form of



placed at each of the legs. To ensure that this recursion relation can be solved, we note that, by construction we have assumed that not exactly one of the k_i is nontrivial (i.e. not simply $|$), and so there is no self reference. In fact, we can actually see that $D(K_1, ..., \pi)$ must have more crossings that each K_i. In the first case, it contains at least two K_i which themselves have crossings, so the sum of their crossing is more than either individually. In the second case, all of the K_i are trivial, and since the diagram is connected, there must be at least one crossing (excepting the case when there are two trivial semi-knots, and no others, which is our base case). As such we can filter the set of semi-knots by number of crossings, and find that this recursion relation moves us up this filtration, hence it has a solution. □

Remark 2.3.4. *Of course, it would be preferable to determine these coefficients explicitly (so far in this work the only one we have determined if that of* $\oplus$*, since all others could be generated in more than one way). In one direction (that of strong semi-knots), the difficulty is in determining when two semi-links are equivalent. If we instead choose to consider weak or fragile semi-knots, then there is additional complexity in the patterns of coefficients brought about by the proliferation of distinct diagrams. Perhaps worse, including distinct weak semi-knots that are equivalent strong semi-knots might make the question of convergence of the sum more opaque.*

Remark 2.3.5. *This result is in principle only one step away from showing that* $q\mathcal{G}_n$ *contains a copy of* $L(\mathbb{F}_n)$*. All that is required is showing that the sums for* $\tau(f^n)$ *converge. Once this is known, then we would in fact have that* f *itself converges in* $q\mathcal{G}_n$*, because convergence of the first 2 moments would imply convergence of* f *in* L^2*. Then convergence of higher moments would ensure that this element is in fact in* L^p *for all* p*, and thus is in the algebra.*

Remark 2.3.6. *There is no issue obvious to the author that would prevent a similar program in reverse, finding a formula for q-Gaussians in* $L(\mathbb{F}_n)$*. Although it has not yet been performed, on account of deadlines, this is an obvious next consideration.*

2.4 Ancillary Diagrammatic Calculations

As addition partial progress toward understanding the formula generated in the previous section, we can at least identify a simpler characterization of the diagrams that appear.

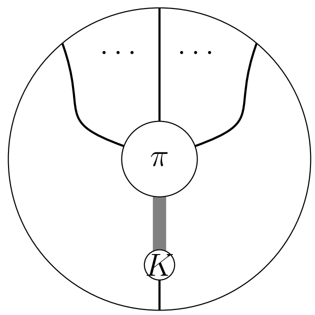
Lemma 2.4.1 (Braid Style Construction)**.** *The set of diagrams that appear in the above are those that can be constructed using the following operations, beginning from the trivial diagram, —. Strings should be envisioned vertically, as in a braid diagram.*

1. *We can create a new string in a cup shape* $\cup$ *with the bottom along an existing string.*

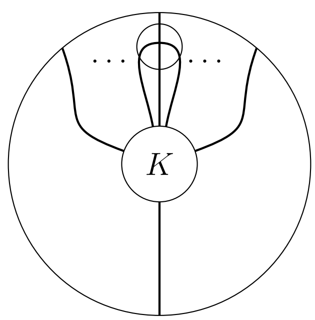
2. *We can permute any existing lines.*

3. *We can cap off two lines that are separated by exactly one line by applying* $\cap$ *with it's vertex placed along the intermediate string.*

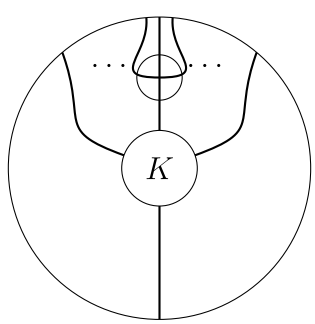
Proof. We first show that all three operations can be obtained via $D(K_1, ..., \pi)$. Given K, the diagram we wish to perform an operation on, we can perform arbitrary permutations via



Since π can be any straight line pairing, it can be any permutation. Next we consider capping off a pair of strings:



I.e. by pairing with a $\oplus$, we can cap off lines. In a fairly similar construction, we can use a pairing between $\oplus$ and two adjacent trivial strings to create a new string around an existing one.

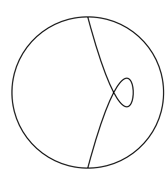


We must now show that all diagrams that can be obtained through D (beginning with only the trivial diagram), can be obtained through these actions. Our hope is to perform an induction, showing that if each K_i is constructible in this manner then so is $D(K_1, ..., \pi)$. The pairing π is not hard to deal with: any pairings between K_i and K_j with $i,\ j \neq 1$ can be handled with cups, and after all appropriate strings have been created, we can then perform arbitrary permutations, thereby realizing π. The only issue for handling the remaining K_i is the fact that they can be constructed with these operations, but a priori only from their base string (the one attached to the outer circle). However, we can deconstruct them in reverse by simply starting with the appropriate number of strings, replacing cups with caps and vice versa, and simply reversing permutations. This allows us to induct on the set of semi-knots obtainable by D to create a family of semi-knots which are closed under D (the minimal such family).

□

This is valuable primarily in that it lets us show that not all diagrams are included in the family. For example:

Corollary 2.4.2. *This family does not contain*



Proof. Any homotopy of this semi-knot must have only one intersection, and must also have in the y coordinate a highest maximum and lowest minimum (we perturb it slightly if there are multiple at the same height). If we suppose that one of these presentations is of the form described by the lemma, then it requires that the local maximum and local minimum both occur at an intersection, a contradiction. □

Chapter 3

Tracial Formulas, Diagrams, and Conjugate Systems

In the end, although the q-Gaussians are interesting objects with several potential applications, they are still a fairly niche subject withing von Neumann algebras. Of greater interest, perhaps, is the question of whether out objects of study (conjugate systems, free Gibbs potentials, transport maps) can be more effectively considered in a more general framework. Following the work of David Jekel on the model theory of von Neumann algebras, we try to find tracial formulas for a few of these objects. Even the rudimentary formulae we find produce some more general results about von Neumann algebras.

Conjugate variables are defined by the equation

$$\langle \xi_i, p \rangle = tr_{\#}(\partial_i p))$$

which we can interpret in a tracial formula setting. By either of two methods we can then pursue a theorem showing that all noncommutative laws near enough the free semi-circle law will have a conjugate system.

Theorem 3.0.1. *Given a tracial von Neumann algebra $W^*(X_1, ..., x_n)$, if the law of $\{X_i\}$ is* *less than* 1 *away from the semi-circle law (in operator norm with respect to the Chebyshev basis), and if* $tr_{\#}(\partial_i T_w)$ *is* l^2 *for each* i*, then a conjugate system exists. Moreover, it is given by a fixed tracial formula.*

Proof. The main idea of the proof is simply to formally invert the metric as an operator. As such, we have several options to proceed, most prominently Neumann series or Graham Schmidt. To play more nicely with our above hypotheses, we will use the Neumann Series. We consider the inner product on $L^2(W^*(X))$ as an operator, g, on $l^2(W)$, where W is the set of words on $\{1, ..., n\}$, and then taking

$$g\left(\sum_w a_w T_w\right) = \sum_v tr(T_w^* T_v) T_v$$

At the semi-circle laws, this operator is the identity, since the Chebyshev polynomials are an o.n. basis for $L^2(L(\mathbb{F}_n))$. Hence, the second hypothesis on the law of X is exactly that this operator is less than 1 away from identity in operator norm. Thus it can be inverted by a Neumann series. Thus we see that the tracial formula

$$\sum_{n\geq 0} \sum_{w,v\in W} (1-g)^n tr_{\#}(\partial_i T_w) T_v$$

The second hypothesis then guarantees that $tr_{\#}(\partial_i T_w) \in l^2(W)$, and so this sum converges in $L^2(W^*(X))$. This gives a tracial formula for ξ with generators that are nearly semicircular simply by leaving g as a matrix of trace polynomials $g_{wv} = tr(T_w^* T_v)$.

