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Galilean theory of dispersion for kinetic equations

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Abstract

This paper introduces a new notion of dispersion for kinetic equations solely based on the conservation laws and independent of the specific type of interactions. We present new a-priori estimates for kinetic PDEs and improve the Bony-type functional approach.

1. Introduction

In this paper, we will study solutions of partial differential equations obtaining the form below, where $(x, \xi) \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}^+$ or $t \in \mathbb{R}$:

$$\begin{cases} \partial_t f(x,\xi,t) + \xi \cdot \nabla_x f(x,\xi,t) = I(f,x,\xi,t) \\ f(x,\xi,0) = f_0(x,\xi) \end{cases} \tag{1.1}$$

The local conservation laws of mass, momentum, and energy for the interaction term I take the following form:

$$\int_{\mathbb{R}^n} I(f,x,\xi,t) \begin{pmatrix} 1 \\ \xi \\ |\xi|^2 \end{pmatrix} d\xi = 0 \tag{1.2}$$

The left-hand side of equation (1.1) implies that particles will be transported along the trajectory of their velocities, while the right-hand side represents changes through possibly non-linear inter-

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actions of particles. It is possible to integrate the effects of interactions along the characteristics of linear transport to obtain a mild form of the equation:

$$f(x,\xi,t) = f_0(x - t\xi, \xi) + \int_0^t I(f, x + (s-t)\xi, \xi, s)\, ds, \tag{1.3}$$

Definition 1.

1. A function $f \in C^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times [0,T)) \cap C^1(\mathbb{R}^n_x \times [0,T))$ is a classical solution of equation (1.1) with initial value f_0, if and only if it satisfies equation (1.1) and conservation laws (1.2) point wise.
2. A function $f \in C^0([0,T); L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi))$ is a mild solution of equation (1.1) with initial value f_0, if and only if it satisfies the integral equation (1.3) for almost every $(x,\xi,t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,T)$ and conservation laws (1.2) for almost every $(x,t) \in \mathbb{R}^n \times [0,T)$.

For these solutions, the following conservation laws hold:

$$\begin{gathered} \phi(\xi) = a|\xi|^2 + b \cdot \xi + c \quad (a,b,c) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \\ \iint_{\mathbb{R}^{2n}} f(x,\xi,t)\phi(\xi)\, dxd\xi = \iint_{\mathbb{R}^{2n}} f_0(x,\xi)\phi(\xi)\, dxd\xi \end{gathered} \tag{1.4}$$

We obtain the notation below to represent total mass, momentum, and energy:

$$\begin{aligned} (mass) \quad \mathbf{M} &= \iint_{\mathbb{R}^{2n}} f_0(x,\xi)\, dxd\xi \\ (momentum) \quad \mathbf{P} &= \iint_{\mathbb{R}^{2n}} f_0(x,\xi)\xi\, dxd\xi \\ (energy) \quad \mathbf{E} &= \iint_{\mathbb{R}^{2n}} f_0(x,\xi)\frac{|\xi|^2}{2}\, dxd\xi \end{aligned} \tag{1.5}$$

Let $\theta(x,\xi) \in [0,\pi]$ for $|x| > 0$ and $|\xi| > 0$ represent the angle between the specified vectors, and let $B(x,R) \subset \mathbb{R}^n$ be the ball of radius R centered at x. When necessary, we will use the notation $B_{x/\xi}$ to distinguish between subsets of the spatial variable x and the velocity variable ξ.

Note that conservation laws (1.2) have no dependency on the specific structure of interactions. The general formalism of this paper is applicable to kinetic PDEs such as the Boltzmann equation with an arbitrary collision kernel or the Landau kinetic equation. A major difficulty in the analysis of kinetic PDEs is due to grazing interactions [7]. We will introduce a new approach to overcome this challenge.

The main results of this paper are formulated via four theorems that will be discussed and compared in this introduction. Section 1.1 and Section 1.2 introduce new estimates for kinetic equations based on conservation laws. Section 1.3 develops a new notion of dispersion and demonstrates a relation between the Landau kinetic equation and other kinetic PDEs in the context of grazing interactions. Section 1.4 and Section 1.5 discuss the technical ideas behind the proofs. The proofs follow in their appropriate order in Section 2.

1.1. Morawetz estimates

We begin with two estimates for solutions of (1.1). These estimates are formulated for ensembles of particles and interactions, and they are analogous to Morawetz-type estimates in the context of wave or Schrodinger equations [17,18].

Theorem 1. *Assume $n \geq 2$, $R > 0$ and $f(x,\xi,t) \geq 0$ is a classical solution of (1.1) such that:*

$$\begin{gathered} f_0(x,\xi) \in L^1_{(1+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \\ \partial_t f, \nabla_x f \in L^1_{loc}(\mathbb{R}^+_t; L^1_{(1+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)) \end{gathered} \tag{1.6}$$

Then for some C_n, depending only on the dimension, we have:

$$\frac{1}{R}\int_0^\infty \int_{\mathbb{R}^n} \int_{B_x(0,R)} f(x,\xi,t)|\xi|^2 \, dx d\xi dt < C_n \sqrt{\mathbf{ME}} \tag{1.7}$$

And:

$$\frac{1}{R}\int_0^\infty \int \cdots \int_{\mathbb{R}^{3n}\times B_x(x_0,R)} f(x,\xi,t) f(x_0,\xi_0,t)|\xi-\xi_0|^2 \, dx d\xi dx_0 d\xi_0 dt < C_n \mathbf{M}\sqrt{(2\mathbf{ME} - |\mathbf{P}|^2)} \tag{1.8}$$

Remarks.

1. Bony [3] proved an estimate similar to (1.8) for a class of 1-dimensional discrete models. His work was subsequently extended to 1-dimensional continuous models by Cercignani [5,6], who used it to prove the existence of global weak solutions to the 1-dimensional Boltzmann equation. Ha and Noh [10] extended the Bony functional approach to higher dimensions; for instance, they proved the following a priori estimate for the 3-dimensional Boltzmann equation without any assumption on the smallness of mass:

$$\frac{1}{R}\int_0^\infty \int \cdots \int_{\mathbb{R}^9\times B_x(x_0,R)} f(x,\xi,t) f(x_0,\xi_0,t)|\xi-\xi_0|^2 \sin^2\theta(x-x_0,\xi-\xi_0) \, dx d\xi dx_0 d\xi_0 dt < \infty$$

   The previous theorem improves the result above and provides two new estimates, likewise without small-data assumptions. Another extension of the Bony functional approach can be found in the recent work of Serre on symmetric divergence-free tensors and compensated integrability, see Theorem 3.3 in [15].

2. Morawetz-type methods can be described as follows: multiplying both sides of a PDE by an appropriate initial expression and integrating over space-time, followed by integration by parts. The goal is to choose the initial expression such that it yields terms with a definite sign. A key observation is that selecting an initial expression resembling the radial derivative[2], allows us to achieve this goal. In the context of kinetic equations, functionals based on the expression below play a comparable role:

$$\frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|} \tag{1.9}$$

   This expression can be interpreted as a measure of dispersion. The proof of Theorem 1 builds on the idea above and brings forth a new concept, namely the **blind cone of an observer**. This concept will be discussed thoroughly by the end of this introduction and it will be employed to formulate a new notion of dispersion.
3. Estimates (1.7) and (1.8) can be seen roughly as the gain of integrability via conservation laws; in contrast, the propagation of regularity for kinetic equations, which are first-order hyperbolic, does not follow from a-priori bounds. However, there are methods such as the velocity-averaging lemmas, initiated by Golse, Perthame, Lions, and Sentis [9], that prove macroscopic quantities (e.g., $\int f(x,\xi,t)\phi(\xi)\,d\xi$) gain regularity and smoothness properties even when the underlying distribution does not.

