UNIVERSITY OF CALIFORNIA, SAN DIEGO

Analysis and Numerical Treatment of Elliptic Equations with Stochastic Data

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by

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EPIGRAPH

Hope is a good breakfast,
but it is a bad supper.
—Francis Bacon

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ABSTRACT OF THE DISSERTATION

Analysis and Numerical Treatment of Elliptic Equations with Stochastic Data

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Doctor of Philosophy in Mathematics with a Specialization in Computational Science

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Professor Michael Holst, Chair

Many science and engineering applications are impacted by a significant amount of uncertainty in the model. Examples include groundwater flow, microscopic biological system, material science and chemical engineering systems. Common mathematical problems in these applications are elliptic equations with stochastic data. In this dissertation, we examine two types of stochastic elliptic partial differential equations(SPDEs), namely nonlinear stochastic diffusion reaction equations and general linearized elastostatic problems in random media.

We begin with the construction of an analysis framework for this class of SPDEs, extending prior work of Babuska [3] in 2010. We then use the framework both for establishing well-posedness of the continuous problems and for posing Galerkin-type numerical methods. In order to solve these two types of problems, single integral weak formulations and stochastic collocation methods are applied. Moreover, *a priori* error estimates for stochastic collocation methods are derived, which imply that the rate of convergence is exponential, along with the order of polynomial increasing in the space of random variables. As expected, numerical experiments show the exponential rate of convergence, verified by *a posterior* error analysis. Finally, an adaptive

strategy driven by *a posterior* error indicators is designed.

Chapter 1

Introduction

Many engineering applications are affected by a relatively large amount of uncertainty in the input data. Among them, stochastic elliptic equations are an important area of research, such as groundwater flow with uncertain conductivity and pressure in soils or in- and out-flow boundary conditions, microscopic systems of brownian motion, stochastic chemical and environmental engineering systems. Applications with stochastic elliptic equations in these areas are discussed in [16, 14, 24, 23]. In material science, many stochastic mechanical behaviors of random media are relevant to various engineering fields, such as composite materials, geotechnical engineering and biomechanics. The reason a stochastic treatment is needed for many phenomena of material science is insufficient data from material properties. It is more realistic to treat them as random media rather than to approximate. One example of random media is simulating the fault formation in an earthquake, where stochastic treatment of ground surface is applied which consists of several layers of not fully known properties and structures. Precise discussion on how to describe random media in engineering examples stochastically are introduced in [1] and [28].

To study stochastic elliptic equations, a general model of stochastic elliptic equation is given as: Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the stochastic elliptic problem: find $u : \Omega \times \overline{D} \to \mathbb{R}^{d'}$, such that P-a.e.

in Ω the following equation holds:

$$\begin{aligned} \mathcal{L}(\omega, x; u) + \mathcal{G}(\omega, x; u) &= f(\omega, x), \quad \text{on } D, \\ \text{subject to} & \\ \mathcal{B}(\omega, x; u) &= g(\omega, x), \quad \text{on } \partial D, \end{aligned} \tag{1.0.1}$$

where $x = (x_1, \ldots, x_d) \in D \subset \mathbb{R}^d$, $\mathcal{L}$ is a linear differential operator, $\mathcal{G}$ is a nonlinear differential operator, and $\mathcal{B}$ is a boundary operator. The operator $\mathcal{B}$ includes Dirichlet boundary conditions on Dirichlet boundary segments and Neumann boundary conditions on Neumann boundary segments. In the most general setting, the operators $\mathcal{L}$ and $\mathcal{G}$, as well as f and g are all able to contain randomness. Additionally, the work [3] in 2010 is a complete introduction and analysis for solving a basic stochastic linear elliptic problem theoretically and numerically by stochastic collocation approach. Based on the general problem setting and the framework of stochastic linear elliptic problem, a nonlinear stochastic diffusion reaction elliptic equation and a general linearized elastostatic problem in random media are analyzed in this work.

How to describe uncertainty as random data input is not major issue discussed in this work, however it is quite important for the motivation of stochastic PDEs. Several ways to describe uncertainty, and worst-case scenario analysis, fuzzy set theory, evidence theory and probabilistic setting are introduced in [2, 11].

The framework of this paper is as follows. A new group of solution spaces $V^p \triangleq L^p(\Omega, W_0^{1,p}(D))$ is constructed to analyze well-posedness of weak form obtained from stochastic elliptic problems, equiped with the norm

$$\|v\|_{V^p} = \left(\int_\Omega \int_D |\nabla_x v|^p d\mu(x) dP(\omega) \right)^{1/p}.$$

It is a generalization of the solution space $V_{P,a}$ in [3].

Based on several assumptions, the well-possedness of the stochastic diffusion reaction elliptic problem and general linearized elasticity in random media are able to be proved in the solution space V^p. Many useful theorems and examples of showing well-possedness can be found in books of nonlinear functional analysis [22, 15] and a book of elasticity foundations [17].

In this work, numerical stochastic collocation approach is employed to find the weak solution u of single integral formulation in subspace $\mathcal{P}_p(\Gamma) \otimes H_h(D)$, and the numerical solution is denoted as $u_{p,h}$. This numerical technique employs standard finite element approximations in domain D and polynomial approximation in the probability domain Γ. The collocation points chosen are zeros of tensor product orthogonal polynomials with respect to the auxiliary probability density $\hat{\rho}$. For instance, if $\hat{\rho} = 1$, then the tensor product of zeros of one dimension legendre polynomial chaos would be selected as collocation points. [25, 27] illustrates basic ideas of polynomial chaos and properties of different types of polynomial chaos. [26, 3] describe the motivation and theoretically analysis of Stochastic collocation approach. Stochastic collocation approach has already been applied for some problem simulations shown in [18, 24]. The other two well developed numerical method for solving stochastic PDEs, Monte Carlo method and stochastic Galerkin methods are also introduced and compared with collocation approach in this work. The reason why we employ stochastic collocation method is it has several advantages as follow compared to other methods:

- It naturally results system of uncoupled deterministic problems.
- It is efficient for the case of dependent random variables by introducing auxiliary density function $\hat{\rho}$.
- It can deal with unbound random variables, such as Gaussian variables by setting the proper density function in space $\mathcal{P}_p(\Gamma)$.

More analysis of Monte Carlo simulation and stochastic Galerkin method is introduced in [4, 9]. Other recent works related to numerical methods for stochastic partial differential equations are [19, 21, 10].

Another Main result of this paper is *a priori* error estimates with respect to nonlinear stochastic diffusion reaction problem and linearized elasticity in random media. These estimates indicate stochastic collocation approach on the target models achieves exponential convergence rate as the order of space $\mathcal{P}_p$ increasing in each dimension p_n, with input data are infinitely differentiable with respect to random variables and other assumptions on growth rate of derivatives. These results of priori error estimates actually stay with the same assumptions in [3] of linear case without introducing extra regularity assumptions of randomness.

The way to prove exponentially convergent error estimates is splitting the error $\|u - u_{h,p}\|_{V_\rho^2}$ into two parts $\|u - u_h\| + \|u_h - u_{h,p}\|$, where u_h is the projection of u onto semilinear subspace $L^p(\Gamma, H_h(D))$. The bound of former term is a generalization of finite element error analysis applied on the new solution space V^p. The ideas of finite element error analysis for nonlinear problems are described in [22, 5, 7]. The second term is actually interpolating discrete solution along the random variables in space Γ, theorems and details of finding interpolation bound in random space are discussed in [6, 8, 11, 20].

Finally, *a posteriori* error indicator is given in order to measure the exponential rate of convergence of the error from the randomness. Based on *a posteriori* error indicator, numerical experiments of a few stochastic linear, nonlinear and linearized elastostatic problems are tested by overwriting the deterministic solver MCLite developed by Prof.Michael Holst as a stochastic collocation solver. Papers describing the mathematical framework used in the MCLite implementation can be found at [12, 13]. All the experiments admit the exponentially convergence as the theorems shown. An adaptive algorithm for real computing is introduced at last.

The outline of this dissertation is as follows: In chapter 2, we introduce the generalized solution space V^p for stochastic elliptic problems. The well-posedness of stochastic elliptic target problems are analyzed in chapter 3. In chapter 4 and 5, how to solve the weak forms of stochastic elliptic problems by collocation method are introduced. Then, we derive *a priori* error estimates of stochastic collocation methods in chapter 6 and 7. In the last chapter 8, *a posteriori* error indicator, an adaptive algorithm and numerical experiments of testing the exponential rate of convergence of the error form randomness are provided.

Chapter 2

V^p Space: Solution Space

2.1 V^p Space

The solution space V^p is a extension of Babuska's work in [3]. Here, we introduce a group of spaces in order to analyze solutions of nonlinear stochastic problem. Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0,1]$ is a probability measure.

Definition 2.1. *Let* $V^p \triangleq L^p(\Omega, W_0^{1,p}(D))$, equipped with the norm

$$\|v\|_{V^p} = \left(\int_\Omega \int_D |\nabla_x v|^p d\mu(x) dP(\omega) \right)^{1/p}.$$

Remark, since all the derivatives discussed about are taken in $D \subset \mathbb{R}$, then $\nabla_x u$ is denoted as ∇u from now on.
Furthermore, one can define the inner product for space V^2 as,

$$(v,u)_{V^2} = \left(\int_\Omega \int_D \nabla v \nabla u d\mu(x) dP(\omega) \right).$$

Now, some properties of this group of constructed spaces are introduced.

Theorem 2.2. *For* $1 \leq p < \infty$*,* V^p *is Banach space.*

Proof. It is not hard to verify V^p is normed vector space for $1 \leq p < \infty$. So, only need to show any Cauchy sequence in V^p converges.

Let $\{u_k\}$ is Cauchy in V^p, by Poincare Inequality (A.8), it implies both $\{u_k\}$

and $\{\nabla u_k\}$ are Cauchy in $L^p(\Omega\times D)$. Since $L^p(\Omega\times D)$ are complete for any $1 \le p < \infty$, there exist $u_k \to u$ and $\nabla u_k \to v$ in $L^p(\Omega \times D)$ for some $u, v \in L^p(\Omega \times D)$.
Now claim $\nabla u = v$, i.e $\forall \phi \in C_0^\infty(D)$ such that $\int_D u\nabla\phi\, d\mu(x) = -\int_D v\phi\, d\mu(x)$ a.e in Ω. To show this, since for any k, $\int_D u_k\nabla\phi\, d\mu(x) = -\int_D \nabla u_k\phi\, d\mu(x)$ holds, consider

$$\begin{aligned} \left|\int_D (u\nabla\phi + v\phi)d\mu(x)\right| &\le \int_D |u\nabla\phi - u_k\nabla\phi + v\phi - \nabla u_k\phi| d\mu(x) \\ &\le \int_D |u-u_k||\nabla\phi|d\mu(x) + \int_D |\nabla u_k - v||\phi|d\mu(x) \\ &\le \|u-u_k\|_{L^p(D)}\|\nabla\phi\|_{L^q(D)} + \|\nabla u_k - v\|_{L^p(D)}\|\phi\|_{L^q(D)}, \end{aligned}$$

where $1/p+1/q = 1$. Notice that u_k and ∇u_k go to u and v in $L^p(\Omega\times D)$ respectively, which implies $u_k \to u$ a.e in Ω and $\nabla u_k \to v$ a.e in Ω in $L^p(D)$. Also since $\phi \in C_0^\infty(D) \subset L^q(D)$ for any q. Therefore, $|\int_D (u\nabla\phi + v\phi)d\mu(x)| \to 0$ a.e in Ω, i.e $\nabla u = v$ a.e in Ω.

