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A quantum hindsight on density functional theory for computation of materials properties

Lu J. Sham

The following article is based on the Materials Theory Award presentation given by Lu J. Sham “for pioneering contributions to the quantum theory of molecules and solids, especially the Kohn-Sham formulation of density functional theory,” at the 2019 MRS Fall Meeting in Boston, Mass.

Fundamental materials properties are determined by electrons under the potential energy from the nuclei, the electron mass, and their mutual repulsion. The variable from material to material is the ion potential. The logical procedure of computing electronic properties is to go from the potential to the electron distribution. This enables practical computation of the material properties ranging from atoms and molecules to solids. This method has blossomed due to the effort of numerous people. The concept is analogous to changing prediction of human population distribution from the landscape of hills and dales to determination of the landscape from a population distribution. In atomic systems, quantum quirkiness allows this switch, but dictates that it is only one slice in the tomography of the quantum state. The author shares his experience in the development from this slice, but hews close to the powerful concept of switching the landscape with the population.

Introduction

For 55 years, density functional theory (DFT), founded by the fundamental Hohenberg–Kohn (HK) theorem¹ and the practical Kohn–Sham (KS) equation,² has flourished, due to advances in computation methods and applications to materials properties. Advances in computation have been fueled by tremendous innovations in formulating the exchange and correlation functional potentials in the KS equation empirically for large classes of systems and open computation program apps in the free spirit of the internet in its pioneering days. Further development of the theory and applications does not seem to be flattening.

In the early 1960s, single-electron-band structure computations had become reliable³ and the interacting electron gas treated by field-theoretic methods went beyond the Hartree–Fock approximation to include correlation effects by, for example, the random phase approximation⁴ or the ring diagrams.⁵ There was the dichotomy of the former looking for a reliable effective potential for the single electron and the latter being limited to the jellium (a uniform background of positive charges) extended somewhat to simple metals. DFT then built a bridge to bring the exchange and correlation effects to the band calculation.

The DFT field has become too vast to review in a single article. I will use a thread of what I have been involved in to examine the field from a quantum perspective. This spans from the old quantum criterion of state coherence to the quantum information era requirement of entanglement. The former drives, for example, density oscillations due to an impurity in a metal or the electron density shell structure in atoms, while the latter is concerned with what phenomena entanglement can drive. We examine how DFT elucidates such quantum phenomena in various systems.

Hohenberg–Kohn theorem and the state topology

The HK theorem¹ forms the foundation of DFT. It allows the ground-state energy of a many-electron state to be determined by the knowledge of the electron-density distribution as a function of electron position instead of its potential. The use of the variational principle of the ground-state energy as a function of the density allows it to be determined for a given single-electron potential. This work reaches the pinnacle of the improvement of the Thomas–Fermi theory. The important point is the introduction of the Legendre transformation from the potential $v(r)$ to the density $n(r)$. The

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transformation has the convenience for computation that can be extended to other duality pairs. An example is next presented.

The complete characterization of a quantum state in an experiment requires tomography (i.e., a series of measurements of non-commutative observables).^{6,7} From the quantum viewpoint, the dependence on the density is akin to a slice of the tomography. The variational equation of the ground-state energy in terms of the density distribution alone cannot recover the loss of phase information in the electron-density distribution. Even in the case where the electron–electron interaction is neglected, attempts with gradient expansion cannot remedy this phase loss, as the history of numerous attempts to improve the Thomas–Fermi equations by adding higher-order gradient terms to find the shell structure of the electron in an atom had shown.

The problem does not lie in the slowly varying approximation. The quantum effect of density oscillations as a function of position has been demonstrated by computation using the slowly varying WKB approximation of the wave function of the Schrödinger equation.⁸ The quantum effect is driven by the confinement effect to produce a standing wave from the turning point (where the potential equals the electron energy). Interestingly, the wave behavior is determined mostly by the energy states close to the highest occupied states. There, the connection between the oscillatory functions and the decaying functions are not determined by sinusoidal and the exponential waves, but more accurately, by the corresponding oscillating and decaying Bessel function solutions of the linear portion of the potential bracketing the turning point.⁹ The method of finding oscillations also applies to nuclear matter density.¹⁰ This is also the cause of the Friedel oscillations¹¹ due to an impurity potential in a metal being the state of energy just below the Fermi level.

Quantum phase transition via DFT

The quantum phase transition (QPT)^{12,13} is a critical change in the ground-state properties of a material system. It can be observed at very low temperatures. Consider a material's Hamiltonian,

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{ext}} = \sum \lambda_l \hat{A}_l, \quad (1)$$

where l is the lattice site and $\hat{A}$ is an observable driven by the external field parameter λ_l . QPT is usually characterized by the change of the ground-state energy as a nonanalytic point on varying the field parameters (i.e., external forces), in the Hamiltonian, (e.g., a first-order QPT at a finite discontinuity and a second-order one at a discontinuous derivative).

