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Cohomological Kernels for Cyclic, Dihedral and Other Extensions

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NATHANIEL BRYANT SCHLEY

In memory of Mary Repetski and Wolcott Schley

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Abstract

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Let F be a field and E an extension of F with $[E : F] = d$ where the characteristic of F is zero or prime to d. We assume $\mu_{d^2} \subset F$ where μ_{d^2} are the d^2th roots of unity. This thesis studies the problem of determining the cohomological kernel $H^n(E/F) := \ker(H^n(F, \mu_d) \to H^n(E, \mu_d))$ (Galois cohomology with coefficients in the dth roots of unity) when the Galois closure of E is a semi-direct product of cyclic groups. The main result is a six-term exact sequence determining the kernel as the middle map and is based on tools of Positelski [Pos05]. When $n = 2$ this kernel is the relative Brauer group $\mathrm{Br}(E/F)$, the classes of central simple algebras in the Brauer group of F split in the field E. In the case where E has degree d and the Galois closure of E, $\widetilde{E}$ has Galois group $\mathrm{Gal}(\widetilde{E}/F)$ a dihedral group of degree $2d$, then work of Rowen and Saltman (1982) [RS82] shows every division algebra D of index d split by E is cyclic over F (that is, D has a cyclic maximal subfield.) This work, along with work of Aravire and Jacob (2008, 2018) [AJ08] [AJO18] which calculated the groups $H^n_{p^m}(E/F)$ in the case of semi-direct products of cyclic groups in characteristic p, provides motivation for this work.

CONTENTS

1. INTRODUCTION

This thesis studies cohomological kernels of field extensions. Historically, such kernels have played a key role in the computation of relative Brauer groups, although the results presented here apply more generally to higher cohomology. Similarly, the computation of analogous cohomological kernels have played important roles in the development of the algebraic theory of quadratic forms.

The Brauer group and central simple division algebras played a key role in the development of local and global class field theory in the first half of the 20th century. One corollary of that work was the discovery that finite dimensional division algebras over local and global fields were all cyclic algebras (e.g. they had cyclic Galois splitting fields), and indeed, one of the major accomplishments of global class field theory is the determination of the splitting fields of a Brauer class in terms of its local data. Similar local-global principles are known for quadratic forms. These accomplishments led to the investigation of these same problems over general fields, not just local and global fields. By mid century a large body of work in the general case began to develop, with A.A. Albert playing a major role in the study of central simple algebras and E. Witt in the case of quadratic forms.

The second half of the 20th century witnessed progress on two main problems considered by Albert, where in each case the role of understanding relative Brauer groups proved to be essential. These questions involve determining the maximal subfields of division algebras and determining generators for the Brauer group. The question of determining maximal subfields is the question of determining those field extensions of degree equal to the index of the splitting fields. Amitsur's discovery of non-crossed product division algebras is probably the most famous from this period. Albert's question of whether or not division algebras of prime index are cyclic (that is, have a cyclic Galois maximal subfield) remains open. Although the work in this thesis does not consider this question directly, it does study cohomological kernels of non-Galois extensions which could at some point provide an approach.

In 1980, the work of Merkurjev and Suslin on the conjecture of Albert showed that in the presence of roots of unity, the Brauer group is generated by classes of cyclic algebras. In terms of cohomology, this means that the cup product map $H^1(F,\mu_d)\times H^1(F,\mu_d)\to H^2(F,\mu_d)\cong \mathrm{Br}_d(F)$ is surjective. Their work was dependent upon detailed analyses of the K-theory of Severi-Brauer varieties and the relationship between the Milnor K-theory of a field and its Galois cohomology. Initially in the context of quadratic forms, Milnor (1970) had asked if his map $s_n : K_nF/2K_nF \to H^n(F,\mathbb{Z}/2\mathbb{Z})$ was an isomorphism. He also wondered if there existed a well-defined map $e_n : I^nF/I^{n+1}F \to H^n(F,\mathbb{Z}/2\mathbb{Z})$ (here, IF is the ideal of even-dimensional forms in the Witt ring WF.) Building upon the work of Merkurjev and Suslin who had extended their work to the case of $n = 3$ (1986), and the well-definition of e_4, the fourth cohomological invariant for quadratic forms (Jacob and Rost, 1989), Veovodski (1996) proved the Milnor Conjecture that s_n and e_n are isomorphisms for all n, linking Milnor K-theory, Galois cohomology mod 2 and the graded Witt Ring. Some time after that the Bloch-Kato conjecture generalized the relationship between Milnor K-theory of a field and Galois cohomology in 2003. The proof of the Bloch-Kato conjecture is essential to applications of the the results of Positselski's work that are used in this thesis. The work on the Milnor problems and the Bloch-Kato conjecture all involve the study of cohomological kernels of function field extensions and as such use the arithmetic geometry of the associated varieties.

In 2005, Positselki studied cohomological kernels of biquadratic extensions and certain degree 8 extensions [Pos05] using a four-term exact sequence of Galois group modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_4 \longrightarrow 0$$

with homotopy maps and some other properties to produce a six-term exact sequence of cohomology. Prior to his work, it was known by work analyzing the Witt ring that if $E = F(\sqrt{a}, \sqrt{b})$ is biquadratic then the kernel of the map $H^2(F, \mathbb{Z}/2\mathbb{Z}) \to H^2(E, \mathbb{Z}/2Z)$ is generated by the images of "expected elements" $(a) \smile (x)$ and $(b) \smile (y)$ for $x, y \in F$ [EL76] and that the analogue of this expected result for triquadratic extensions was false [ART79]. The question of determining the kernel of $H^n(F, \mathbb{Z}/2\mathbb{Z}) \to H^n(E, \mathbb{Z}/2\mathbb{Z})$ for $n \geq 3$ in the separable biquadratic case was considered by a number of researchers (Merkurjev, Tignol, Kahn), and it is this problem that Positselski solved with his tools. Positselski's tools also applied to dihedral extensions of degree 8, indicating the applicability of these techniques to the non-Galois case. Characteristic p versions of Positselski's machinery have been constructed by Aravire and Jacob (2012), for the separable biquadratic case and the dihedral and quaternion cases in characteristic 2 (2016), and more generally for the cyclic by cyclic semi-direct product cases in characteristic $p > 2$ by Aravire-Jacob-O'Ryan (2018). It is these latter constructions that this thesis generalizes to the case where the characteristic is prime to the field degree d and the d^2th roots of unity are present in the field.

The key to this work is determining the appropriate modules M_3 and M_4 (see below for the set-up) and establishing the requisite homotopies necessary to apply Positselski's tools. This is spelled out in Chapter 4. Chapters 2 and 3 develop the background as well as provide details necessary for the application of Positselski's results that are not clearly spelled out in his paper. In particular, the "connecting map" $\eta : H^n(\mathcal{G}, M_4) \to H^{n+1}(\mathcal{G}, M_1)$ needs to be carefully computed. This concludes with interpretations of the results.

1.1 Notation and Further Background

Let F be a field, $d \in \mathbb{N}$ with $d > 1$, we will assume that $\mathrm{char}(F) = 0$ or $(\mathrm{char}(F), d) = 1$, and that $\mu_{d^2} \subseteq F$, where μ_{d^2} are the d^2 distinct d^2th roots of unity. Let F_{sep} denote the separable closure of F, $\mathcal{G} = \mathrm{Gal}(F_{\mathrm{sep}}/F)$, and $H^n(\mathcal{G}, M)$, the nth cohomology groups for any $\mathbb{Z}[G]$-module M [Ser97]. Let E/F be an extension of degree d, $\mathcal{H} \subseteq \mathcal{G}$ be $\mathrm{Gal}(F_{\mathrm{sep}}/E)$. We will also use the notation $H^n(F, M) = H^n(\mathcal{G}, M)$, so $H^n(E, M) = H^n(\mathcal{H}, M)$. We also denote by $H^n(E/F, M) := \ker(H^m(F, M) \to H^n(E, M))$.

The groups $H^0(F, \mu_d)$ and $H^1(F, \mu_d)$ have an interpretation from Kummer theory. Consider the following short exact sequence of $\mathbb{Z}[\mathcal{G}]$-modules

$$0 \longrightarrow \mu_d \xrightarrow{\subseteq} F_{\mathrm{sep}}^{\times} \xrightarrow{\cdot d} F_{\mathrm{sep}}^{\times} \longrightarrow 0$$

where the second map is multiplication by d over the $\mathbb{Z}[\mathcal{G}]$-modules. It is surjective because F_{sep} is separably closed. This short exact sequence of $\mathbb{Z}[\mathcal{G}]$-modules yields a long exact sequence of cohomology [Rot09].

$$
\begin{array}{l}
0 \longrightarrow H^0(\mathcal{G}, \mu_d) \xrightarrow{\subseteq} H^0(\mathcal{G}, F_{\text{sep}}^\times) \xrightarrow{\cdot d} H^0(\mathcal{G}, F_{\text{sep}}^\times) \\
\xrightarrow{\partial} H^1(\mathcal{G}, \mu_d) \xrightarrow{\subseteq} H^1(\mathcal{G}, F_{\text{sep}}^\times) \xrightarrow{\cdot d} H^1(\mathcal{G}, F_{\text{sep}}^\times) \\
\xrightarrow{\partial} H^2(\mathcal{G}, \mu_d) \xrightarrow{\subseteq} H^2(\mathcal{G}, F_{\text{sep}}^\times) \xrightarrow{\cdot d} H^2(\mathcal{G}, F_{\text{sep}}^\times) \\
\xrightarrow{\partial} H^3(\mathcal{G}, \mu_d) \longrightarrow \cdots
\end{array}
$$

Note that $\text{Fix}_{\mathcal{G}}(\mu_d) = \mu_d$ and $\text{Fix}_{\mathcal{G}}(F_{\text{sep}}^\times) = F^\times$ by Galois theory. Furthermore, $H^1(\mathcal{G}, F_{\text{sep}}^\times)$ is trivial by the cohomological version of Hilbert's Theorem 90. This information gives the long exact sequence.

$$
\begin{array}{l}
0 \longrightarrow \mu_d \xrightarrow{\subseteq} F^\times \xrightarrow{\cdot d} F^\times \\
\xrightarrow{\partial} H^1(\mathcal{G}, \mu_d) \xrightarrow{\subseteq} 0 \xrightarrow{\cdot d} 0 \\
\xrightarrow{\partial} H^2(\mathcal{G}, \mu_d) \xrightarrow{\subseteq} H^2(\mathcal{G}, F_{\text{sep}}^\times) \xrightarrow{\cdot d} H^2(\mathcal{G}, F_{\text{sep}}^\times) \\
\xrightarrow{\partial} H^3(\mathcal{G}, \mu_d) \longrightarrow \cdots
\end{array}
$$

In particular we have the following three results:

1. $H^0(F, \mu_d) \cong \mathbb{Z}/dZ$.

2. $H^1(F, \mu_d) \cong F^\times/F^{\times d}$ and

3. $H^2(F, \mu_d)$ is the d-torsion of $H^2(F, F_{\text{sep}}^{\times})$.

For $a \in F^{\times}$ we use $(a) \in H^1(F, \mu_d)$ to denote the class that $aF^{\times d} \in F^{\times}/F^{\times d}$ corresponds to in the second identification. Since $H^2(F, F_{\text{sep}}^{\times}) \cong \text{Br}(F)$ is the Brauer group (the cohomological Brauer group and the Brauer group agree for fields), the third result will be of particular importance because it means $H^2(F, \mu_d)$ picks out the d-torsion in $\text{Br}(F)$. We use $\smile$ to denote the cup product: $\smile : H^r(F, \mu_d) \times H^s(F, \mu_d) \to H^{r+s}(F, \mu_d)$, which makes sense in our context because $\mu_d \subset F$ and therefore has trivial $\mathcal{G}$-action so $\mu_d^{\otimes 2} \cong \mu_d$ as $\mathcal{G}$-modules.

1.2 The Problem Studied

The problem studied in this thesis is that of determining the kernels of scalar extension (restriction in group cohomology),

$$\text{res}_{\text{E/F}} : \text{H}^{\text{n}}(\text{F}, \mu_{\text{d}}) \longrightarrow \text{H}^{\text{n}}(\text{E}, \mu_{\text{d}})$$

for various extensions E/F of degree d. The case where E/F is cyclic Galois is basic. Of course, by definition, in the $n = 2$ case if the Brauer class of an F-division algebra D of index d lies in $H^2(E/F, \mu_d)$, then this D is a cyclic algebra (with maximal subfield E.) More specifically, we know that when $E = F(\sqrt[d]{a})$ (recall $\mu_d \subset F$) we have $H^2(E/F, \mu_d) = (a) \smile H^1(F, \mu_d)$. The first basic result proved in this thesis, Theorem 13, is that in the cyclic case this is valid for all n, namely $H^{n+1}(E/F, \mu_d) = (a) \smile H^n(F, \mu_d)$. The next cases generalize this situation, where either the Galois closure of E is dihedral or E is an extension of degree d that becomes a cyclic extension when F is extended by a cyclic extension of degree prime to d. In this latter case the Galois group of the Galois closure of E is a cyclic by cyclic semi-direct product. In these latter cases one cannot describe the cohomological kernel as a cup product by a class (a) (indeed, $H^1(E/F, \mu_d) = 0$), but one does have the connecting map η from Positselski's theory to capture the kernel.

In order to compute these kinds of kernels in his work, Positselski used 4-term exact sequences of Galois group modules with homotopy maps and some other properties

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow M_4 \longrightarrow 0$$

to produce a six-term exact sequence of cohomology. Here with $M_1 \cong \mu_d$ and M_2 an appropriately selected induced module with $H^n(\mathcal{G}, M_2) \cong H^n(E, \mu_n)$, so that the six-term sequence can be used to compute the cohomological kernel $H^n(E/F, \mu_d)$. Aravire and Jacob [AJ08] and Aravire, Jacob and O'Ryan [AJO18] have developed a variant of this machinery to compute cohomologial kernels in characteristic p for E/F of prime degree $d = p$ with Galois closure having Galois group a semi-direct product of two cyclic groups of order p and s, where $s|(p-1)$. This thesis gives an analogous result when E/F is degree d, F has characteristic prime to d, and the Galois closure of E/F has Galois group a semi-direct product of cyclic groups of order d and s, with $s|\phi(d)$ (the Euler ϕ-function) and $\mathbb{Z}/s\mathbb{Z}$ acting faithfully on $\text{Aut}(\mathbb{Z}/d\mathbb{Z})$.

1.3 The Brauer Group

We briefly review here basic properties of central simple algebras and the Brauer group. The Brauer group of a field F, denoted $\text{Br}(F)$, is the group of isomorphism classes of finite dimensional central division algebras over F with a binary operation induced by the tensor product of algebras. However, it may be easier to view the Brauer group as a partition of finite dimensional central simple algebras (FDCSAs) over F as follows: For any FDCSA A over F, $A \cong M_n(D)$ from the Artin-Wedderburn theorem for unique $n \in \mathbb{N}$ and a unique division algebra D up to isomorphism. Two FDCSAs A_1 and A_2 are identified if and only if they are isomorphic to matrices over the same division algebra, i.e. if and only if $A_1 \underset{F}{\otimes} M_n(F) \cong A_2 \underset{F}{\otimes} M_m(F)$ for some $m, n \in \mathbb{N}$. In this sense, the isomorphism classes of division algebras over F are thought of as favored representatives of each Brauer group element. The binary operation for the Brauer group is

$$[A_1] \cdot [A_2] = [A_1 \underset{F}{\otimes} A_2]$$

with F representing the identity and inverses given by opposite algebras, from the result that $A \underset{F}{\otimes} A^{\mathrm{op}} \cong M_n(F)$ with n being the dimension of A over F.

For any field extension E/F, there is an induced homomorphism, called scalar extension, $\otimes_F E : \mathrm{Br}(F) \longrightarrow \mathrm{Br}(E), \quad [A] \mapsto [A \otimes_F E]$. The kernel of $\otimes_F E$ is denoted $\mathrm{Br}(E/F)$ and is commonly referred to as the relative Brauer group. The dimension of the smallest field E for which an element of the Brauer group is sent to the identity is called the index, and the order of an element of the Brauer group always divides its index [Jac80].

It is often useful to study the Brauer group using Galois cohomology, making use of the isomorphism

$$\mathrm{Br}(F) \cong H^2(\mathcal{G}, F_{\mathrm{sep}})$$

where $\mathcal{G} = \mathrm{Gal}(F_{\mathrm{sep}}/F)$. This isomorphism is one through which the extension of scalars homomorphism $\otimes_F E$ commutes with the restriction map in the diagram below [Ser97]

$$\begin{array}{ccc}
\mathrm{Br}(F) & \xrightarrow{\otimes_F E} & \mathrm{Br}(E) \\
\updownarrow \cong & & \updownarrow \cong \\
H^2(\mathcal{G}, F_{\mathrm{sep}}) & \xrightarrow{\mathrm{res}} & H^2(\mathcal{H}, F_{\mathrm{sep}})
\end{array}.$$

Because the order of an element of the Brauer group divides its index, E/F having degree d means that the Brauer kernel is contained in its d-torsion, i.e. $\mathrm{Br}(E/F) \subseteq \mathrm{Br}_d(F)$.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Br}(E/F) & \xrightarrow{\subseteq} & \mathrm{Br}_d(F) & \xrightarrow{\otimes_F E} & \mathrm{Br}_d(E) \\
 & & \updownarrow \mathrm{id} & & \downarrow \subseteq & & \downarrow \subseteq \\
0 & \longrightarrow & \mathrm{Br}(E/F) & \xrightarrow{\subseteq} & \mathrm{Br}(F) & \xrightarrow{\otimes_F E} & \mathrm{Br}(E)
\end{array}$$

Thus, the kernel of the map $\mathrm{Br}_d(F) \xrightarrow{\otimes_F E} \mathrm{Br}_d(E)$ will be studied, and more directly its cohomological version $H^2(\mathcal{G}, \mu_d) \xrightarrow{res} H^2(\mathcal{H}, \mu_d)$.

2. ARASON'S THEOREM

In his thesis, Arason [Ara] proved that the third cohomological invariant, e_3 of quadratic forms is well-defined. To accomplish this, he determined the cohomological kernel of a quadratic extension away from characteristic two (an equivalent result in group cohomology was proved independently by D.L. Johnson [Joh] at the same time.) We discuss Arason's results here because the approach he took provides a model for understanding the work of Positselski, and the computation of his connecting map lays a conceptual framework for the computation of Positselksi's connecting map η.

2.1 THE THEOREM

Let F be a field, $\mathrm{char}(F) \neq 2$, $E = F(\sqrt{a})$ a quadratic extension, and F_{sep} the separable closure of F. Let μ_2 be the square roots of unity ± 1; clearly $\mu_2 \subseteq F$. This result of Arason [Ara] is the following, a result which has a critical role in the algebraic theory of quadratic forms. It is a cohomological analogue of an exact sequence for the Witt ring (see [Lam05] chap. 7 Sec. 3). We also sketch the proof.

Theorem 1. *Let F be a field, $char(F) \neq 2$, $E = F(\sqrt{a})$ a quadratic extension. There is a long exact restriction/corestriction sequence*

$$
\begin{array}{l}
0 \longrightarrow H^0(F,\mu_2) \xrightarrow{\text{res}} H^0(E,\mu_2) \xrightarrow{cor} H^0(F,\mu_2) \xrightarrow{\partial} \\
\quad H^1(F,\mu_2) \xrightarrow{\text{res}} H^1(E,\mu_2) \xrightarrow{cor} H^1(F,\mu_2) \xrightarrow{\partial} \\
\quad H^2(F,\mu_2) \xrightarrow{\text{res}} H^2(E,\mu_2) \xrightarrow{cor} H^2(F,\mu_2) \xrightarrow{\partial} \\
\quad \cdots
\end{array}
$$

.

where the connecting map ∂ is the cup product with the character function $\chi_E \in H^1(F, \mu_2)$, which corresponds to the class $(a) \in F^\times / F^{\times 2}$.

