

UC Irvine

UC Irvine Electronic Theses and Dissertations

Title

Three-Mode Degeneracies in an RLC Circuit and a Nonlinear Coupled-Mode-Theory Model

Permalink

<https://escholarship.org/uc/item/4st750xk>

ISBN

9798193222231

Author

yan, weijian

Publication Date

2026-10-05

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA,
IRVINE

THREE-MODE DEGENERACIES IN AN RLC CIRCUIT AND A NONLINEAR
COUPLED-MODE-THEORY MODEL

THESIS

submitted in partial satisfaction of the requirements
for the degree of

MASTER OF SCIENCE
in Electrical and Computer Engineering

by

Weijian Yan

Thesis Committee:
Professor Filippo Capolino, Chair
Professor Ozdal Boyraz
Professor Hamidreza Aghasi

TABLE OF CONTENTS

	Page
LIST OF FIGURES	iv
LIST OF TABLES	v
ACKNOWLEDGMENTS	vi
ABSTRACT OF THE THESIS	vii
1 Introduction	1
1.1 Research Background	1
1.2 Scope	5
1.3 Thesis Organization	6
2 Linear Third-Order EPD in a Coupled-Resonator Circuit	7
2.1 Circuit Topology and Parameters	7
2.2 State-Space Formulation	9
2.3 Eigenvalue Problem	11
2.4 EPD ₃ Synthesis Constraints	13
2.5 Design Branches and Candidate Selection	14
2.6 Eigenfrequency Splitting Under Component Perturbations	17
2.7 Time-Domain Response in Linear Region	20
2.8 Frequency-Domain Response in the Linear Regime	21
3 Op-Amp NIC as a Negative Resistance	23
3.1 Ideal NIC as a Negative Resistance	24
3.2 DC Saturation of the NIC	26
3.3 Small-Signal Frequency Dependence	27
3.4 Saturated Oscillatory Region of the Three Coupled Resonator Circuit	30
4 Nonlinear Coupled-Mode Theory for Three Modes	34
4.1 Three Coupled Modes and the Synchronized State	34
4.2 Complex Determinant Condition	36
4.3 Invertibility of the Passive-mode Block	38
4.4 Elimination of the Saturated Gain	38
4.5 Polynomial Order and Gain Law	39
5 Sensitivity and Stability Analysis	41
5.1 Operational Definition and Numerical Design of SS-EPD ₅	41
5.2 Differential Sensitivity	43
5.3 Numerical Evaluation of Sensitivity	45

5.4	Local Transverse Linear Stability at a Nearby Operating Point	49
6	Conclusion	58
	REFERENCES	60

LIST OF FIGURES

	Page
1 Ideal three-resonator circuit used in EPD ₃ synthesis.	8
2 EPD ₃ branch.	16
3 Eigenfrequency trajectories on the lower- k branch.	18
4 Eigenfrequency trajectories on the higher- k branch.	19
5 Capacitance- and inductance-induced splitting near the triple root.	19
6 Linear time-domain response of the reference design.	20
7 Frequency-domain response of the selected ideal circuit.	22
8 Complete three-resonator circuit with an op-amp NIC.	24
9 External NIC topology and port variables.	25
10 NIC input conductance under DC saturation.	27
11 Small-signal NIC input admittance near the triple-root design frequency. . .	29
12 Complete-circuit ADS transient response with the output-limited NIC. . . .	30
13 Late-time voltage response and normalized spectra of the complete circuit. .	31
14 Effective NIC conductance G_{eff} and susceptance B_{eff} extracted from the late-time port-voltage and port-current spectra. The dashed line marks the oscillation frequency 1.178 MHz.	33
15 Scaled perturbed steady-state function $10^5 p(\omega, X)$ versus normalized oscillation frequency ω for negative, zero, and positive active-mode detuning X . At $X = 0$, the function has a fivefold root at $\omega = 0$. Negative and positive detuning shift the selected real root to negative and positive frequencies, respectively. The triangles indicate the corresponding real roots.	47
16 Fifth-root frequency shift on the positive real branch. The solid curve is calculated from $p(\omega, X) = 0$ using the candidate parameters. The dashed curve is Eq. (113).	48
17 Differential sensitivity on the positive real branch. The solid curve is calculated from Eq. (108) using the candidate parameters. The dashed curve is Eq. (114). The marker denotes $X = 10^{-5}$	48

LIST OF TABLES

	Page
1 Representative designs satisfying the algebraic EPD ₃ conditions for $L = 10 \mu\text{H}$ and $L_a = 12 \mu\text{H}$	16
2 Normalized parameters for the candidate SS-EPD ₅ design.	42
3 Centered finite-difference check at $X = 10^{-5}$	46

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Filippo Capolino, for his guidance and support throughout this work. I also thank Professor Ozdal Boyraz and Professor Hamidreza Aghasi for serving on my thesis committee and for their time and comments on the manuscript.

ABSTRACT OF THE THESIS

Three-Mode Degeneracies in an RLC Circuit and a Nonlinear Coupled-Mode-Theory Model

By

Weijian Yan

Master of Science in Electrical and Computer Engineering

University of California, Irvine, 2026

Professor Filippo Capolino, Chair

This thesis examines two distinct three-mode degeneracy problems. First, a third-order exceptional point of degeneracy (EPD_3) is synthesized in a linear three-resonator RLC circuit. Kirchhoff's laws yield a six-state model and two physical coupling branches. An op-amp negative-impedance converter (NIC) is then substituted for the ideal negative resistance. Finite gain, bandwidth, output resistance, loading, and saturation cause the simulated circuit to depart from the designed linear EPD_3 condition. In the transient simulation, the circuit develops a finite-amplitude oscillation with a dominant spectral peak near 1.178 MHz rather than at the designed frequency of 1.45 MHz.

Second, an independent normalized nonlinear coupled-mode-theory model with saturable gain is analyzed. Eliminating the modal amplitudes and saturated gain gives a real-fifth order steady-state frequency equation. A numerical design produces a fivefold real root, defined here as an SS-EPD_5 . Active-mode frequency detuning exhibits fifth-root frequency scaling. The stability of the steady state oscillation frequency is analyzed using the Lyapunov coefficients and a linearization. This leads to one neutral common-phase eigenvalue (Lyapunov coefficient) and five other eigenvalues with negative real parts, indicating local transverse linear stability of the steady state solution.

Chapter 1

Introduction

1.1 Research Background

Exceptional points are defective degeneracies of non-Hermitian systems. The terms exceptional point (EP) and exceptional point of degeneracy (EPD) are used in the literature to describe the same underlying singularity. At an order- m EPD, m eigenvalues coincide and their associated eigenvectors coalesce into a single eigenvector. The repeated eigenvalue has algebraic multiplicity m and geometric multiplicity one, and the corresponding operator contains a Jordan block of size m [1, 2]. This distinguishes an EPD from an ordinary degeneracy, at which repeated eigenvalues may retain linearly independent eigenvectors. For a perturbation that couples to the relevant Jordan chain, the leading eigenfrequency splitting can follow an m th-root dependence on the perturbation. EPDs with $m > 2$ are therefore commonly classified as higher-order EPDs.

Electrical resonator circuits provide a direct platform for constructing and studying EPDs. Kirchhoff's laws yield an explicit state matrix whose eigenvalues and eigenvectors can be calculated from the circuit parameters. Capacitances and inductances determine the uncoupled resonances and coupling strengths, while positive and negative resistances represent dissipation and amplification. The component values can consequently be adjusted

to locate a defective eigenvalue, and the resulting degeneracy can be examined through eigenfrequency trajectories, eigenvector correlations, time-domain responses, or impedance measurements. Circuit studies of higher-order degeneracies include finite coupled-resonator systems [5, 9], frequency-dependent impedance formulations [11], periodic electrical networks [12, 13], and nonlinear steady-state models [17–19, 23]. Although these studies use related terminology, their governing equations and the objects that coalesce are not necessarily the same.

A detailed treatment of a third-order circuit EPD was presented in the 2018 M.S. thesis of Alshetaiwi [5]. The circuit consisted of three capacitively coupled resonators: a central LC resonator and two outer RLC resonators providing amplification and attenuation. A six-dimensional state vector was formed from the three capacitor voltages and three inductor currents, and the corresponding state matrix was derived from the first-order circuit equations. By varying the gain–loss parameter, Alshetaiwi obtained two second-order EPDs and one third-order EPD in the eigenfrequency dispersion. The state-variable result was independently reproduced using a transverse-resonance formulation.

Eigenvector coalescence in that circuit was evaluated using the hyperdistance and complex correlation coefficients. The hyperdistance reached its minimum at the EPD_3 , while the magnitudes of the correlation coefficients approached unity as the eigenvectors became parallel. ADS simulations were then used to study the time-domain response. Away from the EPD, the circuit exhibited either bounded or exponentially growing oscillations, depending on whether the eigenfrequencies were real or complex. At the EPD_3 , the simulated voltage envelope followed a quadratic-in-time dependence even though the degenerate eigenfrequency was real, consistent with a length-three Jordan chain. The work was primarily analytical and numerical; physical implementation and the effects of component tolerances and thermal and temporal drift remained outside its scope [5].

Other finite resonator circuits have provided experimental evidence of higher-order EPDs. Yin *et al.* studied three inductively coupled RLC resonators containing one ampli-

fying and two dissipative resonators [9]. Their theoretical analysis and PCB measurements demonstrated an EP_3 , and a capacitive perturbation of the middle resonator produced the expected cube-root eigenfrequency response. The result showed that an experimentally observable EP_3 does not require a fully symmetric three-resonator configuration.

Higher-order degeneracies can also arise from governing equations that differ from a fixed finite-dimensional state matrix. Zhang *et al.* used frequency-dependent gain to obtain an EP_3 in a coupled-LC dimer and an EP_5 in a trimer [11]. These orders resulted from a frequency-dependent input-impedance condition and therefore exceeded the numbers of resonators. In periodic networks, Sloan, Othman, and Capolino predicted a fourth-order degenerate band edge in a passive LC double ladder [12], while Yazdi *et al.* experimentally reconstructed a sixth-order degenerate band edge from the transfer matrix of a triple-ladder microwave structure [13]. In these periodic systems, the coalescing states are Floquet–Bloch modes rather than eigenvectors of a finite circuit state matrix.

Nonlinear gain provides another route to higher-order singularities with relatively few resonators. Bai *et al.* showed theoretically and experimentally that saturable gain in two coupled resonators can produce a higher-order nonlinear exceptional nexus [17]. Stable and unstable steady-state branches coalesced at the nexus, and the frequency response followed different root laws for different perturbation directions, including a cube-root dependence for resonance detuning. In a related three-resonator model, one stable and four auxiliary steady states coalesced at a fifth-order nonlinear exceptional point, producing a fifth-root frequency response [19]. A later two-resonator experiment directly observed a nonlinear exceptional point with a third-root frequency response [18]. Unlike a conventional linear EPD, the instantaneous Hamiltonian at this nonlinear singularity retained a complete eigenbasis. These results show that the order of a nonlinear EPD may refer to coalescing steady-state branches rather than to eigenvectors of a fixed linear operator.

Bradshaw, Hakimi, and Capolino developed a steady-state EPD formulation for two coupled oscillators with saturable nonlinear gain [23]. Using coupled-mode theory, they ana-

lyzed a third-order steady-state EPD (SS-EPD₃) associated with cubic-root-like oscillation-frequency sensitivity. Their study considered the root law together with saturated gain, stored energy, stability, bistability, and the perturbation range over which high sensitivity could be maintained. Increasing the loss extended the high-sensitivity range while reducing the stored energy, and the useful cubic-root-like regime occurred under weak coupling and near, rather than exactly at, the SS-EPD₃. The formulation was also applied to electronic circuits, illustrating practical limitations of the steady-state sensitivity model [23].

Higher-order degeneracies have also been investigated in photonic systems. Hodaei *et al.* experimentally realized an EP₃ in three coupled microring resonators and measured a cube-root spectral response [6]. Periodic coupled-resonator optical waveguides have likewise been shown to support a third-order stationary inflection point, a fourth-order degenerate band edge, and a sixth-order degenerate band edge [7, 8]. These examples demonstrate the broader applicability of higher-order degeneracies, although their governing eigenvalue problems differ from those of finite lumped circuits.

The term “higher-order EPD” therefore covers several mathematically distinct situations. The coalescing objects may be eigenmodes of a fixed circuit state matrix, Floquet–Bloch states of a periodic transfer matrix, roots of a frequency-dependent impedance equation, or branches of a nonlinear steady-state problem. A comparison of EPD orders must consequently identify the governing operator or equation, the states or roots that coalesce, the perturbation channel, and the resulting root law. Moreover, enhanced eigenfrequency splitting does not by itself establish improved measurement resolution, which also depends on linewidth, noise, drift, calibration, and the frequency-estimation procedure [14–16]. These distinctions provide the background for the linear circuit EPD and nonlinear steady-state EPD studied in this thesis.

1.2 Scope

This thesis contains two independent three-mode studies under the common theme of higher-order degeneracy. The first examines a conventional linear circuit EPD_3 , whereas the second examines root multiplicity and stability in a nonlinear steady-state problem. The two models address related questions, but they are not successive stages of one model, and no quantitative parameter mapping between them is assumed.

The linear study covers the synthesis and analysis of an ideal three-resonator EPD_3 and the deviations introduced when the ideal negative resistance is replaced by an op-amp NIC. The NIC analysis includes DC saturation, small-signal frequency dependence, and complete-circuit transient behavior. This part does not claim that the saturated NIC circuit retains an exact EPD_3 and does not include experimental validation.

The nonlinear study covers synchronized steady states, derivation of a reduced quintic frequency equation, numerical construction of a fivefold real root, its response to active-mode detuning, and local stability of a nearby synchronized state. Here, SS-EPD_5 denotes the multiplicity of the reduced scalar steady-state root and not a conventional matrix EP_5 .

For the linear three-resonator circuit, this thesis derives the algebraic EPD_3 synthesis conditions and shows that fixing the outer-resonator inductance L and the center-resonator inductance L_a produces two finite coupling branches. The eigenfrequency, time-domain, and frequency-domain responses of a selected design are then characterized. The analysis further examines how DC saturation and frequency dependence of an op-amp NIC move the complete circuit away from the ideal triple-root design.

For the nonlinear three-mode CMT model, this thesis derives a gain-eliminated scalar quintic frequency polynomial and identifies a parameter set whose coefficients reproduce the prescribed fivefold-root target to the reported numerical tolerance. For the target polynomial, active-mode detuning produces fifth-root scaling of the synchronized-frequency shift. Along the neighboring simple-root branch, the absolute differential sensitivity increases as the branch approaches the fivefold root, whereas the ordinary derivative is undefined at the

multiple root itself. At a selected nearby operating point, the stability Jacobian has one neutral common-phase eigenvalue and five eigenvalues with negative real parts, indicating local linear stability transverse to the common-phase direction.

1.3 Thesis Organization

Chapter 2 develops the three-resonator RLC model, derives the algebraic EPD₃ synthesis conditions, and examines the resulting eigenfrequency splitting and the ideal circuit response in the time and frequency domains.

Chapter 3 introduces the op-amp NIC implementation. It develops static saturation and small-signal frequency-dependent models and then examines how replacing the ideal negative resistance affects the complete circuit response.

Chapter 4 develops a separate normalized nonlinear three-mode CMT model and derives its scalar steady-state frequency function.

Chapter 5 evaluates the fifth-root perturbation response of the SS-EPD₅ design and the local linear stability of a nearby synchronized state. Chapter 6 summarizes the results and limitations.

Chapter 2

Linear Third-Order EPD in a Coupled-Resonator Circuit

2.1 Circuit Topology and Parameters

Figure 1 shows three capacitively coupled LC resonators. The node voltages are v_1 , v_a , and v_2 . The shunt capacitors are C_1 , C_a , and C_2 , and the shunt inductors are L_1 , L_a , and L_2 . The capacitor C_{12} connects nodes 1 and a , C_{23} connects nodes a and 2, and C_{13} connects nodes 1 and 2. Node 1 is loaded by $-R$, node a has no resistive load, and node 2 is loaded by R , where $R > 0$.

Parity-time symmetry combines spatial reflection with time reversal [3]. The component values are chosen as

$$C_1 = C_a = C_2 = C, \quad L_1 = L_2 = L, \quad C_{12} = C_{23} = C_{13} = C_c. \quad (1)$$

Parity exchanges the two outer resonators and leaves the center resonator unchanged. Time reversal changes t to $-t$, leaves the voltage signs unchanged, and reverses the current signs.