Dispersive properties of kinetic transport equation have been studied extensively [13]. Consider the equation below for $g(x,\xi,t)$, where $(x,\xi)\in\mathbb{R}^{2n}$ and $t\in\mathbb{R}$:

$$\begin{cases}\partial_t g(x,\xi,t)+\xi\cdot\nabla_x g(x,\xi,t)=0\\ g(x,\xi,0)=g_0(x,\xi)\end{cases} \tag{1.10}$$

This situation is equivalent to having $I=0$ in equation (1.1). We will use the notation g exclusively for this linear case to avoid any confusion with the non-linear setting, which is the main subject of this paper. The explicit solution $g(x,\xi,t)=g_0(x-t\xi,\xi)$ solves equation (1.10), nonetheless many properties of this equation are far from trivial. Bardos and Degond [1] demonstrated a notion of dispersion for solutions of the linear equation in terms of the following decay:

$$\Big|\int\limits_{\mathbb{R}^n} g(x,\xi,t)\,d\xi\Big| < \frac{1}{t^n}\|g_0(x,\xi)\|_{L^1(\mathbb{R}^n_x;L^\infty(\mathbb{R}^n_\xi))} \tag{1.11}$$

A more intricate approach for the mathematical demonstration of dispersion has been done by Castella and Perthame [4,13] providing estimates over the whole space in $L^p_t(L^q_x(L^r_v))$, for solutions of the linear equation and a specific range of exponents. These estimates are analogous to a collection of inequalities developed by Strichartz for the Schrodinger equation and they have been the subject of active research in the contexts of kinetic, Schrodinger and wave equations [2,11,18]. Another dispersive estimate for the linear equation is by Lions and Perthame [13]:

[2] For instance, $\dfrac{x\cdot\nabla u}{|x|}$ is the radial derivative for solutions of the Schrodinger equation.

$$\frac{1}{R}\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\int_{B_x(0,R)} |g(x,\xi,t)||\xi|\; dxd\xi dt \le 2\|g_0(x,\xi)\|_{L^1(\mathbb{R}^n_x\times\mathbb{R}^n_\xi)} \tag{1.12}$$

The equality holds as R goes to infinity. Note that estimates (1.11) and (1.12) are for the linear equation and are not directly applicable to the non-linear case. However, they are powerful tools in the small-data setting, where they can be used for the corresponding inhomogeneous linear PDE. In this paper, we will not make any small-data assumptions.

1.2. New a-priori estimates

Adopt the following notation:

$$\begin{aligned}
A(t) &= \iint_{\mathbb{R}^{2n}} f(x,\xi,t)x\cdot\xi\; dxd\xi \\
A_I(t) &= \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)(x-x_0)\cdot(\xi-\xi_0)\; dxd\xi dx_0 d\xi_0 \\
X(t) &= \iint_{\mathbb{R}^{2n}} f(x,\xi,t)|x|^2\; dxd\xi \\
X_I(t) &= \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)|x-x_0|^2\; dxd\xi dx_0 d\xi_0
\end{aligned} \tag{1.13}$$

We introduce a notion of **uncertainty** associated with particles and interactions:

$$U(t) = \iint_{\mathbb{R}^{2n}} f(x,\xi,t)|x||\xi|\; dxd\xi \tag{1.14}$$

$$U_I(t) = \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)|x-x_0||\xi-\xi_0|\; dxd\xi dx_0 d\xi_0 \tag{1.15}$$

The justification for the terminology above will be discussed after the next theorem. The interesting observation is that the following quantities are a-priori bounded:

$$\mathbf{G} = \sup_t (U(t) - A(t)) \tag{1.16}$$

$$\mathbf{G_I} = \sup_t (U_I(t) - A_I(t)) \tag{1.17}$$

Theorem 2. *Assume $n \ge 1$ and $f(x,\xi,t) \ge 0$ is a mild solution of (1.1) such that:*

$$\begin{aligned}
f_0(x,\xi) &\in L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi) \\
I &\in L^1_{loc}(\mathbb{R}^+_t; L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x\times\mathbb{R}^n_\xi))
\end{aligned} \tag{1.18}$$

Then:

$$\mathbf{G} \leq \sqrt{2X(0)\mathbf{E}} - A(0) \tag{1.19}$$

$$\mathbf{G_I} \leq \sqrt{2X_I(0)(2\mathbf{ME} - |\mathbf{P}|^2)} - A_I(0) \tag{1.20}$$

Moreover:

$$A(t) = A(0) + 2t\mathbf{E} \tag{1.21}$$

$$A_I(t) = A_I(0) + 2t(2\mathbf{ME} - |\mathbf{P}|^2) \tag{1.22}$$

$$X(t) = X(0) + 2tA(0) + 2t^2\mathbf{E} \tag{1.23}$$

$$X_I(t) = X_I(0) + 2tA_I(0) + 2t^2(2\mathbf{ME} - |\mathbf{P}|^2) \tag{1.24}$$

Remarks.

1. To the best of the author's knowledge, the a-priori estimates (1.19) and (1.20) have not been previously studied. These estimates will be employed to formulate a new notion of dispersion.
2. Equation (1.23) has been explored within the framework of conservation laws in diverse contexts; refer to Proposition 2.11 in [13] for kinetic transport equation, and for non-linear equations like the Boltzmann equation, see [12]. This equation is similar to Morawetz and Virial identities, which can be found in both classical and quantum settings; consider chapter 1.5 and equation (2.38) in [18]. Equations (1.21) to (1.24) are closely related to the following well-known observations:

$$X(0) = \iint\limits_{\mathbb{R}^{2n}} f(x, \xi, t)|x - t\xi|^2 \, dxd\xi \tag{1.25}$$

$$X_I(0) = \iiiint\limits_{\mathbb{R}^{4n}} f(x, \xi, t) f(x_0, \xi_0, t)|x - x_0 - t(\xi - \xi_0)|^2 \, dxd\xi dx_0 d\xi_0 \tag{1.26}$$

3. An interesting implication of Theorem 2 is as follows: since A and A_I go to infinity with time while $\mathbf{G}$ and $\mathbf{G_I}$ remain bounded, we have:

$$\begin{aligned} &\lim_{t\to\infty} U(t) = +\infty \\ &\lim_{t\to\infty} U_I(t) = +\infty \end{aligned} \tag{1.27}$$

It is possible to interpret the uncertainty defined in (1.14) and (1.15) akin to its quantum mechanics counterpart, highlighting a fundamental limit on the precision of physical measurements. As in Fig. 1, consider an observer located at the origin and assume the speed of light is C. For each particle located at x there is a minimum delay of $T = C^{-1}|x|$ between the actual time of the measurement and observation. The quantity $Tf(x, \xi, t)|\xi| = C^{-1} f(x, \xi, t)|x||\xi|$ represents the uncertainty of measurement relative to an idle observer at the origin, due to this interval of delay for a particle at position x with velocity ξ. For a different formulation of uncertainty in the context of kinetic theory consider chapter I.5 in [7].

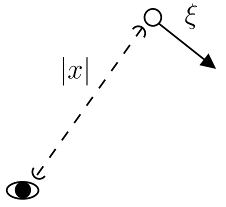

Fig. 1. The uncertainty associated with a particle is proportional to both its velocity and its distance from the observer.

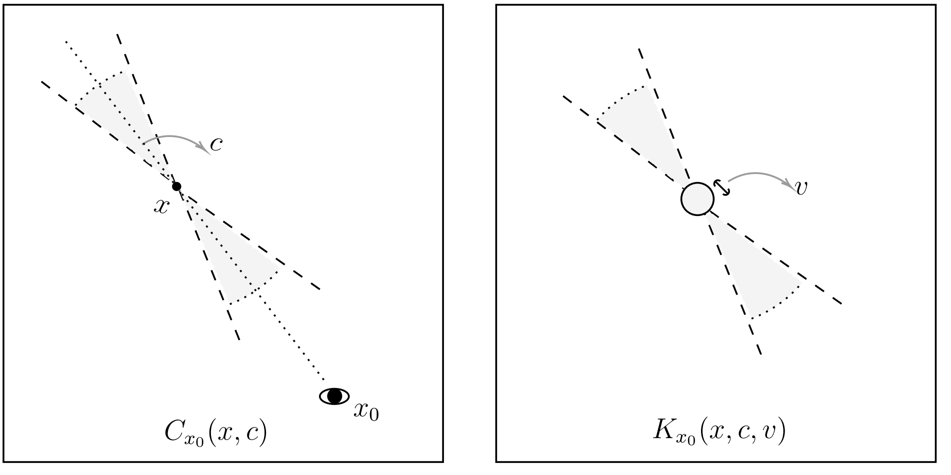

Fig. 2. The blind cone $C_{x_0}(x,c) \subset \mathbb{R}^n$ is a subset of the space of velocities at point x. This illustration includes a blind cone at x relative to an idle observer at x_0 and $K_{x_0}(x,c,v) \subset \mathbb{R}^n$.

1.3. A new notion of dispersion

We need to introduce a few concepts before explaining our main theorems. Define the **blind cone** with apex angle $c > 0$ at point $x \in \mathbb{R}^n$ relative to an **idle observer** at $x_0 \in \mathbb{R}^n$ for $|x - x_0| > 0$ as below:

$$C_{x_0}(x,c) = \{\xi \in \mathbb{R}^n \big| \, \theta(x - x_0, \xi) \notin [c/2, \pi - c/2] \text{ and } |\xi| > 0\} \tag{1.28}$$

If the velocities of two particles at point x belong to the blind cone $C_{x_0}(x,c)$, then the angle of deflection between the particles is bounded by the apex angle of the cone. By including the velocities of idle particles in this set, i.e., particles with very small velocities, we will obtain the set of velocities associated with particles at a point in space whose interactions are almost grazing, that is either their angle of deflection or their relative velocity is very small:

$$K_{x_0}(x,c,v) = C_{x_0}(x,c) \cup B_\xi(0,v) \quad v > 0 \tag{1.29}$$

Consider Fig. 2 and note that both the angles of deflection and the velocities of particles depend on the observer. The expression above is based on an idle observer at point x_0.