□

Moreover, we have an alternate formula that we can consider as a necessary condition for the existence of ξ. We begin by defining the family of tracial formulas $\tilde{T}_w$ as follows. First, choose an ordering on the words $w \in W$. For now, we will choose an arbitrary (well) ordering such that $|w| > |v|$ implies $w > v$. Then take $\tilde{T}_\emptyset = 1$ and inductively define

$$\tilde{T}_w = T_w - \sum_{v<w} \frac{tr(\tilde{T}_v^* T_w)}{tr(\tilde{T}_v^* \tilde{T}_v)} \tilde{T}_v$$

In any von Neumann algebra these will now evaluate to orthogonal element of the algebra (taking the convention that $(\tilde{T}_v(X_1, ..., X_n) = 0$ simply removes that term from the sum). This orthogonality allows the formula

$$\xi_i = \sum_w \frac{tr_\#(\partial_i \tilde{T}_w)}{tr(\tilde{T}_w^* \tilde{T}_w)} \tilde{T}_w$$

This formula is a priori more difficult to garner estimates from, however, it allows for the following proposition:

For any X such that $W^*(X)$ has a conjugate system, the above tracial formula must converge in L^2.

Proof. The above formula is simply a construction of an orthogonal family of elements of any $W^*(X)$, which further span $L^2(W^*(X))$, since their span is equal to that of the T_w, which includes all noncommutative polynomials. We will adopt the convention that $\tilde{T}_v(X) = 0$ simply removes the term from the sum, since the projection onto the zero vector is zero. Removing these terms therefore produces an orthogonal basis for $L^2(W^*(X))$ and the terms that remain are exactly the coordinates of ξ in this basis, and so must converge. □

One might reasonably wonder if the above formula depends on the ordering we choose on words. This is answered at least in part by

Lemma 3.0.2. *The tracial formula above is independent of finite reorderings of W.*

Proof. Consider two orderings of W which agree after some initial segment ω. Then since ω is finite, we can compute the coefficient of T_w over terms $w \in \omega$ simply using Cramer's rule. □

Chapter 4

Moment measures and Optimal Transport

4.1 Main Result for One Variable

In the case of a single (non-commutative) random variable X in the nc probability space (M, τ), the law of X can be given as a functional on the space of polynomials in a single variable by letting the law $\tau_X(p)$ for a polynomial $p(z)$ be $\tau_X(p) = \tau(p(X))$. Alternatively we can view the law of X as a probability measure μ, using positivity and the Riesz-Markov theorem.

Suppose X has law τ with corresponding measure μ. Then if τ is a free moment law, there exists Y with law τ_u such that $X = (\mathcal{D}u)(Y)$. As the cyclic gradient of a function in one variable is equal to the ordinary derivative, the pushforward condition is equivalently $X = (u')(Y)$. If μ is the measure corresponding to τ and ν_u is the measure corresponding to τ_u, then we have $\mu = (u')_{\#}\nu_u$.

We refer to the measures associated to free Gibbs laws in one dimension as free Gibbs measures. The authors emphasize that the idea of free Gibbs measures is not wholly novel; Indeed, free Gibbs laws were defined earlier (see Def 1), it was known (see [Voi02],[Voi94]) that $\chi(\tau)$ reduces in the single variable case to log energy, and minimizers of $-\chi(\tau) + \tau(U)$ have already been studied, e.g. in ([dPS95]).

Definition 4.1.1. The free Gibbs measure ν_u associated to the convex function $u : \mathbb{R} \to \mathbb{R}$ is the measure corresponding to the free Gibbs law τ_u, if it exists. In other words, ν_u is the minimizer of

$$\iint \log |s - t| \, d\mu(s) \, d\mu(t) + \int u(s) \, d\mu(s)$$

if it exists.

Definition 4.1.2. A real probability measure μ is a free moment measure if

$$\mu = (u')_{\#}\nu_u$$

for some convex function $u : \mathbb{R} \to \mathbb{R}$.

Our main result in this section is Theorem 4.1.8 which implies that if μ is a probability measure on $\mathbb{R}$ other than δ_0 with finite second moment and barycenter zero, there exists a convex $u : \mathbb{R} \to \mathbb{R}$ such that $\mu = (u')_{\#}\rho$ and $\rho = \nu_u$, where ν_u is the free Gibbs measure associated to the potential u. Observe also that if μ is centered, then u must have a derivative which changes signs, and so $u(x) \to \infty$ as both $x \to \pm\infty$. Through prior understanding of free Gibbs measures, we'll also have that ρ is absolutely continuous with respect to Lebesgue measure and $2\pi H(\rho)(x) = u'(x)$ for any $x \in \operatorname{supp}(\rho)$. Here $H\rho$ is the Hilbert transform of ρ, given by the principal value integral

$$H\rho(t) = \frac{1}{\pi} \, \mathrm{PV}\!\int_{\mathbb{R}} \frac{1}{t - x} \, d\rho(x).$$

For a brief computational guide to solving $2\pi(H\rho)(x) = u'(x)$ for $x \in \operatorname{supp}(\rho)$ for a fixed u, see the appendix. See also [dPS95] for more examples.

In the classical case of moment measures, we are searching for $\rho = \frac{1}{Z}e^{-u}\,dx$, the log concave Gibbs measure with real convex potential u satisfying $(\nabla u)_{\#}\rho = \mu$ for some μ. Here Z is the constant that makes ρ a probability measure.

It is possible to find such ρ when μ has barycenter zero and is not supported on a hyperplane (which for $\mathbb{R}^1$ only means it isn't δ_0) [CEK13]. The measure ρ can be found by considering the functional

$$\begin{aligned}\int \rho \log \rho\, dx + \tfrac{1}{2}\int x^2 \rho(x)\, dx + \tfrac{1}{2}\int x^2\, d\mu - \tfrac{1}{2}W_2^2(\rho,\mu) &= \int \rho \log \rho\, dx + T(\rho,\mu)\\ &=: \mathcal{E}(\rho) + T(\rho,\mu)\end{aligned}$$

where W_2 is the Wasserstein distance between ρ and μ, $T(\rho,\mu)$ is the maximal correlation functional defined as follows and $\mathcal{E}$ is the negative differential entropy, $\mathcal{E}(\rho\, dx) = \int \rho \log \rho\, dx$.

The measure ρ satisfying $(\nabla u)_{\#}\rho = \mu$ and $\rho = \frac{1}{Z}e^{-u}\,dx$ is then the minimizer of $\mathcal{E}(\rho) + T(\rho,\mu)$ when such a ρ exists [San15a].

Definition 4.1.3 ([San15a])**.** The maximal correlation functional $T(\rho,\mu)$ is given by

$$\begin{aligned}T(\rho,\mu) &= \sup\left\{\int x\cdot y\, d\gamma \mid \gamma \in \Pi(\rho,\mu)\right\}\\ &= \frac{1}{2}\int x^2\, d\rho + \frac{1}{2}\int x^2\, d\mu - \frac{1}{2}W_2^2(\rho,\mu)\end{aligned}$$

where $\Pi(\rho,\mu)$ is the set of transport plans, i.e. probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals ρ and μ.

We replace the entropy term of $\mathcal{E}(\rho) + T(\rho,\mu)$ with free entropy, which in the 1-D case is the log energy (up to a constant) [Voi93]:

$$L(\rho) = \iint -\log|s-t|\, d\rho(s)\, d\rho(t).$$

This is justified by the following proposition:

Proposition 4.1.4 ([San15a] p. 14). *For V convex, the minimizer of the functional*

$$\mathcal{E}(\rho) + \int V\rho\, dx = \int \rho \log \rho + V\rho\, dx$$

over ρ probability measures with finite second moment is the density of the Gibbs measure $\rho = \frac{1}{Z}e^{-V}$.

As free entropy in the 1-D case is log energy up to a constant, we recall that the minimizer of the functional

$$L(\rho) + \int V\, d\rho$$

is the free Gibbs measure ν_V if it exists. Thus we see how $\mathcal{E}$ and L play analogous roles for Gibbs measures and free Gibbs measures.

Following this analogy, we define the following functional:

$$\mathcal{F}(\rho) = L(\rho) + T(\rho, \mu). \tag{4.1}$$

Throughout this section, ρ will be assumed to have finite second moment unless otherwise specified.