Proof: (Sketch) This long exact sequence of Galois cohomology is induced from a short exact sequence of Galois modules

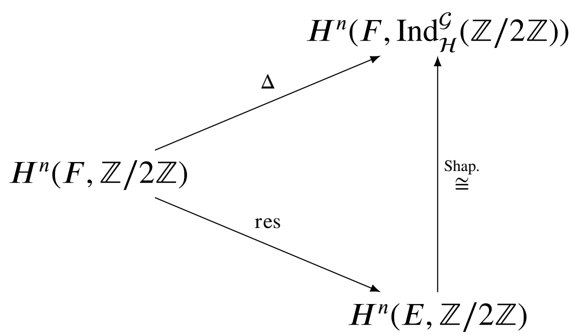
$$0 \longrightarrow \mu_2 \longrightarrow \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mu_2) \longrightarrow \mu_2 \longrightarrow 0$$

that will give the restriction and corestriction once we replace the middle module with μ_2 on E using the Shapiro isomorphism. Note that $\mathcal{G} = \mathrm{Gal}(F_{\mathrm{sep}}/F)$ and $\mathcal{H} = \mathrm{Gal}(F_{\mathrm{sep}}/E)$ So the induced module in the middle of the sequence is just the induced module of E over F for μ_2. To make things easier, we will work additively, with the sequence

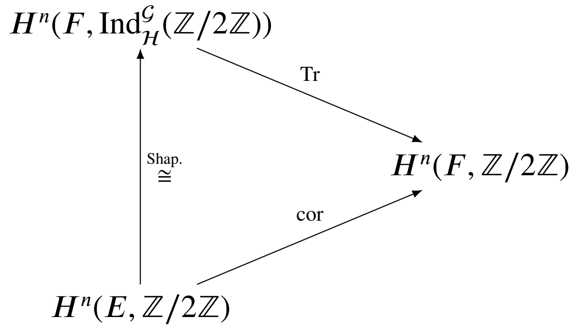
$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Here, $\mathbb{Z}/2\mathbb{Z}$ is a trivial $\mathcal{G}$-module, since the field elements (± 1) are fixed by the Galois group. And the induced module has a $\mathcal{G}$-action of permuting the two entries for any $g \in \mathcal{G} \setminus \mathcal{H}$, and trivial action for any $h \in \mathcal{H}$. Then the maps are the diagonal map $1 \mapsto 1 \oplus 1$ and the trace $x \oplus y \mapsto x + y$, respectively, which happen to be the only non-zero choices of $\mathbb{Z}[\mathcal{G}]$-module homomorphisms.

We need to show that the two $\mathcal{G}$-maps in the short exact sequence induce maps that commute with the restriction and corestriction through the Shapiro isomorphism, and we need to show that the snake-lemma connecting map ∂ is the cup product with the character function χ_E. This latter fact is pulled out as Theorem 2 below. The Shapiro isomorphism gives the following,



The composition of the restriction with the Shapiro map is induced by the identity map on $\mathbb{Z}/2\mathbb{Z}$, followed by the diagonal map, which agrees with the induced map on the top.



The Shapiro homomorphism on $\mathbb{Z}[G]$-modules is the diagonal map. When composed with the trace 1 is sent to the sum of 1 over every coset. This is also the map that induces the corestriction, and therefore the diagrams commute on cohomology. This concludes the proof sketch of Theorem 1 □

It remains to interpret the connecting map in the long exact sequence. Arason describes this as the cup product with the character function, and this will be shown next by direct computation.

2.2 The Connecting Map

Theorem 2. *The connecting map ∂ in Theorem 1 (Arason's Theorem) sends a cocycle σ to $\chi_E \smile \sigma$, the cup product with the character function, where the character function χ_E is the unique non-trivial homomorphism from Gal(E/F) onto μ_2.*

Proof: For the connecting map, we will compute it directly here. Let $\sigma \in Z^{n-1}(F, \mathbb{Z}/2\mathbb{Z})$. We pick the lifting ℓ of the trace map that sends 1 to $1 \oplus 0$. So $\ell(\sigma)(g_1, \dots, g_{n-1}) = (\sigma(g_1, \dots, g_{n-1})) \oplus 0$. Note that though ℓ is not a $\mathcal{G}$-map (if it were, then the connecting map would be zero), ℓ is an abelian group homomorphism. We will use this fact in the next step of the computation.

The next step after the lifting ℓ is the chain map δ, which can be computed through the bar resolution:

$$\begin{aligned}\delta(\ell(\sigma))(g_1,\dots,g_n) &= g_1\cdot\ell(\sigma(g_2,\dots,g_n)) - \ell(\sigma(g_1g_2,\dots,g_n)) + \cdots + (-1)^n\ell(\sigma(g_1,\dots,g_{n-1}))\\ &= g_1\cdot\ell(\sigma(g_2,\dots,g_n)) - \ell(g_1\cdot\sigma(g_2,\dots,g_n)) + \ell(\delta(\sigma)(g_1,\dots,g_n))\\ &= g_1\cdot\ell(\sigma(g_2,\dots,g_n)) - \ell(g_1\cdot\sigma(g_2,\dots,g_n))\end{aligned}$$

Note that everything is 2-torsion here, and $\mathbb{Z}/2\mathbb{Z}$ has a trivial $\mathcal{G}$-action. So

$$g_1\cdot\ell(\sigma(g_2,\dots,g_n)) - \ell(g_1\cdot\sigma(g_2,\dots,g_n)) = (g_1+1)\ell(\sigma(g_2,\dots,g_n)).$$

Furthermore, $(h+1)$ annihilates all of $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ whenever $h\in\mathcal{H}$, and for every $g_1\notin\mathcal{H}$, $(g_1+1)(1\oplus 0) = 1\oplus 1$. Thus

$$(g_1+1)\ell(\sigma(g_2,\dots,g_n)) = \begin{cases}\sigma(g_2,\dots,g_n)\oplus\sigma(g_2,\dots,g_n) & \text{if } g_1\notin\mathcal{H}\\ 0\oplus 0 & \text{if } g_1\in\mathcal{H}\end{cases}$$

This is the diagonal image of the function

$$\chi_{\mathcal{H}}(g_1)\cdot\sigma(g_2,\dots,g_n)) = \begin{cases}\sigma(g_2,\dots,g_n) & \text{if } g_1\notin\mathcal{H}\\ 0 & \text{if } g_1\in\mathcal{H}\end{cases}$$

which completes the computation of the connecting map ∂. Therefore $\partial(\sigma) = \chi_{\mathcal{H}} \smile \sigma$, the cup product with the character function $\chi_{\mathcal{H}}$, completing the proof of Theorem 2. $\square$

These ideas in the proof of Theorem 2 will be generalized in the sections that follow when we compute Positselski's connecting map η.

3. Positselski's 6-Term Cohomological Sequence

If we attempt to construct a short exact sequence in the fashion of Arason's theorem for a cyclic extension of degree $d > 2$, then the dimensions of the modules over $\mathbb{Z}/d\mathbb{Z}$ would be 1, d, 1, which makes exactness (and hence this approach) impossible. However, with the right machinery we can create exact sequences with 4 terms with a connecting map whose image is the cohomological kernel. Positselski's theorem offers exactly this machinery. But we need some additional conditions to be met [Pos05]. We collect these in the following definition.

Definition 3. Positselski's Hypotheses*: Let $\mathcal{G}$ be a pro-finite group, let $d, n \in \mathbb{Z}$ with $d \geq 2$ and $n \geq 0$, and let*

$$0 \longrightarrow A_2 \xrightarrow{d_1} B_2 \xrightarrow{d_2} C_2 \xrightarrow{d_3} D_2 \longrightarrow 0$$

be a 4-term exact sequence of free $\mathbb{Z}/d^2\mathbb{Z}$-modules with a discrete action of $\mathcal{G}$. Let h_1, h_2, h_3 be homotopy maps

$$A_2 \xleftarrow{h_1} B_2 \xleftarrow{h_2} C_2 \xleftarrow{h_3} D_2.$$

Furthermore, let A_1, B_1, C_1, D_1 be the d-torsion of A_2, B_2, C_2, D_2 (respectively).

If the homotopy maps satisfy the "prism" condition, that $d_i h_i + h_{i+1} d_{i+1} = d \cdot id$ for all $i \in \{0, 1, 2, 3\}$, and if the Bochstein maps are 0 for all 4 modules, i.e. the kernel/cokernel short exact sequences

$$0 \longrightarrow A_1 \xrightarrow{\subseteq} A_2 \xrightarrow{\cdot d} \overline{A_2} \longrightarrow 0$$

have 0 connecting maps $\partial : H^n(\mathcal{G}, \overline{A_2}) \longrightarrow H^{n+1}(\mathcal{G}, A_1)$ *and the same thing applies for* $\boldsymbol{B}, \boldsymbol{C}$ *and* $\boldsymbol{D}$ *alike, then the four modules and their associated maps satisfy the Positselski Hypotheses.*

Given the definition we can now state Positselksi's main result [Pos05] Theorem 6.

Theorem 4. *(Positselski) Given a 4 term exact sequence of* $\mathcal{G}$*-modules that satisfies the Positselski hypotheses, there is a 6-term exact sequence of* $\mathcal{G}$*-cohomology*

$$H^n(B_1 \oplus D_1) \xrightarrow{d_2 + h_3} H^n(C_1) \xrightarrow{\quad d_3 \quad} H^n(D_1) \xrightarrow{\ \eta\ } H^{n+1}(A_1) \xrightarrow{\quad d_1 \quad} H^{n+1}(B_1) \xrightarrow{h_1 \oplus d_2} H^{n+1}(A_1 \oplus C_1)$$

with the connecting map η *defined as follows:*

$$\eta(\sigma) = \left((d_1)^{-1} h_2 \delta \ell\right)(\sigma)$$

where ℓ *is any lifting of* d_3 *(not necessarily a homomorphism) and* $\delta : C^n(\mathcal{G}, C_1) \longrightarrow C^{n+1}(\mathcal{G}, C_1)$ *is the group cohomology coboundary map.*

The reader may note that the only difference between this definition of η and the definition of ∂ from the snake lemma is h_2 in the composition. h_2 connects the two middle terms of the 4-term exact sequence, where the snake lemma has only 1 middle term in a 3-term exact sequence. So in this sense they are as close as can be, given the different number of modules in the exact sequence.

We will spend the remainder of this section setting up the framework for this 6-term exact sequence as an exposition (and a few extra details) of how Positselski builds the framework in his paper [Pos05] as well as providing the tools we need for computing the map η in our applications. This process begins with the following two technical lemmas and ends with a proof of Theorem 4. This first lemma is based on Positselski's Lemma 5 [Pos05].

Lemma 5. *(Posetselski) Let X be a $\mathcal{G}$-module that is also a free $\mathbb{Z}/d^2\mathbb{Z}$-module. Then we have a natural short exact sequence: $0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \overline{X_2} \longrightarrow 0$ of $\mathcal{G}$-modules, where X_2 is d^2-torsion, X_1 is the d-torsion of X_2, and $\overline{X_2}$ is the quotient group, must also be d-torsion and isomorphic to X_1. Suppose Y, Z are short exact sequences defined in the same way, and that*

$$0 \longrightarrow X \xrightarrow{d_1} Y \xrightarrow{d_2} Z \longrightarrow 0$$

is a short exact sequence of these short exact sequences with homotopy maps

$$X \xleftarrow{h_1} Y \xleftarrow{h_2} Z$$

that satisfy the prism condition $d_i h_i + h_{i+1} d_{i+1} = d \cdot id$ for all $i \in \{0, 1, 2\}$. Let $\Phi_X : \overline{X_2} \longrightarrow X_1$ be the isomorphism defined as follows: For any $\overline{x} \in \overline{X_2}$, find a (non-unique) $x \in X_2$ such that $\pi(x) = \overline{x}$. Then multiply it by d. $d \cdot x$ is now unique, since any two such x's differ by a multiple of d. Furthermore, $d \cdot x$ is a d-torsion element of X_2 and is therefore an element of X_1. Let $\mathcal{B}_X, \mathcal{B}_Y, \mathcal{B}_Z$ denote the Bochstein homomorphisms, which are the snake-lemma connecting maps: $\mathcal{B}_X : H^n(\overline{X_2}) \longrightarrow H^{n+1}(X_1)$, etc. Finally, let $\overline{\partial} : H^n(\overline{Z_2}) \longrightarrow H^{n+1}(\overline{X_2})$. Then

1. *There are well-defined homomorphisms $\widehat{h_2} : Z_1 \longrightarrow X_1$ and $\widehat{\overline{h_2}} : \overline{Z_2} \longrightarrow \overline{X_2}$ both defined as $d_1^{-1} h_2 = -h_1 f$, for any lifting f of d_2.*

2. *$\Phi_X \overline{\partial} = \widehat{h}_2 \mathcal{B}_Z - \mathcal{B}_X \widehat{h}_2$*

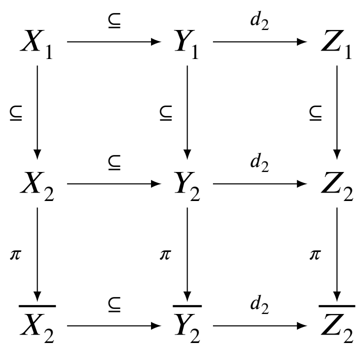
Proof:

1. Let f be a lifting of d_2, which must exist because d_2 is surjective, and let $z \in Z_1$ or $\overline{Z_1}$. Then

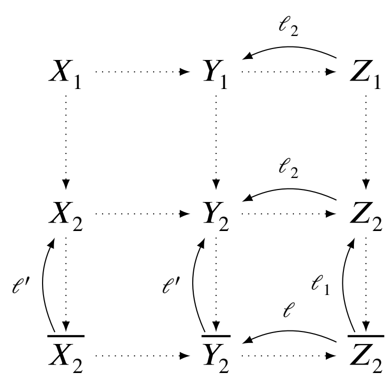
$$\begin{aligned} h_2(z) &= h_2(d_2(f((z)))) \\ &= (h_2 d_2) f(z) \\ &= d \cdot f(z) - (d_1 h_1)(f(z)) \\ &= -d_1(h_1(f(z))) \\ &= d_1(h_1(-f(z))) \end{aligned}$$

which is in the image of d_1. Furthermore, $\ker(d_2) = \mathrm{im}(d_1) \subseteq \ker(h_1)$ for both Y_1 and $\overline{Y_2}$, since both modules are d-torsion and $h_1 d_1 = \cdot d$. Thus, any two liftings of d_2 differ by an element in the kernel of h_1.

2. We assume without loss of generality that $X \subseteq Y$ and d_1 is the inclusion map. This allows the the snake-lemma connecting maps to be computed by applying the coboundary map δ to any pre-image of a given cocycle. Computation will be done this way for both Bochstein maps $\mathcal{B}_X$, $\mathcal{B}_Z$ and for $\overline{\partial}$. The inclusions also make $h_1 = -h_2 d_2$ in Y_1 and $\overline{Y_2}$, and furthermore h_1 restricted to $X \subseteq Y$ is multiplication by d. Below is a diagram of the exact square.



Careful choices for liftings makes the computations easier. Let ℓ be any lifting of $d_2 : \overline{Y_2} \longrightarrow \overline{Z_2}$, and let ℓ' be a lifting of $\pi : Y_2 \longrightarrow \overline{Y_2}$ that maps $\overline{X_2}$ into X_2. We define a lifting for $\pi : \overline{Z_2} \longrightarrow Z_2$ by $\ell_1 := d_2\ell'\ell$, and a lifting for $d_2 : Y_2 \longrightarrow Z_2$ by $\ell_2 := \ell'\ell\pi$. Let $\sigma_z \in Z^n(\overline{Z_2})$, and let $\sigma_y = \ell'(\ell(\sigma_z)) = \ell_2(\ell_1(\sigma_z)) \in C^n(Y_2)$. The liftings can be seen in the following diagram, with the liftings in the bottom right square commuting.



Direct computation and an application of the prism condition yield the desired result as follows, with (1) indicating part 1 of this lemma and (c) indicating for the commuting of maps; either π with d_2 or δ with $\mathcal{G}$-module homomorphism.

$$-\mathcal{B}_X(\widehat{h}_2(\sigma_z)) = -\mathcal{B}_X(h_2(\sigma_z)) = \mathcal{B}_X(-h_2(\sigma_z)) \stackrel{(1)}{=} \mathcal{B}_X(h_1\ell(\sigma_z))$$

$$= \mathcal{B}_X(h_1\pi(\sigma_y)) \stackrel{(c)}{=} \mathcal{B}_X(\pi h_1(\sigma_y)) = \mathcal{B}_X\pi(h_1(\sigma_y))$$

$$= \delta(h_1(\sigma_y)) \stackrel{(c)}{=} h_1(\delta(\sigma_y))$$

$$\widehat{h}_2(\mathcal{B}_Z(\sigma_z)) = \widehat{h}_2(\mathcal{B}_Z(\pi d_2(\sigma_y))) = \widehat{h}_2(\mathcal{B}_Z\pi(d_2(\sigma_y))) = \widehat{h}_2(\delta\pi^{-1}\pi d_2(\sigma_y)))$$

$$= \widehat{h}_2(\delta(d_2(\sigma_y))) \stackrel{(c)}{=} (h_2d_2)(\delta(\sigma_y))$$

$$\Phi_X(\overline{\partial}(\sigma_z)) = \Phi_X(\delta(\ell(\sigma_z))) = \Phi_X(\delta(\pi(\sigma_y))) = \Phi_X(\pi(\delta(\sigma_y)))$$

$$= (\Phi_X\pi)(\delta(\sigma_y))) = (\cdot d\pi^{-1}\pi)(\delta(\sigma_y)) = d\cdot(\delta(\sigma_y)))$$

From the prism condition at Y_1, it follows that

$$\Phi_X\overline{\partial} = \widehat{h}_2\mathcal{B}_Z - \mathcal{B}_X\widehat{h}_2.$$

This concludes the proof of Lemma 5. □

This next lemma is critical to the description of the map η and uses a splitting of the four term sequence into two three term exact sequences. It shows how η is related to the connecting maps of these short exact sequences via the homotopies provided by the Positeselski Hypotheses.

Lemma 6. *Using the language of the previous lemma and viewing the 4-term exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

as two short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow \Delta \longrightarrow 0$$

$$0 \longrightarrow \Delta \longrightarrow C \longrightarrow D \longrightarrow 0$$

where $\Delta = \mathrm{im}(d_2) = \ker(d_3) \subseteq C$*, we have the following equivalent definitions of* η*:*

$$-\widehat{h}_2 \partial_{\Delta_1 D_1} = \eta = \partial_{A_1 \Delta_1} \widehat{h}_3$$

where $\widehat{h}_2 := d_1^{-1} h_2 : C \longrightarrow A$ *for the first short exact sequence as in the previous lemma, and* $\widehat{h}_3 := h_3 : D \longrightarrow \Delta$ *is the analogue of* $\widehat{h}_2$ *for the second short exact sequence.*

Proof: The first equality follows immediately from the definition of η. The second equality follows from direct computation.

$$\begin{aligned} -\widehat{h}_2 \partial_{\Delta_1 D_1} &= -(h_1 d_2^{-1})(\delta d_3^{-1}) = (-h_1 d_2^{-1})(\delta d_3^{-1}) = (d_1^{-1} h_2)(\delta d_3^{-1}) \\ &= d_1^{-1} h_2 \delta d_3^{-1} = d_1^{-1} \delta h_2 d_3^{-1} = d_1^{-1} \delta (d_2^{-1} d_2) h_2 d_3^{-1} \\ &= (d_1^{-1} \delta d_2^{-1})(d_2 h_2 d_3^{-1}) = \partial_{A_1 \Delta_1} \widehat{h}_3 \end{aligned}$$

The third equality above already follows from the prism condition, the fifth equality from the commutativity of the chain map with the homotopies, the sixth from the fact that with the right choice of lifting (namely d_2^{-1}), applying d_2 and then the lifting is the same as the identity (though any lifting of d_2 would suffice up to coboundaries). The last equality follows from the definitions of the connecting map $\partial_{A_1 \Delta_1}$ and the homomorphism $\widehat{h}_3$ from D to Δ. Note that it may seem natural to define $\widehat{h_3}$ as $h_2 d_3^{-1}$, but h_2 is a homotopy from C to B in the four-term exact sequence, not from C to $\Delta \subseteq C$. The first homotopy we need for the short exact sequence

$$0 \longrightarrow \Delta \xrightarrow{\subseteq} C \xrightarrow{d_3} D \longrightarrow 0$$

is $d_2 h_2 : C \longrightarrow \Delta$, and the fact that it satisfies the prism condition follows from the prism condition being satisfied for the four-term exact sequence. This concludes the proof of Lemma 6. □

With Lemmas 5 and 6 proved, we move on to the proof of Positselski's theorem.