Then, show the limit function $u \in V^p$. This is easy to see because its weak derivative is $v \in L^p(\Omega\times D)$ a.e in Ω, i.e $\int_\Omega\int_D |\nabla u|^p d\mu(x)dP(\omega) < \infty$. Finally, consider $\|u-u_k\|_{V^p}^p = \int_\Omega\int_D |\nabla u - \nabla u_k|^p d\mu(x)dP(\omega) = \int_\Omega\int_D |v-\nabla u_k|^p d\mu(x)dP(\omega) \to 0$, hence $\{u_k\}$ converges in V^p implying V^p is complete for $1 \le p < \infty$. □

Theorem 2.3. *For $p = 2$, V^p is Hilbert space.*

Proof. This is a straightforward result by definition of Hilbert space and Thm 2.2. □

Theorem 2.4. V^p **Imbedding Theorem I.** *If $\Omega \times D$ has finite measure and $0 < p < q < \infty$, then $V^q \hookrightarrow V^p$ and $\|u\|_{V^p} \le C\|u\|_{V^q}$ where $C = (P(\Omega)\mu(D))^{1/p-1/q}$.*

Proof. If $u \in V^q$, then $\nabla u \in L^q(\Omega \times D)$, thus simply apply Theorem A.1 on ∇u. □

Theorem 2.5. *For $2 \le p < \infty$, V^p is reflexive Banach space.*

Proof. It is not hard to see V^2 is Hilbert space, and hence by Theorem A.10 V^2 is reflexive Banach space. Since for V^p Imbedding Theorem 2.4, one has V^p are subspaces of V^2 for $2 < p < \infty$. Additionally, any Banach space is closed, thus all of them are closed subspaces of V^2. Now apply Theorem A.11, for any $2 < p < \infty$, V^p is reflexive. □

Theorem 2.6. V^p **Imbedding Theorem II.** *For $1 \le p < \infty$, D is a bounded set in $\mathbb{R}^d$, then*

$$\|u\|_{L^p(\Omega\times D)} \le C\|u\|_{V^p}, \quad \text{for any } u \in V^p. \tag{2.1.1}$$

Proof. For a.e in Ω, $u(\omega,\cdot)$ is a group of functions in $W_0^{1,p}(D)$, thus by Poincare inequality (A.2.5), one has $\|u(\omega,\cdot)\|_{L^p(D)} \le C|u(\omega,\cdot)|_{W^{1,p}(D)}$ a.e in Ω. Therefore,

$$\begin{aligned}\|u\|_{L^p}^p &= \int_\Omega \int_D u^p d\mu(x) dP(\omega) \\ &\le C^p \int_\Omega \int_D |\nabla u|^p \mu(x) dP(\omega) \\ &= \bar{C}\|u\|_{V^p}^p.\end{aligned}$$

□

2.2 $\mathbf{V}^p$ Space

In order to analyze problems in vector field, V^p space can be generalized as $\mathbf{V}^p$ space of vector field functions as following,

Definition 2.7. *Let* $V^p \triangleq [V^p]^d$, equipped with the norm defined in the sense of Euclidean distance

$$\|v\|_{\mathbf{V^p}} = (\sum_{i=1}^{d} \|v_i\|_{V^p}^p)^{1/p}.$$

The inner product for space $\mathbf{V}^2$ is defined as

$$(v,u)_{\mathbf{V^2}} = \sum_{i=1}^{d} (v_i, u_i)_{V^2}.$$

Theorem 2.8. *For* $1 \le p < \infty$, $\mathbf{V}^p$ *is Banach space, particularly* $\mathbf{V}^2$ *is a Hilbert space.*

Proof. Analogous prove as Thm 2.2 with the norm defined above. □

The two imbedding Theorems 2.4 and 2.6 can be generalized as following.

Theorem 2.9. $\mathbf{V}^p$ Imbedding Theorem I. *If* $\Omega \times \mathscr{D}$ *has finite measure and* $0 < p < q < \infty$, *then* $\mathbf{V}^q \hookrightarrow \mathbf{V}^p$ *and* $\|u\|_{\mathbf{V}^p} \le C\|u\|_{\mathbf{V}^q}$ *where* $C = (P(\Omega)\mu(\mathscr{D}))^{1/p-1/q}$.

Theorem 2.10. $\mathbf{V}^p$ Imbedding Theorem II. *For* $1 \le p < \infty$, $\mathscr{D}$ *is a bounded set in* $\mathbb{R}^d$, *then*

$$\|u\|_{\mathbf{L}^p(\Omega\times\mathscr{D})} \le C\|u\|_{\mathbf{V}^p}, \quad \text{for any } u \in \mathbf{V}^p. \tag{2.2.1}$$

Chapter 3

Existence and Uniqueness of Target Problems

3.1 General Stochastic Elliptic Differential Equations

The problems analyzed in [3] can be generalized as: Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0,1]$ is a probability measure. Consider the stochastic elliptic problem: find $u : \Omega \times \overline{D} \to \mathbb{R}^{d'}$, such that P-a.e. in Ω the following equation holds:

$$\begin{aligned} \mathcal{L}(\omega, x; u) + \mathcal{G}(\omega, x; u) &= f(\omega, x), \quad \text{on } D, \\ \text{subject to} \qquad & \\ \mathcal{B}(\omega, x; u) &= g(\omega, x), \quad \text{on } \partial D, \end{aligned} \tag{3.1.1}$$

where $x = (x_1, \ldots, x_d) \in D \subset \mathbb{R}^d$, $\mathcal{L}$ is a linear differential operator, $\mathcal{G}$ is a nonlinear differential operator, and $\mathcal{B}$ is a boundary operator. The operator $\mathcal{B}$ includes Dirichlet boundary conditions on Dirichlet boundary segments and Neumann boundary conditions on Neumann boundary segments. In the most general setting, the operators $\mathcal{L}$ and $\mathcal{G}$, as well as f and g are all able to contain randomness. In the following, three types of typical stochastic partial differential equations are analyzed.

3.2 Linear Stochastic Poisson Problem (Babuska's work)

To study the stochastic elliptic equations, we start with introducing the basic Stochastic Poisson problem which has been analyzed by the work of Babuska in [3]. Yet many engineering applications are affected by a relatively large amount of uncertainty in the input data. Among them, the stochastic elliptic equations is such an important area of research, such as groundwater flow with uncertain conductivity and pressure in soils or in- and out-flow boundary conditions, microscopic system of brownian motion, stochastic chemical and environmental engineering systems. A few cases of applications with stochastic equations in these areas are discussed in [16, 14, 24, 23].

3.2.1 Problem setting

Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the stochastic Poisson boundary value problem: find $u : \Omega \times \overline{D} \to \mathbb{R}$, such that P-a.e. in Ω the following equation holds:

$$\begin{aligned} -\nabla(a(\omega, x) \cdot \nabla u(\omega, x)) &= f(\omega, x) \quad \text{on } D, \\ u(\omega, x) &= 0 \quad \text{on } \partial D. \end{aligned} \tag{3.2.1}$$

3.2.2 Existence and Uniqueness

As long as $a(\omega, \cdot)$ is uniformly bounded from below and there exists $a_{min} > 0$, and $f \in V^2$, then the weak from of (3.2.1),

$$\int_D \int_\Omega a \nabla u \cdot \nabla v \;=\; \int_D \int_\Omega f v \quad \forall v \in V^2. \tag{3.2.2}$$

Admits a unique solution $u \in V^2$, and is able to shown by Lax-Milgram which is discussed in Sec 1 of [3].

3.3 Nonlinear Stochastic Diffusion-Reaction Elliptic Equation

Based on the model linear stochastic problem introduced earlier, we look at a more complicated nonlinear problem, the nonlinear Stochastic Diffusion-Reaction problem in this section. Our goal is to extend the results in [3] to nonlinear problems.

3.3.1 Problem setting

Let D be a convex bounded polygonal domain in $\mathbb{R}^d$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0,1]$ is a probability measure. Consider the stochastic Poisson boundary value problem: Find $u : \Omega \times \overline{D} \to \mathbb{R}$, such that P-a.e. in Ω the following equation holds:

$$\begin{aligned} -\nabla(a(\omega,x)\nabla u(\omega,x)) + b(\omega,x)u(\omega,x) + c(\omega,x)u^{\alpha}(\omega,x) &= f(\omega,x) \quad \text{on } D, \\ u(\omega,x) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{3.3.1}$$

where a, b, and c are in $L^{\infty}(\Omega \times D)$, here ∇u indicates $\sum_j \frac{\partial u}{\partial x_j}$ and the power of nonlinear term is assumed $\alpha > 1$.

Let $q = \alpha + 1 > 2$, the weak Galerkin formulation of (3.3.1) is: Find $u \in V^q$, such that

$$\begin{aligned} &\int_{\Omega}\int_{D}(a\nabla u\nabla v + buv + cu^{\alpha}v)d\mu(x)dP(\omega) \\ &\qquad = \int_{\Omega}\int_{D} fv d\mu(x)dP(\omega), \quad \forall v \in V^q, \end{aligned} \tag{3.3.2}$$

where a, b, and c are in $L^{\infty}(\Omega \times D)$, $f \in V^{q'}$ and $\alpha > 1$.

Theorem 3.1. *Well-defined Weak form. The weak formulation (3.3.2) is well-defined, i.e. any term is less than infinity.*

Proof. We look at weak formulation (3.3.2) term by term. For the first two terms, it is nothing more than Theorem 2.4 and 2.6. For any u and v in V^q with $\alpha > 1$ and $q = \alpha + 1$,

$$
\begin{aligned}
|\int_\Omega\int_D a\nabla u\nabla v d\mu(x)dP(\omega)| &\le \|a\|_{L^\infty}\int_\Omega\int_D |\nabla u||\nabla v|d\mu(x)dP(\omega)\\
&\le \|a\|_{L^\infty}\|u\|_{V^2}\|v\|_{V^2}\\
&\le C\|a\|_{L^\infty}\|v\|_{V^q}\|v\|_{V^q}\\
&< \infty.
\end{aligned}
$$

And,

$$
\begin{aligned}
|\int_\Omega\int_D buv d\mu(x)P(\omega)| &\le \|b\|_{L^\infty}\int_\Omega\int_D |u||v|d\mu(x)P(\omega)\\
&\le \|b\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2}\\
&\le \|b\|_{L^\infty}\|u\|_{V^q}\|v\|_{V^q}\\
&< \infty.
\end{aligned}
$$

Next let's look at the nonlinear term, notice $1/q + \alpha/q = 1$,

$$
\begin{aligned}
|\int_\Omega\int_D cu^\alpha v d\mu(x)P(\omega)| &\le \|c\|_{L^\infty}\int_\Omega\int_D |u^\alpha||v|d\mu(x)P(\omega)\\
&\le \|c\|_{L^\infty}\|u\|_{V^q}^\alpha\|v\|_{V^q}\\
&< \infty.
\end{aligned}
$$

Finally, since f is already in the dual of V^q, thus the RHS is bounded. □

3.3.2 Existence and Uniqueness with odd α

Now consider the energy functional $J : V^q \to \overline{\mathbb{R}}$

$$
J(u) = \int_\Omega\int_D (\frac{a}{2}|\nabla u|^2 + \frac{b}{2}u^2 + \frac{c}{\alpha+1}u^{\alpha+1} - fu)d\mu(x)dP(\omega). \tag{3.3.3}
$$

Theorem 3.2. *Energy functional (3.3.3) is G-differentiable.*

Proof. It is not hard to show $\lim_{t\to 0}\frac{1}{t}|J(u+th) - J(u) - t\frac{dJ(u+th)}{dt}|_{t=0}| = 0$, hence G-differentiable. □

We are interested in the following global minimization problem:

$$
\text{Find } u_0 \in V^q \text{ s.t } J(u_0) = \inf_{u\in V^q} J(u). \tag{3.3.4}
$$

Theorem 3.3. *The solution of minimization problem (3.3.4) is also a solution to the weak formulation (3.3.2).*

Proof. It is not hard to see the weak formulation (3.3.2) is equivalent with $\frac{dJ(u+tv)}{dt}|_{t=0} = 0$. Since any global minimizer in V^q will have zero G-derivative, thus a solution of the weak formulation. □

Lemma 3.4. *If $a \geq r > 0$ a.e, $c > 0$ a.e on $\Omega \times D$ and α is odd, then functional J in (3.3.3) is proper, i.e. $J(u_j) \to +\infty$ as $\|u_j\|_{V^q} \to +\infty$, $\forall u_j \in V^q$.*

Proof. Assume there exists a sequence $\{u_j\}$ with $\|u_j\|_{V^q} \to +\infty$, such that there exists a subsequence that $J(u_{j_k}) \leq C$ for all k. For convenience, the subsequence is indexed by j (subsequence will be indexed by j automatically afterwards). Consider the rescaled sequence $\{v_j\}$ where $v_j = u_j/\|u_j\|_{V^q}$, thus a bounded sequence. Now divide $\|u_j\|_{V^q}^{\alpha+1}$ to the sequence $J(u_j)$ (3.3.3), one has

$$\int_\Omega \int_D (\frac{a}{2}\frac{|\nabla u_j|^2}{\|u_j\|_{V^q}^{\alpha+1}} + \frac{b}{2}\frac{u_j^2}{\|u_j\|_{V^q}^{\alpha+1}} + \frac{c}{\alpha+1}\frac{u_j^{\alpha+1}}{\|u_j\|_{V^q}^{\alpha+1}} - f\frac{u_j}{\|u\|_{V^q}^{\alpha+1}})d\mu(x)dP(\omega) \;\leq\; \frac{C}{\|u_j\|_{V^q}^{\alpha+1}}.$$