Following [San15a], we can rewrite $T(\rho, \mu)$ a few ways. First, we use the maximal correlation formulation:

$$T(\rho, \mu) = \sup \left\{ \int x \cdot y\, d\gamma(x, y) \;\middle|\; \gamma \in \Pi(\rho, \mu) \right\}$$

where $\Pi(\rho, \mu) = \{\gamma \in \mathcal{P}(\mathbb{R} \times \mathbb{R}) \mid (\pi_x)_{\#}\gamma = \rho, (\pi_y)_{\#}\gamma = \mu\}$ is the space of measures with marginals ρ and μ. Here $\mathcal{P}(X)$ denotes the space of probability measures on X.

This maximization problem has an equivalent dual problem, a minimization with the same optimal value:

$$T(\rho,\mu) = \min\left\{ \int u\, d\rho + \int u^*\, d\mu \;\middle|\; u \text{ convex, lower semicontinuous} \right\}.$$

This lets us rewrite (4.1) as

$$\mathcal{F}(\rho) = \min\left\{ \underbrace{\iint -\log|s-t|\, d\rho(s)\, d\rho(t) + \int u\, d\rho + \int u^*\, d\mu}_{\mathcal{G}(\rho,u)} \right\} \tag{4.2}$$

minimizing over the set where $\rho \in \mathcal{P}(\mathbb{R})$, $\mathbb{E}_\rho(|x|) < \infty$, and u is convex and lower semicontinuous. Here u^* denotes the Legendre transform

$$u^*(y) = \sup_x (x \cdot y - u(x)).$$

We'll define $\mathcal{G}(\rho,u) = \iint -\log|s-t|\, d\rho(s)\, d\rho(t) + \int u\, d\rho + \int u^*\, d\mu$ and so $\mathcal{F}(\rho) = \min_u \mathcal{G}(\rho,u)$.

By minimizing $\mathcal{G}(\rho,u)$ first in u for each ρ, we can appeal to Santambrogio's analysis of the maximal correlation functional and deduce that $(u')_{\#}\rho = \mu$ [San15a]. Next, for optimal u, minimizing in ρ lets us rely on [dPS95] to see that $\rho = \nu_u$, the free Gibbs measure associated to u. This is explained in further detail in Theorem 4.1.8.

We now adapt the proof from [San15a] to show that $\mathcal{F}$ has a minimizer. First we prove weak lower semicontinuity of the $L(\rho)$ term and show that it's bounded below by an expression involving the first moment of ρ, a bound we will combine with a known bound on $T(\rho,\mu)$. We then prove a kind of convexity of $L(\rho)$ in the Wasserstein space $\mathbb{W}_2$. We use this to deduce the existence and uniqueness of the minimizer of $\mathcal{F}(\rho) = L(\rho) + T(\rho,\mu)$.

Lemma 4.1.5. *Assume that ρ is a probability measure with finite first moment. Then the log energy $L(\rho)$ satisfies the bound $L(\rho) \geq -\sqrt{2\int |s|\, d\rho(s)}$.*

Furthermore, when ρ_n and ρ are probability measures with $\rho_n \rightharpoonup \rho$ weakly and $\int |x|\, d\rho_n \leq C$ for some $C > 0$ and all $n \in \mathbb{N}$, then $L(\rho) \leq \liminf_{n\to\infty} L(\rho_n)$. In short, weak lower *semi-continuity of L if the first moments are uniformly bounded.*

Proof. To bound $L(\rho)$, we split it into three terms with a method inspired by [San15a]. In that paper, Santambrogio splits up the integrand of the entropy term into three parts using a Legendre transform of $x \log x$ for a key inequality.

We need an analogous inequality:

$$-1 + \log\left(\frac{1}{h}\right) - \log|x| \geq -|x|h$$

for any $x \neq 0$ and $y > 0$. This inequality can be derived from the Legendre transform of $-\log x$, the analogous term in our case, but it is more easily derived from an application of $1 + \log a \leq a$ where $a = |x|h$.

With this inequality, we consider the decomposition:

$$\begin{aligned} L(\rho) = &\iint -1 + \log\left(\frac{1}{h}\right) - \log|s-t| + h|s-t|\, d\rho(s)\, d\rho(t) \\ &\iint -\log\left(\frac{1}{h}\right)\, d\rho(s)\, d\rho(t) + \iint 1 - |s-t|h\, d\rho(s)\, d\rho(t) \\ = &\ \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

While this decomposition holds regardless of $h > 0$, we'll select h inspired by the proof in [San15a]. We choose

$$h(s,t) = e^{-\sqrt{|s-t|}}.$$

Observe that term (I) has a positive integrand by the inequality mentioned above. Since the integrand is continuous and bounded below, we have that (I) is lower semi-continuous with respect to weak convergence of measures.

Next, we bound the second term

$$
\begin{aligned}
II &= \iint -\log\left(\frac{1}{e^{-\sqrt{|s-t|}}}\right) d\rho(s)\, d\rho(t) = \iint -\sqrt{|s-t|}\, d\rho(s)\, d\rho(t) \\
&\geq -\sqrt{\iint |s-t|\, d\rho(s)\, d\rho(t)} \\
&\geq -\sqrt{\iint |s|+|t|\, d\rho(s)\, d\rho(t)} = -\sqrt{2\int |s|\, d\rho(s)}
\end{aligned}
$$

where the first inequality follows by Cauchy-Schwarz and the fact that ρ is a probability measure.

Note that $\sqrt{x}/|x| \to 0$ as $x \to \infty$. We'll use this to show that (II) is weakly lower semi-continuous for ρ_n having bounded first moments.

Observe that as $\int |x|\, d\rho_n \leq C$, we have

$$
\left| \int_{[-M,M]^c} -\sqrt{x}\, d\rho_n \right| \leq \frac{\sqrt{M}}{M} \int_{[-M,M]^c} |x|\, d\rho_n \leq \frac{C}{\sqrt{M}}
$$

for any $M > 1$. Fix $\epsilon > 0$. Thus we may choose M so large that $\left|\int_{[-M,M]^c} -\sqrt{x}\, d\rho_n\right| < \epsilon$. We now write

$$
\begin{aligned}
\iint -\sqrt{|s-t|}\, d\rho_n(s)\, d\rho_n(t) &= \iint_{|s-t|>M} -\sqrt{|s-t|}\, d\rho_n(s)\, d\rho_n(t) \\
&\quad + \iint -\sqrt{|s-t|}\chi_{|s-t|\leq M}\, d\rho_n(s)\, d\rho_n(t).
\end{aligned}
$$

The first term is bounded in absolute value by ϵ. As the second term is integration against a lower semi-continuous functions which is bounded from below, it is a lower semi-continuous function with respect to weak convergence of measures.

Combining these facts,

$$
\iint -\sqrt{|s-t|}\, d\rho(s)\, d\rho(t) \leq \liminf_{n\to\infty} \iint -\sqrt{|s-t|}\, d\rho_n(s)\, d\rho_n(t) + 2\epsilon
$$

for any $\epsilon > 0$ and thus we have the desired weak lower semi-continuity of this term.

Finally we write

$$III = \iint 1 - |s-t| e^{-\sqrt{|s-t|}} \, d\rho(s) \, d\rho(t)$$

and observe that the integrand is bounded between 0 and 1, so $0 \leq III \leq 1$. The integrand being continuous and bounded implies that this term is continuous with respect to the weak convergence of measures.

Combining these inequalities, we have

$$L(\rho) = I + II + III \geq 0 - \sqrt{2 \int |s| \, d\rho(s)} + 0$$

as desired.

Furthermore, we have the desired weak lower semi-continuity in each term, and so it holds that $L(\rho) \leq \liminf_{n \to \infty} L(\rho_n)$ when the ρ_n all have bounded first moments. □

We will need another lemma to obtain uniqueness of the minimizer. We'll show that $L(\rho)$ is displacement convex, i.e., convex along geodesics in the Wasserstein space $\mathbb{W}_2$.

