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A Characterization of Bounded Convex Domains in $\mathbb{C}^n$ with Non-Compact Automorphism Group

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Gabrielle Tsai

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Dissertation Committee:
Dr. Bun Wong, Chairperson
Dr. Fred Wilhelm
Dr. Yat Sun Poon

The Dissertation of Gabrielle Tsai is approved:

Committee Chairperson

University of California, Riverside

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ABSTRACT OF THE DISSERTATION

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Gabrielle Tsai

Doctor of Philosophy, Graduate Program in Mathematics
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In the field of several complex variables, the Greene-Krantz Conjecture has been of interest for decades.

Conjecture 0.0.1 (Greene-Krantz Conjecture) Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose there exists $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point $q \in \partial\Omega$ for some $p \in \Omega$. Then $\partial\Omega$ is of finite type at q.

Proof of the conjecture would allow us to start classifying bounded domains in $\mathbb{C}^n$. We already have classification for bounded domains in one dimension.

The Riemann Mapping Theorem states that there are only two simply connected domains. Specifically, every proper, simply connected open subset in $\mathbb{C}$ that is not all of $\mathbb{C}$ is biholomorphic to the disc. While it would be nice to generalize this to higher dimensions, in $\mathbb{C}^2$, the ball and bidisc are not biholomorphic to each other. To be one step closer in classifying all bounded domains in $\mathbb{C}^n$, we will add some restrictions, like studying bounded domains with non-compact automorphism group.

In this paper, we will do that and prove a special case of the Greene-Krantz Conjecture in $\mathbb{C}^2$, which can be extended to the case where we have a domain with smooth boundary in $\mathbb{C}^n$.

Theorem 0.0.2 Let Ω be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose there is a sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then q is of finite type.

Contents

Chapter 1:

Introduction

1.1 Motivation

Conjecture 1.1.1 (Greene-Krantz Conjecture). Let Ω be a smoothly bounded domain in $\mathbb{C}^n$. Suppose there exists $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point $q \in \partial\Omega$ for some $p \in \Omega$. Then $\partial\Omega$ is of finite type at q.

The following result, a special case of the Greene-Krantz Conjecture, was proved by Kaylee Hamann and Bun Wong.

Theorem 1.1.2 (Hamann, Wong [8]) Let Ω be a bounded, convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose that there is a sequence $\{\phi_k\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then $\partial\Omega$ contains no non-trivial analytic variety at q.

Although finite type implies that the boundary does not contain a non-trivial analytic disc, the converse does not hold in general. The following theorem which Dylan Noack proved shows that with the addition of non-tangential convergence of the sequence that accumulates at a boundary point, the boundary point is of finite type.

Theorem 1.1.3 (Noack [16]) Let Ω be a bounded, convex domain with smooth boundary in $\mathbb{C}^n$. Suppose there exists $p \in \Omega$ and $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates and approaches non-tangentially to a boundary point q. Then q is of finite type.

The goal of this paper is to drop the non-tangential convergence condition from Theorem 1.1.3 to give the following result.

Theorem 1.1.4 Let Ω be a bounded, convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose there is a sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then q is of finite type.

In this paper, we give a proof of Theorem 1.1.4, in which the bigger picture of the proof will be similar to that of Theorem 1.1.2 and 1.1.3.

Chapter 2:

Background

2.1 Holomorphic Functions

This chapter will present key definitions, examples, and results that will build the foundation of the main result.

Definition 2.1.1 Let $\Omega \subseteq \mathbb{C}^n$ be an open, connected set. A function $f : \Omega \to \mathbb{C}$ is holomorphic if for each $j = 1, ..., n$ and each fixed $z_1, ..., z_{j-1}, z_{j+1}, ..., z_n$, the function

$$\zeta \mapsto f(z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_n)$$

is holomorphic in the classic one-variable sense on the set

$$\{\zeta \in \mathbb{C} : (z_1, ..., z_{j-1}, \zeta, z_{j+1}, ..., z_n) \in \Omega\}$$

Another way to interpret the definiton of holomorphic functions in several variables is that it must be holomorphic in each variable separately. Those familiar with single-variable complex analysis may be familiar with the following theorem which gives equivalent definitions of a holomorphic function.

Theorem 2.1.2 Let $D_n(z_0, r) = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_j - z_0| < r, 1 \leq j \leq n\}, \Omega \subseteq \mathbb{C}^n$ be an open, connected set and $f : \Omega \to \mathbb{C}$ be continuous in each variable separately. Then the following are equivalent.

1) f is holomorphic
2) f satsifies the Cauchy-Riemann equations in each variable separately
3) For each $z_0 \in \Omega$ there exists an $r = r(z_0) > 0$ such that $\overline{D_n(z_0, r)} \subseteq \Omega$ and f can be written as an absolutely and uniformly convergent power series

$$f(z) = \sum_{\alpha} a_\alpha (z - z_0)^\alpha \text{ for all } z \in D_n(z_0, r)$$

4) For each $w \in \Omega$ there exists $r = r(w) > 0$ such that $\overline{D_n(w, r)} \subset \Omega$ and

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_n - w_n| = r} \cdots \int_{|\xi_1 - w_1| = r} \frac{f(\xi_1, ..., \xi_n)}{(\xi_1 - z_1)...(\xi_n - z_n)} d\xi_1...d\xi_n \text{ for all } z \in D_n(w, r)$$

For the proof of Theorem 2.1.2, we refer the reader to [11].

Definition 2.1.3 Let $\Omega, \Omega' \subseteq \mathbb{C}^n$ be open, connected sets and $f : \Omega \to \Omega'$ a holomorphic function. If f is a bijection then we refer to f as a biholomorphism. We refer to Ω and Ω' being biholomorphic. If $\Omega = \Omega'$, we refer to f as an automorphism.

Definition 2.1.4 $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy if there do not exist non-empty open sets U_1, U_2 with U_2 connected, $U_2 \subsetneq \Omega, U_1 \subseteq U_2 \cap \Omega$ such that for every holomorphic function f on Ω, there is a holomorphic function g on U_2 such that $f = g$ on U_1.

To put the definition using other terms, a domain of holomorphy is a domain that cannot be extended to some larger domain under every holomorphism. This allows us to study maximal domains for some holomorphic function, which make them more interesting as they cannot be extended. The definition for domain of holomorphy is not usually brought up in a single variable complex context because in one variable every domain is a domain of holomorphy. One well-known example that a domain Ω can be extended to some larger domain under every holomorphic function $f : \Omega \to \mathbb{C}$ is the following example of Hartogs.

Example 2.1.5 Consider the domain

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\} - \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$$

We will show that every holomorphic function $f : \Omega \to \mathbb{C}$ extends to the domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\}$$

For z_1 fixed, $|z_1| < 3$ we write

$$f_{z_1}(z_2) = f(z_1, z_2) = \sum_{j=-\infty}^{\infty} a_j(z_1) z_2^j$$

where the cofficients of the Laurent expansion are given by

$$a_j(z_1) = \frac{1}{2\pi i} \int_{|\xi|=2} \frac{f(z_1, \xi)}{\xi^{j+1}} d\xi.$$

Specifically, $a_j(z_1)$ depends holomorphically on z_1 by Morera's theorem. But $a_j(z_1) = 0$ for $j < 0$ and $1 < |z_1| < 3$. Thus by analytic continuation, a_j is identially zero for $j < 0$. But then the series expansion becomes

$$\sum_{j=0}^{\infty} a_j(z_1) z_2^j$$

and this series defines a holomorphic function $\hat{f}$ on $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 3, |z_2| < 3\}$ such that $\hat{f}|_\Omega = f$. Because f was was arbitrary, all holomorphic functions on Ω can be continued to a larger domain, and thus Ω is not a domain of holomorphy.

In several complex variables, not every domain is a domain of holomorphy.

2.2 Convexity and Pseudoconvexity

In this section we give the definitions and some important properties of convex and pseudoconvex domains.

Definition 2.2.1 A bounded domain $\Omega \subset \mathbb{C}^n$ is convex if Ω contains the entire line segment joining any pair of its points.

Holomorphic mappings do not preserve convexity. One example is the unit disc $\Delta \subset \mathbb{C}$, which is convex, but the image under the mapping $f(z) = (4+z)^4$ is not convex. Thus we will need some less rigid geometric condition to characterize bounded domains, preferably one where convexity is biholomorphically invariant.

Definition 2.2.2 Let $\Omega \subseteq \mathbb{R}^n$ be an open set with C^k boundary. A function $\rho : \mathbb{R}^n \to \mathbb{R}$ is said to be a defining function for Ω if ρ is C^k and

1) $\rho(x) < 0$ for all $x \in \Omega$
2) $\rho(x) > 0$ for all $x \notin \Omega$ and
3) $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$.

Definition 2.2.3 Let $\Omega \subset \mathbb{R}^n$ have C^1 defining function ρ. Let $p \in \partial\Omega$. We consider the vector $w = (w_1, ..., w_n)$ to be tangent to $\partial\Omega$ at p if

$$\sum_{j=1}^{n} \left.\frac{\partial \rho}{\partial x_j}\right|_p w_j = 0$$

In this case we write $w \in T_p\partial\Omega$.

Definition 2.2.4 Let $\Omega \subset \mathbb{R}^n$ have C^2 defining function ρ. Let $p \in \partial\Omega$. We say that $\partial\Omega$ is convex at p if

$$\sum_{j,k=1}^{n} \left.\frac{\partial^2 p}{\partial x_j \partial x_k}\right|_p w_j w_k \geq 0$$

for all $w = (w_1, ..., w_n) \in T_p\partial\Omega$. If the inequality is strict we refer to p as a point of strong convexity.

The following are a few convex domains and their defining functions.

1) The unit disk in $\mathbb{C}$ is given by $\Delta = \{z \in \mathbb{C} : |z| - 1 < 0\}$.
2) The half plane in $\mathbb{C}$ given by $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}z > 0\}$.
3) The unit polydisk in $\mathbb{C}^n$ given by $\Delta_n = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_j| < 1 \text{ for all } j \in \mathbb{Z}\}$.
4) The ball in $\mathbb{C}^n$ centered at z_0 of radius r given by $B_r(z_0) = \{(z_1, ..., z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 - 1 < 0\}$.

Definition 2.2.5 Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a C^2 boundary (i.e. the defining function ρ for the boundary is C^2). Then $\partial\Omega$ is pseudoconvex at q if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \geq 0 \qquad \text{for all } w \in T_q^{1,0}(\partial\Omega), \qquad \text{where}$$

$$T_q^{1,0}(\partial\Omega) := \left\{ w \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(q) w_j = 0 \right\}$$

If we have a strict inequality, then q is a point of strong pseudoconvexity. $T_q^{1,0}$ is the complex tangent space to the boundary $\partial\Omega$ at q. Note that $\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}$ can be referenced as the complex Hessian, or Levi form. If every point in the boundary is strongly pseudoconvex, then the domain itself is strongly pseudoconvex.

The following are a few important properties of pseudoconvex domains:

1) Pseudoconvexity is independent of the choice of defining function.

