Title

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by
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and
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STRUCTURAL ENGINEERING LABORATORY
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Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering

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I. Introduction

In this study the response in a transversely isotropic, semi-infinite, elastic rod is found to an input on its end that is time dependent. The material of the rod is arranged so that axes of isotropy are parallel to the axis of the rod.

In developing the response, the method developed by Skalak [1] for isotropic rods is followed closely. Like Skalak, the exact three-dimensional theory of elasticity is used to solve the field equations, using the Laplace transform for the time variable and the sine and cosine transforms for the space variable along the rod, whichever is suitable. As the solutions, in the form of inverse integral transforms are complicated, asymptotic expansions are used which are for "large" values of the space coordinate and for the head of the pulse.

The asymptotic solution found in this paper is not new; it was obtained by Hensel and Curtis [2] for a single-crystal bar. However, because an approximate theory was used throughout, the solution of Hensel and Curtis cannot in the future be refined or improved. Because the solution in transform space developed here was obtained using the three-dimensional theory, the asymptotic solution obtained is open to improvement, albeit with difficulty, by extending the length of the fundamental spectral line used or even by taking into account higher modes.

The problem differs from that of Skalak, (apart from material properties), in the conditions that are imposed at the end of the rod. The choice here is to impose mixed-mixed conditions on the end: uniform normal stress together with zero radial displacement. The rationale behind this choice is that even though these conditions are unlikely to be imposed realistically, the influence of an additional condition needed to make the end condition realistic, (that is, a set of natural radial displacements on the end), would have little effect on the response at a distant station. This assumption is shared by Kaul and McCoy [3] in a study they made of the same problem for isotropic rods. The normal stress on the end of the rod has a step dependency on time. The response to this input is fundamental in that other solutions can be found from the solution contained here using the Duhamel integral.

With the end conditions described, the problem is one of finding the response in the rod. The choice here is to represent the response in terms of two strains: axial strain and tangential or "hoop" strain, both of which can be measured on the surface. Both are established at a station on arbitrarily "long" distance from the end of the rod at $z = z^*$. At such a distant station and for times close to the arrival of the first disturbance, both strains are given in terms of the Airy function.

The responses are shown in Fig. 1, and in the last section these responses are discussed in some detail.

II. Formulation of the Problem

Our study is of semi-infinite, cylindrical rod, of circular cross section made of a transversely isotropic elastic material whose axes of material symmetry are parallel to the axis of the rod.

We refer the rod to a cylindrical coordinate system (r, θ, z) within which the center of the end of the rod is located at the origin and positive z is measured along the axis of the rod. In the formulation of the problem, we employ the three dimensional theory of elasticity and choose mixed-mixed end boundary conditions; namely, uniform step pressure and zero radial displacement because they are mathematically simple to handle.

We now proceed to formulate the problem mathematically. The constitutive relation is

$$\tau_\alpha = c_{\alpha\beta}\ \epsilon_\alpha, \tag{1}$$

where

$$(c_{\alpha\beta}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix} \tag{2}$$

$$\begin{aligned} (\tau_\alpha) &= (\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \tau_{\theta z}, \tau_{zr}, \tau_{r\theta}) \\ (\epsilon_\alpha) &= (\epsilon_{rr}, \epsilon_{\theta\theta}\ \ \epsilon_{zz}, 2\epsilon_{\theta z}, 2\epsilon_{zr}, 2\epsilon_{r\theta}). \end{aligned} \tag{3}$$

Because of the axisymmetric nature of the problem we assume

$$\begin{aligned} u_r &= u_r(r, z, t) \\ u_z &= u_z(r, z, t) \\ u_\theta &\equiv 0, \end{aligned} \tag{4}$$

so that the strain-displacement relations in cylindrical coordinates become

$$\epsilon_{rr} = u_{r,r} \qquad \epsilon_{r\theta} \equiv 0$$
$$\epsilon_{\theta\theta} = \frac{u_r}{r} \qquad \epsilon_{rz} = \tfrac{1}{2}(u_{z,r} + u_{r,z}) \tag{5}$$
$$\epsilon_{zz} = u_{z,z} \qquad \epsilon_{\theta z} \equiv 0,$$

where $(\ )_{,r} = \dfrac{\partial(\ )}{\partial r}$ etc.

