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ABSTRACT

The reactions $\pi^- + p \rightarrow \bar{p} + p + n$ and $\pi^- + p \rightarrow \bar{p} + d$ have been investigated. The calculations are based on third-order perturbation theory with pseudoscalar coupling between nucleons and pions and with a phenomenological treatment of the nucleon-nucleon interaction in the final state. The final-state interactions of the antinucleon are neglected. Cross sections are given in graphical form for the above reactions and for transitions between eigenstates of isotopic spin. The final-state nucleon-nucleon interaction is shown to have a large effect on the cross sections. The cross section for the reaction $\pi^- + p \rightarrow \bar{p} + d$ is found to be relatively large. At an an energy of 10 Mev above threshold in the center-of-momentum system the ratio of this cross section to that for $\pi^- + p \rightarrow \bar{p} + p + n$ is about 5:1. At an energy of 40 Mev above threshold this ratio has decreased to 1:1. The total cross section for the reaction leading to the unbound final state is calculated by assuming a modified Fermi statistical model. At an energy 100 Mev above threshold, this cross section is approximately 0.1 mb. A theoretical expression for the transition amplitude is developed.

I. INTRODUCTION

Until now antinucleons have been produced by bombarding complex nuclei with protons from the Berkeley Bevatron,[1] since the lack of an external proton beam precludes the production by protons on hydrogen. However, there is available a pion beam with momenta ranging up to 5 Bev/c, with which it should be possible to produce antiprotons through the reactions[2]

$$\pi^- + p \rightarrow \bar{p} + p + n \qquad (1.a)$$

and

$$\pi^- + p \rightarrow \bar{p} + d \,. \qquad (1.b)$$

A distinctive feature is the strong and attractive interaction between the two nucleons in the final state, which can lead to a bound state, the deuteron. From the experimental point of view this two-particle final state is a distinct advantage, particularly when compared to the production in nucleon-nucleon collisions,

$$N + N \rightarrow 3N + \bar{N} \,. \qquad (2)$$

It is also possible for the three nucleons in the final state of Reaction (2) to be bound as a He^3 nucleus,[1,3] but the probability for the formation of this bound state is low.

In this thesis we are primarily concerned with Reaction (1). We calculate the cross sections for Reaction (1.a) by using lowest-order perturbation theory with pseudoscalar coupling. We include the interaction of the two nucleons in the final state by using the nucleon-nucleon scattering wave functions or the wave function of the deuteron in evaluating these matrix elements.

The asymptotic form of these wave functions is well known at low energies, but in our case we need to know the detailed behavior of the wave functions for small separations of the two nucleons, since the production occurs within a small volume.

We determine these wave functions by solving the Schrödinger equation for a square-well potential with a hard core. We use only the s-wave part of the scattering wave function. The higher angular momentum states should be important only when the relative momentum of the two final nucleons is greater than 140 Mev/c. This region includes the entire spectrum of the antinucleon for energies less than about 30 Mev above threshold in the center-of-momentum system. However, at higher beam energies the s-wave part of the final-state interaction should be dominant within 200 Mev/c of the maximum momentum of the antinucleon.

The problem is complicated by the large annihilation cross section for the antinucleon. This annihilation is also due to a strong final-state interaction and should be included in the calculation. One can even hypothesize a bound state for the nucleon-antinucleon system, but rough calculations using the known annihilation cross section show that such a bound state would annihilate while traveling a distance comparable to the pion Compton wave length.

The region in which the nucleon-antinucleon interaction is least important is near the end of the antinucleon spectrum. In the center-of-momentum system the two nucleons then have equal momenta and are moving directly away from the antinucleon. This portion of the spectrum is also the region where the nucleon-nucleon interaction is most important. Consequently we ignore the antinucleon interaction in the final state.

The interaction of the pion and nucleon in the initial state should be negligible. The energies of these two particles are so very high that a plane-wave approximation is certainly justified.

We find that the final-state interaction is very important in Reaction (1) up to an energy of 120 Mev above threshold in the center-of-momentum system. The effects of this interaction are seen both in the momentum distribution of the antinucleon in Reaction (1.a) and in the total cross section.

In the momentum spectrum a characteristic peak occurs near the maximum momentum of the antinucleon. It is due simply to the distortion of the wave function and occurs when the relative momentum of the two final nucleons is small and the final-state interactions are the strongest. One may argue that this enhancement should occur only for the singlet spin states of the two final nucleons. One knows that a bound state exists for the triplet spin states and therefore may expect that the probability for the formation of an unbound state is thereby decreased. No such effect is seen in this calculation.

We also find that the magnitude of the final-state interaction depends on the details of the scattering wave function and that only the general shape of the spectrum can be predicted from a knowledge of nucleon-nucleon scattering.

We find that the final-state interaction strongly affects the energy dependence of the total cross section. The low-energy cross section is enhanced, so that the cross section increases essentially linearly with the available energy for energies beyond about 40 Mev above threshold in the center-of-momentum system. We normalize the total cross section to that given by a modified Fermi statistical model. We find that the total cross section for Reaction (1.a) is approximately 0.14 mb at an energy of 100 Mev above threshold in the center-of-momentum system.

The binding of the two final nucleons is due directly to the final-state interaction. We find that the probability for the formation of a deuteron is relatively large. The ratio of the cross section for Reaction (1.b) to that for Reaction (1.a) is approximately 5:1 at 10 Mev above threshold. This number is 1:1 at 40 Mev above threshold.

This ratio also depends on the detailed short-range behavior of the wave function.

In Appendix II we develop a general expression for the transition amplitude for the pion production of antinucleons.

II. THE DESCRIPTION OF THE MODEL

A. Isotopic Spin Ampliltudes

Many properties of the production amplitude for the reaction

$$\pi + N_1 \rightarrow \bar{N}_2 + N_3 + N_4 \tag{3}$$

can be determined from general invariance conditions and the allied conservation laws without reference to a particular interaction. A general transition amplitude is derived in Appendix II, but it is of no particular use to us in the present calculation. The particular model for the basic production amplitude that we use, covariant perturbation theory, already satisfies all constraints derived from known conservation laws.

However, we derive the consequences of the isotropic spin separately. We can then easily emphasize the two-nucelon substate in the final state, and easily include the effects of the final-state interaction between these two particles.

We treat the isotopic spin in the standard manner. We define a three-dimensional isotopic spin space. Under rotations in this space the field representing nucleons and antinucleons transform as a spinor and the pion field as a vector. The generators of rotations, $\underset{\sim}{T}$, in this space must be constructed from the fields in such a manner that $\underset{\sim}{T}$ obeys the same commutation relations as the angular momentum. We can then construct eigenstates of T_3 and $\underset{\sim}{T} \cdot \underset{\sim}{T}$ in the same way as eigenstates of the angular momentum. Physical states are linear combinations of these eigenstates. The charge independence of the pion-nucleon interaction is equivalent to the statement that transitions between eigenstates of isotopic spin cannot depend upon the third component of isotopic spin.

There is a great deal of ambiguity in the choice of the phases of physical states. The relative phases of one-particle states are made clear in the following definitions.

The plane-wave expansions of the nucleon and pion fields are[4]

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3\underset{\sim}{p} \sqrt{\frac{m}{E}} \sum_{r=1}^{2} \left\{ b_p^{\ r}(\underset{\sim}{p}) \begin{vmatrix} u_r(\underset{\sim}{p}) \\ 0 \end{vmatrix} e^{-ip\cdot x} + b_n^{\ r}(\underset{\sim}{p}) \begin{vmatrix} 0 \\ u_r(\underset{\sim}{p}) \end{vmatrix} e^{-ip\cdot x} + c_p^{\ r}(\underset{\sim}{p})^{\dagger} \begin{vmatrix} v_r(\underset{\sim}{p}) \\ 0 \end{vmatrix} e^{ip\cdot x} + c_n^{\ r}(\underset{\sim}{p})^{\dagger} \begin{vmatrix} 0 \\ v_r(\underset{\sim}{p}) \end{vmatrix} e^{ip\cdot x} \right\} \tag{4}$$

and

$$\underset{\sim}{\pi}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{a=1}^{3} \int \frac{d^3\underset{\sim}{k}}{\sqrt{2\omega}} \left\{ \hat{e}_a\, a_a(\underset{\sim}{k})\, e^{-k\cdot x} + \hat{e}_a\, a_a^{\dagger}(\underset{\sim}{k})\, e^{ik\cdot x} \right\} \tag{5}$$

Here the subscripts p, n, and α refer to the proton, neutron, and any of the three components of the pion field.

The physical pion fields are defined as linear combinations of the three components of the vector:

$$\pi_+ (x) = \pi_- (x)^\dagger = \sqrt{\frac{1}{2}} \left[\pi_1 (x) - i\pi_2 (x) \right] \tag{6}$$

and

$$\pi_0 (x) = \pi_3 (x) .$$

The field $\pi_+ (x)$ contains an annihilation operator for a positive meson and a creation operator for a negative meson.

In terms of these conventions the physical one-particle states are expressed in terms of eigenfunctions of nucleon number, total isotopic spin, and the third component of isotopic spin:

$$\begin{aligned}
|\pi^+\rangle &= -\,|N = 0,\ t = 1,\ t_3 = 1\rangle , \\
|\pi^0\rangle &= |0,\ 1,\ 0\rangle . \\
|\pi^-\rangle &= |0,\ 1,\ -1\rangle , \\
|p\rangle &= |1,\ \tfrac{1}{2},\ \tfrac{1}{2}\rangle , \\
|n\rangle &= |1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\rangle , \\
|\bar{p}\rangle &= |-1,\ \tfrac{1}{2},\ -\tfrac{1}{2}\rangle , \\
|\bar{n}\rangle &= -\,|-1,\ \tfrac{1}{2},\ \tfrac{1}{2}\rangle .
\end{aligned} \tag{7}$$

Many particle states can then be constructed by using the phases assumed by Edmond[5] for the addition of angular momenta.

The initial two-particle state can be characterized by two numbers, t and t_3. The final state has three particles and there are several ways to combine the three spins. We choose to first combine the isotopic spins of the two nucleons into singlet and triplet substates, characterized by the total isotopic spin $\bar{t}$ of the substate. The effect of the generalized Pauli principle -- the antisymmetry of the state when the two nucleons are exchange -- is easily seen. The final state is then characterized by the three numbers, t, t_3, and $\bar{t}$.

The general transition amplitudes between eigenstates of isotopic spin are

$$\left\langle t\,,\, t_3\,,\, \bar{t}\,\middle|\, T \,\middle|\, t\,,\, t_3 \right\rangle = T(t, \bar{t}). \tag{8}$$

These amplitudes cannot depend on the orientation of states in isotopic spin space. Furthermore there are only three of them:

$$\left\langle \frac{3}{2}\,,\, t_3\, 1 \,\middle|\, T \,\middle|\, \frac{3}{2}\,,\, t_3 \right\rangle = T\left(\frac{3}{2}\,,\, 1\right),$$

$$\left\langle \frac{1}{2}\,,\, t_3\,,\, 1 \,\middle|\, T \,\middle|\, \frac{1}{2}\,,\, t_3 \right\rangle = T\left(\frac{1}{2}\,,\, 1\right), \tag{9}$$

and

$$\left\langle \frac{1}{2}\,,\, t_3\, 0 \,\middle|\, T \,\middle|\, \frac{1}{2}\,,\, t_3 \right\rangle = T\left(\frac{1}{2}\,,\, 0\right).$$

The two-nucleon substate with $\bar{t} = 0$ cannot be combined with an anti-nucleon state to produce (a state with) $t = \frac{3}{2}$.

The deuteron has isotopic spin zero, so there is one transition amplitude to the deuteron state:

$$\left\langle \frac{1}{2}, t_3 \, d \,\middle|\, T \,\middle|\, \frac{1}{2}, t_3 \right\rangle = T\left(\frac{1}{2}, d\right) . \tag{10}$$

The transition amplitudes $T\left(\frac{1}{2}, 0\right)$ and $T\left(\frac{1}{2}, d\right)$ are related directly by the final-state interaction.

The useful information in this decomposition is the expression of the physical transition amplitudes in terms of eigenamplitudes of isotopic spin. These amplitudes are given in Table I. Once the cross sections for transitions between eigenstates are calculated, the cross sections for any of the transitions given by Eq. (3) may be gotten by using this table.

Table I

T[illegible] coefficients of the expansion of transition amplitudes in terms of the eigenamplitudes of isotopic spin

Transition final state	Amplitude initial state	Eigenamplitude coefficients $T(\frac{3}{2}, 1)$	$T(\frac{1}{2}, 1)$	$T(\frac{1}{2}, 0)$	$T(\frac{1}{2}, d)$
$\bar{n}pp$	$\pi^+ p$	1	0	0	0
$\bar{p}pp$	$\pi^0 p$	$\sqrt{2/9}$	$\sqrt{2/9}$	0	0
$\bar{n}pn$	$\pi^0 p$	$-\sqrt{2/9}$	$\sqrt{1/18}$	$\sqrt{1/6}$	0
$\bar{p}pp$	$\pi^+ n$	-1/3	2/3	0	0
$\bar{n}pn$	$\pi^+ n$	1/3	1/3	$\sqrt{1/3}$	0
$\bar{n}nn$	$\pi^0 n$	$-\sqrt{2/9}$	$-\sqrt{2/9}$	0	0
$\bar{p}pn$	$\pi^0 n$	$\sqrt{2/9}$	$-\sqrt{1/18}$	$\sqrt{1/6}$	0
$\bar{n}nn$	$\pi^- p$	-1/3	2/3	0	0
$\bar{p}pn$	$\pi^- p$	1/3	1/3	$-\sqrt{1/3}$	0
$\bar{p}nn$	$\pi^- n$	1	0	0	0
$\bar{n}d$	$\pi^0 p$	0	0	0	$\sqrt{1/3}$
$\bar{n}d$	$\pi^+ n$	0	0	0	$\sqrt{2/3}$
$\bar{p}d$	$\pi^0 n$	0	0	0	$-\sqrt{1/3}$
$\bar{p}d$	$\pi^- p$	0	0	0	$\sqrt{2/3}$

B. Perturbation Theory

We now turn our attention to the basic matrix elements for the reaction of Eq. (3). Our purpose is to define a model for the basic production process, which is then modified by the inclusion of the final-state interaction.

In Appendix II we derive a general expression for the transition amplitude in terms of 28 arbitrary functions. We use in this derivation the invariance of a cross section under Lorentz transformations and rotations in the isotopic spin space.

We write the transition amplitude in terms of a Lorentz-invariant matrix element M' :[4]

$$T_{fi} = - i \, (2\pi)^4 \, \frac{m^2 M'_{fi}}{\sqrt{2 \, \omega \, E_1 \, E_2 \, E_3 \, E_4}} \quad \delta^4 \, (p_f - p_i) \, , \tag{11}$$

where E_1 refers to the energy of nucleon 1 with momentum $\underset{\sim}{p}_1$ and ω is the energy of the pion with momentum $\underset{\sim}{k}$.

We construct the invariant matrix with four spinors defined by the plane-wave expansion of Eq. (13) ,

$$M = \bar{u}_4 \, 0^{(1)} \, u_1 \, \bar{u}_3 \, 0^{(2)} \, v_2 \, . \tag{12}$$

Each of the functions o is a two-by-two matrix in the isotopic spin space and a four-by-four matrix in the space of the Dirac spinors. In the isotopic spin space we have

$$M = \left[M_a \;\; \tau_k{}^{(1)} + M_b \; \tau^{(2)}{}_k + i\epsilon_{ij} \, k \;\; M_c \;\; \tau_i{}^{(1)} \; \tau_j{}^{(2)} \right] \; \epsilon_k \tag{13}$$

where the superscripts refer to the spinor spaces of Eq. (12), and ϵ_k is a component of the unit vector in the isotopic spin space.

Each of these matrices, M_α, is expanded in terms of the sixteen Dirac matrices. Assuming invariance under time reversal and space reflection, we find that each matrix has the form

$$
\begin{aligned}
M_\alpha = {} & A_\alpha \bar{u}_4 u_1 \bar{u}_3 \gamma_5 v_2 \\
& + B_\alpha \bar{u}_4 \gamma_5 u_1 \bar{u}_3 v_2 \\
& + C_\alpha \bar{u}_4 \gamma \cdot k u_1 \bar{u}_3 \gamma_5 v_2 \\
& + D_\alpha \bar{u}_4 \gamma_5 u_1 \bar{u}_3 \gamma \cdot k v_2 \\
& + E_\alpha \bar{u}_4 \gamma \cdot q u_1 \bar{u}_3 \gamma_5 v_2 \\
& + F_\alpha \bar{u}_4 \gamma_5 u_1 \bar{u}_3 \gamma \cdot p v_2 \\
& + \frac{G_\alpha}{} \bar{u}_4 \gamma_5 \gamma \cdot k u_1 \bar{u}_3 \sigma^{\alpha\beta} k_\alpha p_\beta v_2 \\
& + \frac{H_\alpha}{} \bar{u}_4 \sigma^{\alpha\beta} k_\alpha q_\beta u_1 \bar{u}_3 \gamma_5 \gamma \cdot k v_2 \\
& + \frac{I_\alpha}{} \bar{u}_4 \gamma_5 \gamma \cdot q u_1 \bar{u}_3 \sigma^{\alpha\beta} k_\alpha p_\beta v_2 \\
& + \frac{J_\alpha}{} \bar{u}_4 \sigma^{\alpha\beta} k_\alpha q_\beta u_1 \bar{u}_3 \gamma_5 \gamma \cdot p v_2 \\
& + \frac{K_\alpha}{2} \bar{u}_4 \epsilon^{\alpha\beta\gamma\delta} \sigma_{\alpha\beta} k_\gamma q_\delta u_1 \bar{u}_3 \sigma^{\rho\eta} k_\rho p_\eta v_2
\end{aligned}
$$

$$+ \frac{L_a}{2} \; \bar{u}_4 \sigma^{\rho\eta} k_\rho q_\eta u_1 \bar{u}_3 \epsilon^{\alpha\beta\gamma\delta} \sigma_{\alpha\beta} k_\gamma p_\delta v_2 ,$$

$$+ M \; \bar{u}_4 \gamma_5 \gamma_\alpha u_1 \bar{u}_3 \sigma^{\alpha\beta} k_\beta v_2 ,$$

$$+ N \; \bar{u}_4 \sigma^{\alpha\beta} k_\beta u_1 \bar{u}_3 \gamma_5 \gamma_\alpha v_2 ,$$

$$+ O \; \bar{u}_4 \gamma_5 \gamma_\alpha u_1 \bar{u}_3 \sigma^{\alpha\beta} p_\beta v_2 ,$$

$$+ \underline{P} \; \bar{u}_4 \sigma^{\alpha\beta} q_\beta u_1 \bar{u}_3 \gamma_5 \gamma_\alpha v_2 , \qquad (14)$$

$$+ Q \; \epsilon^{\alpha\beta\gamma\delta} \bar{u}_4 \sigma_{\alpha\beta} u_1 \bar{u}_3 \sigma_{\alpha\delta} v_2 ,$$

where $q = \dfrac{p_3 - p_2}{2}$ and $p = \dfrac{p_1 + p_4}{2}$.

Each of the coefficients is a function of the five independent scalars

$$s = (k + p_1)^2 ,$$

$$t = (k - p_2)^2 ,$$

$$u = (k - p_4)^2 ,$$

$$v = (p_1 - p_3)^2 , \qquad (15)$$

and

$$r = (p_2 + p_3)^2 .$$

We express the exchange symmetries of the transition by defining

$$M' = M - \overline{M} , \quad (16)$$

where $\overline{M}$ is obtained from M by exchanging all the coordinates of particles 3 and 4, the two final nucleons. The exchange of particles 1 and 2 gives the relations

$$B_a (s, t, u, v, r) = + A_b (t, s, \bar{u}, \bar{r}, \bar{v}) ,$$

$$B_b (s, t, u, v, r) = + A_a (t, s, \bar{u}, \bar{r}, \bar{v}) , \quad (17)$$

and

$$B_c (s, t, u, v, r) = - A_c (t, s, \bar{u}, \bar{r}, \bar{v}) ,$$

where

$$\bar{u} = (k - p_3)^2 ,$$

$$\bar{v} = (p_1 - p_4)^2 ,$$

and

$$\bar{r} = (p_2 + p_4)^2 .$$

There are similar relations between

C and D, E and F, and

each of the other pairs of coefficients. For Q we find the relations

$$Q_a (s, t, u, v, r) = + Q_b (t, s, \bar{u}, \bar{r}, \bar{v})$$

and

$$Q_c (s, t, u, v, r) = - Q_c (t, s, \bar{u}, \bar{r}, \bar{v}) . \quad (18)$$

This expansion of the transition amplitude is not unique, but has the advantage that perturbation theory with pseudoscalar coupling gives terms of the same form.