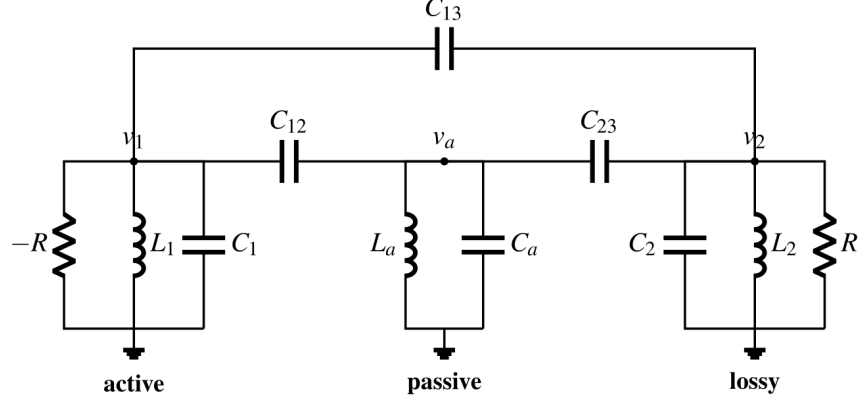


Figure 1: Ideal three-resonator circuit with balanced outer loads $-R$ and R and a passive center resonator. The nominal synthesis sets $C_{12} = C_{23} = C_{13} = C_c$.

With the equal and opposite outer conductances $-1/R$ and $+1/R$, the combined operation leaves the nominal circuit equations unchanged. The nominal circuit is therefore $\mathcal{PT}$ -symmetric [3].

PT symmetry is used because balanced gain and loss can be tuned to a point where modes coalesce [3]. The center resonator supplies a third resonant degree of freedom. The three frequencies coincide only when the remaining circuit parameters also satisfy the algebraic conditions derived in Section 2.4.

Define the dimensionless coupling parameter

$$k \triangleq \frac{C_c}{C + 3C_c}. \quad (2)$$

Solving Eq. (2) for C_c gives

$$C_c = \frac{kC}{1 - 3k}. \quad (3)$$

For $C > 0$ and finite $C_c > 0$, the physical range is $0 < k < 1/3$. At the lower boundary, $k = 0$ corresponds to $C_c = 0$, for which the resonators are uncoupled. The upper boundary

is approached as $C_c \rightarrow \infty$, for which $k \rightarrow 1/3$. Therefore, $k = 1/3$ cannot be realized with a finite coupling capacitor.

The shunt conductances at the active and lossy outer nodes are $-1/R$ and $+1/R$, respectively. Define the gain-loss parameter

$$\alpha \triangleq \frac{1}{RC}. \quad (4)$$

The parameter α has units of s^{-1} . Equivalently, the resistance value is

$$R = \frac{1}{\alpha C}. \quad (5)$$

2.2 State-Space Formulation

The state vector contains the three node voltages and the three shunt-inductor currents. Each inductor current is positive from its node to ground:

$$\Psi(t) \triangleq \begin{bmatrix} v_1(t) & v_a(t) & v_2(t) & i_1(t) & i_a(t) & i_2(t) \end{bmatrix}^T. \quad (6)$$

No independent excitation is applied. Define

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) & v_a(t) & v_2(t) \end{bmatrix}^T, \quad \mathbf{i}(t) = \begin{bmatrix} i_1(t) & i_a(t) & i_2(t) \end{bmatrix}^T. \quad (7)$$

Taking currents leaving each node as positive, Kirchhoff's current law gives

$$\mathbf{C}_{\text{net}} \frac{d}{dt} \mathbf{v}(t) + \mathbf{G} \mathbf{v}(t) + \mathbf{i}(t) = \mathbf{0}. \quad (8)$$

Here, $\mathbf{C}_{\text{net}}$ and $\mathbf{G}$ are the nodal capacitance and conductance matrices. For the node ordering $(1, a, 2)$,

$$\mathbf{G} = \text{diag}\left(-\frac{1}{R}, 0, \frac{1}{R}\right). \quad (9)$$

The nodal capacitance matrix is

$$\mathbf{C}_{\text{net}} = \begin{bmatrix} C + 2C_c & -C_c & -C_c \\ -C_c & C + 2C_c & -C_c \\ -C_c & -C_c & C + 2C_c \end{bmatrix}. \quad (10)$$

A coupling capacitor carries $C_c(\dot{v}_j - \dot{v}_k)$ from node j to node k . Using the nominal capacitances in Eq. (1) and omitting the time argument, Eq. (8) becomes

$$\begin{aligned} \text{node 1 : } & (C + 2C_c)\dot{v}_1 - C_c\dot{v}_a - C_c\dot{v}_2 - \frac{v_1}{R} + i_1 = 0, \\ \text{node } a : & -C_c\dot{v}_1 + (C + 2C_c)\dot{v}_a - C_c\dot{v}_2 + i_a = 0, \\ \text{node 2 : } & -C_c\dot{v}_1 - C_c\dot{v}_a + (C + 2C_c)\dot{v}_2 + \frac{v_2}{R} + i_2 = 0. \end{aligned} \quad (11)$$

The inductor equations are

$$\frac{di_1}{dt} = \frac{1}{L}v_1, \quad \frac{di_a}{dt} = \frac{1}{L_a}v_a, \quad \frac{di_2}{dt} = \frac{1}{L}v_2. \quad (12)$$

Equations (8) and (12) give

$$\frac{d}{dt}\Psi(t) = \mathbf{M}\Psi(t), \quad (13)$$

where the state matrix has the block form

$$\mathbf{M} = \begin{bmatrix} -\mathbf{C}_{\text{net}}^{-1} \mathbf{G} & -\mathbf{C}_{\text{net}}^{-1} \\ \mathbf{L}^{-1} & \mathbf{0} \end{bmatrix}, \quad \mathbf{L} = \text{diag}(L, L_a, L). \quad (14)$$

Using Eq. (2), the inverse of $\mathbf{C}_{\text{net}}$ is

$$\mathbf{C}_{\text{net}}^{-1} = \frac{1}{C} \begin{bmatrix} 1-2k & k & k \\ k & 1-2k & k \\ k & k & 1-2k \end{bmatrix}. \quad (15)$$

Substituting Eqs. (9) and (15) into Eq. (14) gives

$$\mathbf{M} = \begin{bmatrix} \alpha(1-2k) & 0 & -\alpha k & -\frac{1-2k}{C} & -\frac{k}{C} & -\frac{k}{C} \\ \alpha k & 0 & -\alpha k & -\frac{k}{C} & -\frac{1-2k}{C} & -\frac{k}{C} \\ \alpha k & 0 & -\alpha(1-2k) & -\frac{k}{C} & -\frac{k}{C} & -\frac{1-2k}{C} \\ \frac{1}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{L_a} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{L} & 0 & 0 & 0 \end{bmatrix}. \quad (16)$$

2.3 Eigenvalue Problem

Assume a modal solution of Eq. (13) of the form $\Psi(t) = \Psi_0 e^{\lambda t}$, where $\Psi_0 \neq \mathbf{0}$ is constant.

Substitution gives

$$(\mathbf{M} - \lambda \mathbf{I}_6) \Psi_0 = \mathbf{0}. \quad (17)$$

Here, $\mathbf{I}_6$ is the 6×6 identity matrix, λ is an eigenvalue, and Ψ_0 is the corresponding eigenvector. Let $j \triangleq \sqrt{-1}$. Define the generally complex angular eigenfrequency ω_n by

$$\lambda_n = j\omega_n, \quad n = 1, \dots, 6. \quad (18)$$

If $\omega_n = \omega_{r,n} + j\omega_{i,n}$, the time factor is $e^{j\omega_{r,n}t - \omega_{i,n}t}$. Thus, $\omega_{i,n} > 0$ denotes decay, whereas $\omega_{i,n} < 0$ denotes growth.

An exceptional point of degeneracy is a repeated eigenvalue at which the corresponding eigenvectors also coalesce. At an EPD₃, three eigenvalues coincide and only one independent eigenvector remains. For this circuit, the target is a triple eigenvalue at $\lambda_e = j\omega_e$, where $\omega_e > 0$ is real. The EPD₃ requirement therefore has two parts: λ_e must be a triple root of the characteristic equation, and this repeated eigenvalue must have only one independent eigenvector.

Define the characteristic polynomial

$$P(\lambda) \triangleq \det(\mathbf{M} - \lambda\mathbf{I}_6). \quad (19)$$

Equation (17) has a nonzero solution if and only if $P(\lambda) = 0$, which is the dispersion equation. Substituting Eq. (16) gives the even characteristic polynomial

$$P(\lambda) = \lambda^6 + b_4\lambda^4 + b_2\lambda^2 + b_0, \quad (20)$$

where

$$\begin{aligned} b_4 &= \alpha^2(-3k^2 + 4k - 1) + \frac{1 - 2k}{C} \left(\frac{1}{L_a} + \frac{2}{L} \right), \\ b_2 &= -\frac{\alpha^2(1 - 3k)^2}{CL_a} + \frac{6k^2 - 8k + 2}{C^2LL_a} + \frac{3k^2 - 4k + 1}{C^2L^2}, \\ b_0 &= \frac{(1 - 3k)^2}{C^3L^2L_a}. \end{aligned} \quad (21)$$

The coefficients are real, and $P(\lambda)$ contains only even powers of λ . Therefore, if λ is a root, then $-\lambda$, λ^* , and $-\lambda^*$ are also roots with the same multiplicity. In particular, a triple root at $\lambda_e = j\omega_e$ is accompanied by a triple root at $-\lambda_e = -j\omega_e$.

The conditions for λ_e to have algebraic multiplicity at least three are

$$P(\lambda_e) = P'(\lambda_e) = P''(\lambda_e) = 0, \quad (22)$$

where the primes denote differentiation with respect to λ . These conditions make λ_e at least a triple root. Because $P(\lambda)$ is even, $-\lambda_e$ is also at least a triple root. The two roots are distinct because $\omega_e > 0$, and together they account for all six roots of $P(\lambda)$. Hence,

$$P(\lambda) = (\lambda - j\omega_e)^3(\lambda + j\omega_e)^3 = (\lambda^2 + \omega_e^2)^3. \quad (23)$$

Comparing Eqs. (20) and (23) gives

$$b_4 = 3\omega_e^2, \quad b_2 = 3\omega_e^4, \quad b_0 = \omega_e^6. \quad (24)$$

For the polynomial in Eq. (20), these three relations are equivalent to the algebraic triple-root condition. They are necessary but not sufficient for an EPD₃ because they do not establish eigenvector coalescence. Circuit simulations are subsequently used to examine the corresponding coalescence behavior.

2.4 EPD₃ Synthesis Constraints

Since $\omega_e > 0$, define dimensionless constraint functions for the three relations in Eq. (24):

$$F_4 \triangleq \frac{\alpha^2(-3k^2 + 4k - 1)}{\omega_e^2} + \frac{1 - 2k}{C\omega_e^2} \left(\frac{1}{L_a} + \frac{2}{L} \right) - 3. \quad (25)$$

$$F_2 \triangleq -\frac{\alpha^2(1-3k)^2}{CL_a\omega_e^4} + \frac{6k^2-8k+2}{C^2LL_a\omega_e^4} + \frac{3k^2-4k+1}{C^2L^2\omega_e^4} - 3. \quad (26)$$

$$F_0 \triangleq \frac{(1-3k)^2}{C^3L^2L_a\omega_e^6} - 1. \quad (27)$$

The algebraic triple-root conditions are equivalent to

$$F_4 = F_2 = F_0 = 0. \quad (28)$$

Using the expression for b_0 in Eq. (21), $F_0 = 0$ gives

$$\omega_e^6 = \frac{(1-3k)^2}{C^3L^2L_a}. \quad (29)$$

Because $1-3k > 0$ in the physical interval $0 < k < 1/3$, the positive real solution is

$$\omega_e = \frac{(1-3k)^{1/3}}{\sqrt{C}(L^2L_a)^{1/6}}. \quad (30)$$

For fixed k , L , and L_a , Eq. (30) gives $\omega_e \propto C^{-1/2}$.

2.5 Design Branches and Candidate Selection

For the numerical designs, set

$$L = 10\mu\text{H}, \quad L_a = 12\mu\text{H}. \quad (31)$$

Solving $F_0 = 0$ gives ω_e . Substituting this result into $F_4 = 0$ gives α^2 . After these two variables are eliminated, $F_2 = 0$ becomes $F'_2 = 0$. Here, the prime does not denote differen-

tiation. The dimensionless function F'_2 is

$$F'_2 \triangleq \frac{864}{25}(1-k)^2 - 27 \left(\frac{6}{5}\right)^{4/3} (1-k)(1-3k)^{1/3} - \frac{153}{5}(1-2k) + 27 \left(\frac{6}{5}\right)^{2/3} (1-3k)^{2/3}. \quad (32)$$

Each root of $F'_2(k) = 0$ in $0 < k < 1/3$ defines a candidate branch. Because C does not appear in this equation, it remains free. For each root, the corresponding recovered values of α and ω_e are substituted together with k into F_0 , F_4 , and F_2 . Only candidates for which both recovered values are real and positive are retained. For each valid root and any chosen $C > 0$, the eliminated variables are

$$\alpha = \left[\frac{\frac{17}{5}(1-2k) - 3 \left(\frac{6}{5}\right)^{2/3} (1-3k)^{2/3}}{L_a C (1-3k)(1-k)} \right]^{1/2}, \quad (33)$$

$$\omega_e = \frac{(1-3k)^{1/3}}{\sqrt{C} [(10 \mu\text{H})^2 (12 \mu\text{H})]^{1/6}}.$$

The positive square root gives $\alpha > 0$, and the expression for ω_e gives the positive-frequency solution.

The equation $F'_2(k) = 0$ has two roots in $0 < k < 1/3$ that give real, positive α :

$$k_1(\text{lower} - k) \approx 0.2189824530513445, \quad k_2(\text{higher} - k) \approx 0.2874008258571359. \quad (34)$$

For either fixed branch $k = k_i$, the coupling capacitance and resistance vary with C as

$$C_c = \frac{k_i}{1-3k_i} C, \quad (35)$$

$$R = \left[\frac{L_a(1-3k_i)(1-k_i)}{C \left[\frac{17}{5}(1-2k_i) - 3 \left(\frac{6}{5}\right)^{2/3} (1-3k_i)^{2/3} \right]} \right]^{1/2}, \quad i = 1, 2.$$

Thus, $C_c \propto C$ and $R \propto C^{-1/2}$ on each branch.

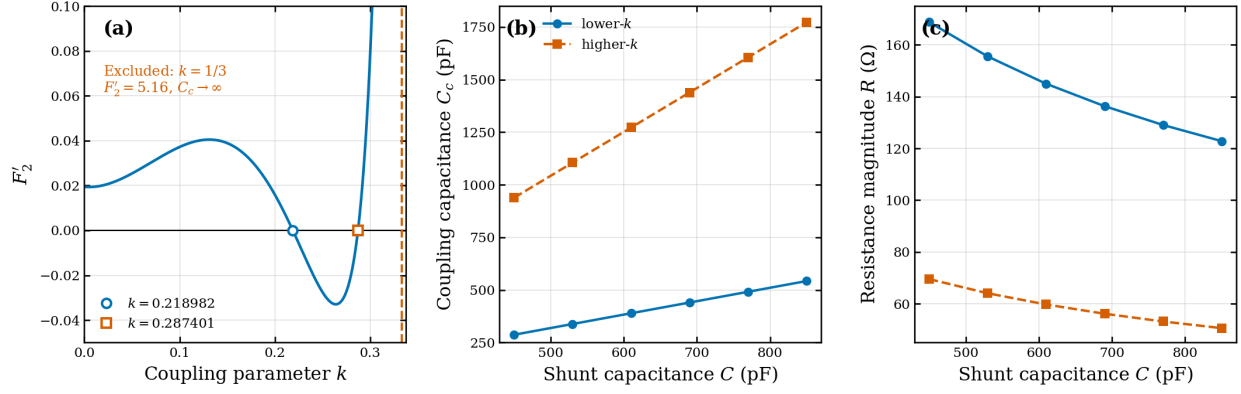


Figure 2: Algebraic EPD₃ synthesis branches for $L = 10 \mu\text{H}$ and $L_a = 12 \mu\text{H}$. (a) The function F'_2 from Eq. (32) (b) Coupling capacitance C_c and (c) resistance R versus the free capacitance C .

Table 1: Representative designs satisfying the algebraic EPD₃ conditions for $L = 10 \mu\text{H}$ and $L_a = 12 \mu\text{H}$.

Case	Branch	k	C (pF)	C_c (pF)	R (Ω)	f_e (MHz)
1	lower- k	0.218982	553	353	152.3	1.45
2	lower- k	0.218982	600	383	146.2	1.3953
3	higher- k	0.287401	712	1485	55.3	0.9451
4	higher- k	0.287401	747	1558	54.0	0.9227

Figure 2(a) plots the function F'_2 and identifies its two valid roots. Figures (b) and (c) show C_c and R , respectively, along the lower- and higher- k branches as the free capacitance C is varied. Representative designs from both branches are listed in Table 1.