Proof: Let ρ and $\tilde{\rho}$ be two defining functions of $\partial\Omega$ in a neighborhood U of q, for $q \in \partial\Omega$. Then there exists a C^1 function h defined in U such that $\tilde{\rho} = h\rho$, where $h(z) > 0$ for all $z \in U$. Hence,

$$\begin{aligned}
\frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) &= \frac{\partial^2 (\rho h)}{\partial z_j \partial \bar{z}_k}(q) \\
&= \frac{\partial}{\partial z_j}\left(\frac{\partial(\rho h)}{\partial \bar{z}_k}(q)\right) \\
&= \frac{\partial}{\partial z_j}\left(\frac{\partial \rho}{\partial \bar{z}_k}(q) \cdot h(q) + \rho(q) \cdot \frac{\partial h}{\partial \bar{z}_k}\right) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) \cdot h(q) + \frac{\partial \rho}{\partial \bar{z}_k}(q) \cdot \frac{\partial h}{\partial z_j}(q) + \frac{\partial \rho}{\partial z_j}(q) \cdot \frac{\partial \rho h}{\partial \bar{z}_k}(q) + \rho(q) \cdot \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}(q) \\
&= \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) \cdot h(q) + \frac{\partial \rho}{\partial \bar{z}_k}(q) \cdot \frac{\partial h}{\partial z_j}(q) + \frac{\partial \rho}{\partial z_j}(q) \cdot \frac{\partial h}{\partial \bar{z}_k}(q)
\end{aligned}$$

where the last equality follows from $\rho(q) = 0$. Therefore

$$\begin{aligned}
\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k &= h(q) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \\
&\quad + \sum_{j,k=1}^{n} \left(\frac{\partial \rho}{\partial \bar{z}_k}(q) \cdot \frac{\partial h}{\partial z_j}(q) + \frac{\partial \rho}{\partial z_j}(q) \cdot \frac{\partial h}{\partial \bar{z}_k}(q)\right) w_j \bar{w}_k \\
&= h(q) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \\
&\quad + 2\mathrm{Re} \sum_{j,k=1}^{n} \left(\frac{\partial \rho}{\partial z_j}(q) \cdot \frac{\partial h}{\partial \bar{z}_k}(q) w_j \bar{w}_k\right) \\
&= h(q) \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \quad \text{if } w \in T_q(\partial\Omega).
\end{aligned}$$

Therefore, since $h(q) > 0, q \in \partial\Omega$ is a pseudoconvex point with respect to ρ if and only if it is a pseudoconvex point with respect to $\tilde{\rho}$. Therefore the pseudoconvexity of a boundary point is not dependent on the choice of defining function.

2) Pseudoconvexity is preserved under biholomorphic mappings.

Proof: Let $\Phi : \Omega \to \mathbb{C}^n$ be biholomorphic onto its image, and let Ω' denote the image $\Phi(\Omega)$. Further, assume that Φ is biholomorphic in a neighborhood of $q \in \partial\Omega$. Then $\Phi(z) = \Phi(z_1, ..., z_n) = (\Phi(z_1), ..., \Phi(z_n)) = (z_1', ..., z_n')$. Let $\rho : U \to \mathbb{R}$ be a local defining function for $\partial\Omega$, for U an open set. Then $\tilde{\rho} := \rho \circ \Phi^{-1}$ is a local defining function for $\partial\Omega'$. Choose $q \in \partial\Omega$ and $w \in T_q(\partial\Omega)$. Then $\Phi(q) \in \partial\Omega'$ and $w' \in T_{\Phi(q)}(\partial\Omega')$, where

$$w' = \begin{pmatrix} w_1' \\ \vdots \\ w_n' \end{pmatrix} = \begin{pmatrix} \frac{\partial\Phi_1}{\partial z_1}(q) & \dots & \frac{\partial\Phi_1}{\partial z_n}(q) \\ \vdots & \ddots & \vdots \\ \frac{\partial\Phi_n}{\partial z_1}(q) & \dots & \frac{\partial\Phi_n}{\partial z_n}(q) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \sum \frac{\partial\Phi_1}{\partial z_j}(q) w_j \\ \vdots \\ \sum \frac{\partial\Phi_n}{\partial z_j}(q) w_j \end{pmatrix}$$

Now, since $\tilde{\rho} := \rho \circ \Phi^{-1}$, one can see that $\rho = \tilde{\rho} \circ \Phi$, implying that

$$\begin{aligned} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) &= \frac{\partial^2 (\tilde{\rho} \circ \Phi)}{\partial z_j \partial \bar{z}_k}(q) \\ &= \frac{\partial}{\partial z_j}\left(\frac{\partial(\tilde{\rho} \circ \Phi)}{\partial \bar{z}_k}(q) \right) \\ &= \sum_{t,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_m' \partial \bar{z}_t'}(\Phi(q)) \cdot \frac{\partial \Phi_m}{\partial z_j}(q) \cdot \frac{\partial \bar{\Phi}_t}{\partial \bar{z}_k}(q) \end{aligned}$$

by the chain rule. Hence,

$$\begin{aligned}\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k &= \sum_{j,k=1}^{n} \left(\sum_{t,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_t}(\Phi(q)) \cdot \frac{\partial \Phi_m}{\partial z_j}(q) \cdot \frac{\partial \bar{\Phi}_t}{\partial \bar{z}_k}(q) \right) w_j \bar{w}_k \\ &= \sum_{j,k=1}^{n} \left(\sum_{t,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_t}(\Phi(q)) \cdot \frac{\partial \Phi_m}{\partial z_j}(q) w_j \cdot \frac{\partial \bar{\Phi}_t}{\partial \bar{z}_k}(q) \bar{w}_k \right) \\ &= \sum_{t,m=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z'_m \partial \bar{z}'_t}(\Phi(q)) w'_m \bar{w}'_t,\end{aligned}$$

which implies that the Levi form is preserved under biholomorphic mappings. In other words, pseudoconvexity is preserved under biholomorphism.

3) If $q \in \partial\Omega$ is a strongly pseudoconvex point, then there exists a neighborhood U containing q such that for all $p \in \partial\Omega \cap U, p$ is strongly pseudoconvex.

Proof: First we will need the following technical lemma from Chapter 3 of Krantz.

Lemma 2.2.6 If Ω is strongly pseudoconvex, then Ω has a defining function $\tilde{\rho}$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \geq C|w|^2$$

for all $q \in \partial\Omega$ and $w \in \mathbb{C}^n$, where $C > 0$.

By lemma 2.2.6, there exists a defining function $\tilde{\rho}$ for Ω such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \geq C|w|^2$$

for all $w \in \mathbb{C}^n$. In particular,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k \geq 0$$

for all $w \neq 0, w \in \mathbb{C}^n$. Since $\tilde{\rho}$ is C^2, the function

$$\Phi : p \mapsto \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq C|w|^2$$

is continuous in a neighbhorhood U of q, which implies that for all $p \in U \cap \partial\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k}(q) w_j \bar{w}_k > 0$$

for all $w \neq 0, w \in \mathbb{C}^n$ by the continuity of Φ. This implies that $p \in U \cap \partial\Omega$ is strongly pseudoconvex.

4) Every domain in $\mathbb{C}$ with a C^2 boundary is vacuously pseudoconvex.

Proof: Let Ω be a domain in $\mathbb{C}$ with C^2 boundary. That is, the defining function ρ is C^2. Then for all $q \in \partial\Omega$,

$$\nabla \rho(q) = \frac{\partial \rho}{dz}(q) \neq 0,$$

which implies that $w \in T_q(\partial\Omega)$ if and only if $w = 0$. This implies that $T_q(\partial\Omega) = \{0\}$, giving that Ω is pseudoconvex, since the condition for pseudoconvexity in one dimension,

$$\frac{\partial^2 \tilde{\rho}}{\partial z \partial \bar{z}}(q) w \bar{w} \geq 0,$$

is always satisfied.

Remark 2.2.7 A small perturbation of the ball remains convex, but in general may no longer be biholomorphic to a ball. If the small perturbation is made in the radial direction, then the domain will be biholomorphic to a ball. A general perturbation of the domain will not remain biholomorphic to the ball.

Remark 2.2.8 Convexity is not preserved under biholomorphism. For example the unit disk in $\mathbb{C}$ can be mapped to a non-convex domain.

Remark 2.2.9 Since any geometrically convex domain in $\mathbb{C}^n$ is pseudoconvex and pseudoconvex domains are domains of holomorphy, all convex domains are domains of holomorphy.

2.3 Finite Type

Definition 2.3.1 Let $\Omega := \{z : \rho(z) < 0\}$ be a smoothly bounded domain in $\mathbb{C}^2$, and let $q \in \partial\Omega$. Then the analytic disc $\phi : \Delta \to \mathbb{C}^2$ is called a non-singular disc tangent to $\partial\Omega$ at q if $\phi(0) = q, \phi'(0) \neq 0$, and $(\rho \circ \phi)'(0) = 0$.

Definition 2.3.2 If we have a holomorphic function $f : \mathbb{C} \to \mathbb{C}$, then the multiplicity of f at P, $\nu_P(f)$, is defined to be the least positive integer k such that the kth derivative does not vanish at P. If f is not differentiable then the multiplicity of f at c is k if and only if k is the smallest integer such that $\lim_{z\to c} \frac{f(z)}{|z-c|^k} \neq 0$.

Definition 2.3.3 Let $\Omega \subset \mathbb{C}^n$ be a smooth domain and $q \in \partial\Omega$. Let ρ be a defining function for Ω in a neighborhood of q. We say that q is of finite type C in the sense of

D'Angelo if

$$\sup_f \left\{ \frac{\nu(\rho \circ f)}{\nu(f)} \right\} = C < \infty$$

where f ranges through non-constant holomorphic curves with $f(0) = q$. Otherwise we refer to q as being infinite type. If every point $q \in \partial\Omega$ is of finite type we say Ω is a finite type domain.

The type of a boundary point measures the maximum order of contact of an analytic disc with said point. A couple of important properties of type:

1) The definition of type is independent of the choice of defining function.

   Proof: Let $\Omega \subset \mathbb{C}^2$ be smooth, with defining function ρ. Let $q \in \partial\Omega$ and $\tilde{\rho}$ be a second defining function for Ω. Then there exists a function h, non-vanishing in a neighborhood of $\partial\Omega$, such that $\tilde{\rho} = h\rho$, and hence $\rho = \frac{1}{h}\tilde{\rho}$. Thus, for any non-singular analytic disc ϕ that is tangent to $\partial\Omega$ at q,

$$|\rho(\phi(\zeta))| = \left| \left(\frac{\tilde{\rho}}{h} \right) (\phi(\zeta)) \right| = \left| \frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))} \right|.$$

   Let $q \in \partial\Omega$ be a point of finite type m with respect to ρ. That is, suppose there exists a non-singular disc ϕ tangent to $\partial\Omega$ at q such that for small $|\zeta|$,

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^m.$$

Then for small $|\zeta|$ one sees that

$$\left|\frac{\tilde{\rho}(\phi(\zeta))}{h(\phi(\zeta))}\right| \leq C|\zeta|^m,$$

i.e.

$$|\tilde{\rho}(\phi(\zeta))| \leq C|h(\phi(\zeta))||\zeta|^m \leq CM|\zeta|^m$$

for small $|\zeta|$, where

$$M := \sup_{\text{small } |\zeta|} |h(\phi(\zeta))|.$$

Thus for small $|\zeta|$,

$$|\tilde{\rho}(\phi(\zeta))| \leq C_1|\zeta|^m.$$

Now suppose there exists a non-singular disc ψ tangent to $\partial\Omega$ at q such that $|\tilde{\rho}(\psi(\zeta))| \leq C|\zeta|^{m+1}$ for small $|\zeta|$. Then,

$$|\rho(\psi(\zeta))| \leq \frac{|\tilde{\rho}(\psi(\zeta))|}{|h(\psi(\zeta))|} \leq \frac{C|\zeta|^{m+1}}{|h(\psi(\zeta))|} \leq \frac{C}{M}|\zeta|^{m+1},$$

where

$$M := \inf_{\text{small } |\zeta|} |h(\psi(\zeta))|.$$

Therefore,

$$|\rho(\psi(\zeta))| \leq C_1|\zeta|^{m+1}$$

for small $|\zeta|$.

