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Simple picture for neutrino flavor transformation in supernovae

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We can understand many recently discovered features of flavor evolution in dense, self-coupled supernova neutrino and antineutrino systems with a simple, physical scheme consisting of two quasistatic solutions. One solution closely resembles the conventional, adiabatic single-neutrino Mikheyev-Smirnov-Wolfenstein (MSW) mechanism, in that neutrinos and antineutrinos remain in mass eigenstates as they evolve in flavor space. The other solution is analogous to the regular precession of a gyroscopic pendulum in flavor space, and has been discussed extensively in recent works. Results of recent numerical studies are best explained with combinations of these solutions in the following general scenario: (1) Near the neutrino sphere, the MSW-like many-body solution obtains. (2) Depending on neutrino vacuum mixing parameters, luminosities, energy spectra, and the matter density profile, collective flavor transformation in the nutation mode develops and drives neutrinos away from the MSW-like evolution and toward regular precession. (3) Neutrino and antineutrino flavors roughly evolve according to the regular precession solution until neutrino densities are low. In the late stage of the precession solution, a stepwise swapping develops in the energy spectra of ν_e and ν_μ/ν_τ. We also discuss some subtle points regarding adiabaticity in flavor transformation in dense-neutrino systems.

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I. INTRODUCTION

Although it was well known that neutrino-neutrino forward scattering could generate sizable flavor refractive indices in dense-neutrino environments [1–6], it was generally thought that the large matter density near the supernova neutrino sphere would suppress neutrino self-coupling effects. However, it recently became apparent that neutrino self-coupling could alter drastically the way neutrinos and antineutrinos evolve in flavor space in the supernova environment [7–9], even for the small neutrino mass-squared differences inferred from experiments [8,9].

Assuming coherent neutrino flavor transformation and the efficacy of a mean field approach [10,11], and using the magnetic spin analogy for 2×2 neutrino oscillations and the technique of the "corotating frame" (discussed later in the text), Ref. [12] was able to show that neutrinos can experience collective flavor transformation even in the presence of a dominant matter field. The first large-scale numerical simulations of neutrino flavor transformation in the late-time supernova environment with correlated neutrino trajectories were discussed in Refs. [13,14]. The results of these simulations clearly revealed the collective nature of neutrino oscillations in supernovae. These effects are not explainable by the conventional Mikheyev-Smirnov-Wolfenstein (MSW) effect [15–17].

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The numerical simulations [13,14] showed that neutrinos of all energies and propagating along all trajectories can experience rapid flavor oscillations on short time/distance scales in the late-time supernova hot-bubble region. This type of collective flavor transformation was actually first discussed in the context of the early universe [18–20]. Reference [21] showed that a uniform monoenergetic neutrino gas initially in pure ν_e and $\bar{\nu}_e$ states is equivalent to a gyroscopic pendulum in flavor space, and that the rapid collective neutrino flavor transformation in supernovae and in the early universe corresponds to the nutation of this flavor pendulum.

The simulations discussed in Refs. [13,14] also revealed a stunning feature of collective neutrino flavor transformation: ν_e's appear to swap their energy spectra with $\nu_{\mu/\tau}$'s at energies below or above (depending on the neutrino mass hierarchy) a transition energy E_C. It was suggested in Ref. [13] that this stepwise swapping in neutrino energy spectra is related to a collective mode in which the relative phases of the two flavor components of neutrinos oscillate at the same rate. Reference [22] argued that this collective mode is related to the regular precession of a flavor pendulum. This was done by solving the equations for the regular precession mode of a gas of monoenergetic neutrinos initially in pure ν_e and $\bar{\nu}_e$ states. Following the suggestion proposed in Ref. [13], Ref. [22] further elucidates why a stepwise swapping in neutrino energy spectra can result from neutrinos evolving in a collective precession mode. Reference [23] found a way to obtain the regular precession solution for neutrino gases with any given neutrino number density and energy spectra.

Reference [23] also pointed out that the transition energy E_C was determined by the conserved lepton number in the regular precession solution. This solution was termed an "adiabatic solution" in Ref. [23]. However, we will show later in this paper that the flavor evolution in the regular precession solution is actually not adiabatic.

Inspired by the method that Ref. [23] used to find the regular precession solution, we show that a quasistatic solution for given electron density and neutrino number density can be derived in a similar way. This solution is very much like the conventional adiabatic MSW solution, except that the background neutrinos contribute an additional refractive index to a given propagating neutrino. We calculate this MSW-like solution as well as the regular precession solution under the same conditions as the single-angle numerical simulations discussed in Refs. [13,14]. The comparison between these solutions and the numerical results renders a simple picture for neutrino flavor transformation in supernovae: Neutrinos initially follow the MSW-like solution near the neutrino sphere before being driven away from this solution by collective flavor transformation in the nutation mode. Subsequently, neutrinos roughly follow the regular precession solution with some nutation. As the neutrino and antineutrino densities decrease, stepwise swapping appears in the neutrino energy spectra.

This paper is organized as follows. In Sec. II we derive the MSW-like solution and discuss the conditions under which flavor evolution in this scenario is adiabatic. In Sec. III we briefly recapitulate the derivation of the regular precession solution and discuss the effects of the matter field on this solution. In Sec. IV we compare the MSW-like solution and the regular precession solution to the numerical results and discuss the origin of stepwise swapping in neutrino energy spectra. We also elaborate on the adiabaticity of collective neutrino flavor evolution. In Sec. V we give our conclusions.

II. MSW-LIKE SOLUTION FOR DENSE-NEUTRINO GASES

A. Equations of motion

Here we briefly recapitulate a discussion of the equations of motion (e.o.m.) for neutrino flavor transformation. We use the notation of neutrino flavor isospin (NFIS) (see Ref. [12] for detailed discussions). A NFIS is defined as the expectation value of the Pauli spin operator of the flavor wave function for a neutrino or antineutrino. The flavor basis wave function for a neutrino in the 2×2 mixing scheme is

$$\psi_\nu \equiv \begin{pmatrix} a_{\nu_e} \\ a_{\nu_\tau} \end{pmatrix}, \tag{1}$$

where a_{ν_e} and a_{ν_τ} are the amplitudes for the neutrino to be in ν_e and another flavor state, say ν_τ, respectively. The corresponding NFIS in the flavor basis is

$$\mathbf{s}_\nu \equiv \psi_\nu^\dagger \frac{\boldsymbol{\sigma}}{2} \psi_\nu = \frac{1}{2} \begin{pmatrix} 2\,\mathrm{Re}(a_{\nu_e}^* a_{\nu_\tau}) \\ 2\,\mathrm{Im}(a_{\nu_e}^* a_{\nu_\tau}) \\ |a_{\nu_e}|^2 - |a_{\nu_\tau}|^2 \end{pmatrix}. \tag{2}$$

The flavor basis wave function for an antineutrino is defined as

$$\psi_{\bar\nu} \equiv \begin{pmatrix} -a_{\bar\nu_\tau} \\ a_{\bar\nu_e} \end{pmatrix}, \tag{3}$$

where $a_{\bar\nu_e}$ and $a_{\bar\nu_\tau}$ are the amplitudes for the antineutrino to be $\bar\nu_e$ and $\bar\nu_\tau$, respectively. The corresponding NFIS in the flavor basis is

$$\mathbf{s}_{\bar\nu} \equiv \psi_{\bar\nu}^\dagger \frac{\boldsymbol{\sigma}}{2} \psi_{\bar\nu} = -\frac{1}{2} \begin{pmatrix} 2\,\mathrm{Re}(a_{\bar\nu_e} a_{\bar\nu_\tau}^*) \\ 2\,\mathrm{Im}(a_{\bar\nu_e} a_{\bar\nu_\tau}^*) \\ |a_{\bar\nu_e}|^2 - |a_{\bar\nu_\tau}|^2 \end{pmatrix}. \tag{4}$$

As one will see later, the special definition of the flavor basis wave function and NFIS for the antineutrino allows one to write the e.o.m. for both neutrinos and antineutrinos in a unified manner.

The flavor evolution of a neutrino is represented by the precession of the corresponding NFIS around an effective external field $\mathbf{H}$. The e.o.m. of NFIS $\mathbf{s}_i$ can be written in a form similar to that for a magnetic spin:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{s}_i = \mathbf{s}_i \times \mathbf{H}_i \tag{5}$$

where the subscript i denotes the initial physical state (flavor species and momentum), or "mode," of the neutrino/antineutrino. As $\mathbf{s}_i$ precesses around $\mathbf{H}_i$, the projection of $\mathbf{s}_i$ on $\hat{\mathbf{e}}_z^{\mathrm{f}}$ oscillates. This represents variation in the flavor content of the neutrino. Here $\hat{\mathbf{e}}_x^{\mathrm{f}}$, $\hat{\mathbf{e}}_y^{\mathrm{f}}$, and $\hat{\mathbf{e}}_z^{\mathrm{f}}$ are the unit vectors in the flavor basis.