For the following circuit design, the lower- k root $k = k_1 \approx 0.218982$ is selected. Because C remains a free scaling parameter on this branch, it is set to 553 pF, which gives a coupling capacitance close to 353 pF. The linear reference design satisfying the algebraic triple-root conditions is

$$\begin{aligned}
C &= 553 \text{ pF}, & C_c &\approx 353 \text{ pF}, \\
L &= 10 \mu\text{H}, & L_a &= 12 \mu\text{H}, \\
R &\approx 152.27 \Omega.
\end{aligned} \tag{36}$$

Its triple-root angular frequency is $\omega_e \approx 9.13 \times 10^6$ rad/s, or

$$f_e \triangleq \frac{\omega_e}{2\pi} \approx 1.45 \text{ MHz.} \quad (37)$$

2.6 Eigenfrequency Splitting Under Component Perturbations

For each design, define the fractional center-capacitance perturbation

$$\varepsilon_C \triangleq \frac{C_a - C}{C}. \quad (38)$$

The horizontal axes in Figures 3 and 4 show $100\varepsilon_C$ in percent. The two figures present cases 1–2 and 3–4 of Table 1, respectively, with all other components fixed at their designed values. For each value of C_a , $\mathbf{C}_{\text{net}}$ is rebuilt and inverted before $\mathbf{M}$ is recomputed from Eq. (14).

At $\varepsilon_C = 0$, the three branches with $\text{Re } \omega > 0$ coalesce at the real eigenfrequency ω_e , for which $\text{Im } \omega = 0$. When $\varepsilon_C \neq 0$, the three eigenfrequencies separate. In each design, one branch remains on the real- ω axis, while the other two form a complex-conjugate pair with equal real parts and opposite imaginary parts. The real parts give the oscillation angular frequencies; under the $e^{j\omega t}$ convention of Eq. (18), positive and negative imaginary parts denote decay and growth, respectively.

For the design on the selected lower- k branch, define the fractional center-inductance perturbation

$$\varepsilon_L \triangleq \frac{L_a - L_{a0}}{L_{a0}}. \quad (39)$$

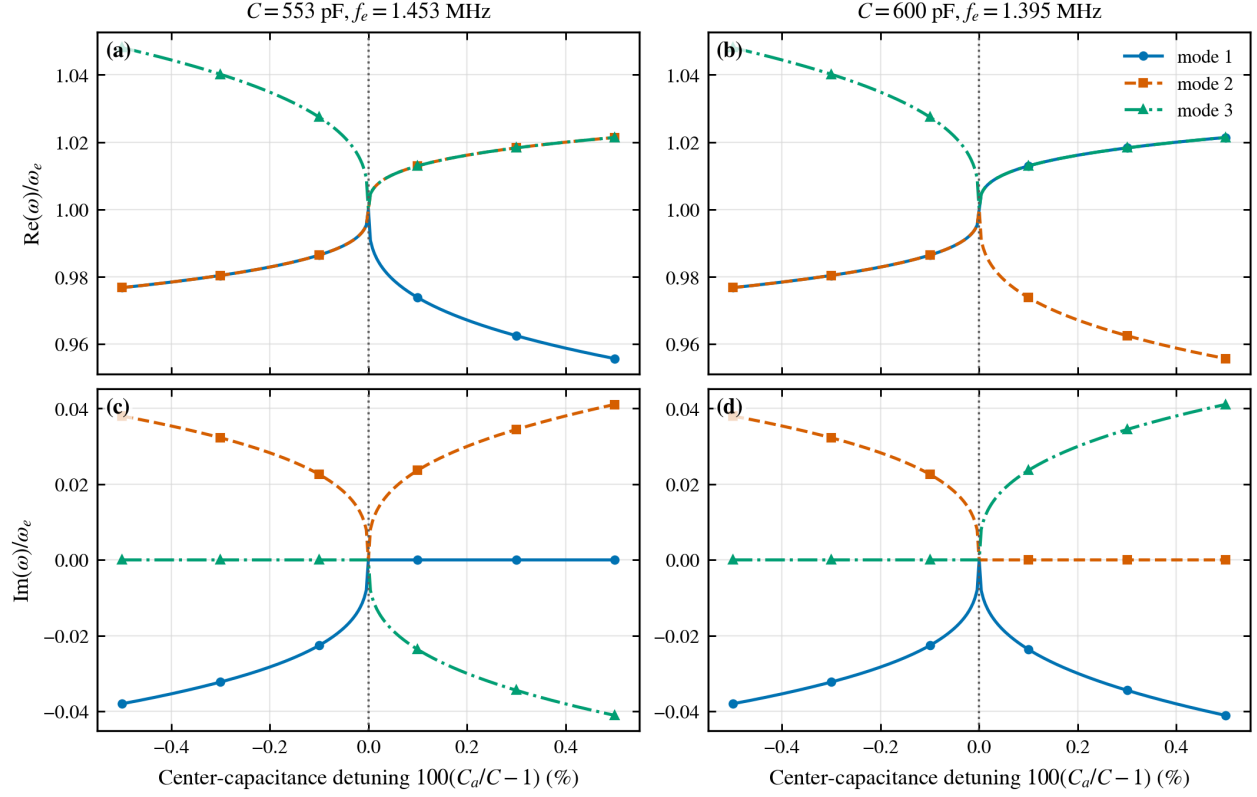


Figure 3: The three eigenfrequency branches with $\text{Re } \omega > 0$ for cases 1 and 2 of Table 1, computed using the designed parameters under the center-capacitance perturbation ε_C defined in Eq. (38). The top and bottom rows show $\text{Re } \omega/\omega_e$ and $\text{Im } \omega/\omega_e$, respectively, over $-0.5\% \leq \varepsilon_C \leq 0.5\%$. The legend entries “mode 1”–“mode 3” denote eigenfrequency branches continued away from the degeneracy, not individual resonators.

Here, $L_{a0} = 12 \mu\text{H}$ is the nominal center inductance. Figure 5 compares the effects of ε_C and ε_L , each swept from -0.2% to 0.2% , with all other components fixed at the values of the design. For the C_a sweep, $\mathbf{C}_{\text{net}}$ is rebuilt and inverted at every perturbation value; the L_a sweep changes only the center entry of $\mathbf{L}^{-1}$. Because the two perturbations enter different blocks of Eq. (14), their eigenfrequency trajectories differ.

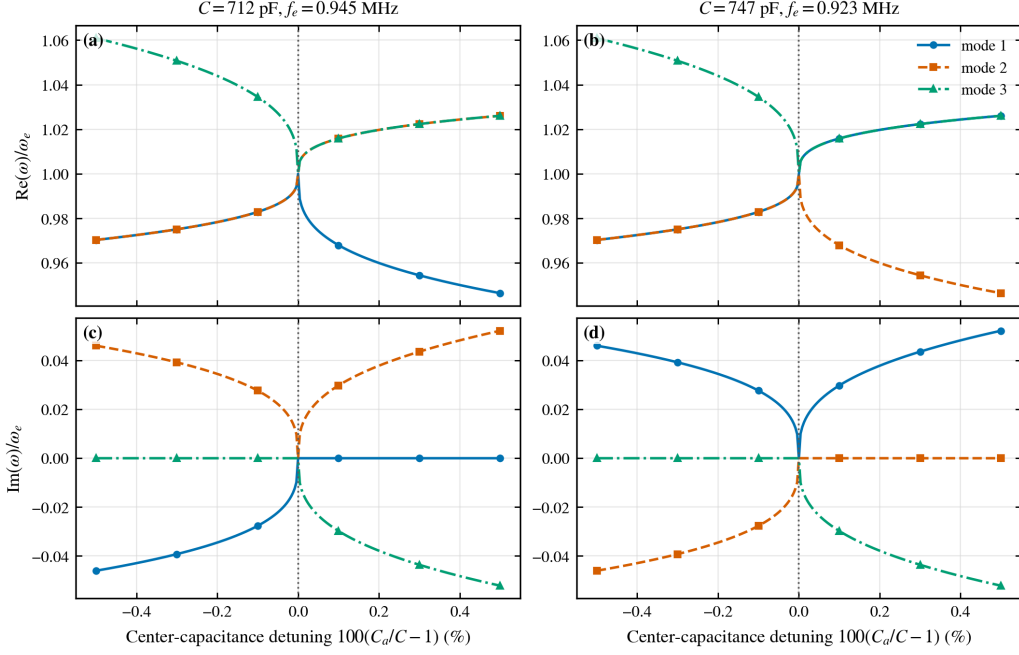


Figure 4: The three eigenfrequency branches with $\text{Re } \omega > 0$ on the higher- k branch, computed using the designed parameters. The left and right columns correspond to cases 3 and 4 of Table 1, respectively. The center-capacitance perturbation ε_C , defined in Eq. (38), spans $-0.5\% \leq \varepsilon_C \leq 0.5\%$. The top and bottom rows show $\text{Re } \omega / \omega_e$ and $\text{Im } \omega / \omega_e$, respectively.

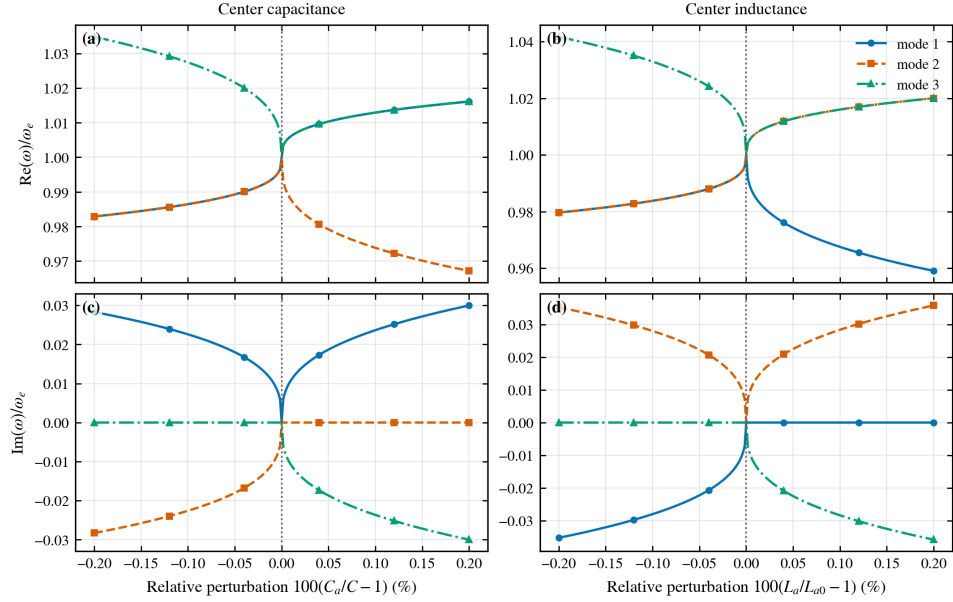


Figure 5: Eigenfrequency splitting for the design on the selected lower- k branch. The left and right columns sweep ε_C and ε_L , defined in Eqs. (38) and (39), respectively, from -0.2% to 0.2% . The top and bottom rows show $\text{Re } \omega / \omega_e$ and $\text{Im } \omega / \omega_e$, respectively, for the three branches with $\text{Re } \omega > 0$.

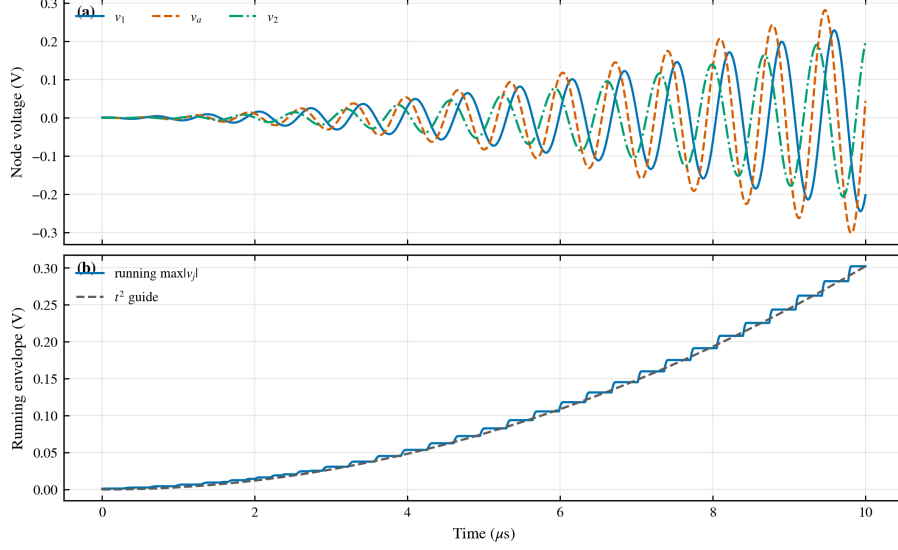


Figure 6: Time-domain response of the designed circuit, for the initial state $\Psi(0) = [1 \text{ mV}, 0, 0, 0, 0, 0]^T$. (a) The three node voltages over $0 \leq t \leq 10 \mu\text{s}$. (b) The running maximum $E(t)$. The dashed t^2 curve is amplitude-scaled for qualitative comparison.

2.7 Time-Domain Response in Linear Region

This section presents ADS transient simulations of the ideal circuit in Figure 1. Case 1 of Table 1 is used here.

Figure 6 shows the node-voltage response for

$$\Psi(0) = \begin{bmatrix} 1 \text{ mV} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T. \quad (40)$$

An EPD_3 can produce an oscillatory response with an envelope that grows approximately as t^2 [5]. Panel (a) of Figure 6 shows the three node voltages $v_1(t)$, $v_a(t)$, and $v_2(t)$. Although the initial voltage is applied only to the active node v_1 , the coupling between the three resonators causes all three node voltages to become nonzero. The three voltages oscillate at approximately the same frequency, while their different amplitudes and phases are determined by the coupled-circuit dynamics and the selected initial condition. Their oscillation amplitudes increase throughout the displayed interval. In panel (b), the running

maximum is defined as

$$E(t) \triangleq \max_{0 \leq \tau \leq t} \max_{m \in \{1, a, 2\}} |v_m(\tau)|.$$

The ADS waveforms in Figure 6 show a similar pattern: the node voltages oscillate with increasing amplitude, and $E(t)$ is close to the amplitude-scaled t^2 guide near the end of the plotted interval. The dashed curve is a visual guide, not a fitted model.

2.8 Frequency-Domain Response in the Linear Regime

The frequency-domain data presented in this section were obtained from a frequency sweep performed in ADS. In the simulation, a sinusoidal current phasor $I_a(\omega)$ is injected into the center node, and $V_m(\omega)$ denotes the resulting voltage phasor at node m . Here, the subscript a in I_a indicates that the excitation is applied at node a . The quantity $I_a(\omega)$ denotes the externally applied source current and should not be confused with the time-domain state variable $i_a(t)$ representing the center-inductor current.

At each sampled frequency, the transfer impedance is extracted from the ADS simulation data as

$$Z_{ma}(\omega) \triangleq \frac{V_m(\omega)}{I_a(\omega)}, \quad m \in \{1, a, 2\}. \quad (41)$$

In Z_{ma} , the first subscript m identifies the node at which the voltage is measured, whereas the second subscript a identifies the excited node. Thus, $V_1 = Z_{1a}I_a$, $V_a = Z_{aa}I_a$, and $V_2 = Z_{2a}I_a$, where the common frequency argument ω is omitted for brevity. Because the circuit is linear, the transfer-impedance ratios are independent of the nonzero excitation amplitude used in the ADS simulation.

Figure 7 shows the frequency response of the unrounded linear reference design. All three transfer-impedance magnitudes have a sharp peak near

$$f_e = \frac{\omega_e}{2\pi} \approx 1.45 \text{ MHz}. \quad (42)$$

The common peak frequency agrees with the triple-root frequency obtained from the eigenvalue calculation. The plotted phase of Z_{aa} , which represents the phase of V_a relative to the injected current I_a , changes by approximately 180° across the same frequency.