From Eqs. (1-4) we see that $\tau_{r\theta} = \tau_{\theta z} \equiv 0$ and that all of the other stresses are functions of r, z and t. Thus, the stress equations of motion without body forces become

$$\tau_{rr,r} + \tau_{rz,z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho\, \ddot{u}_r$$
$$\tau_{rz,r} + \tau_{zz,z} + \frac{\tau_{rz}}{r} = \rho\, \ddot{u}_z\ , \tag{6}$$

where $(\dot{\ }) = \dfrac{\partial(\ )}{\partial t}$. The third equation is satisfied identically. Equation (6) can be expressed in terms of displacements and, when we make use of stress-strain and strain-displacement relations, they are

$$c_{11}\left[\frac{1}{r}(r\, u_r)_{,r}\right]_{,r} + c_{44}\, u_{r,zz} + (c_{13} + c_{44})\, u_{z,rz} = \rho\, \ddot{u}_r$$
$$c_{44}\, \frac{1}{r}(r\, u_{z,r})_{,r} + c_{33}\, u_{z,zz} + (c_{13} + c_{44})\, \frac{1}{r}(r\, u_{r,z})_{,r} = \rho\, \ddot{u}_z . \tag{7}$$

We seek the solution of Eqs. (7) subject to the end boundary conditions

$$\tau_{zz}(o,\ r,\ t) = -\, P_o H(t)$$
$$u_r(o,\ r,\ t) = 0, \tag{8}$$

where P_o is a constant, and the boundary conditions on the lateral surface

$$\tau_{rr}(a, z, t) = 0$$
$$\tau_{rz}(a, z, t) = 0 \ . \qquad (9)$$

The initial conditions are

$$u_r(r, z, o) = \dot{u}_r(r, z, o) = 0$$
$$u_z(r, z, o) = \dot{u}_z(r, z, o) = 0, \qquad (10)$$

Using the constitutive relations, Eqs. (1), and the strain-displacement relations, Eqs. (5), the boundary conditions, Eqs. (8, 9), can be expressed in terms of displacements as

$$c_{33}\, u_{z,z} \Big|_{z=0} = - P_o H(t)$$
$$u_r \Big|_{z=0} = 0 \qquad (11)$$

and

$$c_{11}\, u_{r,r} \Big|_{r=a} + c_{12} \frac{u_r}{r} \Big|_{r=a} + c_{13}\, u_{z,z} \Big|_{r=a} = 0$$
$$u_{z,r} \Big|_{r=a} + u_{r,z} \Big|_{r=a} = 0. \qquad (12)$$

In the derivation of the first of Eqs. (11) we used the fact that

$$u_r \Big|_{z=0} = u_{r,r} \Big|_{z=0} = 0 \ .$$

III. Solutions in Transform Space

For finding the solution for the transient response of the rod, we make use of a double transform technique; namely, sine or cosine transform for axial distance and Laplace transform for time. First for the time variable t we apply Laplace transform to the governing equations, Eqs. (7), and end boundary conditions, Eqs. (11); thus we have respectively

$$c_{11}\left(\frac{1}{r}(r u_r^*)_{,r}\right)_{,r} + c_{44}\, u^*_{r,zz} + (c_{13} + c_{44})\, u^*_{z,rz} = \rho p^2\, \ddot{u}^*_r$$

$$c_{44}(r u^*_{z,r})_{,r} + c_{33}\, u^*_{z,z} + (c_{13} + c_{44})\, \frac{1}{r}(r\, u^*_{r,z})_{,r} = \rho p^2\, \ddot{u}^*_z \qquad (13)$$

and

$$u^*_r(r,\ o,\ p) = 0$$

$$c_{33}\, u^*_{z,z}(r,\ o,\ p) = -\frac{P_o}{p}\ , \qquad (14)$$

where the Laplace transform and its inverse are defined by

$$f^*(p) = \int_o^\infty f(t)\ e^{-pt}\ dt$$

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(p)\ e^{pt}\ dp. \qquad (15)$$

We note that in deriving Eqs. (13) we used the initial conditions, Eqs. (10).