Let $j \to \infty$,

$$\lim \int_\Omega \int_D \frac{c}{\alpha+1} v_j^{\alpha+1} d\mu(x)dP(\omega) \;\leq\; 0.$$

We have $\lim \|v_j\|_{L^p} = 0$, i.e $v_j \to 0$ strongly in L^p. Then ignore the positive $u^{\alpha+1}$ term and divide by $\|u_j\|_{V^q}^2$ to $J(u_j)$, let $j \to \infty$, Since L^p is continuously imbedding into L^2 thus $|\int bv_j^2| \leq \|b\|_\infty \int v_j^2 \to 0$, and hence

$$\frac{r}{2} \;\leq\; 0,$$

which is a contradiction. □

Lemma 3.5. *If $a \geq r > 0$ a.e, $c > 0$ a.e, $b > -C^2 r + \varepsilon$ for some $\varepsilon > 0$ where C is the Poincare constant of domain D, and α is odd, then functional J in (3.3.3) is bounded blow.*

Proof. With those assumptions on the coefficients, and it is not hard to see $L^2(\Omega \times$

$D) \subset V^{p'}$, one has

$$\begin{aligned} J(u) &\geq \int_\Omega \int_D (\frac{a}{2}|\nabla u|^2 + \frac{b}{2}u^2 + \frac{c}{\alpha+1}u^{\alpha+1} - |fu|) d\mu(x) dP(\omega) \\ &\geq \int_\Omega \int_D (\frac{C^2 r}{2} + \frac{b}{2}) u^2 d\mu(x) dP(\omega) - \|f\|_{L^2}\|u\|_{L^2} \\ &\geq \frac{\varepsilon}{2}\|u\|_{L^2}^2 - \|f\|_{L^2}\|u\|_{L^2} \\ &\geq -\frac{\|f\|_{L^2}^2}{2\varepsilon} > -\infty. \end{aligned}$$

□

Lemma 3.6. *Functional J in (3.3.3) is convex with odd α, and nonnegative coefficients a, b and c.*

Proof. Since α is odd, then all the functions $\frac{a}{2}|\nabla u|^2$, $\frac{b}{2}u^2$, $\frac{c}{\alpha+1}u^{\alpha+1}$ and fu are convex, hence the sum of convex functions is still convex. □

Theorem 3.7. *If $a \geq r > 0$ a.e, $b \geq 0$ a.e and $c > 0$ a.e, and α is odd, then the minimization problem (3.3.4) has a solution.*

Proof. Since we find the minimizer in the whole space $V^{\alpha+1}$ which is shown as a reflexive Banach space in Theorem 2.5, and obvious it is closed and convex. Under those assumptions of coefficient functions, the energy functional J in (3.3.3) satisfies Lemma 3.4, 3.5 and 3.6, thus it is proper, bounded below and convex. From Theorem 3.2, we know J is a G-differentiable, thus it is weakly lower semicontinuous by Theorem A.15 which implies lower semicontinuous. Finally, apply Theorem A.13, there exist one solution of problem (3.3.4). □

Lemma 3.8. *Define $\int_\Omega \int_D cu^p v$ as $\int_\Omega \int_D B(u)v$, where B is the nonlinear operator. If the nonlinear operator B is monotonely increasing on V^q, then for any $u_1, u_2 \in V^q$, one has $(B(u_1) - B(u_2))(u_1 - u_2) \geq 0$.*

Proof. $\forall u_1, u_2 \in V^q$, let $U^+ = \{(\omega, x) \,|\, u_1 \geq u_2\}$ and $U^- = \{(\omega, x) \,|\, u_1 \leq u_2\}$. Since B is monotonely increasing, one has $B(u_1) - B(u_2) \geq 0$ on U^+ and $B(u_1) - B(u_2) \leq 0$ on U^-, thus $(B(u_1) - B(u_2))(u_1 - u_2) \geq 0$ on $\Omega \times D$, thus the integral of a nonnegative function is nonnegative. □

Theorem 3.9. Existence and Uniqueness. *If $a \geq r > 0$ a.e, $b \geq 0$ a.e, $c > 0$ a.e, and α is odd, the weak problem (3.3.2) has a unique solution.*

Proof. By Theorem 3.7 and Theorem 3.3, the weak problem (3.3.2) has a solution. Therefore, the last thing remains is to show uniqueness. Assume there exist two distinct solutions u_1 and u_2. Then, let $v = u_1 - u_2$ in the weak form (3.3.2), by Lemma 3.8 one has

$$0 \;=\; \langle A(u_1 - u_2), u_1 - u_2\rangle + (B(u_1) - B(u_2), u_1 - u_2) \;\geq \langle A(u_1 - u_2), u_1 - u_2\rangle \;\geq\; 0.$$

Thus, $u_1 = u_2$ a.e, i.e $u_1 = u_2$ in V^q. □

3.3.3 Existence with even α

Since for even α, the energy functional is not bounded below or above, thus variational method will not applied. Additionally, V^p spaces do not have compact imbedding into any L^p space, therefore fixed point argument does not work. Hence, with more assumptions introduced, one is able to show the existence directly from looking at those solutions of deterministic PDEs, where $u^\omega(x) \in H_0^q$ denotes deterministic solution at ω. Since the deterministic solution may not be unique with respect to each ω, we denote the set of combinations of deterministic solutions by $U = \{u \mid u(\omega, x) = u^\omega(x)\ , \Omega\ a.e\}$. For even α, the nonlinear operator is no longer monotonously increasing, thus uniqueness will be not discussed.

Theorem 3.10. *For $u \in U$, if $\int_D |\nabla u^\omega(x)|^q dx$ is a random variable with finite expectation, then this u is a weak solution of (3.3.2).*

Proof. For any $v \in V^q$, since (Ω, P) is a complete probability space, then almost everywhere v^ω is in H_0^q, thus $\int_D a\nabla u \nabla v + buv + cu^\alpha v = \int_D fv$ a.e in Ω. Since, $\int_D |\nabla u|^q$ has finite expectation, then $u \in V^q$ and it solves (3.3.2). □

Although, for general probability space not many results can be derived, some prettier results are able to obtain with further restriction on probability space.

Theorem 3.11. *Assume (Ω, P) is a complete probability space with finite many elements and the set of each element is measurable, then u is a solution of (3.3.2) if $u \in U$.*

Proof. Here, we only need to show $\int_D |\nabla u^\omega(x)|^q dx$ is a random variable with finite expectation. Let $u_\omega = u^\omega(x)$ at ω, and zero otherwise, thus $\int_D |\nabla u_\omega|^q$ are measurable. Notice that $\int_D |\sum \nabla u_\omega|^q = \sum \int_D |\nabla u_\omega|^q$, since the cross terms are zeros, one has $\int_D |\nabla u|^q$ is a finite sum of $\int_D |\nabla u_\omega|^q$, which is a measurable function, i.e random variable. Since, each $u^\omega \in H_0^q$, then $E(\int_D |\nabla u^\omega(x)|^q dx) \leq \max_{\omega\in\Omega} \int_D |\nabla u^\omega(x)|^q dx < \infty$, thus $u \in V^q$ and by Theorem 3.10, u solves (3.3.2). □

Theorem 3.12. *Assume (Ω, P) is a complete probability space with countable many elements and the set of each element is measurable, if $u \in U$ and $\|u^\omega(x)\|_{H_0^q}$ is uniformly bounded in Ω, then u is a solution of (3.3.2).*

Proof. Similarly as Theorem 3.11, let $u_\omega = u^\omega(x)$ at ω, otherwise zero. It is not hard to see $\int_D |\nabla u|^q = \sum_{i=1}^{\infty} \int_D |\nabla u_{\omega_i}|^q$ is the pointwise limit of $\sum_{i=1}^{n} \int_D |\nabla u_{\omega_i}|^q$, thus $\int_D |\nabla u|^q$ is measurable, i.e a random variable. Since $\|u^\omega(x)\|_{H_0^q} \leq C$ uniformly in Ω, one has $E(\int_D |\nabla u|^q) \leq C < \infty$, thus $u \in V^q$, and by Theorem 3.10, it is a solution of (3.3.2).

□

3.4 Linearized Elastostatic Problem with Random Media

The purpose of this section is to extend Babuska's work to stochastic Elasticity. Stochastic mechanical behaviors of random media is relevant to various of engineering fields, such as composite materials, geotectonic engineering and biomechanics. The reason why a stochastic treatment is needed for many phenomena from these fields, is due to insufficient data from material properties. It is more realistic to treat them as random media rather than to approximate as exact. One example of random media is, in order to simulate the fault formation of earthquake, stochastic treatment on ground surface is applied which consists of several layers of not fully known properties and structures. Precise discussion on how to describe random media in engineering examples stochastically are introduced in [1] and [28]. In this section, a general linearized Elastostatic problem with random media is analyzed. The notations and problem setting follows closely the classical book of Elasticity [17].

3.4.1 Problem setting

The problem setting follows closely the classical book of Elasticity [17], and is generalized to stochastic problems. Let $\mathscr{D}$ be a convex bounded polygonal domain in $\mathbb{R}^3$, and let $(\Omega, \mathcal{F}, P)$ be a complete probability space where $P : \mathcal{F} \to [0, 1]$ is a probability measure. Consider the linearized elastostatic problem: Find the displacement vector $\mathbf{u} : \Omega \times \overline{\mathscr{D}} \to \mathbb{R}^3$, such that P-a.e. in Ω the following system of equations holds for each $i = 1, 2, 3$:

$$\begin{aligned} -\partial_j \sigma_{ij}(\omega, x)) &= f_i(x) \quad \text{on } \mathscr{D}, \\ \sigma_{ij}(\omega, x) &= A_{ijpq}(\omega, x) e_{pq}(\mathbf{u}(\omega, x)) \quad \text{on } \mathscr{D} \quad p, q = 1, 2, 3, \\ n_j \sigma_{ij}(\omega, x) &= g_i(\omega, x) \quad \text{on } \partial_{\mathcal{N}} \mathscr{D}, \\ u_i(\omega, x) &= 0 \quad \text{on } \partial_{\mathcal{D}} \mathscr{D}, \end{aligned} \tag{3.4.1}$$

where σ is stress tensor, $A \in \mathbf{C}^\infty$ is index 4 random tensor field, $e_{pq}(u) = \frac{1}{2}(u_{p,q} + u_{q,p})$ is the linearized strain tensor, f is body force vector, and $g \in \mathbf{L}^\infty$ is vector of the Neumann boundary condition. All the subindex are written in the style of Einstein summation.

Remark 3.13. *One could assume A is homogeneous and isotropic, then it can be written in the simpler form $A_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})$.*

The weak Galerkin formulation of (3.4.1) is derived as: Find $u \in \mathbf{V}^2$, such that

$$\int_\Omega \int_{\mathscr{D}} (\sigma_{ij}(u) \partial_j v_i - f_i v_i) d\mu(x) dP(\Omega) - \int_\Omega \int_{\partial_{\mathcal{N}} \mathscr{D}} g_i v_i d\mu(x) dP(\Omega) = 0 \quad \forall v \in \mathbf{V}^2, \tag{3.4.2}$$

where the A in stress tensor is $\mathbf{C}^\infty$, $g \in \mathbf{L}^\infty(\Omega \times \partial_{\mathcal{N}} \mathscr{D})$, $f \in \mathbf{V}^{2'}$ and the test functions $v \in \mathbf{V}^2$ can be represented as

$$v = \phi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi_1, \phi_2, \phi_3 \in V^2.$$

Furthermore, based on the assumptions of homogeneous and isotropic on A, the (3.4.2) is equivalent to the following symmetric form: Find $u \in \mathbf{V}^2$, such that

$$\int_\Omega \int_{\mathscr{D}} (A_{ijpq} e_{pq}(u) e_{ij}(v) - f_i v_i) d\mu(x) dP(\Omega) - \int_\Omega \int_{\partial_{\mathcal{N}} \mathscr{D}} g_i v_i d\mu(x) dP(\Omega) = 0 \quad \forall v \in \mathbf{V}^2, \tag{3.4.3}$$

where the $e_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i})$, A in stress tensor is $\mathbf{C}^\infty$, $g \in \mathbf{L}^\infty(\Omega \times \partial_{\mathcal{N}}\mathscr{D})$ and $f \in \mathbf{V}^{2'}$.