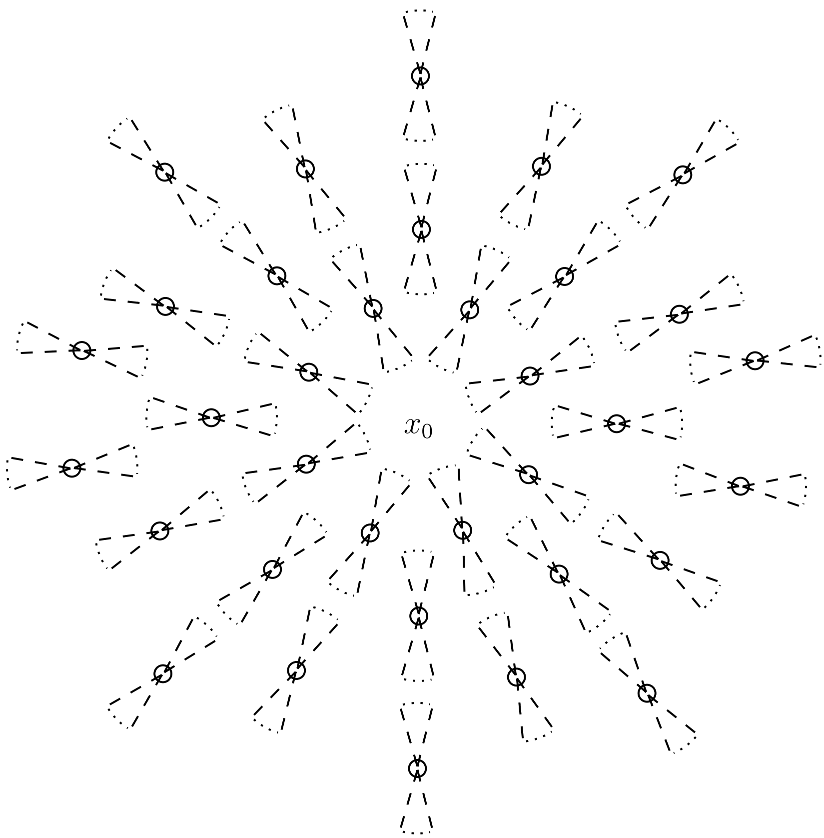

Fig. 3. An illustration of $\Gamma_{x_0}(c, v) \subset \mathbb{R}^{2n}$. In this drawing, the balls have radius v, and cones with an apex angle c are oriented toward the idle observer x_0 at the center.

It is possible to collect the velocities of these almost grazing particles throughout the entire space:

$$\Gamma_{x_0}(c, v) = \{(x, \xi) \in \mathbb{R}^{2n} \,\big|\, \xi \in K_{x_0}(x, c, v)\} \tag{1.30}$$

The set $\Gamma_{x_0}(c, v)$ (refer to Fig. 3) consists of the velocities of all grazing particles relative to an idle observer at x_0. The remarkable observation is the following: when averaged over time, the velocity of almost every particle belongs to this set, irrespective of how small c and v are chosen, and regardless of the specific type of interactions.

Theorem 3. *Assume $n \geq 2$ and $f(x, \xi, t) \geq 0$ is a classical solution of (1.1) such that:*

$$\begin{aligned} &f_0(x, \xi) \in L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \\ &\partial_t f, \nabla_x f \in L^1_{loc}(\mathbb{R}^+_t; L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)) \end{aligned} \tag{1.31}$$

Then for any $c > 0$, $v > 0$ and $x_0 \in \mathbb{R}^n$, we have:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \iint_{\Gamma_{x_0}(c,v)} f(x,\xi,t)\, dxd\xi dt = \mathbf{M} \tag{1.32}$$

Remarks.

1. To the best of the author's knowledge, no similar notion of dispersion exists in the literature of the field.
2. The generality of this approach reveals a mathematical relationship between the Landau kinetic equation and other kinetic PDEs in the context of grazing interactions. We will postpone this discussion until after presenting the next theorem, which extends the results above.

We will extend the previous theorem to the ensemble of interactions using Galilean invariance. Recall the notion of a blind cone with respect to an idle observer (1.28). We will now expand this definition for **moving** observers. Define the blind cone with apex angle $c > 0$ at $x \in \mathbb{R}^n$, relative to an observer at $x_0 \in \mathbb{R}^n$ moving with velocity $\xi_0 \in \mathbb{R}^n$ for $|x - x_0| > 0$ as below:

$$C_{(x_0,\xi_0)}(x,c) = \{\xi \in \mathbb{R}^n \,\big|\, \theta(x - x_0, \xi - \xi_0) \notin [c/2, \pi - c/2] \text{ and } |\xi - \xi_0| > 0\} \tag{1.33}$$

Following the same approach as above, we will extend the definition given in (1.29) to accommodate a moving observer:

$$K_{(x_0,\xi_0)}(x,c,v) = C_{(x_0,\xi_0)}(x,c) \cup B_\xi(\xi_0, v) \quad v > 0 \tag{1.34}$$

Note that $K_{(x_0,\xi_0)}(x,c,v)$ consists of velocities of particles at point x such that their interactions are almost grazing relative to the moving observer. Now, let us define $\Gamma(c,v)$ as the collection of these velocities:

$$\Gamma(c,v) = \{(x_0,\xi_0,x,\xi) \in \mathbb{R}^{4n} \,\big|\, \xi \in K_{(x_0,\xi_0)}(x,c,v)\} \tag{1.35}$$

The set $\Gamma(c,v)$ builds upon $\Gamma_{x_0}(c,v)$ by considering the relative velocity of two interacting particles at different locations, in which the roles of observer and particle are interchangeable. Similar to the previous theorem, we make the following remarkable observation: averaged over time, almost every interaction occurs between particles whose velocities belong to $\Gamma(c,v)$, regardless of how small c and v are chosen.

Theorem 4. *Assume $n \geq 2$ and $f(x,\xi,t) \geq 0$ is a classical solution of (1.1) such that:*

$$\begin{gathered} f_0(x,\xi) \in L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi) \\ \partial_t f, \nabla_x f \in L^1_{loc}(\mathbb{R}^+_t; L^1_{(1+|x|^2+|\xi|^2)}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)) \end{gathered} \tag{1.36}$$

Then for any $c > 0$ and $v > 0$, we have:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \iiiint_{\Gamma(c,v)} f(x,\xi,t) f(x_0,\xi_0,t)\, dxd\xi dx_0 d\xi_0 dt = \mathbf{M}^2 \tag{1.37}$$

Remarks.

1. Theorem 3 and Theorem 4 introduce a new notion of dispersion: as uncertainty increases over time (see (1.27)), particles move away in an asymptotically radial manner from any observer (refer to Fig. 3). These results suggest that as time approaches infinity, it seems almost every particle originates from the observer's location and almost every interaction is grazing relative to that observer, thereby establishing a general notion of dispersion.
2. For $v = +\infty$, equations (1.32) and (1.37) follow directly from the conservation of mass across the entire space. Theorem 3 and Theorem 4 demonstrate that by integrating over smaller subsets $\Gamma_{x_0}(c, v) \subset \mathbb{R}^{2n}$ and $\Gamma(c, v) \subset \mathbb{R}^{4n}$, we can still obtain similar equations for arbitrarily small positive c and v. Note that as c and v approach zero, these sets converge to measure-zero subsets.

Landau introduced his kinetic equation as a variation of the Boltzmann equation for Coulomb forces [7,21]. There are other variations of the main equation that can be obtained as a limit of the Boltzmann equation for potentials corresponding to the domination of grazing interactions [20]. Desvillettes [8] demonstrated this asymptotic relation by obtaining the Fokker–Planck–Landau equation from the Boltzmann equation. Therefore, the Landau equation is not only interesting in itself but also because of its connection to the Boltzmann equation when grazing interactions are not neglected. Theorem 3 and Theorem 4 prove that for any kinetic equation defined over the entire space, grazing interactions dominate irrespective of the specific structure of the interactions, and solely under the assumption of conservation laws.

1.4. Blind cones and grazing interactions

In this section, we will discuss a technical lemma regarding the blind cones that will be used in the proofs. The important observation concerning the blind cones is as follows: for any bounded region of the spatial variable, it is possible to have three observers such that their blind cones intersect trivially at every point in the region.