Proof of Theorem 4: With η defined, it remains to show exactness at four terms of the 6-term sequence.

Exactness of $H^n(B_1 \oplus D_1) \xrightarrow{d_2+h_3} H^n(C_1) \xrightarrow{d_3} H^n(D_1)$: The composition $d_3 d_2$ is the 0 map on the modules, which makes it the zero map on cohomology as well. The composition $d_3 h_3$ is also the 0 map because it is multiplication by d and D_1 is d-torsion.

Now we show that $d_2 + h_3$ maps onto the kernel of d_3. Assume $y \in Z^n(C_1)$ with $d_3(y)$ a coboundary in $C^n(D_1)$. Let $\Delta_1 = \ker(d_3) \subseteq C_1$. Then, adding a coboundary in $C^n(C_1)$ to y if necessary, we may assume that $y \in Z^n(\Delta_1) \subseteq Z^n(C_1)$. Let $y' \in Z^n(\overline{\Delta_1})$ be the corresponding element to y from the isomorphism $\Delta_1 \cong \overline{\Delta_1}$. Then $\beta_C(y')$ is a coboundary in $C^{n+1}(C_1)$ because the Bochstein maps are 0 for A, B, C, D. Therefore $\beta_\Delta(y')$ is a coboundary in $C^n(C_1)$ as well (though not necessarily a coboundary in $C^n(\Delta_1)$), so $\beta_\Delta(y') = \partial_{\Delta D}(x)$ for some $x \in Z^n(D_1)$ from the exactness of the long exact sequence induced by the Snake Lemma from the short exact sequence.

$$0 \longrightarrow \Delta_1 \longrightarrow C_1 \longrightarrow D_1 \longrightarrow 0.$$

With $\beta_\Delta(y')$ being in the image of $\partial_{\Delta_1 D_1}$, use can be made of the lemma 5 applied to the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow \Delta \longrightarrow 0$$

with the following result: Let the $\widehat{h}_2 : \Delta_1 \longrightarrow A_1$ be as in the lemma. Then the following equalities hold up do coboundaries in $C^{n+1}(A_1)$.

$$\partial_{A_1\Delta_1}(y) = \beta_A \widehat{h}_2(y') - \widehat{h}_2\beta_\Delta(y') = -\widehat{h}_2\beta_\Delta(y') = -\widehat{h}_2\partial_{\Delta_1 D_1}(x) = \eta(x)$$

with the last equality following from the definition of η. Furthermore, the equivalent definition of η as $\partial_{A_1\Delta_1}\widehat{h}_3$ yields the equality

$$\partial_{A_1\Delta_1}(y) = \partial_{A_1\Delta_1}\widehat{h}_3(x),$$

where $\widehat{h}_3$ comes from the short exact sequence

$$0 \longrightarrow \Delta_1 \longrightarrow C_1 \longrightarrow D_1 \longrightarrow 0.$$

The desired result follows from this equality, since $\partial_{A_1\Delta_1}\left(y - \widehat{h}_3(x)\right)$ is a coboundary in $C^n(A_1)$ it follows that $\left(y - \widehat{h}_3(x)\right) = d_2(z)$ for some $z \in Z^n(B_1)$. We conclude exactness at $H^n(C_1)$ from the equivalent definition of η as $\partial_{A_1\Delta_1}\widehat{h}_3$ in Lemma 6.

Exactness of $H^n(C_1) \xrightarrow{d_3} H^n(D_1) \xrightarrow{\eta} H^{n+1}(A_1)$: To show that $\eta d_3 = 0$, let $c \in Z^n(C_1)$. Then

$$\eta d_3(c) = \widehat{h}_2\partial_{A_1\Delta_1} d_3(c) = \widehat{h}_2\delta d_3^{-1} d_3(c) = \widehat{h}_2\delta(c) = \widehat{h}_2(0) = 0$$

where $\delta(c) = 0$ because c is a cocycle.

Now suppose $y \in Z^n(D_1)$ with $\eta(y) = \delta(a)$ for some $a \in C^n(A_1)$. We know that $d_3^{-1}(y) \in C^n(C_1)$ exists because d_3 is surjective. We just need to show that $d_3^{-1}(c)$ is a cocycle for some choice of lifting of d_3. Because $\eta(y) = \partial_{A_1\Delta_1}\widehat{h}_3(y)$ is a coboundary, $\widehat{h}_3(y) = d_2(b)$ for some $b \in C^n(B_1)$, namely $b = d_1(a)$ suffices. Then from the commuting of the Bochstein maps we have the following equalities up to coboundaries.

$$d_2\beta_B(b) = \beta_\Delta d_2(b) = \beta_\Delta(\widehat{h}_3(y))$$

From Lemma 5 the equalities

$$\beta_\Delta\widehat{h}_3(y) = \beta_\Delta\widehat{h}_3(y) - \widehat{h}_3\beta_D(y) = \partial_{\Delta_1 D_1}(y)$$

also follow up to coboundaries. Similarly, $\partial_{\Delta_1 D_1}(y) = \beta_A d_2(b) = \beta_A d_2 d_1(a) = 0$ hold up to coboundaries as well, and therefore $y + B^n(D_1) = d_3(x)$ for some $x \in H^n(D_1)$.

Exactness of $H^n(D_1) \xrightarrow{\eta} H^{n+1}(A_1) \xrightarrow{d_1} H^{n+1}(B_1)$: To show that $d_1\eta = 0$, let $y \in Z^n(D_1)$. Then, using Lemma 6 to define η,

$$d_1\eta(y) = d_1(-\partial_{A_1\Delta_1}\widehat{h}_3)(y) = d_1(-d_1^{-1}\delta d_2^{-1})\widehat{h}_3(y) = -d_1 d_1^{-1}\delta(d_2^{-1}\widehat{h}_3(y)) = -\delta(d_2^{-1}\widehat{h}_3(y))$$

and therefore $d_1\eta(y)$ is a coboundary.

Now suppose $a \in Z^{n+1}(A_1)$ and $d_1(a) = \delta(b)$ for some $b \in C^n(B_1)$. From Lemma 5,

$$a = d_1^{-1}(\delta(b)) = \partial_{A_1\Delta_1} d_2(b) \overset{(5)}{=} \beta_A\widehat{h}_2(d_2(b) - \widehat{h}_2\beta_\Delta(d_2(b)) = -\widehat{h}_2\beta_\Delta(d_2(b)).$$

Let $c = d_2(b)$. Then $a = \partial_{A_1\Delta_1}(c) = -\widehat{h}_2\beta_\Delta(c)$ from the above equalities. Now, $\beta_\Delta(c)$ may not be a coboundary in $C^{n+1}(\Delta_1)$, but it is a coboundary in $C^{n+1}(C_1)$ because in $C^{n+1}(C_1)$, $\beta_\Delta(c) = \beta_C(c)$ with C having a vanishing Bochstein. Thus, $\beta_\Delta(c) = \partial_{\Delta_1 D_1}(y)$ for some $y \in Z^n(D_1)$. So $a = -\widehat{h}_2\beta_\Delta(c) = -\widehat{h}_2\partial_{\Delta_1 D_1}(y)$ for some $y \in Z^n(D_1)$, as desired.

<u>Exactness of $H^{n+1}(A_1) \xrightarrow{d_1} H^{n+1}(B_1) \xrightarrow{d_2 \oplus h_1} H^n(A_1 \oplus C_1)$</u>: Both of the compositions d_2d_1 and h_1d_1 are 0 maps. This comes from exactness in the first case, and from A_1 being d-torsion in the second case. It remains to show that for any $t \in Z^{n+1}(B_1)$ such that $d_2(t)$ and $h_1(d)$ are coboundaries of $C^{n+1}(C_1)$ and $C^{n+1}(A_1)$ respectively, that up to a coboundary t is in the image of d_1. To do this, we will show that $d_2(x)$ a coboundary in $C^{n+1}(\Delta_1)$.

It may benefit the reader to clarify that though $d_2(t) \in Z^n(\Delta_1)$ because Δ_1 is the image of d_2, the hypothesis only states that $d_2(t) = \delta(y)$ for some $y \in C^n(C_1)$ rather than some $y \in C^n(\Delta_1)$. To show that such a $y \in C^n(\Delta_1)$ exists, we need to argue as follows:

Due to the fact that $d_2(t) \in Z^{n+1}(\Delta_1)$ is a coboundary in $C^{n+1}(C_1)$, $t = \partial_{\Delta_1 D_1}(x)$ for some $x \in Z^n(D_1)$ up to a coboundary. Then the following equalities also hold up to coboundaries from Lemma 6.

$$\partial_{A_1\Delta_1}\widehat{h}_3(x) = -h_2\partial_{\Delta_1 D_1}(t) = -\widehat{h}_2 d_2(t) = -h_1(t) = 0$$

This is to say that $\widehat{h}_3(x)$ is in the kernel of $\partial_{A_1\Delta_1}$, so up to coboundaries $\widehat{h}_3(x) = d_3(z)$ for some $z \in Z^n(B_1)$. Finally, from Lemma 5 it follows that up to coboundaries,

$$d_2(t) = \partial_{\Delta_1 D_1}(x) = \beta_\Delta\widehat{h}_3(x) - \widehat{h}_3\beta_D(x) = \beta_\Delta\widehat{h}_3(x) = \beta_\Delta d_2(z) = d_2\beta_B(z) = 0.$$

Lemma 4 is applied to the short exact sequence.

$$0 \longrightarrow A \longrightarrow B \longrightarrow \Delta \longrightarrow 0$$

which means $d_2(t)$ is a coboundary in $C^{n+1}(\Delta_1)$, as desired. This completes the proof of Theorem 4.

□

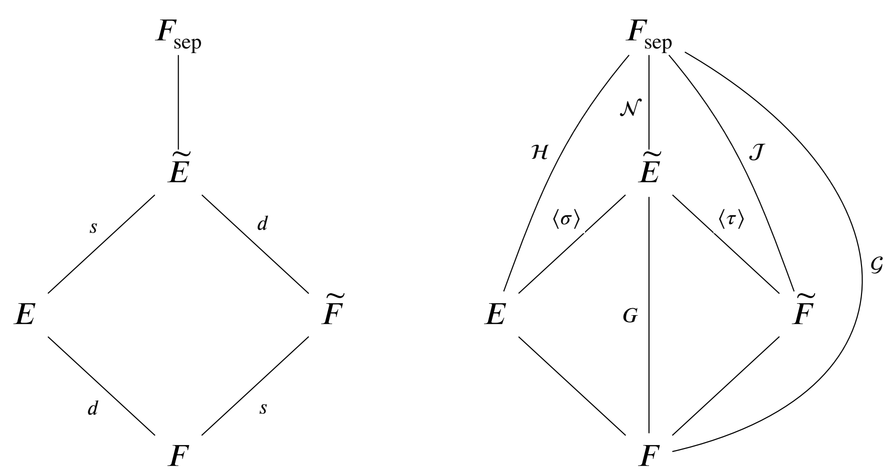
Remark: Though the Bochstein maps were all required to be zero for the six-term sequence to be exact, the proof of exactness at each term only required some of these to be zero. Exactness at $H^n(C_1)$ only required that β_A be zero, at $H^n(D_1)$ that β_D be zero, at $H^n(A_1)$ that β_A and β_C be zero, and at $H^n(B_1)$ that β_D and β_B be zero.

4. The General Setup and the Bochstein Maps

4.1 General Setup

For the sections that follow we adopt the following notation:

Let G be a semi-direct product of $\langle\tau\rangle$ by $\langle\sigma\rangle$ with $|\tau| = d$, $|\sigma| = s$ and $\langle\sigma\rangle$ has a faithful action on $\langle\tau\rangle$. We will be using this group for $\mathcal{G} = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ for some field F whose degree d extension E is the extension we wish to study. Let $\widetilde{E}$ be the Galois closure of E/F, $\mathcal{N} = \mathrm{Gal}(F^{\mathrm{sep}}/\widetilde{E}) \lhd \mathcal{G}$. Assume further that $G \cong \mathcal{G}/\mathcal{N}$, $E = F(\beta)$, $\widetilde{E} = F(\alpha,\beta)$, and let $\widetilde{F}$ be the cyclic extension of F given by $\mathrm{Fix}(\langle\tau\rangle)$ in this setup. Let $\mathcal{H} = \mathrm{Gal}(F^{\mathrm{sep}}/E)$, $\mathcal{J} = \mathrm{Gal}(F^{\mathrm{sep}}/F_0)$, $H = \langle\sigma\rangle$ and $J = \langle\tau\rangle$ so that $\sigma(\beta) = \beta$ and $\tau(\alpha) = \alpha$. We also assume that the dth roots of unity $\mu_d \subseteq F$, and $\mathrm{char}(F)$ does not divide d so that μ_d contains d distinct roots of unity.



The following three sections will cover the cases of E/F in increasing generality. In the next section E/F will be a cyclic extension so that $\widetilde{E} = E$, $\widetilde{F} = F$ and $s = |\sigma| = 1$. In the section that follows we assume $s = |\sigma| = 2$ so that G is a dihedral group with d odd. Lastly we let $s = |\sigma|$ be any even positive integer with $s|(d-1)$, hence d is still odd.

We denote by $\theta : \{0, 1, \ldots, d-1\} \to \{0, 1, \ldots, d-1\}$ the conjugation in $\langle \tau \rangle$ by σ, that is, $\sigma\tau^i\sigma^{-1} = \tau^{\theta(i)}$. We define θ_j by $\sigma^j\tau\sigma^{-j} = \tau^{\theta_j}$ (in fact $\theta_j = \theta^j(1)$ where the latter is the j'th iterate of θ, but the notation θ_j is less cumbersome.) We assume that θ has order s, that is, conjugation by σ on $\langle \tau \rangle$ has order s. As τ has odd order this means $\sigma^{\frac{s}{2}}\tau^i\sigma^{-\frac{s}{2}} = \tau^{-i}$ for all i. From this, $\theta_{j+\frac{s}{2}} \equiv -\theta_j \pmod{d}$ and since $0 < \theta_j < d$ we must have $\theta_j + \theta_{j+\frac{s}{2}} = d$.

4.2 The Bochstein Maps

Part of the Positselski Hypotheses is the requirement that the Bochstein maps are zero for the four modules in the exact sequence, and we will show that this is indeed the case for the next three sections. We start by examining the long exact sequence over which the Bochstein map is defined for the module μ_{d^2} for a given field F that contains those roots of unity. We will be using $M_1 = \mathbb{Z}$ as a trivial $\mathcal{G}$-module with $M_1 2 M_1 = \mathbb{Z}/d^2\mathbb{Z} \cong \mu_{d^2}$ in all three of the following sections, and M_2 will be an induced module with the same cohomology when taken over a slightly larger field.

The long exact sequence associated with $0 \to \mu_d \to \mu_{d^2} \to \mu_{d^2}/\mu_d \to 0$ is the following

$$\cdots \longrightarrow H^n(F, \mu_d) \xrightarrow{i} H^n(F, \mu_{d^2}) \xrightarrow{\pi} H^n(F, \mu_{d^2}/\mu_d) \xrightarrow{\beta_{\mu_{d^2}}} H^{n+1}(F, \mu_d) \longrightarrow \cdots$$

and the relevant Bochstein map is the connecting map labelled $\beta_{\mu_{d^2}}$ in this sequence. In this case the vanishing of the Bochstein map is given next.

Lemma 7. *Suppose F is a field with $\mu_{d^2} \subset F$. Then the Bochstein map $\beta_{\mu_{d^2}}$ associated with the short exact sequence $0 \to \mu_d \to \mu_{d^2} \to \mu_{d^2}/\mu_d \to 0$ is zero.*

Proof. The Bloch-Kato Conjecture, proved by Veovodski in 2003, states that the norm residue homomorphisms below are surjective since $(\mathrm{char}(F), d) = 1$. Furthermore, the identity map commutes with the canonical quotient map through the norm residue homomorphism. This means we have a commutative diagram.

$$
\begin{array}{ccccccc}
 & & K_n^M & \xrightarrow{\mathrm{id}} & K_n^M & & \\
 & & \downarrow{\scriptstyle s_{n,d^2}} & & \downarrow{\scriptstyle s_{n,d}} & & \\
H^n(F, \mu_d) & \xrightarrow{\subseteq} & H^n(F, \mu_{d^2}) & \xrightarrow{\pi} & H^n(F, \mu_{d^2}/\mu_d) & \xrightarrow{\beta_{\mu_{d^2}}} & H^{n+1}(F, \mu_d)
\end{array}
$$

Therefore the map π is surjective as well, so that $\beta_{\mu_{d^2}} = 0$ by exactness. This proves the lemma.

□

With the Bochstein map for $M_1 = \mathbb{Z}$ checked, we move on to M_2. $M_2 \cong \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$ in every case, which means that $H^n(\mathcal{G}, M_2/d^2M_2) \cong H^n(\mathcal{H}, \mu_{d^2})$ from Shapiro's lemma, and the field E may be used instead of F with Lemma 7 to obtain the result of β_{M_2} being 0. Now for just the cyclic case, $M_3 = M_2$ and $M_4 = M_1$, so all the Bochstein maps are zero. For the dihedral and semi-direct cases, the lemma below will be used to show that M_3 and M_4 have zero Bochstein maps in their modular framework sections.

Lemma 8. *Let $\mathcal{J} \subseteq \mathcal{G}$ be an index s subgroup, X_2 a free $\mathbb{Z}/d^2\mathbb{Z}$-module with a discrete action of $\mathcal{G}$ with $(s,d) = 1$. If $\beta_{\mathcal{J},X} = 0$ then $\beta_{\mathcal{G},X} = 0$ as well.*

Proof: The restriction map res : $H^n(\mathcal{G}, X_2) \longrightarrow H^n(\mathcal{J}, X_2)$ is injective because cor$\circ$res $= \cdot s$, which is invertible because $(s,d) = 1$. Now, we have the following commutative diagram

$$\begin{array}{ccccccc} H^n(\mathcal{G}, X_1) & \xrightarrow{i} & H^n(\mathcal{G}, X_2) & \xrightarrow{\pi} & H^n(\mathcal{G}, \overline{X_2}) & \xrightarrow{\beta_{\mathcal{G},X}} & H^{n+1}(\mathcal{G}, X_1) \\ \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} \\ H^n(\mathcal{J}, X_1) & \xrightarrow{i} & H^n(\mathcal{J}, X_2) & \xrightarrow{\pi} & H^n(\mathcal{J}, \overline{X_2}) & \xrightarrow{\beta_{\mathcal{J},X}} & H^{n+1}(\mathcal{J}, X_1) \end{array}$$

in which case

$$\text{res}\circ\beta_{\mathcal{G},X} = \beta_{\mathcal{J},X}\circ\text{res} = 0\circ\text{res} = 0$$

means $\beta_{\mathcal{G},X} = 0$ because the restriction is injective. This concludes the proof of Lemma 8. □

With the previous discussion and the lemma proved, we have the framework necessary to show that all four modules have zero Bochstein maps for all three cases considered in this thesis.