Lemma 4.1.6. *The functional $L(\rho)$ is displacement convex. Specifically, if ρ_t is any geodesic connecting ρ_0 to ρ_1 in the Wasserstein space $\mathbb{W}_2$, then $L(\rho_t)$ is convex.*

Furthermore, L is strictly displacement convex for measures which are not translates. That is, if ρ_0 and ρ_1 are not translates of each other, by which we mean one is not the pushforward of the other under a map of the form $x \mapsto x + c$, then $L(\rho_t) < (1-t)L(\rho_0) + tL(\rho_1)$.

Proof. Let ρ_0 and ρ_1 to be two measures with finite second moments (so that they're in $\mathbb{W}_2$). Then let γ be the optimal transport plan between them (see [San15b] or [Vil08] for a thorough introduction to these ideas), and consider $\rho_t = \pi_{t\#}(\gamma)$ where $\pi_t(x, y) = (1-t)x + ty$. Note that ρ_t is the geodesic connecting ρ_0 and ρ_1 in $\mathbb{W}_2$, and all geodesics have this form

[San15b, Chap. 5]. We then observe

$$\begin{aligned}
L(\rho_t) &= \iint -\log|s-r|\,d\rho_t(s)\,d\rho_t(r)\\
&= \iint -\log\left|(1-t)x+ty-(1-t)x'-ty'\right|\,d\gamma(x,y)\,d\gamma(x',y')\\
&= \iint -\log\left|t(y-y')+(1-t)(x-x')\right|\,d\gamma(x,y)\,d\gamma(x',y')
\end{aligned}$$

By the convexity of $-\log$, the integrand is strictly less than

$$-\left((1-t)\log|x-x'|+t\log|y-y'|\right)$$

unless $x-y=x'-y'$. Thus $L(\rho_t)$ is strictly less than $(1-t)L(\rho_0)+tL(\rho_1)$ unless γ is supported on a translate of the diagonal, which can only occur if ρ_0 and ρ_1 are translates of one another. □

We aim to minimize $\mathcal{F}$, but we need to show now that the minimizer will have finite second moment.

Proposition 4.1.7. *Let $u:\mathbb{R}\to\mathbb{R}$ be convex and have a minimum so that $u(x)\geq a|x|+b$ for some $a>0$ and real b.*

Suppose ρ is the free Gibbs measure associated to u and has finite first moment. Then ρ is compactly supported and absolutely continuous with respect to Lebesgue measure. Furthermore, $2\pi H\rho=u'$ on the support of ρ.

Proof. By [dPS95, Remark 3] and noting that the function u satisfies their condition (1.2), Theorem 1 of [dPS95] guarantees that ρ is absolutely continuous with respect to Lebesgue measure and that the support of ρ is contained in the set of points such that

$$h(x)=\int -\log|x-y|\,d\rho(y)+u(x)$$

is minimal. We can also see this by taking a first variation of the functional $\iint -\log|s-$

$t|\,d\rho(s)\,d\rho(t) + \int u(t)\,d\rho(t)$ and considering the optimality conditions. Theorem 1 of that paper also guarantees that $2\pi H\rho = u'$ on the support of ρ, noting that $\beta = 2$ for our case in [dPS95, Eqn. 1.17], although using absolute continuity we could also get this by considering optimality conditions for the functional defining ν_u and differentiating under the integral.

Since $U(x) \geq a|x| + b$, $-\log$ is non-increasing, and $z \mapsto \log(1+z)$ is subadditive on the positive reals, we have

$$\begin{aligned}
h(x) &\geq \int -\log|x-y|\,d\rho(y) + a|x| + b \\
&\geq \int -\log(|x| + |y| + 1)\,d\rho(y) + a|x| + b \\
&\geq \int -\log(|x| + 1)\,d\rho(y) + \int -\log(|y| + 1)\,d\rho(y) + a|x| + b \\
&\geq -\log(|x| + 1) + \int -\log(|y| + 1)\,d\rho(y) + a|x| + b.
\end{aligned}$$

Note that the finite first moment of ρ implies $\int -\log(|y|+1)\,d\rho(y) > -\infty$, since log has sublinear growth at ∞. Thus $h(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Note that $h(x)$ isn't constantly ∞ as its integral gives the functional minimized by ρ. Therefore the set where h is its minimum value is compact, so $\operatorname{supp}(\rho)$ is compact. □

We now show the existence of a minimizer of $\mathcal{F}$ and prove the main theorem of this section.

Theorem 4.1.8. *Let $\mu \neq \delta_0$ be a probability measure with finite second moment. The functional $\mathcal{F}(\rho) = L(\rho) + T(\rho, \mu)$ has a minimizer in $\mathcal{P}_2$, the space of probability measures with finite second moment, which is unique up to translation, i.e., unique up to a pushforward by the map $x \mapsto x + c$.*

The minimizer $\hat{\rho}$ is also absolutely continuous with respect to Lebesgue measure, has compact support, and satisfies $2\pi H\hat{\rho} = u'$ on its support.

Furthermore, the following are equivalent:

1. *$\hat{\rho}$ is the unique centered minimizer of $\mathcal{F}(\rho)$*

2. $\hat{\rho}$ *satisfies* $\hat{\rho} = \nu_u$ *for some convex* u *and* $(u')_\# \hat{\rho} = \mu$.

Proof. First we'll show that $\mathcal{F}$ has a minimizer unique up to translation.

Let ρ_n be a minimizing sequence of probability measures with finite first moment. Note that without loss of generality we may assume that the ρ_n are centered, as $\mathcal{F}$ is invariant under translation.

By [San15a], we have that $T(\rho_n, \mu) \geq c \int |x| \, d\rho_n(x)$ for some $c > 0$ depending only on μ, since μ is not supported on a hyperplane, which here means $\mu \neq \delta_0$. Applying Lemma 4.1.5, we have $L(\rho_n) \geq -\sqrt{2 \int |x| \, d\rho_n(x)}$. Combining these yields a uniform bound on the first moment of the ρ_n, which implies the sequence is tight. By passing to a subsequence, we can assume that $\rho_n \rightharpoonup \hat{\rho}$ weakly for some probability measure $\hat{\rho}$. Note also that $\hat{\rho} \in \mathcal{P}_1$, the space of probability measures with finite first moment. This is because integration against $|x|$, a lower semi-continuous function bounded from below, is a weakly lower semi-continuous functional.

By weak convergence of $\rho_n \rightharpoonup \hat{\rho}$ and a uniform bound on the first moments, Lemma 4.1.5 gives us that $L(\hat{\rho}) \leq \liminf_{n \to \infty} L(\rho_n)$. As we know that $T(\rho, \mu)$ is weakly lower semi-continuous in ρ by [San15a], we have that $\hat{\rho}$ is a minimizer of $\mathcal{F}$.

We know that $\hat{\rho}$ has finite first moment, but we need to show now that it has finite second moment as well. As part of showing this, we'll see that it must satisfy $\hat{\rho} = \nu_u$ for some convex u with $(u')_\# \hat{\rho} = \mu$, so we'll have (1) implies (2). Afterwards we will show uniqueness of the minimizer of $\mathcal{F}$ and then prove (2) implies (1).

Take u to be a convex lower semi-continuous function which realizes the dual formulation of $T(\hat{\rho}, \mu)$, that is, $T(\hat{\rho}, \mu) = \int u \, d\hat{\rho} + \int u^* \, d\mu$. Additionally, we know that $(u')_\# \hat{\rho} = \mu$ [San15a].

Simplifying $\mathcal{F}$ using u now yields

$$\mathcal{F}(\hat{\rho}) = \iint -\log|s - t| \, d\hat{\rho}(s) \, d\hat{\rho}(t) + \int u \, d\hat{\rho} + \int u^* \, d\mu$$

We consider a new functional

$$\mathcal{G}(\rho) = \iint -\log|s-t|\, d\rho(s)\, d\rho(t) + \int u\, d\rho + \int u^*\, d\mu$$

and observe that since the first term is $L(\rho)$ latter two terms are larger than $T(\rho, \mu)$, we must have $\mathcal{G}(\rho) \geq \mathcal{F}(\hat{\rho})$. Therefore $\hat{\rho}$ minimizes $\mathcal{G}$.