For the "z" dependency, we apply sine and cosine transforms to the first and second of Eqs. (13), respectively, which gives

$$u^{*s}_{r,rr} + \frac{1}{r}\, u^{*s}_{r,r} - (m^2 + \frac{1}{r^2})\, u^{*s}_r - s\, u^{*c}_{z,r} = 0$$

$$u^{*c}_{z,rr} + \frac{1}{r}\, u^{*c}_{z,r} - n^2\, u^{*c}_z + q(u^{*s}_{r,r} + \frac{1}{r}\, u^{*s}_r) = -c_{44}\frac{P_o}{p}\ , \qquad (16)$$

where sine and cosine transforms and their inverses are defined by

$$f^s(\alpha) = \int_o^\infty f(x)\ \sin\alpha x\ dx$$

$$f(x) = \frac{2}{\pi}\int_o^\infty f^s(\alpha)\ \sin\alpha x\ d\alpha \qquad (17)$$

$$f^c(\alpha) = \int_o^\infty f(x)\ \cos\alpha x\ dx$$

$$f(x) = \frac{2}{\pi}\int_o^\infty f^c(\alpha)\ \cos\alpha x\ d\alpha, \qquad (18)$$

and

$$m^2 = \frac{\alpha^2 c_{44} + \rho p^2}{c_{11}} ; \qquad s = \frac{c_{13}+c_{44}}{c_{11}} \alpha ;$$

$$n^2 = \frac{\alpha^2 c_{33} + \rho p^2}{c_{44}} ; \qquad q = \frac{c_{13}+c_{44}}{c_{44}} \alpha . \tag{19}$$

In derivation of Eqs. (16) use is made of the end boundary conditions, Eqs. (14). The general solution of Eqs. (16) can be written as

$$u_r^{*s} = (u_r^{*s})_h + (u_r^{*s})_p$$
$$u_z^{*c} = (u_z^{*c})_h + (u_z^{*c})_p \tag{20}$$

where $(\ )_h$ and $(\ )_p$ denotes homogeneous and particular solutions respectively.

For the particular solution we assume the trial solution

$$(u_r^{*s})_p = Ar^2 + Br + C$$
$$(u_z^{*c})_p = Dr^2 + Er + F . \tag{21}$$

The constants introduced in Eqs. (21) can be determined from the condition that the assumed form of the particular solution must satisfy Eqs. (16); thus we obtain

$$(u_r^{*s})_p = 0$$
$$(u_z^{*c})_p = \frac{P_o}{p(\alpha^2 c_{33} + \rho p^2)} . \tag{22}$$

For the homogeneous solution we choose the trial solution

$$(u_r^{*s})_h = A\ I_1(hr)$$
$$(u_z^{*c})_h = B\ I_0(hr) , \tag{23}$$

where I_1 and I_0 are modified Bessel functions of the first kind, and h is a constant to be determined. We note that the modified Bessel functions of second kind K_1 and K_0 are unacceptable on physical grounds because their values at r = 0 are infinite.

Substitution of Eqs. (23) into the homogeneous part of Eqs. (16) gives

$$\begin{bmatrix} h^2-m^2 & -sh \\ qh & h^2-n^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{24}$$

For a non trivial solution, the determinant of the coefficient matrix of Eq. (24) must vanish. This determines the values of h. We also note that the coefficients A and B are related through the eigenvectors of Eq. (24). Thus, the homogeneous solution becomes

$$\begin{aligned} (u_r^{*s})_h &= A\ I_1(fr) + B\ I_1(\ell r) \\ (u_z^{*c})_h &= A\left(\frac{f^2-m^2}{sf}\right) I_o(fr) + B\left(\frac{\ell^2-m^2}{s\ell}\right) I_o(\ell r), \end{aligned} \tag{25}$$

where

$$\left.\begin{aligned} h_1^2 &= f^2 \\ h_2^2 &= \ell^2 \end{aligned}\right| = \tfrac{1}{2}\{(m^2 + n^2 - qs) \pm [(m^2+n^2-qs)^2-4m^2n^2]^{\frac{1}{2}}\}. \tag{26}$$

Combining homogeneous and particular solutions, we obtain the general solution

$$\begin{aligned} u_r^{*s} &= AI_1(fr) + BI_1(\ell r) \\ u_z^{*c} &= A\ \frac{f^2-m^2}{sf}\ I_o(fr) + B\ \frac{\ell^2-m^2}{s\ell}\ I_o(\ell r) \\ &\quad + \frac{P_o}{p(\alpha^2 c_{33} + \rho p^2)} . \end{aligned} \tag{27}$$