This contradicts the fact that q is of finite type m with respect to ρ. Therefore, q is a point of finite type m with respect to $\tilde{\rho}$, which completes the proof.

2) The condition of finite type is preserved under biholomorphism.

Remark 2.3.4 A strongly pseudoconvex boundary point is always of type 2. For example, every boundary point on the unit ball is of finite type 2.

Consider the boundary point $q = (1, 0)$ on the unit ball B_2. Observe that

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ \bar{z}_2 \end{pmatrix} \implies \nabla\rho(q) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

which implies that any curve tangent to ∂B_2 at q must be of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)),$$

after a re-parametrization.

Consider the disc $\phi(\zeta) = (1, \zeta)$. It has order of contact 2 with the boundary of B_2 at q because

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |\zeta|^2.$$

So what is the maximum order of contact when ϕ is of the form $\phi(\zeta) = (1+O(\zeta^2), \zeta+O(\zeta^2))$?

Observe the following computation:

$$\begin{aligned}\rho(\phi(\zeta)) &= |1+O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^2 - 1\\ &= |1+O(\zeta^2)|^2 + |\zeta|^2 \cdot |1+O(\zeta)|^2 - 1\\ &\leq C|\zeta|^2\end{aligned}$$

for small $|\zeta|$, since

$$|1+O(\zeta^2)|^2 \to 1 \text{ as } |\zeta| \to 0$$

$$\text{and } |1+O(\zeta)|^2 \to 1 \text{ as } |\zeta| \to 0.$$

Therefore $q = (1,0) \in \partial B_2$ is a point of finite type 2. In general, it can be shown that a strongly pseudoconvex boundary point is always of type 2.

Remark 2.3.5 Finite type implies there does not exist a non-trivial analytic disc in the boundary.

Example 2.3.6 The analytic ellipsoid, or also known as the egg domain, $E_m = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^{2m} - 1 < 0\}$ has the boundary point $(1,0)$ which is of finite type $2m$.

Consider the boundary point $q = (1,0)$. In order to calculate the type at q, notice that

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ m z_2^{m-1} \bar{z}_2^{m-1} \end{pmatrix} \implies \nabla\rho(q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which implies that, after a re-parametrization, a non-singular analytic disc ϕ that intersects ∂E_m at q is of the form

$$\phi(\zeta) = (1 + O(\zeta^2), \zeta + O(\zeta^2)).$$

What is the maximum order of contact of such a curve with the boundary? First, consider the simple case wherein $\phi(\zeta) = (1, \zeta)$. This curve has order of contact $2m$ at the boundary point q, because

$$\phi(\rho(\zeta)) = |\zeta|^{2m}.$$

One must now ask, can the order of contact improve? For an arbitrary curve ϕ as described above,

$$\begin{aligned}\rho(\phi(\zeta)) &= |1 + O(\zeta^2)|^2 + |\zeta + O(\zeta^2)|^{2m} - 1 \\ &= |1 + O(\zeta^2)|^2 + |\zeta|^{2m} \cdot |1 + O(\zeta)|^{2m} - 1 \\ &\leq C|\zeta|^{2m}\end{aligned}$$

for small $|\zeta|$, since

$$|1 + O(\zeta^2)|^2 \to 1 \text{ as } |\zeta| \to 0$$

$$\text{and } |1 + O(\zeta)|^{2m} \to 1 \text{ as } |\zeta| \to 0.$$

Therefore, the maximum order of contact of any non-singular analytic disc tangent to ∂E_m at $q = (1, 0)$ is $2m$, which implies that q is a point of finite type $2m$.

We will now look at a few examples of domains in $\mathbb{C}^2$ that have boundary points of infinite type.

Example 2.3.7 An exponentially flat domain $E_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + 2\exp(-|z_2|^{-2}) - 1 < 0\}$ is of infinite type at the point $(1, 0)$.

Consider the point $q = (1, 0) \in \partial E_\infty$. Then

$$\nabla \rho = \begin{pmatrix} \bar{z}_1 \\ \frac{2e^{-1/|z_2|^2}}{z_2^2 \bar{z}_2} \end{pmatrix} \implies \nabla \rho(q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

And regarding the curve $\phi(\zeta) = (1, \zeta)$ which is tangent to ∂E_∞ at q, one sees that

$$\rho(\phi(\zeta)) = 2e^{-1/|\zeta|^2},$$

which implies that

$$\frac{|\rho(\phi(\zeta))|}{|\zeta|^m} = \frac{2e^{-1/|\zeta|^2}}{|\zeta|^m} \to 0 \text{ as } \zeta \to 0$$

by l'Hopital's rule, since

$$\frac{d^k}{d\zeta^k}(2e^{-1/|\zeta|^2})|_{\zeta=0} = 0 \ \ \forall k \in \mathbb{Z}^+.$$

Since this is true for any $m \in \mathbb{Z}^+$,

$$|\rho(\phi(\zeta)) \leq C|\zeta|^m$$

as $|\zeta| \to 0$ for all $m \in \mathbb{Z}^+$, implying that $q = (1, 0)$ is a point of infinite type.

Example 2.3.8 The bidisc $\Delta_2 = \{(z_1, z_2) \in \mathbb{C}^2 : \rho_1(z_1, z_2) = |z_1|^2 - 1 < 0 \text{ and } \rho_2(z_1, z_2) = |z_2|^2 - 1 < 0\} \subset \mathbb{C}^2$ is of infinite type at $(1, 0)$.

Consider the point $q = (1, 0) \in \partial\Delta_2$. In a neighborhood U of q, let $\rho(z) = |z_1| - 1$ be a local defining function for the boundary defined inside $U \cap \partial\Delta_2$. Then

$$\nabla\rho = \begin{pmatrix} \bar{z}_1 \\ \\ 0 \end{pmatrix} \implies \nabla\rho(q) = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix}$$

and hence the non-singular analytic disc $\phi(\zeta) = (1, \zeta)$ is tangent to $\partial\Delta_2$ at q. Thus

$$\rho(\phi(\zeta)) = \rho(1, \zeta) = |1| - 1 = 0 \ \ \forall\zeta \in \Delta_2,$$

which implies that

$$|\rho(\phi(\zeta))| \leq C|\zeta|^m$$

as $|\zeta| \to 0$ for all $m \in \mathbb{Z}^+$. Therefore $q = (1, 0) \in \partial\Delta_2$ is a point of infinite type.

The previous examples illustrate the greater the type at a boundary point, the flatter the boundary is in a neighborhood of that point.

If there is an analytic variety in the boundary of some domain $\Omega \subset \mathbb{C}^n$ passing through $q \in \partial\Omega$, then q is a point of infinite type. Observe that this is the contrapositive of Remark 2.3.5.

The converse, "if a boundary point is of infinite type, then there is an analytic variety in the boundary," is not true. When $\Omega \subset \mathbb{C}^n$ is a convex domain that contains a boundary point q of infinite type, it does not necessarily mean that $\partial\Omega$ contains a disc. For example, $E_\infty = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + 2\exp(-|z_2|^{-2}) - 1 < 0\}$ is of infinite type at the point $(1, 0)$, but the ∂E_∞ does not contain a disc.

Definition 2.3.9 Suppose that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ where ρ is a defining function. We say a point $x \in \partial\Omega$ has finite line type L if

$$\sup\{\nu(\rho \circ t) | t : \mathbb{C} \to \mathbb{C}^n \quad \text{is a non-trivial affine map and} \quad t(0) = x\} = L$$

where $L < \infty$. If $L = \infty$ we say that x has infinite line type.

Theorem 2.3.10 (McNeal [13]) If a convex domain has finite line type, then it is of finite type in the sense of D'Angelo.

Remark 2.3.11 The converse of 2.3.10 says that if we have a point on the boundary of a convex domain that is of infinite type, then the line type is infinite. This will be used later in the proof of Lemma 5.2.1.

Lemma 2.3.12 [21] Let $\Omega \subseteq \mathbb{C}^n$ be bounded, convex domain with smooth boundary such that $0 \in \partial\Omega$, the positive imaginary z_1-axis points normally inward with all other directions tangent. Let $f : \mathbb{R} \times \mathbb{C}^{n-1} \to \mathbb{R}$ be a smooth, non-negative and convex function and U be a neighborhood of 0. Ω is described locally around 0 by:

$$\Omega \cap U = \{(z_1, ..., z_n) \in U : f(\mathrm{Re} z_1, z_2, ..., z_{n-1}, z_n) < \mathrm{Im} z_1\}$$

If $0 \in \partial\Omega$ is a point of infinite line type then there exists a change of coordinates so for all $k > 0$,

$$\lim_{w \to 0} \frac{f(0, w, ..., 0, 0)}{|w|^k} = 0$$

Proof: By definition of infinite line type and the converse of Theorem 2.3.10 McNeal [13], there exists a sequence of linear maps t_k such that $\nu(f \circ t_k) \geq k$. By compactness of the sphere, $t_k \to t$ in subsequence after choosing the correct parametrizations. By continuity of the defining function f, $\nu(f \circ t) = \infty$. Let us choose our coordinates so t is the z_2 axis. Then the conclusion follows.

Definition 2.3.16 For a domain $\Omega \subseteq \mathbb{C}^n$ with C^1 boundary, a sequence $\{q_j\} \subset \Omega$ and a point $q \in \partial\Omega$, we say that $q_j \to q$ non-tangentially if for all $j > 0$,

$$q_j \in \Gamma_\alpha(q) = \{z \in \Omega : ||z - q|| \leq \alpha \mathrm{dist}(\partial\Omega, z)\}$$

for some $\alpha > 1$. We say that $q_j \to q$ normally if the q_j's approach q along the real normal line to $\partial\Omega$ at q.

We can visualize a sequence in the domain converging to the boundary non-tangentially as a conical-shaped region where points in the region cannot approach the boundary along a tangential curve.

Chapter 3:

Automorphism Groups

3.1 Domains with non-compact automorphism groups

Throughout this dissertation, a bounded domain in $\mathbb{C}^2$ will be denoted by Ω, and the automorphism group will be denoted by $\text{Aut}(\Omega)$. An element $\phi \in \text{Aut}(\Omega)$ is a biholomorphic map from Ω onto itself. $\text{Aut}(\Omega)$ is a topological group, the topology being given by the compact-open topology. Furthermore, Henri Cartan showed that $\text{Aut}(\Omega)$ is a Lie group.

Theorem 3.1.1 (H. Cartan) [14] Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. Then $\text{Aut}(\Omega)$ is a Lie group.

Our focus will be on domains whose automorphism groups are non-compact.

Definition 3.1.2 Let G be a topological group and X a topological Hausdorff space. G acts on X if there exists a continuous map $\sigma : G \times X \to X, \sigma(g, x) = gx$, such that $\sigma(e, x) = ex = x$ for all $x \in X$ and $\sigma(gg', x) = \sigma(g, \sigma(g', x))$ for all $g, g' \in G$ and $x \in X$.

Definition 3.1.3 Let G and X be as in the previous definition. The orbit of $x \in X$ under the action of G is the set $\{\sigma(g,x) : g \in G\} \subset X$.

Definition 3.1.4 A map $f : \Omega \to \bar{\Omega}$, where $\Omega \subset \mathbb{C}^n$ and $\bar{\Omega} \subset \mathbb{C}^m$, is called proper if for any compact set $\bar{K} \subset \bar{\Omega}$, the set $f^{-1}(\bar{K})$ is compact in Ω.