However, the final term does not depend on the measure, so $\mathcal{K}(\rho) = L(\rho) + \int u\, d\rho$ is still minimized at $\hat{\rho}$. Thus $\hat{\rho} = \nu_u$ by definition of ν_u. And as $\hat{\rho}$ has finite first moment, Proposition 4.1.7 implies that $\hat{\rho}$ has compact support, and thus all its moments are finite and in particular $\hat{\rho} \in \mathcal{P}_2$. We also get that $2\pi H\hat{\rho} = u'$ on the support of $\hat{\rho}$.

Thus we now have that $\mathcal{F}$ has a minimizer with finite second moment, and (1) implies (2). Let's now show that the minimizer to $\mathcal{F}$ is unique.

To show uniqueness up to translation, and thus uniqueness of a centered minimizer, we invoke the displacement convexity of both L using Lemma 4.1.6 and T using [San15a, Prop. 3.3]. Combining these will give displacement convexity of $\mathcal{F}$. Note that displacement convexity of T in [San15a, Prop. 3.3] is shown between two measures which are absolutely continuous with respect to Lebesgue measure, but the result holds just as well with no modifications when the initial measure is non-atomic and thus optimal transport maps from it still exist in the space $\mathbb{W}_2$.

Furthermore, by Lemma 4.1.6, we have strict displacement convexity of L except between translates. In particular, if ρ_0 and ρ_1 are minimizers and not translates of each other, then on the geodesic between them, there is some ρ_t with a strictly smaller value of L and a value of T no larger than that of ρ_0 or ρ_1. This is a contradiction, so any two minimizers of $\mathcal{F}$ must be translates of each other.

Finally, let's show (2) implies (1). Let $\hat{\rho}$ satisfy $\hat{\rho} = \nu_u$ with u convex and $(u')_{\#}\hat{\rho} = \mu$. We intend to show that $\hat{\rho}$ is a minimizer of $\mathcal{F}(\rho)$, where we note that uniqueness up to translation is already guaranteed. Also by the functional that defines ν_u not being $+\infty$, we know that $\hat{\rho}$ is non-atomic.

With $\hat{\rho}$ as above, let ρ be another probability measure with finite second moment, and f be the transport map between $\hat{\rho}$ and ρ and let $\rho_t = (f_t)_{\#}\hat{\rho}$ where $f_t = (1-t)I + tf$.

The map $t \mapsto \mathcal{F}(\rho_t)$ is convex, so it is enough to show that its derivative at zero is non-negative. We will compute the derivative of the log-energy term and borrow Santambrogio's calculation for T, which we observe does not require absolute continuity but only the existence of an optimal transport map [San15a, Prop. 3.3]. We calculate

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} L(\rho_t) &= \frac{d}{dt}\Big|_{t=0} \iint -\log|x-y|\, d\rho_t(x)\, d\rho_t(y) \\
&= \iint -\frac{d}{dt}\Big|_{t=0} \log|tf(x) + x - tx - tf(y) - y + ty|\, d\hat{\rho}(x)\, d\hat{\rho}(y) \\
&= -\iint \frac{f(x) - x - (f(y) - y)}{x-y}\, d\hat{\rho}(x)\, d\hat{\rho}(y) \\
&= -\iint 2\frac{f(x)}{x-y} - 1\, d\hat{\rho}(x)\, d\hat{\rho}(y) \\
&= 1 - 2\pi \int f(x) H\hat{\rho}(x)\, d\hat{\rho}(x) \\
&= 1 - \int f(x) u'(x)\, d\hat{\rho}(x).
\end{aligned}$$

The last line follows by recalling $2\pi H\hat{\rho} = u'$ on $\mathrm{supp}(\hat{\rho})$.

Note that for the T term, we have that $\frac{d}{dt}\big|_{t=0} T(\rho_t, \mu)$ is bounded below by $\int (f(x) - x)u'(x)\, d\hat{\rho}(x)$ [San15a, Prop. 3.3]. Thus combining these two terms, we find that

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}(\rho_t) \geq 1 - \int x u'(x)\, d\hat{\rho}(x) \geq 0$$

where the final inequality follows immediately from Schwinger-Dyson for $\hat{\rho} = \nu_u$ (in particular, $\tau(xu') = \tau \otimes \tau(1)$, which is an application of $2\pi H\hat{\rho} = u'$ on the support of $\hat{\rho}$). Thus, using the convexity of the functional and noting that the above holds for any ρ, we see that $\hat{\rho}$ minimizes $\mathcal{F}$. □

We include some examples of free moment measures.

The semicircular distribution μ equals $\nu_{\frac{1}{2}x^2}$, so μ is a free moment measure with potential $u(x) = \frac{1}{2}x^2$, just as the Gaussian is a (classical) moment measure with quadratic potential. This is not surprising, as the semicircle law plays an analogous role in free probability to the Gaussian law in classical probability.

The next simplest example is $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ which has the potential $u(x) = \frac{1}{2}|x|$, since ν_u is necessarily centered when u is even, and thus $(u')_{\#}\nu_u = \mu$. In this particular case, the corresponding measure is $\nu_u(x) = \frac{1}{\pi}\log\left|\frac{1+\sqrt{1-x^2}}{x}\right|$ (supported on $[-1, 1]$).

Given the potential $x^4/4$, we calculate the free Gibbs measure to be

$$\nu_u(x) = \frac{r^3}{4\pi}(2x^2+1)\sqrt{1-\left(\frac{x}{r}\right)^2}dx$$

where

$$r = \frac{2}{\sqrt[4]{3}}$$

is the radius of the support. When we then push this forward by $u' = x^3$, we get

$$\mu(x) = \frac{3r^3}{4\pi}(2+x^{-2/3})\sqrt{1-\frac{x^{2/3}}{r^2}}dx$$

Thus μ is a free moment measure with potential $x^4/4$.

Note that translations $u(x+c)+d$ of a potential yield the same free moment measure as u does.

Suppose μ has potential u such that $(u')_{\#}\nu_u = \mu$. Let's consider $u(x/c)$ for $c > 0$. We'd like to find the corresponding free moment measure. First, let's find the free Gibbs measure.

If $f(x)$ is the density for an optimizer for $\mathcal{F}_u(\rho)$, then $cf(cx)$ is the density for the optimizer of $\mathcal{F}_{u(cx)}(\rho)$, and vice-versa. To see this we change variables

$$\begin{aligned}\iint -\log|s-t|cf(cs)cf(ct)\,ds\,dt + \int u(ct)cf(ct)\,dt \\ = \iint -(\log|x-y| - \log c)f(x)f(y)\,dx\,dy + \int u(x)f(x)\,dx \\ = \mathcal{F}_u(f(x)\,dx) + \text{constant}\end{aligned}$$

and note that the constant $\log c$ is irrelevant to maximization or minimization. This tells us that if u is replaced with $u(cx)$, the corresponding free Gibbs measure $\nu_u = f(x)\,dx$ is replaced with $cf(cx)\,dx$.

As a consequence, for $v(x) = u(cx)$, we have that for any g

$$\begin{aligned}\int g(x)\,d\left(v'_{\#}\nu_v\right) &= \int g(cu'(cx))cf(cx)\,dx \\ &= \int g(cu'(t))\,d\nu_u(t)\end{aligned}$$

and so $(v')_{\#}\nu_v = c_{\#}\left((u')_{\#}\nu_u\right)$. Thus the new measure is a dilated copy of the old measure, scaled by a factor of c.

4.2 Multivariable Case

Instead of generalizing the variational argument, we will be applying the methods of Shlyakhtenko and Guionnet in [GS12]. These methods will allow us to deal with free Gibbs laws which are near the free semicircular law (which is the free Gibbs law for the potential $\frac{1}{2}(X_1^2 + ... + X_n^2)$). In order to state out main theorem, we recall the norms $||\cdot||_A$ defined on nc power series as

$$\Big|\Big|\sum_I a_I X_I\Big|\Big|_A = \sum_I |a_I| A^{|I|}$$

where I ranges over multi-indices, and $|I|$ is the length of I (see [GMS07]).