The constants A and B in Eqs. (27) will be determined from the lateral boundary conditions, Eq. (12). Upon applying Laplace and sine transforms to the first of Eqs. (12), and Laplace and cosine transforms to the second of Eqs. (12), the lateral boundary conditions take the forms:

$$c_{11}\, u^{*s}_{r,r}\Big|_{r=a} + c_{12}\frac{u^{*s}_r}{r}\Big|_{r=a} - \alpha c_{13}\, u^{*c}_z\Big|_{r=a} = 0$$
$$u^{*c}_{z,r}\Big|_{r=a} + \alpha\, u^{*s}_r\Big|_{r=a} = 0. \qquad (28)$$

Substituting Eq. (27) into (28) and solving for A and B we find

$$A = \frac{d_{22}}{\Delta}\,\alpha c_{13}\,\frac{P_o}{p(\alpha^2 c_{33}+\rho p^2)}$$
$$B = -\frac{d_{21}}{\Delta}\,\alpha c_{13}\,\frac{P_o}{p(\alpha^2 c_{33}+\rho p^2)}, \qquad (29)$$

where

$$\Delta = d_{11}\, d_{22} - d_{12}\, d_{21}$$
$$d_{11} = (c_{11}f - c_{13}\,\alpha\,\frac{f^2-m^2}{sf})\, I_o(fa) + \frac{c_{12}-c_{11}}{a}\, I_1(fa)$$
$$d_{12} = (c_{11}\ell - c_{13}\,\alpha\,\frac{\ell^2-m^2}{s\ell})\, I_o(\ell a) + \frac{c_{12}-c_{11}}{a}\, I_1(\ell a) \qquad (30)$$
$$d_{21} = (\frac{f^2-m^2}{s} + \alpha)\, I_1(fa)$$
$$d_{22} = (\frac{\ell^2-m^2}{s} + \alpha)\, I_1(\ell a).$$

We note that if p is replaced by $(i\omega)$, then $\Delta = 0$ will be the same as the frequency equation for transversely isotropic rods. Finally, substituting Eqs. (29) into Eqs. (27) we find

$$u_r^{*s} = \frac{\alpha c_{13} P_o}{p(\alpha^2 c_{33}+\rho p^2)\Delta} \{ (\frac{\ell^2-m^2}{s} + \alpha) I_1(\ell a) I_1(fr) - (\frac{f^2-m^2}{s} + \alpha) I_1(fa) I_1(\ell r) \}$$

$$u_z^{*c} = \frac{\alpha c_{13} P_o}{p(\alpha^2 c_{33}+\rho p^2)\Delta} \{ (\frac{\ell^2-m^2}{s} + \alpha) (\frac{f^2-m^2}{sf}) I_1(\ell a) I_o(fr) - (\frac{f^2-m^2}{s} + \alpha) (\frac{\ell^2-m^2}{s\ell}) I_1(fa) I_0(\ell r) \} + \frac{P_o}{p(\alpha^2 c_{33}+\rho p^2)} . \tag{31}$$

In anticipation of possible comparison with experimental results, ϵ_{zz} and $\epsilon_{\theta\theta}$ are chosen as a measure of the response. In transform space they are

$$\epsilon_{\theta\theta}^{*s} = \frac{u_r^{*s}}{r}$$

$$\epsilon_{zz}^{*s} = - \alpha u_z^{*c} . \tag{32}$$

Finally, using inversion formulas we find

$$\epsilon_{\theta\theta} = \frac{1}{i\pi^2} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \epsilon_{\theta\theta}^{*s} \sin \alpha z \, e^{pt} \, d\alpha dp$$

$$\epsilon_{zz} = \frac{1}{i\pi^2} \int_{c-i\infty}^{c+i\infty} \int_0^{\infty} \epsilon_{zz}^{*s} \sin \alpha z \, e^{pt} \, d\alpha dp , \tag{33}$$

where $\epsilon_{\theta\theta}^{*s}$ and ϵ_{zz}^{*s} are given by Eqs. (32).