Definition 3.1.5 If G is a topological group and X is a topological Hausdorff space where G acts on X and both are locally compact, then the action of G on X is proper if the map $G \times X \to X \times X$, defined by $(g,x) \mapsto (\sigma(g,x), x)$, is proper.

Definition 3.1.6 We say $q \in \partial\Omega$ is a boundary accumulation point for the action of $\text{Aut}(\Omega)$ on Ω if there exists a point $p \in \Omega$ and a sequence $\{\phi_j\} \subset \text{Aut}(\Omega)$ such that $\phi_j(p) \to q$ as $j \to \infty$.

Claim 3.1.7 If $\partial\Omega$ contains a boundary accumulation point q, then $\text{Aut}(\Omega)$ is non-compact.

Proof: Assume towards a contradiction that $\text{Aut}(\Omega)$ is compact. Then for any sequence $\{\phi_j\} \subset \text{Aut}(\Omega)$, there exists a subsequence $\{\phi_{j_v}\} \subset \{\phi_j\}$ such that $\phi_{j_v} \to \phi \in \text{Aut}(\Omega)$. Consider the sequence $\{g_{j_v}\} \subset \text{Aut}(\Omega)$. By assumption, there exists $\{g_{j_v}\} \subset \{g_j\}$ such that $g_{j_v} \to g \in \text{Aut}(\Omega)$ as $v \to \infty$. In particular $g(p) = q \in \Omega$ for some $p \in \Omega$, which implies that $q \in \Omega \cap \partial\Omega$, contradicting that Ω is open. Therefore $\text{Aut}(\Omega)$ must be non-compact.

Example 3.1.8 $\text{Aut}(E_m)$, where $E_m := \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^{2m} - 1 < 0\}$ is the polynomial ellipsoid, is non-compact.

$$\text{Aut}(E_m) = \left\{(z_1, z_2) \mapsto \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \left(\frac{\sqrt{1-|a|^2}}{1 - \bar{a}z_1}\right)^{1/m} z_2\right) : |a| < 1\right\}$$

The point $(1, 0) \in \partial E_m$ is a boundary orbit accumulation point for the action of $\text{Aut}(E_m)$ on E_m, and thus non-compact.

Proof: Choose a_j, $0 \leq a_j \leq 1$, such that as $j \to \infty, a_j \to 1$. Let $z = (z_1, z_2) \in E_m$ and $\phi_{a_j} \in \text{Aut}(E_m)$. Then $(1, 0)$ is a boundary orbit accumulation point of the action of $\text{Aut}(E_m)$ on E_m, since $\phi_{a_j}(z) \to (1, 0)$ as $j \to \infty$. This implies that $\text{Aut}(E_m)$ is non-compact.

Proposition 3.1.9 Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with a transitive automorphism group, i.e. let Ω be homogeneous. Then $\text{Aut}(\Omega)$ is non-compact.

Proof: Let Ω be a bounded domain in $\mathbb{C}^n$ with a transitive automorphism group. That is, given any two points $x, y \in \Omega$, there exists $\phi \in \text{Aut}(\Omega)$ such that $\phi(x) = y$. Let $z \in \Omega$. By the transitivity of $\text{Aut}(\Omega)$, the orbit of z is

$$\{w \in \Omega : w = \phi(z), \text{ for some } \phi \in \text{Aut}(\Omega)\} = \Omega$$

Since Ω is open, it is not compact, hence the orbit of z is non-compact, and hence $\text{Aut}(\Omega)$ is non-compact.

Proposition 3.1.9 can now be used to show that Aut(Δ) is non-compact because it is known that Aut(Δ) is transitive.

Example 3.1.10 Aut(Δ) where $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc in $\mathbb{C}$ is non-compact. Then

$$\text{Aut}(\Delta) = \{e^{i\theta}\frac{z-a}{1-\bar{a}z} : a \in \Delta, \theta \in [0, 2\pi]\}$$

Specifically, given any two points $a, b \in \Delta$, let

$$\phi_a(z) := \frac{z-a}{1-\bar{a}z} \text{ and } \phi_{-b}(z) := \frac{z+b}{1+\bar{b}z}$$

Then both ϕ_a and ϕ_{-b} are in Aut(Δ). Furthermore, $\phi_{-b} \circ \phi_a(a) = \phi_{-b}(0) = b$, which implies that Aut(Δ) is transitive. Thus, by Proposition 3.1.9, Aut(Δ) is non-compact.

Remark 3.1.11 Since Aut(Δ_n) and Aut(B_n) are transitive, the automorphism groups are non-compact.

Claim 3.1.12 Aut(Δ_n) is transitive and thus non-compact.

Proof: Let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in \Delta_n$. Consider the following automorphisms of Δ_n:

$$\phi_{-b}(z) = \left(\frac{z_1+b_1}{1+\bar{b_1}z_1}, ..., \frac{z_n+b_n}{1+\bar{b_n}z_n}\right) \text{ and } \phi_a(z) = \left(\frac{z_1-a_1}{1-\bar{a_1}z_1}, ..., \frac{z_n-a_n}{1-\bar{a_n}z_n}\right)$$

Then $(\phi_{-b} \circ \phi_a)(a) = \phi_{-b}(0) = b$, which implies that $\text{Aut}(\Delta_n)$ is transitive. Thus by Proposition 3.1.9, $\text{Aut}(\Delta_n)$ is non-compact.

Claim 3.1.13 $\text{Aut}(B_n)$ is transitive and thus non-compact.

Proof: Choose $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in B_n$. Then there exists $\Phi_a \in U(n)$ such that $\Phi_a(a) = (a_1, 0, ..., 0)$, i.e. Φ_a rotates the a onto the z_1-axis. Choose $\Phi_{-b} \in U(n)$ such that $\Phi_{-b}(b_1, 0, ..., 0) = b$, i.e. Φ_{-b} is the inverse of Φ_b. We define automorphisms of the ball to be $\phi_a(z_1, ..., z_n) := \left(\frac{z_1 - a}{1 - \bar{a}z_1}, \frac{\sqrt{1-|a|^2}}{1-\bar{a}z_1} z_2, ..., \frac{\sqrt{1-|a|^2}}{1-\bar{a}z_1} z_n\right)$. Let ϕ_{a_1}, ϕ_{-b_1} be the automorphisms of the ball as described. Note that $\phi_{-b_1} = (\phi_{b_1})^{-1}$. Then $(\Phi_{-b} \circ \phi_{-b} \circ \phi_a \circ \Phi_a)(a) = \Phi_{-b}(\phi_{-b}(\phi_a(\Phi_a(a)))) = \Phi_{-b}(\phi_{-b}(\phi_a(a_1, 0, ..., 0))) = \Phi_{-b}(\phi_{-b}(0)) = \Phi_{-b}(b_1, 0, ..., 0) = b$. Hence $\text{Aut}(B_n)$ is transitive. Therefore by Proposition 3.1.9, $\text{Aut}(B_n)$ is non-compact.

Example 3.1.14 $\text{Aut}(\Delta_n)$ where $\Delta_n := \{z = (z_1, ..., z_n) : |z_j| < 1 \text{ for all } 1 \leq j \leq n\}$ denotes the unit polydisc in $\mathbb{C}^n$, is non-compact.

$$\text{Aut}(\Delta_n) = \left\{\psi(z) = \psi(z_1, ..., z_n) := \left(e^{i\theta_1} \frac{z_{\sigma(1)} - a_1}{1 - \bar{a}_1 z_{\sigma(1)}}, ..., e^{i\theta_n} \frac{z_{\sigma(n)} - a_n}{1 - \bar{a}_n z_{\sigma(n)}}\right)\right\}$$

where $\psi \in \text{Aut}(\Omega)$ such that $\psi(z_1) = z_2$ given any two points $z_1, z_2 \in \Omega$, $a \in \Delta_n, 0 \leq \theta_k \leq 2\pi$, and $\sigma \in S_n$, where S_n is the symmetric group on n letters.

Example 3.1.15 $\text{Aut}(B_n)$ where $B_n := \{z = (z_1, ..., z_n) \in \mathbb{C}^n : ||z|| := \sum_{j=1}^n |z_j|^2 < 1\}$ is the unit ball in $\mathbb{C}^n$, is non-compact. $\text{Aut}(B_n)$ is the group generated by $U(n) := \{A \in M_n(\mathbb{C}) : A\bar{A}^t = \bar{A}^t A = I\}$, the Lie group under matrix multiplication of unitary matrices, and $\{\phi_a(z_1, ..., z_n) := \left(\frac{z_1 - a}{1-\bar{a}z_1}, \frac{\sqrt{1-|a|^2}z_2}{1-\bar{a}z_1}, ..., \frac{\sqrt{1-|a|^2}z_n}{1-\bar{a}z_1}\right),$ for $|a| < 1\}$, a collection of maps. That is, every automorphism of the ball is a composition of elements from $U(n)$ or $\{\phi_a\}$.

After examining these examples, you may wonder if any of these domains are biholomorphic. Could there be a higher-dimensional Riemann Mapping theorem for the set of bounded domains with non-compact automorphism group? Unfortunately without consideration of additional conditions upon the domains, no result holds. The following Poincare theorem demonstrates this.

3.2 Automorphism groups in $\mathbb{C}^n$

Theorem 3.2.1 (Poincare's Theorem) The ball B_n is not biholomorphic to the polydisc Δ_n for $n \geq 2$.

Definition 3.2.2 For a bounded domain Ω, $\mathrm{Aut}(\Omega)$ becomes a topological group by defining a distance between the two automorphisms, for $z \in \Omega$, as

$$d(\phi_1, \phi_2) := \sup|\phi_1(z) - \phi_2(z)|.$$

Then let $\mathrm{Aut}^{\mathrm{Id}}(\Omega)$ denote the subgroup of all automorphisms in the connected component of the identity. Further, given $a \in \Omega$, let $\mathrm{Aut}_a(\Omega)$ denote the subgroup of automorphisms which leave a invariant.

Lemma 3.2.3 (Poincare) If Ω_1 is biholomorphic to Ω_2, then the respective automorphism groups $\mathrm{Aut}(\Omega_1)$ and $\mathrm{Aut}(\Omega_2)$ are isomorphic groups. Furthermore, if there are $a_1 \in \Omega_1$ and $a_2 \in \Omega_2$ for which there exists a biholomorphic map $f : \Omega_1 \to \Omega_2$ with $f(a_1) = a_2$, then $\mathrm{Aut}_{a_1}(\Omega_1)$ and $\mathrm{Aut}_{a_2}(\Omega_2)$ are isomorphic groups. In addition, $\mathrm{Aut}^{\mathrm{Id}}(\Omega_1)$ and $\mathrm{Aut}^{\mathrm{Id}}(\Omega_2)$ are isomorphic groups, as are $\mathrm{Aut}^{\mathrm{Id}}_{a_1}(\Omega_1)$ and $\mathrm{Aut}^{\mathrm{Id}}_{a_2}(\Omega_2)$.

Proof: Let $\phi : \Omega_1 \to \Omega_2$ be a biholomorphic map from Ω_1 to Ω_2. Then

$$\phi \mapsto f \circ \phi \circ f^{-1}$$

is a group homomorphism from $\mathrm{Aut}(\Omega_1)$ to $\mathrm{Aut}(\Omega_2)$. Because the map is invertible, it is a group isomorphism.

Proposition 3.2.4 $\mathrm{Aut}^{\mathrm{Id}}_0(B_n)$ is non-abelian.