Before looking into this weak formulation, one needs to show it is well-defined.

Theorem 3.14. *Well-defined Weak form. The weak formulation (3.4.3) is well-defined, i.e. any term is less than infinity.*

Proof. We look at weak formulation (3.4.3) term by term. For any u and v in $\mathbf{V}^2$, apply the Cauchy-Schwarz Inequality (A.4), one has

$$\begin{aligned}
|\int_\Omega \int_{\mathscr{D}} A_{ijpq} e_{pq}(u) e_{ij}(v) d\mu(x) dP(\Omega)| &\leq \|A\|_{\mathbf{C}^\infty} \| \int_\Omega \int_{\mathscr{D}} \sum_{p,q} |e_{pq}(u)| \sum_{i,j} |e_{ij}(v)| \\
&\leq \|A\|_{\mathbf{C}^\infty} \| \sum_{p,q} |e_{pq}(u)| \|_{L^2} \| \sum_{i,j} |e_{ij}(v)| \|_{L^2} \\
&\leq \frac{1}{2} \|A\|_{\mathbf{C}^\infty} \| \sum_{p,q} (|u_{p,q}| + |u_{q,p}|) \|_{L^2} \| \sum_{i,j} (|v_{i,j}| + |v_{j,i}|) \|_{L^2} \\
&\leq C(i,j,p,q) \|A\|_{\mathbf{C}^\infty} \|u\|_{\mathbf{V}^2} \|v\|_{\mathbf{V}^2} \\
&< \infty,
\end{aligned}$$

where the constant $C(i,j,p,q) > 0$ from Cauchy-Schwarz Inequality only depends on those four indexes.

The second term is straightforward, since f is already in the dual space of $\mathbf{V}^2$. For the third term, $g \in \mathbf{L}^\infty$, simply apply Holder Inequality for Vector-valued functions A.6

$$\begin{aligned}
|\int_\Omega \int_{\partial_{\mathcal{N}}\mathscr{D}} g_i v_i d\mu(x) dP(\Omega)| &\leq \|g\|_{\mathbf{L}^\infty} \int_\Omega \int_{\mathscr{D}} |v_i| d\mu(x) dP(\Omega) \\
&\leq C \|g\|_{\mathbf{L}^\infty} \|v\|_{\mathbf{V}^2} \\
&< \infty.
\end{aligned}$$

□

3.4.2 Existence and Uniqueness

In order to show the solution existence and uniqueness, a few assumptions on the equations are introduced as following

Assumption 3.15. *The index four random tensor field A is homogeneous and isotropic, that is,*

$$A_{ijpq}(\omega) = \lambda(\omega)\delta_{ij}\delta_{pq} + \mu(\omega)(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}),$$

where λ and μ are called the Lame moduli. Furthermore, we can define Young's modulus as $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ and Poisson's ratio as $\nu = \lambda/2(\mu + \lambda)$.

Actually, we can generalize the above assumption as,

Assumption 3.16. *The index four random tensor field A is isotropic in the constitutive law, that is,*

$$A_{ijpq}(\omega, x) = \lambda(\omega, x)\delta_{ij}\delta_{pq} + \mu(\omega, x)(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}),$$

where the Lame moduli λ and μ are functions of x. Furthermore, we can define Young's modulus as $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ and Poisson's ratio as $\nu = \lambda/2(\mu + \lambda)$.

Assumption 3.17. Uniformly Pointwise Stable. *Let A_{ijpq} be a classical elasticity tensor. It is uniformly pointwise stable if there exists an constant $\eta > 0$ such that*

$$\frac{1}{2}A_{ijpq}e_{ij}e_{pq} \;\geq\; \eta\|e\|^2,$$

where the $\|\cdot\|$ of e is the matrix norm $(\sum_{i,j}|e_{ij}|^2)^{\frac{1}{2}}$.

The Assumption 3.15 and 3.16 have been applied to derive the symmetric weak formulation (3.4.3) by the symmetry of indexes i and j. It also a stronger condition of hyperelasticity which is $A_{ijpq} = A_{pqij}$. Thus, the bilinear form of $\langle Au, v\rangle = \int A_{ijpq}e_{pq}(u)e_{ij}(v)$ is symmetric. Although, the generalized Lax-Milgram theorem no longer requires symmetry from the bilinear form, the symmetry for bilinear form is still needed for many properties of A as an elliptic operator and the equivalence between weak solutions and strong solutions. This work is not for analysis of operator A, however, because this symmetry implies the fundamentals of weak formulation, we list the assumption implying to it at beginning.

Second Assumption 3.17 is actually a stronger condition of strong ellipticity of A which is there is an $\epsilon > 0$ such that

$$A_{ijpq}\xi_i\xi_p\eta_j\eta_q \;\geq\; \epsilon\|\xi\|^2\|\eta\|^2$$

for all $\xi, \eta \in \mathbb{R}^d$. The strong ellipticity also plays an important role for many results of A as an elliptic operator like the hyperelasticity mentioned in the pervious paragraph. The reason the stronger assumption uniform pointwise stability is listed is for the purpose of showing the well-posedness for the symmetric weak formulation (3.4.3). More works and results based on the weaker assumptions hyperelasticity and strong ellipticity on A are introduced in [17].

Lemma 3.18. First Korn's Inequality in $\mathbf{V}^2$. *For $u \in \mathbf{V}^2$ satisfying displacement boundary conditions on $\partial_{\mathcal{D}} \subset \partial\mathscr{D}$, we have*

$$\int_{\Omega\times\mathscr{D}} \|e\|^2 \;\geq\; C\|u\|^2_{\mathbf{V}^2},$$

for a suitable constant $C > 0$ independent of u, where the norm $\|e\|^2$ is the matrix norm $\sum_{i,j} e_{ij}^2$.

Proof. The proof of this lemma is actually a generalized version of First Korn's Inequality in $\mathbf{H}^2$. Consider u is a smooth displacement in $\mathbb{R}^3$ with compact support. Let $\hat{u}(\xi)$ be the Fourier transform of u. Thus

$$\hat{e}(\xi) \;=\; \frac{1}{2}(\xi \otimes \hat{u} + \hat{u} \otimes \xi).$$

By Plancerel's theorem for Fourier transforms of tensor fields,

$$\int_{\Omega\times\mathscr{D}} \|e\|^2 \;=\; \int_{\Omega\times\mathscr{D}} \|\hat{e}(\xi)\|^2 \;=\; \frac{1}{4}\int_{\Omega\times\mathscr{D}} \sum_{i,j}(\xi_i\hat{u}_j + \xi_j\hat{u}_i)^2.$$

Where,

$$\begin{aligned}\sum_{i,j}(\xi_i\hat{u}_j + \xi_j\hat{u}_i)^2 \;&=\; \sum_{i,j}(\xi_i^2\hat{u}_j^2 + \xi_j^2\hat{u}_i^2 + 2\xi_i\hat{u}_i\xi_j\hat{u}_j)\\ &=\; 4\sum_{i=j}\xi_i^2\hat{u}_i^2 + \sum_{i\neq j}(\xi_i^2\hat{u}_j^2 + \xi_j^2\hat{u}_i^2) + 2\sum_{i\neq j}\xi_i\hat{u}_i\xi_j\hat{u}_j\end{aligned}$$

By the inequality $2\xi_i^2\hat{u}_i^2\xi_j^2\hat{u}_j^2 \geq -(\xi_i^2\hat{u}_i^2 + \xi_j^2\hat{u}_j^2)$, one has

$$\begin{aligned}\sum_{i,j}(\xi_i\hat{u}_j + \xi_j\hat{u}_i)^2 \;&=\; 2\sum_{i=j}\xi_i^2\hat{u}_i^2 + \sum_{i\neq j}(\xi_i^2\hat{u}_j^2 + \xi_j^2\hat{u}_i^2)\\ &=\; 2\sum_{i,j}\xi_i^2\hat{u}_j^2.\end{aligned}$$

Taking integral on both sides,

$$\int_{\Omega\times\mathscr{D}} \|e\|^2 \;\geq\; \frac{1}{2}\int_{\Omega\times\mathscr{D}} \|Du\|^2 \;=\; \frac{1}{2}\|u\|^2_{\mathbf{V}^2},$$

which implies First Korn's Inequality in $\mathbf{V}^2$. $\square$

Before showing the existence and uniqueness of the symmetric weak formulation (3.4.3), let's review the the following method for showing them.

Theorem 3.19. Generalized Lax-Milgram Theorem. *Let H be a real Hilbert space, let the bilinear form $a(u, v)$ be bounded and coercive on $H \times H$, and let $f(u)$ be a bounded linear functional on H. Then there exists a unique solution to the problem: Find $u \in H$ such that*

$$a(u, v) \;=\; f(v), \quad \forall v \in H.$$

Where,

(i) Boundedness of a: $a(u, v) \le M\|u\|\|v\|$, $\forall u, v \in H$.

(ii) Coerciveness of a: $a(u, v) \ge m\|u\|^2$, $\forall u \in H$.

(iii) Boundedness of f: $f(v) \le L\|v\|$, $\forall v \in H$.

Remark 3.20. *Notice that this generalization of Lax-Milgram Theorem on longer needs symmetry of bilinear form $a(u, v)$, and only a single Hilbert space H is involved.*

Theorem 3.21. Existence and Uniqueness. *If the assumptions 3.15(or 3.16) and 3.17 hold, then the symmetric weak formulation (3.4.3) has a unique solution $u \in \mathbf{V}^2$.*

Proof. Firstly, let's transform the weak form (3.4.3) into the following form, Find $u \in \mathbf{V}^2$, such that

$$a(u, v) \;=\; f(v) \quad \forall v \in \mathbf{V}^2,$$

where

$$\begin{aligned} a(u, v) &= \int_\Omega \int_{\mathscr{D}} A_{ijpq} e_{pq}(u) e_{ij}(v) d\mu(x) dP(\Omega), \\ f(v) &= \int_\Omega \int_{\mathscr{D}} f_i v_i d\mu(x) dP(\Omega) + \int_\Omega \int_{\partial_{\mathcal{N}} \mathscr{D}} g_i v_i d\mu(x) dP(\Omega). \end{aligned}$$

Since $\mathbf{V}^2$ is a Hilbert space by Thm 2.8. Then, one only needs to verify those three conditions listed in Thm 3.19 which implies this theorem directly.

(i) Boundedness of $a(u, v)$
This is shown in Thm 3.1 as the first term.

(ii) Coerciveness of $a(u, v)$

According to the assumption 3.17 and First Korn's Inequality in $\mathbf{V}^2$ (3.18), we have the following,

$$\begin{aligned} a(u,u) &\geq 2\eta \int_{\Omega\times\mathscr{D}} \|e_{ij}(u)\|^2 \\ &\geq 2C\eta\|u\|^2_{\mathbf{V}^2} \quad \text{for some constant } \eta, C > 0, \quad \forall u \in \mathbf{V}^2. \end{aligned}$$

(iii) Boundedness of $f(v)$

Since $f \in \mathbf{V}^{2'}$ and $g \in \mathbf{L}^\infty$, one has

$$f(v) \leq (\|f\|_{\mathbf{V}^{2'}} + C\|g\|_{\mathbf{L}^\infty})\|v\|_{\mathbf{V}^2}.$$

□

Chapter 4

Assumption On Randomness and Single Integral Formulation

4.1 Finite−Dimensional Noise Assumption

In many cases, randomness can be approximated by only a small number of random variables, sometimes uncorrelated or independent. A well-known approach to approximate the randomness is truncated KL expansion which is a finte sum of n uncorrelated orthogonal random variables. Therefore, we make the finite-dimensional noise assumption, which is a generation of Babuska's assumption in [3].

Assumption 4.1. Finite-Dimensional Noise. *A function $r(\omega, \cdot)$ with randomness on Ω has the form*

$$r(\omega, \cdot) \;=\; r(Y_1(\omega), \ldots, Y_N(\omega), \cdot) \quad \text{on } \Omega,$$

where N is a finite positive integer and $\{Y_n\}_1^N$ are real-valued random variables with mean zero and unit variance.