An extension of the Hohenberg–Kohn theorem is to make a Legendre transformation of the ground-state energy as a function of the field parameters λ_l to its conjugate pairs, viz. the set of ground-state expectation values $a_l = \langle \hat{A} \rangle$. If we regard the set of $\{a_l\}$ as the results of simultaneous quantum

measurements of the observables $\hat{A}$, the set of the observables has to be commutative. The rule permits a certain amount of latitude, such as having noncommutative pairs, the spin components $\hat{S}_i^x \hat{S}_i^y$, or two-site operators $\{\hat{A}_l \hat{A}_{l+c}\}$ for all l sites and a constant, c . A simple connection between the λ_l representation and the $\{a_l\}$ representation is by the Hellmann–Feynman theorem for the ground-state energy E ,

$$\frac{\partial E}{\partial \lambda_l} = \left\langle \frac{\partial \hat{H}}{\partial \lambda_l} \right\rangle = \langle \hat{A} \rangle = a_l, \quad (2)$$

where $\langle \dots \rangle$ denotes a ground-state expectation value.

One method to detect the QPT point is to use a measure of the quantum entanglement of the material system. The entanglement between two qubits, for example, two states of each of a pair of spins, is the correlation between two spins at any distance apart. Consider the singlet or one triplet state of the two spins,

$$|\Psi_{\mp}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\downarrow\rangle_2 \mp |\downarrow\rangle_1 |\uparrow\rangle_2). \quad (3)$$

The two spins are entangled. If the first spin is measured to be up, then the second spin has to be down and vice versa. This can happen even if one spin is half a world apart from the other, described famously by Einstein as “spooky action at a distance.” In a many-body system, the entanglement may be considered to be bipartite (between two parts) or multipartite (between multiple parts). The bipartite can be even just one qubit with the rest of the system. The quantitative entanglement measure may be the entropy or simplified to just an appropriate log function as a function of the field parameters $\{\lambda_l\}$,

$$L_l^{(2)} = 1 - \nabla_l E \cdot \nabla_l E. \quad (4)$$

The entanglement measure with the field parameter variation may be used to detect QPT.¹⁴ We showed¹² that the use of the conjugate partner $\{a_l\}$ in the Legendre transformation was not only possible, but could cover the whole range, rather than being confined to the critical region near the phase transition point as was done with field parameters. Equation 4 was used to express the entanglement measure.

A model with existing theory and experiment is the transverse-field Ising model in which $\hat{H}_0$ is the Ising model of anti-parallel spin pairs in one direction and $\hat{H}_l$ has the magnetic field normal to that direction. The density functional numerical computation led to a clear indication of the critical point at the linear model with 1000 sites. The use of other entanglement measures also confirmed the QPT point.

The KS equation

The missing observable of importance in the density functional variational equation is the momentum that does not commute with the position. The KS equation² can thus be seen as an attempt to restore this quantum feature while

the famous Wigner probability function in terms of the sum and difference of two points in position.

A derivation of the density functional exchange and correlation potential

Numerous authors have written that the exchange-correlation functional $E_{xc}[n]$ is “unknown.” In fact, a general expression may be derived from the many body theory for $E_{xc}[n]$.¹⁸ A coefficient λ is attached to the interaction term in the Hamiltonian,

$$\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{H}_1, \quad (13)$$

so that as λ is varied from 0 to 1, the total Hamiltonian will move from the independent particle one to the one with the full electron interaction. The noninteracting particle part $\hat{H}_0$ is designed to treating the full KS equation for the independent electrons under the influence of the total effective potential $v_{\text{eff}}(\mathbf{r})$ in Equation 8. The interaction part $\hat{H}_1$ includes the usual interaction between two electrons minus the effective single-particle potential term driven by $v_{\text{eff}}(\mathbf{r}) - v(\mathbf{r})$ so that the interaction effect in $\hat{H}_0$ is not double counted.

Following the field-theoretic procedure of Luttinger and Ward,¹⁹ the ground-state energy is given by the coupling constant integral,

$$E = E_0 + \int_0^1 \frac{d\lambda}{\lambda} \langle \lambda \rangle_\lambda, \quad (14)$$

where E_0 is the ground-state energy of $\hat{H}_0$ and the angular brackets denote the exact ground-state expectation. From the infinite perturbation series in terms of Green's function in the Feynman diagrams,²⁰ the integral leads to,

$$E_{xc}[n] = i\text{Tr}[\ln(1 - \Sigma G_0) + \Sigma G] + Y_{xc}, \quad (15)$$

where Tr denote the trace and Y_{xc} is the exchange-correlation part of the energy diagrams of the dressed Green function, G . The formula for the exchange and correlation $v_{xc}[n]$ then follows. Its validity depends on the validity of the perturbation series. The simplicity of the LDA is gone, but the complexity is necessary in a number of cases. An example is shown next.