We take as basic matrix elements the first nonvanishing terms in the Feynman-Dyson expansion[6] of the transition amplitude. We assume the interaction Hamiltonian,

$$H_{int} = G \int d^3 x \, \overline{\Psi}(x) \, \gamma_5 \, \vec{\tau} \cdot \vec{\pi}(x) \, \psi(x) \, . \qquad (19)$$

Certainly the next few terms in the expansion contribute significantly to the amplitude; we take the lowest order only for the sake of a simple, definite model for the process.

This perturbation theory gives terms of the same form as the general expansion. These terms contain simple poles in the scalar invariants with residues related to the pion-nucleon coupling constant. The locations of these poles can be determined without reference to perturbation theory. They occur whenever it is possible for a pair of the external particles to form an intermediate state with a definite mass, but they are located outside the physical range of the variables.

Perturbation theory corresponds to the choice of coefficients:

$$C_b = \frac{G^3}{(2\pi)^{15/2}} \frac{1}{r - \mu^2} \left[-\frac{1}{s - m^2} + \frac{1}{u - m^2} \right] ,$$

$$C_c = \frac{G^3}{(2\pi)^{15/2}} \frac{1}{r - \mu^2} \left[-\frac{1}{s - m^2} - \frac{1}{u - m^2} \right] ,$$

$$D_a = \frac{G^3}{(2\pi)^{15/2}} \frac{1}{\bar{v} - \mu^2} \left[-\frac{1}{t - m^2} + \frac{1}{\bar{u} - m^2} \right] ,$$

and

$$D_c = \frac{G^3}{(2\pi)^{15/2}} \frac{1}{\bar{v} - \mu^2} \left[+ \frac{1}{t - m^2} + \frac{1}{\bar{u} - m^2} \right] . \tag{20}$$

None of the other terms contributes.

The analyticity properties of transition amplitudes involving five particles have not been investigated systematically, but one expects that these poles will be present in any future theory of many-particle interactions. There is of course one pole that does not appear in perturbation theory. The two nucleons in the final state can form a deuteron, which should be represented by a pole in the variable $\bar{x} = (p_3 + p_4)^2$; the pole is located at the $\bar{x} = M^2$ with a residue proportional to the deuteron normalization constant. The effect of this intermediate state is precisely what we are calculating by explicit integration over the scattering wave functions for the two final nucleons.

The third-order contribution to the transition amplitude consists of eight matrix elements, M_j :

$$M'_{fi} = \frac{G^3}{(2\pi)^{15/2}} \sum_{j=1}^{8} C_j M_j . \tag{21}$$

The dependence of the matrix elements upon the isotopic spin is contained in the coefficients C_j . In order to easily substitute scattering-wave functions for plane waves we write the matrix elements in a partially integrated form as a function of the variables,

$$\underset{\sim}{p} = \frac{1}{2} \underset{\sim}{p}_3 - \frac{1}{2} \underset{\sim}{p}_4 \quad \text{and} \quad \underset{\approx}{P} = \underset{\sim}{p}_3 + \underset{\sim}{p}_4 . \tag{22}$$

The integration variables $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are linearly related to the variables occuring naturally in perturbation theory. They are regarded as the relative separation and relative momentum of nucleons 3 and 4 in the intermediate state.

Corresponding to the Feynman diagram of Fig. 1a is the matrix element

$$M_1(3,4) = -\frac{1}{(2\pi)^3}\int d^3\underset{\sim}{r}\; d^3\underset{\sim}{q}\;\frac{u_4\,\gamma\cdot k\,u_1}{(p_1+k)^2 - m^2 + i\epsilon} \tag{23}$$

$$\times\frac{u_3\,\gamma_5\,v_2\,e^{-i\underset{\sim}{r}\cdot(\underset{\sim}{p}-\underset{\sim}{q})}}{(E_2+E_3)^2 - (\underset{\sim}{q}+\underset{\sim}{p}_2+\frac{1}{2}\underset{\sim}{P})^2 - \mu^2 + i\epsilon}\,.$$

We have used here the defining equations for the spinors:

$$(\gamma\cdot p - m)\,u(\underset{\sim}{p}) = 0 \tag{24}$$

and

$$(\gamma\cdot p + m)\,v(\underset{\sim}{p}) = 0\,.$$

We can roughly determine the angular distributions that would result from this term alone by examining the denominators of the matrix element.

We are interested in the angular distribution of the antinucleon as a function of $\cos\theta = \hat{k}\cdot\hat{p}_2$. The integration over $\underset{\sim}{r}$ and $\underset{\sim}{q}$ corresponds to replacing $\underset{\sim}{q}$ by $\underset{\sim}{p}$. In the center-of-momentum system we have

$$\underset{\sim}{p}_1 = -\underset{\sim}{k} \quad\text{and}\quad \underset{\sim}{p}_2 = -\underset{\sim}{P}\,.$$

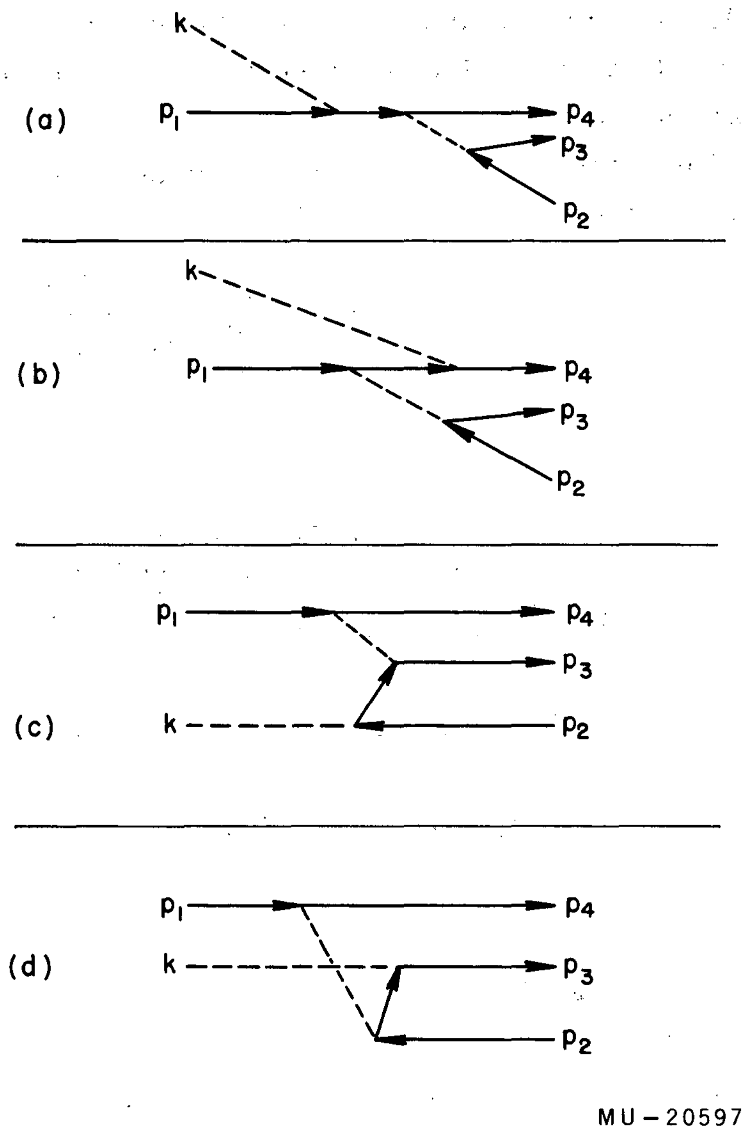

Fig. 1. The diagrams that contribute to the production of antinucleons in the lowest order.

We see that the first denominator, $s - m^2$, is completely isotropic. The second denominator,

$$r - \mu^2 = (E_2 + E_3)^2 - (\underset{\sim}{p} + \tfrac{1}{2}\underset{\sim}{p}_2)^2 - \mu^2 \ , \tag{25}$$

is also isotropic at energies near threshold, where $\underset{\sim}{p}$ and $\underset{\sim}{p}_2$ are both small. We conclude that this term alone gives an isotropic distribution at low energies.

This conclusion agrees with the physical picture of the interaction given by the Feynman diagram. The initial nucleon and the pion come together with equal and opposite momenta. The pion is absorbed, leaving an excited nucleon at rest. There is now no preferred direction in space for the nucleon, so it emits a pion in a direction not correlated with any direction of the initial state. The pion decays into a nucleon-antinucleon pair with an isotropic distribution.

One cannot readily determine the angular distribution at higher energies without actually squaring the matrix element and integrating over the phase space. When one does this one finds that the angular distrubition is indeed almost isotropic at low-energies, and is peaked in the forward direction, $\cos\theta = 1$, at higher energies.

The second diagram, Fig. 1b, goes with the matrix element

$$M_2(3,4) = \frac{1}{(2\pi)^3}\int d^3\underset{\sim}{r}\, d^3\underset{\sim}{q}\ \frac{\bar{u}_4\left[\gamma \cdot k - \underset{\sim}{\gamma}\cdot(\underset{\sim}{q} - \underset{\sim}{p})\right] u_1}{(\omega - E_4)^2 - (\underset{\sim}{q} + \underset{\sim}{k} - \frac{1}{2}\underset{\approx}{P})^2 - m^2\, i\epsilon}$$

$$\times \frac{\bar{u}_3\, \gamma_5\, v_2\, e^{-ir\cdot(\underset{\sim}{p} - \underset{\sim}{q})}}{(E_2 + E_3)^2 - (\underset{\sim}{q} + \underset{\sim}{p}_2 + \frac{1}{2}\underset{\approx}{P})^2 - \mu^2 + i\epsilon} \tag{26}$$

Again we can draw conclusions about the angular distributions by examining the diagram. We have the pion and nucleon coming together with equal and opposite momenta, but the nucleon emits a pion before the collision. This pion then decays in a nucleon-antinucleon pair, and each of these two particles has acquired some of the momentum of the initial nucleon. We expect then that the antinucleon will be emitted in the same direction as the incoming nucleon or predominantly backward relative to the incoming pion.

The matrix element is the same as M_1 except for the first denominator,

$$u - m^2 = \mu^2 - 2\omega E_4 - 2\,\underset{\sim}{k}\cdot\underset{\sim}{p} - \underset{\sim}{k}\cdot\underset{\sim}{p}_2 . \qquad (27)$$

We see that this denominator is smallest at $\cos\theta = -1$.

We can compare directly the magnitudes of M_1 and M_2 at threshold:

$$\left.\frac{M_1}{M_2}\right]_{\text{threshold}} = -\frac{\mu^2 - 2\omega m}{W^2 - m^2} \cong \frac{1}{3} . \qquad (28)$$

Associated with the third diagram (Fig. 1 c) is the matrix element

$$M_3\,(3,4) = \frac{-1}{(2\pi)^3}\int d^3\underset{\sim}{r}\; d^3\underset{\sim}{q}\; \frac{\bar{u}_4\,\gamma_5\,u_1}{(E_1 - E_4)^2 - (\underset{\sim}{q} + \underset{\sim}{p}_1 - \frac{1}{2}\underset{\sim}{P})^2 - \mu^2 + i\epsilon}$$

$$\times\, \frac{\bar{u}_3\,\gamma\cdot k\, v_2\, e^{-i\underset{\sim}{r}\cdot(\underset{\sim}{p} - \underset{\sim}{q})}}{(k - p_2)^2 - m^2 + i\epsilon} . \qquad (29)$$

From the diagram we see at once that the incoming meson gives a portion of its momentum directly to the antinucleon. We expect the angular distribution of the antinucleon to be peaked in the forward direction.

We confirm this supposition by looking at the two denominators in the center-of-momentum system,

$$t - m^2 = \mu^2 - 2\omega E_2 + 2\underset{\sim}{k} \cdot \underset{\sim}{p}_2 \tag{30}$$

and

$$\bar{v} - \mu^2 = 2m^2 - \mu^2 - 2 E_1 E_4 + 2 \underset{\sim}{k} \cdot \underset{\sim}{p} + \underset{\sim}{k} \cdot \underset{\sim}{p}_2 .$$

Both are smallest at $\cos\theta = 1$.

The fourth and last diagram contributing to the transition in the lowest order is shown in Fig. 1 d, with the matrix element

$$M_4(3,4) = + \frac{1}{(2\pi)^3} \int d^3\underset{\sim}{r}\, d^3\underset{\sim}{q} \; \frac{\bar{u}_4 \gamma_5 u_1}{(E_1 - E_4)^2 - (\underset{\sim}{q} + \underset{\sim}{p}_1 - \frac{1}{2}\underset{\sim}{P})^2 - \mu^2 + i\epsilon}$$

$$\frac{\bar{u}_3 \left[\gamma \cdot k + \underset{\sim}{\gamma} \cdot (\underset{\sim}{q} - \underset{\sim}{p})\right] v_2\, e^{-i\underset{\sim}{r} \cdot (\underset{\sim}{p} - \underset{\sim}{q})}}{(E_3 - \omega)^2 - (\underset{\sim}{q} - \underset{\sim}{k} + \frac{1}{2}\underset{\sim}{P})^2 - m^2 + i\epsilon} \tag{31}$$

We conclude from the denominators,

$$\bar{u} - m^2 = \mu^2 - 2\omega E_3 + 2\,\underset{\sim}{k} \cdot \underset{\sim}{p} + \underset{\sim}{k} \cdot \underset{\sim}{p}_2$$

and (32)

$$\bar{v} - \mu^2 = 2m^2 - \mu^2 - 2E_1 E_2 + 2\,\underset{\sim}{k} \cdot \underset{\sim}{p} + \underset{\sim}{k} \cdot \underset{\sim}{p}_2 \;,$$

that this matrix element also gives an angular distribution peaked in the forward direction.

At threshold the last two matrix elements are related by

$$\left.\frac{M_3}{M_4}\right|_{\text{threshold}} = -\frac{\bar{u} - m^2}{t - m^2} = -\frac{\mu^2 - 2\omega m}{\mu^2 - 2\omega m} = -1 . \qquad (33)$$

The other four matrix elements are gotten by exchanging the coordinates of particles 3 and 4:

$$M_5\,(3,\,4) = -\,M_1\,(4,\,3)\,,$$

$$M_6\,(3,\,4) = -\,M_2\,(4,\,3)\,,$$

$$M_7\,(3,\,4) = -\,M_3\,(4,\,3)\,, \qquad (34)$$

and

$$M_8\,(3,\,4) = -\,M_4\,(4,\,3)\,.$$

The behavior of the individual matrix elements at low energies is not altered to any appreciable extent by the exchange.

A crossing relation becomes apparent when we note that diagram No. 3 can be obtained from diagram No. 1 and diagram No. 4 from diagram No. 2 by exchanging the external lines of particle 1 with those of particle 2 and the external lines of particle 3 with those of particle 4. From the structure of the scattering matrix it is easily seen that for

$$p_1 \leftrightarrow - p_2$$

and

$$u_1 \leftrightarrow v_2$$

then

$$M_1 \,(3,\ 4) \leftrightarrow - M_7 \,(4,\ 3) \tag{35}$$

and

$$M_2 \,(3,\ 4) \leftrightarrow - M_8 \,(4,\ 3)\ .$$

The angular dependence of a cross section in perturbation theory is determined by the coefficients C_j of Eq. (21), the relative weights of the matrix elements M_j. These coefficients are given in Table II for transitions between eigenstates of isotopic spin.

Table II

The relative contributions of perturbation theory matrix elements to the transition amplitudes

Transition	Coefficients							
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
T(3/2, 1)	0	$\sqrt{8}$	$\sqrt{2}$	$-\sqrt{2}$	0	$\sqrt{8}$	$\sqrt{2}$	$-\sqrt{2}$
T(1/2, 1)	$\sqrt{9/2}$	$-\sqrt{1/2}$	$\sqrt{1/2}$	$\sqrt{25/2}$	$\sqrt{9/2}$	$-\sqrt{1/2}$	$\sqrt{1/2}$	$\sqrt{25/2}$
T(1/2, 0)	$\sqrt{27/2}$	$-\sqrt{3/2}$	$-\sqrt{27/2}$	$\sqrt{3/2}$	$-\sqrt{27/2}$	$\sqrt{3/2}$	$\sqrt{27/2}$	$-\sqrt{3/2}$
T $(\pi + p \rightarrow \bar{p} + p + n)$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{8}$	0	$\sqrt{8}$	0	$-\sqrt{2}$	$\sqrt{2}$

C. The Final-State Interaction

The main problem that we are considering is the effect upon a simple model for antinucleon production of an interaction that causes a scattering and binding of the particles in the final state. The general formalism for such final-state interactions has been developed by Watson,[7, 8, 9] using the operator methods of Lippman and Schwinger.[10] He found that it is legitimate to separate the basic production reaction from the scattering in the final state when certain conditions are satisfied.

First the primary interaction must have a short range. Then it is possible to think of the production of particles by the primary interaction, followed by a distortion of production amplitude by a secondary interaction which also has a short range. A second condition is that the final-state interaction must be strong and attractive. We may understand this by considering the reaction as proceeding backward in time. We have several particles scattering from one another. The interaction causing the scattering must be attractive in order to guide these particles into the small region in which the primary interaction is effective. In general we can say that the scattering cross section must be greater than the effective cross section of the primary interaction. As a corollary to these two conditions we expect the final-state interaction to be important only at low relative energies of the emerging particles.

These conditions are certainly satisfied in the production of antinucleons if we ignore the interactions of the antinucleon in the final state.

We proceed on the assumption that the interaction Hamiltonian, V, of the system can be broken into two parts,

$$H = H_0 + V \quad \text{and} \quad V = V_1 + V_2 \quad . \tag{36}$$

We assume V_1 to be responsible for the production of particles and V_2 to cause the scattering and binding in the final state. This separation occurs naturally in some cases, such as in beta decay, where a point interaction causes the emission of an electron and a neutrino, and a Coulomb interaction causes a scattering of the electron in the field of the residual nucleus. In the production of antinucleons the same interaction causes both production and scattering, so we symbolically include in V_2 a projection operator so that it acts only on the final state.

Two complete sets of eigenstates of the unperturbed Hamiltonian are defined for the initial and the final state:

$$H_0 \left| \chi_i \right\rangle = E \left| \chi_i \right\rangle$$

and

$$H_0 \left| \chi_f \right\rangle = E \left| \chi_f \right\rangle . \qquad (37)$$

The assumption that V_2 produces scattering only in the final state is expressible as

$$V_2 \left| \chi_i \right\rangle = 0 . \qquad (38)$$

The state vector of the system can be expanded in terms of one of these complete sets:

$$\left| \psi_i^{(+)} \right\rangle = \left| \chi_i \right\rangle + \frac{1}{E - H_0 + i\epsilon} (V_1 + V_2) \left| \psi_i^{(+)} \right\rangle . \qquad (39)$$

The small positive imaginary part of the denominator insures that this solution contains only outgoing waves in addition to the plane wave. At the end of the calculation ϵ is to go to zero.