In the absence of ordinary matter and other neutrinos (in vacuum), the effective field experienced by NFIS $\mathbf{s}_i$ is

$$\mathbf{H}_i = \omega_i \mathbf{H}_{\mathrm{V}}, \tag{6}$$

where the vacuum field is

$$\mathbf{H}_{\mathrm{V}} \equiv -\hat{\mathbf{e}}_x^{\mathrm{f}} \sin 2\theta_{\mathrm{v}} + \hat{\mathbf{e}}_z^{\mathrm{f}} \cos 2\theta_{\mathrm{v}}, \tag{7}$$

and ω_i is

$$\omega_i \equiv \pm \frac{\delta m^2}{2E_i}. \tag{8}$$

In this notation $\theta_{\mathrm{v}} \in (0, \pi/2)$ is the effective 2×2 vacuum mixing angle, $\delta m^2 > 0$ is the mass-squared difference between two neutrino vacuum mass eigenstates, and E_i is the energy of the neutrino. The plus sign in Eq. (8) is for neutrinos, and the minus sign is for antineutrinos. Throughout this paper we will assume that $\sin 2\theta_{\mathrm{v}} = \sin 2\theta_{13} \ll 1$. In this convention one has $\theta_{\mathrm{v}} \ll 1$ for the normal neutrino mass hierarchy and $\theta_{\mathrm{v}} \simeq \pi/2$ for the inverted one. Equations (5) and (6) for NFIS $\mathbf{s}_i$ can be pictured as the precession of $\mathbf{s}_i$ around $\mathbf{H}_\mathrm{V}$ with angular velocity ω_i.

In the presence of ordinary matter, the forward scattering of neutrinos on electrons induces different refractive indices for neutrinos of electron flavor from those for other flavors. As a result, in the presence of matter, NFIS $\mathbf{s}_i$ will also tend to precess around the vector field

$$\mathbf{H}_e \equiv -\sqrt{2}G_\mathrm{F} n_e \hat{\mathbf{e}}_z^\mathrm{f}. \tag{9}$$

Here G_F is the Fermi constant and n_e is the net electron number density.

For dense-neutrino gases, forward neutrino-neutrino scattering couples the flavor evolution of different neutrino modes. With all the contributions taken into account, the effective field experienced by NFIS $\mathbf{s}_i$ is

$$\mathbf{H}_i = \omega_i \mathbf{H}_\mathrm{V} + \mathbf{H}_e + \mu_\nu \sum_j n_j \mathbf{s}_j, \tag{10}$$

where

$$\mu_\nu \equiv -2\sqrt{2}G_\mathrm{F} \tag{11}$$

is the coupling coefficient of two neutrino modes for isotropic neutrino gases, and n_j is the population (number density) of neutrino mode j. We define the normalized distribution function as

$$\tilde{f}_i \equiv \frac{n_i}{n_\nu^\mathrm{tot}}, \tag{12}$$

where

$$n_\nu^\mathrm{tot} \equiv \sum_i n_i \tag{13}$$

is the total neutrino number density.

B. MSW-like solution

In the NFIS notation, the instantaneous light (heavy) mass eigenstate of a neutrino is represented by a NFIS completely aligned (antialigned) with its effective field. (Note that the effective energy of NFIS $\mathbf{s}_i$ is $\varepsilon_i^\mathrm{eff} = -\mathbf{s}_i \cdot \mathbf{H}_i$.) If the flavor evolution of neutrinos is fully adiabatic, a neutrino initially in the light (heavy) mass eigenstate will stay in the same instantaneous mass eigenstate, and a NFIS initially aligned (antialigned) with its effective field will stay aligned (antialigned) with the effective field. In such a limit one has

$$\mathbf{s}_i = \epsilon_i \frac{\mathbf{H}_i}{2H_i}, \tag{14}$$

where the alignment factor $\epsilon_i = +1\ (-1)$ if NFIS $\mathbf{s}_i$ is aligned (antialigned) with its effective field $\mathbf{H}_i$.

It is convenient to work in the vacuum mass basis in which the unit vectors $\hat{\mathbf{e}}_{x(y,z)}^\mathrm{v}$ are related to $\hat{\mathbf{e}}_{x(y,z)}^\mathrm{f}$ by

$$\hat{\mathbf{e}}_x^\mathrm{v} = \hat{\mathbf{e}}_x^\mathrm{f}\cos 2\theta_\mathrm{v} + \hat{\mathbf{e}}_z^\mathrm{f}\sin 2\theta_\mathrm{v}, \tag{15a}$$

$$\hat{\mathbf{e}}_y^\mathrm{v} = \hat{\mathbf{e}}_y^\mathrm{f}, \tag{15b}$$

$$\hat{\mathbf{e}}_z^\mathrm{v} = \mathbf{H}_\mathrm{V} = -\hat{\mathbf{e}}_x^\mathrm{f}\sin 2\theta_\mathrm{v} + \hat{\mathbf{e}}_z^\mathrm{f}\cos 2\theta_\mathrm{v}. \tag{15c}$$

Using Eq. (10) we can express Eq. (14) in the explicit component form in the vacuum mass basis:

$$s_{i,x} = \frac{\epsilon_i}{2H_i}(H_{e,x} + \mu_\nu S_x), \tag{16a}$$

$$s_{i,y} = \frac{\epsilon_i}{2H_i}\mu_\nu S_y, \tag{16b}$$

$$s_{i,z} = \frac{\epsilon_i}{2H_i}(\omega_i + H_{e,z} + \mu_\nu S_z). \tag{16c}$$

Here $S_{x(y,z)}$ are the components of the total NFIS

$$\mathbf{S} \equiv \sum_i n_i \mathbf{s}_i = n_\nu^\mathrm{tot} \sum_i \tilde{f}_i \mathbf{s}_i \tag{17}$$

in the same basis, and

$$H_i = \sqrt{(H_{e,x} + \mu_\nu S_x)^2 + (\mu_\nu S_y)^2 + (\omega_i + H_{e,z} + \mu_\nu S_z)^2}. \tag{18}$$

Summing Eq. (16) over index i with weight $n_i = n_\nu^\mathrm{tot}\tilde{f}_i$, we obtain

$$S_x = \frac{n_\nu^\mathrm{tot}}{2}(H_{e,x} + \mu_\nu S_x)\sum_i \frac{\epsilon_i \tilde{f}_i}{H_i}, \tag{19a}$$

$$S_y = \frac{n_\nu^\mathrm{tot}}{2}\mu_\nu S_y \sum_i \frac{\epsilon_i \tilde{f}_i}{H_i}, \tag{19b}$$

$$S_z = \frac{n_\nu^\mathrm{tot}}{2}\sum_i \frac{\epsilon_i \tilde{f}_i}{H_i}(\omega_i + H_{e,z} + \mu_\nu S_z). \tag{19c}$$

Equations (19a) and (19b) imply $S_y = 0$ if $H_e \neq 0$. In this case, and for any given n_ν^tot and n_e, one can solve for S_x and S_z using the equations

$$S_x = \frac{n_\nu^\mathrm{tot}}{2}(H_{e,x} + \mu_\nu S_x) \times \sum_i \frac{\epsilon_i \tilde{f}_i}{\sqrt{(H_{e,x} + \mu_\nu S_x)^2 + (\omega_i + H_{e,z} + \mu_\nu S_z)^2}}, \tag{20a}$$

$$S_z = \frac{n_\nu^\mathrm{tot}}{2}\sum_i \frac{\epsilon_i \tilde{f}_i(\omega_i + H_{e,z} + \mu_\nu S_z)}{\sqrt{(H_{e,x} + \mu_\nu S_x)^2 + (\omega_i + H_{e,z} + \mu_\nu S_z)^2}}. \tag{20b}$$

The components of NFIS $\mathbf{s}_i$ can then be obtained from Eq. (16). We note that the counterpart of Eq. (20) in the flavor basis was discussed in connection with adiabatic flavor transformation of supernova neutrinos in Ref. [24].