Proof: Consider the non-abelian special unitary group $SU(n)$ of all $n \times n$ matrices A such that $AA^* = \mathrm{Id}_n$ and $\det(A) = 1$. $SU(n)$ is a subgroup of $\mathrm{Aut}^{\mathrm{Id}}_0(B_n)$, because $A \in SU(n)$ defines a biholomorphism $z \mapsto Az$ on B_n which leaves 0 invariant.

Proposition 3.2.5 For every $a \in \Delta_n$, $\mathrm{Aut}^{\mathrm{Id}}_a(\Delta_n)$ is abelian.

The proof of the previous Proposition follows directly from the following results due to Cartan:

Proposition 3.2.6 (Cartan Uniqueness Theorem) Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain and let $a \in \Omega$. If $f \in \mathrm{Aut}_a(\Omega)$ satisfies $f'(a) = 1$, then $f(z) = z$ for all $z \in \Omega$.

Proof: It can be assumed that $a = 0$ after a change of coordinates, replacing Ω with $\Omega - a$, if necessary. Then since Ω is bounded, $\bar{\Omega} \subset \Delta_n(0, R)$ for some $R > 0$. Recall that every $f \in \mathrm{Aut}_0(\Omega)$ has a Taylor expansion centered at the origin, $f(z) = \sum_n a_n z^n$. Cauchy's estimate gives that $|a_n| \leq M r^{-n}$, where r is such that $\Delta_n(0, r) \subset \Omega$ and $M = \sup_{z \in \bar{\Omega}} |f(z)|$. Then by assumption, f has a Taylor expansion

$$f(z) = z + f_N(z) + ...$$

where f_k are n-tuples of homogeneous polynomials of degree k, and where N is chosen to be the smallest possible. Then the kth iterate $f^k = f \circ ... \circ f$ of f has Taylor expansion

$$f^k(z) = z + k\dot{f}_N(z)$$

which violates the Cauchy estimate for large k unless $f_N = 0$. But if $f(z) = z$ in $\Delta_n(0, r)$, then $f(z) = z$ in Ω by the principle of analytic continuation.

Definition 3.2.7 A bounded domain $\Omega \subseteq \mathbb{C}^n$ is called a circular domain if $z \in \Omega$ implies that $k_\theta(z) = e^{i\theta}$ for all $z \in \Omega$ and all $\theta \in \mathbb{R}$.

Corollary 3.2.8 (Cartan) Let Ω be a bounded circular domain in $\mathbb{C}^n$ and assume that $0 \in \Omega$ and $f \in \mathrm{Aut}_0(\Omega)$. Then f is linear.

Proof: Assuming Ω is a circular domain and $0 \in \Omega$, one has that $k_\theta \in \text{Aut}_0(\Omega)$. Define

$$g = k_{-\theta} \circ f^{-1} \circ k_\theta \circ f.$$

Then $g'(0) = k'_{-\theta}(0) \circ (f^{-1})'(0) \circ k'_\theta(0) \circ f'(0) = \text{Id}$, so that by the previous proposition $g(z) = z$. This implies that $k_\theta \circ f = f \circ k_\theta$. If $f = (f_1, f_2, ..., f_n)$, then $f_j(e^{i\theta} z) = e^{i\theta} f_j(z)$. Let $f_j(z) = \sum_k a_k z^k$. Then $e^{i\theta} a_k = e^{i|k|\theta} a_k$, implying that $a_k = 0$ for all $|k| \geq 1$.

Corollary 3.2.9 Every $f = (f_1, f_2, ..., f_n) \in \text{Aut}(\Delta_n)$ has the form

$$f_j(z) = e^{i\theta_j} \frac{z_{p(j)} - a_j}{1 - \bar{a}_j z_{p(j)}},$$

where $\theta_j \in \mathbb{R}, a \in \Delta_n$, and p is a permutation of the multi-index $j = (j_1, j_2, ..., j_n)$.

Proof: The map $f_j(z)$ is an automorphism. Denote f_j by σ_a if $\theta_j = 0$ and $p = \text{Id}$. Then given $f \in \text{Aut}(\Delta_n)$, the automorphism $\sigma_a \circ f$ leaves 0 invariant. One can therefore assume that $f \in \text{Aut}_0(\Delta_n)$. Because $f(\Delta_n) \subset \Delta_n$, we have $\sum_{k=1}^n |A_{kj}| \leq 1$. However, by choosing sequences $z^{(n)} = (0, ..., 0, 1 - \frac{1}{n}, 0, ..., 0)$ converging to the distinguished boundary $\mathbb{T}^n$ of Δ_n, one sees that the sequence

$$f(z^{(n)}) = (1 - \frac{1}{n})(A_{1j}, ..., A_{2j})$$

converges to the distinguished boundary of Δ_n. Therefore

$$|A_{q(j)j}| := \max_{k=1,...,n} |A_{kj}| = 1$$

Then since $\sum_{k=1}^{n} |A_{kj}| \leq 1$, one has that A_{jk} is a permutation matrix which has non-vanishing entries of norm 1 only at entries $A_{q(j)j}$. If p is the inverse permutation of q, then $f_k(z) = A_{k,p(k)} z_{p(k)}$ with $|A_{k,p(k)}| = 1$.

Theorem 3.2.10 (Poincare) The ball B_n is not biholomorphic to the polydisc Δ_n for $n \geq 2$.

Proof (of Poincare's Theorem): Assume by contradiction that f is a bihomorphic map between $B_n(0,1)$ and $\Delta_n(0,1)$. From Lemma and transitivity of $\mathrm{Aut}(\Delta_n(0,1))$, we can conclude that $\mathrm{Aut}_0^{\mathrm{Id}}(B_n(0,1))$ and $\mathrm{Aut}_{f(0)}^{\mathrm{Id}}(\Delta_n(0,1))$ are isomorphic. But by Proposition 3.2.4, we have $\mathrm{Aut}_0^{\mathrm{Id}}(B_n(0,1))$ is non-abelian, while $\mathrm{Aut}_{f(0)}^{\mathrm{Id}}(\Delta_n(0,1))$ is abelian by Proposition 3.2.5, which results in a contradiction.

Theorem 3.2.11 Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. $\mathrm{Aut}(\Omega)$ admits a boundary orbit accumulation point if and only if $\mathrm{Aut}(\Omega)$ is non-compact.

Proof: We refer the reader to the proof of Claim 3.1.7 for the forward direction, that if $\mathrm{Aut}(\Omega)$ admits a boundary accumulation point, the automorphism group is non-compact. For the reverse direction, suppose $\mathrm{Aut}(\Omega)$ is non-compact. Then there exists a sequence $\{\phi_j\}$ such that $\phi_j \to \phi \notin \mathrm{Aut}(\Omega)$. It is a theorem of Cartan that either $\phi \in \mathrm{Aut}(\Omega)$ or $\phi(\Omega) \subseteq \partial\Omega$. See Narasimhan for more details. Therefore, for any $z \in \Omega$, $\lim_{j\to\infty} \phi_j(z) = \phi(z) \in \partial\Omega$.

So far we have been working with bounded domains in $\mathbb{C}^n$ with non-compact automorphism group. Next we will consider adding the constraint of pseudoconvexity upon the domain to obtain classification. The following theorems classify the set of bounded domains with the restriction of strong pseudoconvexity and suggest the Greene-Krantz conjecture to be true.

Theorem 3.2.12 (Wong [19]) Let Ω be a strongly pseudoconvex, bounded domain with smooth boundary in $\mathbb{C}^n$ with non-compact automorphism group. Then Ω is biholomorphic to the unit ball B_n.

Theorem 3.2.13 Let $\Omega \subset \mathbb{C}^n$ be any bounded domain with a strongly pseudoconvex boundary point $q \in \partial\Omega$. Suppose further that there exist $K \subset \Omega$, $\{z_j\} \in K$, and $\{\phi_j\} \in \text{Aut}(\Omega)$ such that $\{\phi_j(z_j)\} \to p$. Then Ω is biholomorphic to the unit ball $B_n \subset \mathbb{C}^n$.

Theorem 3.2.14 (Wong [20]) Let $\Omega \subseteq \mathbb{C}^2$ be a bounded domain and $\{\phi_j\} \subset \text{Aut}(\Omega)$ be such that

1) $W = \{\lim_{j\to\infty} \phi_j(\Omega)\}$ is a complex variety of positive dimension contained in $\partial\Omega$
2) W is contained in an open subset $U \subseteq \partial\Omega$ such that ∂U is C^1 and there is an open set $N \subset \mathbb{C}^2$ such that $N \cap \partial\Omega = U$ and $N \cap \Omega$ is convex
3) There exists a point $x \in \Omega$ such that $\{\phi_j(x)\}$ converges to $q \in W \subseteq \partial\Omega$ non-tangentially.

Then Ω is biholomorphic to Δ_2.

Theorem 3.2.15 (Kim [9]) Suppose that $\Omega \subset \mathbb{C}^2$ is a bounded, convex domain with piecewise-smooth Levi flat boundary. If $\text{Aut}(\Omega)$ is non-compact then Ω is biholomorphic to Δ_2.

Could we obtain any classification if the case of strong pseudoconvexity is relaxed to pseudoconvexity? Under the addition of further conditions upon the domain, yes. This classification is given in the following result from Bedford and Pinchuk [2] for a higher dimensional cases:

Theorem 3.2.16 Let Ω be a smoothly bounded, pseudoconvex domain in $\mathbb{C}^{n+1}$ of finite type with non-compact automorphism group such that the Levi form of $\partial\Omega$ has no more than one zero eigenvalue at any point. Then Ω is biholomorphic to the ellipsoid $E_m \subset \mathbb{C}^{n+1}$ for some integer $m > 1$.

We have another useful classification from Bedford and Pinchuk [1]:

Theorem 3.2.17 Suppose Ω is a bounded, convex domain with smooth boundary and finite type in the sense of D'Angelo. Then $\text{Aut}(\Omega)$ is non-compact if and only if Ω is biholomorphic to a polynomial ellipsoid.

Chapter 4:

Invariant Metrics and Measures

Definition 4.1.1 The Kobayashi and Caratheodory metrics on $\Omega \subseteq \mathbb{C}^n$ at $x \in \Omega$ in the direction of $\xi \in \mathbb{C}^n$, denoted by $F_K^\Omega(x,\xi)$ and $F_C^\Omega(x,\xi)$, respectively, are defined by

$$F_K^\Omega(x,\xi) = \inf\{\frac{1}{\alpha} : \phi \in H(\Delta,\Omega) \text{ such that } \phi'(0) = \alpha\xi, \alpha > 0\}$$

$$F_C^\Omega(x,\xi) = \sup\{|\sum_{j=1}^{n} \frac{\partial f(x)}{\partial z_j}\xi_j| : \exists f \in H(\Omega,\Delta) \text{ such that } f(x) = 0\}$$

If $z, w \in \Omega$, then the Kobayashi and Caratheodory pseudo-distances on Ω between z and w, denoted by $d_K^\Omega(z,w)$ and $d_C^\Omega(z,w)$, respectively, are given by

$$d_K^\Omega(z,w) = \inf_\gamma \int_0^1 F_K^\Omega(\gamma(t),\gamma'(t))dt \quad \text{and} \quad d_C^\Omega(z,w) = \sup_f \rho(f(z),f(w)),$$

where $\gamma : [0,1] \to \Omega$ is a piecewise C^1 curve connecting z and w, and where $\rho(p,q)$ is the Poincare distance on Δ between $p, q \in \Delta$. The supremum in the Caratheodory pseudo-distance is taken over all holomorphic mappings $f : \Omega \to \Delta$.

The Kobayashi and Caratheodory metrics satisfy the following non-increasing property under holomorphism.