From now on, Γ_n denotes the range of $Y_n(\omega)$ on Ω, and $\Gamma \triangleq \Pi_{n=1}^N \Gamma_n$ with elements $y \in \Gamma$, and furthermore it is assumed that these random variables $[Y_1, \ldots, Y_N]$ have a joint probability density function $\rho : \Gamma \to \mathbb{R}^+$ with $\rho \in L^\infty(\Gamma)$.

4.2 Single Integral Formulation

By taking the advantage of the finite dimensional noise assumption on randomness and existence and uniqueness of weak forms, the outer integral of random domain Ω is able to be removed from the weak forms of targeting problems, the simplified form is defined as single integral formulation of the original weak form, which is introduced in [3], and generalized to a wider range of problems in this section by us.

4.2.1 Nonlinear Stochastic Diffusion-Reaction Elliptic Equation

With Assumption 4.1, the coefficients and f used in the computations have the form

$$\begin{aligned} a(\omega, x) &= a(Y_1(\omega), \ldots, Y_N(\omega), x), \\ b(\omega, x) &= b(Y_1(\omega), \ldots, Y_N(\omega), x), \\ c(\omega, x) &= c(Y_1(\omega), \ldots, Y_N(\omega), x) \quad \text{and} \\ f(\omega, x) &= f(Y_1(\omega), \ldots, Y_N(\omega), x) \quad \text{on } \Omega \times \overline{\mathrm{D}}, \end{aligned}$$

Remark 4.2. *Usually those coefficients and f are independent in terms of $a(Y_a(\omega), x)$, $b(Y_b(\omega), x)$, $c(Y_c(\omega), x)$ and $f(Y_f(\omega), x)$, then Y can be defined as $[Y_a, Y_b, Y_c, Y_f]$.*

The solution $u(\omega, x)$ of stochastic problem (3.3.2) can be described by $u(Y_1(\omega), \ldots, Y_N(\omega), x) : \Omega \times D \to \mathbb{R}$ and furthermore $u(y, x) : \Gamma \times D \to \mathbb{R}$. Thus, our weak formulation (3.3.2) has an equivalent form which is, Find $u \in V_\rho^{\alpha+1}$ where the space $V_\rho^{\alpha+1}$ is the analogue of $V^{\alpha+1}$ with $(\Omega, \mathcal{F}, P)$ replaced by $(\Gamma, \mathcal{B}^N, \rho dy)$,

$$\begin{aligned} &\int_\Gamma \rho(y) \int_D (a\nabla u \nabla v + buv + cu^\alpha v) d\mu(x) dy \\ &\qquad = \int_\Gamma \rho(y) \int_D fv d\mu(x) dy, \quad \forall v \in V_\rho^{\alpha+1}. \end{aligned} \tag{4.2.1}$$

For preference, one integral from the above formulation (4.2.1) can be removed and the single integral formulation is, Find $u(y) : \Gamma \to W_0^{1,q}(D)$,

$$\int_D (a(y)\nabla u(y)\nabla\phi + b(y)u(y)\phi + c(y)u(y)^\alpha \phi)d\mu(x) \\ = \int_D f(y)\phi d\mu(x), \quad \forall \phi \in W_0^{1,q}(D),\ \rho - \text{a.e. in } \Gamma. \tag{4.2.2}$$

Remark 4.3. *As long as the single integral formulation (4.2.2) have unique solutions almost everywhere in* Γ*, and the original weak form (4.2.1) has unique solution, they are equivalent to each other. The reason is because if there is a unique solution for original weak form, it must be the unique solution for single integral formulation* Γ *almost everywhere. And, the discussion of existence and uniqueness of this kind of deterministic single integral semilinear equations with the assumptions of Theorem 3.9 is omitted since it has been well understood.*

4.2.2 Linearized Elastostatic Problem with Random Media

The assumption of [3] is generalized to stochastic Elasticity in this section. With Assumption 4.1, the index 4 random tensor field A used in the computations has the form

$$A(\omega, x) = A(Y_1(\omega), \ldots, Y_N(\omega), x).$$

The solution $u(\omega, x) \in \mathbf{V}^2$ of stochastic linearized elastostatics (3.4.3) can be described by $u(Y_1(\omega), \ldots, Y_N(\omega), x) : \Omega \times \mathscr{D} \to \mathbb{R}^3$ and furthermore $u(y, x) : \Gamma \times \mathscr{D} \to \mathbb{R}^3$. Thus, the weak formulation (3.4.3) has an equivalent form which is, Find $u \in \mathbf{V}_\rho^2$ where the space $\mathbf{V}_\rho^2$ is the analogue of $\mathbf{V}^2$ with $(\Omega, \mathcal{F}, P)$ replaced by $(\Gamma, \mathcal{B}^N, \rho dy)$,

$$\int_\Gamma \rho(y) \int_{\mathscr{D}} (A_{ijpq} e_{pq}(u) e_{ij}(v) - f_i v_i) d\mu(x) dy - \int_\Gamma \rho(y) \int_{\partial\mathscr{D}} g_i v_i d\mu(x) dy = 0, \quad \forall v \in \mathbf{V}_\rho^2. \tag{4.2.3}$$

For simplicity, one integral from the above formulation (4.2.3) can be removed and the single integral formulation is, Find $u(y) : \Gamma \to \mathbf{H}_0^1(\mathscr{D})$,

$$\int_{\mathscr{D}} (A_{ijpq}(y) e_{pq}(u(y)) e_{ij}(\phi) - f_i \phi_i) d\mu(x) \\ - \int_{\partial\mathscr{D}} g_i(y) \phi_i d\mu(x) = 0, \quad \forall \phi \in \mathbf{H}_0^1(\mathscr{D}), \quad \rho - \text{a.e. in } \Gamma. \tag{4.2.4}$$

Remark 4.4. *The reason why the single integral formulation (4.2.4) is equivalent to the weak form (4.2.3) is the same as the previous case nonlinear diffusion-reaction elliptic equation.*

If we suppose all coefficients, forcing term f and index 4 random tensor field A mentioned above admit a smooth extension on ρdy-zero measure sets. Then the single integral formulations (4.2.2) and (4.2.4) can be extended a.e. on Γ with Lebesgue measure instead of ρdy. Now, the stochastic equations become deterministic parametric equations as a result of Assumption 4.1 and the well-posedness of targeting problems.

Remark 4.5. *The single integral formulations imply (4.2.1) and (4.2.3) are able to be seen as a group of deterministic equations almost everywhere in* Γ*, which actually motivates the stochastic collocation method in chapter 5.*

Chapter 5

Introduction to Stochastic Collocation Method

5.1 Generalized Stochastic Collocation Method

We generalize the Stochastic Collocation method introduced in [3] in this section as follows. Generally speaking, to approximate the weak solution $u \in \mathcal{P}(\Gamma) \otimes X(D)$ where $X(D)$ is a Banach space on physical domain $D \in \mathbb{R}^d$ and $\mathcal{P}(\Gamma)$ is the space of randomness with assumption 4.1. We solve the weak solution numerically in a subspace $\mathcal{P}_p(\Gamma) \otimes X_h(D) \subset \mathcal{P}(\Gamma) \otimes X(D)$ for $u_{h,p} \in \mathcal{P}_p(\Gamma) \otimes X_h(D)$, where

- $X_h(D) \subset X(D)$ is just the finite element space discredited on D. For example, the piecewise polynomials defined on regular triangulations $\mathcal{T}_h$ that have a maximum mesh spacing parameter $h > 0$.

- $\mathcal{P}_p(\Gamma) \subset L^2_\rho(\Gamma)$ or $\mathbf{P}_p(\Gamma) \subset \mathbf{L}^2_\rho(\Gamma)$ is the span of tensor product polynomials with degree at most $p = (p_1, \cdots, p_N)$, i.e. $\mathcal{P}_p(\Gamma) = \otimes_1^N \mathcal{P}_{p_n}(\Gamma_n)$, where $\mathcal{P}_{p_n}(\Gamma_n) = \text{span}(y_n^m, m = 0, \ldots, p_n)$, $n = 1, \ldots, N$. Hence, the dimension of $\mathcal{P}_p$ is $N_p = \Pi_1^N (p_n + 1)$.

  Remark 5.1. *This $\mathcal{P}_p(\Gamma)$ will generate a fully tensor grid. It is computationally expensive when N is large. Therefore, some new approaches known as sparse grid methods are developed to reduce number of grid points by considering a subspace of fully tensor product space.*

The generalized main steps of Stochastic Collocation Method are listed in order as below,

- Generate collocation points $\{y_i\}s$.
  Collocation points are actually zeros of orthogonal polynomials in $\mathcal{P}_p(\Gamma)$ with respect to auxiliary probability density function $\hat{\rho} : \Gamma \to \mathbb{R}^+$ that can be seen as the joint probability of N independent random variables factorized as

$$\hat{\rho}(y) \ = \ \Pi_1^N \hat{\rho}_n(y_n), \ \forall \mathrm{y} \in \Gamma, \text{ such that } \|\frac{\rho}{\hat{\rho}}\|_{\mathrm{L}^\infty(\Gamma)} \ < \ \infty. \tag{5.1.1}$$

  The standard choices of $\hat{\rho}$ are constant or Gaussian corresponding to Legendre zeros or Hermite zeros, etc. For instance, consider in each dimension $n = 1, \ldots, N$, the orthogonal polynomial $q_{q_n}(y_n)$ where $1 \leq q_n \leq p_n$ such that $\int_{\Gamma_n} q_{q_n}(y_n) r(y_n) \hat{\rho}_n(y_n) dy_n = 0$ for any $r \in \mathcal{P}_{q_n - 1}(\Gamma_n)$. Let $\{y_n^{k_n}\}_{k_n=1}^{q_n}$ be the roots of q_{q_n}, then $y_k = [y_1^{k_1}, \ldots, y_N^{k_N}]$ are collocation points chosen as. The reason of choosing collocation points as zeros of polynomial basis corresponding to auxiliary probability density functions is to take advantages of their well known roots and quadratrue weights which have been calculated with full accuracy.

- Solve semidiscrete approximation $u_h(y) : \Gamma \to X_h(D)$ on collocation points $\{y_k\}s$.
  $u_h(y)$ is the projection of solution $u(y) : \Gamma \to X(D)$ onto subspace $X_h(D)$. The problem needs to be solved is find $u_h(y_k)$ by solving the single integral formulation with testing functions $\phi \in X_h(D)$.

- Construct the discrete solution $u_{h,p} \in \mathcal{P}_p(\Gamma) \otimes X_h(D)$ by interpolating $u(y)$ on collocation points $\{y_k\}$s.
  Consider the Lagrange interpolation operator

$$u_{h,p}(y, x) = \mathcal{I}_{\hat{\rho}} u_h(y, x) = \sum_k u_h(y_k, x) l_k(y), \tag{5.1.2}$$

  with the weights $l_k(y) = \Pi_{n=1}^N l_{k,n}(y_n)$, where $l_{k,n}(y_n)$ is one-dimensional weight function of Lagrange form on nodes $y_n^{k_n}$.

Remark 5.2. *Someone may ask the single integral formulations (4.2.2) and (4.2.4) are satisfied a.e on* Γ*, and the set of collocation points has measure zero,* *thus the interpolation derived from solutions on collocation points does not make sense if some of y_k are outside almost everywhere. However, this problem is able to avoid by making a few simply reasonable assumptions on regularity of randomness which are discussed in chapter 7.*

- Compute the mean and variance of $u_{h,p}$ by quadrature.

  The first moment and second moment can be approximated by,

$$\begin{aligned} E(u)(x) &\approx E(u_{h,p})(x) = \int_\Gamma u_{h,p}(y,x)\rho dy, \\ E(u^2)(x) &\approx E(u_{h,p}^2)(x) = \int_\Gamma u_{h,p}^2(y,x)\rho dy. \end{aligned} \tag{5.1.3}$$

  Next, one can employ suitable quadrature to calculate the integrals. Moreover, if $\frac{\rho}{\hat{\rho}}$ is smooth, it is able to take advantage of y_k by using Gauss quadrature on $\frac{\rho}{\hat{\rho}}u_{h,p}$ and $\frac{\rho}{\hat{\rho}}u_{h,p}^2$ as,

$$\begin{aligned} \int_\Gamma u_{h,p}(y,x)\rho dy &= \sum_k w_k(\frac{\rho}{\hat{\rho}}u_{h,p})(y_k,x), \\ \int_\Gamma u_{h,p}^2(y,x)\rho dy &= \sum_k w_k(\frac{\rho}{\hat{\rho}}u_{h,p})(y_k,x), \end{aligned} \tag{5.1.4}$$

  where the weights w_ks of quadrature with respect to $\hat{\rho}$ have been well calculated.