We also note that the vanishing of the z component of $\mathbf{S}$ in the flavor basis, i.e.,

$$S_z^{\mathrm{f}} = S_z \cos 2\theta_{\mathrm{v}} + S_x \sin 2\theta_{\mathrm{v}} = 0, \tag{21}$$

corresponds to a MSW-like resonance at which the total NFIS represents a maximally mixed state. Using Eqs. (20a) and (20b) we rewrite the above equation as

$$\left(\sum_i \frac{\epsilon_i \tilde{f}_i}{H_i} \omega_i\right) \cos 2\theta_{\mathrm{v}} + \left(\sum_i \frac{\epsilon_i \tilde{f}_i}{H_i}\right) H_{e,z}^{\mathrm{f}} = 0, \tag{22}$$

where $H_{e,z}^{\mathrm{f}} = -\sqrt{2} G_{\mathrm{F}} n_e$. In the limit $n_\nu^{\mathrm{tot}} \to \infty$, all NFIS's are either aligned or antialigned with the total NFIS:

$$\mathbf{s}_i = \epsilon_i \frac{\mathbf{H}_i}{2H_i} \simeq -\epsilon_i \frac{\mathbf{S}}{2S} \tag{23}$$

(note that $\mu_\nu < 0$). In the same limit, Eq. (20a) simply gives a normalization condition

$$\frac{\mu_\nu n_\nu^{\mathrm{tot}}}{2} \sum_i \frac{\epsilon_i \tilde{f}_i}{H_i} \simeq \frac{n_\nu^{\mathrm{tot}}}{2S} \sum_i (-\epsilon_i) \tilde{f}_i \simeq 1, \tag{24}$$

and Eq. (22) reduces to

$$\omega_{\mathrm{sync}} \cos 2\theta_{\mathrm{v}} + H_{e,z}^{\mathrm{f}} \simeq 0, \tag{25}$$

where

$$\omega_{\mathrm{sync}} \equiv \frac{1}{S^2} \sum_i \omega_i n_i \mathbf{s}_i \cdot \mathbf{S} \simeq \frac{n_\nu^{\mathrm{tot}}}{2S} \sum_i (-\epsilon_i) \tilde{f}_i \omega_i \tag{26}$$

is the synchronization frequency [25]. Equation (25) implies that all neutrinos go through an MSW-like resonance at approximately the radius where a neutrino with energy $E = \delta m^2 / 2|\omega_{\mathrm{sync}}|$ would for a conventional, single-neutrino MSW resonance.

Equations (19a) and (19b) are not independent if $H_e = 0$. Assuming $\mathbf{S}$ is static with given n_ν^{tot}, one can still choose a new coordinate system in which S_y vanishes. Naively, it might seem practical to solve for S_x and S_z using Eqs. (20a) and (20b). However, from Eqs. (5) and (10) one can show that the lepton number

$$\mathcal{L} = 2 \sum_i \tilde{f}_i \mathbf{s}_i \cdot \mathbf{H}_{\mathrm{V}} = \frac{2S_z}{n_\nu^{\mathrm{tot}}} \tag{27}$$

is conserved for $H_e = 0$ [21]. This is true even if n_ν^{tot} is a function of time [22]. Therefore, S_x and S_z are overconstrained by Eqs. (20a), (20b), and (27). In this case, a new unknown is expected. In fact, a regular precession solution exists in the absence of the matter field. In such a solution, $\mathbf{S}$ precesses around $\mathbf{H}_{\mathrm{V}}$ and the precession angular velocity ω_{pr} is the new unknown variable to be found (see Sec. III).

C. Adiabatic condition

The MSW-like solution inherent in Eqs. (20a) and (20b) is a quasistatic solution for any given n_e and n_ν^{tot}. In this solution, as $\mathbf{H}_i$ changes with varying n_e and n_ν^{tot}, $\mathbf{s}_i$ can lag behind. In other words, as the ensemble of neutrinos evolves, misalignment between $\mathbf{s}_i$ and $\mathbf{H}_i$ can develop. Consequently, $\mathbf{s}_i$ will tend to precess around $\mathbf{H}_i$ with angular speed H_i. Therefore, in order for the flavor evolution of neutrino mode i to follow the MSW-like solution, one must have

$$\gamma_i \equiv H_i^{-1} \left| \frac{\mathrm{d}\vartheta_i}{\mathrm{d}t} \right| \ll 1, \tag{28}$$

where ϑ_i is the angle between $\mathbf{H}_i$ and $\mathbf{H}_{\mathrm{V}}$ in the MSW-like solution. Equation (28) is the condition for the MSW-like solution to be adiabatic.

We note that the MSW-like solution becomes more adiabatic with larger neutrino densities. This is because H_i is the energy gap between the instantaneous light and heavy mass eigenstates of neutrino mode i, and it increases with n_ν^{tot}.

We also note that Eq. (28) only gives the necessary condition for the neutrino system to adiabatically follow the MSW-like solution. This is because in deriving Eq. (28) we have assumed that $\mathbf{H}_i$ is described by the MSW-like solution in the first place. This is true for the conventional MSW solution where $n_\nu^{\mathrm{tot}} = 0$. If n_ν^{tot} is large, however, all NFIS's can move in a collective manner, even in the presence of the matter field [12]. As a result, the total NFIS $\mathbf{S}$ and the effective field $\mathbf{H}_i = \omega_i \mathbf{H}_{\mathrm{V}} + \mathbf{H}_e + \mu_\nu \mathbf{S}$ may not follow the MSW-like solution at all. We will further elaborate on this point in Sec. IV D.

III. REGULAR PRECESSION SOLUTION FOR DENSE-NEUTRINO GASES

A. Regular precession solution

In the absence of a matter field ($H_e = 0$), the e.o.m. of NFIS's possess cylindrical symmetry around $\mathbf{H}_{\mathrm{V}}$. A solution with the same symmetry is expected to exist for any given n_ν^{tot}. In this solution, which we term the "regular precession solution," all NFIS's must precess steadily around $\mathbf{H}_{\mathrm{V}}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{s}_i = \mathbf{s}_i \times \omega_{\mathrm{pr}} \mathbf{H}_{\mathrm{V}}. \tag{29}$$

For a gas of monoenergetic neutrinos initially in pure ν_e and $\bar{\nu}_e$ states, this represents the regular precession of the gyroscopic pendulum in flavor space [22].

Combining Eqs. (5), (10), and (29), we have

$$\mathbf{s}_i \times [(\omega_i - \omega_{\mathrm{pr}}) \mathbf{H}_{\mathrm{V}} + \mu_\nu \mathbf{S}] = 0, \tag{30}$$

which means that $\mathbf{s}_i$ is either aligned or antialigned with

$$\tilde{\mathbf{H}}_i \equiv (\omega_i - \omega_{\mathrm{pr}}) \mathbf{H}_{\mathrm{V}} + \mu_\nu \mathbf{S}. \tag{31}$$

We note that $\tilde{\mathbf{H}}_i$ is the effective field experienced by NFIS $\mathbf{s}_i$ in a frame whose unit vectors $\hat{\mathbf{e}}_{x(y,z)}^{\mathrm{v}}$ rotate in the static frame according to

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{\mathbf{e}}_{x(y,z)}^{\mathrm{v}} = \hat{\mathbf{e}}_{x(y,z)}^{\mathrm{v}} \times \omega_{\mathrm{pr}} \mathbf{H}_{\mathrm{V}}. \tag{32}$$

Therefore, the regular precession solution can be derived by means similar to those used to get the MSW-like solution [23].