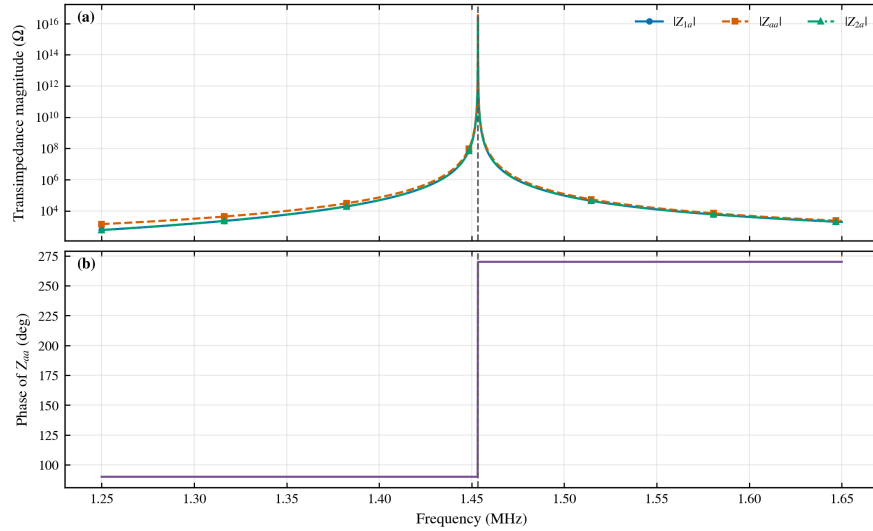


Figure 7: Transfer impedances of the designed circuit under center-node current excitation, extracted from ADS frequency-domain simulation data using Eq. (41) over $f \in [1.25, 1.65]$ MHz. (a) Magnitudes $|Z_{1a}|$, $|Z_{aa}|$, and $|Z_{2a}|$ on a logarithmic scale. (b) Phase of Z_{aa} relative to the injected current. The dashed line marks the algebraic triple-root frequency f_e .

In the ideal linear model, the transfer impedances have a pole at f_e , so no finite sinusoidal steady-state response exists exactly at that frequency. Since ADS evaluates the response at a discrete set of frequency points, the finite sampled maxima shown in the figure represent values close to, rather than exactly at, the pole. Their numerical values therefore depend on the frequency-grid spacing and numerical precision.

Chapter 3

Op-Amp NIC as a Negative Resistance

The linear synthesis assumes an ideal negative resistance $-R$ that is independent of frequency and signal amplitude. A practical op-amp negative-impedance converter (NIC) differs from this idealization because it has finite open-loop gain and bandwidth, finite output resistance and loading, and a limited output-voltage range. This chapter first establishes the ideal NIC port relation, then introduces a static output-limiting model and a frequency-dependent small-signal model, and finally examines the complete circuit with the nonideal NIC.

Figure 8 shows the three-resonator circuit after the ideal negative resistance is replaced by an op-amp NIC. The noninverting input is connected to v_1 . The resistor R connects v_1 to the output v_o , R_f connects v_o to the inverting input, and R_g connects that input to ground. The ADS implementation uses a generic behavioral op-amp model. Selected parameter values, including the open-loop gain, gain-bandwidth product, output resistance, and output-voltage limits, are chosen with reference to common values.

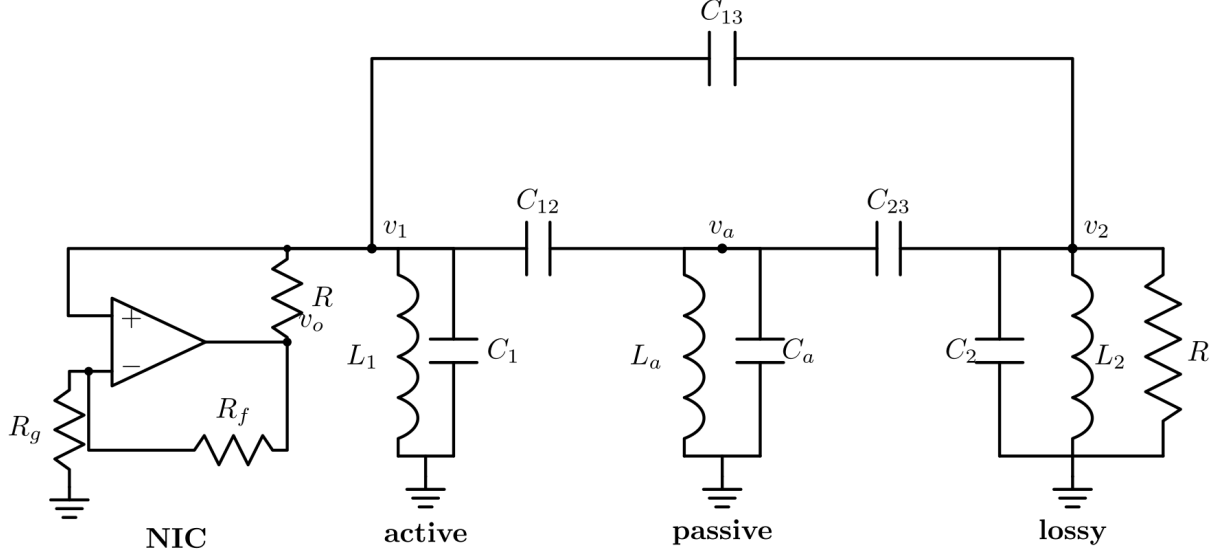


Figure 8: Complete three-resonator circuit with an op-amp NIC at the active node.

3.1 Ideal NIC as a Negative Resistance

Figure 9 shows the NIC circuit. All node voltages are measured with respect to ground. Let $v \equiv v_i$ denote the port voltage, and let v_o denote the NIC output voltage. Define i_{in} as positive into the NIC through the resistor R , from v_i toward v_o . The input admittance is

$$i_{\text{in}} = \frac{v - v_o}{R}, \quad Y_{\text{in}} \triangleq \frac{i_{\text{in}}}{v} = \frac{1 - v_o/v}{R}. \quad (43)$$

The input admittance is written as

$$Y_{\text{in}} = G_{\text{in}} + jB_{\text{in}}, \quad (44)$$

where G_{in} is the input conductance, B_{in} is the input susceptance, and $j = \sqrt{-1}$.

The feedback network consists of R_f between v_o and the inverting input and R_g between the inverting input and ground. Let v_+ and v_- denote the noninverting- and inverting-input

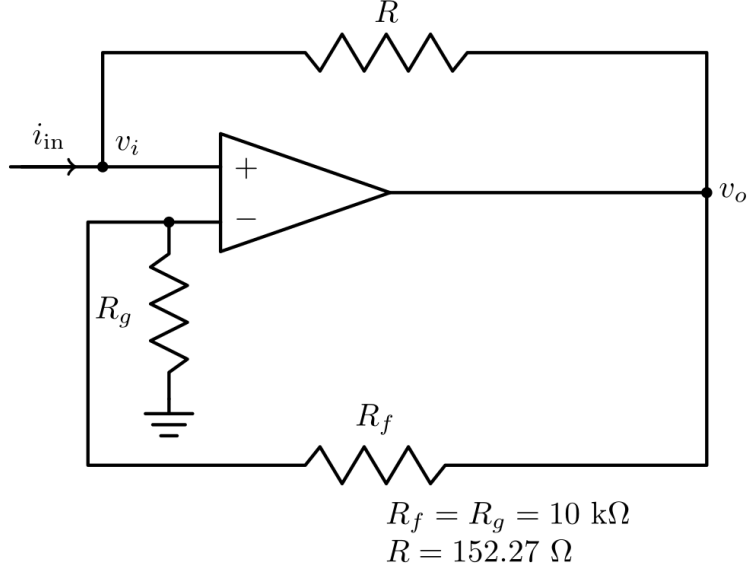


Figure 9: External NIC topology used to define the port variables. The schematic shows the external resistor and feedback network only. The internal controlled source v'_o and the finite output resistance R_{out} introduced in the small-signal model are analytical op-amp model elements and are not explicitly drawn.

voltages, respectively. For an ideal op-amp, the input currents are zero, so

$$v_- = \frac{R_g}{R_f + R_g} v_o. \quad (45)$$

With $R_f = R_g$, $v_- = v_o/2$. Under negative feedback, an ideal op-amp satisfies $v_+ = v_-$.

Since the noninverting input is connected to the port, $v_+ = v$, and therefore $v_o = 2v$.

Substitution into Eq. (43) gives

$$Y_{\text{in}} = G_{\text{in}} = -\frac{1}{R}, \quad B_{\text{in}} = 0, \quad Z_{\text{in}} \triangleq \frac{v}{i_{\text{in}}} = \frac{1}{Y_{\text{in}}} = -R. \quad (46)$$

For the design on the selected lower- k branch, the required NIC input impedance is

$$Z_{\text{in}} = -R \approx -152.268 \text{ }\Omega. \quad (47)$$

3.2 DC Saturation of the NIC

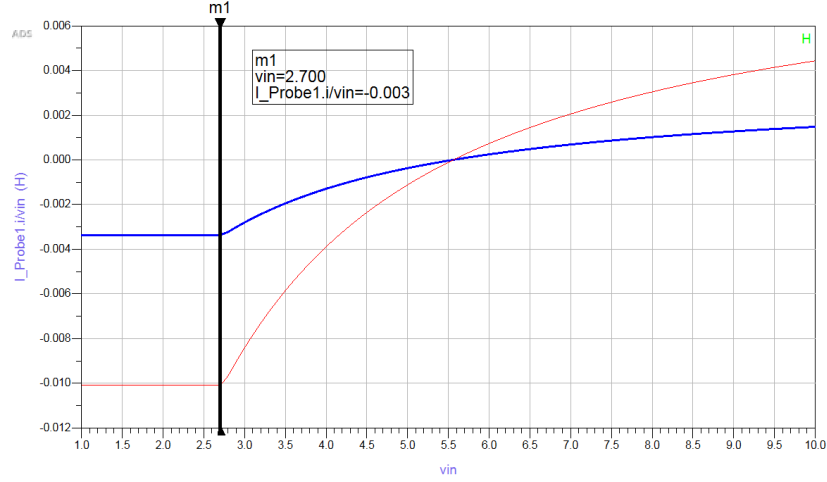
The static model assumes symmetric output limits $\pm V_{\text{lim}}$, with $V_{\text{lim}} \approx 5.5$ V. In the hard-limit model, the unsaturated closed-loop gain is two, and output loading and dynamic effects are omitted. Thus, $v_o = 2v$ for $|v| < V_{\text{lim}}/2$ and $v_o \simeq \text{sgn}(v)V_{\text{lim}}$ for $|v| \geq V_{\text{lim}}/2$, where $\text{sgn}(v)$ denotes the sign of v . At DC, $B_{\text{in}} = 0$, so $Y_{\text{in}} = G_{\text{in}}$. Substitution into Eq. (43) gives

$$G_{\text{in}}(|v|) \simeq \begin{cases} -1/R, & |v| < V_{\text{lim}}/2, \\ \frac{1}{R} \left(1 - \frac{V_{\text{lim}}}{|v|}\right), & |v| \geq V_{\text{lim}}/2. \end{cases} \quad (48)$$

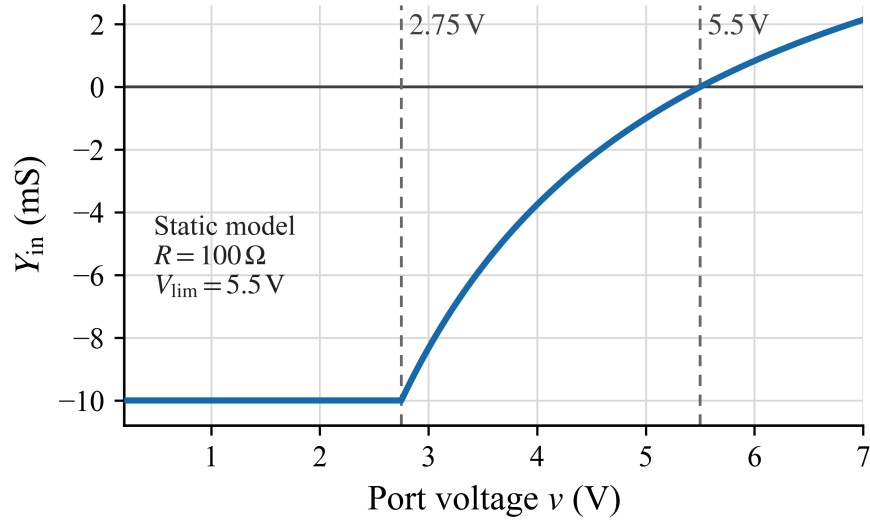
Equation (48) is continuous at the clipping threshold. For the static input magnitude $|v|$, it gives $G_{\text{in}} = -1/R$ up to $|v| = V_{\text{lim}}/2 \approx 2.75$ V, a zero crossing at $|v| = V_{\text{lim}} \approx 5.5$ V, and $G_{\text{in}} \rightarrow +1/R$ as $|v| \rightarrow \infty$. Once clipping begins, the NIC no longer acts as the constant negative resistance used in the linear EPD₃ design.

Figure 10 compares an ADS simulation DC sweep with Eq. (48). Panel (a) uses $R = 100 \, \Omega$ and $300 \, \Omega$, on either side of the selected value $R \approx 152.27 \, \Omega$; panel (b) shows the $100 \, \Omega$ model result. The low-input plateaus agree with $-1/R$, approximately -10 mS and -3.33 mS for the two resistances. After clipping begins, the ADS sweeps and the static-model curve begin to deviate from their nearly constant low-voltage values and cross zero near $|v| = V_{\text{lim}}$. Both panels show positive input voltage only.

The comparison is qualitative because the static model omits the op-amp and loading details present in the ADS sweep. Both show that the conductance leaves its negative constant value region after clipping.



(a) DC sweep(ADS default plotting)



(b) DC model calculation

Figure 10: Static NIC input conductance versus positive port voltage v . (a) ADS DC sweeps over $1 \text{ V} \leq v \leq 10 \text{ V}$ for $R = 100 \Omega$ and 300Ω . (b) Evaluation of Eq. (48) for $R = 100 \Omega$ and $V_{\text{lim}} = 5.5 \text{ V}$ over approximately $0.2 \text{ V} \leq v \leq 7 \text{ V}$.

3.3 Small-Signal Frequency Dependence

For frequency-domain small-signal analysis about an unsaturated operating point, the op-amp output is represented by a Thevenin model: an internal controlled voltage source $v'_o = A(\omega)(v_+ - v_-)$ drives the external output node v_o through the series output resistance R_{out} , where ω is angular frequency and $A(\omega)$ is the open-loop voltage gain. As noted in Figure 9, these two internal model elements are not explicitly drawn in the external NIC schematic.

For the numerical evaluation below, $R_{\text{out}} = 50 \Omega$ is adopted as a nominal small-signal model parameter. It is not an additional physical resistor in the feedback network. Assuming negligible current into the op-amp input terminals, the feedback divider gives

$$\beta \triangleq \frac{R_g}{R_f + R_g}, \quad v_- = \beta v_o, \quad (49)$$

where β is the feedback factor. Since $v_+ = v$, the internal output voltage is $v'_o = A(\omega)(v - v_-)$.

Here, the prime does not denote differentiation. KCL at the loaded output node gives

$$\frac{v_o - v'_o}{R_{\text{out}}} + \frac{v_o - v}{R} + \frac{v_o}{R_f + R_g} = 0. \quad (50)$$

Solving for the loaded closed-loop voltage ratio $H(\omega) \triangleq v_o/v$ yields

$$H(\omega) = \frac{A(\omega) + R_{\text{out}}/R}{1 + \beta A(\omega) + R_{\text{out}}/(R_f + R_g) + R_{\text{out}}/R}. \quad (51)$$

Using $i_{\text{in}} = (v - v_o)/R$, the corresponding input admittance is

$$Y_{\text{in}}(\omega) = \frac{1 - H(\omega)}{R} = G_{\text{in}}(\omega) + jB_{\text{in}}(\omega). \quad (52)$$

For $R_f = R_g$, $\beta = 1/2$. To evaluate Eq. (51), we adopt the single-pole open-loop model

$$\begin{aligned} A(\omega) &= \frac{A_0}{1 + j\omega/\omega_p}, & A_{0,\text{dB}} &\equiv 20 \log_{10} A_0 = 47.18 \text{ dB}, \\ A_0 &= 10^{47.18/20} \approx 228.6, & f_p &= \frac{f_t}{A_0}, & \omega_p &= 2\pi f_p. \end{aligned} \quad (53)$$

Here, A_0 is the dimensionless low-frequency open-loop voltage gain, $A_{0,\text{dB}}$ is its value expressed in decibels, f_p is the open-loop pole frequency, ω_p is the corresponding angular frequency, and f_t is the gain-bandwidth product. Within this model, the assumed gain-bandwidth product is $f_t = A_0 f_p = 2.7 \text{ GHz}$. This gives $f_p \approx 11.8 \text{ MHz}$. The remaining parameters are $R_f = R_g = 10 \text{ k}\Omega$, $R_{\text{out}} = 50 \Omega$, and $R \approx 152.268 \Omega$.