5. The Cyclic Case

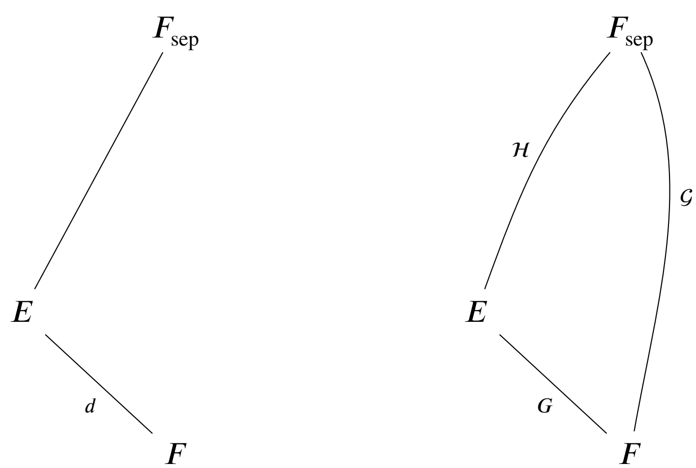
We begin our analysis with the Cyclic case, namely when E/F is cyclic Galois. In this case the kernel $H^2(E/F, \mu_d)$ has been understood since the early days of Class Field Theory. For suppose $E = F(\sqrt[d]{a})$. Then one has the well-known exact sequence

$$H^1(E, \mu_d) \xrightarrow{\mathrm{N}_{E/F}} H^1(F, \mu_d) \xrightarrow{(a)\smile} H^2(F, \mu_d) \xrightarrow{i_{E/F}} H^2(E, \mu_d)$$

which describes cohomology classes in $H^2(F, \mu_d)$ that vanish in E as those corresponding to "symbol algebras" of the form $(a, b)_F$ for some $b \in F$. This result also encodes the classes $(a, b)_F$ which vanish in $H^2(F, \mu_d)$ as those where $b \in \mathrm{N}_{E/F}(E^\times)$, that is b is a norm from E. This section generalizes this classical information to all higher cohomology. It may be that this result is known in some circles via folklore, but we do not know of a reference and because it requires the Positselski machinery for its proof we believe the result is new. One interesting feature of the proof is that the four term module sequence used is nothing other than a module formulation of the classical Hilbert 90 sequence. Of course, the H^2 result just mentioned is a consequence of Hilbert's Theorem 90 so this is not a surprise. Moreover, the generalization of Hilbert's Theorem 90 to higher K-theory is essential to Voevodsky's work, so this is also to be expected.

5.1 The 4-Term Exact Sequence with Homotopies

Let E/F be a cyclic extension of degree d, with dth roots of unity $\mu_d \subseteq F$ and $\mathrm{Gal}(E/F) = G \cong \mathcal{G}/\mathcal{H} = \langle \tau \rangle$. Let F_{sep} denote the separable closure of F. Because E/F is cyclic, the Galois closure $\widetilde{E}$ of E/F is E, and $\widetilde{F} = F$, simplifying the general setup.



It would be nice to use the restriction, corestriction sequence in Arason's theorem, but for a sequence of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Delta} \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \xrightarrow{Tr} \mathbb{Z} \longrightarrow 0$$

to be exact, $d - 2 = 0$ is a necessary condition because the $\mathbb{Z}$-dimensions of the $\mathcal{G}$-modules are 1, d, and 1 respectively. And this fails for all $d > 2$. We remedy this by applying Positselski's machinery to the following 4-term exact sequence of $\mathcal{G}$-modules,

$$0 \longrightarrow \mathbb{Z}/d^2\mathbb{Z} \xrightarrow{\Delta} \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}/d^2\mathbb{Z}) \xrightarrow{(1-\tau)\cdot} \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}/d^2\mathbb{Z}) \xrightarrow{\mathrm{Tr}} \mathbb{Z}/d^2\mathbb{Z} \longrightarrow 0$$

where the maps Δ and Tr are defined in Definition 10 below.

In the notation of this sequence the trivial $\mathcal{G}$-module $\mathbb{Z}/d^2\mathbb{Z}$ acts as a vessel for the short exact sequence

$$0 \longrightarrow \mu_d \xrightarrow{\subseteq} \mu_{d^2} \xrightarrow{\pi} \mu_{d^2}/\mu_d \longrightarrow 0$$

making the $\mathcal{G}$-module homomorphisms chain maps for these short exact sequences. To do this, we identify the roots-of-unity short exact sequence with the short exact sequence

$$0 \longrightarrow d\mathbb{Z}/d^2\mathbb{Z} \xrightarrow{\subseteq} \mathbb{Z}/d^2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/d\mathbb{Z} \longrightarrow 0.$$

Further, because all the arithmetic works without needing to mod out by $d^2\mathbb{Z}$, we use $\mathbb{Z}$ as a vessel for $\mathbb{Z}/d^2\mathbb{Z}$. We need a characterization of the induced module in order to facilitate computations in the four-term sequence given in Definition 10 below. This is the subject of the next lemma.

Lemma 9. *We denote by $\tilde{\tau}$ be a lifting of τ to F_{sep}, and let $G = \langle \tau \rangle = \langle \tilde{\tau}\mathcal{H} \rangle = \mathcal{G}/\mathcal{H}$. Define $\phi : Ind_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \longrightarrow \mathbb{Z}[G]$ as follows: For any $f : \mathcal{G} \longrightarrow \mathbb{Z}$ with the property $f(hg) = h \cdot f(g)$ for every $g \in \mathcal{G}, h \in \mathcal{H}$,*

$$\phi(f) = \sum_{i=0}^{d-1} f(\tilde{\tau}^i)\tau^{-i}$$

Then ϕ is an isomorphism of $\mathcal{G}$-modules, that is, $Ind_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \cong \mathbb{Z}[G]$.

Proof: ϕ is a bijection between $\mathbb{Z}$-basis elements of the two $\mathcal{G}$-modules, so it is a $\mathbb{Z}$-module isomorphism. We need only check that the action is preserved. Every $g \in \mathcal{G}$ can be expressed as $g = h\tilde{\tau}^k$, where $h \in \mathcal{H}$. Let $h_i = \tilde{\tau}^i h \tilde{\tau}^{-i} \in \mathcal{H}$ for each $i \in \{0, \ldots, d-1\}$ (these will be used to “hop over” the $\tilde{\tau}$ terms). Then

$$\begin{aligned}\phi(h\tilde{\tau}^k \cdot f) &= \sum_{i=0}^{d-1}(h\tilde{\tau}^k \cdot f)(\tilde{\tau}^i)\tau^{-i} = \sum_{i=0}^{d-1} f(\tilde{\tau}^i h\tilde{\tau}^k)\tau^{-i} = \sum_{i=0}^{d-1} f(h_i\tilde{\tau}^i\tilde{\tau}^k)\tau^{-i} \\ &= \sum_{i=0}^{d-1} h_i \cdot f(\tilde{\tau}^i\tilde{\tau}^k)\tau^{-i} = \sum_{i=0}^{d-1} f(\tilde{\tau}^i\tilde{\tau}^k)\tau^{-i} = \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\tau^{-(j-k)} \\ &= \tau^k \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\tau^{-j} = \tilde{\tau}^k h \cdot \phi(f)\end{aligned}$$

This concludes the proof of Lemma 9. □

We next give the four term sequence in the cyclic case. We will define the homotopies and verify the computational conditions for Positselski’s 6-term sequence in the group ring $\mathbb{Z}[G]$ in Theorem 11 below.

Definition 10. *Let G be as above with $\mathbb{Z}$ a trivial G-module and with $\mathbb{Z}[G]$ a G-module (and therefore a $\mathcal{G}$-module) via multiplication on the left. We define G-module maps*

$$\Delta : \mathbb{Z} \longrightarrow \mathbb{Z}[G], \quad n \mapsto \bigoplus_{g\in G} ng, \text{and}$$

$$Tr : \mathbb{Z}[G] \longrightarrow \mathbb{Z}, \quad \bigoplus_{g\in G} c_g g \mapsto \sum_{g\in G} c_g.$$

The Positselski modules M_1, M_2, M_3, M_4 and maps d_1, d_2, d_3 for the cyclic case are defined as follows:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z}[G] \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G] \xrightarrow{\text{Tr}} \mathbb{Z} \longrightarrow 0.$$

The homotopies h_1, h_2, h_3 are defined as follows:

$$\mathbb{Z} \xleftarrow{\text{Tr}} \mathbb{Z}[G] \xleftarrow{\cdot \sum -i\tau^i} \mathbb{Z}[G] \xleftarrow{\Delta} \mathbb{Z}.$$

Remark: The $\mathcal{G}$-module homomorphism $d_2 = (1-\tau) : \text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \longrightarrow \text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$ is multiplication on the left by $(1-\tau)$ in the group ring $\mathbb{Z}[G]$ for the Hilbert 90 sequence, which is also multiplication on the right by $(1-\tau)$ since $\langle\tau\rangle$ is abelian. For the more general cases, we will use this last convention, which can also be thought of as the unique $\mathcal{G}$-module homomorphism that sends 1 to $(1-\tau)$. And more generally, multiplication on the right is always a $\mathcal{G}$-module homomorphism. So we adopt this convention now and do the same with $h_2 = \cdot \sum -i\tau^i$.

Theorem 11. *The d_is are exact, and the h_is satisfy the prism condition.*

<u>**Proof**</u>: Exactness follows from the usual Hilbert's Theorem 90 projective resolution argument for a cyclic extension. To show the prism condition,

$$\begin{aligned} h_2 d_2 = d_2 h_2 = (1-\tau)\sum_{i=0}^{d-1} -i\tau^i \ &= \sum_{i=0}^{d-1} -i\tau^i + \sum_{i=0}^{d-1} i\tau^{i+1} \\ &= -0\tau^0 + \sum_{i=1}^{d-1} -i\tau^i + \sum_{i=1}^{d-1}(i-1)\tau^i + (d-1)\tau^d \ = \sum_{i=1}^{d-1} -\tau^i + d \end{aligned}$$

while $d_1 h_1 = h_3 d_3 = \sum_{i=1}^{d-1} \tau^i$. Therefore $d_1 h_1 + h_2 d_2 = h_2 d_2 + d_3 h_3 = d\cdot$. Furthermore, the fact that

$$\text{Tr}\Delta = d\cdot$$

is the prism condition for $h_2 d_1 = d_3 h_3$, which follows from $d = [\mathcal{G}, \mathcal{H}]$.

This concludes the proof for Theorem 11. $\square$

The only remaining requirement to check for the Positselski hypotheses is the Bochstein maps being zero. In the cyclic case, all four modules are either $\mathbb{Z}$ or $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$, with each modded out by $d^2\mathbb{Z}$ to represent μ_{d^2} with a trivial action, both of which have been shown to have a zero Bochstein map in Lemma 8. Therefore the Positselski Hypotheses are satisfied by the 4 term exact sequence with homotopies defined in this section.

5.2 The Connecting Map for the Cyclic Case

Now we compute the connecting map η. Let $\overline{M}_i := M_i/dM_i$ for $i \in \{1, 2, 3, 4\}$ and let $c \in Z^{n-1}(F, \overline{M}_4)$. We define a lifting ℓ of d_3 by sending $x \in \overline{M}_4$ to $x = x \cdot 1_G \in \overline{M}_3$. The connecting map η applied to c is the composition $d_1^{-1} h_2 \delta \ell$, where δ is the chain complex map from Galois cohomology, for which the Bar Resolution is used. The next lemma will greatly reduce the complexity of the computation of η.

Lemma 12. *The following computational results are true.*

1. $\delta(\ell(c))(g_1, \ldots, g_n) = -d_2 \left(\sum_{i=0}^{k-1} \tau^i \cdot \ell(c(g_2, \ldots, g_n)) \right)$,
   where $k \in \{0, \ldots, d-1\}$ *such that* $g_1 = \tau^k$.

2. $h_2 d_2 \equiv -d_1 h_1$ *(mod* dM_2*).*

Proof: (2) follows from the prism condition at M_2. To prove (1), we will start by using a similar argument to that used in the connecting map ∂ for Arason's theorem, using the fact that ℓ is an abelian group homomorphism to make the computation of $\delta(\ell(\sigma))$ easier:

$$\begin{aligned}
\delta(\ell(c))(g_1, \ldots, g_n) &= g_1 \cdot \ell(c(g_2, \ldots, g_n)) - \ell(g_1 \cdot c(g_2, \ldots, g_n)) \\
&= \tau^k \cdot \ell(c(g_2, \ldots, g_n)) - \ell(c(g_2, \ldots, g_n)) \\
&= (\tau^k - 1) \cdot \ell(c(g_2, \ldots, g_n)) \\
&= -(1 - \tau)\left(\sum_{i=0}^{k-1} \tau^i \cdot \ell(c(g_2, \ldots, g_n))\right) \\
&= -d_2\left(\sum_{i=0}^{k-1} \tau^i \cdot \ell(c(g_2, \ldots, g_n))\right).
\end{aligned}$$

□

This next Theorem describes the connecting map in the cyclic case.

Theorem 13. *Let* $\chi_{\mathcal{H}} : \mathcal{G} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ *denote the character function that factors through the isomorphism* $\mathcal{G}/\mathcal{H} \xrightarrow{\cong} \mathbb{Z}/d\mathbb{Z}$*, namely* $\chi_{\mathcal{H}}(\tau^k) = k$*. Let* $\smile$ *denote the cup product. Then*

$$\eta(c) = \chi_{\mathcal{H}} \smile c.$$

Proof: Since $\ell(c(g_2, \ldots, g_n) = c(g_2, \ldots, g_n)$, equality

$$\tau^i \cdot c(g_2, \ldots, g_n) = c(g_2, \ldots, g_n)\tau^i$$

follows. The rest is making use of parts (1) and (2) of Lemma 12 as follows.

$$\begin{aligned}
\eta(c)(g_1,\dots,g_n) = d_1^{-1}h_2\delta\ell(c)(g_1,\dots,g_n) &\overset{(1)}{=} -d_1^{-1}h_2d_2\left(\sum_{i=0}^{k-1}\tau^i\cdot\ell(c(g_2,\dots,g_n))\right)\\
&= -d_1^{-1}h_2d_2\left(\sum_{i=0}^{k-1}c(g_2,\dots,g_n)\tau^i\right) \overset{(2)}{=} d_1^{-1}d_1h_1\left(\sum_{i=0}^{k-1}c(g_2,\dots,g_n)\tau^i\right)\\
&= h_1\left(\sum_{i=0}^{k-1}c(g_2,\dots,g_n)\tau^i\right) = Tr\left(\sum_{i=0}^{k-1}c(g_2,\dots,g_n)\tau^i\mathcal{H}\right)\\
&= \sum_{i=0}^{k-1}c(g_2,\dots,g_n) = kc(g_2,\dots,g_n) = (\chi_{\mathcal{H}}\smile c)(g_1,\dots,g_n).
\end{aligned}$$

This concludes the proof of Theorem 13. □

In view of Theorems 11 and 13 the machinery in Theorem 4 gives the following result.

Theorem 14. *In the cyclic case we have the following 6-term exact sequence.*

$$H^n(E,\mu_d)\oplus H^n(F,\mu_d)\xrightarrow{d_2+h_3}H^{n+1}(E,\mu_d)\xrightarrow{d_3}H^n(F,\mu_d)\xrightarrow{\eta}H^{n+1}(F,\mu_d)\xrightarrow{d_1}H^{n+1}(E,\mu_d)\xrightarrow{h_1\oplus d_2}H^n(F,\mu_d)\oplus H^n(E,\mu_d)$$

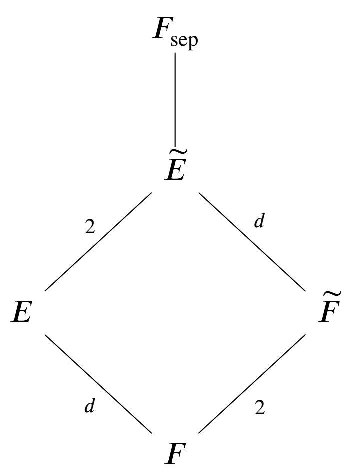
where d_3 is the norm, $\eta(c)=\chi\smile c$ and d_1 is scalar extension.

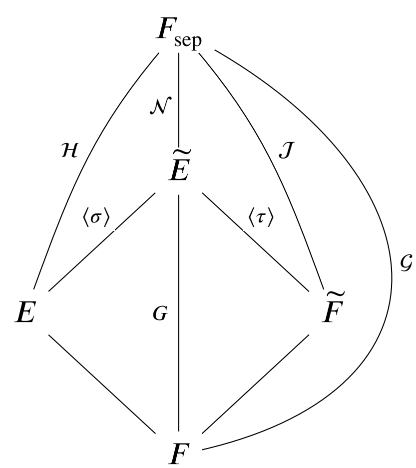
6. The Dihedral Case

We noted in the introduction that the case where $[E : F] = 4$ and $\mathrm{Gal}(\widetilde{E}/F)$ is the dihedral group of order 8 was handled by Positselski in [Pos05]. In this section we turn to the dihedral case where $[E : F] = d$ is odd.

6.1 The 4 Term Exact Sequence with Homotpies

For this section, we have the following notation. Let G be a dyhedral group, $G = \langle \sigma, \tau \rangle$ with $|\tau| = d$, $|\sigma| = 2$ and the relation $\sigma\tau = \tau^{-1}\sigma$. We use the notation described earlier. In this case the diagrams of fields and groups are as follows.





Before the module structure is set up for the Positselski machinery, the author would like to comment about the exact sequences of modules up until this point in order to communicate some motivation for the generalizations in this section and the next.

Remark: Each 4-term exact sequence can be seen as being made from 2 short exact sequences. In the previous cyclic case, when $G = \langle \tau \rangle$, we can make a connection to the following: There is a projective resolution for $\mathbb{Z}$ commonly used in a proof of Hilbert's Theorem 90 for cyclic Galois extensions that has a repeating pattern

$$\cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G] \xrightarrow{\cdot T_\tau} \mathbb{Z}[G] \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G] \xrightarrow{\cdot T_\tau} \mathbb{Z}[G]T_\tau \longrightarrow 0$$

where T_τ is the "trace" element $1 + \tau + \cdots + \tau^{d-1}$ that yields the norm map when acting on a multiplicative field element. This long exact sequence can be viewed as two alternating short exact sequences:

$$0 \longrightarrow \mathbb{Z}[G]T_\tau \xrightarrow{\subseteq} \mathbb{Z}[G] \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G] \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}[G](1-\tau) \xrightarrow{\subseteq} \mathbb{Z}[G] \xrightarrow{\cdot T_\tau} \mathbb{Z}[G]T_\tau \longrightarrow 0$$

These short exact sequences are exactly the two that make up the 4-term exact sequence in the cyclic case. For the more specific case where $d = 2$ with μ_2 being our roots of unity, both sequences are the same, allowing us to both begin and end with $\mathbb{Z}/2\mathbb{Z}$ as a trivial G-module and avoid the need for Positselski's machinery altogether. This is one way of viewing Arason's sequence.