It is convenient to define the wave operator, $\Omega^{(+)}$:

$$|\psi_i^{(+)}\rangle = \Omega^{(+)} |\chi_i\rangle = \left[1 + \frac{1}{E + i\epsilon - H_0 - V_1 - V_2} (V_1 + V_2) \right] |\chi_i\rangle . \tag{40}$$

In terms of $\Omega^{(+)}$ the transition matrix for the system can be written as

$$T_{fi} = \langle \chi_f | (V_1 + V_2) \Omega^{(+)} | \chi_i \rangle = \langle \chi_f | (V_1 + V_2 | \psi_i^{(+)} \rangle . \tag{41}$$

In the same way a solution of the scattering problem involving only V_2 can be defined. A solution of $H' = H_0 + V_2$ that corresponds to an incoming spherical wave and a plane wave is

$$|\phi_f^{(-)}\rangle = |\chi_f\rangle + \frac{1}{E - H_0 - i\epsilon} V_2 |\phi_f^{(-)}\rangle$$

$$= \left[1 + \frac{1}{E - H_0 - V_2 - i\epsilon} V_2 \right] |\chi_f\rangle \tag{42}$$

$$= \omega^{(-)} | \chi_f \rangle .$$

It is then a straightforward (though tedious) matter to show

$$(V_1 + V_2)\, \Omega^{(+)} = \omega^{(-)\dagger} V_1\, \Omega^{(+)} + \omega^{(-)\dagger}\, V_2 \; . \tag{43}$$

The matrix elements of this operator form the transition matrix:

$$T_{fi} = \left\langle \chi_f \,\middle|\, (V_1 + V_2)\, \Omega^{(+)} \,\middle|\, \chi_i \right\rangle = \left\langle \phi_f^{(-)} \,\middle|\, V_1 \,\middle|\, \psi_f^{(+)} \right\rangle \; . \tag{44}$$

Here Eq. (38) has been used to eliminate the second term of Eq. (43).

Comparing Eqs. (41) and (44), we are led to the following conclusions. We are directed to calculate the basic production process using only V_1. However, instead of using plane waves for the final state we use solutions of the scattering problem defined by Eq. (42). These solutions are known at low energies.

In antinucleon production there are three particles in the final state, and the many-body problem is no less difficult in quantum mechanics than in classical mechanics. A first approximation is to let one of the particles go free and consider only the scattering of the other two. This is what we will do: consider only the scattering of the two nucleons in Reaction (1), and assume the antinucleon in the final state to be a plane wave.

The next approximation would be the inclusion of a correction factor to account for the interaction of the antinucleon. In the analogous problem of the production of pions in nucleon-nucleon collisions Mandelstam has followed such a procedure.[11] He considered the most important final-state interaction to be the scattering of the pion with one of the final nucleons. The nucleon-nucleon interaction in the final state is represented by a correction factor.

The interaction of an antinucleon with a nucleon is not as simple but involves the annihilation into many pions. We will simply ignore it.

D. The Modified Wave Functions

We must now replace the plane waves of perturbation theory with the wave functions for nucleon-nucleon scattering. For energies up to 10 Mev in the two-nucleon center-of-momentum system the scattering is almost entirely in states of zero angular momentum. The scattering at low energies is described very well by the effective range approximation,[12] in which the phase shifts are given in terms of two parameters for each spin state, the scattering length a_s , and the effective range r_{0s} .

However, the scattering cross sections depend only on the asymptotic form of the wave function, and very little is known about its detailed behavior close to the origin. For our purposes these details are important, since parts of the basic interaction, the interaction responsible for the production of particles, have a range of the order of the Compton wave length of the nucleon. We take the point of view that the range of an interaction is determined by the masses of intermediate states in perturbation theory.

We use a wave function describing a plane wave plus a distorted wave that corresponds to incoming particles. The plane-wave part reproduces the original perturbation matrix elements of our model. We consider only s waves for the distorted wave function. This should be a good approximation for kinetic energies of the nucleon-nucleon system up to 20 Mev.

To determine the s-wave part of the wave function we solve the Schrödinger equation for a square-well potential of radius R and depth V. We include a hard core[13] of radius b. This hard core is needed to describe scattering at energies of around 200 Mev and is also present in potentials calculated from meson theory.[14,15]

We completely neglect the noncentral parts of the potential. Such terms would be needed to describe the polarization of nucleons and to explain the quadrupole moment of the deuteron, but are interested primarily in the binding effects, for which the central potential is adequate.

In the center-of-momentum system of the two-nucleon substate, where the relative momentum of particles 3 and 4 is $\underset{\sim}{p}'$, we modify perturbation theory by the substitution

$$u_3 u_4 \, e^{i\underset{\sim}{p}' \cdot \underset{\sim}{r}} \rightarrow \phi^{(-)}(3,4) ,$$

where

$$\phi^{(-)}(3,4) = \chi^{s}(3,4) \left[e^{i\underset{\sim}{p}' \cdot \underset{\sim}{r}} + \phi_{F.S.}(r) \right] . \qquad (45)$$

The explicit solution of the Schrödinger equation for a square-well potential with an infinite core is then

$$\phi_{FS}(r) = - \frac{\sin(p' r)}{p' r} \quad \text{when} \quad 0 < r < b ,$$

$$\phi_{FS}(r) = \frac{\sin[\beta(r-b)] \, e^{-i\delta} \sin(p' R + \delta)}{p' r \sin[\beta(R-b)]} - \frac{\sin(p' r)}{p' r} , \qquad (46)$$

for $b < r < R$,

and

$$\phi_{FS}(r) = \frac{e^{-i\delta} \sin\delta \, e^{-ip' r}}{p' r} \quad \text{for} \quad r > R.$$

The parameters β and δ are determined by the equations

$$\beta^2 = m V + (p')^2$$

and

$$p' \cot (p' R + \delta) = \beta \cot [\beta (R - b)] . \qquad (47)$$

The effective range expansion is

$$p' \cot \delta = - \frac{1}{a} + \frac{1}{2} r_0 (p')^2 , \qquad (48)$$

where the parameters are given in terms of the potential by

$$a = R - \frac{\tan [\sqrt{mV} (R - b)]}{\sqrt{m V}} \qquad (49)$$

and

$$r_0 = \frac{2a}{3} \left[1 - (1 - \frac{R}{a})^3 \right] - \frac{a - b}{a^2 mV} - (R - b)(1 - \frac{R}{a})^2 .$$

The spin functions, $\chi^s (3, 4)$, are defined in terms of the generalized Pauli spinors, χ, which have four components:

$$\frac{1 + \gamma_0}{2} \chi = \chi$$

and

$$\frac{1 - \gamma_0}{2} \chi = 0 . \qquad (50)$$

We use the projection operators for the singlet and triplet spin states to define

$$\chi^1(3,4) = \frac{3 + \underset{\approx}{\sigma_3} \cdot \underset{\approx}{\sigma_4}}{4} \chi_3 \chi_4$$

and

$$\chi^0(3,4) = \frac{1 - \underset{\approx}{\sigma_3} \cdot \underset{\approx}{\sigma_4}}{4} \chi_3 \chi_4 . \qquad (51)$$

When the two final nucleons are bound as a deuteron we use the wave function,

$$u_3 u_4 e^{i\underset{\approx}{p}' \cdot \underset{\approx}{r}} \rightarrow (2\pi)^{3/2} \phi_d(r) \chi'(3,4) ,$$

where $\phi_d(r) = 0$ for $0 < r < b$, (52)

$$\phi_d(r) = N \frac{\sin[\gamma(r - b)]}{r} \quad \text{for } b < r < R ,$$

and $\phi_d(r) = N \sin[\gamma(R - b)]\, e^{-\alpha(r-R)}$ for $r > R$.

An additional parameter is the binding energy B of the deuteron, which is related to the well parameters by

$$\alpha^2 = mB ,$$

$$\gamma^2 = m(V - B) ,$$

$$\gamma \cot[\gamma(R - b)] = -\alpha ,$$

and

$$N^2 = \frac{\alpha}{2\pi} \frac{1}{1 + \alpha(R - b)} . \qquad (53)$$

We assume that the well radius R has a value of about the Compton wave length of the pion and that the core radius b is about one-third of this value, as is indicated by experiments at high energy and by the meson theory of nuclear forces. The actual values of the parameters b, R, and V are adjusted to fit the effective range expansion.

The accepted values of a_s, r_{0s}, and B are[12]

$$a_0 = -23.7\,(1 \pm 0.003)\,(10)^{-13}\ \text{cm},$$

$$r_{00} = 2.49\,(1 \pm 0.01)\,(10)^{-13}\ \text{cm},$$

$$a_1 = 5.38\,(1 \pm 0.004)\,(10)^{-13}\ \text{cm},$$

$$r_{01} = 1.69\,(1 \pm 0.017)\,(10)^{-13}\ \text{cm}, \qquad (54)$$

and

$$B = 2.225 \pm 0.002\ \text{Mev}.$$

We choose for b, R, and V the following values:

$$R_0 = 1.42\,(10)^{-13}\ \text{cm},$$

$$b_0 = 0.506\,(10)^{-13}\ \text{cm},$$

$$V_0 = 144\ \text{Mev}, \qquad (55)$$

$$R_1 = 1.64\,(10)^{-13}\ \text{cm},$$

$$b_1 = 0.69\,(10)^{-13}\ \text{cm},$$

and

$$V_1 = 106 \text{ Mev.}$$

These wave functions are to be used in the evaluation of the matrix elements given in Eqs. (23), (26), (29), and (31). The integration over r corresponds to taking the Fourier transform of the complex conjugate-wave functions. For convenience in the later integrations, we write the transform as a sum of four terms--two delta functions, one of which reproduces the perturbation theory, and a positive and a negative frequency part:

$$\phi^* (\underset{\sim}{q}) = \frac{1}{(2\pi)^3} \int d^3 \underset{\sim}{r}\, e^{i\underset{\sim}{q} \cdot \underset{\sim}{r}}\, \phi^{(-)*}(3,4)$$

$$= \left[\chi^s (3,4) \right]^* \left\{ \delta^3 (\underset{\sim}{p}' - \underset{\sim}{q}) + \frac{i e^{i\delta^s} \sin \delta^s}{2\pi q^2}\, \delta (p' - q) \right.$$

$$\left. + \quad \phi^- (q) + \phi^+ (q) \right\} , \qquad (56)$$

where

$$\phi^- (q) = \phi^+ (-q)$$

and

$$\phi^+ (q) = \frac{e^{i\delta^s}}{(2\pi)^2} \left\{ e^{iqR} \left[\frac{\sin (p'R + \delta^s)}{p'} + \frac{i \cos (p'R + \delta^s)}{q} \right] \right.$$

$$\times \left[\frac{1}{q^2 - (p')^2} - \frac{1}{q^2 - \beta^2} \right]$$

$$+ \frac{ie^{iqb}}{q} \frac{\cos(p'R + \delta^s)}{\cos[\beta(R-b)]} \frac{1}{q^2 - \beta^2} \Bigg\}$$

For the deuteron we find

$$\phi_d^*(q) = \left[\phi_d^+(q) + \phi_d^-(q) \right] \left[\chi'(3,4) \right]^* \tag{57}$$

where

$$\phi_d^+(q) = \frac{N}{(2\pi)^2} \Bigg\{ \left(1 - \frac{i\alpha}{q}\right) e^{iqR} \left[\frac{1}{q^2 + \alpha^2} - \frac{\gamma}{\sqrt{mV}} \frac{1}{q^2 - \gamma^2} \right]$$

$$+ \frac{ie^{iqb}}{q} \frac{\gamma}{q^2 - \gamma^2} \Bigg\}$$

E. The Basic Integrals

There are three basic integrals for the each of the bound and unbound cases. The first has the form

$$I_1 \, [\underset{\sim}{B}, C] = \int d^3\vec{q} \; \frac{\phi^*(\underset{\sim}{q})}{q^2 + 2\underset{\sim}{q} \cdot \underset{\sim}{B} - C - i\epsilon} \; . \qquad (58)$$

The integration over the magnitude of q is performed easily. Two of the terms in ϕ^* $(\underset{\sim}{q})$ contain delta functions. The integration of the term containing $\phi^+(q)$ is done in the complex plane. The range of integration is extended to the entire real axis and the contour is closed above. The small imaginary part of the denominator of Eq. (58) has been retained from the causal propagator of perturbation theory and serves to further define the contour. The only poles that occur on the real q axis are those in ϕ^+ (q) . These are to be treated by integrating above and below them and taking the average. One finds that the residues from these poles cancel and do not contribute to the integral.

The result of this integration and the integration over the aximuthal angle is

$$I_1 \, [\underset{\sim}{B}, C] = \int_{-1}^{1} dZ \; I_1 \, [\underset{\sim}{B}, C, Z] \; ,$$

where

$$I_1 \, [\underset{\sim}{B}, C, Z] = \frac{1}{2} \; \frac{1}{p^2 + 2\underset{\sim}{p} \cdot \underset{\sim}{B} - C}$$

$$+ \frac{ie^{i\delta} \sin\delta}{(p')^2 + 2p' BZ - C} + \frac{i[q_1(Z)]^2}{2[q_1(Z) - BZ]} \; \phi^+ [q_1(Z)] \; , \qquad (59)$$

$$q_1(Z) = BZ + \sqrt{C + B^2 Z^2},$$

and

$$Z = \hat{B} \cdot \hat{q}.$$

The integration over the variable Z was done numerically, by using the IBM650.

The other integrals are

$$I_2 \begin{bmatrix} \underset{\sim}{B}, & C \\ \underset{\sim}{D}, & E \end{bmatrix} = \int d^3\underset{\sim}{q} \; \frac{\phi^*(\underset{\sim}{q})}{q^2 + 2\underset{\sim}{q} \cdot \underset{\sim}{B} - C - i\epsilon} \; \frac{1}{q^2 + 2\underset{\sim}{q} \cdot \underset{\sim}{D} - E - i\epsilon}$$

and (60)

$$I_3 \begin{bmatrix} \underset{\sim}{B}, & C \\ \underset{\sim}{D}, & E \end{bmatrix} = \int d^3 q \; \frac{\phi^*(\underset{\sim}{q})}{q^2 + 2\underset{\sim}{q} \cdot \underset{\sim}{B} - C - i\epsilon} \; \frac{\underset{\sim}{\gamma} \cdot \underset{\sim}{q}}{q^2 + 2\underset{\sim}{q} \cdot \underset{\sim}{D} - E - i\epsilon} .$$

The results are, except for a final integration,

$$I_2 \begin{bmatrix} \underset{\sim}{B}, & C \\ \underset{\sim}{D}, & F \end{bmatrix} Z = \frac{1}{2} \; \frac{1}{p^2 + 2\underset{\sim}{p} \cdot \underset{\sim}{B} - C} \; \frac{1}{p^2 + 2\underset{\sim}{p} \cdot \underset{\sim}{D} - E}$$

$$+ \frac{ie^{i\delta} \sin\delta}{(p')^2 + 2p' BZ - C} \; \frac{1}{\sqrt{[E - (p')^2 - 2p\, DZ \cos\beta]^2 - 4(p')^2 D^2 (1 - Z^2) \sin^2\beta}}$$

(61)

$$+ \frac{i}{2} \; \frac{[q_1(Z)]^2}{[q_1(Z) - BZ]} \; \frac{\phi^+[q_1(Z)]}{\sqrt{\{[q_1(Z)^2 - E - 2q_1(Z)\, DZ \cos\beta\}^2 - 4[q_1(Z)]^2 D^2 \sin^2\beta(1 - Z^2)}}$$

$$+\frac{i}{2}\,\frac{[q_2(Z)]^2}{[q_2(Z)-DZ]}\;\frac{\phi^+[q_2(Z)]}{\sqrt{\{[q_2(Z)]^2-C-2q_2(Z)BZ\cos\beta\}^2-4[q_2(Z)]^2B^2\sin^2\beta\,(1-Z^2)}},$$

and

$$I_3\begin{bmatrix} \underset{\sim}{B},\ C & \\ & ,\ Z \\ \underset{\sim}{D},\ E & \end{bmatrix} = \frac{1}{2}\;\frac{\underset{\sim}{\gamma}\cdot\underset{\sim}{p}'}{p^2+2\underset{\sim}{p}\cdot\underset{\sim}{B}-C}\;\frac{1}{p^2+2\underset{\sim}{p}\cdot\underset{\sim}{D}-E}$$

$$+\frac{i}{2}\,\frac{e^{i\delta}\sin\delta}{(p')^2+2p'BZ-C}\;\frac{Zp'\,\underset{\sim}{\gamma}\cdot\hat{B}}{\sqrt{[E-(p')^2-2pDZ\cos\beta]^2-4(p')^2D^2(1-Z^2)\sin^2\beta}}$$

$$+\frac{i}{2}\,\frac{e^{i\delta}\sin\delta}{(p')^2+2p'DZ-E}\;\frac{Zp'\,\underset{\sim}{\gamma}\cdot\hat{D}}{\sqrt{[C-(p')^2-2pBZ\cos\beta]^2-4(p')^2B^2(1-Z^2)\sin^2\beta}}$$

$$+\frac{i}{2}\,\frac{[q_1(Z)]^3}{[q_1(Z)-BZ}\;\frac{Z\,\underset{\sim}{\gamma}\cdot\hat{B}\;\phi^+[q_1(Z)]}{\sqrt{\{[q_1(Z)]^2-C\,2q_2(Z)DZ\cos\beta\}^2-4[q_1(Z)]^2D^2\sin^2\beta(1-Z^2)}}$$

$$+\frac{i}{2}\,\frac{[q_2(Z)]^3}{[q_2(Z)-DZ}\;\frac{Z\,\underset{\sim}{\gamma}\cdot\hat{D}\;\phi^+[q_1(Z)]}{\sqrt{\{[q_1(Z)]^2-C-2q_2(Z)BZ\cos\beta\}^2-4[q_2(Z)]^2B^2\sin^2\beta(1-Z^2)}}$$

The symbols not previously defined are

$$q_2(Z) = DZ + \sqrt{E + D^2 Z^2}$$

and (62)

$$\cos\beta = \hat{B} . \hat{D} .$$

Whenever $q_1(Z)$ or $q_2(Z)$ is complex--which is the case for some negative values of C and E--one must take the imaginary parts to be positive.

We need not give explicitly the corresponding integrals over the deuteron wave function. They have exactly the same form except that the terms coming from the delta functions are not present.

The relative importance of the terms in these integrals can be estimated roughly. The first term comes directly from perturbation theory. The next term is of the same order of magnitude at energies near threshold. The coefficient of this term is $ie^{i\delta}\sin\delta = e^{i\delta}\cos\delta - 1,$

(63)

so that when the phase shift is close to 90 deg the first two terms interfere destructively. At threshold, or whenever the momentum p' is zero, we know that the phase shift for triplet spin states is slightly greater than 90 deg and the phase shift for singlet spin states is slightly less. Therefore we can say that the first two terms are smaller at very low energies, or near the end of the antinucleon spectrum at moderate energies.

The positions of the poles $q_1(Z)$ and $q_2(Z)$ vary with the integration variable Z over a wide range, but in general they are of the order of magnitude of a few nucleon mass units. We would expect then that the terms coming from the integration over the function $\phi^+(q)$ are of the same order of magnitude as the perturbation-theory term. However, occurring in $\phi^+(q)$, Eq. (54), is the term

$$\frac{1}{q^2 - (p')^2} - \frac{1}{q^2 - \beta^2} = \frac{-mV}{[q^2 - (p')^2][q^2 - \beta^2]} , \quad (64)$$

which is proportional to V, and is therefore small since in both cases V is around one-tenth of a nucleon mass unit. This term can become large only when the relative momentum p' becomes small. Then the factor $\frac{\sin(p'R+\delta)}{p'}$ is the important one. We expect then that the term in each integral containing $\phi^+(q)$ becomes dominant at the end of the antinucleon spectrum, and that it is relatively unimportant elsewhere.

So at moderate energies we expect that the momentum distribution of the antinucleon is given essentially by perturbation theory whenever the antinucleon momentum is small. Near the maximum antinucleon momentum the final-state interaction becomes dominant, producing a characteristic peak in the distribution.

The interference effects mentioned in an earlier section can occur, but it seems likely that they are masked by other oscillating terms.