Equation (31) shows that the effective fields for all NFIS's, and, therefore, all NFIS's themselves, reside in a static plane in the corotating frame which is spanned by $\mathbf{H}_\mathrm{V}$ and $\mathbf{S}$. Without loss of generality, we assume this corotating plane to be the $\hat{\tilde{\mathbf{e}}}_x^\mathrm{v}$–$\hat{\tilde{\mathbf{e}}}_z^\mathrm{v}$ plane, and write out the components of NFIS $\mathbf{s}_i$:

$$s_{i,x} = \frac{\epsilon_i}{2}\frac{\mu_\nu S_x}{\sqrt{(\omega_i - \omega_\mathrm{pr} + \mu_\nu S_z)^2 + (\mu_\nu S_x)^2}}, \tag{33a}$$

$$s_{i,z} = \frac{\epsilon_i}{2}\frac{\omega_i - \omega_\mathrm{pr} + \mu_\nu S_z}{\sqrt{(\omega_i - \omega_\mathrm{pr} + \mu_\nu S_z)^2 + (\mu_\nu S_x)^2}}, \tag{33b}$$

where subscripts x and z indicate the projections of the vectors onto $\hat{\tilde{\mathbf{e}}}_x^\mathrm{v}$ and $\hat{\tilde{\mathbf{e}}}_z^\mathrm{v}$, respectively, and the alignment factor is $\epsilon_i = +1\ (-1)$ for $\mathbf{s}_i$ aligned (antialigned) with $\tilde{\mathbf{H}}_i$. We sum Eq. (33a) with weight $n_\nu^\mathrm{tot}\tilde{f}_i$ and obtain

$$1 = \frac{\mu_\nu n_\nu^\mathrm{tot}}{2}\sum_i \frac{\epsilon_i \tilde{f}_i}{\sqrt{(\omega_i - \omega_\mathrm{pr} + \mu_\nu S_z)^2 + (\mu_\nu S_x)^2}}. \tag{34}$$

Summing Eq. (33b) with weight $n_\nu^\mathrm{tot}\tilde{f}_i$ and using Eq. (34), we obtain

$$\omega_\mathrm{pr} = \frac{\mu_\nu n_\nu^\mathrm{tot}}{2}\sum_i \frac{\epsilon_i \omega_i \tilde{f}_i}{\sqrt{(\omega_i - \omega_\mathrm{pr} + \mu_\nu S_z)^2 + (\mu_\nu S_x)^2}}. \tag{35}$$

For any given values of total neutrino number density n_ν^tot and lepton number $\mathcal{L}$, Eq. (27) specifies S_z. One can then obtain S_x and ω_pr from Eqs. (34) and (35). The components of each NFIS in the corotating frame are determined by Eq. (33).

B. Pseudoregular precession condition

Because the angle between $\mathbf{s}_i$ and $\mathbf{H}_\mathrm{V}$ varies with n_ν^tot, it is not possible for $\mathbf{s}_i$ to stay in the regular precession solution. In this case, $\mathbf{s}_i$ will tend to precess around $\tilde{\mathbf{H}}_i$ in the corotating frame. This precession corresponds to nutation in the static frame. Reference [22] has argued that the regular precession solution can be an excellent approximation to the actual evolution of the system if the inverse of the nutation time scale is much larger than $|\mathrm{d}\vartheta_i/\mathrm{d}t|$, where ϑ_i is the angle between $\mathbf{s}_i$ and $\mathbf{H}_\mathrm{V}$ in the regular precession solution. As $\mathbf{s}_i$ is parallel to $\tilde{\mathbf{H}}_i$, ϑ_i is equivalent to the angle between $\tilde{\mathbf{H}}_i$ and $\mathbf{H}_\mathrm{V}$ for the solution. If this is the case, the amplitude of the nutation will be small. This behavior is similar to the pseudoregular precession of a gyroscope.

We note that the time scale of the nutation of $\mathbf{s}_i$ in the static frame is the same as the period of the precession of $\mathbf{s}_i$ around $\tilde{\mathbf{H}}_i$ in the corotating frame, which is $\tilde{H}_i$. Therefore, the condition for NFIS $\mathbf{s}_i$ to be in the pseudoregular precession is

$$\tilde{\gamma}_i \equiv \tilde{H}_i^{-1}\left|\frac{\mathrm{d}\vartheta_i}{\mathrm{d}t}\right| \ll 1. \tag{36}$$

We note that Eq. (36) only gives the necessary condition for neutrino systems to follow the regular precession solution. This is because in deriving Eq. (36) we have assumed that $\tilde{\mathbf{H}}_i$ follows the regular precession solution and rotates much slower than does $\mathbf{s}_i$. This can be true if the nutation of all NFIS's is not correlated. In collective flavor transformation, however, the motion of the total NFIS $\mathbf{S}$ and, therefore, $\tilde{\mathbf{H}}_i$ is correlated with individual NFIS $\mathbf{s}_i$. As a result, $\tilde{\mathbf{H}}_i$ moves at a rate comparable to that of $\mathbf{s}_i$, and the flavor evolution may not exactly follow the regular precession solution.

The pseudoregular precession condition in Eq. (36) was first proposed in Ref. [23] as an "adiabatic condition." The notion "adiabatic condition" here can be confusing or even misleading. Because the effective field for a NFIS in the static frame can be different from that in a corotating frame, the alignment/antialignment of a NFIS with its effective field in one frame does not guarantee the alignment/antialignment in the other frame. We note that the flavor evolution of a neutrino is considered to be adiabatic if, e.g., it stays in the light mass eigenstate or, equivalently, the corresponding NFIS stays aligned with its effective field in the *static* frame. In contrast, Eq. (36) is a necessary condition for all NFIS's to remain aligned or antialigned with their effective fields in the appropriate *corotating* frame. As we will show at the end of Sec. IV D, the flavor evolution of neutrinos is not adiabatic in the conventional sense if they stay in the regular precession mode, which satisfies the condition in Eq. (36).

C. Effect of the matter field

Reference [22] argues that, so long as $\sin 2\theta_\mathrm{v}$ is small, ordinary matter has no effect other than changing the precession angular velocity ω_pr of the system. This is because, if $\sin\vartheta_i \gg \sin 2\theta_\mathrm{v}$, NFIS $\mathbf{s}_i$ essentially sees $\mathbf{H}_e$ as parallel to $\mathbf{H}_\mathrm{V}$, and

$$\mathbf{H}_i = \omega_i \mathbf{H}_\mathrm{V} + (H_{e,x}\hat{\mathbf{e}}_x^\mathrm{v} + H_{e,z}\hat{\mathbf{e}}_z^\mathrm{v}) + \mu_\nu \mathbf{S}, \tag{37a}$$

$$\simeq \omega_i' \mathbf{H}_\mathrm{V} + \mu_\nu \mathbf{S}, \tag{37b}$$

where

$$\omega_i' \equiv \omega_i + H_{e,z}. \tag{38}$$

Equation (37b) takes a form similar to that in the e.o.m. of NFIS $\mathbf{s}_i$ in the absence of the matter field. We note that $\omega_i' - \omega_\mathrm{pr}' = \omega_i - \omega_\mathrm{pr}$, where

$$\omega_\mathrm{pr}' \equiv \omega_\mathrm{pr} + H_{e,z}, \tag{39}$$

and ω_pr is the precession velocity in the absence of the matter field. Therefore, Eq. (30) would hold and the precession solution would obtain in the corotating frame with angular velocity ω_pr' if $H_{e,x}$ were indeed ignorable.

In the case of regular precession, $\mathbf{s}_i$ would be aligned or antialigned with its effective field $\tilde{\mathbf{H}}_i$ in the frame which rotates around $\mathbf{H}_\mathrm{V}$ with angular velocity ω'_pr. In this corotating frame, the NFIS would experience an additional field $\tilde{\mathbf{H}}_e$ which has magnitude $|H_{e,x}|$ and precesses around $\mathbf{H}_\mathrm{V}$ with angular velocity $-\omega'_\mathrm{pr}$. As long as

$$|H_{e,x}| \ll |\omega'_\mathrm{pr}| = |\omega_\mathrm{pr} + H_{e,z}|, \tag{40}$$

$\tilde{\mathbf{H}}_e$ would only introduce a small perturbation in the motion of $\mathbf{s}_i$ and the regular precession solution is a good approximation. The regular precession approximation fails if

$$\omega_\mathrm{pr} + H_{e,z} \simeq 0, \tag{41}$$

which approximately describes the conventional MSW resonance condition for a neutrino with energy $E = \delta m^2/2|\omega_\mathrm{pr}|$.

In addition to the above effect of the matter field on the collective precession in the regime of high neutrino number density, the matter field can also cause individual neutrino modes to deviate from the collective precession mode at their MSW resonances if the neutrino number density is low. This can be understood as follows. The effective field $\mathbf{H}_i$ for NFIS $\mathbf{s}_i$ in the static frame can be written as the sum of two fields:

$$\mathbf{H}_\mathrm{MSW} = (\omega_i + H_{e,z} + \mu_\nu S_z)\hat{\mathbf{e}}_z^\mathrm{v} + H_{e,x}\hat{\mathbf{e}}_x^\mathrm{v} \tag{42}$$

and $\mu_\nu \mathbf{S}_\perp$, where $\mathbf{S}_\perp$ is the component of the total NFIS $\mathbf{S}$ perpendicular to $\hat{\mathbf{e}}_z^\mathrm{v}$. As n_ν^tot and n_e decrease, $\mathbf{H}_\mathrm{MSW}$ will rotate, and, consequently, neutrino mode i will encounter an MSW-like resonance when

$$\omega_i + H_{e,z} + \mu_\nu S_z \simeq 0. \tag{43}$$

At the same time, $\mathbf{S}_\perp$ precesses around $\mathbf{H}_\mathrm{V}$ with angular velocity ω_pr. If the rotation speed of $\mathbf{H}_\mathrm{MSW}$ is slow enough for the MSW resonance to be adiabatic while at the same time $|\mu_\nu|S_\perp \lesssim \omega_\mathrm{pr}$, NFIS $\mathbf{s}_i$ will follow $\mathbf{H}_\mathrm{MSW}$ through the resonance and, therefore, will deviate from the collective precession mode. Obviously, the lepton number $\mathcal{L}$ is not conserved in this process.