At the designed frequency $f_e \approx 1.45$ MHz, Eqs. (51) and (53) give

$$G_{\text{in}}(2\pi f_e) \approx -6.435 \text{ mS}, \quad B_{\text{in}}(2\pi f_e) \approx 0.0161 \text{ mS}. \quad (54)$$

For comparison, the ideal target values are

$$G_{\text{in}} = -\frac{1}{R} \approx -6.567 \text{ mS}, \quad B_{\text{in}} = 0. \quad (55)$$

The normalized deviations, evaluated with the unrounded values, are

$$\frac{|G_{\text{in}}(2\pi f_e) + 1/R|}{1/R} \approx 2.02\%, \quad \frac{|B_{\text{in}}(2\pi f_e)|}{1/R} \approx 0.245\%. \quad (56)$$

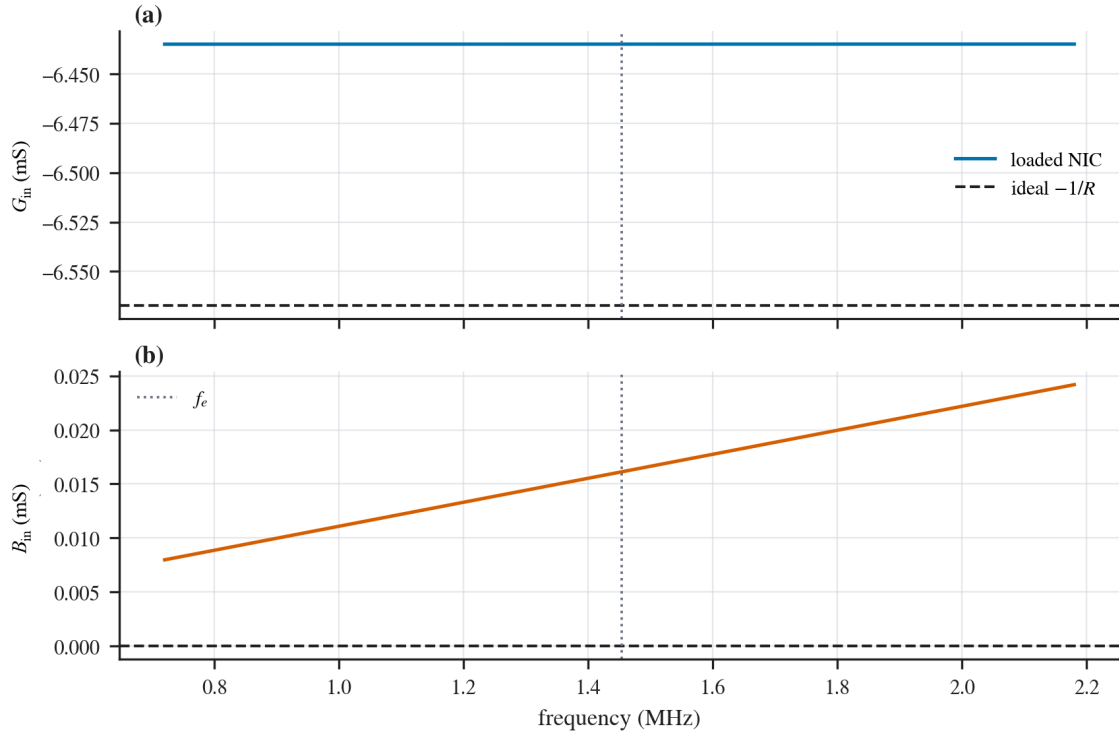


Figure 11: Input admittance calculated from the analytical single-pole small-signal model in Eqs. (51) and (53) over approximately 0.7–2.2 MHz using the parameters stated in the text.

3.4 Saturated Oscillatory Region of the Three Coupled Resonator Circuit

The complete circuit is simulated in ADS with the ideal negative resistance replaced by an output-limited op-amp NIC. The component values are

$$C = 553 \text{ pF}, \quad C_c = 353 \text{ pF}, \quad L = 10 \text{ } \mu\text{H}, \quad L_a = 12 \text{ } \mu\text{H}, \quad R = 152.2678 \text{ } \Omega. \quad (57)$$

Figure 12 shows the transient response. In this run, the NIC output v_o first reaches its clipping limit at $27.11 \text{ } \mu\text{s}$. The envelope of v_1 then grows and becomes nearly constant after approximately $50 \text{ } \mu\text{s}$, as shown in panel (a). Panel (b) shows v_1 , v_a , v_2 , and v_o during the final displayed $8 \text{ } \mu\text{s}$.

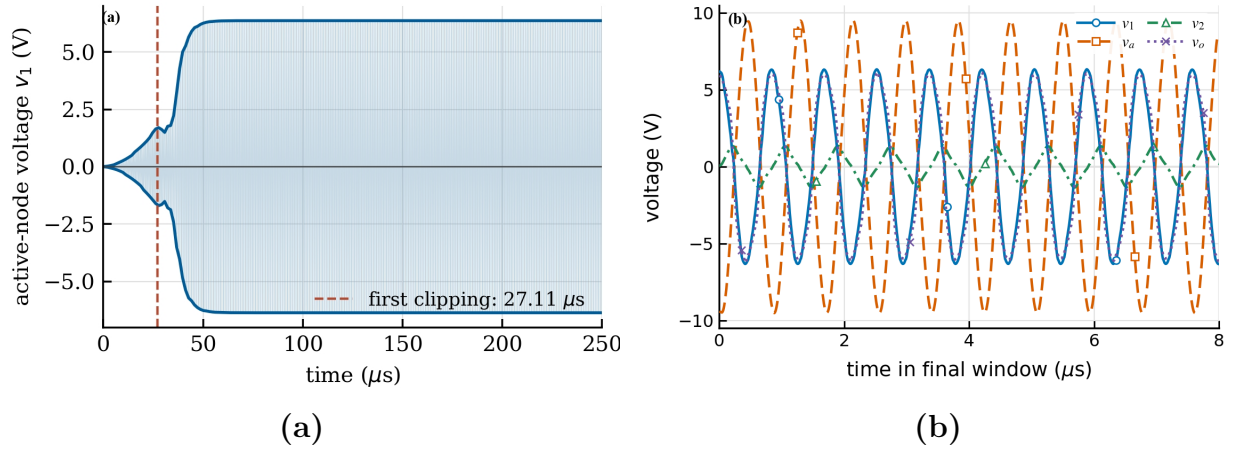
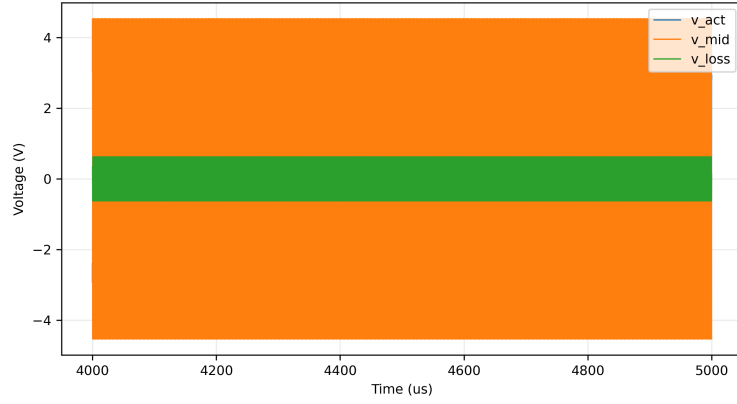
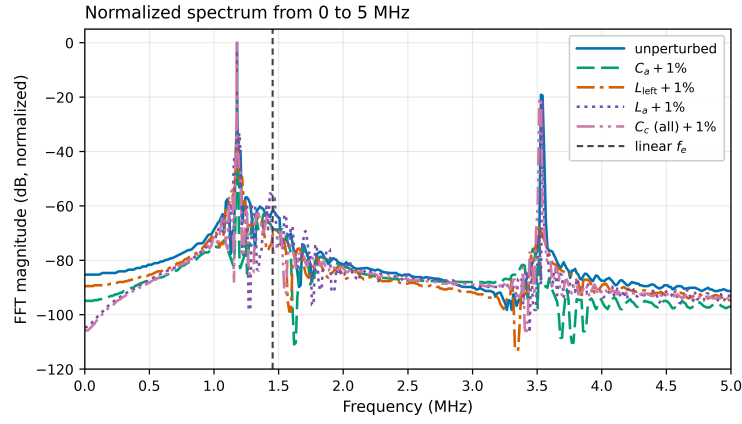


Figure 12: ADS transient response of the rounded circuit. (a) v_1 during the first $250 \text{ } \mu\text{s}$. The dashed line marks the first clipping of v_o at $27.11 \text{ } \mu\text{s}$. (b) v_1 , v_a , v_2 , and v_o during the final displayed $8 \text{ } \mu\text{s}$.

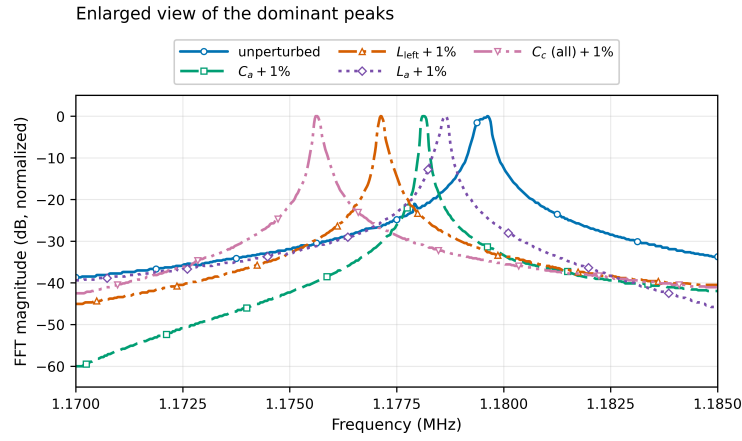
Figure 13 combines the late-time voltage response and its frequency spectra. Panel (a) shows the three resonator-node voltages from 4000 to $5000 \text{ } \mu\text{s}$. During this interval, the circuit exhibits a saturated periodic response.



(a) Late-time resonator-node voltages



(b) Spectrum from 0 to 5 MHz



(c) Enlarged view of the dominant peaks

Figure 13: Late-time voltage response and normalized spectra of the complete circuit. (a) Resonator-node voltages from 4000 to 5000 μs . (b) Active-node voltage spectra from 0 to 5 MHz for the unperturbed circuit and separate +1% changes in C_a , L_1 , L_a , and all three coupling capacitors C_c . The dashed line marks the linear EPD₃ design frequency f_e . (c) Enlarged view of the dominant peaks between 1.170 and 1.185 MHz.

The spectra in panels (b) and (c) were obtained from the active-node voltage v_1 over the same interval using the built-in ADS `fs()` function from 4000 to 5000 μs . Each spectrum was normalized to its own dominant peak. Panel (b) compares the unperturbed circuit with four separate +1% perturbations: C_a , L_1 , L_a , and all three coupling capacitors C_c . The quantity L_1 is labeled L_{left} in the figure. Panel (c) enlarges the frequency range around the dominant peaks.

The unperturbed circuit has its dominant peak near 1.178 MHz. All four component perturbations shift the peak to a lower frequency, as shown in Figure 13(c). Increasing all three coupling capacitors produces the largest shift. The weaker peak near 3.54 MHz in panel (b) is the third harmonic of the saturated oscillation.

The unperturbed oscillation frequency is approximately 0.275 MHz, or 18.95%, below the linear design frequency $f_e \approx 1.4534$ MHz. The different peak positions show the frequency shifts produced by the four 1% component perturbations.

The late-time NIC port voltage and current are transformed using the ADS `fs()` function from 4000 to 5000 μs . Their frequency-domain values are denoted by $V_{\text{sat}}(f)$ and $I_{\text{sat}}(f)$, respectively. Positive current is defined as flowing into the NIC. The effective NIC admittance is

$$Y_{\text{eff}}(f) = \frac{I_{\text{sat}}(f)}{V_{\text{sat}}(f)} = G_{\text{eff}}(f) + jB_{\text{eff}}(f), \quad (58)$$

where G_{eff} is the effective conductance and B_{eff} is the effective susceptance.

At the oscillation frequency of 1.178 MHz shown in Figure 13, the effective conductance and susceptance are approximately

$$G_{\text{eff}}(1.178 \text{ MHz}) \approx -0.1432 \text{ mS}, \quad B_{\text{eff}}(1.178 \text{ MHz}) \approx 0.56 \text{ mS}. \quad (59)$$

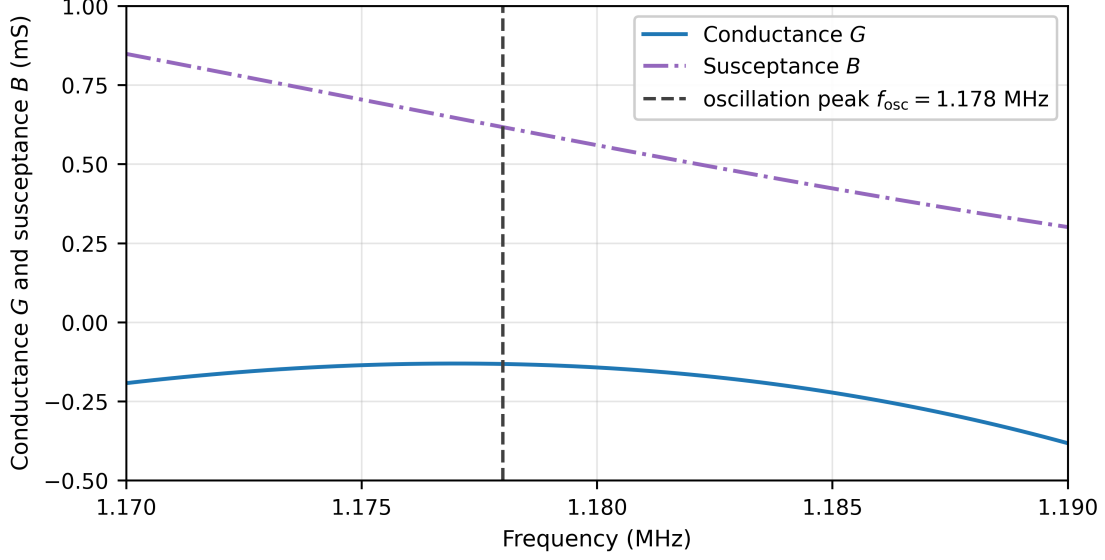


Figure 14: Effective NIC conductance G_{eff} and susceptance B_{eff} extracted from the late-time port-voltage and port-current spectra. The dashed line marks the oscillation frequency 1.178 MHz.

The smaller magnitude of G_{eff} shows that NIC saturation reduces the supplied negative conductance.

The previously evaluated small-signal frequency dependence of the NIC introduces only a limited admittance deviation near the design frequency. Therefore, the observed frequency shift is attributed primarily to the nonlinear saturation of the NIC.

Chapter 4

Nonlinear Coupled-Mode Theory for Three Modes

The earlier chapters use physical circuit variables, component values, and a six-state linear matrix, followed by a complete nonlinear ADS circuit when the NIC saturates. By contrast, this chapter introduces an independent, normalized three-mode coupled-mode-theory (CMT) model with an abstract saturable-gain law. The CMT study asks how we find exceptional point behavior when considering Nonlinearity.

4.1 Three Coupled Modes and the Synchronized State

All parameters in the following CMT model are normalized and dimensionless. Let $a_m(t)$ denote the complex dimensionless envelope amplitude of Mode m , where $m \in \{1, 2, 3\}$. Mode 1 is active. Its effective net gain rate is the real function $g(|a_1|^2)$, which depends on the active-mode amplitude square. A positive value of g represents amplification, and no separate linear loss term is assigned to this mode. Modes 2 and 3 are linear and passive, with loss rates $\gamma_2, \gamma_3 > 0$. In this normalized model, ω_m denotes the uncoupled modal frequency relative to the chosen frequency origin. The reciprocal coupling coefficients $\kappa_{mn} = \kappa_{nm}$ are real, where $m, n \in \{1, 2, 3\}$ and $m \neq n$. The variable t denotes normalized time. With these

conventions, the coupled-mode equations are

$$\frac{da_1}{dt} = [j\omega_1 + g(|a_1|^2)] a_1 - j\kappa_{12}a_2 - j\kappa_{13}a_3, \quad (60)$$

$$\frac{da_2}{dt} = -j\kappa_{12}a_1 + (j\omega_2 - \gamma_2) a_2 - j\kappa_{23}a_3, \quad (61)$$

$$\frac{da_3}{dt} = -j\kappa_{13}a_1 - j\kappa_{23}a_2 + (j\omega_3 - \gamma_3) a_3. \quad (62)$$

Eqs. (60)–(62) describe the temporal evolution of the three modal amplitudes. We seek synchronized steady-state solutions in which the three modes eventually oscillate at the same real normalized angular-frequency detuning ω , while their modal intensities and relative phases remain constant in time. Such a solution is represented by a nonzero constant complex vector $\tilde{\mathbf{a}} = [\tilde{a}_1, \tilde{a}_2, \tilde{a}_3]^T$ as

$$a_m(t) = \tilde{a}_m e^{j\omega t}, \quad m = 1, 2, 3. \quad (63)$$

Here, the tilde identifies a time-independent synchronized steady-state amplitude, and the superscript T denotes transpose. The subscript m identifies the resonator index. The non-linear gain at the synchronized steady state therefore takes the constant value $g_s \equiv g(|\tilde{a}_1|^2)$, where the subscript s denotes evaluation at the synchronized steady state. Substituting Eq. (63) into Eqs. (60)–(62) and cancelling the common factor $e^{j\omega t}$ gives

$$\omega \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \omega_1 - jg_s & -\kappa_{12} & -\kappa_{13} \\ -\kappa_{12} & \omega_2 + j\gamma_2 & -\kappa_{23} \\ -\kappa_{13} & -\kappa_{23} & \omega_3 + j\gamma_3 \end{bmatrix}}_{\mathbf{H}(g_s)} \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix}. \quad (64)$$

The matrix $\mathbf{H}(g_s)$ is the steady-state coupled-mode matrix.