Lemma 4.1.2 Let $\Omega \subseteq \mathbb{C}^n$ and $\widehat{\Omega} \subseteq \mathbb{C}^m$, and suppose there exists a holomorphism between them $\Psi : \Omega \to \widehat{\Omega}$. Then for $p \in \Omega$ and $\xi \in \mathbb{C}^n$,

$$F_K^{\Omega}(p,\xi) \geq F_K^{\widehat{\Omega}}(\Psi(p), \Psi_*(p)\xi) \text{ and } F_C^{\Omega}(p,\xi) \geq F_C^{\widehat{\Omega}}(\Psi(p), \Psi_*(p)\xi)$$

Proof: Beginning with the Kobayashi case, let $\phi \in \text{Hol}(\Delta, \Omega)$ such that $\phi(0) = p$ and $\phi'(0) = \alpha\xi$. Then consider $\Psi \circ \phi \in \text{Hol}(\Delta, \widehat{\Omega})$. $(\Psi \circ \phi)(0) = \Psi(p)$ and $(\Psi \circ \phi)'(0) = \Psi_*(\phi(0))\phi'(0) = \Psi_*(p)\alpha\xi = \alpha\Psi_*(p)\xi$. Thus

$$F_K^{\widehat{\Omega}}(\Psi(p), \Psi_*(p)\xi) \leq \frac{1}{\alpha}$$

Now taking the infimum over all possible ϕ yields

$$F_K^{\Omega}(p,\xi) \geq F_K^{\widehat{\Omega}}(\Psi(p), \Psi_*(p)\xi)$$

For the Caratheodory case, let $\phi \in \text{Hol}(\widehat{\Omega}, \Delta)$ such that $\phi(\Psi(p)) = 0$. Then consider $\phi \circ \Psi \in \text{Hol}(\Omega, \Delta)$. $(\phi \circ \Psi)(p) = \phi(\Psi(p)) = 0$, hence $F_C^{\Omega}(p,\xi) \geq |(\phi \circ \Psi)_*(p)\xi| = |\phi_*(\Psi(p))(\Psi_*(p)\xi)|$. So taking the supremum over all possible ϕ yields

$$F_C^{\Omega}(p,\xi) \geq F_C^{\widehat{\Omega}}(\Psi(p), \Psi_*(p)\xi).$$

Since the Kobayashi and Caratheodory metrics are invariant under biholomorphism, the inequalities become equalities. We can extend the definitions of Kobayashi and Carathodory metrics to define respective measures.

Definition 4.2.1 Let $\Omega \subseteq \mathbb{C}^n$ be a domain, $p \in \Omega$, and $\xi_1, \xi_2, ..., \xi_m \in T_p^{\mathbb{C}}\Omega$ be linearly independent vectors on the complex tangent space to Ω at p for $1 \leq m \leq n$. One can find an (m, m) volume form M on Ω such that $M(\xi_1, \xi_2, ..., \xi_m, \bar{\xi_1}, ..., \bar{\xi_m}) = 1$. Let $U = B_{m-j} \times \Delta_j$ for $0 \leq j \leq m$, and let $\mu_m = \prod_{j=1}^{m}(\frac{1}{2}dz_j \wedge d\bar{z}_j)$. We define the Kobayashi and Caratheodory m-measures with respect to U as follows:

$$K_U^{\Omega}(p; \xi_1, ..., \xi_m) = \inf\{\frac{1}{\alpha} : \Phi \in H(U, \Omega) \text{ such that } \Phi(0) = p \text{ and } \Phi^*(0)M = \alpha\mu_m, \text{ for some } \alpha > 0\}$$

$$C_U^{\Omega}(p; \xi_1, ..., \xi_m) = \sup\{\beta : \Phi \in H(\Omega, U) \text{ such that } \Phi(p) = 0 \text{ and } \Phi^*(p)\mu_m = \beta M, \text{ for some } \beta > 0\}$$

Both measures satisfy the non-increasing property under holomorphic mappings as well. I.e.,

Proposition 4.2.2 Let $\Omega_1 \subseteq \mathbb{C}^n$ and $\Omega_2 \subseteq \mathbb{C}^{n'}$ be domains, and let $U = B_{m-j} \times \Delta_j$, for $0 \leq j \leq m$ and $m \leq \min\{n, n'\}$. Let $p \in \Omega_1$, and for $j = 1, ..., m$, let $\xi_j \in T_p^{\mathbb{C}}\Omega_1$ be linearly independent. If $\phi \in H(\Omega_1, \Omega_2)$, then

$$K_U^{\Omega_1}(p; \xi_1, ..., \xi_m) \geq K_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, ..., \phi_*\xi_m)$$

$$\text{and } C_U^{\Omega_1}(p; \xi_1, ..., \xi_m) \geq C_U^{\Omega_2}(\phi(p); \phi_*(p)\xi_1, ..., \phi_*(p)\xi_m)$$

Proof: Let M be an (m, m) volume form on Ω_1 such that $M(\xi_1, ..., \xi_m, \bar{\xi_1}, ..., \bar{\xi_m}) = 1$. And let $\Phi : U \to \Omega_1$ be a holomorphic mapping such that $\Phi(0) = p$ and $\Phi^*(0) = M = \alpha\mu_m$. Consider $h = \phi \circ \Phi : U \to \Omega_2$. Let M' be an (m, m) volume form on Ω_2 such that

$\phi * (p)M' = M$. Then $h(0) = \phi(p)$ and

$$h * (0)M' = \Phi^*(0)(\phi^*(p)M') = \Phi(0)M = \alpha\mu_m.$$

Hence $\frac{1}{\alpha} \geq K_U^{\Omega_2}(\phi(p), M)$ and $\inf\frac{1}{\alpha} \geq K_U^{\Omega_2}(\phi(p), M)$. The second inequality follows similarly.

Remark 4.2.3 Let $\Omega \subseteq \mathbb{C}^n, p \in \Omega,$ and $\xi_1, ..., \xi_m \in T_p^{\mathbb{C}}\Omega$ where $1 \leq m \leq n$, be linearly independent vectors. If $U = B_{m-j} \times \Delta_j$, for $0 \leq j \leq m$, then

$$\frac{C_U^{\Omega}(p; \xi_1, ..., \xi_m)}{K_U^{\Omega}(p; \xi_1, ..., \xi_m)} \leq 1.$$

Remark 4.2.4 Let $\Omega \subseteq \mathbb{C}^n, p \in \Omega,$ and $\xi_1, ..., \xi_m \in T_p^{\mathbb{C}}\Omega$ where $1 \leq m \leq n$, be linearly independent vectors. Let $U = B_{m-j} \times \Delta_j$. We have

$$\frac{C_U^{\Omega}(p; \xi_1, ..., \xi_m)}{K_U^{\Omega}(p; \xi_1, ..., \xi_m)} = 1.$$

if and only if Ω is biholomorphic to U.

Remark 4.2.5 Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded convex domain and let $p \in \partial\Omega$ be a strongly pseudoconvex boundary point. Let V be a neighborhood of p. Then we have

$$\frac{K_U^{\Omega}(z; \xi_1, ...\xi_m)}{K_U^{\Omega\cap V}(z; \xi_1, ...\xi_m)} \to 1, \frac{C_U^{\Omega}(z; \xi_1, ...\xi_m)}{C_U^{\Omega\cap V}(z; \xi_1, ...\xi_m)} \to 1, \text{as } z \to p.$$

Remark 4.2.6 Let Ω be a smoothly bounded, convex domain. The domain Ω near a strongly pseudoconvex boundary point can be approximated by ellipsoids which are biholomorphic to balls. Since B_m and $B_{m-j} \times \Delta_j$, for $j \geq 1$, are not biholomorphic and since the Kobayashi and Caratheodory measures are localizable near a strong pseudoconvex boundary point by the previous remark, we have

$$\frac{C_U^{\Omega}(z;\xi_1,...,\xi_m)}{K_U^{\Omega}(z;\xi_1,...,\xi_m)} < L < 1, U = B_{m-j} \times \Delta_j, j \geq 1$$

$$\frac{C_U^{\Omega}(z;\xi_1,...,\xi_m)}{K_U^{\Omega}(z;\xi_1,...,\xi_m)} \to 1, U = B_m$$

Theorem 4.2.7 (Theorem E in [11]) Let Ω be a bounded domain. Suppose there exists a point $x \in \Omega$ such that $|M_{\Omega}^E(x)| = |M_{\Omega}^C(x)|$, where M^E and M^C are defined with respect to the unit polydisc $\Delta_n \subset \mathbb{C}^n$, then Ω is biholomorphic to the polydisc, Δ_n. If M^E and M^C are defined with respect to the unit ball $B_n \subset \mathbb{C}^n$ and the condition of the lemma is satisfied with respect to these measures, then Ω is biholorphic to the unit ball, B_n.

Here $M_{\Omega}^E(x)$ is the differential Eisenman-Kobayashi measure on Ω, (an (n,n)-form on Ω) and recall that it is given by

$$M_{\Omega}^E(z) = |M_{\Omega}^E(z)| \prod_{j=1}^{n} (\frac{i}{2} dz_j \wedge d\bar{z_j})$$

where $|M_\Omega^E|$ is a local function on Ω defined by

$$|M_\Omega^E| = \inf\{|\det f'(0)|^{-2} : f : \Delta_n \to \Omega, \text{ a holomorphism with } f(0) = z\}$$

and $M_\Omega^C(x)$ is the differential Caratheodory measure on Ω, (an (n, n)-form on Ω) given by

$$M_\Omega^C(z) = |M_\Omega^C(z)| \prod_{j=1}^{n} (\frac{i}{2} dz_j \wedge d\bar{z}_j)$$

where $|M_\Omega^C|$ is a local function on Ω defined by

$$|M_\Omega^C| = \sup\{|\det f'(z)|^2 : f : \Omega \to \Delta_n, \text{ a holomorphism with } f(z) = 0\}$$

From these definitions, we obtain the following facts:

Lemma 4.2.8 The following are true:

1) $|M_\Omega^E| \geq |M_\Omega^C|$

2) Given a holomorphism between two complex manifolds $f : \Omega_1 \to \Omega_2$, we have

$$f^*(M_{\Omega_2}^E) \leq M_{\Omega_1}^E \text{ and } f^*(M_{\Omega_2}^C) \leq M_{\Omega_1}^C$$

Consequently, both of these measures are preserved under biholomorphism.

3) Let $\tilde{\Omega}$ be the universal covering of Ω and let $\pi : \tilde{\Omega} \to \Omega$ be the covering projection. Then $M^E_{\tilde{\Omega}} = \pi^*(M^E_{\Omega})$. Consequently, for all $z \in \Omega_2 \subset \Omega_1$, we have $|M^C_{\Omega_1}(z)| \leq |M^C_{\Omega_2}(z)|$ and $|M^E_{\Omega_1}(z)| \leq |M^E_{\Omega_2}(z)|$.

Proposition 4.2.9 Let $\Omega \subset \mathbb{C}^n$ be a smoohtly bounded convex domain. Suppose $\Delta_q^{\partial\Omega}$ is not trivial for some $q \in \partial\Omega$. If there exists $\{g_j\} \subset \mathrm{Aut}(\Omega)$ such that $g_j(z) \to \Delta_q^{\partial\Omega}$ for all $z \in \Omega$, then $\Delta_q^{\partial\Omega}$ is biholomorphic to a complex m-ball, where m is the complex dimension of $\Delta_q^{\partial\Omega}$.

Chapter 5:

Proof of Main Theorem

5.1 One-Parameter Translation Groups

We will need a theorem from Kim [10], which comes from a derived version of a result from Frankel [6].