IV. NEUTRINO FLAVOR TRANSFORMATION IN SUPERNOVAE

The problem of neutrino flavor transformation under realistic supernova conditions is very difficult to solve. This is largely a result of the anisotropic nature of the neutrino and antineutrino distribution functions in the supernova environment. However, it has been demonstrated numerically that most of the qualitative features of the supernova neutrino oscillation problem are captured by the so-called "single-angle" simulations (see below) [13,14,26]. In this section we will show explicitly that the results from single-angle simulations can be explained by the combination of the MSW-like solution and the regular precession solution discussed in the preceding sections.

A. Supernova model

In the single-angle simulations discussed in Refs. [13,14], the flavor evolution of all neutrinos is assumed to be the same as that of the neutrinos propagating along radial trajectories. Under this assumption, the neutrino-neutrino coupling strength is taken to be $\mu_\nu = -2\sqrt{2}G_\mathrm{F}$ as in isotropic neutrino gases. The neutrino-neutrino intersection angle dependence in the current-current weak interaction is partially taken into account by introducing an effective number density of neutrinos [1,5]. The effective number density n_ν^eff of neutrinos of energy E at radius r is defined as

$$n_\nu^\mathrm{eff}(E, r) \equiv \frac{D(r/R_\nu)L_\nu f_\nu(E)}{2\pi R_\nu^2\langle E_\nu\rangle}. \tag{44}$$

In Eq. (44), R_ν is the radius of the neutrino sphere (here taken to be $R_\nu = 11$ km), the geometric factor

$$D(r/R_\nu) \equiv \frac{1}{2}\left[1 - \sqrt{1 - \left(\frac{R_\nu}{r}\right)^2}\right]^2 \tag{45}$$

incorporates the geometric coupling and dilution of anisotropic neutrino beams as a function of radius r, L_ν and $\langle E_\nu\rangle$ are the luminosity and average energy of the neutrino species, respectively, and $f_\nu(E)$ is the (normalized) energy distribution function for neutrinos.

In the single-angle simulations, luminosities for all neutrino species are taken to be the same, and two typical late-time supernova neutrino luminosity values, $L_\nu = 5\times 10^{51}$ erg/s and 10^{51} erg/s, have been used. The energy distribution function $f_\nu(E)$ for neutrinos is taken to be of the Fermi-Dirac form with two parameters (T_ν, η_ν),

$$f_\nu(E) \equiv \frac{1}{F_2(\eta_\nu)}\frac{1}{T_\nu^3}\frac{E^2}{\exp(E/T_\nu - \eta_\nu) + 1}, \tag{46}$$

where we take the degeneracy parameter to be $\eta_\nu = 3$, T_ν is the neutrino temperature, and

$$F_k(\eta) \equiv \int_0^\infty \frac{x^k \mathrm{d}x}{\exp(x - \eta) + 1}. \tag{47}$$

The values of T_ν for various neutrino species are determined from our chosen average energies: $\langle E_{\nu_e}\rangle = 11$ MeV, $\langle E_{\bar\nu_e}\rangle = 16$ MeV, and $\langle E_{\nu_\mu}\rangle = \langle E_{\bar\nu_\mu}\rangle = \langle E_{\nu_\tau}\rangle = \langle E_{\bar\nu_\tau}\rangle = 25$ MeV [6].

The simulations use a simple density profile. The net electron density at radii sufficiently above the neutrino sphere is taken to be [9]

$$n_e = Y_e\frac{2\pi^2}{45}g_s\left(\frac{M_\mathrm{NS}m_N}{m_\mathrm{Pl}^2}\right)^3 S^{-4}r^{-3}, \tag{48}$$

where Y_e is the electron fraction (here taken to be

$Y_e = 0.4$), $g_s = 11/2$ is the statistical weight in relativistic particles, $M_{\rm NS} = 1.4 M_\odot$ is the mass of the neutron star, m_N is the mass of the nucleon, $m_{\rm Pl}$ is the Plank mass, and $S = 140$ is the entropy in units of Boltzmann's constant per baryon.

The matter density near the neutrino sphere is much larger than what Eq. (48) calculates. The simulations discussed in Refs. [13,14] have adopted an exponential density profile in the region near the neutrino sphere. We note that a large n_e will keep neutrinos in their initial flavor eigenstates. For the MSW-like solution, the n_e profile in Eq. (48) is large enough to keep neutrinos in their initial flavor states to $r \gg R_\nu$, and the exponential density profile near the neutrino sphere will not affect our analysis below.

B. Comparison of numerical and analytical solutions

For 2×2 flavor mixing, neutrinos starting in pure ν_e ($\bar\nu_e$) and ν_τ ($\bar\nu_\tau$) states with the same energy evolve in the same way and can be viewed as the same neutrino mode. This is because NFIS $\mathbf{s}_i$ and $-\mathbf{s}_i$ follow the same e.o.m. As all neutrinos are assumed to be in pure flavor eigenstates at the neutrino sphere, a neutrino mode i is uniquely designated by ω_i [defined in Eq. (8)], and we can take $\sum_i \to \int_{-\infty}^{\infty} \mathrm{d}\omega$.

We define the total neutrino number density at radius r as

$$n_\nu^{\rm tot}(r) = \int_0^\infty |n_{\nu_e}^{\rm eff}(E,r) - n_{\nu_\tau}^{\rm eff}(E,r)|\mathrm{d}E + \int_0^\infty |n_{\bar\nu_e}^{\rm eff}(E,r) - n_{\bar\nu_\tau}^{\rm eff}(E,r)|\mathrm{d}E. \tag{49}$$

Note that $n_\nu^{\rm tot}$ defined above takes advantage of the equivalent flavor evolution of ν_e ($\bar\nu_e$) and ν_τ ($\bar\nu_\tau$), and is not the simple sum of number densities of all neutrinos. We define the NFIS mode distribution function as

$$\tilde f_\omega = \frac{|\mathrm{d}E/\mathrm{d}\omega|}{n_\nu^{\rm tot}(R_\nu)} \times \begin{cases} |n_{\nu_e}^{\rm eff}(E,R_\nu) - n_{\nu_\tau}^{\rm eff}(E,R_\nu)| & \text{if } \omega > 0, \\ |n_{\bar\nu_e}^{\rm eff}(E,R_\nu) - n_{\bar\nu_\tau}^{\rm eff}(E,R_\nu)| & \text{if } \omega < 0. \end{cases} \tag{50}$$

Because $\mu_\nu < 0$ and ν_e's are dominant in number in supernovae, the vector $\mu_\nu \mathbf{S}$ is initially in the direction opposite to $\hat{\mathbf{e}}_z^{\rm f}$. Taking this into account, we have

$$\frac{\mathbf{H}_\omega}{H_\omega} \simeq \frac{\mathbf{H}_e + \mu_\nu \mathbf{S}}{H_\omega} \simeq -\hat{\mathbf{e}}_z^{\rm f} \tag{51}$$

at the neutrino sphere, where $\mathbf{H}_\omega$ is the total effective field for neutrino mode ω. Equation (51) is true for any neutrino mode ω.

Noting that $\nu_e/\bar\nu_\tau$ and $\bar\nu_e/\nu_\tau$ are represented by NFIS's aligned and antialigned with $\hat{\mathbf{e}}_z^{\rm f}$, respectively, we find the alignment factor ϵ_ω for neutrino mode ω to be

$$\epsilon_\omega = \begin{cases} -\mathrm{sgn}(n_{\nu_e}^{\rm eff}(E,R_\nu) - n_{\nu_\tau}^{\rm eff}(E,R_\nu)) & \text{if } \omega > 0, \\ \mathrm{sgn}(n_{\bar\nu_e}^{\rm eff}(E,R_\nu) - n_{\bar\nu_\tau}^{\rm eff}(E,R_\nu)) & \text{if } \omega < 0, \end{cases} \tag{52}$$

where $\mathrm{sgn}(\xi) \equiv \xi/|\xi|$ is the sign of ξ.