Lemma 1. *Assume $R > 0$, $n \geq 2$ and $O_i \in \partial B(0, R) \subset \mathbb{R}^n_x$ are 3 distinct points for $1 \leq i \leq 3$. Then, there exists $c_0 > 0$ such that for all $0 < c < c_0$ we have:*

$$C_{O_1}(x, c) \cap C_{O_2}(x, c) \cap C_{O_3}(x, c) = \emptyset \quad \forall x \in B(0, R) \tag{1.38}$$

And in general for any $(x_0, \xi_0) \in \mathbb{R}^{2n}$:

$$C_{(x_0+O_1,\xi_0)}(x, c) \cap C_{(x_0+O_2,\xi_0)}(x, c) \cap C_{(x_0+O_3,\xi_0)}(x, c) = \emptyset \quad \forall x \in B(x_0, R) \tag{1.39}$$

An intuitive interpretation of this lemma is that three observers are enough to cover any bounded region of the spatial variable. In other words, almost every interaction is non-grazing relative to at least one of the three observers, providing a method to address the challenge of grazing interactions. It is worth noting that three moving observers serve the same purpose for a bounded moving region. This lemma will be used in the proof of Theorem 1, and the concept of the blind cone is central to the formulation of dispersion in Theorem 3 and Theorem 4, as discussed in the previous section.

1.5. Galilean invariance and uncertainty

The introduction of blind cones is based on the intuition behind the notions of Galilean invariance and uncertainty. Galilean invariance is the assumption that the laws of mechanics remain unchanged relative to different inertial frames of reference. In classical mechanics, Galilean invariance is a consequence of the conservation laws of kinetic energy and momentum. However, it is also possible to start with Galilean invariance as an axiom to foster a different perspective. For example, consider two colliding billiard balls with masses m and m_*, initially moving with velocities $(V, V_*) \in \mathbb{R}^6$ before the collision and $(V', V'_*) \in \mathbb{R}^6$ after the collision. The conservation laws of kinetic energy and momentum imply:

$$\begin{aligned} m|V|^2 + m_*|V_*|^2 &= m|V'|^2 + m_*|V'_*|^2 \\ mV + m_*V_* &= mV' + m_*V'_* \end{aligned} \tag{1.40}$$

It is possible to rephrase these axioms in terms of the conservation of energy and Galilean invariance. In other words, kinetic energy is conserved relative to any observer moving with an arbitrary velocity ξ_0:

$$m|V-\xi_0|^2 + m_*|V_*-\xi_0|^2 = m|V'-\xi_0|^2 + m_*|V'_*-\xi_0|^2 \quad \forall \xi_0 \in \mathbb{R}^3 \tag{1.41}$$

The two formulations (1.40) and (1.41) are equivalent, and they can be derived from each other. Nonetheless, the second formulation, based on Galilean invariance, leads to the following ideas:

1. The angle of deflection before and after an interaction depends on the observer.
2. Even if we assume that there is a maximum magnitude for the velocity of an observer, i.e., if we assume $|\xi_0|$ is bounded by the speed of light, the two formulations (1.40) and (1.41) remain equivalent. This suggests the existence of a notion of uncertainty inherent solely in the conservation laws.

The observations above can be used to study irreversibility in the context of conservation laws. The theories based on conservation laws are advantageous due to their considerable generality; they remain valid independently of the specific type of interactions. However, Boltzmann-type entropy estimates are lost since they depend on the structure of the Boltzmann collision operator. Nevertheless, it is possible to develop other notions of entropy to study irreversibility based on the conservation laws. For interesting discussions on this subject, including a thorough comparison with the works of Peter Lax, consider the lecture notes by Tartar [19]. For instance, entropy can be defined as the average uncertainty in a system. This formulation of entropy is closely related to the following fundamental question: What are the differential constraints upon which Jensen's inequality holds true for a non-concave function (see section 1.2 in [15]).

An important attribute of Boltzmann entropy, besides its monotonicity, is its exclusive dependence on interactions, which makes it suitable for the study of irreversibility. In other words, it remains constant for the linear equation (1.10). The notion of uncertainty discussed in this paper shares a comparable property. Consider the following computation:

$$U(T) - A(T) = U(0) - A(0) + \int_0^T \frac{d}{dt}(U(t) - A(t))\, dt$$
$$= U(0) - A(0) + \int_0^T \frac{d}{dt}\Big(\iint_{\mathbb{R}^{2n}} f(x+t\xi, \xi, t)(|x+t\xi||\xi| - (x+t\xi)\cdot\xi)\, dx d\xi \Big)\, dt$$

Thus, at least in a formal sense, we obtain:

$$U(T) - A(T) = U(0) - A(0) + \underbrace{\int_0^T \iint_{\mathbb{R}^{2n}} I(x,\xi,t)|x||\xi|\, dx d\xi dt}_{i(f,T)}$$
$$- \underbrace{\int_0^T \iint_{\mathbb{R}^{2n}} f(x,\xi,t)|\xi|^2(1-\cos(\theta(x,\xi)))\, dx d\xi dt}_{e(f,T)}$$

Equivalently:

$$i(f,T) = U(T) - A(T) - U(0) + A(0) + e(f,T) \tag{1.42}$$

Theorem 2 yields the following bounds:

$$-U(0) + A(0) + e(f,T) \le i(f,T) \le \mathbf{G} - U(0) + A(0) + e(f,T) \tag{1.43}$$

Note that although $i(f,T)$ is explicitly defined via the interaction term I, the right-hand side of equation (1.42) depends only on f. In the case of a non-negative solution to the linear transport equation g as in (1.10), we have $i(g,T) = 0$, therefore (1.43) implies:

$$0 \le \underbrace{\int_0^\infty \iint_{\mathbb{R}^{2n}} g(x,\xi,t)|\xi|^2(1-\cos(\theta(x,\xi)))\, dx d\xi dt}_{\lim_{T\to\infty} e(g,T)}$$
$$\le \iint_{\mathbb{R}^{2n}} g_0(x,\xi)|x||\xi|(1-\cos(\theta(x,\xi)))\, dx d\xi \tag{1.44}$$

In this linear case, the difference between $U(T)$ and $A(T)$ is monotone and it reaches a minimum as $T \to \infty$.

In the general non-linear case (1.1), the functional $i(f,T)$ is non-zero and it can be interpreted as a measure of irreversibility without any reference to the specific type of interactions. The

Bony-type functionals based on (1.9) can be seen as a measure of dispersion. Similarly, the functionals based on the expression below can be used to study irreversibility via uncertainty:

$$|x - x_0||\xi - \xi_0| - (x - x_0) \cdot (\xi - \xi_0) \tag{1.45}$$

The estimates discussed in this paper are valid independently of the specific structure of interactions and they are solely based on the conservation laws. Another comparable approach is the study of divergence-free symmetric tensors that model many physical settings. For instance, the recent theory of compensated integrability developed by Serre utilizes the determinant map over the space of symmetric tensors to demonstrate estimates for time-space integrals based on conservation laws [16]. These estimates are applicable to a variety of equations including the Boltzmann equation, the BGK model, discrete velocity models, wave equation, Maxwell equations, and Vlasov–Poisson equation [14,15].

We will conclude this introduction with a scenario where Galilean invariance does not hold. Consider a system of an ideal gas confined in a finite-volume box, with no energy exchange with the external environment. In this setting, particles experience specular reflections at the boundary. For this configuration, momentum is not conserved in order to maintain the boundary. In other words, this model is not Galilean invariant because energy is only conserved for an observer at rest with the box. In contrast, equation (1.1) spans the entire space and is Galilean invariant.

2. Proofs

This section provides proofs, in the appropriate order, for the results discussed in the introduction, starting with the technical lemma discussed in Section 1.4.

Proof of Lemma 1. For an arbitrary $x \in B(0, R)$ define P as below:

$$P(x, c) = \partial B(0, R) \cap C_{O_1}(x, 2c)$$

For each x, the set $P(x, c)$ is contained in $\partial B(0, R)$ and it consists of two path connected components (see Fig. 4). Consider the longest geodesic path on each component, set $K(x)$ to be the maximum length of the two and let K_0 be the supremum of $K(x)$ for all $x \in B(0, R)$. For any R, it is possible to choose c small enough (independent of x) such that K_0 becomes as small as desired. Now set c small enough such that K_0 becomes smaller than the length of geodesic path on the sphere between any pair of the three O_i. The pigeon hole principle implies that, since each path connected component of $P(x, c)$ can only contain maximum one of the three observers, there exists an O_i such that $O_i \notin P(x, c)$. Therefore, for every $x \in B(0, R)$, there exists an O_i such that the blind cones $C_{O_i}(x, c)$ and $C_{O_1}(x, c)$ have an empty intersection:

$$C_{O_i}(x, c) \cap C_{O_1}(x, c) = \emptyset$$

This completes the proof of (1.38). To prove (1.39), note that the equation above implies that for any $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and all $x \in B(x_0, R)$, there exists an O_i such that:

$$C_{(x_0+O_i,\xi_0)}(x, c) \cap C_{(x_0+O_1,\xi_0)}(x, c) = \emptyset \quad \square$$