F. Cross Sections

We can now write the modified matrix element for any of the reactions of Eq. (3) in terms of the six integrals. Each matrix element is in general a function of five variables when the two final nucleons are not bound:

$$M = M(W, p_2, \theta\ \eta\ \phi) . \tag{65}$$

In the over-all center-of=momentum system, W is the total energy, p_2 the momentum of the antinucleon, and the angles are taken to be

$$\left.\begin{aligned} \hat{K} \cdot \hat{p}_2 &= \cos\theta , \\ \hat{p} \cdot \hat{p}_2 &= \cos\eta , \\ \text{and}\quad \hat{p} \cdot K &= \cos\theta \cos\eta + \sin\theta \sin\eta \cos\phi . \end{aligned}\right\} \tag{66}$$

The transition probability per unit space-time volume into a state with each final momentum between $\underset{\sim}{p}$ and $\underset{\sim}{p} + d\underset{\sim}{p}$ is given by

$$d\lambda(\bar{N}NN) = (2\pi)^4 \left| t_{fi} \right|^2 \delta^4 (p_f - p_i)\, d^3\underset{\sim}{p}_2 d^3\underset{\sim}{p}_3 d^3\underset{\sim}{p}_4 . \tag{67}$$

To find a differential cross section we divide by the total flux and integrate. The flux is

$$\frac{1}{(2\pi)^6}\left(\frac{K}{E_1}+\frac{K}{\omega}\right)=\frac{1}{(2\pi)^6}\frac{KW}{E_1\omega} \quad . \qquad (68)$$

Using Eq. (20) and introducing the relative momentum p, we find the cross section for the production of an antinucleon in association with two unbound nucleons:

$$d\sigma(\overline{N}NN)=\left(\frac{G^2}{4\pi}\right)^3\frac{8m^4}{KWE_2}\frac{d^3\underset{\sim}{p}_2}{4\pi}\frac{p'}{W-E_2}\int\frac{d\Omega p}{4\pi}\frac{\sum_{\text{spin states}}|M|^2}{\left[1-\frac{p_2^{\,2}\cos^2\eta}{(W-E_2)^2}\right]^{3/2}} , \qquad (69)$$

where

$$(p')^2 = \frac{W(E_{2m}-E_2)}{2} ,$$

$$E_{2m} = \frac{W^2-3m^2}{2W} ,$$

$$E_1 = \frac{W^2+m^2-\mu^2}{2W} ,$$

$$E_3-E_4 = -\frac{2\underset{\sim}{p}\cdot\underset{\sim}{p}_2}{W-E_2} , = -\frac{2\,p\,p_2\cos\eta}{W-E_2}$$

and

$$p = \frac{p'}{\left[1 - \frac{p_2^2 \cos^2 \eta}{(W-E_2)^2}\right]^{1/2}} .$$

In the nonrelativistic approximation the transformation from the center of momentum of the two-nucleon substate, where the matrix elements have been evaluated, to the over-all center-of-momentum system is a trival one. We also expand the matrix elements in powers of p. By taking the first term in this expansion we are neglecting terms of the order of $\frac{p^2}{m^2}$.

We define the matrix elements for the two spin states for the two final nucleons with the help of the spin projection operators:

$$M = \sum_{s=0,1} P_s M_s \quad . \tag{70}$$

The matrix elements for the unbound final state are then

$$\begin{aligned} M' = {} & \alpha\, \chi_4^* \gamma \cdot k\, v_1\, \chi_3^* \gamma_5\, v_2 + \bar{\alpha}\, \chi_3^* \gamma \cdot k\, v_1\, \chi_4^* \gamma_5\, v_2 \\ & + \beta \chi_4^* \gamma_5\, v_1\, \chi_3^* \gamma \cdot k\, v_2 + \bar{\beta}\, \chi_3^* \gamma_5\, v_1\, \chi_4^* \gamma \cdot k\, v_2 \\ & + \chi_4^* \underset{\sim}{\gamma} \cdot \underset{\sim}{\xi}\, v_1\, \chi_3^* \gamma_5\, v_2 + \chi_3^* \underset{\sim}{\gamma} \cdot \bar{\underset{\sim}{\xi}}\, v_1\, \chi_4^* \gamma_5\, v_2 \\ & + \chi_4^* \gamma_5\, v_1\, \chi_3^* \underset{\sim}{\gamma} \cdot \underset{\sim}{\lambda}\, v_2 + \chi_3^* \gamma_5\, v_1\, \chi_4^* \underset{\sim}{\gamma} \cdot \bar{\underset{\sim}{\lambda}}\, v_2 \end{aligned} \tag{71}$$

where

$$\begin{Bmatrix} \alpha \\ \\ -\bar{\alpha} \end{Bmatrix} = \begin{Bmatrix} C_{11} \\ \\ C_5 \end{Bmatrix} I_1 \left[\underset{\sim}{p}_2 + \frac{1}{2}\,\underset{\sim}{P}\,,\ (p_2 + \frac{P}{2})^2 - \mu^2 \right] \frac{1}{(p_1 + k)^2 - m^2}$$

$$+ \begin{Bmatrix} C_2 \\ \\ C_6 \end{Bmatrix} I_2 \begin{bmatrix} \underset{\sim}{p}_2 + \frac{1}{2}\,\underset{\sim}{P}\,,\ (p_2 + \frac{P}{2})^2 - \mu^2 \\ \\ \underset{\sim}{k} - \frac{1}{2}\,\underset{\sim}{P}\,,\ (k - \frac{P}{2})^2 - m^2 \end{bmatrix} ,$$

(72)

$$\begin{Bmatrix} \beta \\ \\ -\bar{\beta} \end{Bmatrix} = - \begin{Bmatrix} C_3 \\ \\ C_7 \end{Bmatrix} I_1 \left[\underset{\sim}{p}_1 - \frac{1}{2}\,\underset{\sim}{P}\,,\ (p_1 - \frac{P}{2})^2 - \mu^2 \right] \frac{1}{(k - p_2)^2 - m^2}$$

$$- \begin{Bmatrix} C_4 \\ \\ C_8 \end{Bmatrix} I_2 \begin{bmatrix} \underset{\sim}{p}_1 - \frac{1}{2}\,\underset{\sim}{P}\,,\ (p_1 - \frac{P}{2})^2 - \mu^2 \\ \\ -\underset{\sim}{k} + \frac{1}{2}\,\underset{\sim}{P}\,,\ (k - \frac{P}{2})^2 - M^2 \end{bmatrix} ,$$

$$\left\{ \begin{matrix} \underset{\sim}{\xi} \\ \\ -\underset{\sim}{\bar{\xi}} \end{matrix} \right\} = -\left\{ \begin{matrix} C_2 \\ \\ C_6 \end{matrix} \right\} \left\{ (\underset{\sim}{p}_2 + \frac{1}{2}\underset{\sim}{P}) \; I_{3B} \begin{bmatrix} \underset{\sim}{p}_2 + \frac{1}{2}\underset{\sim}{P}, \; (p_2 + \frac{P}{2})^2 - \mu^2 \\ \\ \underset{\sim}{k} - \frac{1}{2}\underset{\sim}{P}, \; (k - \frac{P}{2})^2 - m^2 \end{bmatrix} \right.$$

$$+ (\underset{\sim}{k} - \frac{1}{2}\underset{\sim}{P}) \; I_{3D} \begin{bmatrix} \underset{\sim}{p}_2 + \frac{1}{2}\underset{\sim}{P}, \; (p_2 + \frac{P}{2})^2 - \mu^2 \\ \\ \underset{\sim}{k} - \frac{1}{2}\underset{\sim}{P}, \; (k - \frac{P}{2})^2 - m^2 \end{bmatrix} \tag{73}$$

and

$$\left\{ \begin{matrix} \underset{\sim}{\lambda} \\ \\ -\underset{\sim}{\bar{\lambda}} \end{matrix} \right\} = -\left\{ \begin{matrix} C_4 \\ \\ C_8 \end{matrix} \right\} \left\{ (\underset{\sim}{p}_1 - \frac{1}{2}\underset{\sim}{P}) \; I_{3B} \begin{bmatrix} \underset{\sim}{p}_1 - \frac{1}{2}\underset{\sim}{P}, \; (p_1 - \frac{P}{2})^2 - \mu^2 \\ \\ -\underset{\sim}{k} + \frac{1}{2}\underset{\sim}{P}, \; (k - \frac{P}{2})^2 - m^2 \end{bmatrix} \right.$$

$$+ (-\underset{\sim}{k} + \frac{1}{2}\underset{\sim}{P}) \; I_{3D} \begin{bmatrix} \underset{\sim}{p}_1 - \frac{1}{2}\underset{\sim}{P}, \; (p_1 - \frac{1}{2}P)^2 - \mu^2 \\ \\ -\underset{\sim}{k} + \frac{1}{2}\underset{\sim}{P}, \; (k - \frac{P}{2})^2 - m^2 \end{bmatrix}.$$

For comparison we also calculate the contribution from the pole terms, lowest-order perturbation theory:

$$M^{pole} = \delta\, \bar{u}_4\, \gamma \cdot k\, v_1\, \bar{u}_3\, \gamma_5\, v_2 + \bar{\delta}\, u_3\, \gamma \cdot k\, v_1\, \bar{u}_4\, \gamma_5\, v_2$$
$$+ \epsilon\, \bar{u}_4\, \gamma_5\, v_1\, \bar{u}_3\, \gamma \cdot k\, v_2 + \bar{\epsilon}\, \bar{u}_3\, \gamma_5\, v_1\, \bar{u}_4\, \gamma \cdot k\, v_2 \ , \tag{74}$$

where

$$\delta = \frac{1}{(p_2 + p_3)^2 - \mu^2} \left[- \frac{C_1}{(p_1 + k)^2 - m^2} + \frac{C_2}{(p_4 - k)^2 - m^2} \right],$$

$$\epsilon = \frac{1}{(p_1 - p_4)^2 - \mu^2} \left[\frac{C_3}{(k - p_2)^2 - m^2} - \frac{C_4}{(p_3 - k)^2 - m^2} \right],$$

$$\bar{\delta} = \frac{1}{(p_2 + p_4)^2 - \mu^2} \left[\frac{C_5}{(p_1 + k)^2 - m^2} - \frac{C_6}{(p_3 - k)^2 - m^2} \right],$$

and

$$\bar{\epsilon} = \frac{1}{(p_1 - p_3)^2 - \mu^2} \left[- \frac{C_7}{(k - p_2)^2 - m^2} + \frac{C_8}{(k - p_4)^2 - m^2} \right]. \tag{75}$$

These matrix elements are proportional to the probability amplitudes for a transition from either of the two initial spin states to any of the eight final spin states. We sum over these states in the usual way by taking traces over the γ matrices.

We use the following relations and definitions:

$$\underset{\sim}{\sigma} = i \gamma_5 \gamma_0 \underset{\sim}{\gamma},$$

$$\sum_{\alpha=1}^{4} \bar{u}_\alpha^{\,r} (\underset{\sim}{p})\, u_\alpha^{\,s} (\underset{\sim}{p}) = \delta_{rs},$$

$$\sum_{\alpha=1}^{4} \bar{v}_\alpha^{\,r} (\underset{\sim}{p})\, v_\alpha^{\,s} (\underset{\sim}{p}) = - \delta_{rs},$$

and

$$\sum_{s=1}^{2} \left[u_\alpha^{\,s} (\underset{\sim}{p})\, \bar{u}_\beta^{\,s} (\underset{\sim}{p}) - v_\alpha^{\,s} (\underset{\sim}{p})\, \bar{v}_\beta^{\,s} (\underset{\sim}{p}) \right] = \delta_{\alpha\beta} . \tag{76}$$

Here the superscripts r and s have two values corresponding to the two spin states. The subscripts α and β refer to the four components of the spinors.

The sum over spin states is then

$$\sum_{\substack{\text{spin} \\ \text{states}}} | M'_s |^2 =$$

$$\frac{(2s+1)}{4} \; | a \pm \bar{a} |^2 \; \frac{E_2 + m}{m^2} \; (\mu^2 m - \mu^2 E_1 + 2\omega\, k \cdot p_1)$$

$$+ \frac{(2S+1)}{4} \left| \beta \pm \overline{\beta} \right|^2 \frac{(E_1 - m)}{m^2} \left(2\omega\, k \cdot p_2 - E_2\, \mu^2 - m\mu^2 \right)$$

$$+ \frac{(2s+1)}{4} \left| \vec{\xi} \pm \vec{\overline{\xi}} \right|^2 \frac{(E_2 + m)(E_1 - m)}{m^2}$$

$$+ \frac{(2s+1)}{4} \left| \vec{\lambda} \pm \vec{\overline{\lambda}} \right|^2 \frac{(E_2 + m)(E_1 - m)}{m^2}$$

$$+ \frac{(-1)^{s+1}}{2} \mathrm{Re} \left[(\alpha \pm \overline{\alpha})(\beta \pm \overline{\beta})^* \right] \frac{1}{m^2} \Big[- m^2\, \mathbf{k} \cdot \mathbf{k}$$

$$+ m \left(\omega \mathbf{k} \cdot \mathbf{p}_2 - \omega \mathbf{k} \cdot \mathbf{p}_1 - E_2\, \mathbf{k} \cdot \mathbf{k} + E_1\, \mathbf{k} \cdot \mathbf{k} \right)$$

$$+ E_1 E_2\, \mathbf{k} \cdot \mathbf{k} + \omega^2\, \mathbf{p}_1 \cdot \mathbf{p}_2 - E_2\, \omega \mathbf{k} \cdot \mathbf{p}_1 - E_1\, \omega\, \mathbf{k} \cdot \mathbf{p}_2 \Big]$$

$$+ \frac{(2s+1)}{4} \mathrm{Re} \left[(\alpha \pm \alpha)\, (\boldsymbol{\xi} \pm \overline{\boldsymbol{\xi}})^* \right] \cdot \frac{E_2 + m)}{m^2} \left[(E_1 - m)\, \mathbf{k} - \omega\, \mathbf{p}_1 \right]$$

$$+ \frac{(2s+1)}{4} \mathrm{Re} \left[(\beta \pm \overline{\beta})\, (\boldsymbol{\lambda} \pm \overline{\boldsymbol{\lambda}})^* \right] \cdot \frac{(E_1 - m)}{m^2} \left[(E_2 + m)\, \mathbf{k} - \omega\, \mathbf{p}_2 \right]$$

$$+ \frac{(-1)^{s+1}}{4} \mathrm{Re} \left[(\alpha \pm \overline{\alpha})\, (\boldsymbol{\lambda} \pm \overline{\boldsymbol{\lambda}})^* \right] \frac{(E_2 + m)}{m^2} \left[(E_1 - m)\, \mathbf{k} - \omega\, \mathbf{p}_1 \right]$$

$$+ \frac{(-1)^{s+1}}{4} \mathrm{Re} \left[(\beta \pm \overline{\beta})\, (\boldsymbol{\xi} \pm \overline{\boldsymbol{\xi}})^* \right] \frac{(E_1 - m)}{m^2} \left[(E_2 + m)\, \mathbf{k} - \omega\, \mathbf{p}_2 \right]$$

$$+ \frac{(-1)^{s+1}}{2} \mathrm{Re} \left[(\boldsymbol{\xi} \pm \overline{\boldsymbol{\xi}})\, (\boldsymbol{\lambda} \pm \overline{\boldsymbol{\lambda}})^* \right] \frac{(E_2 + m)(E_1 - m)}{m^2} \qquad (77)$$

The plus sign between coefficients refers to the singlet spin state.

The sum of spins for the pole terms is evaluated (pion mass neglected):

$$\sum_{\substack{\text{spins}\\ \text{states}}} |M^{\text{pole}}|^2 = \frac{2|\delta|^2}{m^4} \; k\cdot p_1 \, k\cdot p_4 \, (p_2\cdot p_3 + m^2)$$

$$+ \frac{2|\epsilon|^2}{m^4} \; k\cdot p_2 \, k\cdot p_3 \, (p_1\cdot p_4 - m^2)$$

$$+ \frac{1}{2} \frac{\mathrm{Re}\,[\delta\bar{\delta}^*]}{m^4} \; k\cdot p_1 \left[m^2 k\cdot p_1 + k\cdot p_4 \, p_2\cdot p_3 + k\cdot p_2 \, p_2\cdot p_4 - k\cdot p_2 \, p_3\cdot p_4 \right]$$

$$+ \frac{1}{2} \frac{\mathrm{Re}[\epsilon\bar{\epsilon}^*]}{m^4} \; k\cdot p_2 \left[m^2 k\cdot p_2 + k\cdot p_4 \, p_1\cdot p_3 - k\cdot p_1 \, p_3\cdot p_4 + \right.$$

$$\left. + k\cdot p_3 \, p_1\cdot p_4 \right]$$

$$+ \frac{\mathrm{Re}[\bar{\delta}\bar{\epsilon}^*]}{m^4} \; k\cdot p_4 \left[m^2 k\cdot p_4 + p_1\cdot p_3 \, k\cdot p_2 - p_1\cdot p_2 \, k\cdot p_3 + \right.$$

$$\left. + p_2\cdot p_3 \, k\cdot p_1 \right] . \tag{78}$$

To this should be added five more terms gotten by the replacement

$$p_3 \longleftrightarrow p_4 \quad \epsilon \longleftrightarrow \bar{\epsilon} , \quad \delta \longleftrightarrow \bar{\delta} .$$

The differential cross section for the production of an anti-nucleon in association with two unbound nucleons is then

$$4\pi \frac{d\sigma(\bar{N}NN)}{d\Omega_2 dp_2} = 4\left(\frac{G^2}{4\pi}\right)^3 \frac{m^3}{KW} \frac{p_2^2 p'}{E_2} \sum_{s=0,1} \sum_{\substack{\text{spin}\\ \text{states}}} |M_s|^2 . \tag{79}$$

When the two final nucleons are bound as a deuteron we evaluate the matrix elements at $p = 0$ and project out states of spin one only. We find

$$4\pi \frac{d\sigma(\bar{N}d)}{d\Omega_2} = 8\pi^2 \left(\frac{G^2}{4\pi}\right)^3 \frac{p_2}{KmW^2} \sum_{\substack{\text{spin}\\ \text{state}}} |M_{1d}|^2 . \tag{80}$$

III. RESULTS AND CONCLUSIONS

The cross sections for Reaction (1) and for each of the transitions between eigenstates of isotopic spin are shown graphically in Figs. 2-21. They are all evaluated in the center-of-momentum system. The unit of energy is the nucleon mass, and cross sections are given in terms of the basic cross sections, which occurs naturally in perturbation theory

$$\sigma_0 = \left(\frac{G^2}{4\pi}\right)^3 \frac{1}{m^2} = 0.442 \left(\frac{G^2}{4\pi}\right)^3 \text{mb.} \qquad (81)$$

The total cross section for Reaction (1.a) is normalized at an energy of $W/m = 3.1$ to the total cross section given by the statistical model of Appendix I. This corresponds to taking $\sigma_0 = 6.4$ mb.

The total cross section of Reaction (1) is plotted as a function of the total energy in Fig. 2. The perturbation-theory result is also given. At energies less than $W/m = 3.04$, which corresponds to an available energy of 40 Mev, the cross section increases as the square of the available energy. Beyond this point the curve is essentially linear and approaches the perturbation-theory result. This enhancement of low energies is characteristic of the final-state interaction.

The total cross section for the production of a deuteron in conjunction with an antiproton is given in Fig. 3. This cross section increases as the square root of the available energy, the dependence expected from phase space alone.

The ratio of the cross section of Reaction (1.a) to that of Reaction (1.b) is shown in Fig. 4. This ratio is unity at about $W/m = 3.04$, 40 Mev above threshold.