Using Eqs. (20), (49), (50), and (52) we are able to find the MSW-like solution in the same supernova model as adopted in the single-angle simulations. Taking $\delta m^2 = 3\times 10^{-3}$ eV^2 (close to the value associated with atmospheric neutrino oscillations), $\theta_{\rm v} = 0.1$ (normal neutrino mass hierarchy) or $\pi/2 - 0.1$ (inverted neutrino mass hierarchy), $L_\nu = 5\times 10^{51}$ erg/s, we compute the average NFIS

$$\langle \mathbf{s}\rangle \equiv \frac{\mathbf{S}}{n_\nu^{\rm tot}} \tag{53}$$

in the MSW-like solution. We plot $\langle s_\perp\rangle = |\langle s_x\rangle|$ and $\langle s_z\rangle$, the perpendicular and z components, respectively, of the average NFIS in the vacuum mass basis, as dot-dashed lines in Fig. 1.

We also obtain the regular precession solution under the same conditions. For the inverted mass hierarchy case, we determine the value of $\langle s_z\rangle$ in the regular precession solution using the lepton number $\mathcal{L} = 2\langle s_z\rangle = 2S_z/n_\nu^{\rm tot}$ at the neutrino sphere [see discussions around Eq. (54)]. For the normal mass hierarchy case, we determine the value of $\langle s_z\rangle$ using the transition energy $E_{\rm C}$ [see discussions around Eq. (61)]. Note that the swapping of energy spectra above/below $E_{\rm C}$ (stepwise swapping) is a natural result of the regular precession solution. With $S_z = n_\nu^{\rm tot} s_z$ we solve Eqs. (34) and (35) for S_x (or $S_\perp$ in the static frame) and $\omega_{\rm pr}$. The values of $\langle s_\perp\rangle = S_\perp/n_\nu^{\rm tot}$ and $\langle s_z\rangle$ corresponding to this solution for both the inverted and normal mass hierarchies are plotted as solid lines in Fig. 1.

For comparison with the MSW-like and regular precession solutions, we extract the values of $\langle s_\perp\rangle$ and $\langle s_z\rangle$ in the single-angle simulations presented in Ref. [13]. These numerical results are plotted as dashed lines in Fig. 1.

For the inverted mass hierarchy case [Figs. 1(a) and 1(b)] we observe that $\langle s_\perp\rangle$ in the numerical simulation follows the MSW-like solution until radius $r_{\rm X} \simeq 88$ km, but then abruptly jumps out of the MSW-like solution. Thereafter, $\langle s_\perp\rangle$ roughly follows the regular precession solution. The reason for this change in the behavior of the NFIS's is discussed extensively in Ref. [22]. Briefly speaking, supernova neutrinos naturally form a "bipolar system" with the neutrinos forming two groups of NFIS's pointing in opposite directions in flavor space [13]. This bipolar system is roughly equivalent to a gyroscopic pendulum in flavor space [21]. For the inverted mass hierarchy case, the gyroscopic pendulum is initially in the highest position, and $(n_\nu^{\rm tot})^{-1}$ plays the role of (variable) gravity. For a given value of internal spin (determined by ν_e–$\bar\nu_e$ asymmetry), the gyroscopic pendulum will not fall until the "gravity" is large enough and $n_\nu^{\rm tot}$ drops below some critical value $n_\nu^{\rm c}$. If the gravity increases very slowly, so

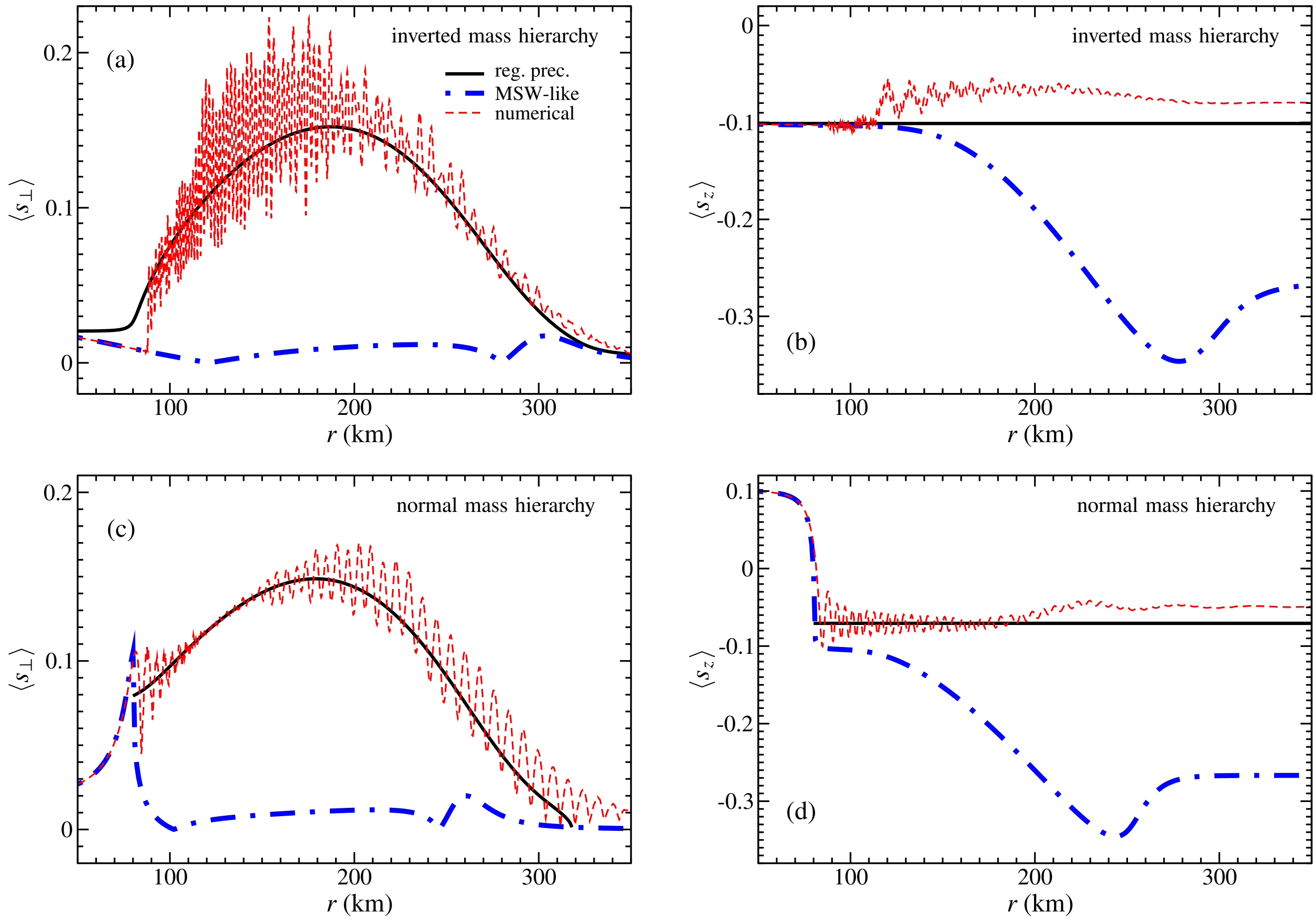

FIG. 1 (color online). Plots of $\langle s_\perp(r)\rangle$ (left panels) and $\langle s_z(r)\rangle$ (right panels) for both the inverted (upper panels) and normal (lower panels) neutrino mass hierarchies. The dashed lines show numerical simulation results from Ref. [13]. The solid lines show the regular precession solution. The dot-dashed lines show the MSW-like solution. The luminosity for all neutrino species is taken to be $L_\nu = 5 \times 10^{51}$ erg/s.

that the symmetry around $\mathbf{H}_\mathrm{V}$ is preserved, the gyroscopic pendulum will precess steadily. This corresponds to the regular precession solution. However, the symmetry is usually broken near the point where $n_\nu^\mathrm{tot}(r) \simeq n_\nu^\mathrm{c}$. Below this critical density, the pendulum will develop significant nutation in addition to the precession.