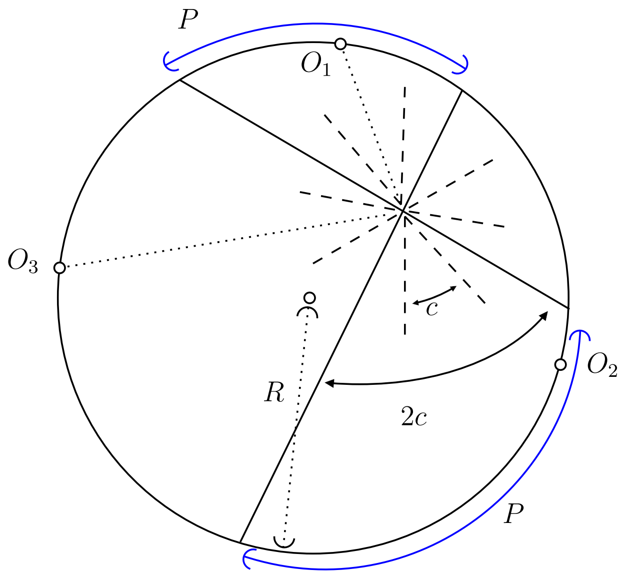

Fig. 4. A possible configuration of the three observers. This drawing includes blind cones $C_{O_1}(x,c)$, $C_{O_1}(x,2c)$, $C_{O_3}(x,c)$ and ball $B_x(0,R)$. In this example, we have $O_3 \notin P$ and $C_{O_1}(x,c) \cap C_{O_3}(x,c) = \emptyset$.

Now we are ready to prove Theorem 1 using the previous lemma and (1.9). In the following proof, we will discuss how to use the blind cones to overcome the difficulties due to grazing particles and interactions.

Proof of Theorem 1. Multiply both sides of equation (1.1) with $\dfrac{(x-x_0)\cdot\xi}{|x-x_0|}$ for $x_0 \in \mathbb{R}^n$ to get:

$$\partial_t f(x,\xi,t)\frac{(x-x_0)\cdot\xi}{|x-x_0|} + \xi\cdot\nabla_x f(x,\xi,t)\frac{(x-x_0)\cdot\xi}{|x-x_0|} = I(f,x,\xi,t)\frac{(x-x_0)\cdot\xi}{|x-x_0|}$$

Integrate over x and ξ, followed by a change of variables:

$$\partial_t \iint\limits_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)\frac{x\cdot\xi}{|x|}\,dxd\xi + \iint\limits_{\mathbb{R}^{2n}} \xi\cdot\nabla_x f(x+x_0,\xi,t)\frac{x\cdot\xi}{|x|}\,dxd\xi$$

$$= \iint\limits_{\mathbb{R}^{2n}} I(f,x+x_0,\xi,t)\frac{x\cdot\xi}{|x|}\,dxd\xi = 0$$

The last equality is due to (1.2). Continue with an integration by parts with respect to x, we get:

$$\partial_t \iint\limits_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)\frac{x\cdot\xi}{|x|}\,dxd\xi = \iint\limits_{\mathbb{R}^{2n}} \frac{1}{|x|}|\xi|^2\sin^2(\theta(x,\xi))f(x+x_0,\xi,t)\,dxd\xi$$

The positive derivative appearing in the previous equation is the time derivative of a bounded quantity:

$$\Big|\iint_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)\frac{x\cdot\xi}{|x|}\,dxd\xi\Big| \le \iint_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)|\xi|\,dxd\xi$$
$$\le \Big(\iint_{\mathbb{R}^{2n}} f(x,\xi,t)|\xi|^2\,dxd\xi\Big)^{1/2}\Big(\iint_{\mathbb{R}^{2n}} f(x,\xi,t)\,dxd\xi\Big)^{1/2} = \sqrt{2\mathbf{ME}}$$

The boundedness of this quantity and positivity of its derivative imply that for an arbitrary $x_0 \in \mathbb{R}^n$ the following limit exists:

$$\lim_{t\to\infty}\iint_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)\frac{(x\cdot\xi)}{|x|}\,dxd\xi \le \sqrt{2\mathbf{ME}} \tag{2.1}$$

Furthermore:

$$\int_0^\infty\iint_{\mathbb{R}^{2n}} f(x+x_0,\xi,t)\frac{1}{|x|}|\xi|^2\sin^2(\theta(x,\xi))\,dxd\xi dt$$
$$= \int_0^\infty\iint_{\mathbb{R}^{2n}} f(x,\xi,t)\frac{1}{|x-x_0|}|\xi|^2\sin^2(\theta(x-x_0,\xi))\,dxd\xi dt \le 2\sqrt{2\mathbf{ME}} \tag{2.2}$$

Recall equation (1.28) for the blind cone $C_{x_0}(x,c)$. We can change the domain of integration in (2.2) to a new set, such that it has a lower bound for $\sin^2(\theta(x-x_0,\xi))$ and $|x-x_0|^{-1}$ in terms of R and c, i.e., inside a ball of radius R in x and outside of blind cones in ξ:

$$\sin^2(c)\frac{1}{R}\int_0^\infty\int_{B_x(x_0,R)}\int_{\mathbb{R}^n\setminus C_{x_0}(x,c)} f(x,\xi,t)|\xi|^2\,d\xi dxdt$$
$$\le \int_0^\infty\iint_{\mathbb{R}^{2n}} f(x,\xi,t)\frac{1}{|x-x_0|}|\xi|^2\sin^2(\theta(x-x_0,\xi))\,d\xi dxdt \le 2\sqrt{2\mathbf{ME}}$$

Let $W_{x_0}(R,c)$ represent the domain of integration on the left hand side of the inequality above:

$$W_{x_0}(R,c) = \{(x,\xi)\in B(x_0,R)\times\mathbb{R}^n \,\big|\, \xi\in(\mathbb{R}^n\setminus C_{x_0}(x,c))\} \tag{2.3}$$

We can re-write the previous estimate as:

$$\int_0^\infty\iint_{W_{x_0}(R,c)} f(x,\xi,t)|\xi|^2\,d\xi dxdt \le \frac{2R}{\sin^2(c)}\sqrt{2\mathbf{ME}} \tag{2.4}$$

Recall Lemma 1 and Fig. 4. Fix any 3 distinct points $O_i \in \partial B_x(0,R)$ and continue with the following definitions for J_1, J_2 and J_3:

$$J_i = \int_0^\infty \iint_{W_{O_i}(2R,c)} f(x,\xi,t)|\xi|^2 \, d\xi dx dt \le \frac{4R}{\sin^2(c)}\sqrt{2\mathbf{ME}} \tag{2.5}$$

Consider the following two observations. First, Lemma 1 implies we can choose c small enough such that for any $x \in B(0, R)$ the 3 blind cones at x (relative to the 3 observers) intersect trivially:

$$C_{O_1}(x,c) \cap C_{O_2}(x,c) \cap C_{O_3}(x,c) = \emptyset \quad \forall x \in B(0,R)$$

The second observation is that:

$$B(0,R) \subset B(O_1, 2R) \cap B(O_2, 2R) \cap B(O_3, 2R)$$

These observations imply that for sufficiently small c, any subset of $B_x(0,R) \times \mathbb{R}^n$ is covered at least once in the domains of integration for J_1, J_2 and J_3:

$$B_x(0,R) \times \mathbb{R}^n \subset \cup_{i=1}^3 W_{O_i}(2R,c)$$

Therefore, the positivity of integrands implies:

$$\int_0^\infty \int_{B_x(0,R)} \int_{\mathbb{R}^n} f(x,\xi,t)|\xi|^2 \, d\xi dx dt \le J_1 + J_2 + J_3 \le \frac{12R}{\sin^2(c)}\sqrt{2\mathbf{ME}}$$

Note that we can select three O_i in a way that maximizes the minimum central angle between any pair of them (dependent only on the dimension). Consequently, we can choose a small c that is independent of R. This completes the proof of (1.7).