With the above remark in mind, we will introduce the modular framework for the dihedral case. It will be a 4-term exact sequence comprised of the two short exact sequences.

$$0 \longrightarrow \mathbb{Z}[G]T_\sigma T_\tau \xrightarrow{\subseteq} \mathbb{Z}[G]T_\sigma \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G]T_\sigma(1-\tau) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z}[G](1-\tau)\text{Б} \xrightarrow{\subseteq} \mathbb{Z}[G]\text{Б} \xrightarrow{\cdot T_\tau} \mathbb{Z}[G]T_\tau\text{Б} \longrightarrow 0$$

where T_σ is the corresponding "trace" for σ, $(1+\sigma)$, and $\text{Б} = (1-\sigma\tau) \in \mathbb{Z}[G]$. Of course, we need $\mathbb{Z}[G]T_\sigma(1-\tau) \cong \mathbb{Z}[G](1-\tau)\text{Б}$ for the two short exact sequences to build a 4-term exact sequence, and in fact $T_\sigma(1-\tau) = (1-\tau)\text{Б}$. Thus, the 4-term exact sequence is

$$\mathbb{Z} \xrightarrow{\Delta} \text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G]\text{Б} \xrightarrow{\cdot T_\tau} \mathbb{Z}[G]\text{Б}T_\tau.$$

Our first lemma describes the induced module $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}$ needed for this case and relates it to $\mathbb{Z}[G]T_\sigma$ for computation.

Lemma 15. *Let $\tilde{\sigma}$ and $\tilde{\tau}$ be liftings of σ and τ to $\mathcal{G}$. Then for the trivial $\mathbb{Z}[\mathcal{G}]$-module $\mathbb{Z}$, the map $\phi : Ind_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \cong \mathbb{Z}[G]T_\sigma$ given by*

$$\phi(f) = \sum_{i=0}^{d-1} f(\tilde{\tau}^i)\tau^{-i}T_\sigma$$

is an isomorphism of $\mathbb{Z}[G]$-modules.

Proof: The proof is similar to that for the cyclic case. Note that this type of proof would not work using $\mathbb{Z}[\langle\tau\rangle]$ instead of $\mathbb{Z}[G]T_\sigma$ because in this case $\mathcal{H}$ is not a normal subgroup of $\mathcal{G}$.

However, $\mathcal{N}$ is normal in $\mathcal{G}$ and we will use this fact. The general bijection statement will be skipped and we proceed to check the compatibility of $\mathcal{G}$-actions. Every element of $\mathcal{G}$ has a unique expression $n\tilde{\sigma}^m\tilde{\tau}^k$, $n \in \mathcal{N}$. Let $\theta_m : \mathbb{Z}/d\mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ the σ^m-conjugation automorphisms so that $\sigma^m\tau^i = \tau^{\theta_m(i)}\sigma^m$ and $\tau^i\sigma^m = \sigma^m\tau^{\theta_{-m}(i)}$. And let $n_{i,m} = \tilde{\tau}^i n\tilde{\sigma}^m\tilde{\tau}^{\sigma_{-m}(-i)}\tilde{\sigma}^{-m} \in \mathcal{N}$ so that $\tilde{\tau}^i n\tilde{\sigma}^m = n_{i,m}\tilde{\sigma}^m\tilde{\tau}^{\theta_{-m}(i)}$. Then

$$\begin{aligned}
\phi(n\tilde{\sigma}^m\tilde{\tau}^k \cdot f) &= \sum_{i=0}^{d-1}(n\tilde{\sigma}^m\tilde{\tau}^k \cdot f)(\tau^i)\tau^{-i}T_\sigma \;=\; \sum_{i=0}^{d-1} f(\tilde{\tau}^i n\tilde{\sigma}^m\tau^k)\tau^{-i}T_\sigma \\
&= \sum_{i=0}^{d-1} f(n_{i,m}\tilde{\sigma}^m\tilde{\tau}^{\theta_{-m}(i)}\tau^k)\tau^{-i}T_\sigma \;=\; \sum_{i=0}^{d-1}(n_{i,m}\tilde{\sigma}^m)\cdot f(\tilde{\tau}^{\theta_{-m}(i)}\tau^k)\tau^{-i}T_\sigma \\
&= \sum_{i=0}^{d-1} f(\tilde{\tau}^{\theta_{-m}(i)+k})\tau^{-i}T_\sigma \;=\; \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\tau^{-\theta_m(j-k)}T_\sigma \;=\; \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\tau^{-\theta_m(j-k)}\sigma^m T_\sigma \\
&= \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\sigma^m\tau^{-(j-k)}T_\sigma \;=\; \sum_{j=0}^{d-1} f(\tilde{\tau}^j)\sigma^m\tau^k\tau^{-j}T_\sigma \;=\; (\sigma^m\tau^k)\cdot\phi(f) \\
&= (n\sigma^m\tau^k)\cdot\phi(f).
\end{aligned}$$

This completes the proof of Lemma 15. □

Remark: The fact that $|\sigma| = 2$ was not used in this proof. And indeed, this same proof can be used in the analogous claim in the more general semi-direct case later. We will therefore refer to this lemma for the Semi-direct case as well.

With $\mathbb{Z}[G]T_\sigma \cong \mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$ established, we move on to defining the homomorphisms.

Definition 16. *Let d_1, d_2, d_3 be the maps*

$$0 \longrightarrow \mathbb{Z}[G]T_\sigma T_\tau \xrightarrow{\subseteq} \mathbb{Z}[G]T_\sigma \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[G]\text{Б} \xrightarrow{\cdot T_\tau} \mathbb{Z}[G]\text{Б}T_\tau \longrightarrow 0 \tag{6.1}$$

Let h_1, h_2, h_3 be the homotopy maps

$$\mathbb{Z}[G]T_\sigma T_\tau \xleftarrow{\cdot T_\tau} \mathbb{Z}[G]T_\sigma \xleftarrow{h_2} \mathbb{Z}[G]\text{Б} \xleftarrow{\supseteq} \mathbb{Z}[G]\text{Б}T_\tau \tag{6.2}$$

where

$$h_2(\text{Б}) = \sum_{i=0}^{d-1} \left(\frac{d-1}{2} - i\right) \tau^i T_\sigma.$$

<u>**Observation**</u>: $\mathbb{Z}[G]T_\sigma T_\tau = \mathbb{Z}[G]\left(\sum_{g\in G} g\right) \cong \mathbb{Z}$ is a trivial $\mathbb{Z}[G]$-module, while $\mathbb{Z}[G]\text{Б}T_\tau$ has a trivial τ-action but σ and $\sigma\tau$ act as multiplication by (-1).

The main result needed for applying Positselski's machinery is given next.

Theorem 17. *The sequence (6.1) is an exact sequence, and the homotopy maps in sequence (6.2) satisfy the prism condition. Furthermore, $d_1, d_2, d_3, h_1, h_2, h_3$ are $\mathbb{Z}[G]$-module homomorphisms.*

Proof: The exactness of the d_i's follows from the remark at the beginning of this section, so we move on to checking the prism condition. For the first and last modules, we have multiplication by d because $T_\tau T_\tau = dT_\tau$. For the second module, $d_1 h_1(T_\sigma) = T_\tau T_\sigma$ and

$$\begin{aligned} h_2 d_2(T_\sigma) &= h_2(T_\sigma(1-\tau)) \;=\; h_2((1-\tau)\text{Б}) \;=\; (1-\tau)\sum_{i=0}^{d-1}\left(\frac{d-1}{2} - i\right)\tau^i T_\sigma \\ &= (1-\tau)\sum_{i=0}^{d-1}(-i)\,\tau^i T_\sigma \;=\; (d - T_\tau)T_\sigma \end{aligned}$$

Therefore $(d_1 h_1 + h_2 d_2)(T_\sigma) = T_\tau T_\sigma + (d - T_\tau)T_\sigma = dT_\sigma$. To verify the prism condition for the third module, $h_3 d_3(Б) = БT_\tau = T_\tau Б$ and

$$\begin{aligned} h_2 d_2(Б) &= d_2\left(\sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i T_\sigma\right) = \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i T_\sigma(1-\tau) \\ &= \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i(1-\tau)Б = \sum_{i=0}^{d-1}(-i)\,\tau^i(1-\tau)Б \\ &= (d - T_\tau)Б \end{aligned}$$

Therefore $(h_3 d_3 + h_2 d_2)(Б) = T_\tau Б + (d - T_\tau)Б = dБ$.

Every one of these maps except for h_2 is defined by left multiplication in the group ring $\mathbb{Z}[G]$, so it remains to check that h_2 preserves action by $\mathcal{G}$. h_2 is also map from a cyclic $\mathbb{Z}[G]$-module to a cyclic $\mathbb{Z}[G]$-module defined by sending one generator to another, it need only be checked that the annihilator of Б, which is $\mathbb{Z}[G](1 + \sigma\tau)$, also annihilates the image of h_2. We will check this by showing that h_2 preserves the action of $\sigma\tau$.

$$\begin{aligned} \sigma\tau \cdot h_2(Б) &= \sigma\tau \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i T_\sigma = \sigma \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^{i+1} T_\sigma \\ &= \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^{d-(i+1)}\sigma T_\sigma = \sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^{(d-1)-i} T_\sigma \\ &= \sum_{i=0}^{d-1}\left(-\frac{d-1}{2}+((d-1)-i)\right)\tau^{(d-1)-i} T_\sigma = \sum_{j=0}^{d-1}\left(-\frac{d-1}{2}+j\right)\tau^i T_\sigma \\ &= -\sum_{j=0}^{d-1}\left(\frac{d-1}{2}-j\right)\tau^i T_\sigma = h_2(-Б) = h_2(\sigma\tau \cdot Б) \end{aligned}$$

This concludes the proof of Lemma 17. ◻

The only remaining requirement to check for the Positselski hypotheses is that the Bochstein homomorphisms are zero. The first two modules are isomorphic to $\mathbb{Z}$ and the induced module $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$ respectively. Both of these modules were shown to have zero Bochstein maps in the general setup section. For M_3 and M_4, we will show that as $\mathcal{J}$-modules, $M_3 \cong \mathbb{Z}[J] \cong \mathrm{Ind}_{\mathcal{N}}^{\mathcal{J}}(\mathbb{Z})$ and that $M_4 \cong \mathbb{Z}$ as a trivial $\mathcal{J}$-module. Then Lemma 8 will imply that the Bochstein map to vanishes for M_3 and M_4. The isomorphisms to be defined send Б $\mapsto 1 \in \mathbb{Z}[\mathcal{J}]$ and Б$T_\tau \mapsto 1 \in \mathbb{Z}$ respectively.

Lemma 18. *As $\mathcal{J}$-modules,*

1. $M_3 = \mathbb{Z}[G]$Б $\cong Ind_{\mathcal{N}}^{\mathcal{J}}(\mathbb{Z})$
2. $M_4 = \mathbb{Z}[G]$Б$T_\tau \cong Ind_{\mathcal{J}}^{\mathcal{J}}(\mathbb{Z})$

Proof: There are two steps.

1. We begin with the observation that

$$M_3 = \mathbb{Z}[G] \cdot (1 - \sigma\tau) = \mathbb{Z}[\langle \tau \rangle] \cdot (1 - \sigma\tau).$$

The second equality above follows from

$$\tau^i \sigma(1 - \sigma\tau) = \tau^{i+1} \cdot \sigma\tau(1 - \sigma\tau) = -\tau^{i+1}(1 - \sigma\tau)$$

and hence

$$\left(\sum_{i=0}^{d-1} c_{i,0}\tau^i + \sum_{i=0}^{d-1} c_{i,1}\tau^i\sigma \right)(1 - \sigma\tau) = \left(\sum_{i=0}^{d-1} (c_{i,0} - c_{i,1}\tau)\tau^i \right)(1 - \sigma\tau) \in \mathbb{Z}[\langle \tau \rangle](1 - \sigma\tau).$$

Furthermore, the set $\{\tau^i(1-\sigma\tau \mid 0 \leq i \leq d-1\}$ is a $\mathbb{Z}$-basis for M_3, with $\tau^j \cdot \tau^i(1-\sigma\tau) = \tau^{j+i}(1-\sigma\tau)$.

2. Similarly

$$M_4 = \mathbb{Z}[G](1-\sigma\tau)T_\tau = \mathbb{Z}[\langle\tau\rangle](1-\sigma\tau)T_\tau = \mathbb{Z}[\langle\tau\rangle]T_\tau(1-\sigma\tau) = \mathbb{Z}T_\tau(1-\sigma\tau) = \mathbb{Z}(1-\sigma\tau)T_\tau$$

and $\mathbb{Z}(1-\sigma\tau)T_\tau \cong \mathbb{Z}$ via the $\mathcal{J}$-module isomorphism $(1-\sigma\tau)T_\tau \mapsto 1$.

This concludes the proof of Lemma 18. □

Now, with M_3 isomorphic to $\mathbb{Z}[J]$ and M_4 isomorphic to $\mathbb{Z}$ as $\mathcal{J}$-modules, the fact that $\widetilde{E}/\widetilde{F}$ is a cyclic extension allows the application of Lemma 8 to reduce the problem to the cyclic case. This makes the Bochstein maps zero for M_3 and M_4 as well as M_1 and M_2. Therefore the Positselski Hypotheses are satisfied by the four module exact sequence with homotopies defined in this section.

6.2 The Connecting Map for the Dihedral Case

Now we compute the connecting map η. Let $\overline{M}_i := M_i/dM_i$ for $i \in \{1,2,3,4\}$. In this section we will use the exact sequence of modules with homotopies defined in the previous section to describe the connecting map $\eta : H^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow H^n(\mathcal{G}, \overline{M}_1)$ given by Positselski's machinery.

Definition 19. *: Given the above notation we define the following.*

1. *$\ell : \overline{M}_4 \longrightarrow \overline{M}_3$, the d_3-lifting defined as $\ell(z\text{Б}T_\tau) = z\text{Б}$ for any $z \in \mathbb{Z}/d\mathbb{Z}$.*

2. *$\delta : C^{n-1}(\mathcal{G}, \overline{M}_3) \longrightarrow C^n(\mathcal{G}, \overline{M}_3)$ the cochain map from the bar resolution.*

3. $\widetilde{\eta}_\ell = \widetilde{\eta} : Z^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow Z^n(\mathcal{G}, \overline{M}_1),\ \widetilde{\eta}(c) := d_1^{-1} h_2 \delta \ell(c).$

4. $\eta : H^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow H^n(\mathcal{G}, \overline{M}_1),\ \eta([c]) := [\widetilde{\eta}(c)].$

Observation: Our choice of lifting ℓ is a $\mathbb{Z}$-module homomorphism, though it is not a $\mathbb{Z}[G]$-module homomorphism. Note, for $x \in \operatorname{im}(d_1)$, we let $d_1^{-1}(x)$ denote the unique preimage element.

We collect some basic properties of these maps next.

Lemma 20. *Let $c \in Z^{n-1}(\mathcal{G}, \overline{M}_4)$ be a cocycle, $g_1, \dots, g_n \in \mathcal{G}$, and let $c' \in \mathbb{Z}/d\mathbb{Z}$ such that $c(g_2, \dots, g_n) = c' \cdot \text{Б} T_\tau$. Let $(\sigma\tau)^j \tau^i$, an element of $\mathcal{G}/\mathcal{N}$, be the coset of g_1. Then*

1. $\delta(\ell(c))(g_1, \dots, g_n) = g_1 \cdot \ell(c(g_2, \dots, g_n)) - \ell(g_1 \cdot c(g_2, \dots, g_n))$
   $= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1-\tau)\text{Б}.$

2. *For any $x \in C^n(\mathcal{G}, \overline{M}_2)$ such that $\delta(\ell(c)) = d_2(x)$, $\widetilde{\eta}(c) = -h_1(x)$.*

Proof: For the first part, by definition,

$$\delta(\ell(c))(g_1, \dots, g_n) = g_1 \cdot \ell(c(g_2, \dots, g_n)) - \ell(c(g_1 g_2, \dots, g_n)) + \dots \pm \ell(c(g_1, \dots, g_{n-1}))$$

We will next subtract an expanded form of $\ell(\delta(c))(g_1, \dots, g_n)$ from the right side of this equation. This term is 0 because ℓ is a $\mathbb{Z}$-homomorphism and c is a cocycle. Subtracting the expanded form will leave us with only two remaining terms. The expansion is as follows.

$$
\begin{aligned}
&\ell(\delta(c))(g_1, \dots, g_n) \\
&= \ell(\delta(c)(g_1, \dots, g_n)) \\
&= \ell\left(g_1 \cdot c(g_2, \dots, g_n) - c(g_1 g_2, \dots, g_n) + \cdots \pm c(g_1, \dots, g_{n-1})\right) \\
&= \ell(g_1 \cdot c(g_2, \dots, g_n)) - \ell(c(g_1 g_2, \dots, g_n)) + \cdots \pm \ell(c(g_1, \dots, g_{n-1})))
\end{aligned}
$$

We can now subtract and simplify

$$
\begin{aligned}
&\delta(\ell(c))(g_1, \dots, g_n) \\
&\quad - 0 \\
&= \delta(\ell(c))(g_1, \dots, g_n) \\
&\quad - \ell(\delta(c))(g_1, \dots, g_n) \\
&= g_1 \cdot \ell(c(g_2, \dots, g_n)) - \ell(c(g_1 g_2, \dots, g_n)) + \cdots \pm \ell(c(g_1, \dots, g_{n-1})) \\
&\quad - \ell(g_1 \cdot c(g_2, \dots, g_n)) + \ell(c(g_1 g_2, \dots, g_n)) - \cdots \mp \ell(c(g_1, \dots, g_{n-1})) \\
&= g_1 \cdot \ell(c(g_2, \dots, g_n)) \\
&\quad - \ell(g_1 \cdot c(g_2, \dots, g_n))
\end{aligned}
$$

This shows the first equality in part (1) of the lemma.

For the next equality, we replace g_1 with $\sigma^j \tau^i$, g_1's coset representative in $\mathcal{G}/\mathcal{N}$, and use the fact that ℓ preserves the action by $\sigma\tau$, which is also multiplication by (-1) for both $Б \in \overline{M}_3$

and Б$T_\tau \in \overline{\boldsymbol{M}}_4$.

$$\begin{aligned}
\delta(\ell(c))(g_1, \dots, g_n) &= g_1 \cdot \ell(c(g_2, \dots, g_n)) - \ell(g_1 \cdot c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \tau^i \cdot \ell(c(g_2, \dots, g_n)) - \ell((\sigma\tau)^j \tau^i \cdot c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \tau^i \cdot \ell(c(g_2, \dots, g_n)) - \ell((\sigma\tau)^j c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \tau^i \cdot \ell(c(g_2, \dots, g_n)) - \ell((-1)^j c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \tau^i \cdot \ell(c(g_2, \dots, g_n)) - (-1)^j \ell(c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \tau^i \cdot \ell(c(g_2, \dots, g_n)) - (\sigma\tau)^j \ell(c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j \left(\tau^i \cdot \ell(c(g_2, \dots, g_n)) - \ell(c(g_2, \dots, g_n))\right) \\
&= (\sigma\tau)^j (\tau^i - 1) \cdot \ell(c(g_2, \dots, g_n)) \\
&= (\sigma\tau)^j (\tau^i - 1) \cdot c' \text{Б} \\
&= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right)(1 - \tau) \cdot c' \text{Б} \\
&= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1 - \tau) \text{Б}
\end{aligned}$$

This proves the first statement of the lemma.

The second part of the lemma follows from the prism condition. Mod d,

$$d_1 h_1 + h_2 d_2 = \cdot d = 0$$

This means

$$h_2 d_2 = -d_1 h_1$$

And hence

$$
\begin{aligned}
\eta(c) &= (d_1^{-1})(h_2(\delta(\ell(c))) \\
&= (d_1^{-1})(h_2(d_2(x))) \\
&= (d_1^{-1})(-d_1(h_1(x))) \\
&= -h_1(x)
\end{aligned}
$$

This concludes the proof of Lemma 20. □

As an application we obtain a description of the connecting map η in this case. Although it is not a cup product, its description is almost one.