We note that the fast oscillations of $\langle s_\perp \rangle$ around the regular precession solution correspond to the nutation mode. We also note that $\langle s_z \rangle$ roughly stays constant in the region $r \lesssim 115$ km and is

$$\langle s_z \rangle \equiv \frac{\mathbf{S}\cdot\mathbf{H}_\mathrm{V}}{n_\nu^\mathrm{tot}}, \tag{54a}$$

$$\simeq -\frac{\mathbf{S}\cdot\hat{\mathbf{e}}_z^\mathrm{f}}{n_\nu^\mathrm{tot}} = \frac{1}{2}\int_{-\infty}^{\infty} \epsilon_\omega \tilde{f}_\omega \mathrm{d}\omega, \tag{54b}$$

$$\simeq -0.1. \tag{54c}$$

This is because both the collective precession and nutation modes conserve the lepton number $\mathcal{L} = 2\langle s_z \rangle$. The value of $\langle s_z \rangle$ is subsequently changed because of matter effects, which do not conserve $\mathcal{L}$. However, $\langle s_\perp \rangle$ in the numerical solution still roughly follows that of the regular precession solution, even after $\langle s_z \rangle$ starts to evolve. This seems to imply that the neutrino system is still described by the regular precession solution, except for the neutrino modes which have dropped out of the collective precession.

For the normal mass hierarchy case [Figs. 1(c) and 1(d)] we observe that $\langle s_\perp \rangle$ and $\langle s_z \rangle$ in the numerical simulation follow the MSW-like solution until radius $r \simeq 83$ km. At this point, $\langle \mathbf{s} \rangle$ has almost completely flipped its direction. This corresponds to the complete flavor transformation in both $\nu_e \leftrightharpoons \nu_\tau$ and $\bar{\nu}_e \leftrightharpoons \bar{\nu}_\tau$ channels. In the flavor pendulum analogy, the gyroscopic pendulum has been raised from the lowest position to the highest position, and one expects it to fall when $n_\nu^\mathrm{tot}(r) \lesssim n_\nu^\mathrm{c}$. Indeed, for large enough radii, both $\langle s_\perp \rangle$ and $\langle s_z \rangle$ leave the MSW-like solution and start to oscillate around the regular precession solution with $\langle s_z \rangle \simeq -0.07$.

We have also obtained the three kinds of solutions for a smaller neutrino luminosity, $L_\nu = 10^{51}$ erg/s, with other parameters unchanged. These results are plotted in Fig. 2. Comparing Figs. 1 and 2, one can see that the small

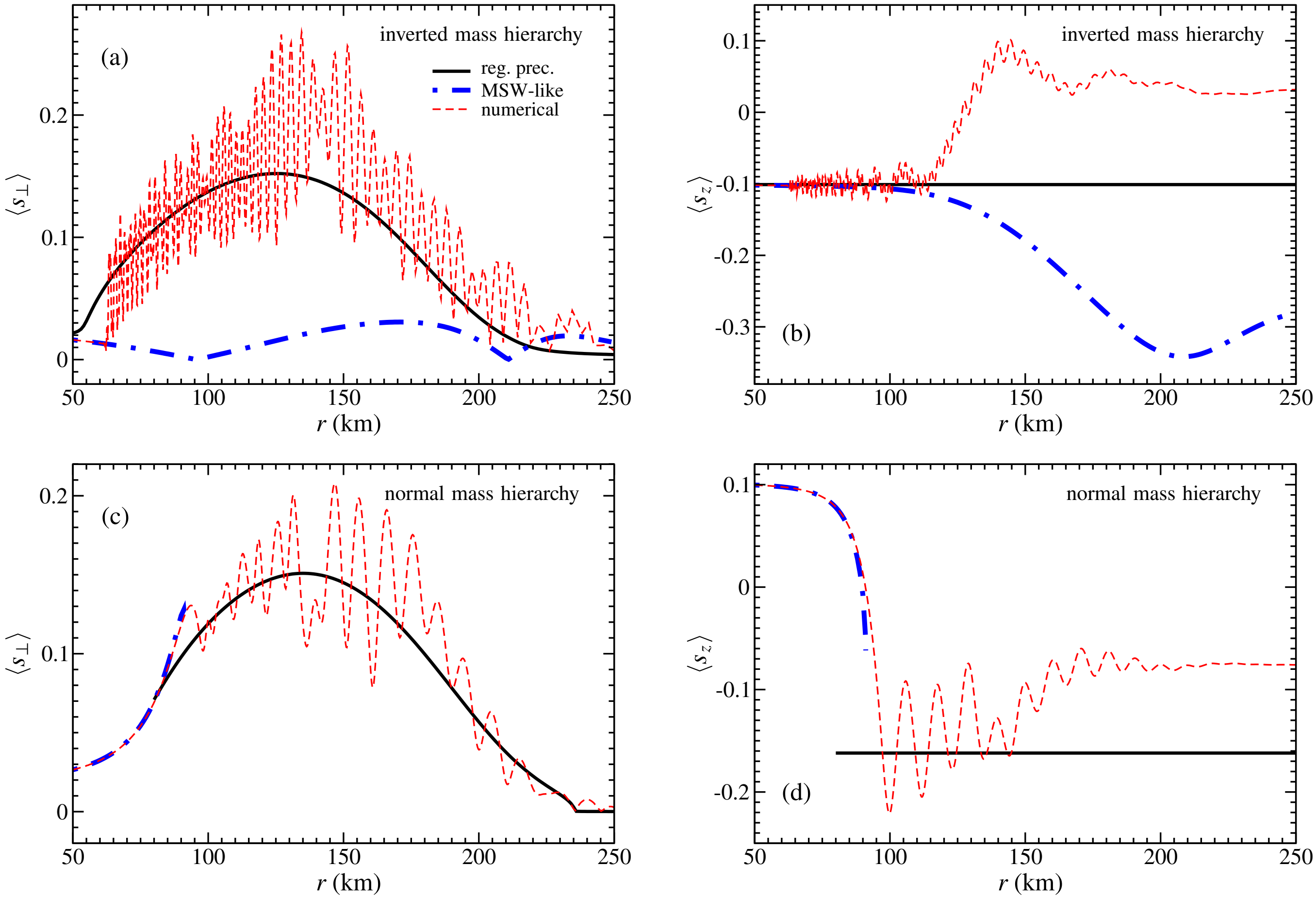

FIG. 2 (color online). Same as Fig. 1 except that the luminosity for all neutrino species is taken to be $L_\nu = 10^{51}$ erg/s in this figure.

luminosity cases are very similar to their large luminosity counterparts. However, we note that, for the inverted mass hierarchy case, $\langle s_\perp \rangle$ leaves the MSW-like solution at $r_X \simeq 63$ km, which is smaller than the r_X value in the large luminosity case. This is because with smaller L_ν the total neutrino number density $n_\nu^{\rm tot}$ reaches $n_\nu^{\rm c}$ earlier. For the normal mass hierarchy case, our calculation procedure cannot find the MSW-like solution in a small radius interval immediately beyond $r \simeq 91$ km. We have taken $\langle s_z \rangle \simeq -0.16$ for the regular precession solution in this case. We also note that the value of $\langle s_z \rangle$ changes significantly in both small luminosity cases. This is because, with smaller neutrino number densities, the matter field drives more neutrinos or antineutrinos off the track of the collective precession solution.

C. Stepwise swapping in neutrino energy spectra

As we have seen in Sec. IV B, supernova neutrinos evolve initially according to the MSW-like solution, but subsequently roughly follow the regular precession solution. In the latter solution, all NFIS's are aligned or antialigned with effective fields in the appropriate corotating frame (see Sec. III A). It was first suggested in Ref. [13] and further explained in Ref. [22] that, if this alignment in the corotating frame is maintained, ν_e and ν_τ will swap their energy spectra for energy smaller (larger) than a transition energy $E_{\rm C}$ in the normal (inverted) mass hierarchy case. This phenomenon is also referred to as a "spectral split" and the transition energy is determined by the conservation of lepton number $\mathcal{L}$ [23]. This stepwise swapping of neutrino energy spectra has been observed in the numerical simulations presented in Refs. [13,14].