IV. An Asymptotic Solution

It is not possible in general to integrate Eqs. (33). In this section we obtain an asymptotic solution which is valid for large distances from the end of the rod and for the head of the pulse. Detailed solutions will be displayed only for $\epsilon_{\theta\theta}$.

First, we note that in Eqs. (30) we have

$$\Delta(-\alpha,p) = -\Delta(\alpha,p). \tag{34}$$

Then, from the first of Eqs. (32) and the first of Eqs. (31) it follows that

$$\epsilon^{*s}_{\theta\theta}(a,-\alpha,p) = -\epsilon^{*s}_{\theta\theta}(a,\alpha,p). \tag{35}$$

We have therefore,

$$\begin{aligned} \epsilon^{*}_{\theta\theta} &= \frac{2}{\pi}\int_{0}^{\infty} \epsilon^{*s}_{\theta\theta} \sin \alpha z d\alpha \\ &= -\frac{i}{\pi}\int_{-\infty}^{\infty} \epsilon^{*s}_{\theta\theta}\, e^{i\alpha z}\, d\alpha\,. \end{aligned} \tag{36}$$

Because we are seeking an asymptotic solution which is valid for large distances from the end of the rod and for the head of pulse we assume that $\alpha \ll 1$, $p \ll 1$. Then, from Eqs. (26) it follows that $f \ll 1$, $\ell \ll 1$. Hence, if we use the approximate formulas of I_1 and I_0 for small arguments, namely:

$$\begin{aligned} I_0(x) &\cong 1 + \frac{x^2}{4} \\ I_1(x) &\cong \frac{x}{2} + \frac{x^3}{16}\,, \end{aligned} \tag{37}$$

from the first of Eqs. (32) we obtain

$$\epsilon^{*s}_{\theta\theta} \cong c_{13}\, P_o\, \frac{\alpha}{p\Delta^1}\,, \tag{38}$$

where

$$\begin{aligned} \Delta^1 &= \bar{a}\alpha^2 + bp^2 + \frac{a^2}{4}(L\alpha^4 + Mp^2\alpha^2 + Np^4) \\ L &= \frac{1}{2}\,\bar{a}\,\bar{c} \\ M &= \frac{1}{2}\,\bar{c}\,b + \frac{1}{2}\,d\,\bar{a} + \rho c_{33} \end{aligned} \tag{39}$$

(cont'd)

$$N = \frac{1}{2} d\, b + \rho^2$$

$$\bar{a} = (c_{12} + c_{11})\, c_{33} - 2c_{13}^2$$

$$b = \rho(c_{12} + c_{11}) \qquad (39)\ \text{(cont'd)}$$

$$\bar{c} = \frac{c_{33}\, c_{11} - 2c_{13}\, c_{44} - c_{13}^2}{c_{11}\, c_{44}}$$

$$d = \frac{\rho(c_{11} + c_{44})}{c_{11}\, c_{44}} .$$

We substitute Eq. (38) into (36) and carry out integration with respect to α. We choose the semi-circle of radius R on the upper half of the α complex plane as the path of integration. It can be shown that the value of the integral along the semi-circle approaches zero as R tends to infinity. Then, using the residue theorem we obtain

$$\epsilon_{\theta\theta}^{*} \cong 2\, c_{13} \frac{P_o}{p} \left\{ \frac{\alpha}{\frac{\partial \Delta^1}{\partial \alpha}} e^{iaz} \right\}_{\alpha=\alpha_1(p)} \qquad (40)$$

where

$$\alpha_1(p) = \frac{ip}{v_e} \left(1 - \frac{a^2 \eta^2}{4v_e^2} p^2\right)$$

$$v_e = \left\{ \frac{c_{33}(c_{11} + c_{12}) - 2c_{13}^2}{\rho(c_{11} + c_{12})} \right\}^{\frac{1}{2}} \qquad (41)$$

$$\eta = \frac{c_{13}}{c_{11} + c_{12}} .$$

We also note that $\alpha = \alpha_1(p)$ is the simple pole of the integrand of Eq. (36) located on the upper half of the complex α plane for $\alpha \ll 1$, $p \ll 1$. After some manipulation, Eqs. (40) can be written as

$$\epsilon_{\theta\theta}^{*} \cong c_{13} \frac{P_o}{\bar{a} p} e^{\left(- \frac{zp}{v_e} + z \frac{a^2 \eta^2}{4v_e^3} p^3 \right)} . \qquad (42)$$