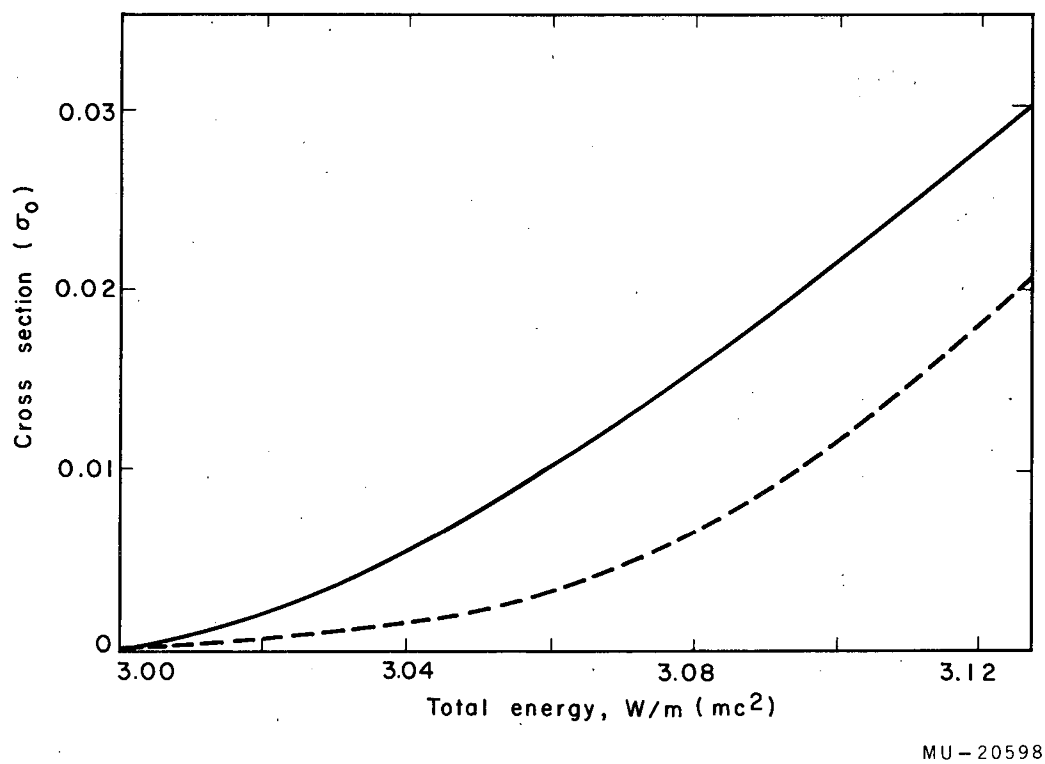

Fig. 2. The total cross section for $\pi^- + p \rightarrow \bar{p} + p + n$ as a function of energy. Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

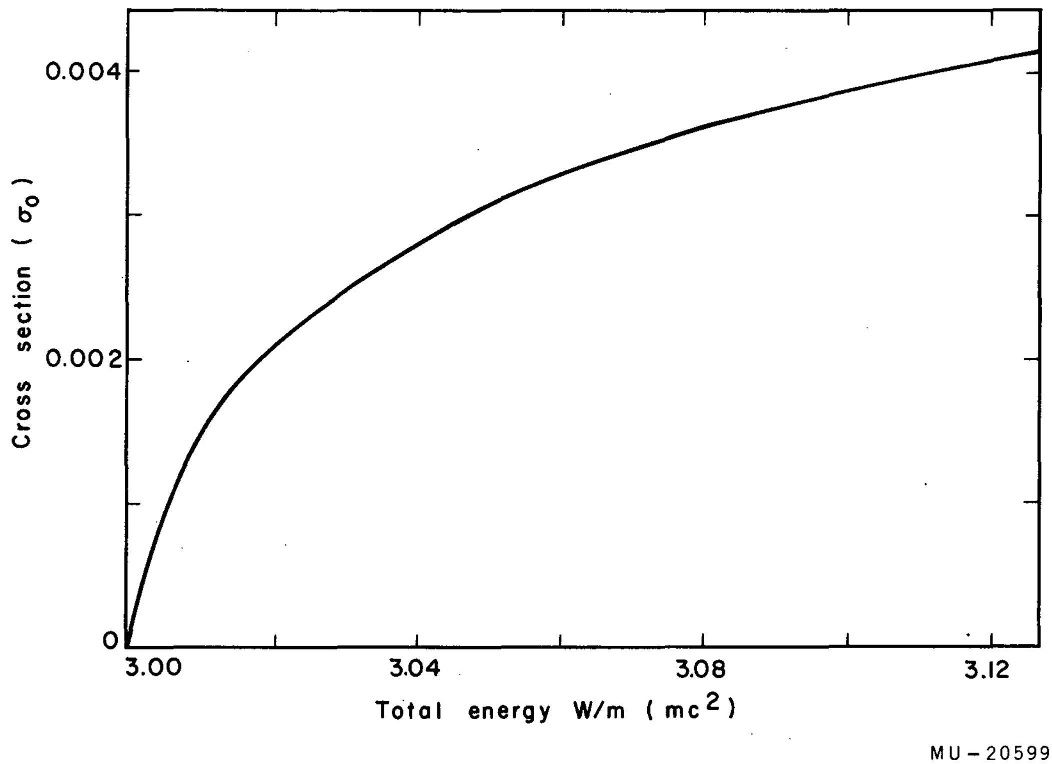

Fig. 3. The total cross section for $\pi^- + p \rightarrow \bar{p} + d$ as a function of energy.

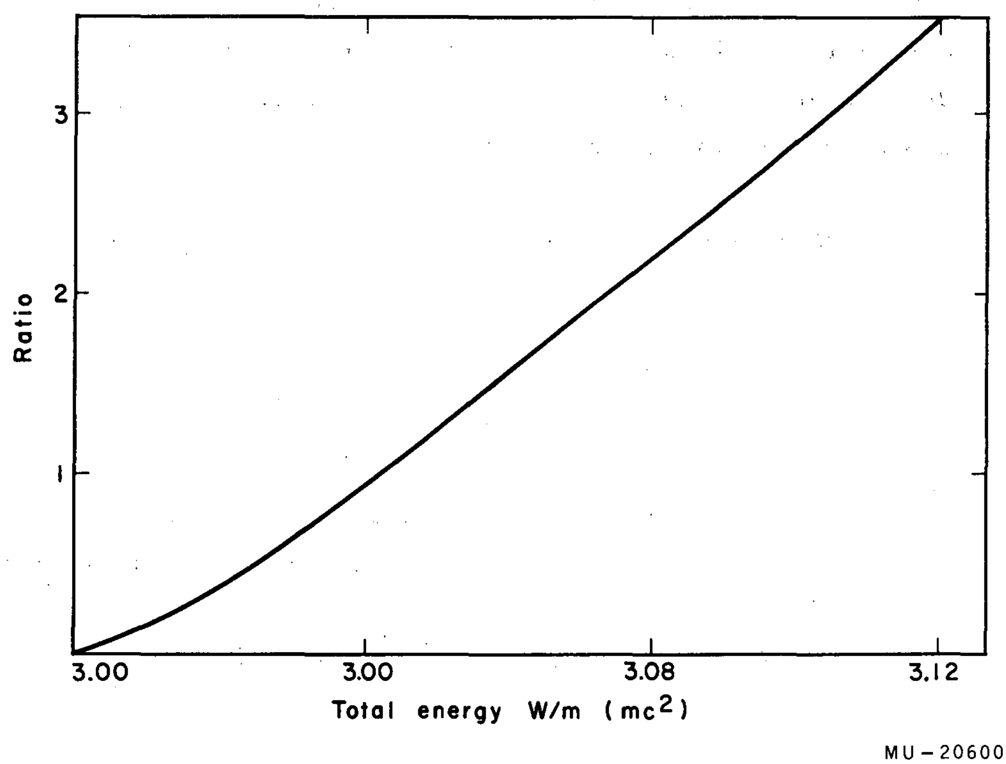

Fig. 4. The ratio $\frac{\sigma_T(\pi^- + p \rightarrow \bar{p} + p + n)}{\sigma_T(\pi^- + p \rightarrow \bar{p} + d)}$ as a function of energy.

The total cross sections for transitions between eigenstates of isotopic spin are given in Figs. 5-7. The cross sections for any of the transitions of Eq. (3) may be obtained from these three cross sections. One must realize that the cross terms between eigenamplitudes give no contribution when one integrates over the solid angle of the antinucleon. Total cross sections and momentum distributions then depend only on the squares of the coefficients given in Table I. For example, we find the relation

$$\sigma(\pi^- + p \rightarrow \bar{p} + p + n) = \sigma(\pi^+ + n \rightarrow \bar{n} + p + n) \tag{82}$$

$$= \frac{1}{9}\,\sigma(\tfrac{3}{2}, 1) + \frac{1}{9}\,\sigma(\tfrac{1}{2}, 1) + \frac{1}{3}\,\sigma(\tfrac{1}{2}, 0) .$$

Angular distributions cannot be obtained in this simple way.

At an energy of $W/m = 3.08$ we find the approximate relation

$$\sigma(\tfrac{1}{2}, 1) : \sigma(\tfrac{3}{2}, 1) : \sigma(\tfrac{1}{2}, 0) \cong 1 : 1.5 : 3.9 . \tag{83}$$

The antinucleon momentum spectra for each of these transitions are plotted in Figs. 8-13 and compared to perturbation theory. The effect of the final-state interactions is plainly seen in the peak at the end of each spectrum. There is no discernible difference in the shape of the spectra for the singlet and the triplet spin states of the nucleon-nucleon system. Whenever the relative momenta of the two final nucleons are greater than about 150 Mev/c one expects that the p-wave nucleon-nucleon scattering becomes important. The region where the s-wave scattering is dominant corresponds roughly to the region where the antinucleon momentum p_2 is within 200 Mev/c of its maximum value.

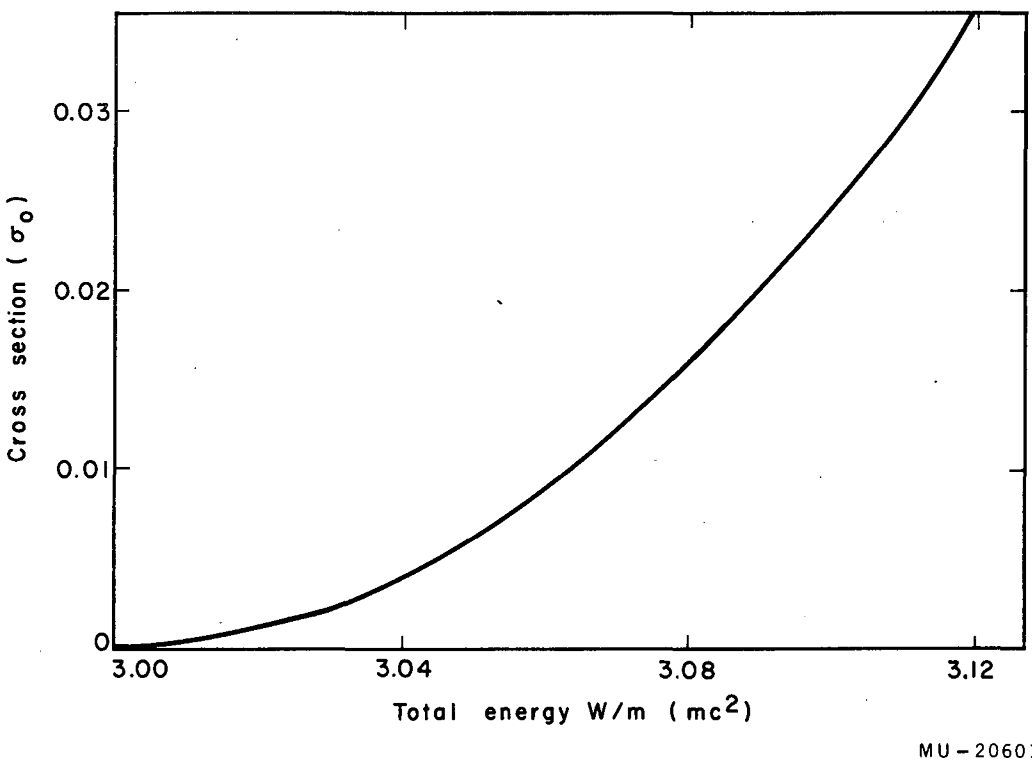

Fig. 5. The total cross section $\sigma\left(\frac{3}{2}, 1\right)$ as a function of energy.

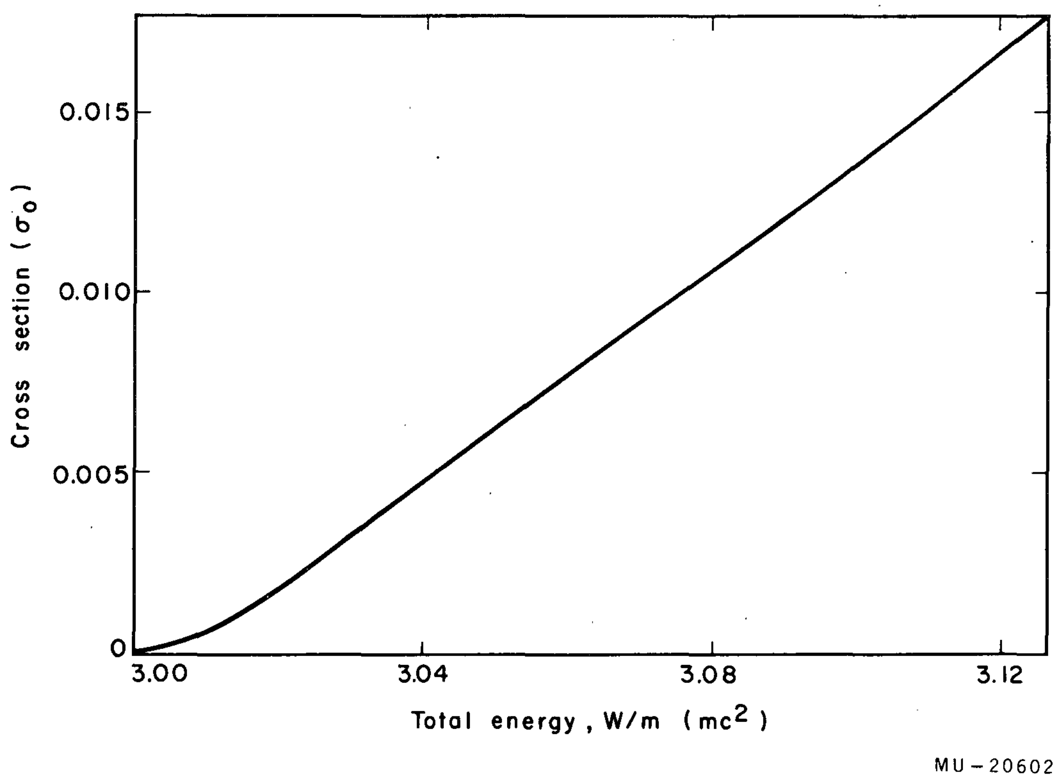

Fig. 6. The total cross section $\sigma\left(\frac{1}{2}, 1\right)$ as a function of energy.

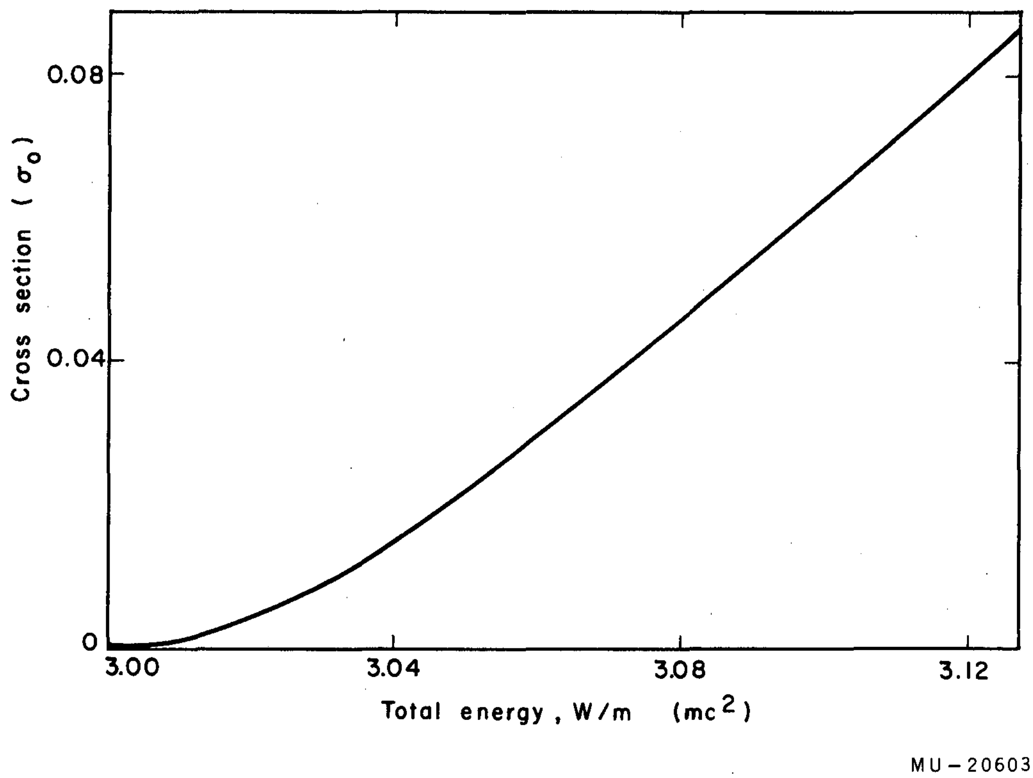

Fig. 7. The total cross section $\sigma\left(\frac{1}{2}, 0\right)$ as a function of energy.

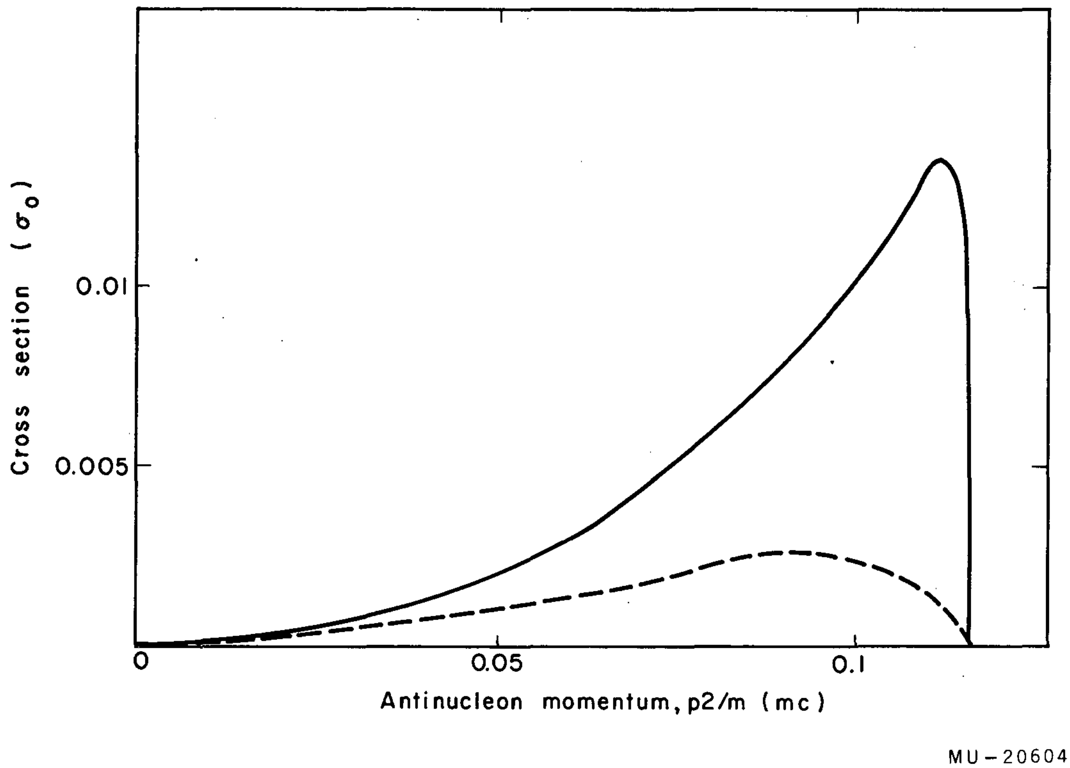

Fig. 3. The momentum distribution

$m\frac{d\sigma}{dp_2}$ ($\pi^- + p \rightarrow \bar{p} + p + n$) as a function of anti-nucleon momentum. $\frac{W}{m}$ = 3.01 . Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