We continue with a proof for (1.8). Let $Z(t)$ be defined as:

$$Z(t) = \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|} \, dx d\xi dx_0 d\xi_0$$

We have:

$$\begin{aligned}|Z(t)| &\le \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t)|\xi-\xi_0| \, dx d\xi dx_0 d\xi_0 \\ &\le \Big(\iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t)|\xi-\xi_0|^2 \, dx d\xi dx_0 d\xi_0\Big)^{1/2} \\ &\qquad \times \Big(\iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t) \, dx d\xi dx_0 d\xi_0\Big)^{1/2}\end{aligned}$$

Consider that:

$$\iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t)\ |\xi - \xi_0|^2\ dx d\xi dx_0 d\xi_0 = 2(2\mathbf{ME} - |\mathbf{P}|^2) \tag{2.6}$$

Therefore the following bound holds true:

$$|Z(t)| \leq \mathbf{M}\sqrt{2(2\mathbf{ME} - |\mathbf{P}|^2)} \tag{2.7}$$

Continue with differentiating $Z(t)$:

$$\begin{aligned} \frac{d}{dt} Z(t) = & \iint_{\mathbb{R}^{2n}} f(x_0,\xi_0,t) \underbrace{\iint_{\mathbb{R}^{2n}} \partial_t f(x,\xi,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx d\xi}_{Z_1}\ dx_0 d\xi_0 \\ & + \iint_{\mathbb{R}^{2n}} f(x,\xi,t) \underbrace{\iint_{\mathbb{R}^{2n}} \partial_t f(x_0,\xi_0,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx_0 d\xi_0}_{Z_2}\ dx d\xi \end{aligned} \tag{2.8}$$

Using the equation (1.1) we get:

$$Z_1 = \iint_{\mathbb{R}^{2n}} I(f,x,\xi,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx d\xi - \iint_{\mathbb{R}^{2n}} \xi \cdot \nabla_x f(x,\xi,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx d\xi$$

From the conservation laws (1.2) we know that the first term on the right hand side of the equation above is zero, therefore:

$$Z_1 = -\iint_{\mathbb{R}^{2n}} \xi \cdot \nabla_x f(x,\xi,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx d\xi$$

Similarly for Z_2 we get:

$$Z_2 = -\iint_{\mathbb{R}^{2n}} \xi_0 \cdot \nabla_x f(x_0,\xi_0,t) \frac{(x-x_0)\cdot(\xi-\xi_0)}{|x-x_0|}\ dx_0 d\xi_0$$

We will continue with an integration by parts with respect to x for Z_1:

$$Z_1 = \iint_{\mathbb{R}^{2n}} f(x,\xi,t) \frac{\xi\cdot(\xi-\xi_0)|x-x_0|^2 - ((x-x_0)\cdot\xi)((x-x_0)\cdot(\xi-\xi_0))}{|x-x_0|^3}\ dx d\xi$$

Similarly for Z_2 we get:

$$Z_2 = \iint_{\mathbb{R}^{2n}} f(x_0,\xi_0,t) \frac{-\xi_0\cdot(\xi-\xi_0)|x-x_0|^2 + ((x-x_0)\cdot\xi_0)((x-x_0)\cdot(\xi-\xi_0))}{|x-x_0|^3}\ dx_0 d\xi_0$$

Substitute the computations above for Z_1 and Z_2 in (2.8) to obtain:

$$\frac{d}{dt}Z(t) = \iiiint\limits_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t) \frac{1}{|x-x_0|} |\xi-\xi_0|^2 \sin^2(\theta(x-x_0,\xi-\xi_0))\, d\xi dx d\xi_0 dx_0$$

Since the time derivative of $Z(t)$ is positive, from estimate (2.7) we get:

$$\int\limits_0^\infty \iiiint\limits_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t) \frac{1}{|x-x_0|} |\xi-\xi_0|^2 \times \sin^2(\theta(x-x_0,\xi-\xi_0))\, d\xi dx d\xi_0 dx_0 dt \le 2\mathbf{M}\sqrt{2(2\mathbf{ME}-|\mathbf{P}|^2)}$$

Recall the definition of a blind cone with respect to a moving observer from (1.33). By removing blind cones $C_{(x_0,\xi_0)}(x,c)$ from the space of velocities and integrating within $B_x(x_0,R)$, we can impose lower bounds for $|x-x_0|^{-1}$ and $\sin^2(\theta(x-x_0,\xi-\xi_0))$, hence the inequality above leads to:

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \int\limits_{B_x(x_0,R)} \int\limits_{\mathbb{R}^n \setminus C_{(x_0,\xi_0)}(x,c)} f(x,\xi,t) f(x_0,\xi_0,t) \times |\xi-\xi_0|^2\, d\xi dx d\xi_0 dx_0 dt \le \frac{2R}{\sin^2(c)}\mathbf{M}\sqrt{2(2\mathbf{ME}-|\mathbf{P}|^2)}$$

Obtain the following notation:

$$W_{(x_0,\xi_0)}(R,c) = \{(x,\xi) \in B(x_0,R)\times\mathbb{R}^n \,\big|\, \xi \in \mathbb{R}^n \setminus C_{(x_0,\xi_0)}(x,c)\}$$

We can re-write the previous estimate as:

$$\int\limits_0^\infty \int\limits_{\mathbb{R}^{2n}\times W_{(x_0,\xi_0)}(R,c)} \!\!\cdots\!\! \int f(x,\xi,t) f(x_0,\xi_0,t) |\xi-\xi_0|^2\, d\xi dx d\xi_0 dx_0 dt \le \frac{2R}{\sin^2(c)}\mathbf{M}\sqrt{2(2\mathbf{ME}-|\mathbf{P}|^2)} \tag{2.9}$$

Recall Lemma 1 and (1.39). Choose 3 distinct points $O_i \in \partial B_x(0,R)$ and define J_1, J_2 and J_3 as below:

$$J_i = \int\limits_0^\infty \int\limits_{\mathbb{R}^{2n}\times W_{(x_0+O_i,\xi_0)}(2R,c)} \!\!\cdots\!\! \int f(x,\xi,t) f(x_0,\xi_0,t) |\xi-\xi_0|^2\, d\xi dx d\xi_0 dx_0 dt \tag{2.10}$$

The following bounds are consequences of (2.9):

$$J_i \leq \frac{4R}{\sin^2(c)}\mathbf{M}\sqrt{2(2\mathbf{ME}-|\mathbf{P}|^2)} \tag{2.11}$$

For convenience, obtain the notation below for the domain of integration of J_i in (2.10):

$$Y_i = \{(x_0, \xi_0, x, \xi) \in \mathbb{R}^{4n} |\ (x, \xi) \in W_{(x_0+O_i,\xi_0)}(2R, c)\} \tag{2.12}$$

Continue with the following two observations. First, from (1.39) we know that it is possible to choose the apex angle c small enough such that for any point $x \in B(x_0, R)$ the 3 blind cones $C_{(x_0+O_i,\xi_0)}(x, c)$ intersect trivially:

$$C_{(x_0+O_1,\xi_0)}(x, c) \cap C_{(x_0+O_2,\xi_0)}(x, c) \cap C_{(x_0+O_3,\xi_0)}(x, c) = \emptyset \quad \forall x \in B(x_0, R)$$

And second observation is that:

$$B(x_0, R) \subset B(x_0 + O_1, 2R) \cap B(x_0 + O_2, 2R) \cap B(x_0 + O_3, 2R)$$

These two observations imply:

$$Y = \{(x_0, \xi_0, x, \xi) \subset \mathbb{R}^{4n} |\ x \in B(x_0, R)\} \subset \cup_{i=1}^{3} Y_i \tag{2.13}$$

In other words, any subset of Y is covered at least once in the domains of integration for J_1, J_2 and J_3. Therefore, the positivity of integrands completes the proof:

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{B_x(x_0,R)}\int_{\mathbb{R}^n} f(x,\xi,t)f(x_0,\xi_0,t)|\xi-\xi_0|^2\, d\xi dx d\xi_0 dx_0 dt &\leq J_1 + J_2 + J_3 \\ &\leq \frac{12R}{\sin^2(c)}\mathbf{M}\sqrt{2(2\mathbf{ME}-|\mathbf{P}|^2)}\end{aligned}$$

Note that small c can be chosen independent of R, similar as before. □

We continue with the proof of Theorem 2 for mild solutions using conservation laws. The new a-priori bounds (1.20) and (1.19) are the main results of this theorem.

Proof of Theorem 2. First, we will prove equations for A, A_I, X and X_I, and then we continue with proofs of the main results of the theorem for $\mathbf{G}$ and $\mathbf{G_I}$.