Theorem 4.2.1. *There exist a C and an ϵ such that, if $W(X_1,...,X_n)$ is a self adjoint nc power series containing only terms of even degree, and $||W||_C < \epsilon$, then there is a corresponding power series $V(Y_1,...,Y_n)$ such that, when Y has the free Gibbs law associated to $\frac{1}{2}|Y|^2 + V$, then $Y + \mathcal{D}_Y V(Y)$ has the free Gibbs law associated to $\frac{1}{2}|X|^2 + W$.*

This is precisely the condition that the free Gibbs law for $\frac{1}{2}|Y|^2 + V(Y)$ pushes forward to that of $\frac{1}{2}X^2 + W(X)$ along $\mathcal{D}(\frac{1}{2}|Y|^2 + V(Y))$.

In fact, we must take this opportunity to elaborate on the existence of free Gibbs laws. In this perturbative regime, we cannot rely on convexity to ensure the existence of solutions to Schwinger-Dyson, no matter how small the perturbation. Indeed, consider the single variable case and $W = \epsilon X^3$. The functional to minimize in τ is $\chi(\tau) + \tau(X^2 + W)$. The value can be reduced by taking any measure which has finite free entropy and translating it left, reducing $\int W$. Since there is no limit to how far we can translate it, and since this effect will eventually overpower the increase in $\int X^2$, we find that there can be no minimum. Instead, we must artificially institute a cutoff, requiring that the norm of our random variable is less than $T > 2$. Specifically, we invoke a slight modification of ([GMS06]):

Proposition 4.2.2. *For each cutoff $T > 2$, we have that there is an $R > 0$ such that $||W||_T < R$ implies that there exists a unique solution, τ, to the bounded Schwinger-Dyson equation*

$$\tau(P \cdot (X + \mathcal{D}W(X))) = \tau \otimes \tau \times Tr(JP)$$
$$|\tau(X_{i_1},...,X_{i_k})| \leq T^k$$

We will split the proof of Theorem 4.2.1 into two main steps—deriving a differential equation for V in which all terms are cyclic derivatives, and then "integrating" that equation to find a map to which we can apply the contraction mapping theorem to find a solution. Following the proof, we will compare the restrictions in this result to those in the commutative case and discuss potential directions for extension.

The first step is to rephrase the Schwinger-Dyson equation from an integral equation to a differential equation. To do so, it will be useful to define inner products associated to τ:

$$\langle a, b\rangle_M = \tau(a^* b)$$

$$\langle a\otimes b, c\otimes d\rangle_{M\otimes M^{op}} = \tau(a^* c)\tau(b^* d) = \tau\otimes\tau((a\otimes b)^* c\otimes d)$$

$$\langle A, B\rangle_{M_n(M\otimes M^{op})} = \tau\otimes\tau\,(Tr(A^* B))$$

We will omit the subscripts if the ambient space can be inferred. Thus the Schwinger-Dyson equation can be written as

$$\langle \mathcal{D}U, P\rangle = \langle 1, JP\rangle,\ i.e.$$

$$\langle \mathcal{D}U, P\rangle = \langle J^*(1), P\rangle,\ i.e.$$

$$\Rightarrow \mathcal{D}U = J^*(1)$$

We will also need some additional operators on nc power series, $\mathcal{S}$, $\mathcal{N}$, Σ (the inverse of $\mathcal{N}$), and Π. These are linear operators on power series in Y, which act on monomials as follows. The cyclic symmetrization operator, S, is given by

$$\mathcal{S}(x_{i_1}...x_{i_n}) = \frac{1}{n}\sum_{j=1}^{n} x_{i_j}...x_{i_n}x_{i_1}x_{i_{j-1}},$$

on constant terms it acts as the identity. The number operator N is given by

$$\mathcal{N}(x_{i_1}...x_{i_n}) = n x_{i_1}...x_{i_n},$$

Finally,

$$\Sigma(x_{i_1}...x_{i_n}) = \frac{x_{i_1}...x_{i_n}}{n},$$

is defined on power series with no constant term and is the inverse of $\mathcal{N}$ on that space. Π is the projection onto power series with no constant term.

With these operators defined, we may state the following lemma.

Lemma 4.2.3. *V satisfies the conclusion of Theorem 3.1 if and only if*

$$\mathcal{S}\Pi\Big[W(Y+\mathcal{D}V)+(\mathcal{N}-1)V+\frac{|\mathcal{D}V|^2}{2}-(1\otimes\tau+\tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\Big]=0$$

Proof. Our aim is to express the Schwinger-Dyson equation of the pushforward as a single cyclic derivative. For the purpose of keeping our derivatives clear, we will define the variable $X=Y+\mathcal{D}V(Y)$. We then have that

$$Y+\mathcal{D}_YV(Y)=J_Y^*(1) \tag{4.3}$$

and want to understand what condition on V ensures Schwinger-Dyson for X, i.e.

$$X+\mathcal{D}_XW(X)=J_X^*(1). \tag{4.4}$$

Substituting the definition of X into (4.4) gives

$$Y+\mathcal{D}_YV(Y)+\mathcal{D}_XW(Y+\mathcal{D}_YV(Y))=J_X^*(1),$$

to which we apply the chain rule found in [GS12, Lemma 3.1],

$$J_X^*(1)=J_Y^*\left(\frac{1}{1+J_Y\mathcal{D}_YV(Y)}\right),$$

to arrive at the equation

$$Y+\mathcal{D}_YV(Y)+\mathcal{D}_XW(Y+\mathcal{D}_YV(Y))=J_Y^*\left(\frac{1}{1+J_Y\mathcal{D}_YV(Y)}\right). \tag{4.5}$$

Similarly, we apply the chain rule for the cyclic derivative:

$$\mathcal{D}_Y = (1 + J_Y \mathcal{D}_Y V)\mathcal{D}_X,$$

obtaining

$$Y + \mathcal{D}_Y V(Y) + (1 + J_Y \mathcal{D}_Y V)^{-1} \mathcal{D}_Y W(Y + \mathcal{D}_Y V(Y)) = J_Y^* \left(\frac{1}{1 + J_Y \mathcal{D}_Y V(Y)} \right). \tag{4.6}$$

In this equation, 1 is the identity matrix in $M_n(M \otimes M^{op})$, the $n \times n$ matrix with $1 \otimes 1$ in all its diagonal entries. We know that $1 + J_Y \mathcal{D}_Y V$ is invertible in this space provided that $J_Y \mathcal{D}_Y V$ has norm less than 1. In our next step, we will be restricting V to a smaller set still, so invertibility is guaranteed.

As X has been removed from our equation and all derivatives are with respect to Y now, we will assume this going forwards and neglect the subscripts. We expand the right hand side of (4.5) as

$$\begin{aligned} J^* \left(\frac{1}{1 + J\mathcal{D}V} \right) &= J^*(1) - J^* \left(\frac{J\mathcal{D}V}{1 + J\mathcal{D}V} \right) \\ &= Y + \mathcal{D}V - J^* \left(\frac{J\mathcal{D}V}{1 + J\mathcal{D}V} \right), \end{aligned}$$

Performing the resulting cancellation and multiplying (4.6) by $(1 + J\mathcal{D}V)$ gives

$$\mathcal{D}W(Y + \mathcal{D}V) = -(1 + J\mathcal{D}V) J^* \left(\frac{J\mathcal{D}V}{1 + J\mathcal{D}V} \right) \tag{4.7}$$

We will expand the right hand side of (4.7) and then simplify with the following identity from [GS12, Lemma 3.4]:

$$\frac{1}{m+1} \mathcal{D} \left[(\tau \otimes 1 + 1 \otimes \tau) Tr(Jf^{m+1}) \right] = -J^*(Jf^{m+1}) + JfJ^*(Jf^m). \tag{4.8}$$

Expanding the right hand side of (4.7) yields

$$\begin{aligned}
&\sum_{n=1}^{\infty}(-1)^n J^*(J\mathcal{D}V^n) + (-1)^n J\mathcal{D}V J^*(J\mathcal{D}V^n) \\
&\quad = -J^*(J\mathcal{D}V) + \sum_{n=1}^{\infty}(-1)^n (J\mathcal{D}V J^*(J\mathcal{D}V^n) - J^*(J\mathcal{D}V^{n+1})) \\
&\quad = -J\mathcal{D}V J^*(1) + \mathcal{D}\left[(1\otimes\tau + \tau\otimes 1)Tr\left(J\mathcal{D}V + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}J\mathcal{D}V^{n+1}\right)\right] \\
&\quad = -J\mathcal{D}V J^*(1) + \mathcal{D}\left[(1\otimes\tau + \tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\right] \\
&\quad = -J\mathcal{D}V\cdot Y - J\mathcal{D}V\cdot\mathcal{D}V + \mathcal{D}\left[(1\otimes\tau + \tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\right]
\end{aligned}$$