5.2 Comparison with other Methods

There are several numerical methods can be employed to solve SPDEs. Here, t-wo well developed and popular methods, Monte Carlo method and Stochastic Galerkin method, will be briefly introduced.

5.2.1 Monte Carlo Method

Monte Carlo method is one of the most developed methods for solving the strong form of SPDEs directly. It generalizes K independent realizations of random variables, and solve these K deterministic problems or K decoupled system by any deterministic solver, although the process is straightforword, it converges asymptotically $\frac{1}{\sqrt{K}}$ only introduced in [9], i.e., quadrupling the number of sampled realizations halves the error . Steps of Monte Carlo Method are as following,

- For a prescribed number of realizations K, generate independent and identically distributed random variables $y_k = \{Y^i(w_k)\}_{i=1}^N$, for $k = 1, \ldots, K$.
- For each $k = 1, \ldots, K$, solve a deterministic problem with $y_k = \{Y^i(w_k)\}_{i=1}^N$ for a solution $u_k = u(y_k, x)$.
- Postprocess the results to evaluate the statistics of solution, for example $E[u] = \frac{1}{K}\sum_{k=1}^K u_k$.

Notice that, Monte Carlo method requires few regularity on random variables.

5.2.2 Stochastic Galerkin Method

Another alternative approach to solve SPDEs is Stochastic Galerkin method discussed in [4] . This approach is a high order method. It is firstly to find a Galerkin approximation $u^G = \sum_{n=1}^{N_h}\sum_{k=1}^{N_p} c_{kn}\phi(x_n)\psi(y_k)$ with $N_h \times N_p$ unknown by solving a coupled $N_h \times N_p$ by $N_h \times N_p$ system, where $\phi(x_n)$s are basis functions of $X_h(D)$ and $\psi(y_k)$s are basis functions of $\mathcal{P}(\Gamma)$. To solve this system, highly efficient strategies and parallel computing are demanded. Then, postprocess the Galerkin approximation to evaluate the statistics of solution.

5.2.3 Strengths of Stochastic Collocation Method

With the assumptions of randomness introduced in chapter 4 which are actually generalized conditions for many cases, Stochastic Collocation method has the following advantages compared to the other two methods:

- Stochastic Collocation method is higher order method. Its convergence rate is much higher than Monte Carlo method whose convergence rate is only $\frac{1}{\sqrt{K}}$, and is at least as fast as the rate of Stochastic Galerkin method. With more assumptions of regularity introduced in chapter 6, its convergence rate may be exponential shown in Thm 7.8 and 7.17.
- Stochastic Collocation method has less computational complexity. By using Stochastic Collocation method, N_p systems with size N_h by N_h need to be solved, while one has to solve N_h by N_h system K times by Monte Carlo method

where $K >> N_p$ such that Monte Carlo method is able to reach the same level of accuracy as Collocation method. Additionally, Stochastic Galerkin method requires to solve a $N_p \times N_h$ by $N_p \times N_h$ system. Assume the Gaussian elimination is applied, Galerkin method requires $N_p^3 N_h^3$ flops compared to $N_p N_h^3$ flops needed only by Collocation method, not to mention the later may have better accuracy.

- Stochastic Collocation method can deal with unbounded random variables more easily than Stochastic Galerkin method, see Thm 7.8 and 7.17 for details.

5.3 Stochastic Collocation Method for Nonlinear Stochastic Diffusion-Reaction Elliptic Equation

We restrict the generale stochastic collocation method to nonlinear diffusion reaction problem. To approximate the weak solution $u \in V_\rho^{\alpha+1}$ or in other words $u \in L_\rho^{\alpha+1}(\Gamma) \otimes W^{1,\alpha+1}(D)$ of (4.2.2), we solve numerically in a subspace $V_{p,h}^{\alpha+1} \triangleq \mathcal{P}_p(\Gamma) \otimes H_h(D) \subset V_\rho^{\alpha+1}$ for $u_{h,p} \in V_{p,h}^{\alpha+1}$, where

- $H_h(D) \subset W_0^{1,\alpha+1}(D) \subset H_0^1(D)$ is just the finite element space discretized on D.
- $\mathcal{P}_p(\Gamma) \subset L_\rho^{\alpha+1}(\Gamma) \subset L_\rho^2(\Gamma)$ is the span of tensor product polynomials with degree at most $p = (p_1, \cdots, p_N)$, i.e. $\mathcal{P}_p(\Gamma) = \otimes_1^N \mathcal{P}_{p_n}(\Gamma_n)$, where $\mathcal{P}_{p_n}(\Gamma_n) = \text{span}(y_n^m, m = 0, \ldots, p_n)$, $n = 1, \ldots, N$.

The main steps of Stochastic Collocation Method are listed in order below,

- Generate collocation points $\{y_i\}s$.
  Collocation points are actually zeros of orthogonal polynomials in $\mathcal{P}_p(\Gamma)$ with respect to auxiliary probability density function $\hat{\rho} : \Gamma \to \mathbb{R}^+$.

- Solve semidiscrete approximation $u_h(y) : \Gamma \to H_h(D)$ on collocation points $\{y_k\}$s.
  $u_h(y)$ is the projection of solution $u(y)$ in (4.2.2) onto subspace $H_h(D)$. The

problem needs to be solved is: Find $u_h(y_k)$ for each k, such that

$$\int_D (a(y_k)\nabla u_h(y_k)\nabla\phi + b(y_k)u_h(y_k)\phi + c(y_k)u_h(y_k)^\alpha\phi)d\mu(x) = \int_D f(y_k)\phi d\mu(x), \quad \forall\phi \in H_h(D). \tag{5.3.1}$$

- Construct the discrete solution $u_{h,p} \in \mathcal{P}_p(\Gamma) \otimes H_h(D)$ by interpolating $u(y)$ on collocation points $\{y_k\}$s, which is shown in (5.1.2).

- Compute the mean and variance of $u_{h,p}$ by quadrature, i.e. apply suitable quadrature on (5.1.3).

5.4 Stochastic Collocation Method for Linearized Elastostatic Problem with Random Media

We restrict the generale stochastic collocation method to linearized elastostatic problem with random media. To approximate the weak solution $u \in \mathbf{V}^2_\rho$ or in other words $u \in \mathbf{L}^2_\rho(\Gamma) \otimes \mathbf{H}^1_0(\mathscr{D})$ of (4.2.4), we solve numerically in a subspace $\mathbf{V}^2_{p,h} \triangleq \mathbf{P}_p(\Gamma) \otimes \mathbf{H}_h(\mathscr{D}) \subset \mathbf{V}^2_\rho$ for $u_{h,p} \in \mathbf{V}^2_{p,h}$, where

- $\mathbf{H}_h(\mathscr{D}) \subset \mathbf{H}^1_0(\mathscr{D})$ is just the finite element space discretized on D.

- $\mathbf{P}_p(\Gamma) \subset \mathbf{L}^2_\rho(\Gamma)$ is the span of tensor product polynomials with degree at most $p = (p_1, \cdots, p_N)$, i.e. $\mathbf{P}_p(\Gamma) = \otimes_1^N \mathcal{P}_{p_n}(\Gamma_n)$, where $\mathcal{P}_{p_n}(\Gamma_n) = \text{span}(y_n^m, m = 0, \ldots, p_n)$, $n = 1, \ldots, N$.

The main steps of Stochastic Collocation Method are listed in order below,

- Generate collocation points $\{y_i\}s$.
  Collocation points are actually zeros of orthogonal polynomials in $\mathbf{P}_p(\Gamma)$ with respect to auxiliary probability density function $\hat{\rho} : \Gamma \to \mathbb{R}^+$.

- Solve semidiscrete approximation $u_h(y) : \Gamma \to \mathbf{H}_h(D)$ on collocation points $\{y_k\}$s.

$u_h(y)$ is the projection of solution $u(y)$ in (4.2.4) into subspace $\mathbf{H}_h(D)$. The problem need to be solved is: Find $u_h(y_k)$ for each k, such that

$$\int_{\mathscr{D}} (A_{ijpq}(y_k) e_{pq}(u(y_k)) e_{ij}(\phi) - f_i \phi_i) d\mu(x) \\ - \int_{\partial\mathscr{D}} g_i(y_k) \phi_i d\mu(x) \; = \; 0, \quad \forall \phi \in \mathbf{H}_h(\mathscr{D}). \tag{5.4.1}$$

- Construct the discrete solution $u_{h,p} \in \mathbf{P}_p(\Gamma) \otimes \mathbf{H}_h(D)$ by interpolating $u(y)$ on collocation points $\{y_k\}$s, which is shown in (5.1.2).
- Compute the mean and variance of $u_{h,p}$ by quadrature, i.e. apply suitable quadrature on (5.1.3) for each component of vector $u_{h,p}$.

Chapter 6

Regularity Analysis

In this chapter, based on the regularity results in [3], we derive the regularity results of the two new types of SPDEs, nonlinear stochastic diffusion reaction equations and general linearized elastostatic problems. Before go through proof of convergence, some regularity results are needed for f and ρ. Here, f represents the right hand side of weak form for Dirichlet problem. For continence, we denote the right hand side of mixed boundary problem as $\bar{f}$ which is the sum of Dirichlet boundary condition f and Neumann boundary condition gI_D, and I_D is the indicator function of domain D.
Firstly, we introduce a weight function $\sigma(y) = \Pi_n^N \sigma_n(y_n) \leq 1$, where

$$\sigma_n(y_n) = \begin{cases} 1 & & \text{if } \Gamma_n \text{ is bounded,} \\ e^{-\beta_n |y_n|} & \text{for some } \beta_n > 0 & \text{if } \Gamma_n \text{ is unbounded.} \end{cases} \tag{6.0.1}$$

Then, we define a space of continuous functions whose growth at infinity are at most exponential,

$$C^0_\sigma(\Gamma; V) \triangleq \{v : \Gamma \to V,\, v \text{ continuous in } y,\, \max_{y \in \Gamma} \|\sigma(y) v(y)\|_V < \infty\},$$

with a Banach space V defined on D.

Assumption 6.1. Growth at Infinity. *Suppose f and $\bar{f}$ are continuous along y and its growth at infinity is at most exponential, and the joint probability density function ρ behaves like Gaussian weight at infinity.*
(i) $f \in C^0_\sigma(\Gamma; L^2(D))$, *or* $\bar{f} \in C^0_\sigma(\Gamma; \mathbf{L}^2(D))$

(ii) $\rho(y) \leq C_\rho e^{-\sum_{n=1}^{N}(\delta_n y_n)^2}$ *for any* $y \in \Gamma$, *with constant* $C_\rho > 0$ *and* $\delta_n > 0$ *if* Γ_n *is unbounded and zero otherwise.*

Notice that one can select the auxiliary density $\hat{\rho}(y) = \Pi_1^N \hat{\rho}_n(y_n)$ satisfies for $n = 1, \dots, N$,

$$C_{min}^n e^{-(\delta_n y_n)^2} \leq \hat{\rho}_n(y_n) \leq C_{max}^n e^{(\delta_n y_n)^2} \quad \forall y_n \in \Gamma_n,$$

with some positive constants C_{min}^n and C_{max}^n are independent on y_n, therefore $\|\frac{\rho}{\hat{\rho}}\|_{L^\infty(\Gamma)} \leq \frac{C_\rho}{\Pi_1^N C_{min}^n} < \infty$ is satisfied. From this assumption, the following inclusions hold,

Theorem 6.2. $C_\sigma^0(\Gamma; V) \subset L_{\hat{\rho}}^2(\Gamma; V) \subset L_\rho^2(\Gamma; V)$.