To prove (1.21), start with a change of variables:

$$\begin{aligned}A(t) = \iint_{\mathbb{R}^{2n}} f(x,\xi,t)x\cdot\xi\, dxd\xi &= \iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)(x+t\xi)\cdot\xi\, dxd\xi \\ &= \iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)x\cdot\xi\, dxd\xi + t\iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)|\xi|^2\, dxd\xi\end{aligned}$$

Consequently from equation (1.3) we get:

$$A(t) = \iint\limits_{\mathbb{R}^{2n}} f(x,\xi,0)x\cdot\xi \, dxd\xi + \int\limits_0^t \iint\limits_{\mathbb{R}^{2n}} I(x+s\xi,\xi,s)x\cdot\xi \, dxd\xi ds + t\iint\limits_{\mathbb{R}^{2n}} f(x,\xi,t)|\xi|^2 \, dxd\xi$$

Hence using (1.2) leads to:

$$A(t) = \iint\limits_{\mathbb{R}^{2n}} f(x,\xi,0)x\cdot\xi \, dxd\xi + \int\limits_0^t \iint\limits_{\mathbb{R}^{2n}} I(x,\xi,s)(x-s\xi)\cdot\xi \, dxd\xi ds + t\iint\limits_{\mathbb{R}^{2n}} f_0(x,\xi)|\xi|^2 \, dxd\xi = A(0) + 2t\mathbf{E}$$

To prove (1.22), start with a change of variables:

$$\begin{aligned} A_I(t) &= \iiiint\limits_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)\ (x-x_0)\cdot(\xi-\xi_0)\ dxd\xi dx_0 d\xi_0 \\ &= \iiiint\limits_{\mathbb{R}^{4n}} f(x+t\xi,\xi,t)f(x_0+t\xi_0,\xi_0,t)\ (x-x_0)\cdot(\xi-\xi_0)\ dxd\xi dx_0 d\xi_0 \\ &\quad + t\iiiint\limits_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)\ |\xi-\xi_0|^2\ dxd\xi dx_0 d\xi_0 \end{aligned}$$

Using (2.6) we get:

$$A_I(t) = \underbrace{\iiiint\limits_{\mathbb{R}^{4n}} f(x+t\xi,\xi,t)f(x_0+t\xi_0,\xi_0,t)\ (x-x_0)\cdot(\xi-\xi_0)\ dxd\xi dx_0 d\xi_0}_{J} + 2t(2\mathbf{ME} - |\mathbf{P}|^2)$$

Substitute the equation below in J:

$$f(x+t\xi,\xi,t)f(x_0+t\xi_0,\xi_0,t) = (f_0(x,\xi) + \int\limits_0^t I(x+s\xi,\xi,s)ds)(f_0(x_0,\xi_0) + \int\limits_0^t I(x_0+z\xi_0,\xi_0,z)dz)$$

After this substitution, from the symmetry between (x,ξ) and (x_0,ξ_0), and the conservation laws (1.2) we get:

$$J = A_I(0) + 2\int_0^t \iiiint_{\mathbb{R}^{4n}} I(x+s\xi,\xi,t) f_0(x_0,\xi_0)(x-x_0)\cdot(\xi-\xi_0)\, dx d\xi dx_0 d\xi_0 ds$$
$$+ \int_0^t \int_0^t \iiiint_{\mathbb{R}^{4n}} I(x+s\xi,\xi,s) I(x_0+z\xi_0,\xi_0,z)(x-x_0)\cdot(\xi-\xi_0)\, dx d\xi dx_0 d\xi_0 ds dz = A_I(0)$$

Therefore:

$$A_I(t) = A_I(0) + 2t(2\mathbf{ME} - |\mathbf{P}|^2)$$

We continue with a proof for (1.23):

$$X(t) = \iint_{\mathbb{R}^{2n}} f(x,\xi,t)|x|^2\, dx d\xi = \iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)|x+t\xi|^2\, dx d\xi$$
$$= \iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)(|x|^2 + t^2|\xi|^2 + 2t(x\cdot\xi))\, dx d\xi$$

Using the equation for $A(t)$, conservation laws (1.2) and equation (1.25), we get:

$$X(t) = X(0) + 2t^2\mathbf{E} + 2t\iint_{\mathbb{R}^{2n}} f(x,\xi,t)((x\cdot\xi) - t|\xi|^2)\, dx d\xi = X(0) + 2tA(0) + 2t^2\mathbf{E}$$

To prove (1.24), start with the following observation:

$$X_I(t) = \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t)$$
$$\times \big(|x-x_0-t(\xi-\xi_0)|^2 - t^2|\xi-\xi_0|^2 + 2t(x-x_0)\cdot(\xi-\xi_0)\big)\, dx d\xi dx_0 d\xi_0$$

Using equation proved previously for $A_I(t)$ and equations (2.6) and (1.26), we can conclude:

$$X_I(t) = X_I(0) + 2tA_I(0) + 2t^2(2\mathbf{ME} - |\mathbf{P}|^2)$$

Now we are ready to prove estimates (1.19) and (1.20). Start with the equation below:

$$U(t) - A(t) = \iint_{\mathbb{R}^{2n}} f(x,\xi,t)|x||\xi|\, dx d\xi - \iint_{\mathbb{R}^{2n}} f(x,\xi,t) x\cdot\xi\, dx d\xi$$
$$= \iint_{\mathbb{R}^{2n}} f(x+t\xi,\xi,t)|x+t\xi||\xi|\, dx d\xi - A(0) - 2t\mathbf{E}$$

Continue with the triangle inequality and conservation of energy:

$$U(t) - A(t) \leq \iint_{\mathbb{R}^{2n}} f(x+t\xi, \xi, t)(|x||\xi| + t|\xi|^2)\, dxd\xi - A(0) - 2t\mathbf{E}$$

$$\leq \iint_{\mathbb{R}^{2n}} f(x+t\xi, \xi, t)|x||\xi|\, dxd\xi - A(0)$$

The Cauchy–Schwarz leads to:

$$U(t) - A(t) \leq (\iint_{\mathbb{R}^{2n}} f(x+t\xi, \xi, t)|x|^2\, dxd\xi)^{1/2} (\iint_{\mathbb{R}^{2n}} f(x+t\xi, \xi, t)|\xi|^2\, dxd\xi)^{1/2} - A(0)$$

Finally, using (1.25) and the conservation of energy we get:

$$U(t) - A(t) \leq \sqrt{2X(0)\mathbf{E}} - A(0)$$

The right hand side of the inequality above is independent of time, therefore:

$$\mathbf{G} = \sup_t (U(t) - A(t)) \leq \sqrt{2X(0)\mathbf{E}} - A(0)$$

This concludes (1.19). To prove (1.20), implement the change of variables $(x + t\xi, x_0 + t\xi_0)$ followed by the triangle inequality to get:

$$U_I(t) - A_I(t) = \iiiint_{\mathbb{R}^{4n}} f(x, \xi, t) f(x_0, \xi_0, t) \Big(|x - x_0||\xi - \xi_0| - (x - x_0) \cdot (\xi - \xi_0) \Big)\, d\xi dx d\xi_0 dx_0$$

$$\leq \iiiint_{\mathbb{R}^{4n}} f(x+t\xi, \xi, t) f(x_0 + t\xi_0, \xi_0, t) \Big(|x - x_0||\xi - \xi_0| + t|\xi - \xi_0|^2$$

$$- (x - x_0 + t(\xi - \xi_0)) \cdot (\xi - \xi_0) \Big)\, d\xi dx d\xi_0 dx_0$$

Therefore we have:

$$U_I(t) - A_I(t) \leq \underbrace{\iiiint_{\mathbb{R}^{4n}} f(x+t\xi, \xi, t) f(x_0 + t\xi_0, \xi_0, t) |x - x_0||\xi - \xi_0|\, d\xi dx d\xi_0 dx_0}_{Z_1}$$

$$- \underbrace{\iiiint_{\mathbb{R}^{4n}} f(x+t\xi, \xi, t) f(x_0, \xi_0 + t\xi_0, t)(x - x_0) \cdot (\xi - \xi_0)\, d\xi dx d\xi_0 dx_0}_{Z_2}$$

We estimate Z_1 using the Cauchy–Schwarz:

$$Z_1 \leq \Big(\iiiint_{\mathbb{R}^{4n}} f(x+t\xi,\xi,t) f(x_0+t\xi_0,\xi_0,t)|x-x_0|^2 \, d\xi dx d\xi_0 dx_0 \Big)^{1/2}$$
$$\times \Big(\iiiint_{\mathbb{R}^{4n}} f(x+t\xi,\xi,t) f(x_0+t\xi_0,\xi_0,t)|\xi-\xi_0|^2 \, d\xi dx d\xi_0 dx_0 \Big)^{1/2}$$

Therefore, (1.26) and (2.6) lead to:

$$Z_1 \leq \sqrt{2X_I(0)(2\mathbf{ME} - |\mathbf{P}|^2)}$$

To estimate Z_2 we will use (2.6) and (1.22) to get:

$$Z_2 = \iiiint_{\mathbb{R}^{4n}} f(x,\xi,t) f(x_0,\xi_0,t)\big((x-x_0)\cdot(\xi-\xi_0) - t|\xi-\xi_0|^2\big) d\xi dx d\xi_0 dx_0$$
$$= A_I(t) - 2t(2\mathbf{ME} - |\mathbf{P}|^2) = A_I(0)$$

Therefore we get:

$$U_I(t) - A_I(t) \leq Z_1 - Z_2 \leq \sqrt{2X_I(0)(2\mathbf{ME} - |\mathbf{P}|^2)} - A_I(0)$$

This completes the proof of the theorem:

$$\mathbf{G_I} = \sup_t (U_I(t) - A_I(t)) \leq \sqrt{2X_I(0)(2\mathbf{ME} - |\mathbf{P}|^2)} - A_I(0) \quad \square$$

We will use the previous results to prove Theorem 3 and Theorem 4. The following proofs demonstrate a new notion of dispersion using the blind cones.