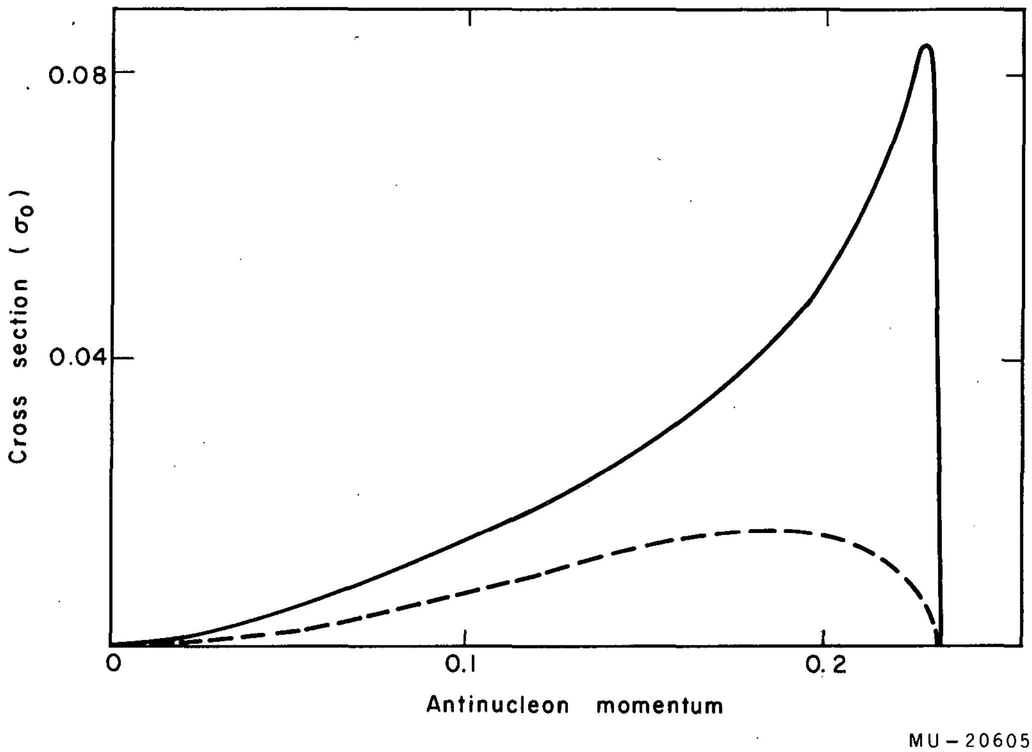

Fig. 9. The momentum distribution

$m\frac{d\sigma}{dp_2}$ $(\pi^- + p \rightarrow \overline{p} + p + n)$ as a function of antinucleon momentum. $\frac{W}{m} = 3.04$. Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

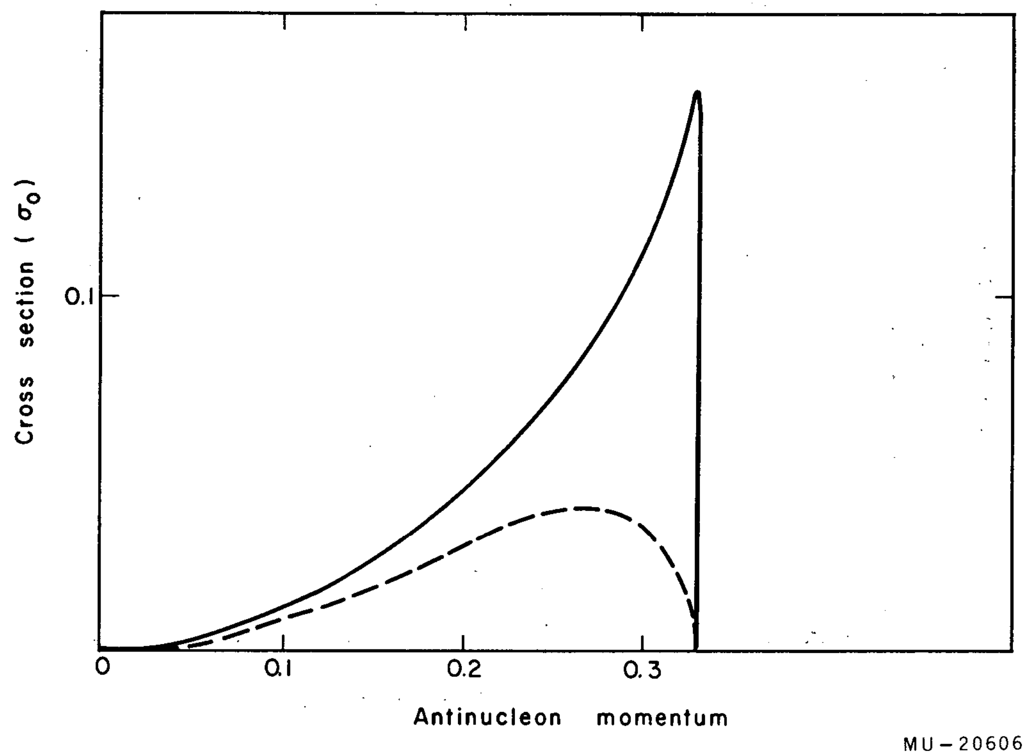

Fig. 10. The momentum distribution

$m\frac{d\sigma}{dp_2}$ ($\pi^- + p \rightarrow \bar{p} + p + n$) as a function of antinucleon momentum. $\frac{W}{m} = 3.08$. Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

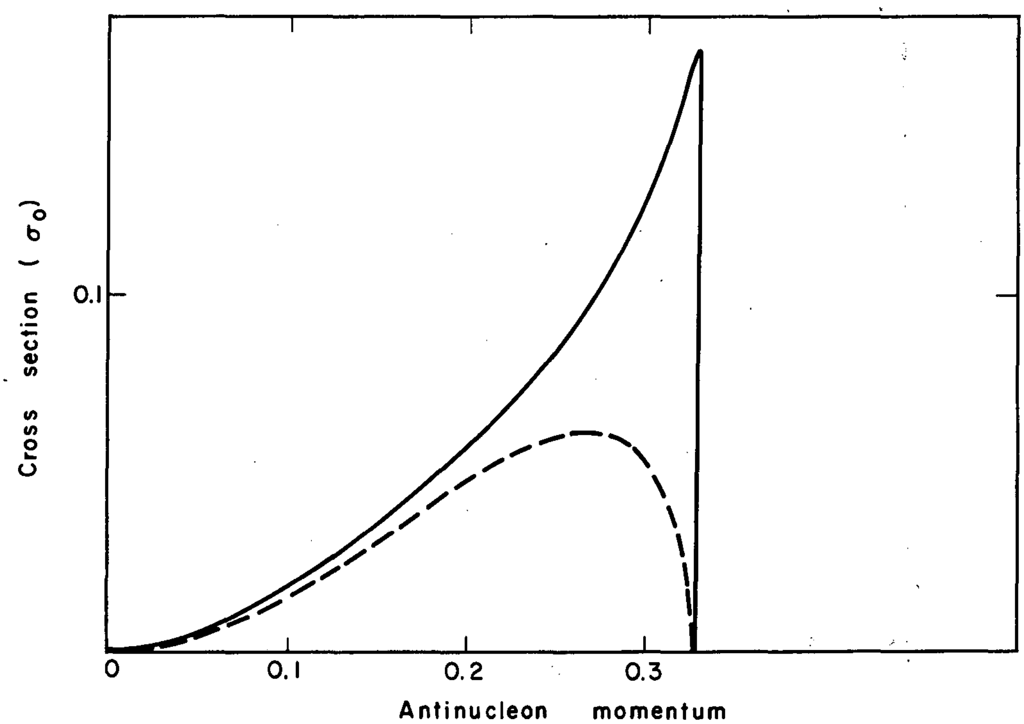

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Fig. 11. The momentum distribution $m \frac{d\sigma}{dp_2} (\frac{3}{2}, 1)$ as a function of antinucleon momentum. $\frac{W}{m} = 3.08$. Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

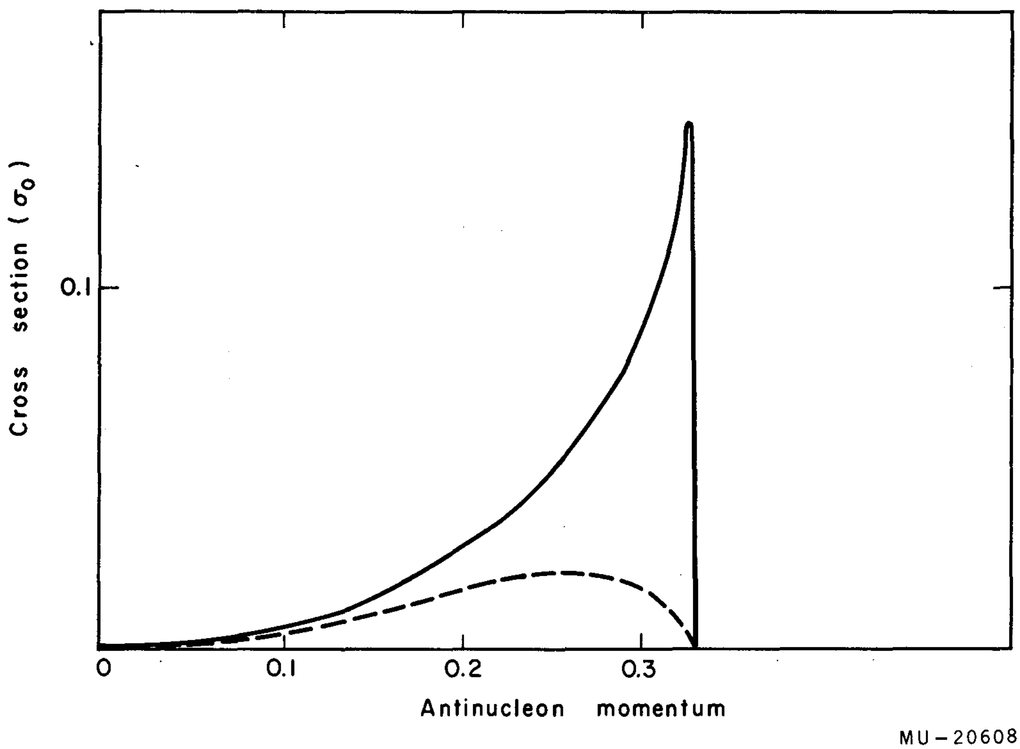

Fig. 12. The momentum distribution $\frac{md\sigma}{dp_2}(\frac{1}{2}, 1)$ as a function of antinucleon momentum. $\frac{W}{m} = 3.08$.
Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

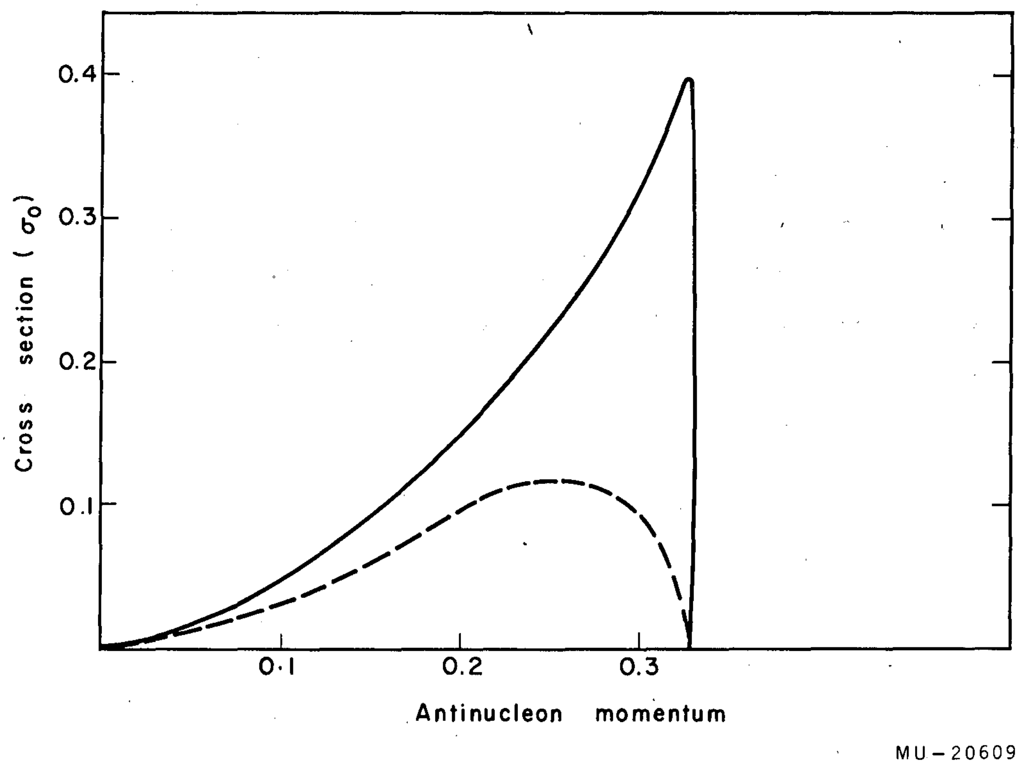

Fig. 13. The momentum distribution $m \frac{d\sigma}{dp_2} (\frac{1}{2}, 0)$ as a function of antinucleon momentum. $\frac{W}{m} = 3.08$. Solid line: prediction of the theory when final state interactions are included. Dashed line: prediction of perturbation theory.

Representative angular distributions are given in Figs. 14-21. Whenever the final state interaction is important, for relatively large momenta, there is a peak at $\cos\theta = \hat{k} \cdot \hat{p}_2 = -1$. At lower momenta the angular distributions are similar to those given by perturbation theory.

These cross sections show that the effect of the final state interaction is important at energies where the s-wave nucleon-nucleon interaction is dominant. Unfortunately the results depend critically upon the details of the nucleon-nucleon scattering wave function and upon the parameters of the nucleon potential.

We find that the magnitude of the effect of the final state interaction is determined essentially by the depth of the potential well. The magnitude of the peak in the momentum distributions changes linearly with the potential depth. This effect could have been predicted by an examination of the wave function of Eq. (54). One of the terms is inversely proportional to the relative momentum of the two final nucleons and directly proportional to the potential depth. Whenever the momentum p' is small enough this term is dominant and accounts for the characteristic peak in the momentum distribution.

Variation of the other parameters seems to have no great effect.

We find then that we cannot predict quantitatively with accuracy any of the cross sections since we are doubtful both of the basic production model and of the details of the nucleon-nucleon wave functions. Qualitatively we can say that the effect of the final state interaction is large, and that there is a large probability for the two final nucleons to be bound as a deuteron. The general shape of the momentum spectra is also certainly correct.

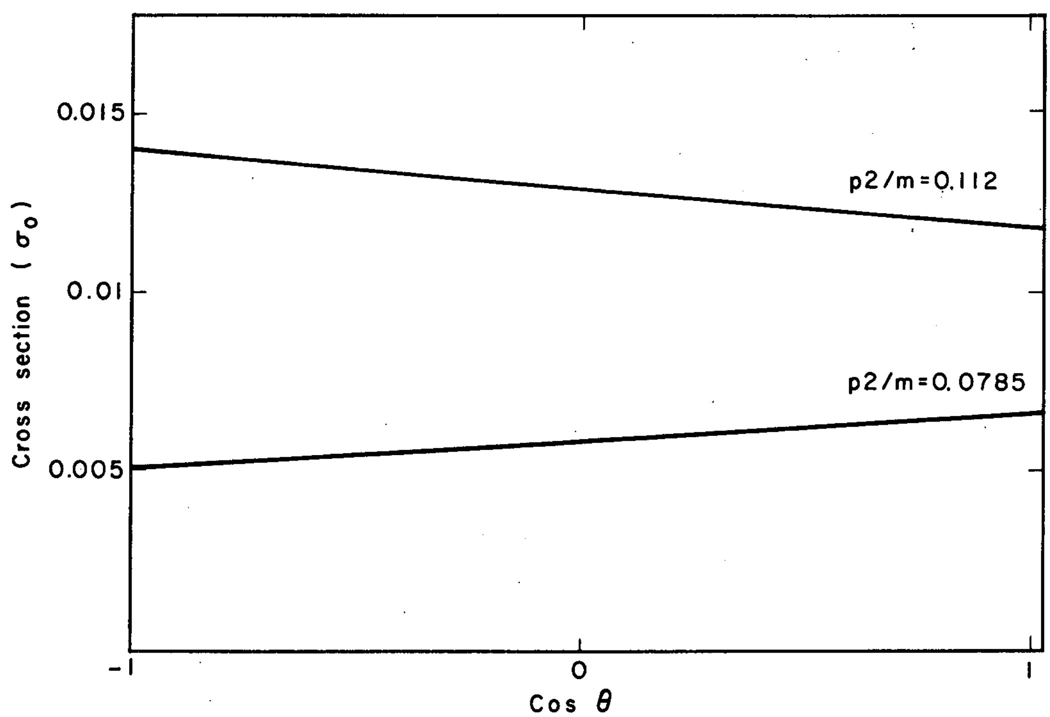

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Fig. 14. Angular distributions $4\pi m \frac{d\sigma}{d\Omega_2 dp_2}$ $(\pi^- + p \rightarrow \overline{p} + p + n)$. $\frac{W}{m} = 3.01$.

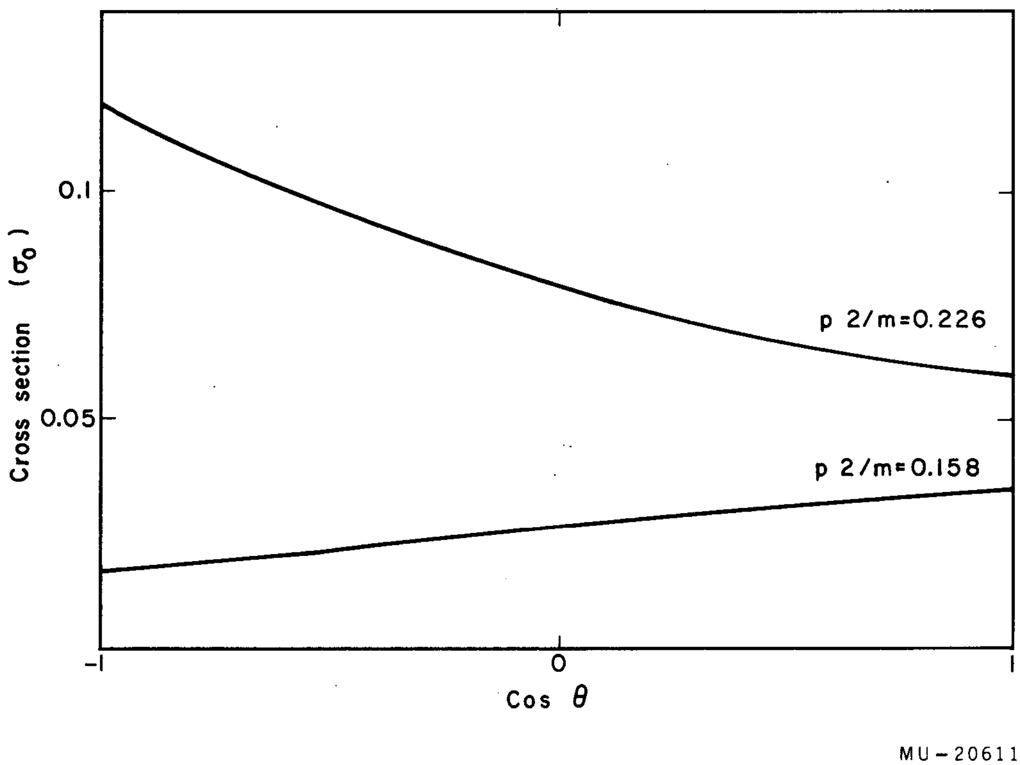

Fig. 15. Angular distributions

$$4\pi\, m \frac{d\sigma}{d\Omega_2 dp_2} \quad (\pi^- + p \rightarrow \bar{p} + p + n)\,. \quad \frac{W}{m} = 3.04.$$