Proof. The brief proof is given in [3]. Here is a complete proof.
(i) For $v \in L_{\hat{\rho}}^2(\Gamma; V)$

$$\|v\|_{\hat{\rho}}^2 = \int_\Gamma \hat{\rho}(y)\|v(y)\|_V^2 dy \leq \|v\|_{C_\sigma^0}^2 \int_\Gamma \frac{\hat{\rho}(y)}{\sigma^2(y)} dy \leq \|v\|_{C_\sigma^0}^2 \Pi_1^N I_n,$$

where

$$\begin{aligned} I_n &\leq C_{max}^n |\Gamma_n| && \text{if } \Gamma \text{ is bounded,} \\ I_n &\leq \textstyle\int_{\Gamma_n} (e^{2\beta_n |y_n|}) C_{max}^n e^{-(\delta_n y_n)^2} dy_n \\ &\leq C_{max}^n e^{(\frac{\beta_n}{\delta_n})^2} \textstyle\int_{\Gamma_n} e^{-(\delta_n y_n - \frac{\beta_n}{\delta_n})^2} dy_n = C_{max}^n e^{(\frac{\beta_n}{\delta_n})^2} \sqrt{\frac{2\pi}{\delta_n}} && \text{if } \Gamma \text{ is unbounded.} \end{aligned}$$

(ii) For $v \in L_\rho^2(\Gamma; V)$,

$$\|v\|_{L_\rho^2(\Gamma;V)} \leq \|\frac{\rho}{\hat{\rho}}\|_{L^\infty}^{\frac{1}{2}} \|v\|_{L_{\hat{\rho}}^2} \leq \sqrt{\frac{C_\rho}{\Pi_1^N C_{min}^n}} \|v\|_{L_{\hat{\rho}}^2}.$$

□

Actually, one can take $\sigma_n = e^{(-\delta_n y_n)^2/8}$ instead of the one defined in (6.0.1), by doing this the space $C_\sigma^0(\Gamma; V)$ becomes wider.

6.1 Regularity of Nonlinear Stochastic Diffusion-Reaction Elliptic Equation

Based on the previous assumption, further more, we need to assume,

Assumption 6.3. Continuity of coefficients. *Suppose the coefficient functional a b and c are continuous and $a, b, c \in C^0_{loc}(\Gamma; L^\infty(D))$.*

From these two assumptions 6.1 and 6.3, the first regularity result can be derived,

Lemma 6.4. *For the weak solution in (4.2.2), one has $\|u(y)\|_{H^1_0(D)} \leq C_{f_y}$, $\|u(y)\|_{L^{\alpha+1}(D)} \leq \overline{C}_{f_y}$, where C_{f_y} and $\overline{C}_{f_y}$ are two constants only depend on $\|f(y)\|_{L^2(D)}$ and $|D|$.*

Proof. Let the test function $\phi = u$, since α is odd, so each term on the left is positive. At last by Holder's inequality the results can be obtained. $\square$

Theorem 6.5. Regularity of solution *Assume 6.1 and 6.3, and the assumptions in Theorem 3.9, the weak solution in (4.2.2) is satisfied*
$u \in C^0_\sigma(\Gamma; H^1_0(D))$.

Proof. We proof this theorem term by term. Take $y_1 \neq y_2$ with any $y_2 \in \mathcal{B}_\delta(y_1)$ such that $\|a(y_1) - a(y_2)\|_{L^\infty(D)} \leq \varepsilon$ as well as b and c and $\|f(y_1) - f(y_2)\|_{L^2(D)} \leq \varepsilon$, and let $\phi = u(y_1) - u(y_2)$, then subtract two equations. For convenient, the functional u on y_1 denotes as u_1, and so does u_2.
First look at $I_1 \triangleq \int_D (a_1 \nabla u_1 \nabla (u_1 - u_2) - a_2 \nabla u_2 \nabla (u_1 - u_2))$.

$$\begin{aligned} I_1 &= \int_D a_2 [\nabla (u_1 - u_2)]^2 + \int_D (a_1 - a_2) \nabla u_1 \nabla (u_1 - u_2) \\ &\geq r \|u_1 - u_2\|^2_{H^1_0(D)} - \int_D |(a_1 - a_2) \nabla u_1 \nabla (u_1 - u_2)| \\ &\geq r \|u_1 - u_2\|^2_{H^1_0(D)} - \|a_1 - a_2\|_{L^\infty(D)} \|u_1\|_{H^1_0(D)} \|u_1 - u_2\|_{H^1_0(D)} \\ &\geq r \|u_1 - u_2\|^2_{H^1_0(D)} - \varepsilon C_{f_1} \|u_1 - u_2\|_{H^1_0(D)}. \end{aligned}$$

Similarly, the following results are able to obtained,

$$\begin{aligned} I_2 &\triangleq \int_D (b_1 u_1 (u_1 - u_2) - b_2 u_2 (u_1 - u_2)) \\ &\geq -\varepsilon C_{f_1} C_1 \|u_1 - u_2\|_{H^1_0(D)}, \quad \text{where } C_1 \text{ is the constant of Poincare's Inequality.} \end{aligned}$$

and since u^α is monotone increasing,

$$\begin{aligned} I_3 &\triangleq \int_D (c_1 u_1^\alpha (u_1 - u_2) - c_2 u_2^\alpha (u_1 - u_2)) \\ &\geq -\varepsilon \|u_1\|^\alpha_{L^{\alpha+1}(D)} \|u_1 - u_2\|_{L^{\alpha+1}(D)} \\ &\geq -\varepsilon \overline{C}^\alpha_{f_1} C_2 \|u_1 - u_2\|_{H^1_0(D)}, \end{aligned}$$

where C_2 is the constant of Sobolev imbedding theorem.

And the right hand side can be bounded as $\int_D (f_1 - f_2)(u_1 - u_2) \leq \varepsilon C_1 \|u_1 - u_2\|_{H^1_0(D)}$. Therefore, by moving the negative terms from lower bound of I_1, I_2 and I_3 to the right, the final inequality is shown as following,

$$\begin{aligned} r\|u_1 - u_2\|^2_{H^1_0(D)} &\leq (2C_{f_1} + \overline{C}^\alpha_{f_1} + 2C_1 + C_2)\varepsilon \|u_1 - u_2\|_{H^1_0(D)} \\ \Longrightarrow \|u_1 - u_2\|_{H^1_0(D)} \leq C\varepsilon, &\quad \text{with } C = \frac{2C_{f_1} + \overline{C}^\alpha_{f_1} + 2C_1 + C_2}{r}. \end{aligned}$$

Hence, u is continuous functional. The last task is to show $\max_{y\in\Gamma} \|\sigma(y)u(y)\|_{H^1_0(D)} < \infty$, which is easy to show as follows since $f \in C^0_\sigma(\Gamma; L^2(D))$,

$$\begin{aligned} \|\sigma(y^\star)u(y^\star)\|_{H^1_0(D)} &= \sigma(y^\star)\|u(y^\star)\|_{H^1_0(D)} \\ &\leq C_{f_{y^\star}} \sigma(y^\star) \|f(y^\star)\|_{L^2(D)} \\ &= C_{f_{y^\star}} \|\sigma(y^\star) f(y^\star)\|_{L^2(D)}, \end{aligned}$$

then take max on both sides, one has desired result. □

Until now, the weak form (4.2.2) is able to be defined on everywhere on Γ in the sense of $\|\cdot\|_{V^2}$, and the interpolation of semidiscrete solution of collocation method (5.3.1) becomes reasonable.
In order to analyze the error of interpolation, the following assumption is required.

Assumption 6.6. Bounded derivatives. *Assume there exists n positive $\gamma_n < \infty$, such that*

$$\|\tfrac{\partial^k_{y_n} a(y)}{a(y)}\|_{L^\infty}(D) \leq \gamma_n^k k! \quad \textit{and} \quad \tfrac{\|\partial^k_{y_n} f(y)\|^2_L(D)}{1+\|f(y)\|^2_L(D)} \leq \gamma_n^k k!,$$

for any $y \in \Gamma$.

Then, let $\Gamma^\star_n \triangleq \Pi^N_{j\neq n}\Gamma_j$ with $y^\star_n \in \Gamma^\star_n$, and $\sigma^\star_n \triangleq \Pi^N_{j\neq n}\sigma_j$, the second regularity result needed is

Theorem 6.7. Analytic Extension of u. *Under assumption 6.6 and assumption in theorem 3.9, the solution* $u(y_n, y_n^\star, x)$ *as a function of* y_n, $u : \Gamma_n \to C^0_{\sigma_n^\star}(\Gamma_n^\star; H_0^1(D))$ *admits an analytic extension* $u(z, y_n^\star, x)$, $z \in \mathbb{C}$, *in the region of the complex plane* $\Sigma(\Gamma_n; \tau_n) \triangleq \{z \in \mathbb{C},\ \mathrm{dist}(z, \Gamma_n) \le \tau_n\}$ *with* $0 < \tau_n < \frac{1}{2\gamma_n}$. *Moreover, for all* $z \in \Sigma(\Gamma_n; \tau_n)$,

$$\|\sigma_n(Rez)u(z)\|_{C^0_{\sigma_n^\star}(\Gamma_n^\star; H_0^1(D))} \le \frac{C_1 e^{\alpha_n \tau_n}}{a_{min}(1 - 2\tau_n\gamma_n)}(2\|f\|_{C^0_\sigma(\Gamma; H_0^1(D))} + 1),$$

with the poincare constant C_1.

Proof. This proof for the linear poisson problem is given in [3]. To extend his proof into targeting nonlinear problem is straightforward. Let the bilinear form $B(y; u, v) = \int_D a(y)\nabla u \cdot \nabla v$, then

$$\begin{aligned} B(y; \partial^k_{y_n} u, \partial^k_{y_n} u) &= -\sum_{l=1}^{k} (\frac{k}{l}) \partial^l_{y_n} B(y; \partial^{k-l}_{y_n} u, \partial^k_{y_n} u) - \int_D b(y)(\partial^k_{y_n} u)^2 \\ &\quad - \int_D c(y)(\partial^k_{y_n} u)^{\alpha+1} + (\partial^k_{y_n} f, v) \\ &\le -\sum_{l=1}^{k} (\frac{k}{l}) \partial^l_{y_n} B(y; \partial^{k-l}_{y_n} u, \partial^k_{y_n} u) + (\partial^k_{y_n} f, v),\ \forall v \in H_0^q(D) \subset H_0^1(D). \end{aligned}$$

The result inequality above is the same as the one in Lemma 3.2 of [3], thus the rest of proof is exactly the same of his. □

6.2 Regularity of Linearized Elastostatic Problem with Random Media

A further assumption is required for Linearized Elastostatic Problem, that is

Assumption 6.8. Continuity of coefficients. *Suppose the coefficient functional* A *continuous and* $A \in \mathbf{C}^0_{loc}(\Gamma; \mathbf{L}^\infty(\mathscr{D}))$.

From these two assumptions 6.1 and 6.8, the first regularity result can be derived,

Lemma 6.9. *For the weak solution in (4.2.4), one has* $\|u(y)\|_{\mathbf{H}_0^1(D)} \le C_{\bar{f}_y}$, $\|u(y)\|_{\mathbf{L}^2(\mathscr{D})} \le \overline{C}_{\bar{f}_y}$, *where* $C_{\bar{f}_y}$ *and* $\overline{C}_{\bar{f}_y}$ *are two constants only depend on* $\|\bar{f}(y)\|_{\mathbf{L}^2(\mathscr{D})}$ *and* $|\mathscr{D}|$.

Proof. Let the test function $\phi = u$, Apply the Holder's inequality for the right hand side and First Korn's Inequality (3.18) for the left hand side, then the results can be obtained. $\square$

Theorem 6.10. Regularity of solution *Assume 6.1 and 6.8, and the assumptions in Theorem 3.21, the weak solution in (4.2.4) is satisfied* $u \in C^0_\sigma(\Gamma; \mathbf{H}^1_0(D))$.