We're left with

$$\mathcal{D}(W(Y+\mathcal{D}V)) = -J\mathcal{D}V\cdot Y - J\mathcal{D}V\cdot\mathcal{D}V + \mathcal{D}\left[(1\otimes\tau + \tau\otimes 1)Tr\log(1+J\mathcal{D}V)\right] \quad (4.9)$$

which nearly expresses the equation as a total (cyclic) derivative. All that remains is writing the first two terms of the right hand side of (4.7) above as cyclic derivatives. Analyzing the remaining two terms of (4.7), we make use of the operators defined earlier, noticing

$$Jg\cdot Y = \mathcal{N}g$$

for any g. Thus, when $g = \mathcal{D}V$, we get

$$J\mathcal{D}V\cdot Y = \mathcal{N}\mathcal{D}V = \mathcal{D}(\mathcal{N}-1)V$$

We can also see that

$$J\mathcal{D}V\cdot\mathcal{D}V = \mathcal{D}\left(\frac{\mathcal{D}_1V^2 + \mathcal{D}_2V^2 + ... + \mathcal{D}_nV^2}{2}\right) = \mathcal{D}\left(\frac{|\mathcal{D}V|^2}{2}\right).$$

So equation (4.7) can be rewritten as

$$\mathcal{D}\left[W(Y+\mathcal{D}V)+(\mathcal{N}-1)V+\frac{|\mathcal{D}V|^2}{2}-(1\otimes\tau+\tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\right]=0 \quad (4.10)$$

Since $\mathcal{D}$ only sees the cyclically symmetric part of power series, and does not see constants, this is equivalent to the desired equation

$$\mathcal{S}\Pi\Big[W(Y+\mathcal{D}V)+(\mathcal{N}-1)V+\frac{|\mathcal{D}V|^2}{2}-(1\otimes\tau+\tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\Big]=0$$

Thus concludes this lemma as well as the first step in the proof of Theorem 4.2.1, deriving a differential equation for V in which all terms are cyclic derivatives. □

We proceed to the second step in the proof of Theorem 4.2.1, where we "integrate" the above equation to find a map to which we can apply the contraction mapping theorem in order to find a solution.

We rephrase the differential equation in Lemma 4.2.3:

$$\mathcal{S}\Pi\mathcal{N}V=\mathcal{S}\Pi\Big[-W(Y+\mathcal{D}V)+V-\frac{|\mathcal{D}V|^2}{2}+(1\otimes\tau+\tau\otimes 1)Tr(\log(1+J\mathcal{D}V))\Big]. \quad (4.11)$$

It will be more useful to solve for $\tilde{V}=\mathcal{S}\Pi\mathcal{N}V$, which must satisfy

$$\tilde{V}=\mathcal{S}\Pi\left[-W(Y+\mathcal{D}\Sigma\tilde{V})+\Sigma\tilde{V}-\frac{|\mathcal{D}\Sigma\tilde{V}|^2}{2}+(1\otimes\tau+\tau\otimes 1)Tr(\log(1+J\mathcal{D}\Sigma\tilde{V}))\right] \quad (4.12)$$

Whenever necessary, we will denote the right hand side by $F(\Sigma\tilde{V})$. We will show that there is a set on which $F(\Sigma\cdot)$ is a contraction. Along the way, we must prove two lemmas.

Lemma 4.2.4. *$F(\Sigma\cdot)$ preserves evenness of power series. In other words, if U has only terms of even degree, then $F(\Sigma U)$ also has only terms of even degree.*

In turn, proving this requires an easy proposition:

Proposition 4.2.5. *If U is a potential which contains only even terms, then $\tau_U(P) = 0$ for any polynomial P which contains only odd terms.*

Proof. This is a corollary of uniqueness of free Gibbs measures, [Gui06]. In particular, if X has free Gibbs law τ_U, then $Y = -X$ also satisfies

$$\tau(P(Y) \cdot \mathcal{D}_Y U(Y)) = -\tau(P(-X) \cdot \mathcal{D}_X U(X))$$

$$= -\tau \otimes \tau(Tr(J_X P(-X))) = \tau \otimes \tau(Tr(J_Y P(Y)))$$

So by uniqueness, $-X$ has the same law as X, yet $\tau(P(X)) = -\tau(P(-X))$ for any odd polynomial, so this must be zero. □

Proof of Lemma 3.4. We must check that each term preserves evenness. The term $W(Y + \mathcal{D}\Sigma V)$ certainly does, since all terms in W are even, and all terms in $Y + \mathcal{D}\Sigma V$ are odd. The term ΣV is most immediate of all, and every term in $|\mathcal{D}\Sigma V|^2$ is a product of two odd factors. To see that the log term also preserves this, we expand it into its Taylor series

$$\sum_n \frac{(-1)^n}{n}(1 \otimes \tau + \tau \otimes 1) Tr((J\mathcal{D}\Sigma V)^n)$$

Considering now a fixed n, we see that each term in $J\mathcal{D}\Sigma V$ is of the form $a \otimes b$ where the degrees of a and b sum to an even number. The same is thus true of all powers. If $(1 \otimes \tau)$ or $(\tau \otimes 1)$ were to produce a term with odd degree, it would be multiplied by $\tau(a)$ where a also had odd degree, and so, by the proposition, would be zero. □

We note that the $\log(1 + \mathcal{J}\mathcal{D}\Sigma\tilde{V})$ and $W(Y + \mathcal{D}\Sigma V)$ terms produce the requirement that W be even. If V contains any terms of odd degree, both of these terms can produce linear (degree one) terms in $F(\Sigma V)$ on which ΣV is not strictly contractive.

Next, we introduce the sets that we will consider as domains for $F(\Sigma\cdot)$: $E \cap B_{A,R}$ where E is the space of nc power series with only even, positive degree terms, and $B_{A,R}$ is the ball of $||\cdot||_A$ radius R. The previous lemma shows that $F(\Sigma\cdot)$ preserves E. We also need

Lemma 4.2.6. *If $A \geq 1$, $F(\Sigma\cdot)$ has a Lipschitz constant on $B_{A,R} \cap E$ bounded above by*

$$\frac{1}{2} + \left|\left|\sum_i \partial_i W\right|\right|_{B\otimes B} + R + \frac{4R}{A^2 - 2R}$$

where

$$||\sum_{I,J} a_I b_J X_I \otimes X_J||_{A\otimes B} = \sum_{I,J} |a_I||b_j| A^{|I|} B^{|I|}$$

and, in the above bound, $B = A + R$.

Moreover

$$||F(\Sigma\cdot)||_A \leq ||W||_B + ||V||_A \left(\tfrac{1}{2} + R + \frac{4R}{A^2 - 2R}\right)$$

Proof. Most of this proof can be reduced to an appeal to Cor. 3.12 in [GS09]. However, two terms deserve a comment.

Unlike [GS09], our F contains a ΣV:

$$||\Sigma V - \Sigma U||_A \leq \frac{1}{2}||V - U||_A$$

Which follows immediately from the fact that all terms in U and V are of order 2 or greater.

Additionally, our bound for the log term is different from that in [GS09] so we briefly comment on it's proof. We begin by Taylor expanding the log term as

$$\sum_n \frac{(-1)^n}{n}(1 \otimes \tau + \tau \otimes 1) Tr\left((J\mathcal{D}\Sigma V)^n - (J\mathcal{D}\Sigma U)^n\right),$$

a notationally tedious, but otherwise straightforward calculation then shows that this is bounded in norm by

$$\sum_n \frac{2^{n+1}R^n}{A^{2n}}||V - U||_A = ||V - U||_A \frac{4R}{A^2 - 2R},$$

see [GS12, Lemma 3.8] for more detail.