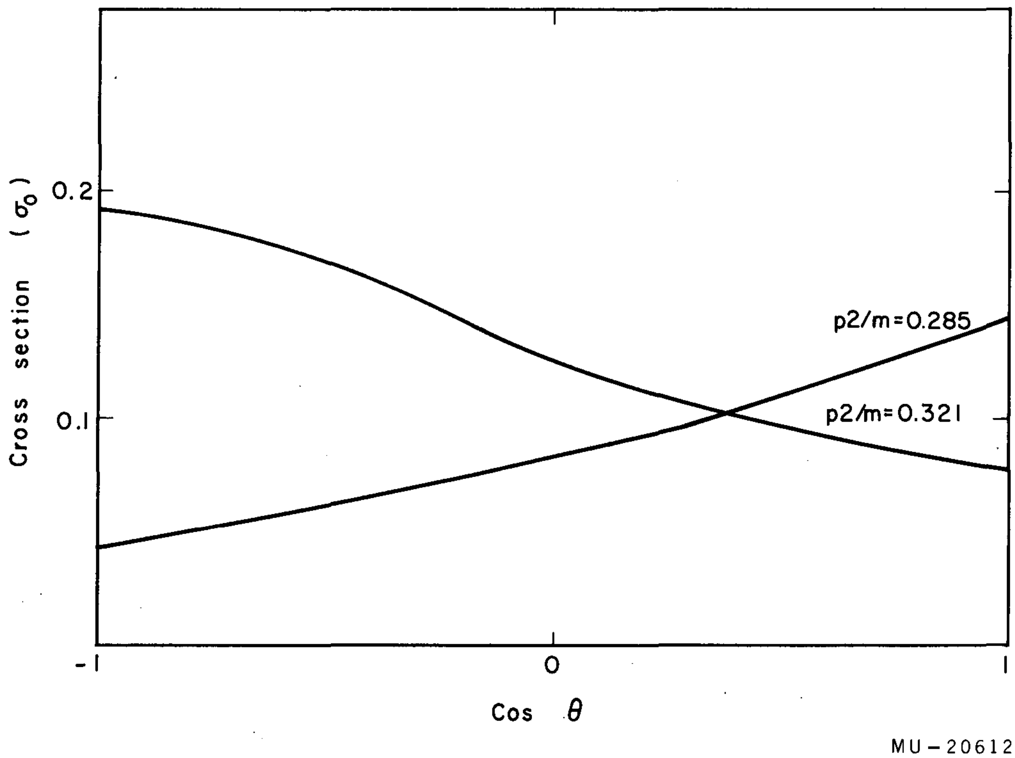

Fig. 16. Angular distributions

$$4\pi\, m \frac{d\sigma}{d\Omega_2 dp_2} \quad (\pi^- + p \to \bar{p} + p + n)\,. \quad \frac{W}{m} = 3.08.$$

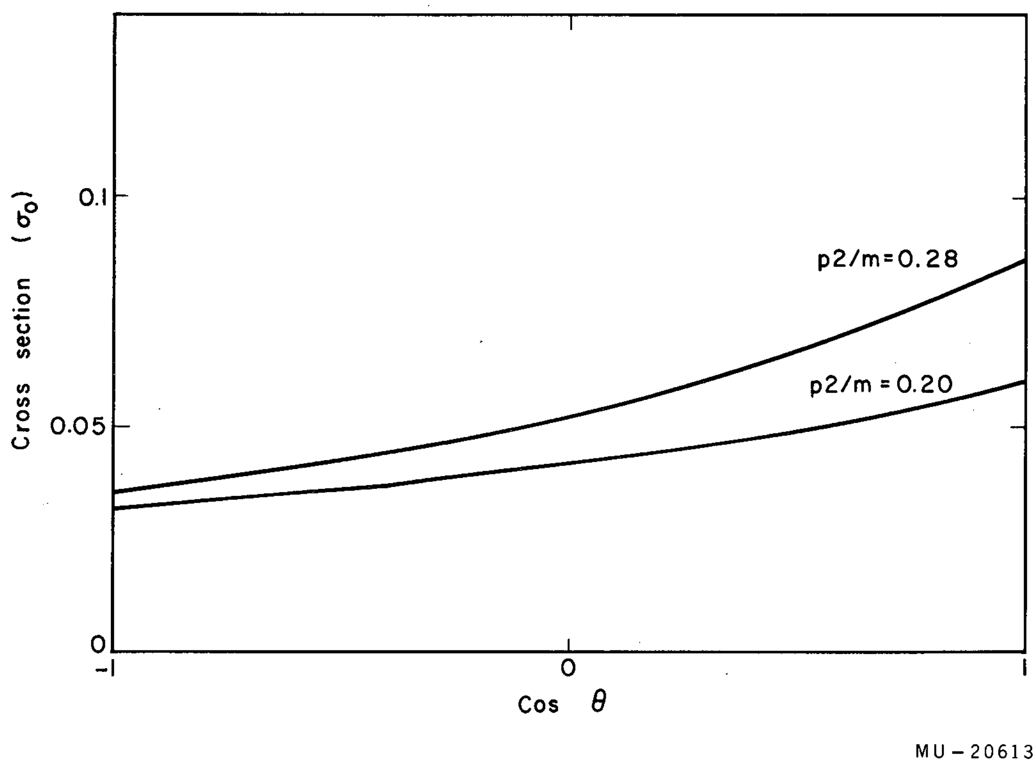

Fig. 17. Angular distributions from perturbation theory.

$$4\pi m \frac{d\sigma}{d\Omega_2 dp_2} \quad (\pi^- + p \rightarrow \bar{p} + p + n) . \quad \frac{W}{m} = 3.08$$

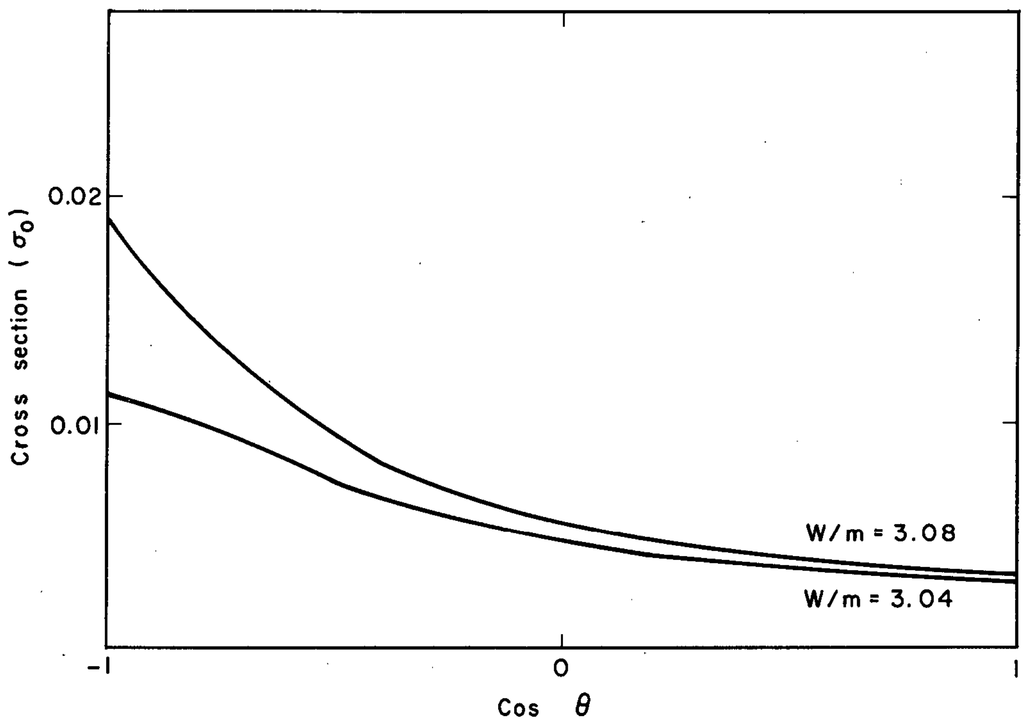

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Fig. 18. Angular distributions

$$4\pi \frac{d\sigma}{d\Omega_2} \quad (\pi^- + p \rightarrow \bar{p} + d) .$$

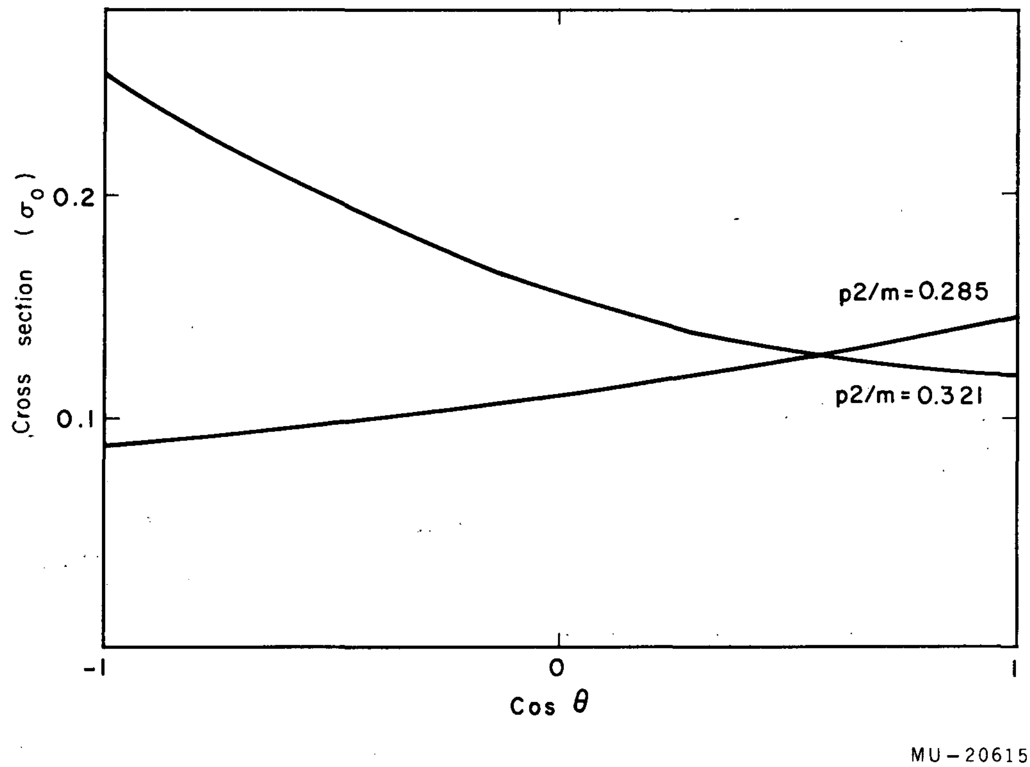

Fig. 19. Angular distributions

$$4\pi m \frac{d\sigma}{d\Omega_2 dp_2} \left(\frac{3}{2}, 1\right) . \quad \frac{W}{m} = 3.08 .$$

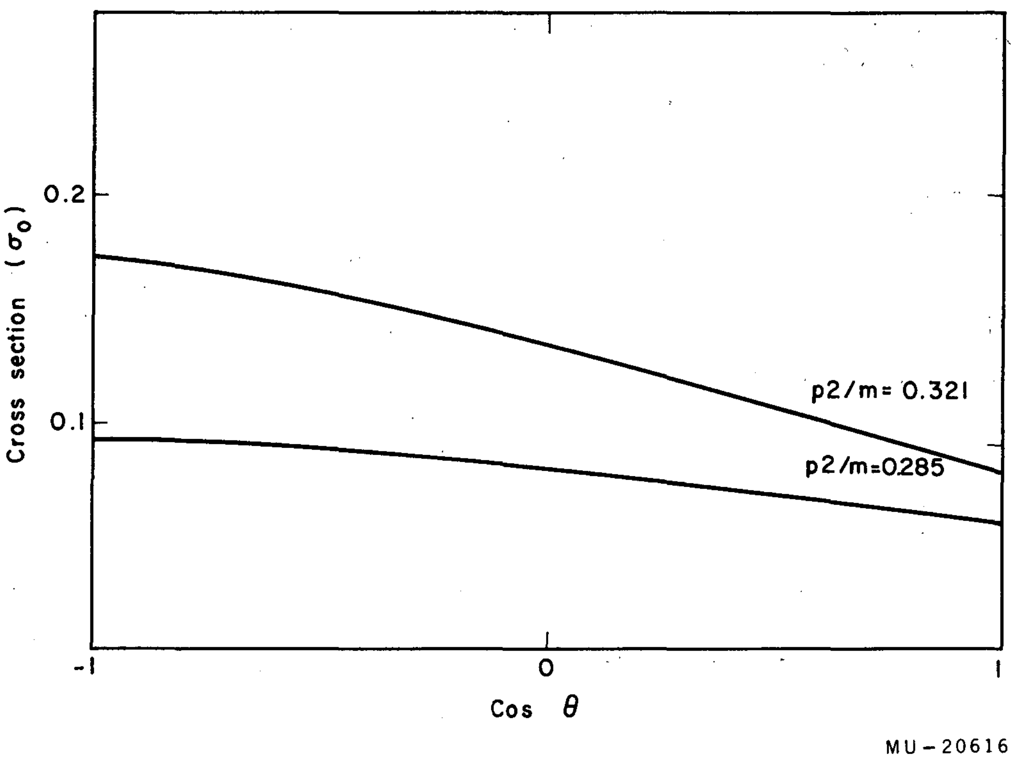

Fig. 20. Angular distributions

$$4\pi m \frac{d\sigma}{d\Omega_2 dp_2} \quad (\frac{1}{2}, 1). \quad \frac{W}{m} = 3.08.$$

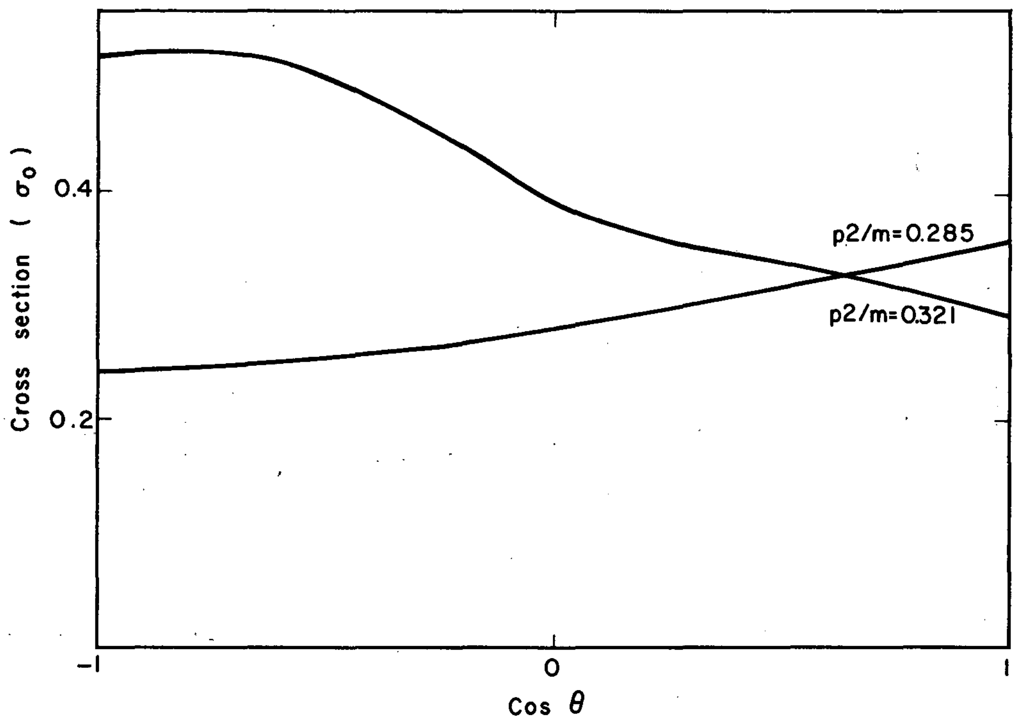

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Fig. 21. Angular distributions
$$4\pi m \frac{d\sigma}{d\Omega_2 dp_2} \left(\frac{1}{2}, 0\right). \quad \frac{W}{m} = 3.08.$$

ACKNOWLEDGMENTS

The problem considered in this thesis was suggested by Dr. Joseph V. Lepore, whose valuable advice throughout the course of this work I greatly appreciate. I also benefited greatly from discussions with Elihu Lubkin and Peter Cziffra. I thank Benjamin Abington for doing a large part of the numerical calculations.

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APPENDIX

I. The Fermi Model

As originally proposed by Fermi,[16] the statistical model consists of the assumption that the probability for the production of n particles in a collision is proportional to the statistical weight S_n:

$$S_n = \mathscr{S}_n \frac{\Omega^{n-1}}{(2\pi)^{3n-3}} \int \prod_{f=1}^{n} d^3 p_f \, \delta^3 \left(\sum_{f=1}^{n} p_f \right) \delta\left(W - \sum_{f=1}^{n} E_f\right) , \qquad (I.1)$$

where W is the total energy of the system, and E_f is the energy of the particle with momentum p_f. The interaction volume Ω is an adjustable parameter and the numerical factor $\mathscr{S}_n$ takes into account the conservation laws not explicitly included.

The cross section for production for these n particles is then given by

$$\sigma_n = \sigma_0 \frac{S_n}{\sum_i S_i} , \qquad (I.2)$$

where σ_0 is the total cross section, the sum of elastic and all inelastic cross sections for a given initial state. The sum is to extend over the statistical weight for all possible final states.

Implicitly contained in this model is the assumption that the particles inside the interaction volume are in statistical equilibrium. This limits the validity of the model to the region of high energies, and to particles whose interactions are very strong. The assumption of statistical equilibrium also makes it difficult to understand how the final state can depend on the initial state in any way except through the rigorous conservation laws.

In accordance with this assumption we assume that the production of antinucleon in pion-nucleon collisions and in nucleon-nucleon collisions are directly related. For each of these reactions we assume

$$\sigma_n = \Omega^{1/3} S_n \tag{I.3}$$

in the region near each threshold.

The one adjustable parameter in the problem is then the interaction volume. We determine this parameter from the known production cross section for antinucleons in nucleon-nucleon collisions. The best known cross section is that for the production of antiprotons by protons with an energy of about 6.1 Bev in the laboratory system incident on carbon nuclei. For antiprotons produced in the forward direction with momentum of 1.2 Bev/c the production cross section is,[1]

$$\left.\frac{d\sigma}{d\bar{p}d\Omega}\right]_{lab} \cong 1.2\,(10)^{-24}\;\frac{cm^2}{Bev/c}\;. \tag{I.4}$$

Published data indicate that the cross section for the production by protons on hydrogen, Eq. (2), is essentially the same,[1] but some recent evidence indicates a smaller valve.[17]

We transform to the center-of-momentum system and find

$$\frac{W}{m} = 4.13$$

and

$$\frac{\bar{p}}{m} = 0.100 \tag{I.5}$$

The four particles in the final state are definitely nonrelativistic but to find a differential cross section it is convenient to use a modification of the phase-space integral, which is a relativistic invariant:

$$S_n^{rel} = \frac{\Omega^{n-1}}{(2\pi)^{3n-3}}\; m^n \int \prod_{f=1}^{n} \left(\frac{d^3 p_f}{E_f}\right) \delta^3\left(\sum_{f=1}^{n} p_f\right) \delta\left(W - \sum_{f=1}^{n} E_f\right)\,. \tag{I.6}$$

This allows us to write the recursion relation,[18]

$$S_n^{rel}(W^2) = \Omega \int_0^{p_n max} \frac{d^3 p_n}{E_n} \; S_{n-1}^{rel}(W^2 + m^2 - 2W E_n) , \qquad (I.7)$$

if all the particles in the final state have the nucleon mass, m.

We can use the nonrelativistic expressions for S_n given by Lepore and Stuart,[19] since there is no difference between S_n and S_n^{rel} in the nonrelativistic limit.

We find, for the differential cross section of Reaction (2),

$$\frac{d\sigma_4}{d\bar{p} d\Omega} = 1.85 \, (10)^{-23} \, \lambda^{10/3} \left(\frac{\bar{p}}{m} \right)^2 \left[\sqrt{\frac{W^2}{m^2} + 1 - 2 \frac{W\bar{E}}{m^2}} - 3 \right]^2 \frac{cm^2}{Bev/c} ; \qquad (I.8)$$

where $\Omega = \left(\frac{4\pi\lambda}{3\mu^3} \right)$.

We determine the parameter λ by comparison with Eq. (I.4),

$$\frac{1}{\lambda} = 9.55. \qquad (I.9)$$

The total cross section for the production of antinucleons by pions is then

$$\sigma_3 = 3.0 \, (10)^{-24} \, \lambda^{7/3} \left(\frac{W}{m} - 3 \right)^2 cm^2$$

$$= 1.4 \, (10)^{-26} \left(\frac{W}{m} - 3 \right)^2 cm^2 . \qquad (I.10)$$

At $W = 3.1$ m this cross section is 0.14 mb. We have normalized the cross section for the reaction of Eq. (1) at this energy.