The bandgap problem

The band structure computations solving the KS equation with the local density approximation² were widely applied to the electron-band structures of materials. By the early 1980s, it was found that the computed semiconductor bandgaps were quite generally about 2/3 of the experimental values. Perdew and Levy²¹ and Sham and Schlüter²² pointed out the discontinuity in the DFT, which was absent in LDA based on the metallic electron gas. The Sham–Schlüter argument was based on the Green's function formulation in Equation 15 and was able to show a model result²³ and, moreover, detailed computations. In collaboration with Lannoo, we studied the correction to the bandgap via a model study.²⁴ With Godby, we computed

the band structures of Group IV and III–V semiconductors beyond LDA, as detailed next.

The energy gap of an insulator by Green's function method

First, we precisely define the bandgap E_g .

$$E_g = \epsilon_c - \epsilon_v = [E_{N+1} - E_N] - [E_N - E_{N-1}], \quad (16)$$

where E_M is the lowest state energy of M electrons. The gap is then divided into two parts,

$$E_g = \epsilon_g + \Delta_{xc}, \quad (17)$$

where ϵ_g is the bandgap defined by the KS eigenvalues and Δ_{xc} , the gap addition,

$$\Delta_{xc} = \int \left[\frac{\delta E_{xc}[n]}{\delta n_+} - \frac{\delta E_{xc}[n]}{\delta n_-} \right] |\psi_{N+1}(N)|^2, \quad (18)$$

where the position $\mathbf{r}$ in the integral is understood. The suffixes + and – denote the functional derivatives above and below N respectively and $\psi_j(N)$ is the j -th KS orbital of the N particle state. The gap addition is solved by the Green's function method.

Model study

For a two-band plane-wave model in one dimension,²³ the Hartree–Fock approximation, including only the exchange term, demonstrates a gap discrepancy comparable to the KS gap. When the correlation is added in the random phase approximation, which yields a dynamically screened interaction between two electrons, the bandgap due to discontinuity Δ_{xc} is shown.²⁴

Computation of band structures of common semiconductors

Computations of the electron-band structures of semiconductors by Godby, Schlüter, and Sham used the Green's function method described above. The dielectric screening and the self-energy were approximated by the random phase approximation.

For Si,²⁵ v_{xc} beyond LDA was calculated. The gap computed with the Green's function method is in good agreement with experiments. If the LDA gap is raised by the calculated discontinuity Δ_{xc} , then the energies at symmetry points of the Brillouin zone are quite similar to the LDA ones. This feature is not shared by the results for the III–V compounds.

For GaAs and AlAs,²⁶ the quasiparticle energies computed at the symmetry points are in general in good agreement with experiments. One exception is the L point conduction minimum of AlAs. The paper suggested,²⁶ with some optical data support, the discrepancy was due to the extrapolation of the experimental data from a range of alloys of $\text{Al}_x\text{Ga}_{1-x}\text{As}$ to pure AlAs.

For the series of semiconductors, ranging from small to large measured bandgaps of Si, GaAs, AlAs, diamond,²⁷ we examined the trends in the self-energies, the exchange-correlation

near the Fermi level. The calculated results showed that the $j = 5/2$ peak covering the Fermi level was dominant over $j = 7/2$ about an eV above (j being the magnitude of the sum of the orbital angular momentum and spin). This determined the trend of the three compounds. It was also confirmed by the similar behavior in the pseudobinary alloys of UPd_3 and URh_3 from the evidence by the de Haas–van Alphen measurement⁴⁰ and the resonant photoelectron emission.⁴¹

Summary

While the paradigm of finding out the landscape by the human population distribution may seem outlandish, since the technology of shaping the landscape is here, the use of the population distribution is not a bad idea. The exchange of the roles between the potential and the density in the HK theorem may be viewed quite logically as the Legendre transformation. To demonstrate its power, I used the example of some quantum phase transitions (namely the phase transition at zero temperature) that make the Legendre transformation from the magnetic field at a spin site to the spin's magnetization to calculate the quantum entanglement as a signature for the phase transition.

By the quantum viewpoint of the density distribution as one slice of the tomography measurement of the system, we see the limitation of the HK theorem and the KS effort in changing a part of the kinetic energy back to the quantum form in terms of the momentum operator that does not commute with the position. KS equation is then a single-particle problem with all the electron interaction effects in the effective DFT potential.

The connection between the single electron without Coulomb interaction to the case with the interaction is expressed in terms of an infinite perturbation series. If the perturbation series is valid, then the effective potential is known and legitimate. A number of problems, such as the bandgap problem and good band structures, have in fact been solved.

In the strong electron interaction case, the theory is postulated to be beyond the perturbation theory. Some of us have made efforts to combination the KS band structure part with strong coupling fluctuations. They have yielded some interesting observations of the existence of the strongly correlated phenomena in some ferromagnets and heavy fermions but not others.

It would be interesting to see how much DFT can contribute to understanding the modern quantum materials.

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