Using the inversion formula for Laplace transform, the second of Eqs. (15), we obtain

$$\epsilon_{\theta\theta} = \frac{c_{13}\ P_o}{\bar{a}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{p}\, e^{\left((t - \frac{z}{v_e})p + z \frac{a^2\eta^2}{4v_e^3} p^3\right)} dp \tag{43}$$

When integration with respect to p is carried out, we find

$$\epsilon_{\theta\theta} = \frac{c_{13}\ P_o}{\bar{a}} (1/3 + \int_o^{B^1} A_i(-B)dB), \tag{44}$$

where

$$B^1 = \left(t - \frac{z}{v_e}\right)\left(\frac{4v_e^3}{3za^2\eta^2}\right)^{1/3} . \tag{45}$$

In Eq. (44), $A_i(x)$ denotes the Airy function [4] defined as

$$A_i(x) = \frac{1}{\pi}\int_o^{\infty} \cos\ (xy + \frac{y^3}{3})\ dy. \tag{46}$$

Similarly, it can be shown that

$$\epsilon_{zz} = - \frac{(c_{11} + c_{12})}{\bar{a}} P_o (1/3 + \int_o^{B^1} A_i(-B)dB) . \tag{47}$$

We note that the solutions found above, Eqs. (44) and (47), take into account only the contribution coming from the fundamental mode in the neighborhood of $\alpha = 0$ and also note that this neighborhood accommodates some dispersion.

V. Comments

With the choice of asymptotic solution, the major part of the response will come at a time corresponding to disturbances propagating with the bar velocity v_e. The "short" interval of time following this arrival, that is the interval of time not affected by the higher modes, is large enough to accommodate the time at which the maximum strains occur.

With the use of Airy function tables [4], the quantities $-\frac{\bar{a}}{(c_{11}+c_{12})P_o}\,\varepsilon_{zz}$ and $\left(\frac{\bar{a}}{c_{13}\,P_o}\right)\varepsilon_{\theta\theta}$, which are proportional to the axial and hoop strains respectively, are calculated and shown in Fig. 1. For a fixed station, say $z = z^*$, and at the time $t = \frac{z^*}{v_e}$, (the time of arrival of the major disturbance), the strains are equal to one third of their static values. The strains increase with increasing time until they reach a maximum value (approximately 1.27 times their static value) and then the magnitudes start to oscillate about their static values. We note that their maximum values will occur at approximately

$$\left(\bar{t} - \frac{\xi}{\hat{v}_e}\right)\left(\frac{4}{3\xi\eta^2}\right)^{1/3}\hat{v}_e = 2.3,$$

$$\left(\bar{t} - \frac{\xi}{\hat{v}_e}\right) = \frac{2.3}{\hat{v}_e}\left(\frac{3\eta^2}{4}\right)^{1/3}\xi^{1/3}\,, \tag{48}$$

where

$$\xi \text{ (dimensionless distance)} = \frac{z}{a}$$

$$\bar{t} \text{ (dimensionless time)} = \frac{tG_{ns}}{a}$$

$$G_{ns} = \left(\frac{c_{44}}{\rho}\right)^{\frac{1}{2}}, \text{ propagation velocity of shear waves}$$

$$\hat{v}_e = \frac{v_e}{G_{ns}}\,.$$

Given material constants $c_{\alpha\beta}$, $\hat{v}_e$ and η are known. They of course, affect the responses. What is more important to note from the second of Eqs. (48) is that the time at which the maximum strains occur is proportional to $z^{1/3}$. We therefore conclude that as the distance of the station from the end of rod decreases, the wave front becomes steeper.

Acknowledgement

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Captions for Figures

Fig. 1 Variations of Axial and Tangential Strains in Time.