II. The General Transition Amplitude

Our purpose is to find a general form of the transition amplitude for the reaction of Eq. (14), using the known physical invariance principles.

A. Lorentz Invariance

The transition amplitude has the general form

$$T_{fi} = -i\,(2\pi)^4 \frac{M_{fi}\,\delta^4\,(p_f - p_i)\,m^2}{\sqrt{2\omega\,E_1 E_2 E_3 E_4}}\,, \qquad \text{(II-1)}$$

where the matrix element M is a Lorentz invariant. The square of this quantity, divided by the volume of space-time and the flux of the incoming particles, and multiplied by a factor $d^3\underset{\sim}{p}$ for each particle in the final state, gives the cross section. This cross section must be a Lorentz invariant.

If one neglects the spin, the invariant matrix M is a function of 15 variables, the components of the five momenta. However, only five of these variables are independent.

In general, for an interaction involving n particles, the number of independent variables is 3n-10. There are 3n components of the momenta. Four relations between these components are found by applying conservation of energy and momentum. Three more relations are due to the conservation of angular momentum, and the other three come from the conservation of the generators of rotations involving the time.

Alternatively one can say that four relations are due to the arbitrary choice of an origin in space-time. The orientation of the complex of vectors in the scattering or production process requires six more numbers, just as the orientation of a rigid body in space is specified by the three Euler angles.

We choose these five variables to be invariant functions of the momenta:

$$s = (k + p_1)^2 ,$$
$$t = (k - p_2)^2 ,$$
$$u = (k - p_4)^2 ,$$
$$v = (p_1 - p_3)^2 , \qquad \text{(II.2)}$$

and

$$r = (p_2 + p_3)^2 .$$

These quantities appear naturally in perturbation theory. Five other variables may be expressed in terms of these:

$$\bar{r} = (p_2 + p_4)^2 = 2\,m^2 + \mu^2 - u - t + v ,$$
$$\bar{t} = (p_1 - p_2)^2 = 3\,m^2 - r - v + u,$$
$$\bar{u} = (k - p_3)^2 = 4\,m^2 + 2\mu^2 - s - t - u,$$
$$\bar{v} = (p_1 - p_4)^2 = 2\,m^2 + \mu^2 - s + r - u, \qquad \text{(II.3)}$$

and

$$\bar{x} = (p_3 + p_4)^2 = m^2 - \mu^2 - r - v + s + t + u .$$

Finally there is the general relation

$$s + t + u + v + r + \bar{r} + \bar{t} + \bar{u} + \bar{v} + \bar{x} = 3\,(4\,m^2 + \mu^2) . \qquad \text{(II.4)}$$

Only one of these variables has a simple physical significance. The square of the total energy in the center-of-momentum system is s.

There are four independent momenta in the collision. We choose these to be

$$\frac{p_1 + p_4}{2} = p , \quad \frac{p_3 - p_2}{2} = q , \quad \frac{p_1 - p_4}{2} = Q, \text{ and } k . \qquad \text{(II.5)}$$

We then construct the invariant matrix, M:

$$M = \bar{u}_4\, O^{(1)}\, u_1 \bar{u}_3\;\; O^{(2)}\, v_2 . \qquad \text{(II.6)}$$

The spinors are the direct product of a Dirac spinor with a spinor in the isotopic spin space as defined in Eq. (13). The matrices Θ are then four-by-four matrices in the space of the Dirac spinors and two-by-two matrices in the isotopic spin space. The order of the spinors has no particular significance.

For the moment we neglect isotopic spin and expand each matrix in terms of the sixteen Dirac matrices. We write M as the sum of three terms. The first term is the product of a scalar in one spin space and a scalar in the other space, the second is the product of two vectors, and the third is the product of two tensors.

We show explicitly the expansion of the scalar part of $\mathcal{O}^{(1)}$ in terms of arbitrary scalar coefficients:

$$\mathcal{O}^{(1)}_{\text{scalar}} = A' + B'\gamma^{\alpha\,(1)} k_\alpha + C'\gamma^{\alpha\,(1)} q_\alpha + \frac{iD'}{2} \sigma^{\alpha\beta} k_\alpha q_\beta$$

$$+ \frac{iE'}{4} \epsilon^{\alpha\beta\gamma\delta} \sigma_{\alpha\beta}^{(1)} k_\gamma q_\delta + iF' (\gamma_5\gamma^\alpha)^{(1)} k_\alpha +$$

$$+ iG' (\gamma_5\gamma^\alpha)^{(1)} q_\alpha + iH' \gamma_5^{(1)} . \tag{II.7}$$

Terms of the form $\gamma \cdot p$ or $i\sigma^{\alpha\beta} k_\alpha p_\beta$ can be reduced to the above form by using the Dirac equation for the spinors,

$$(\gamma \cdot p_i - m)\, u(\underset{\sim}{p_i}) = 0 . \tag{II.8}$$

Terms involving a pseudovector such as $\gamma_\alpha \epsilon^{\alpha\beta\gamma\delta} p_\beta q_\gamma k_\delta$ also are not independent. If one writes

$$\gamma_\alpha \epsilon^{\alpha\beta\gamma\delta} p_\beta q_\gamma k_\delta = \frac{i}{4} \Big[\gamma \cdot k\gamma \cdot q\gamma \cdot p\gamma_5 \tag{II.9}$$

$$-\gamma \cdot k\gamma \cdot p\gamma \cdot q\gamma_5 + \gamma \cdot q\gamma \cdot p\gamma \cdot k\gamma_5 - \gamma \cdot p\gamma \cdot q\gamma \cdot k\gamma_5 \Big]$$

and uses the Dirac equation, one finds that this term is of the above form.

One final point is the requirement that the coefficients of the expansion be scalars under space reflection and time reversal. There is only one pseudoscalar that can be formed from the four independent momenta,

$\epsilon^{\alpha\beta\gamma\delta} k_\alpha q_\beta p_\gamma Q_\delta$. This function can be expressed as a function of four gamma matrices and γ_5:

$$\epsilon^{\alpha\beta\gamma\delta} k_\alpha q_\beta p_\gamma Q_\delta = -\frac{i}{8}\left[\gamma\cdot Q, \left\{\gamma\cdot p, \left[\gamma\cdot q, \gamma\cdot k\right]_-\right\}_+\right]_- \gamma_5 \; . \tag{II.10}$$

It then reduces to a series of terms like those already given.

B. Parity and Time Reversal

The operation of space reflection induces a transformation of the spinors according to

$$u\,(\underset{\sim}{p}) \rightarrow u^s\,(\underset{\sim}{p}) = \gamma_o\, u\,(-\underset{\sim}{p}) \; . \tag{II.11}$$

We assume the invariance of the transition amplitude under this transformation. More exactly the relation is

$$M_{fi} = \langle \underset{\sim}{p}_f \,|M|\, \underset{\sim}{p}_i \rangle \rightarrow \langle \underset{\sim}{p}_f \,|M|\, \underset{\sim}{p}_i \rangle^s = \langle -\underset{\sim}{p}_f \,|M|\, -\underset{\sim}{p}_i \rangle \, . \tag{II.12}$$

Under time reversal of the Wigner type the initial and final states change places:

$$\langle \underset{\sim}{p}_f \mid M \mid \underset{\sim}{p}_i \rangle \rightarrow \langle \underset{\sim}{p}_f \mid M \mid \underset{\sim}{p}_i \rangle^T = \langle -\underset{\sim}{p}_i \mid M \mid -\underset{\sim}{p}_f \rangle \, . \tag{II.13}$$

The transformation for the spinors is[9] $u(\underset{\sim}{p}) \rightarrow u^T(\underset{\sim}{p}) = \gamma_o \gamma_5\, C\bar{u}(-\underset{\sim}{p})$. (II.14)

We must also remember that the pion field changes sign under either time reversal or space reflection.

We now list the seventeen matrix elements that are invariant under the full Lorentz group:

$$
\begin{aligned}
M = {} & A\,\bar{u}_4\, u_1\, \bar{u}_3\, \gamma_5\, v_2 + B\,\bar{u}_4\, \gamma_5\, u_1\, \bar{u}_3\, v_2 \\
& + C\,\bar{u}_4\, \gamma\cdot k\, u_1\, \bar{u}_3\, \gamma_5\, v_2 + D\,\bar{u}_4\, \gamma_5\, u_1\, \bar{u}_3\, \gamma\cdot k\, v_2 \\
& + E\,\bar{u}_4\, \gamma\cdot q\, u_1\, \bar{u}_3\, \gamma_5\, v_2 + F\,\bar{u}_4\, \gamma_5\, u_1\, \bar{u}_3\, \gamma\cdot p\, v_2 \\
& + G\,\bar{u}_4\, \gamma_5\, \gamma\cdot k\, u_1\, \bar{u}_3\, \sigma^{\alpha\beta}\, k_\alpha p_\beta v_2 \\
& + H\,\bar{u}_4\, \sigma^{\alpha\beta}\, k_\alpha\, q_\beta\, u_1\, \bar{u}_3\, \gamma_5\, \gamma\cdot k\, v_2 \\
& + I\,\bar{u}_4\, \gamma_5\, \gamma\cdot q\, u_1\, \bar{u}_3\, \sigma^{\alpha\beta}\, k_\alpha\, p_\beta\, v_2 \\
& + J\,\bar{u}_4\, \sigma^{\alpha\beta}\, k_\alpha\, q_\beta\, u_1\, \bar{u}_3\, \gamma_5\, \gamma\cdot p\, v_2 \\
& + \frac{K}{2}\, \bar{u}_4\, \epsilon^{\alpha\beta\gamma\delta}\, \sigma_{\alpha\beta}\, k_\gamma\, q_\delta\, u_1\, \bar{u}_3\, \sigma^{\rho\eta}\, k_\rho\, p_\eta\, v_2 \\
& + \frac{L}{2}\, \bar{u}_4\, \sigma^{\rho\eta}\, k_\rho q_\eta\, u_1\, \bar{u}_3\, \epsilon^{\alpha\beta\gamma\delta} \sigma_{\alpha\beta}\, k_\gamma p_\delta v_2\,. \\
& + M\,\bar{u}_4\, \gamma_5\, \gamma_\alpha\, u_1\, \bar{u}_3\, \sigma^{\alpha\beta}\, k_\beta\, v_2 \\
& + N\,\bar{u}_4\, \sigma^{\alpha\beta}\, k_\beta\, u_1\, \bar{u}_3\, \gamma_5\, \gamma_\alpha\, v_2 \\
& + o\,\bar{u}_4\, \gamma_5\, \gamma_\alpha\, u_1\, \bar{u}_3\, \sigma^{\alpha\beta}\, p_\beta\, v_2 \\
& + P\,\bar{u}_4\, \sigma^{\alpha\beta}\, q_\beta\, u_1\, \bar{u}_3\, \gamma_5\, \gamma_\alpha\, v_2 \\
& + Q\, \epsilon^{\alpha\beta\gamma\delta}\, \bar{u}_4\, \sigma_{\alpha\beta}\, u_1\, \bar{u}_3\, \sigma_{\alpha\delta}\, v_2\,.
\end{aligned}
\tag{II.15}
$$

These amplitudes are also invariant under charge conjugation.

C. Isotopic Spin

We now find the consequences of invariance under rotations in the isotopic spin space. The matrix elements must be scalars under this transformation. We take a term in the above expansion and expand it in terms of the $\vec{\tau}$ matrices:

$$\begin{aligned} M_A = {} & A_a \bar{u}_4 \tau_k u_1 \bar{u}_3 \gamma_5 v_2 \epsilon_k \\ & + A_b \bar{u}_4 u_1 \bar{u}_3 \tau_k \gamma_5 v_2 \epsilon_k \\ & + i A_c \epsilon_{ijk} \bar{u}_4 \tau_i u_1 \bar{u}_3 \tau_j \gamma_5 v_2 \epsilon_k . \end{aligned} \tag{II.16}$$

The unit vector $\hat{\epsilon}$ occurs in the expansion of the pion field and must appear linearly. The τ matrices obey the same relations as the Pauli spinors:

$$\tau_i \tau_j = \delta_{ij} + i\epsilon_{ijk} \tau_k . \tag{II.17}$$

D. Exchange Properties

According to the generalized Pauli Principle this matrix element must be antisymmetric in all the coordinates of the final two nucleons. This exchange give us

$$\begin{aligned} \overline{M}_A = {} & \overline{A}_a \bar{u}_3 \tau_k u_1 \bar{u}_4 \gamma_5 v_2 \epsilon_k \\ & + \overline{A}_b \bar{u}_3 u_1 \bar{u}_3 \tau_k \gamma_5 v_2 \epsilon_k \\ & + i \overline{A}_c \epsilon_{ijk} \bar{u}_4 \tau_i u_1 \bar{u}_3 \tau_j \gamma_5 v_2 \epsilon_k , \end{aligned} \tag{II.18}$$

where

$$\overline{A}(s,t,u,v,r) = A(s,t,\bar{u},\bar{v},\bar{r}) .$$

This matrix element may be put back into the original form by the use of the formulae

$$f_{\delta\alpha}\, g_{\beta\gamma} = \frac{1}{2} \sum_{i=0}^{4} \tau_{\beta\alpha}{}^{i}\, (f\tau^{i} g)_{\delta\gamma} \qquad \text{(II.19)}$$

and

$$F_{\delta\alpha}\, G_{\beta\gamma} = \frac{1}{4} \sum_{A=1}^{16} \gamma_{\beta\alpha}{}^{A} (F\gamma^{A}\, G)_{\delta\gamma}\, .$$

The functions f and g are two-bytwo matrices, and the sum extends over the three τ matrices and the identity matrix. The functions F and G are four-by-four matrices, and the sum is over the sixteen Dirac matrices, which are adjusted so that their square is unity.

The resulting relations are too complicated to be useful. They may be simplifed by writing the transformation as a 51- by-51 matrix connecting the original matrix elements with the transformed matrix elements. One can then diagonalize this matrix and determine the eigenvectors, those combinations of the orignal matrix elements which have simple transformation properties. One then should find 51 conditions on the matrix elements.

We have not done this. We define the amplitude as the difference of two terms,

$$M'_A = M_A - \overline{M}_A\, .$$

The other terms in the general expansion transform in the same way with

$$q \to \overline{q} = \frac{p_4 - p_2}{2}$$

$$p \to \overline{q} = \frac{p_1 + p_3}{2}\, . \qquad \text{(II.20)}$$

The crossing relation that appears in perturbation theory we will assume to be true in general. The amplitude must be invariant for $p_1 \longleftrightarrow -p_2$ and $u_1 \longleftrightarrow v_2$. We combine this exchange with the exchange of the two final nucleons.

Then we have no change of sign for

$$\overline{M}_A \rightarrow \overline{\overline{M}}_A = \overline{\overline{A}}_a \overline{u}_4 \gamma_5 u_1 \overline{u}_3 \tau_k v_2 \epsilon_k$$

$$+ \overline{\overline{A}}_b \overline{u}_4 \tau_k \gamma_5 u_1 \overline{u}_3 v_2 \epsilon_k \qquad \text{(II.21)}$$

$$+ i \overline{\overline{A}}_c \epsilon_{ijk} \overline{u}_4 \tau_j \gamma_5 u_1 \overline{u}_3 \tau_i v_2 \epsilon_k ,$$

where

$$\overline{A}(s, t, u, v, r) = A(t, s, \overline{u}, \overline{r}, \overline{v}).$$

We recognize this term as the second one in the expansion. We must then have

$$\overline{\overline{A}}_a = + B_b ,$$

$$\overline{\overline{A}}_b = + B_a , \qquad \text{(II.22)}$$

and

$$\overline{\overline{A}}_c = - B_c .$$

The other terms in the transition are also related by pairs in this way, with the exception of the last term, for which the conditions are

$$\overline{\overline{Q}}_a = + Q_b \qquad \text{(II.23)}$$

$$\overline{\overline{Q}}_c = - Q_c .$$

E. Linear Independence

We now have expanded the general transition matrix into a sum of 51 terms. We have found 25 conditions on the coefficients of these terms but have not completely utilized the exchange symmetry. The expansion is general but unwieldy.

It is not at all evident that the separate terms of the expansion are linearly independent. We may find a relation between the terms of the form

$$\sum_A C_A M_A = 0 , \qquad \text{(II.24)}$$

where some of the 51 scalars C_A are nonzero and M_A has the form

$$M_A = \bar{u}_4 \, O_A \, u_1 \, \bar{u}_3 \, O_A \, v_2 \, . \tag{II.25}$$

Let us multiply this relation by each of the quantities $M_B{}^*$, and sum over the possible spin states for the four spinors:

$$\sum_A C_A \sum_{\substack{\text{spin}\\\text{states}}} M_A M_B{}^* = 0 \, . \tag{II.26}$$

This set of 51 equations has a solution only if the determinant,

$$\det \mathcal{M}_{AB} = \det \left[\sum_{\substack{\text{spin}\\\text{states}}} M_A M_B{}^* \right], \tag{II.27}$$

is zero. If this determinant is nonzero the terms M_A are linearly independent.

We may think of each term M_A as a vector in a space of unknown dimensions. The general transition matrix, M, is in this sense a vector formed from the set of basic vectors with arbitrary scalar coefficients.

We have defined the scalar product of two vectors as

$$M_A \cdot M_B = \sum_{\substack{\text{spin}\\\text{states}}} M_A M_B{}^* = (M_B \cdot M_A)^* \tag{II.28}$$

The linear independence of the vector basis is then assured if the determinant of the matrix formed by the scalar products of all the vectors with themselves, Gram's determinant, is nonzero.

One can prove that the number of independent vectors in the basis is simply the rank of the matrix $\mathcal{M}$.[20] The rank of the matrix may be found by transforming it or its determinant to diagonal form. The rank is then the number of nonzero elements along the diagonal.

Once the number of independent vectors is known one can construct the basis.

We will not attempt to carry through this program.

REFERENCES

1. E. Segrè, Ann. Rev. Nuclear Sci. 8, 127 (1958).
2. G. Feldman, Phys. Rev. 95, 1697 (1954).
3. B. Yovtz, Am. J. Phys. 26, 202 (1958).
4. We use units consistent with $\hbar = c = 1$. The other pertinent notation is

$$p \cdot x = p_\mu x^\mu = Et - \underset{\sim}{p} \cdot \underset{\sim}{r},$$

$$\overline{\psi}(x) = \psi^\dagger(x)\,\gamma_0,$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},$$

$$\gamma^{\mu\dagger} = \gamma_0\gamma^\mu\gamma_0,$$

$$\tilde{\gamma}^\mu = -C^{-1}\gamma^\mu C,$$

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5^\dagger = \tilde{\gamma}_5,$$

and

$$\sigma^{\mu\nu} = \frac{1}{2i}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

5. A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton, New Jersey, 1957), Chap. 3, p. 36.
6. S. S. Schweber, H. A. Bethe, and F. DeHoffmann, Mesons and Fields (Row, Peterson and Company, Evanston, Illinois, 1955), Vol. I.
7. K. M. Watson, Phys. Rev. 88, 1163 (1952).
8. K. M. Watson and K. A. Brueckner, Phys. Rev. 83, 1 (1951).
9. K. A. Brueckner, Phys. Rev. 82, 598 (1951).
10. B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).
11. S. Mandelstam, Proc. Roy. Soc. (London) A 244, 491 (1958).
12. H. A. Bethe and P. Morrison, Elementary Nuclear Theory (John Wiley and Sons, Inc., New York, 1956), 2nd ed., Chap. 10.
13. R. Jastrow, Phys. Rev. 81, 165 (1951).
14. M. M. Lévy, Phys. Rev. 88, 725 (1952).

15. K. A. Brueckner and K. M. Watson, Phys. Rev. 92, 1023 (1953).
16. E. Fermi, Progr. Theoret. Phys. (Kyoto) 5, 570 (1950).
17. Tom Elioff (Lawrence Radiation Laboratory), private communication.
18. P. P. Srivastava and G. Sudarshan, Phys. Rev. 110 765 (1958).
19. J. V. Lepore and R. N. Stuart, Phys. Rev. 94, 1724 (1954).
20. G. Birkhoff and S. MacLane, A Survey of Modern Algebra (The Macmillan Company, New York, 1949), Chap. 10, p. 291. This argument was pointed out to me by Peter Cziffra.