Theorem 5.1.1 Let Ω be a convex hyperbolic domain in $\mathbb{C}^n$ and suppose there is a subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ such that

1) Γ is discrete and acts freely,

2) Γ is co-compact in Ω.

Then Ω is biholomorphic to a bounded symemtric domain. Moreover, automorphisms and maps of convex domains can be directly reduced to problems in affine geometry by an elementary technique involving localization near a boundary point. Frankel developed the rescale blow-up to show that Ω admits a one-parameter group of automorphisms $\sigma_t \in \mathrm{Aut}_0(\Omega)$.

Theorem 5.1.2 Let Ω be a convex hyperbolic domain in $\mathbb{C}^n$ and suppose there is a subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ such that Γ is co-compact in Ω. Then $\mathrm{Aut}_0(\Omega)$ is non-trivial, in fact there is a convex holomorphic embedding $\omega : \Omega \to \mathbb{C}^n$ such that $\omega(\Omega)$ is invariant under a one-parameter group of translations.

We have the following theorem which is a result of Frankel that can be found in Kim [10] for which our domain Ω contains a non-compact one-parameter subgroup.

Theorem 5.1.3 (Kim [10]) If a bounded, convex domain in $\mathbb{C}^n$ possesses a non-compact automorphism orbit accumulating at a boundary point with sphere contact inside, then the automorphism group contains a non-compact one-parameter subgroup.

The condition of sphere contact inside means one can draw a ball of radius $\epsilon > 0$ tangent to the boundary accumulation point while it lies completely inside the domain. Visually, we are also able to draw a perpendicular line to the line that is tangent to the boundary point. Note that if the boundary is C^1, then every boundary point has sphere contact inside of Ω. Since Ω has a smooth boundary, we are able to satisfy the interior sphere contact condition, our domain Ω has a non-compact one-parameter subgroup.

5.2 Rescaling the Domain

Rescaling the domain and looking at its limit with respect to the local Hausdorff topology is one of the steps we will need to prove the main result. We can use this result to show that there is a disc in the boundary. This method of rescaling and showing there is a disc in the boundary is used by Zimmer [21]. We will be able to use it in the same way. We first start with the definition of convergence in the local Hausdorff topology.

Definition 5.2.1 Here we define the local Hausdorff topology on the set of all convex domains in $\mathbb{C}^n$. First, define the Hausdorff distance between two compact sets $X, Y \subset \mathbb{C}^n$ by

$$d_H(X,Y) = \max\Big\{\max_{x\in X}\min_{y\in Y}||x-y||, \max_{y\in Y}\min_{x\in X}||x-y||\Big\}$$

To obtain a topology on the set of all convex domains in $\mathbb{C}^n$, we consider the local Hausdorff pseudodistances defined by

$$d_H^{(R)}(X,Y) = d_H(X\cap \overline{B_R(0)}, Y\cap \overline{B_R(0)}), R > 0.$$

Then a sequence of convex domains Ω_j converges to a convex domain Ω if there exists some $R_0 \geq 0$ such that

$$\lim_{j\to\infty} d_H^{(R)}(\overline{\Omega_j}, \overline{\Omega}) = 0$$

for all $R \geq R_0$.

Remark 5.2.2 The Kobayashi metric is continuous with respect to the local Hausdorff topology.

Theorem 5.2.3 Suppose $\Omega_j \subset \mathbb{C}^n$ is a sequence of convex domains and $\Omega = \lim_{j\to\infty} \Omega_j$ in the local Hausdorff topology. Assume the Kobayashi metric is nondegenerate on Ω and each Ω_j. Then

$$d^K_\Omega(z, w) = \lim_{j\to\infty} d^K_{\Omega_j}(z, w)$$

for all $z, w \in \Omega$. Moreover, the convergence is uniform on compact subsets of $\Omega \times \Omega$.

The rescaling method is most useful when the rescaled domain is biholomorphic to the original one. The following convergence theorem by Frankel proves that when we have a convex domain along with a couple of other conditions, our rescaled domain is biholomorphic to the original domain.

Theorem 5.2.4 (Frankel 4.4 [6]) Suppose $\Omega \subset \mathbb{C}^n$ is a convex domain which does not contain a complex line in its boundary. Let $K \subset \Omega$ be compact and $\phi_j \in \mathrm{Aut}(\Omega)$. If there exists a sequence $p_j \in K$ and complex affine maps Λ_j such that

1) $\lim_{j\to\infty} \Lambda_j(\Omega) = \widehat{\Omega}$

2) $\phi_j(p_j) \to p \in \widehat{\Omega}$

where $\widehat{\Omega}$ does not contain a complex line in its boundary, then Ω is biholomorphic to $\widehat{\Omega}$.

Example 5.2.5 Let $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ be a smooth, convex, non-negative defining function, $(z, w) \in \mathbb{C}^2$, $w_j > 0$ converging to 0, and the sequence of linear maps $\Lambda_j(z, w) = (\frac{z}{f(0,w_j)}, \frac{w}{w_j})$. $\Lambda_j(\Omega) = \widehat{\Omega}$ is a sequence of rescaled domains, which will converge to $\widehat{\Omega}$ as $j \to \infty$ because $f(0, w_j) \to 0$ and $w_j \to 0$. We are able to rescale any (z, w) since each component of the coordinate will get bigger for any point as $f(0, w_j) \to 0$ and $w_j \to 0$.

5.3 Disc in the Boundary

We will use the rescaling map from 5.2 to show that when the blow-up domain converges in the local Hausdorff topology to a $\mathbb{C}$-proper convex open domain $\widehat{\Omega}$, $\partial\widehat{\Omega}$ contains a non-trivial holomorphic disc.

Lemma 5.3.1 [21] Let $\Omega \subset \mathbb{C}^2$ be a $\mathbb{C}$-proper convex domain with $q \in \partial\Omega$ which is a point of infinite type and $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ be a smooth, convex, non-negative defining function such that $f(\mathrm{Re}z_1, z_2) < \mathrm{Im}z_1$. Consider the sequence of linear transformations $\Lambda_j(z, w) := (\frac{z}{f(0,w_j)}, \frac{w}{w_j})$ such that $(z, w) \in \mathbb{C}^2$ and $\Lambda_j(\Omega)$ converges in the local Hausdorff topology to a $\mathbb{C}$-proper convex open domain $\widehat{\Omega}$. Then $\partial\widehat{\Omega}$ contains a non-trivial holomorphic disc.

Proof of Lemma 5.3.1: Suppose q is a point of infinite type. By McNeal, q is a point of infinite line type. We can apply an affine transformation to q to let $q = 0 \in \partial\Omega$. We will consider neighborhoods of 0 and

$$\Omega \cap U = \{(z, w) \in U : f(\text{Re}z, w) < \text{Im}z\}$$

we may assume that $\partial\Omega$ does not contain any non-trivial holomorphic discs. In particular, there exists a $w \in \mathbb{C}$ such that $f(0, w) \neq 0$. We are able to apply infinite line type condition to change coordinates since we satisfy the hypotheses of Lemma 2.3.12. Since

$$\lim_{w \to 0} \frac{f(0, w)}{|w|^k} = 0,$$

we can find an $a_k \to 0$ and $w_k \in B_1(0)$ such that $f(0, w_k) = a_k |w_k|^k$ and for all $w \in \mathbb{C}$ with $|w| \leq |w_k|$ we have

$$f(0, w) \leq a_k |w|^k.$$

The above inequality is true because f is a decreasing function. Since $\partial\Omega$ has no non-trivial holomorphic discs, we see that $w_k \to 0$ and hence $f(0, w_k) \to 0$.

Consider the linear transformations

$$\Lambda_k(z, w) := \left(\frac{z}{f(0, w_k)}, \frac{w}{w_k} \right)$$

and let $\Omega_k = \Lambda_k(\Omega)$. Now there exists $\epsilon_1, \epsilon_2 > 0$ such that $B_{\epsilon_1}(\epsilon_2 i, 0) \subset \Omega_k$ for all k. Moreover for any $R > 0$ the set

$$\{\Omega' \quad \text{is open and convex} : B_{\epsilon_1}(\epsilon_2 i, 0) \subset \Omega' \subset B_R(0)\}$$

is compact in the Hausdorff topology. Thus we can pass to a subsequence such that Ω converges in the local Hausdorff topology to a convex open set set $\widehat{\Omega}$. We will consider rescaled neighborhoods of 0. If $U \subset \mathrm{Im}\Lambda_k(U)$ we have

$$\Omega \cap U = \{(z, w) : f_k(\mathrm{Re}z, w) < \mathrm{Im}z\}$$

where the rescaled defining function is

$$f_k(\mathrm{Re}z, w) = \frac{1}{f(0, w_k)} f(f(0, w_k) \cdot \mathrm{Re}z, w_k w)$$

Let $w \in \mathbb{C}$ such that $|w| < 1$. Then if $\mathrm{Re}z = 0$ we have

$$f_k(0, w) = \frac{1}{f(0, w_k)} f(f(0, w_k) \cdot \mathrm{Re}z, w_k w) = \frac{f(0, w_k w)}{f(0, w_k)} \leq \frac{a_k |w_k w|^k}{f(0, w_k)} = \frac{a_k |w_k|^k |w|^k}{f(0, w_k)} = |w|^k \to 0$$

as $k \to \infty$. Since this is true for any $w \in \mathbb{C}$ such that $|w| < 1$, and $f_k(\mathrm{Re}z, w)$ is our defining function, these imply that $0 \times \Delta \subset \partial\widehat{\Omega}$. Thus $\partial\widehat{\Omega}$ contains a non-trivial holomorphic disc.

5.4 Stabilizing the Map and Biholomorphism

Below we have a modified version of Hurwitz theorem that we will need to show there is a biholomorphism between our domain Ω and the rescaled domain $\widehat{\Omega}$. We will need a map to stabilize a point in the blow up domain of Ω.

Let $q = \lim_{k\to\infty} \phi_k(p)$, $\phi_k = \phi_k(p)$ where $\{\phi_k\} \subset \mathrm{Aut}(\Omega)$. By the modification of Frankel's work ([9], using [5]), Ω has a one-parameter translation. Thus we can bring p_k to a point $\tilde{p}_k$ which lies on the hyperplane $\{\mathrm{Re}(z) = 0\}$ with translations, $\tilde{g}_k$. Composing the translations we obtain an orbit $\{\tilde{p}_k\}$ in $\{\mathrm{Re}(z) = 0\}$ converging to q. We choose a point $w_k \in \partial\Omega$ as follows:

Let w_k be a point on the closed curve $c_k = \partial\Omega \cap \{b_k = f(0, w), b_k = \mathrm{Im} z - \tilde{p}_k\} \cap \{\mathrm{Re}(z) = 0\}$ such that $|w_k| = \max_{q_k \in C}\{|w - q_k|\}$. Then we proceed to blow up the domain Ω according to the points $\{w_k\} \to q$.

$$\Lambda_k(z, w) = \left(\frac{z}{f(0, w_k)}, \frac{w}{w_k} \right)$$

As discussed before the blow up domain $\widehat{\Omega} = \lim_{k\to\infty} \Lambda_k(\Omega)$ contains a disc sitting on the boundary $\partial\widehat{\Omega}$ at $(0, 0)$. We notice the following facts

(i) $\partial\widehat{\Omega}$ contains no complex line at $(0, 0)$. We do not have a complex line because we localized around $(0, 0)$.

(ii) Consider the map $f_k = \Lambda_k \circ \tilde{g}_k$, where $\tilde{g}_k$ is the composition of the described translation with g_k.