Corollary 21. *Let $c \in Z^{n-1}(\mathcal{G}, \overline{M}_4)$, $g_1, g_2, \dots, g_n \in \mathcal{G}$, $c' \in \mathbb{Z}/d\mathbb{Z}$ such that $c(g_2, \dots, g_n) = c'\text{Б}T_\tau$. Let $\sigma^i\tau^j$, an element of $\mathcal{G}/\mathcal{N}$, be the coset of g_1. Then*

$$\widetilde{\eta}(c)(g_1, \dots, g_n) = ic' \in \overline{M}_1.$$

Proof: From part (1) of Lemma 20, we know that

$$\delta(\ell(c))(g_1, \dots, g_n) = (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k \right) \cdot c'(1-\tau)\text{Б}.$$

We will first find an $x \in \overline{M}_2$ with this d_2-image, and then use part (2) of Lemma 20 to compute $\widetilde{\eta}(c)$ by finding $-h_1(x)$. This process begins by showing that $(\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1+\sigma)$ is a suitable choice for x.

$$\begin{aligned}
\delta(\ell(c))(g_1, \dots, g_n) &= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1-\tau)\text{Б} \\
&= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1+\sigma)(1-\tau) \\
&= (\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1+\sigma)(1-\tau) \\
&= d_2\left((\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1+\sigma)\right)
\end{aligned}$$

Applying part (b) allows the computation of $\eta(c)(g_1, \dots, g_n)$ as follows.

$$\begin{aligned}
\widetilde{\eta}(c)(g_1, \dots, g_n) &= -h_1\left((\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) \cdot c'(1+\sigma)\right) \\
&= -(\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) c' \cdot h_1\,(1+\sigma) \\
&= -(\sigma\tau)^j \left(-\sum_{k=0}^{i-1} \tau^k\right) c' \cdot 1 \\
&= ic' \cdot 1 \\
&= ic'
\end{aligned}$$

This concludes the proof of Corollary 21 . □

Let $\chi : \mathcal{G} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ be defined by $\chi((\sigma\tau)^j\tau^i\mathcal{N}) = i$. It should be noted that in this case χ is no longer a character because it is not a homomorphism. However, the Corollary shows that the map η can be understood through a cup-product like structure which we denote by $\smile'$ whereby $(\chi \smile' c)(g_1, \dots, g_n) = \chi(g_1) \cdot c(g_2, \dots, g_n)$.

In view of Theorem 17 and Corollary 21 the machinery in Theorem 4 gives the following result.

Theorem 22. *In the dihedral case we have the following 6-term exact sequence.*

$$H^n(E, \mu_d) \oplus H^n(\mathcal{G}, \overline{M}_4) \xrightarrow{d_2 + h_3} H^n(\mathcal{G}, \overline{M}_3) \xrightarrow{d_3} H^n(\mathcal{G}, \overline{M}_4) \xrightarrow{\eta}$$

$$H^{n+1}(F, \mu_d) \xrightarrow{d_1} H^{n+1}(E, \mu_d) \xrightarrow{h_1 \oplus d_2} H^{n+1}(F, \mu_d) \oplus H^{n+1}(\mathcal{G}, \overline{M}_3)$$

where $\eta(c) = \chi \smile' c$ and d_1 is scalar extension.

Corollary 23. *η induces an isomorphism*

$$\frac{H^n(\mathcal{G}, \overline{M}_4)}{d_3 H^n(\mathcal{G}, \overline{M}_3)} \xrightarrow{\cong} H^n(E/F).$$

In Theorem 46, it will be shown that $H^n(\widetilde{F}, \mu_d)$ maps onto $H^n(\mathcal{G}, \overline{M}_4)$ in such a way that the image of the corestriction from $H^n(\widetilde{E}, \mu_d)$, $\mathrm{cor}_{\widetilde{E}/\widetilde{F}}$, maps onto the image of d_3 from $\overline{M}_3$. This will be used to characterize the cohomological kernel for the dihedral setup as follows:

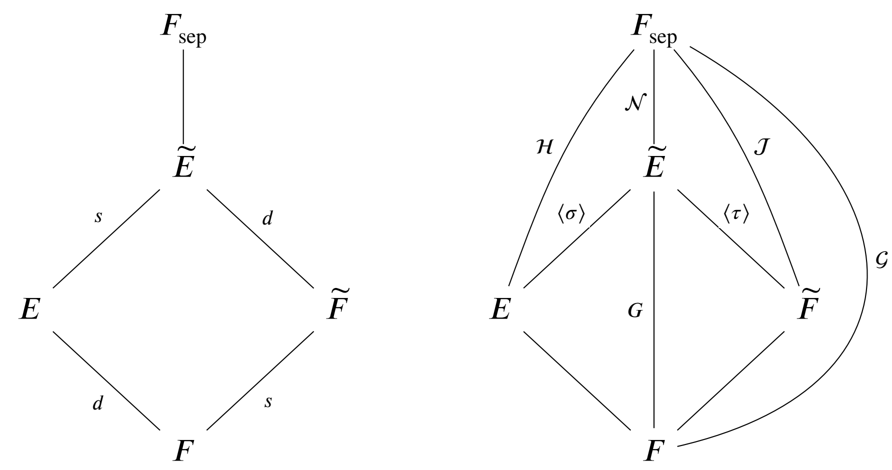
$$\frac{H^n(\widetilde{F}, \mu_d)}{\mathrm{cor}_{\widetilde{E}/\widetilde{F}} H^n(\widetilde{E}, \mu_d)} \cong H^n(E/F).$$

7. The Semi-Direct Case

This section expands the ideas of the previous section. Technically, the previous two sections could be interpreted as applications of the results contained in this section with $T_\sigma = Б = 1$ in the cyclic case, but for this thesis we decided it would be convenient to spell them out to illustrate the development.

7.1 The 4 Term Exact Sequence with Homotopies

For this section we adopt the following notation: G is a semi-direct product of $\langle \tau \rangle$ by $\langle \sigma \rangle$. We assume the order of τ is d, the order of σ is s and we will assume that s is even and divides $d-1$ (so d is odd.) We denote by $\theta : \{0, 1, \ldots, d-1\} \to \{0, 1, \ldots, d-1\}$ be defined by conjugation in $\langle \tau \rangle$ by σ, that is, $\sigma \tau^i \sigma^{-1} = \tau^{\theta(i)}$. We define θ_j by $\sigma^j \tau \sigma^{-j} = \tau^{\theta_j}$ (in fact $\theta_j = \theta^j(1)$ where the latter is the j'th iterate of θ, but the notation θ_j is less cumbersome.) We assume that θ has order s, that is, conjugation by σ on $\langle \tau \rangle$ has order s. As τ has odd order this means $\sigma^{\frac{s}{2}} \tau^i \sigma^{-\frac{s}{2}} = \tau^{-i}$ for all i. From this, $\theta_{j+\frac{s}{2}} \equiv -\theta_j \pmod{d}$ and since $0 < \theta_j < d$ we must have $\theta_j + \theta_{j+\frac{s}{2}} = d$. Here are the diagrams of the fields and Galois groups.



In order to generalize from the dihedral case a special element $\text{Б}_{d,s} = \text{Б} \in \mathbb{Z}[G]$ is the essential tool. It is described next.

Definition 24. *For τ and σ as above,*

$$\text{Б}_{d,s} = \text{Б} = (1 - \sigma^{\frac{s}{2}})\tau^{\frac{d+1}{2}} \sum_{j=0}^{\frac{s}{2}-1} \left(\sum_{i=0}^{\theta_j - 1} \tau^i \right) \sigma^j.$$

We set $T_{\sigma,s} = T_\sigma = 1 + \sigma + \sigma^2 + \cdots + \sigma^{s-1}$ and define

$$C_{d,s,i} = C_i = \tau^i T_\sigma (1 - \tau).$$

We begin with basic properties of these elements.

Lemma 25. *Given the above assumptions and notation we have,*

(i) $\sigma^{\frac{s}{2}} \cdot \text{Б} = -\text{Б}$.

(ii) $(1 - \tau)\text{Б} = C_{\frac{d+1}{2}}$.

Proof: Part (i) is clear by the definition of Б since $\sigma^{\frac{s}{2}}(1-\sigma^{\frac{s}{2}}) = -(1-\sigma^{\frac{s}{2}})$.

For (ii) we begin with some facts that facilitate the proof. Because $\tau\sigma^{\frac{s}{2}} = \sigma^{\frac{s}{2}}\tau^{-1}$, it follows that

$$(1-\tau)(1-\sigma^{\frac{s}{2}}) = 1-\tau+\sigma^{\frac{s}{2}}\tau^{-1}-\sigma^{\frac{s}{2}} = (1+\sigma^{\frac{s}{2}}\tau^{-1})(1-\tau).$$

This allows us to make the substitution

$$(1-\tau)(1-\sigma^{\frac{s}{2}})\tau^{\frac{d+1}{2}} = \tau^{\frac{d+1}{2}}(1+\sigma^{\frac{s}{2}})(1-\tau)$$

Now, multiplying the inner sum in the defintion of Б by $(1-\tau)$ results in

$$(1-\tau)\left(\sum_{i=0}^{\theta_j-1}\tau^i\right) = 1-\tau^{\sigma_j}.$$

These facts allow us to show (ii) as follows.

$$\begin{aligned}
(1-\tau)\text{Б} &= (1-\tau)(1-\sigma^{\frac{s}{2}})\tau^{\frac{d+1}{2}}\sum_{j=0}^{\frac{s}{2}-1}\left(\sum_{i=0}^{\theta_j-1}\tau^i\right)\sigma^j \\
&= \tau^{\frac{d+1}{2}}(1+\sigma^{\frac{s}{2}})(1-\tau)\sum_{j=0}^{\frac{s}{2}-1}\left(\sum_{i=0}^{\theta_j-1}\tau^i\right)\sigma^j \\
&\overset{(1)}{=} \tau^{\frac{d+1}{2}}(1+\sigma^{\frac{s}{2}})\sum_{j=0}^{\frac{s}{2}-1}\left(1-\tau^{\theta_j}\right)\sigma^j \\
&\overset{(2)}{=} \tau^{\frac{d+1}{2}}(1+\sigma^{\frac{s}{2}})\sum_{j=0}^{\frac{s}{2}-1}\sigma^j(1-\tau) \\
&= \tau^{\frac{d+1}{2}}T_\sigma(1-\tau) \\
&= C_{\frac{d+1}{2}}
\end{aligned}$$

with (1) using the geometric series identity $(1-\tau)\sum_{i=0}^{\theta_j-1}\tau^i = 1-\tau^{\theta_j}$ and (2) following from $(1+\sigma^{\frac{s}{2}})\sum_{j=0}^{\frac{s}{2}-1}\sigma^j$ being the sum of every power of σ, i.e. T_σ. This shows directly that $C_{\frac{d+1}{2}} = (1-\tau)Б$ giving (ii). □

We next define the modules we need.

Definition 26. *As in the dihedral case, we define all four modules to be submodules of $\mathbb{Z}[G]$, defined as follows.*

$$
\begin{aligned}
M_1 &= \mathbb{Z}[G]T_\sigma T_\tau \cong \mathbb{Z} \text{ as a trivial } \mathcal{G}\text{-module} \\
M_2 &= \mathbb{Z}[G]T_\sigma = \sum_{i=0}^{d-1}\mathbb{Z}\tau^i T_\sigma \\
M_3 &= \mathbb{Z}[G]Б = \sum_{j=0}^{\frac{s}{2}-1}\mathbb{Z}[\langle\tau\rangle]\sigma^j Б \\
M_4 &= \mathbb{Z}[G]БT_\tau = \sum_{j=0}^{\frac{s}{2}-1}\mathbb{Z}\cdot\sigma^j БT_\tau
\end{aligned}
$$

Lemma 15 (using the remark afterward) implies that $\mathrm{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z}) \cong \mathbb{Z}[G]T_\sigma$. This next lemma gives key properties of the modules.

Lemma 27.

$$\begin{aligned}\mathbb{Z}[G]\text{Б} &= \mathbb{Z}[G](1-\tau)\text{Б} \oplus \mathbb{Z}[\langle\sigma\rangle]\text{Б} \\ &= M_3' \oplus M_{\text{Б}} \\ &\quad \textit{where } M_3' = \sum_{i=0}^{d-1} \mathbb{Z}\cdot\tau^i T_\sigma(1-\tau) \quad \textit{and} \\ &\quad M_{\text{Б}} = \sum_{j=0}^{\frac{s}{2}-1} \mathbb{Z}\cdot\sigma^j \text{Б}_{d,s} \textit{ and the direct sum is that of} \\ &\quad \mathbb{Z}-\textit{modules, not of } \mathcal{G}\textit{-modules.}\end{aligned}$$

Proof: For the (additive) direct summands of M_3, the first identification

$$\mathbb{Z}[G](1-\tau)\text{Б} = \sum_{i=0}^{d-1} \mathbb{Z}\cdot\tau^i T_\sigma(1-\tau)$$

follows from Lemma 25(i) while the second identification

$$\mathbb{Z}[\langle\sigma\rangle]\text{Б} = \sum_{j=0}^{\frac{s}{2}-1} \mathbb{Z}\cdot\sigma^j \text{Б}$$

follows from Lemma 25 (ii). □

We also note that the $\mathbb{Z}$-ranks of M_1, M_2, M_3, and M_4 are, respectively 1, d, $d-1+\frac{s}{2}$, and $\frac{s}{2}$ (although for the latter, one has to check the linear independence of the $\sigma^j\text{Б}$ from M_3'.) We next define the d_i maps, which are essentially the same as those from the dihedral case.

Definition 28. *The $d_i : M_i \to M_{i+1}$ are as follows:*

$d_1 : M_1 \longrightarrow M_2$ is the inclusion $\mathbb{Z}[G]T_\sigma T_\tau \xrightarrow{\subseteq} \mathbb{Z}[G]T_\sigma$.

If we view M_1 as $\mathbb{Z}$, then $d_1(n) = \sum_{i=0}^{d-1} n\tau^i T_\sigma$.

$d_2 : M_2 \to M_3$ is given by $\cdot(1-\tau) : \mathbb{Z}[G]T_\sigma \longrightarrow \mathbb{Z}[G]T_\sigma(1-\tau) \subseteq \mathbb{Z}[G]\text{Б}$.

with "$\subseteq$" coming from the identity $\mathbb{Z}[G]T_\sigma(1-\tau) = \mathbb{Z}[G](1-\tau)\text{Б}$ in Lemma 25 ii).

$d_3 : M_3 \to M_4$ is given by $\cdot T_\tau : \mathbb{Z}[G]\text{Б} \longrightarrow \mathbb{Z}[G]\text{Б}T_\tau$.

We note that each map is a $\mathcal{G}$-module homomorphism. The map d_1, which is an inclusion, can be thought of as the diagonal embedding if M_1 is viewed as $\mathbb{Z}$, with an image that has a trivial $\mathcal{G}$-action. The map d_2 is right mulplitplication by $(1-\tau)$ and hence is a $\mathcal{G}$-map. By construction M_3' is the image of d_2. The map d_3 is right multiplication by T_τ and can be viewed as a trace map on the right summand of M_3. The trace and is therefore trivial on M_3' and need only be defined on $M_\text{Б}$. We finally note that $M_4 \hookrightarrow \mathbb{Z}[G]T_\tau = \mathbb{Z}[G/\langle\tau\rangle] \cong \text{Ind}_{\mathcal{J}}^{\mathcal{G}}(\mathbb{Z})$ by $\text{Б}T_\tau \mapsto (1-\sigma^{\frac{s}{2}})T_\tau \in \mathbb{Z}[G]T_\tau$.

The h_i maps are given next.

Definition 29. *The $h_i : M_{i+1} \to M_i$ are as follows:*

$h_1 : \mathbb{Z}[G]T_\sigma \longrightarrow \mathbb{Z}[G]T_\sigma T_\tau := \cdot T_\tau$ is given by $xT_\sigma \mapsto xT_\sigma T_\tau$.

$h_2 : \mathbb{Z}[G]\text{Б} \longrightarrow \mathbb{Z}[G]T_\sigma$ is given by $h_2(x\text{Б}) = x\sum_{i=0}^{d-1}(\frac{d-1}{2}-i)\tau^i\tau^{\frac{d+1}{2}}T_\sigma$ for every $x \in \mathbb{Z}[G]$.

$h_3 : \mathbb{Z}[G]\text{Б}T_\tau \xrightarrow{\subset} \mathbb{Z}[G]\text{Б}$ is the inclusion.

The next result verifies that the maps just defined are what is needed to satisfy the Positselski hypotheses.

Theorem 30. *In the semi-direct case, given the above definitions we have the following.*

1. *The d_i's are exact*

2. *h_2 is well-defined*

3. *The prism condition is satisfied at all 4 modules.*

Proof: For part (1), exactness follows from extending the Hilbert 90 sequence discussed in the dihedral case,

$$0 \longrightarrow \mathbb{Z}[\langle\tau\rangle]\cdot T_\tau \xrightarrow{\subseteq} \mathbb{Z}[\langle\tau\rangle] \xrightarrow{\cdot(1-\tau)} \mathbb{Z}[\langle\tau\rangle](1-\tau) \longrightarrow 0.$$

Here we replace $\mathbb{Z}[\langle\tau\rangle]$ with $\mathbb{Z}[G]T_\sigma$, which is isomorphic as a $\mathbb{Z}[\langle\tau\rangle]$-module to $\mathbb{Z}[\langle\tau\rangle]$.

For part (2), we will show that the left annihilator of Б is in the left annihilator of h_2(Б). Let $x \in \mathbb{Z}[G]$ such that xБ $= 0$. We will use the direct sum decomposition of $\mathbb{Z}[G]$ to express x as follows:

$$x = x_1(1-\tau) \oplus x_2,$$

where $x_1 \in \mathbb{Z}[G]$ and $x_2 \in \mathbb{Z}[\langle\sigma\rangle]$. We take the direct sum decomposition of $\mathbb{Z}[G]$ into the direct sum composition of M_3:

$$x\text{Б} = x_1(1-\tau)\text{Б} \oplus x_2\text{Б} = 0 \oplus 0.$$

and we will use the facts that x_1Б $= 0$ and x_2Б $= 0$.

Before applying h_2 to each of these direct summands, first note two things.

i) $x_1\tau^{\frac{d+1}{2}}T_\sigma = kT_\tau$ for some $k \in \mathbb{Z}$.

This follows because once $x_1\tau^{\frac{d+1}{2}}T_\sigma$ is multiplied by $(1-\tau)$, we get 0. And the left-annihilator of $(1-\tau)$ in $\mathbb{Z}[G]$ is $\mathbb{Z}[G]T_\tau$. The fact that we get 0 follows from the compuation below:

$$x_1\tau^{\frac{d+1}{2}}T_\sigma(1-\tau) = x_1(1-\tau)\text{Б} = 0$$

with the first equality following from Lemma 25 (ii).

ii) x_2 is a left-multiple of $(1+\sigma^{\frac{s}{2}})$. This follows from direct sum decomposition

$$\mathbb{Z}[\langle\sigma\rangle]\text{Б} = \oplus_{j=0}^{\frac{s}{2}-1}\mathbb{Z}\sigma^j\text{Б}.$$

We first compute $h_2(x_1(1-\tau))\text{Б}$.

$$\begin{aligned}
h_2(x_1(1-\tau)\text{Б} &= x_1(1-\tau)\sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i\tau^{\frac{d+1}{2}}T_\sigma \\
&= x_1\tau^{\frac{d+1}{2}}\left((1-\tau)\sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i\right)T_\sigma = x_1\tau^{\frac{d+1}{2}}\left(d-T_\tau\right)T_\sigma \\
&= x_1\tau^{\frac{d+1}{2}}T_\sigma\left(d-T_\tau\right) \overset{i)}{=} kT_\tau(d-T_\tau) = k(dT_\tau - T_\tau^2) \\
&= k(dT_\tau - dT_\tau) = 0
\end{aligned}$$

And for $h_2(x_2\text{Б})$, we compute the following.

$$\begin{aligned}
h_2(x_2\text{Б}) &\overset{ii)}{=} h_2(x_2'(1+\sigma^{\frac{s}{2}})\text{Б}) = x_2'(1+\sigma^{\frac{s}{2}})\sum_{i=0}^{d-1}\left(\frac{d-1}{2}-i\right)\tau^i\tau^{\frac{d+1}{2}}T_\sigma \\
&= x_2'(d-1)T_\sigma + x_2'(1+\sigma^{\frac{s}{2}})\sum_{i=0}^{d-1}(-i)\,\tau^i\tau^{\frac{d+1}{2}}T_\sigma \\
&= x_2'(d-1)T_\sigma + x_2'\sum_{i=0}^{d-1}(-i)\,\tau^i\tau^{\frac{d+1}{2}}T_\sigma + x_2'\sum_{i=0}^{d-1}(-i)\,\tau^{-i}\tau^{\frac{d-1}{2}}\sigma^{\frac{s}{2}}T_\sigma \\
&\overset{*}{=} x_2'(d-1)T_\sigma + x_2'(-(d-1))T_\sigma = 0
\end{aligned}$$

For the second to last equality above (*), note that the coefficients in the two summands add to $-(d-1)$ for each power of τ. With $x_1(1-\tau)$ and $h_2(x_2)$ both in the left-annihilator of what we call $h_2(\text{Б})$, we have shown that h_2 is well-defined.