4.2 Complex Determinant Condition

Equation (64) has a nonzero solution only when

$$\det[\omega \mathbf{I}_3 - \mathbf{H}(g_s)] = 0. \quad (65)$$

Here, $\mathbf{I}_3$ is the 3×3 identity matrix. For real ω , define the detuning from each uncoupled modal frequency as

$$\delta_m \equiv \omega - \omega_m, \quad m = 1, 2, 3. \quad (66)$$

The diagonal entries of $\omega \mathbf{I}_3 - \mathbf{H}(g_s)$ are then written as

$$x_1 = \delta_1 + jg_s, \quad x_2 = \delta_2 - j\gamma_2, \quad x_3 = \delta_3 - j\gamma_3. \quad (67)$$

Direct expansion gives

$$\begin{aligned} \det[\omega \mathbf{I}_3 - \mathbf{H}(g_s)] &= x_1 x_2 x_3 - \kappa_{23}^2 x_1 - \kappa_{13}^2 x_2 \\ &\quad - \kappa_{12}^2 x_3 + 2\kappa_{12}\kappa_{13}\kappa_{23}. \end{aligned} \quad (68)$$

The lower-right 2×2 submatrix associated with the two passive modes is

$$\mathbf{B}_p(\omega) \equiv \begin{bmatrix} x_2 & \kappa_{23} \\ \kappa_{23} & x_3 \end{bmatrix}. \quad (69)$$

Its determinant is

$$Y(\omega) \equiv \det \mathbf{B}_p(\omega) = x_2 x_3 - \kappa_{23}^2. \quad (70)$$

For later use, define

$$N(\omega) \equiv \kappa_{12}^2 x_3 + \kappa_{13}^2 x_2 - 2\kappa_{12}\kappa_{13}\kappa_{23}. \quad (71)$$

Using these definitions, the determinant condition can be written as

$$x_1 Y(\omega) - N(\omega) = 0. \quad (72)$$

For real ω , all detunings δ_m are real. Therefore, $Y(\omega)$ and $N(\omega)$ can be separated into their real and imaginary parts as

$$Y(\omega) = \Xi(\omega) - j\Lambda(\omega), \quad (73)$$

$$\Xi(\omega) \equiv \delta_2\delta_3 - \gamma_2\gamma_3 - \kappa_{23}^2, \quad (74)$$

$$\Lambda(\omega) \equiv \delta_2\gamma_3 + \delta_3\gamma_2, \quad (75)$$

$$N(\omega) = M(\omega) - jG, \quad (76)$$

$$M(\omega) \equiv \kappa_{12}^2\delta_3 + \kappa_{13}^2\delta_2 - 2\kappa_{12}\kappa_{13}\kappa_{23}, \quad (77)$$

$$G \equiv \kappa_{12}^2\gamma_3 + \kappa_{13}^2\gamma_2. \quad (78)$$

Thus, $\Xi = \text{Re } Y$, $\Lambda = -\text{Im } Y$, $M = \text{Re } N$, and $G = -\text{Im } N$ are real. The constant G is a loss-weighted coupling term and is unrelated to the saturated gain g_s . Here, Re and Im denote the real and imaginary parts, respectively. Define

$$D(\omega) \equiv |Y(\omega)|^2 = \Xi(\omega)^2 + \Lambda(\omega)^2. \quad (79)$$

4.3 Invertibility of the Passive-mode Block

For real ω and $\gamma_2, \gamma_3 > 0$, suppose that $D(\omega) = 0$. Then $\Xi(\omega) = \Lambda(\omega) = 0$. From $\Lambda(\omega) = 0$,

$$\delta_3 = -\frac{\gamma_3}{\gamma_2}\delta_2, \quad \delta_2\delta_3 = -\frac{\gamma_3}{\gamma_2}\delta_2^2 \leq 0. \quad (80)$$

By contrast, $\Xi(\omega) = 0$ requires

$$\delta_2\delta_3 = \gamma_2\gamma_3 + \kappa_{23}^2 > 0, \quad (81)$$

which is a contradiction. Hence,

$$D(\omega) > 0 \quad \text{and} \quad Y(\omega) \neq 0 \quad \text{for every real } \omega. \quad (82)$$

4.4 Elimination of the Saturated Gain

Because Eq. (82) guarantees $Y(\omega) \neq 0$ for real ω , the determinant condition $x_1 Y - N = 0$ can be divided by $Y(\omega)$. To express the resulting ratio in terms of real quantities, multiply its numerator and denominator by $Y(\omega)^* = \Xi(\omega) + j\Lambda(\omega)$, where the superscript $*$ denotes complex conjugation:

$$\delta_1 + jg_s = \frac{N(\omega)}{Y(\omega)} \quad (83)$$

$$= \frac{[M(\omega) - jG][\Xi(\omega) + j\Lambda(\omega)]}{D(\omega)} \quad (84)$$

$$= \frac{M(\omega)\Xi(\omega) + G\Lambda(\omega)}{D(\omega)} + j \frac{M(\omega)\Lambda(\omega) - G\Xi(\omega)}{D(\omega)}. \quad (85)$$

Equating the real parts gives

$$\delta_1 = \frac{M(\omega)\Xi(\omega) + G\Lambda(\omega)}{D(\omega)}, \quad (86)$$

while the imaginary part gives the required value of g_s . Define the real-frequency steady-state function as

$$p(\omega) \equiv \delta_1 D(\omega) - M(\omega)\Xi(\omega) - G\Lambda(\omega). \quad (87)$$

The lowercase $p(\omega)$ denotes the nonlinear steady-state frequency polynomial and is distinct from the uppercase linear characteristic polynomial $P(\lambda)$ defined in Chapter 2. The corresponding steady-state condition is $p(\omega) = 0$. Since $D(\omega) > 0$ for real ω , multiplying the equation by $D(\omega)$ does not change its real solutions. A real zero of p is therefore a candidate synchronized frequency, and the associated gain feasibility must still be checked.

The imaginary part of Eq. (85) determines the saturated gain associated with a candidate real frequency:

$$g_s \equiv \frac{M(\omega)\Lambda(\omega) - G\Xi(\omega)}{D(\omega)}. \quad (88)$$

Because $D(\omega) > 0$ and the numerator is real, g_s is finite and real for every real ω . Therefore, on the real-frequency axis, the complex determinant condition is equivalent to $p(\omega) = 0$ together with Eq. (88), provided that the resulting gain is compatible with the selected saturation law.

4.5 Polynomial Order and Gain Law

The degree of p follows directly from the definitions above. The detuning δ_1 is a monic polynomial of degree one in ω , $\Xi(\omega)$ is a monic polynomial of degree two, and both $\Lambda(\omega)$ and

$M(\omega)$ have degree at most one. Therefore, $D(\omega) = \Xi(\omega)^2 + \Lambda(\omega)^2$ is a monic polynomial of degree four. Consequently, $\delta_1 D(\omega)$ is monic and has degree five, whereas $M(\omega)\Xi(\omega)$ has degree at most three and $G\Lambda(\omega)$ has degree at most one. The lower-degree terms cannot cancel the leading ω^5 term, so

$$p(\omega) = \omega^5 + c_4\omega^4 + c_3\omega^3 + c_2\omega^2 + c_1\omega + c_0. \quad (89)$$

Here, $c_n \in \mathbb{R}$ is the coefficient of ω^n for $n = 0, \dots, 4$. Thus, p is a monic quintic polynomial. The derivation of p and g_s does not assume a particular gain function $g(I)$. A gain function is required to determine whether the saturated gain obtained from the steady-state equations can be realized. In the numerical example we use

$$g(I) = \frac{g_0}{1 + I}, \quad I \equiv |a_1|^2, \quad (90)$$

where $g_0 \equiv g(0) > 0$ is the unsaturated net small-signal gain and I denote the intensity in the active resonator. This law is defined for $I \geq 0$ and has the range $0 < g(I) \leq g_0$. Because every nonzero real-frequency synchronized state has $|\tilde{a}_1| \neq 0$, we define the active-mode steady-state intensity

$$I_s = |\tilde{a}_1|^2. \quad (91)$$

A real root of steady-state function is only admissible under this law precisely when

$$0 < g_s < g_0. \quad (92)$$

For an admissible root, the active-mode steady-state intensity is

$$I_s = \frac{g_0}{g_s} - 1 > 0. \quad (93)$$

Chapter 5

Sensitivity and Stability Analysis

5.1 Operational Definition and Numerical Design of SS-EPD₅

Motivated by prior studies of nonlinear exceptional points and steady-state EPDs [19, 23], this thesis adopts the following operational definition for the present model. An SS-EPD₅ is a real fivefold root ω_e of the scalar steady-state equation $p(\omega) = 0$, where ω_e is a normalized frequency. The subscript e labels quantities evaluated at the SS-EPD₅. For an integer $r \geq 0$, define

$$p^{(r)}(\omega) \equiv \frac{d^r p(\omega)}{d\omega^r}, \quad p^{(0)}(\omega) \equiv p(\omega). \quad (94)$$

The parenthesized superscript (r) is the derivative order. An exact SS-EPD₅ therefore satisfies

$$p^{(r)}(\omega_e) = 0, \quad r = 0, 1, 2, 3, 4, \quad p^{(5)}(\omega_e) = 5! \neq 0. \quad (95)$$

Table 2: Normalized parameters for the candidate SS-EPD₅ design.

Parameter	Value	Parameter	Value
ω_1	-0.2601711705263189	γ_2	0.0264
ω_2	0.2309296178732144	γ_3	0.9736
ω_3	-0.10084403261005493	κ_{23}	0.4970
κ_{12}	0.34133675655216755	κ_{13}	0.4906349073555633

Because p is monic and quintic, these conditions are equivalent to

$$p(\omega) = (\omega - \omega_e)^5. \quad (96)$$

The root is admissible under the chosen saturation law only if

$$0 < g_s < g_0. \quad (97)$$

To construct a numerical design, choose the EPD at a shifted frequency such that $\omega_e = 0$ and choose two given values of γ_1 and γ_2 so that

$$\gamma_2 + \gamma_3 = 1. \quad (98)$$

The numerical search fixed γ_2 , κ_{23} , and γ_3 and varied the other parameters. The polynomial p is

$$p(\omega) = \omega^5 + \sum_{n=0}^4 c_n \omega^n. \quad (99)$$

At $\omega_e = 0$, the fivefold-root condition is $c_0 = \dots = c_4 = 0$. The resulting candidate parameter set is listed in Table 2.

For the displayed values in Table 2,

$$\max_{0 \leq n \leq 4} |c_n| \leq 1.12 \times 10^{-16}. \quad (100)$$

So the displayed values approximate the target $p(\omega) = \omega^5$.

At the target root $\omega_e = 0$, the retained high-precision parameter vector gives

$$D_e \equiv D(\omega_e) \approx 0.137, \quad g_s \approx 0.600. \quad (101)$$

With $g_0 = 1.5$, the strict admissibility condition $0 < g_s < g_0$ holds. The corresponding active-mode intensity is

$$I_s = \frac{1.5}{g_s} - 1 \approx 1.5 > 0. \quad (102)$$

5.2 Differential Sensitivity

The active resonator's natural frequency is perturbed according to

$$\omega_1 \rightarrow \omega_1 + X, \quad X \equiv \Delta\omega_1, \quad (103)$$

while ω_2 and ω_3 are left unperturbed. A positive X increases the active-mode frequency, while all other model parameters remain fixed. This changes δ_1 to $\delta_1 - X$ and leaves the other terms in p unchanged. Define

$$p(\omega, X) \equiv p(\omega) - XD(\omega). \quad (104)$$

The perturbed frequency equation is $p(\omega, X) = 0$ and the real value solutions ω are admissible steady-state eigenfrequencies. At $X = 0$, it reduces to $p(\omega) = 0$.

Consider a real root $\omega(X)$ such that $\omega(X) \rightarrow \omega_e$ as $X \rightarrow 0$. Its frequency shift is

$$\Delta\omega(X) \equiv \omega(X) - \omega_e. \quad (105)$$

Define $p_\omega(\omega, X) \equiv \partial p(\omega, X)/\partial\omega$, where X is held fixed. At the chosen value of X , let $\omega(X)$ be the real root on the selected branch and assume that it is not a repeated root. Then

$$p_\omega(\omega(X), X) = p'(\omega(X)) - XD'(\omega(X)) \neq 0, \quad (106)$$

where $p'(\omega) \equiv \frac{dp(\omega)}{d\omega}$ and $D'(\omega) \equiv \frac{dD(\omega)}{d\omega}$. Thus, both primes denote differentiation with respect to ω . Differentiating $p(\omega(X), X) = 0$ with respect to X along the selected root branch gives

$$\frac{d\omega}{dX} = \frac{D(\omega(X))}{p'(\omega(X)) - XD'(\omega(X))}. \quad (107)$$

The corresponding local differential sensitivity in the normalized variables is

$$S_\omega(X) \equiv \left| \frac{d\omega}{dX} \right| = \left| \frac{D(\omega(X))}{p'(\omega(X)) - XD'(\omega(X))} \right|. \quad (108)$$

Equation (108) is evaluated after solving $p(\omega, X) = 0$ for the selected real admissible branch and applies only where the selected root is simple. Its limiting behavior as $X \rightarrow 0$ is obtained asymptotically below.

Let

$$s \equiv \omega - \omega_e. \quad (109)$$

For the exact target factorization in Eq. (96),

$$p(\omega_e + s) = s^5. \quad (110)$$

Expanding D about ω_e gives

$$D(\omega_e + s) = D_e + D'_e s + \mathcal{O}(s^2), \quad D'_e \equiv \left. \frac{dD}{d\omega} \right|_{\omega=\omega_e}. \quad (111)$$

Here, $\mathcal{O}(s^2)$ denotes terms of order s^2 and higher as $s \rightarrow 0$. Hence, the perturbed target steady-state function is

$$p(\omega_e + s, X) = s^5 - D_e X - D'_e X s + \mathcal{O}(|X|s^2). \quad (112)$$

Because $D_e > 0$, the leading real balance is $s^5 = D_e X$, which gives

$$\Delta\omega(X) = s \sim \text{sgn}(X) (D_e |X|)^{1/5}, \quad X \rightarrow 0. \quad (113)$$

Here, $\text{sgn}(X)$ denotes the sign of X . The symbol $\sim$ means that the ratio of the two sides approaches one as $X \rightarrow 0$.

For sufficiently small real $X \neq 0$, this is the only real branch emerging from the fivefold root. The other four local fifth-root branches form two complex-conjugate pairs and do not represent real synchronized frequencies. Using the leading real-branch relation in Eq. (113) for $X \neq 0$ gives

$$S_\omega(X) \sim \frac{D_e^{1/5}}{5} |X|^{-4/5}, \quad X \rightarrow 0. \quad (114)$$

5.3 Numerical Evaluation of Sensitivity

At $X = 10^{-5}$, using the candidate parameters in Table 2, the selected real root is

$$\omega_0 \equiv \omega(X) \approx 0.065. \quad (115)$$

At this root,

$$\begin{aligned} D(\omega_0) &\approx 0.115, & p'(\omega_0) &\approx 8.854 \times 10^{-5}, \\ D'(\omega_0) &\approx -0.314. \end{aligned} \tag{116}$$

Substitution into Eq. (108) gives

$$S_\omega(X) \approx 1252.932. \tag{117}$$

A centered finite difference with step size h checks this value using the roots at $X - h$ and $X + h$ on the same real branch:

$$S_{\omega,\text{FD}}(h) \equiv \left| \frac{\omega(X+h) - \omega(X-h)}{2h} \right|. \tag{118}$$

For $0 < h < X$, define the relative error as

$$\varepsilon_{\text{rel}}(h) \equiv \frac{|S_{\omega,\text{FD}}(h) - S_\omega(X)|}{S_\omega(X)}. \tag{119}$$

The subscript rel denotes relative error. The subscript FD denotes finite difference.