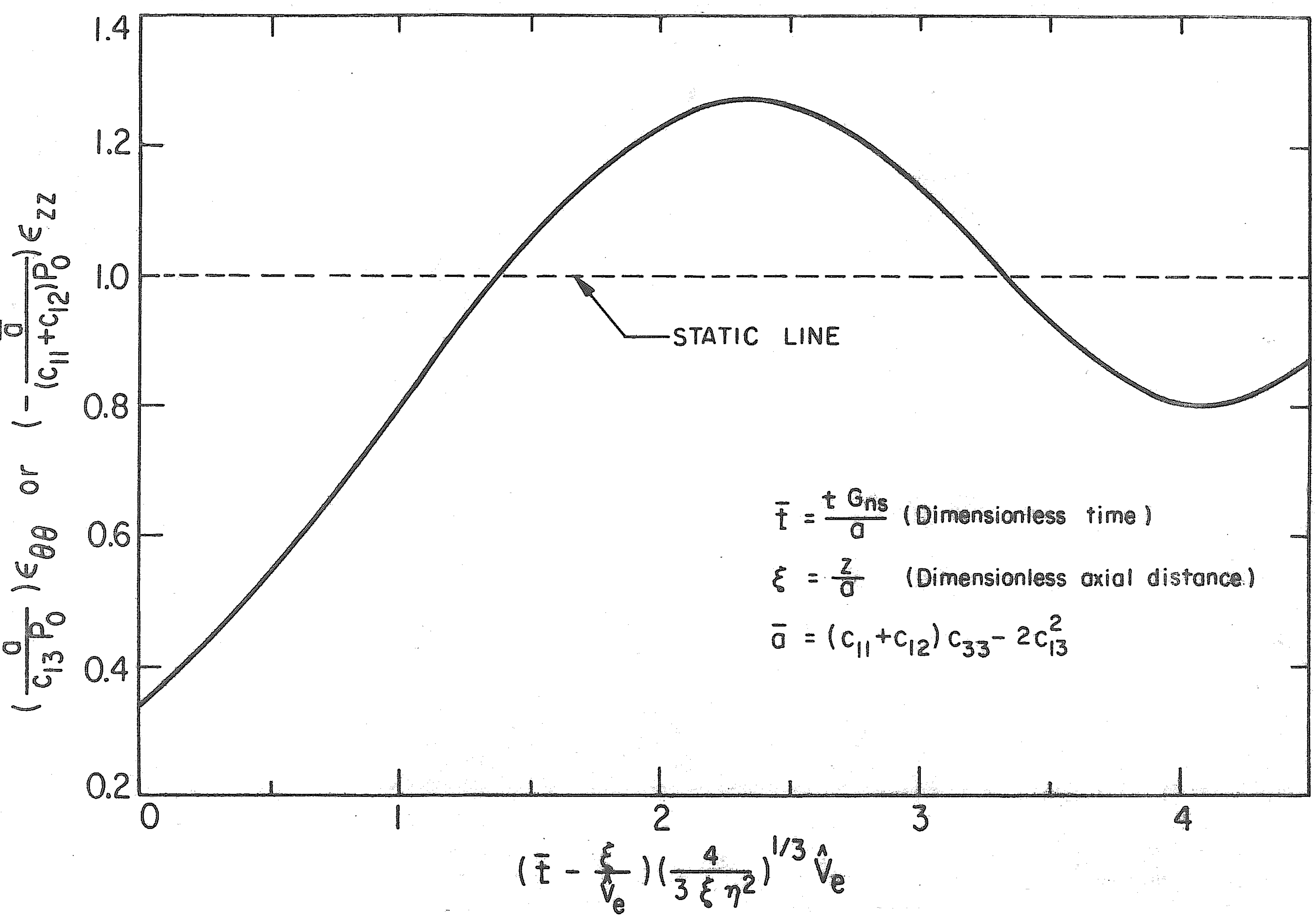
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
1
2
3
4
STATIC LINE
$\bar{t} = \frac{t\, G_{ns}}{a}$ (Dimensionless time)
$\xi = \frac{z}{a}$ (Dimensionless axial distance)
$\bar{a} = (c_{11}+c_{12})\,c_{33} - 2c_{13}^2$
$\left(\frac{\bar{a}}{c_{13} P_0}\right)\epsilon_{\theta\theta}$ or $\left(-\frac{\bar{a}}{(c_{11}+c_{12})P_0}\right)\epsilon_{zz}$
$\left(\bar{t} - \frac{\xi}{\hat{V}_e}\right)\left(\frac{4}{3\,\xi\,\eta^2}\right)^{1/3}\hat{V}_e$

FIG. 1