The sequence of $\{f_k\}$ stabilize the point p within a compact subset of $\widehat{\Omega}$ (i.e. $f_k(p)$ lies in a compact subset of $\widehat{\Omega}$ for large k). Then we apply the convergence theorem 5.2.4. Thus we can conclude that $f = \lim_{k\to\infty} f_k$ exists as a biholomorphism from Ω onto $\widehat{\Omega}$. By Kim [10] we concluded $\mathrm{Aut}(\widehat{\Omega})$ contains a continuous one-parameter group of translations and were able to rescale Ω. In Section 5.3 we saw that there was a disc on $\partial\widehat{\Omega}$. Next we will show there is a biholomorphism between $\widehat{\Omega}$ to the bidisc.

5.5 Blow-Up Domain Biholomorphic to the Bidisc

We would like to show using Kaylee Hamann and Bun Wong's results from [7] and [20] that $\widehat{\Omega}$ is biholomorphic to Δ_2. In Section 5.3 we saw that $\partial\widehat{\Omega}$ contains a disc D in the boundary $\partial\widehat{\Omega}$. Then the induced automorphism will take any point from $\widehat{\Omega}$ to a point in $\bar{D}$, the closure of the disk D. Note that the induced automorphism group may not converge to the disc. Our use of one-parameter group of translations allowed us to make a sequence converging to a point in the disc on the boundary and rescale Ω. Then the same technique of using invariant metrics and measures in [8] give us a biholomorphism between the rescaled domain, $\widehat{\Omega}$, and Δ_2. Note that the theorems from [7] apply to bounded convex domains, but are still applicable to Koybayashi hyperbolic convex domains that are unbounded.

Furthermore, if we have a disc on $\partial\widehat{\Omega}$, then $\widehat{\Omega}$ is biholomorphic to the bidisc. In the next section, we combine all of the previous sections in this chapter to give us the proof of the main result.

5.6 Proof of the Main Result

Recall the statement of the main theorem.

Theorem 5.6.1 Let Ω be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose there is a sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then q is of finite type.

Proof: The proof follows by combining the knowledge of the previous sections in this chapter. We will prove by contradiction that q is a point of finite type. First we assume that q is a point of infinite type. Let $\{\phi_j(p)\} = \{p_j\}$ be a sequence of points which converges to q. We will translate $\{p_j\}$ to $\{\tilde{p}_j\}$ such that $\mathrm{Im}z = f(\mathrm{Re}z, w)$. $\{\tilde{p}_j\}$ converges to q, but on $\mathrm{Re}z = 0$. By Frankel, if we have a convex domain with a non-compact automorphism group, there exists a continuous parameter group of automorphisms $\tilde{g}_j$. $\{\tilde{g}_j(p_j)\}$ hits $\{\mathrm{Re}z = 0\}$ at $\tilde{p}_j$. We can define the linear transformation $\Lambda_j : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\Lambda_j(z, w) := \Big(\frac{z}{f(0, w_j)}, \frac{w}{w_j}\Big)$$

which we will use $\tilde{p}_j$ for the coordinates to construct a rescaled domain. Then $\lim_{j\to\infty} \Lambda_j(\Omega) = \Omega_j$ converges to an unbounded domain $\widehat{\Omega}$ in the local Hausdorff topology, possibly in subsequence. Note that Ω rescales to the unbounded domain $\widehat{\Omega}$ and $\widehat{\Omega}$ is a homogenous domain. We showed that $\widehat{\Omega}$ contained a disc in the boundary and can apply the technique Kaylee Hamann and Bun Wong used to prove that $\widehat{\Omega}$ is biholomorphic to the Δ_2. In section 5.4

we showed that Ω is biholomorphic to $\widehat{\Omega}$. We now consider a modification of the following local version of Bun Wong's theorem.

Theorem 5.6.1 Let $\Omega \subset \mathbb{C}^n$ be any bounded domain with a strongly pseudoconvex boundary point $p \in \partial\Omega$. Suppose further that there exist $K \subseteq \Omega$, $\{z_j\} \in K$, and $\{\phi_j\} \in \mathrm{Aut}(\Omega)$ such that $\{\phi_j(z_j)\} \to p$. Then Ω is biholomorphic to the unit ball $B_n \subset \mathbb{C}^n$.

It follows from the above theorem that Ω is biholomorphic to the unit ball $B_2 \subset \mathbb{C}^2$. Then since there is a biholomorphism between Ω and $\widehat{\Omega}$, this gives us a contradiction because by Poincare's Theorem, when $n > 1$, the unit ball and Δ_n are not biholomorphically equivalent. Therefore we know that there cannot be a biholomorphism between the bidisc and ball. Thus q could not have been a point of infinite type, and must have been of finite type.

Chapter 6:

Conclusion

6.1 Applications

Recall the statement of the Greene-Krantz conjecture.

Conjecture 6.1.1 Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with smooth boundary. If $q \in \partial\Omega$ is a boundary orbit accumulation point for $\mathrm{Aut}(\Omega)$ then $\partial\Omega$ is of finite type at the point q.

Noack's result, combined with some of the results from Zimmer [21] and Bedford and Pinchuk [1] show:

Corollary 6.1.2 Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, convex domain with smooth boundary. If there exists $p \in \Omega$ and a sequence of automorphisms $\{\phi_j\} \in \mathrm{Aut}(\Omega)$ such that $\phi_j(p) \to q \in \partial\Omega$ non-tangentially, then Ω is biholomorphic to a polynomial ellipsoid.

Our result in the previous section proves a special case of the Greene-Krantz conjecture. We were able to drop the non-tangential condition and added the condition of convexity used in results proved in [16] and [8]. Recall the statement of our theorem:

Theorem 6.1.3 Let Ω be a bounded, convex domain in $\mathbb{C}^2$ with C^2 boundary. Suppose there is a sequence $\{\phi_j\} \subset \text{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then q is of finite type.

Between our result and [1], we get the following:

Corollary 6.1.4 Let Ω be a bounded, convex domain in $\mathbb{C}^2$ with C^2 boundary. If there exists a $p \in \Omega$ and a sequence $\{\phi_j\} \subset \text{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point $q \in \partial\Omega$, then Ω is biholomorphic to a polynomial ellipsoid.

These results allow us to classify bounded, convex domains as ellipsoids, $E_m = \{(z_1, z_2) \in \mathbb{C}^2 : \rho(z_1, z_2) = |z_1|^2 + |z_2|^{2m} - 1 < 0\}$.

6.2 Further Results

We were able to remove the hypothesis of non-tangential convergence from the main result in [16]. We are now able to classify smoothly bounded convex domains with non-compact automorphism groups in $\mathbb{C}^2$. We can generalize to higher dimensions and classify smoothly bounded convex domains with non-compact automorphism groups.

Theorem 6.2.1 (Frankel [5]) Suppose $\Omega \subset \mathbb{C}^n$ is a convex domain that does not contain any non-trivial complex lines. If V is a complex affine m-dimensional subspace intersecting Ω, $p_j \in V \cap \Omega$ and $\{\Lambda_j\}$ is a sequence of complex affine maps such that

$$\Lambda_j(\Omega \cap V, p_j) \to (\widehat{\Omega_V}, u)$$

where $\widehat{\Omega_V}$ is a $\mathbb{C}$-proper open domain, then there exists complex affine maps B_j such that

$$B_j(\Omega, p_j) \to (\widehat{\Omega}, u)$$

where $\widehat{\Omega}$ is a $\mathbb{C}$-proper open domain. Furthermore, $\widehat{\Omega} \cap V = \widehat{\Omega_V}$.

With the previous Frankel theorem (Theorem 6.2.1), this will help us in generalizing in $\mathbb{C}^n$ to obtain the following result.

Corollary 6.2.2 Let Ω be a bounded, convex domain in $\mathbb{C}^n$ with smooth boundary. Suppose there is a sequence $\{\phi_j\} \subset \mathrm{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point for some $p \in \Omega$. If $q \in \partial\Omega$ is an orbit accumulation point, then q is of finite type.

Corollary 6.2.2 can then be combined with Theorem 3.2.17 to give us the following result:

Corollary 6.2.3 Let Ω be a bounded, convex domain in $\mathbb{C}^n$ with smooth boundary. If there exists a $p \in \Omega$ and a sequence $\{\phi_j\} \subset \text{Aut}(\Omega)$ such that $\{\phi_j(p)\}$ accumulates at a boundary point $q \in \partial\Omega$, then Ω is biholomorphic to a polynomial ellipsoid.

Beyond that, working to remove convexity to classify smooth domains with non-compact automorphism groups in $\mathbb{C}^n$ would be the next goal.

Bibliography

[1] E. Bedford and S. I. Pinchuk. Convex domains with noncompact groups of automorphisms. *Mat. Sb.*, 185(5):3-26, 1994.

[2] E. Bedford and S.I. Pinchuk. Domains in $\mathbb{C}^{n+1}$ with non-compact automorphism groups. *J. Geom. Anal.*, 1:165-191, 1989.

[3] John P. D'Angelo. Several complex variables and the geometry of real hypersurfaces. *Studies in Advanced Mathematics*, CRC Press, Boca Raton, FL, 1993.

[4] Fusheng Deng, Qian Guan and Liyou Zhang. Some properties of squeezing functions on bounded domains. *Pacific Journal of Mathematics*, 319-341, 2012.

[5] Sidney Frankel. Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure. In *Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989)*, volume 52 of *Proc. Sympos. Pure Math.*, pages 183-208. Amer. Math. Soc., Providence, RI, 1991.

[6] Sidney Frankel. Geometry of comvex domains that cover varieties. *Acta Math.*, 163: 109-149, 1989.

[7] Kaylee Hamann. On the characterization of convex domains with non-compact automorphism group. Dissertation, 2016.

[8] Kaylee Hamann and Bun Wong. Characterization of convex domains in $\mathbb{C}^2$ with non-compact automorphism group. *Sci. China Math.,* 60(6):977-984, 2017.

[9] Kang-Tae Kim. Geometry of bounded domains and the scaling techniques in several complex variables. Volume 13 of *Lecture Notes Series*, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1993.

[10] Kang-Tae Kim. On the automorphism groups of convex domains in $\mathbb{C}^n$. *Adv Geom.*, 4:33-40, 2004.

[11] Steven Krantz. Function theory of several complex variables. AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1992 edition.

[12] Steven Krantz. The automorphism group of a domain with an exponentially flat boundary point. *J. Math. Anal. Appl.*, 823-827, 2012.

[13] Jeffery D. McNeal. Convex domains of finite type. *J. Funct. Anal.*, 108(2):361-373, 1992.

[14] Raghavan Narasimhan. Several complex variables. The University of Chicago Press, Chicago, Ill.-London, 1971. Chicago Lectures in Mathematics.

[15] Nicholas Newsome. Bounded domains on Kobayashi hyperbolic manifolds covering compact complex manifolds. Dissertation, 2022.

[16] Dylan Noack. A characterization of certain bounded, convex domains. Dissertation, 2019.

[17] Joshua Strong. Boundary characterization of iterated automorphism orbits on bounded domains. Dissertation, 2016.

[18] Joshua Strong. Iterated automorphism orbits of bounded convex domains in $\mathbb{C}^n$. *Pacific Journal of Mathematics*, 298(2): 471-481, 2019.

[19] Bun Wong. Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group. *Invent. Math.*, 41(3):253-257, 1077.

[20] Bun Wong. Characterization of bidisc by its automorphism group. *Amer. J. Math.*, 117(2):279-288, 1995.

[21] Andrew M. Zimmer. Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type. *Math. Ann.*, 365(3-4):1425-1498, 2016.