Proof of Theorem 3. Recall definitions of $\Gamma_{x_0}(c,v)$ and $K_{x_0}(x,c,v)$ from (1.29) and (1.30). Continue with the following two definitions for $R > 0$ and $x_0 \in \mathbb{R}^n$:

$$N_{x_0}(R,c,v) = \{(x,\xi) \in \mathbb{R}^{2n} \big| \ \xi \in (\mathbb{R}^n \setminus K_{x_0}(x,c,v)) \text{ and } |x| > R\}$$
$$M_{x_0}(R,c,v) = \{(x,\xi) \in \mathbb{R}^{2n} \big| \ \xi \in (\mathbb{R}^n \setminus K_{x_0}(x,c,v)) \text{ and } |x| \leq R\}$$

Note the following observation due to the conservation of mass:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \iint_{\mathbb{R}^{2n}} f(x,\xi,t) \, dx d\xi dt = \mathbf{M}$$

Consider Fig. 5, we will break the domain of integration above into three non overlapping subsets, namely $N_{x_0}(R,c,v)$, $M_{x_0}(R,c,v)$ and $\Gamma_{x_0}(c,v)$:

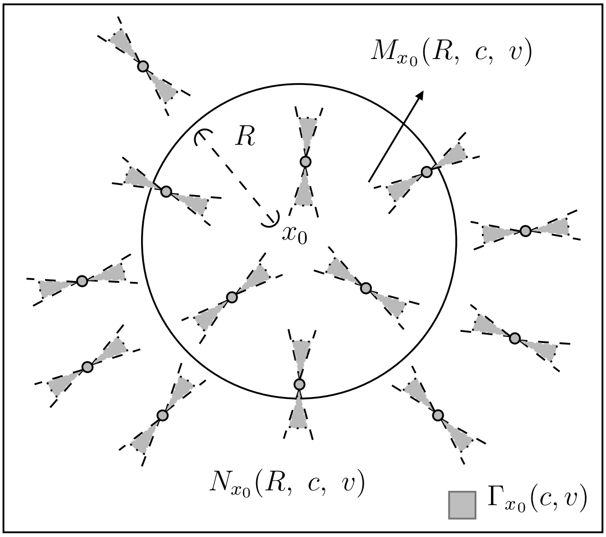

Fig. 5. We have $\mathbb{R}^{2n} = N_{x_0}(R, c, v) \cup M_{x_0}(R, c, v) \cup \Gamma_{x_0}(c, v)$.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\iint\limits_{N_{x_0}(R,c,v)} f(x, \xi, t)\, dxd\xi + \iint\limits_{M_{x_0}(R,c,v)} f(x, \xi, t)\, dxd\xi + \iint\limits_{\Gamma_{x_0}(c,v)} f(x, \xi, t)\, dxd\xi \Big) dt = \mathbf{M} \quad (2.14)$$

We will obtain estimates for integrals over $M_{x_0}(R, c, v)$ and $N_{x_0}(R, c, v)$. As a consequence of Theorem 1 we have:

$$v^2 \int_0^\infty \iint\limits_{M_{x_0}(R,c,v)} f(x, \xi, t)\, dxd\xi dt < \int_0^\infty \int_{\mathbb{R}^n} \int_{B_x(x_0,R)} f(x, \xi, t)|\xi|^2\, dxd\xi dt < \infty$$

From the previous estimate and (2.14) we get:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\iint\limits_{N_{x_0}(R,c,v)} f(x, \xi, t)\, dxd\xi + \iint\limits_{\Gamma_{x_0}(c,v)} f(x, \xi, t)\, dxd\xi \Big) dt = \mathbf{M} \quad (2.15)$$

We continue with an estimation for the integral over $N_{x_0}(R, c, v)$. Theorem 2 allows us to control the total mass within $N_{x_0}(R, c, v)$:

$$\iint\limits_{N_{x_0}(R,c,v)} f(x, \xi, t)(|x||\xi| - x \cdot \xi)\, dxd\xi \le \mathbf{G} \quad (2.16)$$

Note that:

$$Rv(1-\cos(c)) \iint\limits_{N_{x_0}(R,c,v)} f(x,\xi,t)\,dxd\xi \le \iint\limits_{N_{x_0}(R,c,v)} f(x,\xi,t)(|x||\xi| - x\cdot\xi)\,dxd\xi$$

Therefore (2.16) implies:

$$\iint\limits_{N_{x_0}(R,c,v)} f(x,\xi,t)\,dxd\xi \le \frac{\mathbf{G}}{Rv(1-\cos(c))} \tag{2.17}$$

The equation above provides a bound for the total amount of mass within $N_{x_0}(R,c,v)$. Combine (2.17) with equation (2.15) to get:

$$\mathbf{M} - \frac{\mathbf{G}}{Rv(1-\cos(c))} \le \liminf_{T\to\infty} \frac{1}{T}\int_0^T \Big(\iint\limits_{\Gamma_{x_0(c,v)}} f(x,\xi,t)\,dxd\xi\Big)dt \le \mathbf{M}$$

Finally, because R can be arbitrary large and since Theorem 2 proved $\mathbf{G}$ is bounded, we conclude:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(\iint\limits_{\Gamma_{x_0}(c,v)} f(x,\xi,t)\,dxd\xi\Big)dt = \mathbf{M} \quad \square$$

The following proof extends Theorem 3 to the ensemble of interactions via Galilean invariance.

Proof of Theorem 4. Recall definitions of $K_{(x_0,\xi_0)}(x,c,v)$ and $\Gamma(c,v)$ from (1.34) and (1.35). Continue with the following two definitions for $c>0$, $v>0$ and $R>0$:

$$N(R,c,v) = \{(x_0,\xi_0,x,\xi)\in\mathbb{R}^{4n} \mid \xi \in (\mathbb{R}^n \setminus K_{(x_0,\xi_0)}(x,c,v)) \text{ and } |x-x_0|>R\}$$
$$M(R,c,v) = \{(x_0,\xi_0,x,\xi)\in\mathbb{R}^{4n} \mid \xi \in (\mathbb{R}^n \setminus K_{(x_0,\xi_0)}(x,c,v)) \text{ and } |x-x_0|\le R\}$$

Consider the equation below due to the conservation of mass:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \iiiint\limits_{\mathbb{R}^{4n}} f(x,\xi,t)f(x_0,\xi_0,t)d\xi dx d\xi_0 d\xi_0 dt = \mathbf{M}^2$$

It is possible to divide $\mathbb{R}^{4n}$ in the equation above into 3 mutuality exclusive sets:

$$\mathbb{R}^{4n} = N(R,c,v)\cup M(R,c,v)\cup\Gamma(c,v) \tag{2.18}$$

Therefore we have:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(\iiiint_{\Gamma(c,v)} f(x,\xi,t)f(x_0,\xi_0,t)dxd\xi dx_0 d\xi_0 + \iiiint_{M(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)dxd\xi dx_0 d\xi_0 + \iiiint_{N(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)dxd\xi dx_0 d\xi_0\Big)dt = \mathbf{M^2} \quad (2.19)$$

The estimate (1.8) in Theorem 1 implies the bound below for the time integral of interactions within $M(R,c,v)$:

$$v^2\int_0^\infty \iiiint_{M(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0 dt < \infty$$

Consequently from (2.19) we get:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(\iiiint_{\Gamma(c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0 + \iiiint_{N(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0\Big)dt = \mathbf{M^2} \quad (2.20)$$

Now continue with Theorem 2 to estimate interactions within $N(R,c,v)$:

$$Rv(1-\cos(c))\iiiint_{N(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0$$
$$< \iiiint_{N(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\Big(|x-x_0||\xi-\xi_0| - (x-x_0)\cdot(\xi-\xi_0)\Big)\; dxd\xi dx_0 d\xi_0 \le \mathbf{G_I}$$

We get:

$$\iiiint_{N(R,c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0 \le \frac{\mathbf{G_I}}{Rv(1-\cos(c))}$$

Use the inequality above in (2.20) to get:

$$\mathbf{M^2} - \frac{\mathbf{G_I}}{Rv(1-\cos(c))} \le \liminf_{T\to\infty}\frac{1}{T}\int_0^T(\iiiint_{\Gamma(c,v)} f(x,\xi,t)f(x_0,\xi_0,t)\; dxd\xi dx_0 d\xi_0)dt \le \mathbf{M}^2$$

Since the estimate above is valid for an arbitrary $R > 0$ and because Theorem 2 proved $\mathbf{G_I}$ is bounded, we conclude:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \iiiint_{\Gamma(c,v)} f(x,\xi,t) f(x_0,\xi_0,t)\, dx d\xi dx_0 d\xi_0 dt = \mathbf{M^2} \quad \square$$

Data availability

No data was used for the research described in the article.

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