We can then obtain the desired bounds almost immediately from the Lipschitz constants, with the only exception being the W term, for which we use the bound

$$||W(Y + \mathcal{D}\Sigma V)||_A \leq ||W||_B$$

which follows from $\max(||Y_i + \mathcal{D}_i \Sigma V||_A) \leq B$. $\square$

We are now equipped to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. We fix a cutoff $3 \geq T > 2$. We then choose $A = 3$ and an $R < 1/4$ so that $||V||_A < R$ implies the existence of a unique free Gibbs law with support bounded by T. Then we find that the Lipschitz constant of $F(\Sigma \cdot)$ is bounded by:

$$\left|\left|\sum_i \partial_i W\right|\right|_{\frac{13}{4} \otimes \frac{13}{4}} + \frac{1}{4} + \frac{1}{9 - 1/2} \leq ||W||_{\frac{17}{4}} + \frac{59}{68}$$

Where we have used that

$$\left|\left|\sum_i \partial_i W\right|\right|_{A \otimes A} \leq \sum_I |W_I||I|A^{|I|-1} \leq \sum_I |W_I|(1 + A)^{|I|} = ||W||_{A+1}$$

Moreover, $||V||_A < R$ implies that

$$\begin{aligned} ||F(\Sigma V)||_A &\leq ||W||_{\frac{13}{4}} + R\left(\frac{1}{2} + R + \frac{4R}{9 - 2R}\right) \\ &\leq ||W||_{\frac{13}{4}} + \frac{59}{68}R \end{aligned}$$

Since

$$||W||_{\frac{13}{4}} \leq ||W||_{\frac{17}{4}}$$

We find that if $||W||_{\frac{17}{4}} < \frac{9}{68}$ then $F(\Sigma \cdot)$ will be a contraction, and if $||W||_{\frac{17}{4}} < \frac{9}{68}R$, then we can also be assured that it will map $E \cap B_{A,R}$ into itself, so we have a fixed point V. We then find that $\tilde{V} = \Sigma V$ satisfies the conclusion of the theorem, since Σ can only decrease $|| \cdot ||_A$. Thus, we take $C = \frac{17}{4}$ and $\epsilon = \frac{9}{68} R$.

□

Chapter 5

Closing Thoughts and Open Questions

The ideas considered herein produce several tempting open questions, which we shall rank in order of how pressing they seem to the author.

1. Foremost are the questions of convergence from section 3, as well as the determining whether there is a similar diagrammatic formula that produces q-Gaussians from free semicircular variables. These two together could solve the question of whether the finitely generated q-Gaussians are isomorphic to the free group factors, and has been the guiding thrust of this work.

2. Relatedly, there is the question of the optimal transport map from [GS12]. In particular, can a purely diagrammatic formula for this map be found (and is it equal to the formula found by our method)? The program here is quite clear: Find a diagrammatic formula for the operators N, Σ, and S from [GS12]. With this, obtain a formula for the free Gibbs potential W, and then use the transport equation in the same to find f. In particular, one hopes to be able to use the contraction mapping method to get a diagrammatic formula. It is primarily in the second step that this effort has so far been stymied. As with many of these diagrammatic calculations, the problem proceeds almost entirely by pattern recognition, with little in the way of formal methods for calculation.

3. To help solve the issues in the last two items, we think it is worth investigating semi-knots as their own objects. The most obvious question is that of equivalence and classifying invariants. By further restricting the axioms we can find easy invariants, with all three axioms, it is not so clear. The authors also suspect that there are other interesting applications of semi-knots, especially to optimal transport in quantum field theory, and defining a notion of q-Brownian motion.

4. Can the tracial formula in the last section for the conjugate system be made more comprehensible? If so, then, by a similar program to point 2, it might be possible to construct a tracial formula for, e.g. the Wasserstein distance between an arbitrary (finitely generated) von Neumann algebra and $L(\mathbb{F}_n)$. Optimistically, casting some of the familiar free probabilistic objects in this language might make it easier to obtain estimates for their value. There is a relationship between trace polynomials and diagrams that makes it easy to identify what trace polynomial a fully paired diagram can come from (relating to the connected components of the diagram) so solutions to the diagrammatic questions might help inform solutions to this question as well.

5. Can moment measures be framed in this diagrammatic language? The structure of the moment measure result is very similar to that of [GS12], so if the former argument can be adapted, then this should also be possible. There is also the question of whether the moment measure results can be expanded to an approximating result for large random matrices. In particular, we are curious about whether the classical moment measure result applied to large random matrices will approximate the free moment measure potential.

Appendix A

Computing Free Gibbs Laws for Single Variables

This section is intended as a quick overview to methods for solving the equation $2\pi H(\rho) = u'$ among measures on $\mathbb{R}$. We consider the Cauchy transform

$$G_\rho(z) = \int \frac{1}{z-t} d\rho$$

which in particular satisfies

$$\lim_{y \downarrow 0} G_\rho(x+iy) = \pi(H(\rho) - i\rho)(x)$$

We would like to find G using the fact that its real part is known, but we only know this real part on the support of ρ (which is also, a priori, unknown). This is remedied by noticing that G is an analytic function on the Riemann sphere minus the support of ρ. With convex potentials, the support of ρ is connected, so we may assume that G is an analytic function away from some compact subinterval of $\mathbb{R}$. For the sake of brevity, we will assume that the potential is even, so the measure is supported on a symmetric interval $[-r, r]$. We then try to find the Cauchy transform as

$$G(z) = F(R(z))$$

for some F, holomorphic on the interior of the disk, and

$$R(z) = \frac{\sqrt{z^2 - r^2} - z}{r}$$

the Riemann mapping from $S^2 \setminus [-r, r]$ to the disk. We make note of the inverse of this map:

$$S(w) = -\frac{r(1 + w^2)}{2w} = \frac{r\left(\frac{w+1}{1-w}\right)^2 + r}{1 - \left(\frac{w+1}{1-w}\right)^2}$$

The defining equation of ρ now gives that

$$\lim_{z \to e^{i\theta}} F(z) = \tfrac{1}{2}u'(-r\cos(\theta)) - i\pi \mathrm{sgn}(\sin(\theta))\rho(-r\cos(\theta))$$

In particular, the real part is enough to compute the Taylor series for F; if $F = \sum a_n z^n$, then

$$a_n = \frac{1}{\pi} \int u'(-rcos(\theta))e^{-in\theta} d\theta$$

What remains is to fix r; we consider a contour γ_ϵ which traces the rectangle with sides $re(z) = \pm r$ and $im(z) = \pm\epsilon$, oriented clockwise. Since we know the limit of G as z approaches the axis and that ρ is a probability measure, we can see on the one hand that

$$\int_{\gamma_\epsilon} G(z)dz = -2\pi i$$

but on the other that

$$\int_{\gamma_\epsilon} G(z)dz = \int_{\gamma_\epsilon} F(R(z))dz \to \int_{S^1} F(w)S'(w)dw$$

$$= \frac{r}{2} \int_{S^1} F(w) \left(\frac{1}{w^2} - 2w \right) = r\pi i a_1$$

Whence,

$$ra_1 = -2$$

We illustrate the process with the potential $u = x^4/4$ from section 2.5. We have that

$$\mathrm{Re}(F(e^{i\theta})) = -\tfrac{r^3}{2} \cos^3(\theta) = -\tfrac{r^3}{8}(\cos(3\theta) + 3\cos(\theta))$$

so $a_1 = -\frac{3}{8}r^3$ and $a_3 = -\frac{1}{8}r^3$, and all other Taylor coefficients are zero. We then fix r using the equation

$$ra_1 = -2 \Rightarrow r^4 = \tfrac{16}{3}$$

Then we see that

$$-\pi * \nu(-r\cos(\theta)) = \mathrm{Im}(F(e^{i\theta})) = -\tfrac{r^3}{8}(\sin(3\theta) + 3\sin(\theta))$$

so

$$\nu(x) = \tfrac{r^3}{8\pi} \left(4\frac{x^2}{r^2} + 2 \right) \sqrt{1 - \frac{x^2}{r^2}}.$$

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