The phenomenon of the stepwise spectrum swapping (or spectral split) can be understood as follows. In the regular precession solution, NFIS $\mathbf{s}_\omega$ stays aligned or antialigned with $\mathbf{H}_\omega$ and $\mathbf{s}_\omega = \epsilon_\omega \tilde{\mathbf{H}}_\omega / 2\tilde{H}_\omega$. The effective field $\tilde{\mathbf{H}}_\omega$ in the corotating frame becomes

$$\tilde{\mathbf{H}}_\omega = (\omega - \omega_{\rm pr}^0)\mathbf{H}_{\rm V} \tag{55}$$

as $n_\nu^{\rm tot} \to 0$, where $\omega_{\rm pr}^0$ is the value of $\omega_{\rm pr}$ at $n_\nu^{\rm tot} = 0$. Therefore, one has

$$(s_{\omega,z})_{n_\nu^{\rm tot}=0} = \frac{\epsilon_\omega}{2}\,\mathrm{sgn}(\omega - \omega_{\rm pr}^0). \tag{56}$$

In other words, the spectrum of $\epsilon_\omega s_{\omega,z}$ is split into two parts when $n_\nu^{\rm tot} \to 0$: the lower part ($\omega < \omega_{\rm pr}^0$) takes the value $-1/2$ while the upper part ($\omega > \omega_{\rm pr}^0$) takes the value $+1/2$.

For $\sin 2\theta_{\rm v} \ll 1$ the NFIS for mode ω in the limit of $n_\nu^{\rm tot} = 0$ is

$$(\mathbf{s}_\omega)_{n_\nu^{\rm tot}=0} = \mathbf{H}_{\rm V}(s_{\omega,z})_{n_\nu^{\rm tot}=0}, \tag{57a}$$

$$\simeq \frac{\epsilon_\omega}{2}\Xi\,{\rm sgn}(\omega - \omega_{\rm pr}^0)\hat{\mathbf{e}}_z^{\rm f}, \tag{57b}$$

where $\Xi = +1\ (-1)$ for the normal (inverted) mass hierarchy. Equation (57b) means that supernova neutrinos are almost in pure flavor eigenstates when $n_\nu^{\rm tot} \to 0$. We note that neutrinos are in flavor eigenstates at the neutrino sphere and

$$(\mathbf{s}_\omega)_{r=R_\nu} = -\frac{\epsilon_\omega}{2}\hat{\mathbf{e}}_z^{\rm f}. \tag{58}$$

Therefore, the probability for a neutrino mode ω to stay in its original state is

$$P_{\nu\nu}(\omega) = \tfrac{1}{2} + 2(\mathbf{s}_\omega)_{r=R_\nu}\cdot(\mathbf{s}_\omega)_{n_\nu^{\rm tot}=0}, \tag{59a}$$

$$\simeq \tfrac{1}{2}[1 - \Xi\,{\rm sgn}(\omega - \omega_{\rm pr}^0)]. \tag{59b}$$

According to Eq. (59b), a stepwise swapping can occur in the final neutrino energy spectra. For $\omega_{\rm pr}^0 > 0$ and $\theta_{\rm v} \simeq \pi/2$ (inverted mass hierarchy), ν_e and ν_τ will swap their energy spectra for energies above

$$E_{\rm C} = \frac{\delta m^2}{2\omega_{\rm pr}^0}. \tag{60}$$

For $\omega_{\rm pr}^0 > 0$ and $\theta_{\rm v} \ll 1$ (normal mass hierarchy), ν_e and ν_τ will swap their energy spectra for energies below $E_{\rm C}$.

Assuming stepwise spectrum swapping, the value of $\omega_{\rm pr}^0$ can be determined from $\langle s_z\rangle$ without finding the complete regular precession solution [23]:

$$2\langle s_z\rangle = \int_{\omega_{\rm pr}^0}^{\infty}\epsilon_\omega\tilde{f}_\omega{\rm d}\omega - \int_{-\infty}^{\omega_{\rm pr}^0}\epsilon_\omega\tilde{f}_\omega{\rm d}\omega. \tag{61}$$

Using the values of $\langle s_z\rangle$ in the regular precession solutions [Eq. (54)], we obtain $E_{\rm C} \simeq 8.4$ MeV for the inverted mass hierarchy cases. This value agrees well with the numerical simulation results [see Fig. 9(c) in Ref. [13]]. For the normal mass hierarchy scenarios, we actually obtain $\langle s_z\rangle$ in the regular precession solution by using Eq. (61) and the values of $E_{\rm C}$ in the numerical results [Fig. 9(a) in Ref. [13]] which are approximately 7.8 MeV and 9.5 MeV for the large and small neutrino luminosity cases, respectively. As shown in Figs. 1(c) and 2(c), the precession solutions determined by these values of $E_{\rm C}$ agree well with the numerical simulation results. For the parameters we have chosen, the stepwise swapping occurs only in the neutrino sector. This is because ν_e is the dominant (in terms of number flux) neutrino species in supernovae.

D. Adiabaticity

The flavor evolution of neutrino mode ω is adiabatic so long as the angle between $\mathbf{s}_\omega$ and $\mathbf{H}_\omega$ remains constant. It can be shown that the flavor evolution of neutrino mode ω is adiabatic if $\mathbf{H}_\omega$ rotates at a rate much slower than H_ω, or, in other words,

$$\left|\mathbf{H}_\omega \times \frac{{\rm d}\mathbf{H}_\omega}{{\rm d}t}\right| \ll H_\omega^3 \tag{62}$$

(see, e.g., Ref. [12]). Adiabaticity is guaranteed if the condition in Eq. (62) is satisfied. However, when neutrino densities are large, the system can evolve adiabatically even if the condition in Eq. (62) is *not* satisfied.

If neutrinos experience collective flavor transformation, the total NFIS $\mathbf{S}$ and, therefore, effective field $\mathbf{H}_\omega = \omega\mathbf{H}_{\rm V} + \mathbf{H}_e + \mu_\nu\mathbf{S}$ move in correlation with the motion of individual NFIS $\mathbf{s}_\omega$. In this case, $\mathbf{H}_\omega$ can move very fast, with $\mathbf{s}_\omega$ staying aligned or antialigned with $\mathbf{H}_\omega$, and the flavor evolution is, therefore, still adiabatic. We expect this to be the case as neutrinos transition from the MSW-like solution through the collective nutation mode to the regular precession solution. As a result, the alignment factor ϵ_ω for neutrino mode ω remains constant during the transition. This was implicitly assumed in the discussion on stepwise spectrum swapping in Sec. IV C.

On the other hand, we note that flavor evolution through the regular precession solution is *not* adiabatic. Although the pseudoregular precession condition in Eq. (36) is in a form similar to the adiabatic condition in Eq. (62), it simply guarantees that NFIS $\mathbf{s}_\omega$ remains aligned or antialigned with the effective field $\tilde{\mathbf{H}}_\omega$ in the corotating frame, but *not* with $\mathbf{H}_\omega$ in the static frame. Therefore, neutrino mode ω does not stay in the same mass eigenstate throughout its evolution. One may consider two neutrino modes, $\omega_{\rm pr}^0 + \Delta\omega$ and $\omega_{\rm pr}^0 - \Delta\omega$, which are initially in pure ν_e states. If the flavor evolution is adiabatic for both neutrinos, both NFIS's should stay antialigned with their effective fields in the static frame throughout the evolution. However, we know that this is not the case. Because of the stepwise swapping in neutrino energy spectra, one of the two NFIS's will become aligned with its effective field in the static frame when $n_\nu^{\rm tot} \to 0$. Therefore, the flavor evolution of neutrinos in the regular precession solution cannot be adiabatic for all neutrinos.

V. CONCLUSIONS

We have derived a pair of equations from which it is possible to find a quasistatic MSW-like solution for an isotropic neutrino gas with specified electron and neutrino number densities. This solution is the natural extension of the conventional MSW solution, but includes neutrino self-coupling. We have shown that the condition for this MSW-like solution to be adiabatic is a necessary condition for neutrinos to follow such a solution. We have also discussed the previously discovered regular precession solution and the scenarios where it may break down in the presence of ordinary matter.

We have compared the results of the detailed simulations in Refs. [13,14] with the corresponding MSW-like and regular precession solutions. This comparison clearly shows that many features in the numerical simulations, including the stepwise swapping in neutrino energy spectra, can be explained by combinations of these two analytical solutions. We have also discussed the adiabaticity of flavor evolution in dense-neutrino gases.

We emphasize that the discovery of the MSW-like solution and the regular precession solution by no means obviates the need for further numerical simulations. For example, neutrinos only roughly follow the regular precession solution, typically exhibiting significant nutation mode behavior. In fact, the collective nutation mode is responsible for driving neutrinos away from the MSW-like solution and towards the regular precession solution. We note that so far the only method to quantitatively follow neutrino flavor transformation with the nutation mode is through numerical simulations. We also note that, in the normal mass hierarchy scenario, the value of the lepton number, a key parameter for the regular precession solution, is determined numerically. Nevertheless, the combination of the MSW-like solution and the regular precession solution offers a way to gain key insights into the results obtained from numerical simulations.

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