Table 3: Centered finite-difference check at $X = 10^{-5}$.

h	$S_{\omega,\text{FD}}(h)$	$\varepsilon_{\text{rel}}(h)$
1.0×10^{-6}	1256.0241	2.47×10^{-3}
5.0×10^{-7}	1253.7017	6.15×10^{-4}
1.0×10^{-7}	1252.9625	2.46×10^{-5}
5.0×10^{-8}	1252.9394	6.14×10^{-6}
1.0×10^{-8}	1252.9320	2.45×10^{-7}
5.0×10^{-9}	1252.9318	6.14×10^{-8}

Each halving of h reduces the relative error by approximately a factor of four, consistent with the second-order accuracy of the centered difference. The finite-difference values therefore agree with Eq. (117).

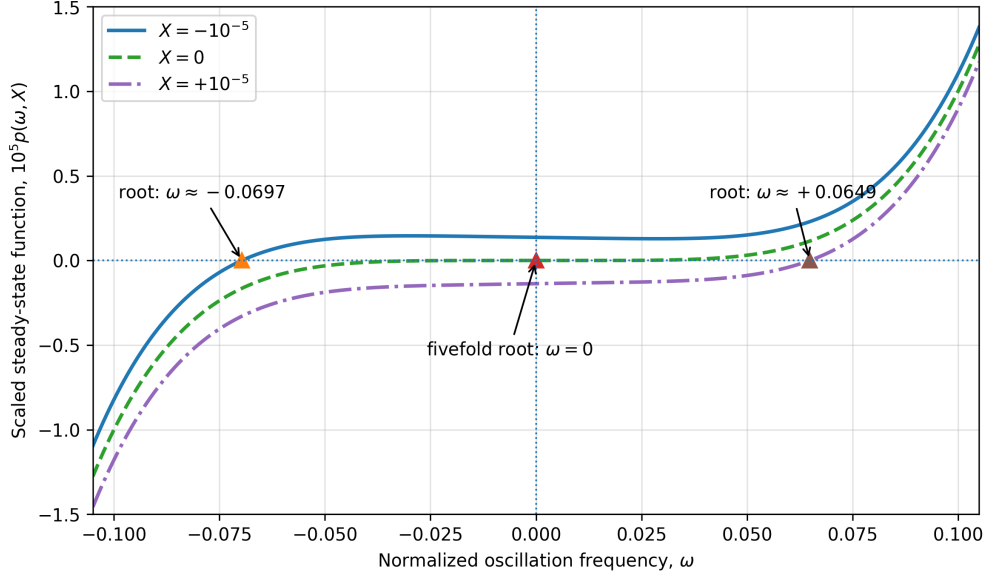


Figure 15: Scaled perturbed steady-state function $10^5 p(\omega, X)$ versus normalized oscillation frequency ω for negative, zero, and positive active-mode detuning X . At $X = 0$, the function has a fivefold root at $\omega = 0$. Negative and positive detuning shift the selected real root to negative and positive frequencies, respectively. The triangles indicate the corresponding real roots.

Equations (113) and (114) predict a $|X|^{1/5}$ frequency shift and a $|X|^{-4/5}$ differential sensitivity. At $X = -10^{-5}$, solving $p(\omega, X) = 0$ gives $\omega \approx -0.070$. At $X = +10^{-5}$, the corresponding root is $\omega \approx 0.065$. The required gains are approximately 0.622 and 0.569, respectively. For both roots, the corresponding values of g_s satisfy $0 < g_s < g_0 = 1.5$, so both roots are admissible.

On the positive real branch, the frequency shift was calculated over $10^{-10} \leq X \leq 10^{-3}$. The numerical value of $|\Delta\omega|$ approaches the fifth-root law in Eq. (113) as X decreases over this range. The required gain remains between 0.505 and 0.597, so the entire plotted branch is admissible for $g_0 = 1.5$. Figure 16 shows this comparison.

The differential sensitivity was calculated on the same branch over $10^{-8} \leq X \leq 10^{-3}$. As X decreases over this range, the numerical result approaches the $|X|^{-4/5}$ law in Eq. (114). At larger X , the higher-order terms omitted from the asymptotic expression produce the difference between the two curves. Figure 17 compares the exact implicit derivative evaluated from the complete steady-state function with the asymptotic $|X|^{-4/5}$ law.

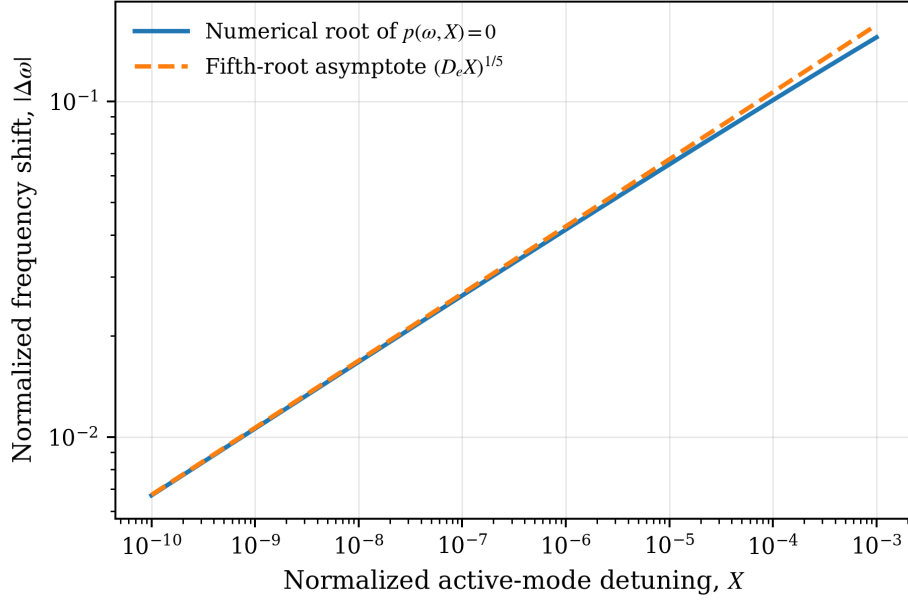


Figure 16: Fifth-root frequency shift on the positive real branch. The solid curve is calculated from $p(\omega, X) = 0$ using the candidate parameters. The dashed curve is Eq. (113).

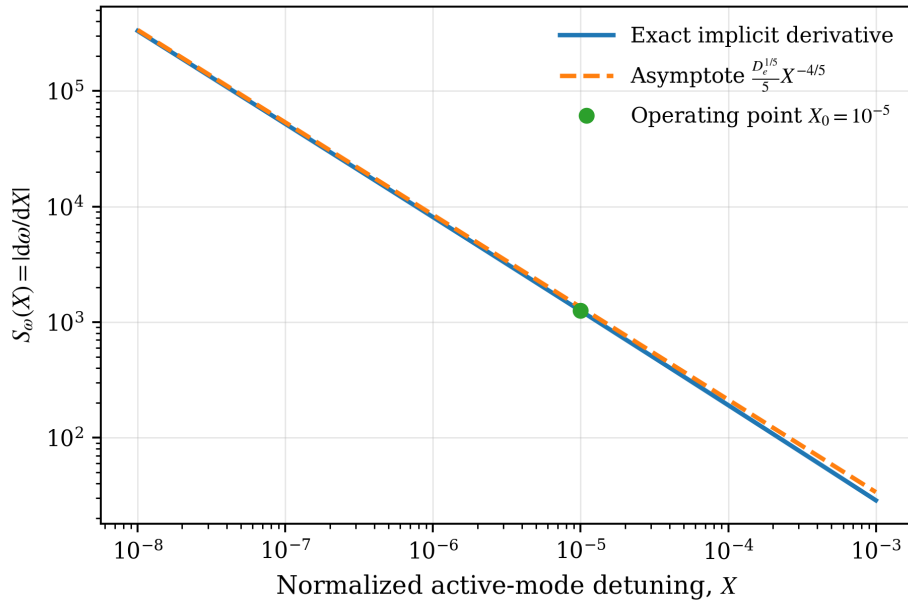


Figure 17: Differential sensitivity on the positive real branch. The solid curve is calculated from Eq. (108) using the candidate parameters. The dashed curve is Eq. (114). The marker denotes $X = 10^{-5}$.

5.4 Local Transverse Linear Stability at a Nearby Operating Point

An admissible real root of the scalar steady-state equation identifies a synchronized steady state, but it does not determine whether that state is dynamically stable. Stability must instead be assessed from the original time-dependent CMT equations. The SS-EPD₅ operating point analyzed in the preceding sections corresponds to $X = 0$, for which the scalar steady-state function $p(\omega, 0)$ has a fivefold root at $\omega = \omega_e = 0$. To examine the local stability of a nearby synchronized state, the resonance frequency of the active resonator is perturbed by replacing ω_1 with $\omega_1 + X$, where

$$X = 10^{-5}. \quad (120)$$

The stability analysis is performed at the selected admissible positive real simple root of $p(\omega, X) = 0$ that lies near the SS-EPD₅ operating point. Solving $p(\omega, X) = 0$, with p defined by Eq. (104), gives three solutions: one real root and two pair of complex conjugate pairs. Only the real root represents a synchronized steady state, so real frequency ω then is selected. Then the g_s is obtained from Eq. (88). The active-mode intensity then follows from the saturation law in Eq. (93). Numerically,

$$\begin{aligned} \omega &\approx 0.0648709, \\ g_s &\approx 0.569288, \\ I_s &= \frac{g_0}{g_s} - 1 \approx 1.63487. \end{aligned} \quad (121)$$

The saturation law fixes the magnitude of the active-mode amplitude. The common phase is chosen such that $\tilde{a}_1$ is real and positive

$$\tilde{\mathbf{a}} \equiv \begin{bmatrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \end{bmatrix}^T, \quad \tilde{a}_1 = \sqrt{I_s} \approx 1.27862. \quad (122)$$

At the selected frequency ω , the quantities defined in Eqs. (67) and (70) are

$$x_2 = \omega - \omega_2 - j\gamma_2, \quad x_3 = \omega - \omega_3 - j\gamma_3, \quad Y(\omega) = x_2 x_3 - \kappa_{23}^2. \quad (123)$$

For real ω , $Y(\omega) \neq 0$, so the passive-mode block is invertible. The second and third rows of the perturbed steady-state equation, which are unchanged from Eq. (64), then give

$$\begin{aligned} \begin{bmatrix} \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix} &= - \begin{bmatrix} x_2 & \kappa_{23} \\ \kappa_{23} & x_3 \end{bmatrix}^{-1} \begin{bmatrix} \kappa_{12} \\ \kappa_{13} \end{bmatrix} \tilde{a}_1 \\ &= \frac{\tilde{a}_1}{Y(\omega)} \begin{bmatrix} \kappa_{13}\kappa_{23} - \kappa_{12}x_3 \\ \kappa_{12}\kappa_{23} - \kappa_{13}x_2 \end{bmatrix}. \end{aligned} \quad (124)$$

Using the parameters in Table 2, together with X and g_0 above, gives

$$\tilde{\mathbf{a}} \approx \begin{bmatrix} 1.27862 \\ -0.043993 - 1.43836j \\ -0.816445 - 0.482924j \end{bmatrix}. \quad (125)$$

A small time-dependent complex perturbation $\rho_m(t)$ is added to the steady-state amplitude of each mode:

$$a_m(t) = [\tilde{a}_m + \rho_m(t)] e^{j\omega t}, \quad m = 1, 2, 3. \quad (126)$$

Here, $\rho_m(t)$ describes a small deviation from the synchronized modal amplitude $\tilde{a}_m$ and generally contains both amplitude and phase perturbations. Setting $\rho_m(t) = 0$ recovers the synchronized steady-state solution. The common factor $e^{j\omega t}$ retains the synchronized oscillation at the selected frequency ω .

Differentiating Eq. (126) with respect to time gives

$$\dot{a}_m(t) = \left[\dot{\rho}_m(t) + j\omega(\tilde{a}_m + \rho_m(t)) \right] e^{j\omega t}, \quad m = 1, 2, 3. \quad (127)$$

Because ω is real, $|e^{j\omega t}| = 1$, and the intensity entering the nonlinear gain is therefore

$$\begin{aligned} |a_1|^2 &= |\tilde{a}_1 + \rho_1|^2 \\ &= I_s + \tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^* + \mathcal{O}(|\rho_1|^2). \end{aligned} \quad (128)$$

Here, the superscript $*$ denotes complex conjugation, and $\mathcal{O}(|\rho_1|^2)$ collects terms that are second order or higher in the perturbation. These higher-order terms are neglected in the local linear stability analysis. Define the local slope of the nonlinear gain with respect to the active-mode intensity as

$$g'_I \equiv \left. \frac{dg(I)}{dI} \right|_{I=I_s}. \quad (129)$$

For the saturation law $g(I) = g_0/(1 + I)$, this slope is

$$g'_I = -\frac{g_0}{(1 + I_s)^2}. \quad (130)$$

Here, the prime denotes differentiation with respect to the intensity I . Using Eq. (128), define

$$\Delta I \equiv |a_1|^2 - I_s = \tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^* + |\rho_1|^2. \quad (131)$$

A Taylor expansion of the nonlinear gain about $I = I_s$ gives

$$\begin{aligned}
g(|a_1|^2) &= g(I_s) + g'_I \Delta I + \frac{1}{2} g''_I (\Delta I)^2 + \mathcal{O}((\Delta I)^3) \\
&= g_s + g'_I (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*) + g'_I |\rho_1|^2 \\
&\quad + \frac{1}{2} g''_I (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*)^2 + \mathcal{O}(|\rho_1|^3) \\
&= g_s + g'_I (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*) + \mathcal{O}(|\rho_1|^2),
\end{aligned} \tag{132}$$

where $g_s = g(I_s)$ and $g'_I \equiv d^2 g(I)/dI^2|_{I=I_s}$. Consequently, the complete nonlinear term in the active-mode equation becomes

$$\begin{aligned}
g(|a_1|^2)(\tilde{a}_1 + \rho_1) &= g_s \tilde{a}_1 + (g_s + g'_I I_s) \rho_1 \\
&\quad + g'_I \tilde{a}_1^2 \rho_1^* + \mathcal{O}(|\rho_1|^2).
\end{aligned} \tag{133}$$

With the active-mode detuning $\omega_1 \rightarrow \omega_1 + X$, substituting Eqs. (126) and (127) into Eq. (60) and cancelling the common factor $e^{j\omega t}$ gives the exact perturbed active-mode equation

$$\begin{aligned}
\dot{\rho}_1 + j\omega(\tilde{a}_1 + \rho_1) &= [j(\omega_1 + X) + g(|\tilde{a}_1 + \rho_1|^2)] (\tilde{a}_1 + \rho_1) \\
&\quad - j\kappa_{12}(\tilde{a}_2 + \rho_2) - j\kappa_{13}(\tilde{a}_3 + \rho_3).
\end{aligned} \tag{134}$$

Moving the term proportional to $j\omega$ to the right-hand side and using the gain expansion in Eq. (132) yield

$$\begin{aligned}
\dot{\rho}_1 &= [j(\omega_1 + X - \omega) + g_s] (\tilde{a}_1 + \rho_1) \\
&\quad + g'_I (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*) (\tilde{a}_1 + \rho_1) \\
&\quad - j\kappa_{12}(\tilde{a}_2 + \rho_2) - j\kappa_{13}(\tilde{a}_3 + \rho_3) + \mathcal{O}(|\rho_1|^2).
\end{aligned} \tag{135}$$

The unperturbed synchronized state is recovered by setting $\rho_m(t) \equiv 0$ in Eq. (134). Using $g(|\tilde{a}_1|^2) = g_s$, this equation reduces to

$$[j(\omega_1 + X - \omega) + g_s] \tilde{a}_1 - j\kappa_{12}\tilde{a}_2 - j\kappa_{13}\tilde{a}_3 = 0. \quad (136)$$

Subtracting Eq. (136) from Eq. (135) cancels the steady-state terms and leaves

$$\begin{aligned} \dot{\rho}_1 = & [j(\omega_1 + X - \omega) + g_s] \rho_1 - j\kappa_{12}\rho_2 - j\kappa_{13}\rho_3 \\ & + g'_I (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*) (\tilde{a}_1 + \rho_1) + \mathcal{O}(|\rho_1|^2). \end{aligned}$$

The remaining nonlinear product can be written as

$$\begin{aligned} & (\tilde{a}_1^* \rho_1 + \tilde{a}_1 \rho_1^*) (\tilde{a}_1 + \rho_1) \\ & = |\tilde{a}_1|^2 \rho_1 + \tilde{a}_1^2 \rho_1^* + \tilde{a}_1^* \rho_1^2 + \tilde{a}_1 |\rho_1|^2 \\ & = |\tilde{a}_1|^2 \rho_1 + \tilde{a}_1^2 \rho_1^* + \mathcal{O}(|\rho_1|^2). \end{aligned}$$

Retaining only the first-order terms gives the linearized active-mode equation

$$\begin{aligned} \dot{\rho}_1 = & [j(\omega_1 + X - \omega) + g_s + g'_I |\tilde{a}_1|^2] \rho_1 \\ & + g'_I \tilde{a}_1^2 \rho_1^* - j\kappa_{12}\rho_2 - j\kappa_{13}\rho_3. \end{aligned} \quad (137)$$

The second and third modal equations are linear and contain no nonlinear gain. Substituting Eq. (126), cancelling the common factor $e^{j\omega t}$, and subtracting the corresponding synchronized steady-state equations give

$$\dot{\rho}_2 = [j(\omega_2 - \omega) - \gamma_2] \rho_2 - j\kappa_{12}\rho_1 - j\kappa_{23}\rho_3, \quad (138)$$

$$\dot{\rho}_3 = [j(\omega_3 - \omega) - \gamma_3] \rho_3 - j\kappa_{13}\rho_1 - j\kappa_{23}\rho_2. \quad (139)$$

To write Eqs. (137)– (139) in a compact form, define

$$\begin{aligned}
A_1 &\equiv j(\omega_1 + X - \omega) + g_s + g'_I I_s, \\
A_2 &\equiv j(\omega_2 - \omega) - \gamma_2, \\
A_3 &\equiv j(\omega_3 - \omega) - \gamma_3, \\
B &\equiv g'_I \tilde{a}_1^2.
\end{aligned} \tag{140}$$

With these definitions, the three linearized perturbation equations become

$$\begin{aligned}
\dot{\rho}_1 &= A_1 \rho_1 + B \rho_1^* - j\kappa_{12} \rho_2 - j\kappa_{13} \rho_3, \\
\dot{\rho}_2 &= -j\kappa_{12} \rho_1 + A_2 \rho_2 - j\kappa_{23} \rho_3, \\
\dot{\rho}_3 &= -j\kappa_{13} \rho_1 - j\kappa_{23} \rho_2 + A_3 \rho_3.
\end{aligned} \tag{141}$$

All coefficients in Eq. (141) are evaluated at the selected synchronized state. Following the paired representation used in Appendix E of Bradshaw et al. [23], write

$$\rho_m(t) = u_m e^{\lambda t} + v_m^* e^{\lambda^* t}, \quad m = 1, 2, 3. \tag{142}$$

Its complex conjugate is

$$\rho_m^*(t) = u_m^* e^{\lambda^* t} + v_m e^{\lambda t}. \tag{143}$$

Here, u_m and v_m are independent, time-independent complex coefficients for mode m , and v_m^* denotes the complex conjugate of v_m . The quantity λ is the complex perturbation exponent.