For part (3), the prism condition at M_1 (which is $h_1 d_1 = \cdot d$) holds because $T_\tau \cdot T_\tau = d \cdot T_\tau$, as $h_1 d_1 = \cdot T_\tau$. The same goes for the prism condition at M_4, since $d_3 h_3 = \cdot T_\tau$ as well. The prism condition at M_2 holds from the following calculations: $(d_1 h_1)(T_\sigma) = T_\sigma \cdot T_\tau$, while

$$(h_2 d_2)(T_\sigma) = h_2(T_\sigma(1-\tau)) = h_2((1-\tau)\tau^{\frac{d-1}{2}}\text{Б}) = (1-\tau)\tau^{\frac{d-1}{2}} \sum_{i=0}^{d-1} \left(\frac{d-1}{2} - i\right) \tau^{\frac{d+1}{2}+i} T_\sigma$$
$$= (1-\tau) \sum_{i=0}^{d-1} (-i)\, \tau^i T_\sigma = -T_\tau T_\sigma + d \cdot T_\sigma.$$

Therefore

$$(d_1 h_1 + h_2 d_2)(T_\sigma) = T_\sigma T_\tau - T_\sigma T_\tau + d \cdot T_\sigma = d \cdot T_\sigma.$$

The prism condition holds at M_3 from the following calculations: $(h_3 d_3)(\text{Б}) = \text{Б} \cdot T_\tau = T_\tau \cdot \text{Б}$, while

$$(d_2 h_2)(\text{Б}) = d_2 \left(\sum_{i=0}^{d-1} \left(\frac{d-1}{2} - i\right) \tau^{\frac{d+1}{2}+i} T_\sigma \right) = \left(\sum_{i=0}^{d-1} \left(\frac{d-1}{2} - i\right) \tau^{\frac{d+1}{2}+i} T_\sigma \right) (1-\tau)$$

$$\sum_{i=0}^{d-1} \left(\frac{d-1}{2} - i\right) \tau^i (1-\tau)\text{Б} = \sum_{i=0}^{d-1} -i\tau^i(1-\tau)\text{Б} = \left(-T_\tau + d\right) \text{Б}.$$

Therefore

$$(d_2 h_2 + h_3 d_3)(\text{Б}) = T_\tau \cdot + \left(-T_\tau + d\right) \text{Б} = d \cdot \text{Б}.$$

And finally, $d_3h_3(\text{Б}T_\tau) = d_3(\text{Б}T_\tau) = \text{Б}T_\tau T_\tau = d\text{Б}T_\tau$. This concludes the proof of Theorem 30. $\square$

The only remaining part to check for the Positselski hypotheses is that the Bochstein homomorphisms are zero. The first two modules are $\mathbb{Z}$ as a trivial $\mathcal{G}$-module and the induced module $\text{Ind}_{\mathcal{H}}^{\mathcal{G}}(\mathbb{Z})$ respectively, each to be modded out by $d^2\mathbb{Z}$ to represent μ_{d^2} with a trivial $\mathcal{G}$-action. Both of these modules were shown to have zero Bochstein maps in the general setup section. For M_3 and M_4, we will show that as $\mathcal{J}$-modules, $M_3/dM_3 \cong \mathbb{Z}[J]/d^2\mathbb{Z}[J] \cong \oplus_{j=0}^{\frac{s}{2}-1}\text{Ind}_{\mathcal{N}}^{\mathcal{J}}(\mathbb{Z}/d^2\mathbb{Z})$ and that $M_4/dM_4 \cong \oplus_{j=0}^{\frac{s}{2}-1}\mathbb{Z}/d^2\mathbb{Z}$ as a trivial $\mathcal{J}$-module. The isomorphisms are defined by $\text{Б} \mapsto 1 \in \mathbb{Z}[\mathcal{J}]$ and $\text{Б}T_\tau \mapsto 1 \in \mathbb{Z}$ respectively.

Lemma 31. *As $\mathcal{J}$-modules,*

1. $M_3 = \mathbb{Z}[G]\text{Б} \cong \oplus_{j=0}^{\frac{s}{2}-1} Ind_{\mathcal{N}}^{\mathcal{J}}(\mathbb{Z})$

2. $M_4 = \mathbb{Z}[G]\text{Б}T_\tau \cong \oplus_{j=0}^{\frac{s}{2}-1} Ind_{\mathcal{J}}^{\mathcal{J}}(\mathbb{Z})$

Proof: Both parts of this lemma follow from the fact that

$$\mathbb{Z}[G]\text{Б} = \oplus_{j=0}^{\frac{s}{2}-1}\sigma^j\mathbb{Z}[\langle\tau\rangle]\text{Б}$$

followed by the arguments in Lemma 18 applied to each direct summand. Note that in the dihedral case, $s = 2$ and therefore $\sigma = \sigma^{\frac{s}{2}}$. So $\sigma^{\frac{s}{2}}$ must be used to apply the same arguments. $\square$

Thus, the Bochstein maps have component maps for each direct summand that are identical to those in the dihedral case. And the dihedral Bochstein maps were shown to be zero. Therefore the Bochstein maps are zero on M_3 and M_4. This completes the verification of the Positselski hypotheses for the 4-term exact sequence of $\mathcal{G}$-modules.

7.2 The Connecting Map for the Semi-Direct Case

Now we compute the connecting map η. Let $\overline{M_i} := M_i/dM_i$ for $i \in \{1,2,3,4\}$. In this section we will use the exact sequence of modules with homotopies defined in the previous section to describe connecting map $\eta : H^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow H^n(\mathcal{G}, \overline{M}_1)$.

Definition 32. *Given the above notation we define the following.*

1. $\ell : \overline{M}_4 \longrightarrow \overline{M}_3$, *the d_3-lifting defined by* $\ell\left(\sum_{j=0}^{\frac{s}{2}-1} x_j\sigma^j \text{Б} T_\tau\right) = \sum_{j=0}^{\frac{s}{2}-1} x_j\sigma^j \text{Б}$ *for* $x_j \in \mathbb{Z}/d\mathbb{Z}$.
2. $\delta : C^{n-1}(\mathcal{G}, \overline{M}_3) \longrightarrow C^n(\mathcal{G}, \overline{M}_3)$ *the cochain map from the bar resolution.*
3. $\widetilde{\eta}_\ell = \widetilde{\eta} : Z^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow Z^n(\mathcal{G}, \overline{M}_1)$, $\widetilde{\eta}(c) := [d_1^{-1}h_2\delta\ell(c)]$.
4. $\eta : H^{n-1}(\mathcal{G}, \overline{M}_4) \longrightarrow H^n(\mathcal{G}, \overline{M}_1)$, $\eta([c]) := \widetilde{\eta}(c)$.

Our choice of lifting ℓ is a $\mathbb{Z}$-module homomorphism, though it is not a $\mathbb{Z}[G]$-module homomorphism. We will see in a moment that ℓ is a module homomorphism for a larger subring of $\mathbb{Z}[G]$ than just $\mathbb{Z}$. Note, for $x \in \text{im}(d_1)$, we let $d_1^{-1}(x)$ denote the unique preimage element.

The analogue to this next lemma used a similar idea in the dihedral case as well, however there was no need to spell it out because σ had order 2. The proof comes down to $\sigma^{\frac{s}{2}}$ acting as $\cdot(-1)$ on Б.

Lemma 33. *ℓ is a $\mathbb{Z}[\langle\sigma\rangle]$-module homomorphism.*

Proof: We will show that ℓ preserves action by σ.

$$\ell\left(\sigma\cdot\sum_{j=0}^{\frac{s}{2}-1}x_j\sigma^j\text{Б}T_\tau\right)=\ell\left(\sum_{j=0}^{\frac{s}{2}-2}x_j\sigma^{j+1}\text{Б}T_\tau+x_{\frac{s}{2}-1}\sigma^{\frac{s}{2}}\text{Б}T_\tau\right)=\ell\left(\sum_{j=0}^{\frac{s}{2}-2}x_j\sigma^{j+1}\text{Б}T_\tau-x_{\frac{s}{2}-1}\text{Б}T_\tau\right)$$
$$=\sum_{j=0}^{\frac{s}{2}-2}x_j\sigma^{j+1}\text{Б}-x_{\frac{s}{2}-1}\text{Б}=\sum_{j=0}^{\frac{s}{2}-2}-x_j\sigma^{j+1}\text{Б}+x_{\frac{s}{2}-1}\sigma^{\frac{s}{2}}\text{Б}$$
$$=\sum_{j=0}^{\frac{s}{2}-1}x_j\sigma^{j+1}\text{Б}=\sigma\cdot\sum_{j=0}^{\frac{s}{2}-1}x_j\sigma^j\text{Б}=\sigma\cdot\ell\left(\sum_{j=0}^{\frac{s}{2}-1}x_j\sigma^j\text{Б}T_\tau\right).$$

This proves the lemma. □

We now can characterize the connecting map in the semi-direct case.

Lemma 34. *Let $c\in Z^{n-1}(\mathcal{G},\overline{M}_4)$ be a cocycle, $g_1,\ldots,g_n\in\mathcal{G}$, and let $c'\in\mathbb{Z}/d\mathbb{Z}$ such that $c(g_2,\ldots,g_n)=c'\cdot\text{Б}T_\tau$. Let $\sigma^j\tau^i$, an element of $\mathcal{G}/\mathcal{N}$, be the coset of g_1. Then*

1. $\delta(\ell(c))(g_1,\ldots,g_n)=g_1\cdot\ell(c(g_2,\ldots,g_n))-\ell(g_1\cdot c(g_2,\ldots,g_n))$

   $=\sigma^j\left(\tau^i-1\right)\cdot\ell(c(g_2,\ldots,g_n))$.

2. *For any $x\in C^n(\mathcal{G},\overline{M}_2)$ such that $\delta(\ell(c))=d_2(x)$, $\widetilde{\eta}(c)=-h_1(x)$.*

Proof: The first equality follows from the fact that ℓ is a $\mathbb{Z}$-module homomorphism, and hence the proof is identical to the analogous proof in the previous section. The second equality follows from ℓ being a $\mathbb{Z}[\langle\sigma\rangle]$-module homomorphism and τ acting trivially on $\overline{M}_4$. Note that τ acts trivially on $\overline{M}_4$ because T_τ is in the center of $\mathbb{Z}[G]$ because $\langle\tau\rangle$ is normal in G, and $\tau T_\tau=T_\tau$.

We use these two reasons in tandem to show the second equality:

$$\begin{aligned} g_1 \cdot \ell(c(g_2,\dots,g_n)) - \ell(g_1 \cdot c(g_2,\dots,g_n)) &= \sigma^j\tau^i\ell(c(g_2,\dots,g_n)) - \ell(\sigma^j\tau^i \cdot c(g_2,\dots,g_n)) \\ &= \sigma^j\tau^i \cdot \ell(c') - \sigma^j \cdot \ell(\tau^i c') \\ &= \sigma^j\tau^i \cdot \ell(c') - \sigma^j \cdot \ell(c') \\ &= \sigma^j(\tau^i - 1) \cdot \ell(c'). \end{aligned}$$

The second part follows from the prism condition, and is also identical to the analogous proof in the previous section. This concludes the proof of Lemma 34. □

As a corollary we obtain a characterization of the connecting map. As in the case for the dihedral extensions, this connecting map behaves like a cup product.

Corollary 35. *Let $g_1, g_2, \dots, g_n, c, c', i, j$ be defined as in Lemma 34, and let $c_m \in \mathbb{Z}/d\mathbb{Z}$ such that $c(g_2,\dots,g_n) = \sum_{m=0}^{\frac{s}{2}-1} c_m\sigma^m \text{Б} T_\tau$. Then*

$$\widetilde{\eta}(c)(g_1,\dots,g_n) = i\sum_{m=0}^{\frac{s}{2}-1} c_m(\theta_{s-m} - 1)T_\sigma T_\tau$$

Proof: From part (1) of Lemma 34, we know that

$$\delta(\ell(c))(g_1,\dots,g_n) = \sigma^j\left(-\sum_{k=0}^{i-1}\tau^k\right)\cdot(1-\tau)c'\text{Б}.$$

We will first find an $x \in \overline{M}_2$ with this d_2-image, and then use part (2) of Lemma 34 to compute $\widetilde{\eta}(c)$ by finding $-h_1(x)$. We begin by showing that $\sigma^j\left(-\sum_{k=0}^{i-1} t^k\right)\sum_{m=0}^{\frac{s}{2}-1} c_m\sigma^m\left(\sum_{r=0}^{\theta_m^{-1}-1}\tau^r\right)\tau^{\frac{d+1}{2}}T_\sigma$ is a suitable choice for x:

$$
\begin{aligned}
\delta(\ell(c))(g_1,\ldots,g_n) &= \sigma^j\left(\tau^i-1\right)\cdot\ell(c(g_2,\ldots,g_n))\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)(1-\tau)\cdot\ell(c')\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)(1-\tau)\cdot\sum_{m=0}^{\frac{s}{2}-2}c_m\sigma^m\text{Б}\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-2}c_m\sigma^m(1-\tau^{\theta_m^{-1}})\text{Б}\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-2}c_m\sigma^m(1-\tau^{\theta_m^{-1}})\text{Б}\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-1}c_m\sigma^m\left(\sum_{r=0}^{\theta_m^{-1}-1}\tau^r\right)(1-\tau)\text{Б}\\
&= \sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-1}c_m\sigma^m\left(\sum_{r=0}^{\theta_m^{-1}-1}\tau^r\right)\tau^{\frac{d+1}{2}}T_\sigma(1-\tau)\\
&= d_2\left(\sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-1}c_m\sigma^m\left(\sum_{r=0}^{\theta_m^{-1}-1}\tau^r\right)\tau^{\frac{d+1}{2}}T_\sigma\right).
\end{aligned}
$$

With a suitable x found, we compute $-h_1(x)$:

$$
\begin{aligned}
-h_1(x) &= -h_1\left(\sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-1}c_m\sigma^m\left(\sum_{r=0}^{\theta_m^{-1}-1}\tau^r\right)\tau^{\frac{d+1}{2}}T_\sigma\right)\\
&= -\sigma^j\left(-\sum_{k=0}^{i-1}t^k\right)\sum_{m=0}^{\frac{s}{2}-1}c_m\sigma^m\left(\sum_{r=0}^{\theta_{s-m}-1}\tau^r\right)\tau^{\frac{d+1}{2}}T_\sigma T_\tau\\
&= -(-i)\sum_{m=0}^{\frac{s}{2}-1}c_m\left(\theta_{s-m}-1\right)T_\sigma T_\tau \;=\; i\sum_{m=0}^{\frac{s}{2}-1}c_m\left(\theta_{s-m}-1\right)T_\sigma T_\tau
\end{aligned}
$$

This concludes the proof of Corollary 35. □

In view of Theorem 30 and Corollary 35 the machinery in Theorem 4 gives the following result.

Theorem 36. *In the semi-direct case we have the following 6-term exact sequence*

$$H^n(E,\mu_d)\oplus H^n(\mathcal{G},\overline{M}_4) \xrightarrow{d_2+h_3} H^{n+1}(\mathcal{G},\overline{M}_3) \xrightarrow{d_3} H^n(\mathcal{G},\overline{M}_4) \xrightarrow{\eta}$$
$$H^{n+1}(F,\mu_d) \xrightarrow{d_1} H^{n+1}(E,\mu_d) \xrightarrow{h_1\oplus d_2} H^n(F,\mu_d)\oplus H^n(\overline{M}_3,\mu_d)$$

where η is as described in Corollary 35 and d_1 is scalar extension.

Corollary 37. *η induces an isomorphism*

$$\frac{H^n(\mathcal{G},\overline{M}_4)}{d_3H^n(\mathcal{G},\overline{M}_3)} \xrightarrow{\cong} H^n(E/F).$$

It will be shown as part of the interpretation that there is an exact sequence

$$\frac{H^n(\mathcal{G},\overline{M}_4)}{\mathrm{cor}_{\widetilde{E}/\widetilde{F}}H^n(\widetilde{E},\mu_d)} \longrightarrow H^n(F,\mu_d) \longrightarrow H^n(E,\mu_d).$$

with the corestriction being mapped from $\widetilde{F}$ into the image of d_3.

7.3 Some Examples

We conclude this section with computation of $\eta : H^n(\mathcal{G}, \overline{M}_4) \longrightarrow H^{n+1}(F, \mu_d)$ for $n = 0$ and $n = 1$.

1. $\underline{\eta \text{ for } n = 0}$

   In the semi-direct case, s is even and the non-trivial action of $\sigma^{\frac{s}{2}}$ on $\overline{M}_4$ is multiplication by (-1). Hence $\overline{M}_4$ has no fixed points with d being odd, and therefore $\eta = 0$ because $H^0(\mathcal{G}, \overline{M}_4)$ is trivial.

   Note that this is not true for the cyclic case, where $s = 1$ is not even. In the cyclic case, $H^0(\mathcal{G}, \overline{M}_4) = H^0(\mathcal{G}, \mu_d) = \mu_d$ and η is the well-known cup product with the character χ_α defined by the extension $E/F = F(\alpha)/F$.

2. $\underline{\eta \text{ for } n = 1}$

   Let $\chi \in Z^1(\mathcal{G}, \overline{M}_4)$ be a crossed homomorphism. Then the identity $\sigma \cdot \chi(\tau) = \theta_1 \chi(\tau)$ may be deduced as from the cocycle condition as follows.

$$\begin{aligned}\sigma \cdot \chi(\tau) = \sigma \cdot \chi(\tau) + \chi(\sigma) - \chi(\sigma) = \chi(\sigma\tau) - \chi(\sigma) = \chi(\tau^{\theta_1}\sigma) - \chi(\sigma)\\ = \tau^{\theta_1} \cdot \chi(\sigma) + \chi(\tau^{\theta_1}) - \chi(\sigma) = \tau^{\theta_1} \cdot \chi(\sigma) + (\tau^{\theta_1 - 1} + \cdots + 1) \cdot \chi(\tau) - \chi(\sigma)\\ \overset{(*)}{=} 1 \cdot \chi(\sigma) + (\theta_1) \cdot \chi(\tau) - \chi(\sigma) = \theta_1 \chi(\tau)\end{aligned}$$

   where the equality $(*)$ comes from τ acting trivially on $\overline{M}_4$. So for a unique $t \in \mathbb{Z}/d\mathbb{Z}$,

$$\chi(\tau) = t \sum_{m=0}^{\frac{s}{2}-1} \theta_1^{-m} \sigma^m \text{Б} T_\tau$$

where t may be thought of as the $\sigma^0 Б T_\tau$-coefficient of $\chi(\tau)$.