Differentiating Eqs. (142) and (143) gives

$$\dot{\rho}_m(t) = \lambda u_m e^{\lambda t} + \lambda^* v_m^* e^{\lambda^* t}, \quad (144)$$

$$\dot{\rho}_m^*(t) = \lambda^* u_m^* e^{\lambda^* t} + \lambda v_m e^{\lambda t}. \quad (145)$$

Substitution of these paired expressions into Eq. (141) allows the coefficients of $e^{\lambda t}$ and $e^{\lambda^* t}$ to be matched separately, producing a closed six-dimensional eigenvalue problem. Because the coupling coefficients κ_{mn} are real, taking the complex conjugate of Eq. (141) changes each factor $-j\kappa_{mn}$ into $+j\kappa_{mn}$. The conjugate perturbation equations are therefore obtained by taking the complex conjugate of each equation in Eq. (141).

$$\begin{aligned} \dot{\rho}_1^* &= A_1^* \rho_1^* + B^* \rho_1 + j\kappa_{12} \rho_2^* + j\kappa_{13} \rho_3^*, \\ \dot{\rho}_2^* &= j\kappa_{12} \rho_1^* + A_2^* \rho_2^* + j\kappa_{23} \rho_3^*, \\ \dot{\rho}_3^* &= j\kappa_{13} \rho_1^* + j\kappa_{23} \rho_2^* + A_3^* \rho_3^*. \end{aligned} \quad (146)$$

Substituting Eqs. (142) and (143) into Eqs. (141) and (146) produces terms proportional to $e^{\lambda t}$ and $e^{\lambda^* t}$. Collecting the coefficients of $e^{\lambda t}$ gives six algebraic equations: the original perturbation system provides the three equations for u_m , and the conjugate system provides the three equations for v_m , where $m = 1, 2, 3$. The resulting six equations are

$$\begin{aligned} \lambda u_1 &= A_1 u_1 + B v_1 - j\kappa_{12} u_2 - j\kappa_{13} u_3, \\ \lambda v_1 &= B^* u_1 + A_1^* v_1 + j\kappa_{12} v_2 + j\kappa_{13} v_3, \\ \lambda u_2 &= -j\kappa_{12} u_1 + A_2 u_2 - j\kappa_{23} u_3, \\ \lambda v_2 &= j\kappa_{12} v_1 + A_2^* v_2 + j\kappa_{23} v_3, \\ \lambda u_3 &= -j\kappa_{13} u_1 - j\kappa_{23} u_2 + A_3 u_3, \\ \lambda v_3 &= j\kappa_{13} v_1 + j\kappa_{23} v_2 + A_3^* v_3. \end{aligned} \quad (147)$$

These six equations can now be assembled into a single 6×6 eigenvalue problem. Collect the six coefficients in the order

$$\mathbf{q} \equiv \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{bmatrix}^T. \quad (148)$$

Using this ordering, Eq. (147) can be written as the 6×6 eigenvalue problem

$$\begin{bmatrix} A_1 & B & -j\kappa_{12} & 0 & -j\kappa_{13} & 0 \\ B^* & A_1^* & 0 & j\kappa_{12} & 0 & j\kappa_{13} \\ -j\kappa_{12} & 0 & A_2 & 0 & -j\kappa_{23} & 0 \\ 0 & j\kappa_{12} & 0 & A_2^* & 0 & j\kappa_{23} \\ -j\kappa_{13} & 0 & -j\kappa_{23} & 0 & A_3 & 0 \\ 0 & j\kappa_{13} & 0 & j\kappa_{23} & 0 & A_3^* \end{bmatrix} \mathbf{q} = \lambda \mathbf{q}. \quad (149)$$

Substituting the selected synchronized state, the gain slope from Eq. (130), and the coupling coefficients in Table 2 directly into Eq. (149) gives the numerical eigenvalue problem

$$\begin{bmatrix} 0.216 - 0.325j & -0.353 & -0.341j & 0 & -0.491j & 0 \\ -0.353 & 0.216 + 0.325j & 0 & 0.341j & 0 & 0.491j \\ -0.341j & 0 & -0.026 + 0.166j & 0 & -0.497j & 0 \\ 0 & 0.341j & 0 & -0.026 - 0.166j & 0 & 0.497j \\ -0.491j & 0 & -0.497j & 0 & -0.974 - 0.166j & 0 \\ 0 & 0.491j & 0 & 0.497j & 0 & -0.974 + 0.166j \end{bmatrix} \mathbf{q} = \lambda \mathbf{q}. \quad (150)$$

Although the entries in Eq. (150) are displayed to three decimal places, the eigenvalues are calculated using the unrounded operating-point values and system parameters.

The complete spectrum is

$$\lambda_1 = 0, \tag{151}$$

$$\lambda_2 \approx -4.22605643 \times 10^{-4}, \tag{152}$$

$$\lambda_{3,4} \approx -0.3124367561 \pm 0.7461436242j, \tag{153}$$

$$\lambda_{5,6} \approx -0.4712930062 \pm 0.1101149347j. \tag{154}$$

The Eq. (151) come from the common-phase invariance $a_m \mapsto e^{j\phi} a_m$ and represents a neutral change in the arbitrary common phase. It is zero because a change in the arbitrary common phase neither grows nor decays and therefore does not represent an instability [23]. All five transverse eigenvalues have negative real parts. Therefore, the selected synchronized state is locally linearly stable transverse to the common-phase orbit. This local result applies only to the selected branch at $X = 10^{-5}$ with the stated gain law.

Chapter 6

Conclusion

This thesis investigated two mathematically distinct routes to higher-order degeneracy in three-mode systems. The first begins with a physical RLC network and a fixed *linear* state matrix. The second begins with synchronized steady states of a *nonlinear* CMT model and a reduced scalar frequency equation. The second method can be used to find the approximation of the steady state solution of the RLC circuit problem.

For the linear RLC circuit, Kirchhoff's laws produced a six-state model and an even sixth-order characteristic polynomial. Matching the polynomial to $(\lambda^2 + \omega_e^2)^3$ yielded the algebraic triple-root conditions associated with an EPD₃. With $L = 10 \mu\text{H}$ and $L_a = 12 \mu\text{H}$ fixed, the synthesis produced two coupling branches. The selected lower- k design had a nominal degenerate frequency of $f_e \approx 1.45 \text{ MHz}$. Component perturbations separated the three eigenfrequencies (the natural frequencies of the three-resonator system), while the ideal frequency- and time-domain calculations characterized the circuit response near the EPD₃.

Replacing the ideal negative resistance with an op-amp NIC showed that the linear design is not preserved automatically in a practical active circuit. Finite op-amp gain, bandwidth, output resistance, and loading make the small-signal NIC admittance saturable. The complete-circuit simulation developed a finite-amplitude oscillation with a dominant spectral

peak near 1.178 MHz over the recorded interval. It shows that matching an ideal negative resistance at one operating condition is insufficient to preserve the designed degeneracy.

For the separate nonlinear CMT model, the synchronized-state equations were reduced to a real monic quintic polynomial equation $p(\omega) = 0$ together with a gain-admissibility condition. A high-precision numerical parameter set matched $p(\omega) = \omega^5$ to the stated residual tolerance, defining the operational SS-EPD₅ used in this thesis. Active-mode detuning produced the asymptotic shift $\Delta\omega \propto \text{sgn}(X)|X|^{1/5}$, and the normalized differential sensitivity on the neighboring real branch scaled as $S_\omega \propto |X|^{-4/5}$. At $X = 10^{-5}$, the sensitivity is $S_\omega \approx 1252.93$, consistent with the finite-difference calculation. Stability of the steady state oscillation is studied via the Lyapunov coefficients. This involved a linearization at the corresponding synchronized steady state that produced one neutral common-phase eigenvalue and five eigenvalues with negative real parts, establishing local transverse steady state stability of the oscillation frequency at that operating point.

The RLC/NIC analysis demonstrates the sensitivity of a linear degeneracy to the physical model of the active element. The nonlinear CMT analysis demonstrates how a self-consistent gain condition can produce a higher-order steady-state frequency root.

Future work should experimentally test the NIC circuit. For the SS-EPD₅ model, the next steps are to map the normalized parameters to a physical platform, continue the admissible steady-state branches, examine stability closer to the multiple root, and include linewidth and noise in a measurement model.

REFERENCES

- [1] W. D. Heiss, “The physics of exceptional points,” *J. Phys. A*, vol. 45, Art. no. 444016, 2012, doi: 10.1088/1751-8113/45/44/444016.
- [2] M.-A. Miri and A. Alù, “Exceptional points in optics and photonics,” *Science*, vol. 363, Art. no. eaar7709, 2019, doi: 10.1126/science.aar7709.
- [3] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, “Experimental study of active LRC circuits with $\mathcal{PT}$ symmetries,” *Phys. Rev. A*, vol. 84, Art. no. 040101(R), 2011, doi: 10.1103/PhysRevA.84.040101.
- [4] W. Cao, C. Wang, W. Chen, S. Hu, H. Wang, L. Yang, and X. Zhang, “Fully integrated parity–time-symmetric electronics,” *Nat. Nanotechnol.*, vol. 17, pp. 262–268, 2022, doi: 10.1038/s41565-021-01038-4.
- [5] M. R. S. Alshetaiwi, “Third order exceptional point of degeneracy in coupled resonators at radio frequencies,” M.S. thesis, Dept. Electr. Eng. Comput. Sci., University of California, Irvine, CA, USA, 2018. [Online]. Available: <https://escholarship.org/uc/item/8458v5bm>
- [6] H. Hodaei *et al.*, “Enhanced sensitivity at higher-order exceptional points,” *Nature*, vol. 548, pp. 187–191, 2017, doi: 10.1038/nature23280.

- [7] M. Y. Nada, M. A. K. Othman, and F. Capolino, “Theory of coupled resonator optical waveguides exhibiting high-order exceptional points of degeneracy,” *Phys. Rev. B*, vol. 96, Art. no. 184304, 2017, doi: 10.1103/PhysRevB.96.184304.
- [8] M. Y. Nada and F. Capolino, “Exceptional point of sixth-order degeneracy in a modified coupled-resonator optical waveguide,” *J. Opt. Soc. Am. B*, vol. 37, no. 8, pp. 2319–2328, 2020, doi: 10.1364/JOSAB.385198.
- [9] K. Yin, X. Hao, Y. Huang, J. Zou, X. Ma, and T. Dong, “High-order exceptional points in pseudo-Hermitian radio-frequency circuits,” *Phys. Rev. Applied*, vol. 20, no. 2, Art. no. L021003, Aug. 2023, doi: 10.1103/PhysRevApplied.20.L021003.
- [10] Z. Xiao, H. Li, T. Kottos, and A. Alù, “Enhanced sensing and nondegraded thermal noise performance based on PT-symmetric electronic circuits with a sixth-order exceptional point,” *Phys. Rev. Lett.*, vol. 123, no. 21, Art. no. 213901, Nov. 2019, doi: 10.1103/PhysRevLett.123.213901.
- [11] X. Zhang, Z. Zhu, Y. Hao, P. Wu, G. Wang, and Y. Cheng, “Observation of higher-order exceptional points using frequency-dependent gain,” *Commun. Phys.*, vol. 9, Art. no. 97, Mar. 2026, doi: 10.1038/s42005-026-02561-3.
- [12] J. T. Sloan, M. A. K. Othman, and F. Capolino, “Theory of double ladder lumped circuits with degenerate band edge,” *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 1, pp. 3–13, Jan. 2018, doi: 10.1109/TCSI.2017.2690971.
- [13] F. Yazdi, D. Oshmarin, T. Mealy, A. T. Almutawa, A. Nikzamir, and F. Capolino, “Sixth-order degenerate band edge in coupled microstrip waveguides,” *Phys. Rev. Applied*, vol. 17, no. 6, Art. no. 064049, Jun. 2022, doi: 10.1103/PhysRevApplied.17.064049.

- [14] H.-K. Lau and A. A. Clerk, “Fundamental limits and non-reciprocal approaches in non-Hermitian quantum sensing,” *Nat. Commun.*, vol. 9, Art. no. 4320, 2018, doi: 10.1038/s41467-018-06477-7.
- [15] W. Langbein, “No exceptional precision of exceptional-point sensors,” *Phys. Rev. A*, vol. 98, Art. no. 023805, 2018, doi: 10.1103/PhysRevA.98.023805.
- [16] J. Wiersig, “Review of exceptional point-based sensors,” *Photonics Res.*, vol. 8, pp. 1457–1467, 2020, doi: 10.1364/PRJ.396115.
- [17] K. Bai, L. Fang, T.-R. Liu, J.-Z. Li, D. Wan, and M. Xiao, “Nonlinearity-enabled higher-order exceptional singularities with ultra-enhanced signal-to-noise ratio,” *Natl. Sci. Rev.*, vol. 10, Art. no. nwac259, 2023, doi: 10.1093/nsr/nwac259.
- [18] K. Bai, T.-R. Liu, L. Fang, J.-Z. Li, C. Lin, D. Wan, and M. Xiao, “Observation of nonlinear exceptional points with a complete basis in dynamics,” *Phys. Rev. Lett.*, vol. 132, Art. no. 073802, 2024, doi: 10.1103/PhysRevLett.132.073802.
- [19] K. Bai, J.-Z. Li, T.-R. Liu, L. Fang, D. Wan, and M. Xiao, “Nonlinear exceptional points with a complete basis in dynamics,” *Phys. Rev. Lett.*, vol. 130, Art. no. 266901, 2023, doi: 10.1103/PhysRevLett.130.266901.
- [20] H. Li *et al.*, “Enhanced sensitivity with nonlinearity-induced exceptional points degeneracy lifting,” *Commun. Phys.*, vol. 7, Art. no. 117, 2024, doi: 10.1038/s42005-024-01609-6.
- [21] D.-Y. Chen, L. Dong, and Q.-A. Huang, “Inductor-capacitor passive wireless sensors using nonlinear parity-time symmetric configurations,” *Nat. Commun.*, vol. 15, Art. no. 9312, 2024, doi: 10.1038/s41467-024-53655-x.

- [22] T. Darcie and J. S. Aitchison, “Responsivity and stability of nonlinear exceptional point lasers with saturable gain and loss,” *Phys. Rev. A*, vol. 111, Art. no. 033517, 2025, doi: 10.1103/PhysRevA.111.033517.
- [23] B. Bradshaw, A. Hakimi, and F. Capolino, “Steady-state exceptional-point degeneracy and sensitivity of nonlinear saturable coupled oscillators,” *Phys. Rev. Applied*, vol. 26, Art. no. 024044, 2026, doi: 10.1103/trjc-5cdn.