The author has not found quite such luck expressing $\chi(\sigma)$, even after considering normalization by different coboundaries. So $\chi(\sigma)$ will be expressed as follows: Let $s : \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z}$ such that $s_{m+\frac{s}{2}} = -s_m$ for every $m \in \mathbb{Z}$ so that

$$\chi(\sigma) = \sum_{m=0}^{\frac{s}{2}-1} s_m \sigma^m Б T_\tau$$

and more generally,

$$\chi(\sigma) = \sum s_{\gamma_m} \sigma^{\gamma_m} Б T_\tau$$

for any choice of representatives $\gamma_1, \ldots, \gamma_{\frac{s}{2}-1} \in \mathbb{Z}$ of the cosets $\mathbb{Z}/(\frac{s}{2})\mathbb{Z}$. Suppose $g_1\mathcal{N} = \sigma^j \tau^i$ and $g_2\mathcal{N} = \tau^k \sigma^\ell$. Then

$$\chi(g_1, g_2) = k \cdot \chi(\tau) + \left(\sum_{\beta=0}^{\ell-1} \sigma^\beta \right) \cdot \chi(\sigma) = \sum_{m=0}^{\frac{s}{2}-1} \left(kt\theta_{-1}^m + \sum_{\beta=0}^{\ell-1} s_{m-\beta} \right) \sigma^m$$

and hence

$$c_m = kt\theta_{-1}^m + \sum_{\beta=0}^{\ell-1} s_{m-\beta}.$$

The connecting map formula in Corollary 35 may be applied to yield

$$\widetilde{\eta}(g_1, g_2) = i \sum_{m=0}^{\frac{s}{2}-1} \left(\theta_1^{-m} - 1\right) c_m = i \sum_{m=0}^{\frac{s}{2}-1} \left(\theta_1^{-m} - 1\right) \left(kt\theta_1^{-m} + \sum_{\beta=0}^{\ell-1} s_{m-\beta} \right).$$

8. INTERPRETING THE SEQUENCES

In this section we record some consequences of the sequences in Theorems 13, 22, and 36. As noted in the introduction, when E/F is cyclic one has the classical description of the relative Brauer group,

$$\frac{F^\times}{N_{E/F}(E^\times)} \cong \ker(\mathrm{Br}_d F \to \mathrm{Br}_d E).$$

As an immediate consequence of Theorem 13, this classical result generalizes as follows.

Theorem 38. *If E/F is cyclic with $[E : F] = d$ and $\mu_{d^2} \subset F$ we have a description of the cohomological kernel for all $n \geq 0$,*

$$\frac{H^n(F, \mu_d)}{\mathrm{cor}_{E/F} H^n(E, \mu_d)} \cong \ker(H^{n+1}(F, \mu_d) \to H^{n+1}(E, \mu_d)).$$

The goal in this section is to find what generalizations of Theorem 38 are possible in the dihedral and semi-direct product cases considered in the previous two sections. We continue to assume the notation of the last section. To better understand M_3 and M_4 we need to introduce more induced modules and subgroups.

Definition 39. *We set $T_2 := J + \sigma^{\frac{s}{2}} J \in \mathrm{Ind}_{\mathcal{J}}^{\mathcal{G}} \mathbb{Z}$ and then set $\mathcal{T}_2 := \sum_{j=0}^{\frac{s}{2}-1} \mathbb{Z} \cdot \sigma^j T_2 \subset \mathrm{Ind}_{\mathcal{J}}^{\mathcal{G}} \mathbb{Z}$. We note that $\mathcal{T}_2$ is a $\mathcal{G}$-submodule of $\mathrm{Ind}_{\mathcal{J}}^{\mathcal{G}} \mathbb{Z}$ with $\sigma^{\frac{s}{2}} T_2 = T_2$. We set $\mathcal{J}'$ to be the unique group in $\mathcal{G}$ of index $\frac{s}{2}$ containing $\mathcal{J}$ and denote by J' the corresponding subgroup of G where one notes $J' = \langle \tau, \sigma^{\frac{s}{2}} \rangle$.*

By Galois theory, since J' has index $\frac{s}{2}$ in G, $\mathcal{J}' = \mathrm{Gal}(F_{sep}/F')$ where $[F' : F] = \frac{s}{2}$ and $F \subset F' \subset \tilde{F} = F(\alpha)$ (since $\tilde{F}/F$ is cyclic, F' is the unique such intermediate extension.) We have the following.

Lemma 40. *Given the above definitions.*

(i) As $\mathcal{G}$-modules, $\mathcal{T}_2 \cong \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z}$.

(ii) As $\mathcal{G}$-modules, $M_4 := M_3/M_3' \cong \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}}\mathbb{Z}/\mathcal{T}_2$.

In particular we have an exact sequence of $\mathcal{G}$-modules.

$$0 \to \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z} \to \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}}\mathbb{Z} \to M_4 \to 0.$$

Proof. We note that $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z} \cong \bigoplus_{j=0}^{\frac{s}{2}-1}\mathbb{Z}\cdot\sigma^j J'$ with σ acting cyclicly and τ acting trivially on the summands. This is exactly how G acts on the summands of $\mathcal{T}_2$ and the map $\sigma^j T_2 \mapsto \sigma^j \mathcal{J}'$ gives the isomorphism required by (i).

For (ii), by definition M_4 is the free $\mathbb{Z}$-module with basis $\overline{Б}$, $\sigma\overline{Б}$,…,$\sigma^{\frac{s}{2}-1}\overline{Б}$, with trivial τ-action and with σ acting cyclically except with $\sigma^{\frac{s}{2}}\overline{Б} = -\overline{Б}$. From this it follows that the map $\sigma^j J \mapsto \sigma^j\overline{Б}$ defines a G-map $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}}\mathbb{Z} \to M_4$ with kernel $\mathcal{T}_2$ (the latter as $J + \sigma^{\frac{s}{2}}J \mapsto \overline{Б} + \sigma^{\frac{s}{2}}\overline{Б} = \overline{Б} - \overline{Б} = 0 \in M_4$.) This gives the lemma. □

Remarks. (i) The $\mathbb{Z}$-ranks of the modules $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z}, \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z}, M_4$ are, respectively, $\frac{s}{2}$, s, and $\frac{s}{2}$.
(ii) Of course, all of these modules can be taken (mod d) and the same results apply.

The cohomology of M_4 can be interpreted using the sequence of Lemma 40. By definition we have $F \subseteq F' \subseteq \widetilde{F}$ where $[\widetilde{F} : F] = s$ and $[F' : F] = \frac{s}{2}$. Computing cohomology in μ_d and using the fact that $(d,s) = 1$ we know that $H^n(F', \mu_d) \to H^n(F, \mu_d)$ must be injective for all n. In particular the long exact sequence in cohomology gives exact sequences

$$0 \to H^n(F', \mu_d) \to H^n(\widetilde{F}, \mu_d) \to H^n(\mathcal{G}, \overline{M}_4) \to 0.$$

This means if we let $\overline{H}^n(F, \mu_d) := \mathrm{cok}(H^n(F', \mu_d) \to H^n(\widetilde{F}, \mu_d))$ then the Positselski connecting map in Theorem 36 gives a map $\overline{\eta} : \overline{H}^n(\widetilde{F}, \mu_d) \to H^{n+1}(F, \mu_d)$ that computes the cohomological kernels as noted next.

Theorem 41. *With the above notation we have an exact sequence,*

$$\overline{H}^n(\widetilde{F}, \mu_d) \xrightarrow{\overline{\eta}} H^{n+1}(F, \mu_d) \to H^{n+1}(E, \mu_d).$$

To understand the kernel of $\overline{\eta}$ one needs to further understand the cohomology of M_3 and how it maps into the cohomology of M_4. For this purpose, if $\pi^* : H^n(\widetilde{F}, \mu_d) \to H^n(\mathcal{G}, \overline{M}_4)$ is the induced map, we shall denote by

$$N_3^n(E/F) := \pi^{*-1}(\mathrm{im}(H^n(\mathcal{G}, \overline{M}_3) \to H^n(\mathcal{G}, \overline{M}_4))) \subseteq H^n(\widetilde{F}, \mu_d)$$

and then Theorem 36 gives the following result.

Theorem 42. *Given the above notation we have the following characterization of the cohomological kernel $H^{n+1}(E/F, \mu_d)$,*

$$\frac{H^n(\widetilde{F}, \mu_d)}{i_{\widetilde{F}/F'} H^n(F', \mu_d) + N_3^n(E/F)} \cong \ker(H^{n+1}(F, \mu_d) \to H^{n+1}(E, \mu_d)).$$

When interpreted loosely, this result can be understood as the analogue of Theorem 38 in the more general case case. (When E/F is cyclic of degree d and $s = 1$ we would have $\widetilde{F} = F$ and $N_3^n(E/F) = \mathrm{cor}_{E/F} H^n(E, \mu_d)$. Also, the subfield F' doesn't exist in the cyclic case.)

Next we turn to $M_3 \subset \mathbb{Z}[G]$. We set $\mathcal{H}' := \langle \mathcal{J}, \sigma\tau \rangle = \mathrm{Gal}(F_{sep}/E')$ where E' is discussed above. We know by Lemma 31 the modules $M_3 = \mathbb{Z}[G] \cdot \text{Б}$, $M_3' = \mathbb{Z}[G] \cdot (1 - \tau)\text{Б}$ and $M_4 = \mathbb{Z}[G] \cdot T_\tau \text{Б}$. By Lemma 40 we have the following.

Theorem 43. *The following diagram of G-modules and G-maps is commutative with exact rows and columns.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}' & \xrightarrow{\subseteq} & \mathcal{K} & \xrightarrow{\cdot T_\tau} & \mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} & & \\
0 & \longrightarrow & \mathbb{Z}[G](1-\tau) & \xrightarrow{\subseteq} & \mathbb{Z}[G] & \xrightarrow{\cdot T_\tau} & \mathbb{Z}[G]T_\tau & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\cdot\text{Б}} & & \downarrow{\scriptstyle\cdot\text{Б}} & & \downarrow{\scriptstyle\cdot\text{Б}} & & \\
0 & \longrightarrow & M_3' & \xrightarrow{\subseteq} & M_3 & \xrightarrow{\cdot T_\tau} & M_4 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The right column is that of Lemma 40, where $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z} = \mathbb{Z}[G] \cdot T_\tau(1 + \sigma^{\frac{s}{2}})$*, and* $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}}\mathbb{Z} = \mathbb{Z}[G] \cdot T_\tau$*. Here* $\mathcal{K} := \ker(\cdot\text{Б} : \mathbb{Z}[G] \to M_3)$ *and* $\mathcal{K}' := \ker(\cdot\text{Б} : \mathbb{Z}[G] \cdot (1 - \tau) \to M_3')$*. Moreover, the* $\mathbb{Z}$*-ranks of* $\mathcal{K}'$*,* $\mathbb{Z}[G] \cdot (1 - \tau)$*,* M_3' *are* $(s - 1)(d - 1)$*,* $s(d - 1)$*,* $(d - 1)$*, resp.,* *the $\mathbb{Z}$-ranks of $\mathcal{K}$, $\mathbb{Z}[G]$, M_3 are $sd - \frac{s}{2} - (d-1)$, sd, $\frac{s}{2} + (d-1)$, resp., and the $\mathbb{Z}$-ranks of $\mathrm{Ind}^{\mathcal{G}}_{\mathcal{J}'}\mathbb{Z}$, $\mathbb{Z}[G] \cdot T_\tau$, M_4 are $\frac{s}{2}$, s, $\frac{s}{2}$, resp.*

Proof. For commutativity, as the first set of rightarrows are inclusions as are the first downarrows, the only question is the lower right square. But T_τ is central in $\mathbb{Z}[G]$ so $\text{Б} \cdot T_\tau = T_\tau \cdot \text{Б}$ and this square commutes.

The bottom row is exact by earlier constructions. Since elements of G can be expressed uniquely in the form $\sigma^j \tau^i$ for $0 \le j < s$ and $0 \le i < d$, together with $(1-\tau)T_\tau = 0$ and T_τ being central, using the usual proof the middle row is exact. Lemma 31 shows the bottom vertical maps are surjective, and therefore by Lemma 40 and the definitions of $\mathcal{K}$ and $\mathcal{K}'$ the rows are exact. It then follows by the usual diagram chase that the first row is exact.

Finally the $\mathbb{Z}$-ranks of the bottom row are given in the remark following Lemma 40. The $\mathbb{Z}$-ranks of the two right columns are clear by previous work, so the ranks of $\mathcal{K}$ and $\mathcal{K}'$ follow by arithmetic. $\square$

As an application of Theorem 43 we can characterize $N_3^n(E/F)$ via the corestriction.

Theorem 44. *Given the above notation we have the following exact sequence calculating the cohomological kernel $H^{n+1}(E/F, \mu_d)$,*

$$\frac{H^n(\widetilde{F}, \mu_d)}{\mathrm{cor}_{\widetilde{E}/\widetilde{F}} H^n(\widetilde{E}, \mu_d)} \to H^{n+1}(F, \mu_d) \to H^{n+1}(E, \mu_d).$$

The first map is injective provided $H^{n+1}(\mathcal{G}, \mathcal{K}) \to H^{n+1}(\widetilde{E}, \mu_d)$ is injective.

Proof. Consider the following diagram.

$$\begin{array}{ccccccc}
H^n(\widetilde{E}, \mu_d) & \xrightarrow{\text{cor}} & H^n(\widetilde{F}, \mu_d) & & & & \\
\downarrow & & \downarrow & & & & \\
H^n(\mathcal{G}, \overline{M}_3) & \longrightarrow & H^n(\mathcal{G}, \overline{M}_4) & \longrightarrow & H^{n+1}(F, \mu_d) & \longrightarrow & H^{n+1}(E, \mu_d) \\
\downarrow & & \downarrow & & & & \\
H^{n+1}(\mathcal{G}, \mathcal{K}) & \longrightarrow & H^{n+1}(F_1, \mu_d) & & & & \\
\downarrow & & \downarrow & & & & \\
H^{n+1}(\widetilde{E}, \mu_d) & \xrightarrow{\text{cor}} & H^{n+1}(\widetilde{F}, \mu_d) & & & &
\end{array}$$

The middle row is exact by Theorem 36. The two columns are exact by the long exact sequence of cohomology applied to the right two columns of the diagram in Theorem 43. The diagram commutes since all maps are those induced by the diagram in Theorem 43. The map $H^{n+1}(F_1, \mu_d) \to H^{n+1}(\widetilde{F}, \mu_d)$ is injective since $[\widetilde{F} : F_1]$ is prime to d. Therefore the map $H^n(\widetilde{F}, \mu_d) \to H^n(\mathcal{G}, \mathcal{M}_4)$ is surjective. This gives a surjective map $H^n(\widetilde{F}, \mu_d) \to \ker(H^{n+1}(F, \mu_d) \to H^{n+1}(E, \mu_d))$. The exactness of the sequence follows by noting the diagram shows $\text{cor}_{\widetilde{E}/\widetilde{F}}(H^n(\widetilde{E}, \mu_d))$ has trivial image in $H^{n+1}(F, \mu_d)$. For the second statement, if $H^{n+1}(\mathcal{G}, \mathcal{K}) \to H^{n+1}(\widetilde{E}, \mu_d)$ is injective then $H^n(\widetilde{E}, \mu_d) \to H^n(\mathcal{G}, M_3)$ is surjective and the result follows by the exactness of the middle column of the diagram in Theorem 43. □

Remark. It is reasonable to conjecture that $H^{n+1}(\mathcal{G}, \mathcal{K}) \to H^{n+1}(\widetilde{E}, \mu_d)$ is injective. But we need to understand $\mathcal{K}$ better. This question will be studied in future work.

The section closes by looking at the case where $s = 2$. We have $(1+\sigma)(1-\tau) = (1+\sigma)(1-\sigma\tau) = (1-\tau)(1-\sigma\tau)$ and we find

$$\begin{aligned} C_i &= \tau^i(1+\sigma)(1-\tau) = \tau^i(1-\sigma)(1-\sigma\tau) = \tau^i(1-\tau)(1-\sigma\tau) \\ \text{Б} &= (1-\sigma)\tau^{\frac{d+1}{2}} = \tau^{\frac{d+1}{2}} - \tau^{-\frac{d+1}{2}}\sigma = \tau^{\frac{d+1}{2}}(1-\tau^{-1}\sigma) = \tau^{\frac{d+1}{2}}(1-\sigma\tau). \end{aligned}$$

From this we find that $M_3 = \mathbb{Z}[J] \cdot (1-\sigma\tau) = \mathbb{Z}[J] \cdot \text{Б}$. Looking at M_3 in this way may make what is going on when $s = 2$ more transparent (in particular the relationship to E'.) Even more, we have noted earlier that both $\mathbb{Z}[J] \cdot (1 \pm \sigma\tau)$ have $\mathbb{Z}$-rank d, and therefore as $(1-\sigma\tau)(1+\sigma\tau) = 0$ we know the kernel of the map $\cdot(1 \pm \sigma\tau)$ is $\mathbb{Z}[J] \cdot (1 \mp \sigma\tau)$. This leads to the following result.

Lemma 45. *When $s = 2$ we have two exact sequences*

$$0 \to M_3 \to \mathbb{Z}[G] \to \text{Ind}_{\mathcal{H}'}^{\mathcal{G}} \to 0$$

and

$$0 \to \text{Ind}_{\mathcal{H}'}^{\mathcal{G}} \to \mathbb{Z}[G] \to M_3 \to 0.$$

The second sequence coincides with the middle row of the diagram of Theorem 43 up to an automorphism of M_3 and therefore $\mathcal{K} \cong \text{Ind}_{\mathcal{H}'}^{\mathcal{G}}$ in this case.

Proof. We know that $\mathbb{Z}[J] \cdot (1+\sigma\tau) \cong \text{Ind}_{\mathcal{H}'}^{\mathcal{G}}$ and $M_3 = \mathbb{Z}[J] \cdot (1-\sigma\tau)$. The exact sequences follow as the kernel of the map $\cdot(1 \pm \sigma\tau)$ is $\mathbb{Z}[J] \cdot (1 \mp \sigma\tau)$. For the second statement, in Theorem 43 the map $\mathbb{Z}[G] \to M_3$ is multiplication $\cdot\text{Б}$ where $\text{Б} = \tau^{\frac{d+1}{2}}(1-\sigma\tau)$, whereas it is multiplication by $(1-\sigma\tau)$ in the lemma. However, multiplication by $\tau^{\frac{d+1}{2}}$ is an autormorphism of M_3 so the result follows. □

In the dihedral case ($s = 2$), the exact sequence of Lemma 45 and the long exact cohomology sequence give the first column of the diagram in the proof of Theorem 44,

$$\cdots \to H^n(E', \mu_d) \to H^n(\widetilde{E}, \mu_d) \to H^n(\mathcal{G}, M_3)$$

$$\to H^{n+1}(E', \mu_d) \to H^{n+1}(\widetilde{E}, \mu_d) \to H^{n+1}(\mathcal{G}, M_3) \cdots .$$

However, $[\widetilde{E} : E'] = 2$ and d is odd, so we know $H^{n+1}(\mathcal{G}, \mathcal{K}) = H^{n+1}(E', \mu_d) \to H^{n+1}(\widetilde{E}, \mu_d)$ is injective. This gives the following application of Theorem 44.

Theorem 46. *In the dihedral case ($s = 2$) the cohomological kernel $H^{n+1}(E/F, \mu_d)$ is given by*

$$\frac{H^n(\widetilde{F}, \mu_d)}{\mathrm{cor}_{\widetilde{E}/\widetilde{F}} H^n(\widetilde{E}, \mu_d)} \xrightarrow{\cong} \ker\left(H^{n+1}(F, \mu_d) \longrightarrow H^{n+1}(E, \mu_d)\right).$$

Proof. Since the map $H^{n+1}(\mathcal{G}, \mathcal{K}) = H^{n+1}(E', \mu_d) \to H^{n+1}(\widetilde{E}, \mu_d)$ is injective the result is immediate by Theorem 44. □

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