Proof. We proof this theorem term by term. Take $y_1 \neq y_2$ with any $y_2 \in \mathcal{B}_\delta(y_1)$ such that $\|A(y_1) - A(y_2)\|_{\mathbf{L}^\infty(\mathscr{D})} \leq \varepsilon$ as well as $\|\bar{f}(y_1) - \bar{f}(y_2)\|_{\mathbf{L}^2(\mathscr{D})} \leq \varepsilon$, and let $\phi = u(y_1) - u(y_2)$, then subtract two equations of weak form. For convenient, the functional u on y_1 denotes as u_1 and $A(y_1)$ denotes as A^1, and so does u_2 and A^2. First look at $I_1 \triangleq \int_{\mathscr{D}} (A^1_{ijpq} e_{pq}(u_1) e_{ij}(u_1 - u_2) - A^2_{ijpq} e_{pq}(u_2) e_{ij}(u_1 - u_2))$. Since, e is linear,

$$\begin{aligned} I_1 &= \int_{\mathscr{D}} A^1_{ijpq} e_{pq}(u_1 - u_2) e_{ij}(u_1 - u_2) + \int_{\mathscr{D}} (A^1_{ijpq} - A^2_{ijpq}) e_{pq}(u_2) e_{ij}(u_1 - u_2) \\ &\geq C_1 \eta \|u_1 - u_2\|^2_{\mathbf{H}^1_0} - \int_{\mathscr{D}} |(A^1_{ijpq} - A^2_{ijpq}) e_{pq}(u_2) e_{ij}(u_1 - u_2)| \\ &\geq C_1 \eta \|u_1 - u_2\|^2_{\mathbf{H}^1_0} - C_2 \|A^1 - A^2\|_{\mathbf{L}^\infty} \|u_2\|_{\mathbf{H}^1_0} \|u_1 - u_2\|_{\mathbf{H}^1_0} \\ &\geq C_1 \eta \|u_1 - u_2\|^2_{\mathbf{H}^1_0(D)} - \varepsilon C_{\bar{f}_y} C_2 \|u_1 - u_2\|_{\mathbf{H}^1_0}, \end{aligned}$$

where C_1 and η are from First Korn's Inequality 3.18 and assumption 3.17, C_2 is from Thm 3.14 and $C_{\bar{f}_y}$ is from pervious lemma 6.9.

And the right hand side can be bounded as $\int_{\mathbf{D}} (\bar{f}_1 - \bar{f}_2)_i (u_1 - u_2)_i \leq \varepsilon C_3 \|u_1 - u_2\|_{\mathbf{H}^1_0}$, Where C_3 is the constant from Poincare's Inequality.

Therefore, by moving the negative terms from lower bound of I_1 to the right, the final inequality is shown as following,

$$\begin{aligned} C_1 \eta \|u_1 - u_2\|^2_{\mathbf{H}^1_0} &\leq (C_{\bar{f}_y} + C_2 + C_3) \varepsilon \|u_1 - u_2\|_{\mathbf{H}^1_0} \\ \Longrightarrow \|u_1 - u_2\|_{\mathbf{H}^1_0} \leq C\varepsilon, &\quad \text{with } C = \frac{C_{\bar{f}_y} + C_2 + C_3}{C_1 \eta}. \end{aligned}$$

Hence, u is continuous functional. The last task is to show $\max_{y \in \Gamma} \|\sigma(y) u(y)\|_{\mathbf{H}^1_0} < \infty$, which is easy to show as follows since $\bar{f} \in C^0_\sigma(\Gamma; \mathbf{L}^2(\mathscr{D}))$,

$$\|\sigma(y^\star) u(y^\star)\|_{\mathbf{H}^1_0} = \sigma(y^\star) \|u(y^\star)\|_{\mathbf{H}^1_0} \leq C_{\bar{f}_{y^\star}} \sigma(y^\star) \|\bar{f}(y^\star)\|_{\mathbf{L}^2} = C_{\bar{f}_{y^\star}} \|\sigma(y^\star) \bar{f}(y^\star)\|_{\mathbf{L}^2},$$

then take max on both sides, one has desired result. $\square$

Until now, the weak form (4.2.4) is able to be defined on everywhere on Γ in the sense of $\|\cdot\|_{\mathbf{V}^2}$, and the interpolation of semidiscrete solution of collocation method (5.3.1) becomes reasonable.

In order to analyze the error of interpolation, the following assumption is required.

Assumption 6.11. Bounded derivatives. *Assume there exists n positive $\gamma_n < \infty$, such that*

$$\|\partial^k_{y_n} A(y)\|_{\mathbf{L}^\infty(\mathscr{D})} \leq \gamma_n^k k! \quad \textit{and} \quad \frac{\|\partial^k_{y_n}\bar{f}(y)\|^2_{\mathbf{L}(D)}}{1+\|f(y)\|^2_{\mathbf{L}(D)}} \leq \gamma_n^k k!,$$

for any $y \in \Gamma$.

Then, let $\Gamma_n^\star \triangleq \Pi_{j\neq n}^N \Gamma_j$ with $y_n^\star \in \Gamma_n^\star$, and $\sigma_n^\star \triangleq \Pi_{j\neq n}^N \sigma_j$, the second regularity result needed is

Theorem 6.12. Analytic Extension of u. *Under assumption 6.11 and assumption in theorem 3.21, the solution $u(y_n, y_n^\star, x)$ as a function of y_n, $u : \Gamma_n \to C^0_{\sigma_n^\star}(\Gamma_n^\star; \mathbf{H}^1_0(D))$ admits an analytic extension $u(z, y_n^\star, x)$, $z \in \mathbb{C}$, in the region of the complex plane $\Sigma(\Gamma_n; \tau_n) \triangleq \{z \in \mathbb{C},\ \mathrm{dist}(z, \Gamma_n) \leq \tau_n\}$ with $0 < \tau_n < \frac{1}{2\gamma_n}$. Moreover, for all $z \in \Sigma(\Gamma_n; \tau_n)$,*

$$\sigma_n(y_n)\|u(z)\|_{C^0_{\sigma_n^*}(\Gamma_n^*,\mathbf{H}^1_0(\mathscr{D}))} \;\leq\; \frac{C_3 e^{\beta_n\tau_n}}{2C_1\eta(1-2\tau_n\gamma_n)}[(1+C_2)\|\bar{f}\|_{C^0_\sigma(\Gamma,\mathbf{L}^2(\mathscr{D})} + 1]. \quad (6.2.1)$$

Where C_1 and η are from First Korn's Inequality 3.18 and assumption 3.17, C_2 is from Thm 3.14 and C_3 is the poincare constant.

Proof. For each point $y \in \Gamma$, the kth derivative of u with respect to y_n satisfies the following equation

$$B(y; \partial^k_{y_n} u, \partial^k_{y_n} u) \;=\; -\sum_{l=1}^{k} (\frac{k}{l}) \partial^l_{y_n} B(y; \partial^{k-l}_{y_n} u, \partial^k_{y_n} u) + (\partial^k_{y_n}\bar{f}, v) \quad \forall v \in \mathbf{H}^1_0(\mathscr{D}),$$

where the bilinear form $B(y; u, v) = \int_{\mathscr{D}} A_{ijpq}(y) e_{pq}(u) e_{ij}(v)$. Hence, one has

$$\begin{aligned} C_1\eta\|\partial^k_{y_n} u_{i,j}\|^2_{\mathbf{L}^2} \;&\leq\; \int_{\mathscr{D}} |A_{ijpq} e_{pq}(\partial^k_{y_n} u) e_{ij}(\partial^k_{y_n} u)| \\ &\leq\; C_2 \sum_{l=1}^{k} (\frac{k}{l}) \|\partial^l_{y_n} A(y)\|_{\mathbf{L}^\infty} \|\partial^{k-l}_{y_n} u_{i,j}\|^2_{\mathbf{L}^2} \\ &+\; C_3 \|\partial^k_{y_n}\bar{f}\|_{\mathbf{L}^2} \|\partial^k_{y_n} u_{i,j}\|_{\mathbf{L}^2}. \end{aligned}$$

Setting $R_k = \|\partial_{y_n}^k u_{i,j}\|_{\mathbf{L}^2}/k!$ and using the bounds of assumption 6.11, a recursive inequality can be obtained

$$R_k \le \frac{C_2}{C_1\eta}\sum_{l=1}^{k}\gamma_n^l R_{k-l} + \frac{C_3}{C_1\eta}\gamma_n^k(1+\|\bar{f}\|_{\mathbf{L}^2}).$$

The generic term R_k admits the bound

$$R_k \le \frac{1}{2}(2\gamma_n)^k(\frac{C_2}{C_1\eta}R_0 + \frac{C_3}{C_1\eta}(1+\|\bar{f}\|_{\mathbf{L}^2})).$$

Because of $R_0 = \|u\|_{\mathbf{H}_0^1} \le C_3\|\bar{f}\|_{\mathbf{L}^2}$, the final estimate on the growth of the derivatives of u is derived as

$$\frac{\|\partial_{y_n}^k u_{i,j}\|_{\mathbf{L}^2}}{k!} \le \frac{C_3}{2C_1\eta}(2\gamma_n)^k((1+C_2)\|\bar{f}\|_{\mathbf{L}^2}+1).$$

Now, we define for every $y_n \in \Gamma_n$ the power series $u : \mathbb{C} \to C^0_{\sigma_n^*}(\Gamma, \mathbf{H}_0^1(\mathscr{D}))$ as

$$u(z, y_n^*, x) = \sum_{k=0}^{\infty}\frac{(z-y_n)^k}{k!}\partial_{y_n}^k u(y_n, y_n^*, x).$$

Therefore,

$$\begin{aligned}
\sigma_n(y_n)\|u(z)\|_{C^0_{\sigma_n^*}(\Gamma_n^*,\mathbf{H}_0^1(\mathscr{D}))} &\le \sum_{k=0}^{\infty}\frac{|z-y_n|^k}{k!}\sigma_n(y_n)\|\partial_{y_n}^k u(y_n)\|_{C^0_{\sigma_n^*}(\Gamma_n^*,\mathbf{H}_0^1(\mathscr{D}))} \\
&\le \frac{C_3}{2C_1\eta}\max_{y_n\in\Gamma_n}\{\sigma_n(y_n)[(1+C_2)\|\bar{f}\|_{C^0_{\sigma_n^*}(\Gamma_n^*,\mathbf{L}^2(\mathscr{D}))}+1]\}\sum_{k=0}^{\infty}(2|z-y_n|\gamma_n)^k \\
&\le \frac{C_3}{2C_1\eta}[(1+C_2)\|\bar{f}\|_{C^0_\sigma(\Gamma,\mathbf{L}^2(\mathscr{D}))}+1]\sum_{k=0}^{\infty}(2|z-y_n|\gamma_n)^k,
\end{aligned}$$

where we employ the fact that $\sigma_n(y_n) \le 1$ for all $y_n \in \Gamma_n$, and the series converges for all $z \in \mathbb{C}$ such that $|z-y_n| \le \tau_n < 1/(2\gamma_n)$. Moreover, in the ball $|z-y_n| \le \tau_n$, by (6.0.1), one has $\sigma_n(\mathrm{Re}\, z) \le e^{\beta_n\tau_n}\sigma_n(y_n)$, and then

$$\sigma_n(y_n)\|u(z)\|_{C^0_{\sigma_n^*}(\Gamma_n^*,\mathbf{H}_0^1(\mathscr{D}))} \le \frac{C_3 e^{\beta_n\tau_n}}{2C_1\eta(1-2\tau_n\gamma_n)}[(1+C_2)\|\bar{f}\|_{C^0_\sigma(\Gamma,\mathbf{L}^2(\mathscr{D})}+1].$$

Hence, the power series converges for every $y_n \in \Gamma_n$ as $\tau_n \to 0$. By a continuation argument, the function u can be extended analytically on the whole region $\sum(\Gamma_n;\tau_n)$ with estimation (6.2.1) □

Chapter 7

Convergence Analysis

In this chapter, based on the results in [3] for the linear stochastic poisson problem, we analyze the convergence of solution and as well as the first two moments for nonlinear stochastic diffusion-reaction elliptic equations and linearized Elastostatic problems with random media. Before showing any result for each cases, a few important lemmas are introduced for later use. In these lemmas, the space V is defined as same as it in (6.0.1) which is a general Banach space.

Lemma 7.1. *The operator $\mathcal{I}_p : C^0_\sigma(\Gamma; V) \to L^2_\rho(\Gamma; V)$ is continuous.*

Proof. This proof is based on the orthogonality of polynomials, and the details of proof are introduced in Lemma 4.2 of [3]. □

Lemma 7.2. *Given a function $v \in C^0(\Gamma; V)$ which admits an analytic extension in the region of the complex plane $\sum(\Gamma; \tau) = \{z \in \mathbb{C}, \operatorname{dist}(z, \Gamma) \leq \tau\}$ for some $\tau > 0$, it holds that*

$$\min_{w \in \mathcal{P}_p \otimes V} \|v - w\|_{C^0(\Gamma; V)} \leq \frac{2}{\varrho - 1} e^{-p \log \varrho} \max_{z \in \sum(\Gamma; \tau)} \|v(z)\|_V,$$

where,

$$1 < \varrho = \frac{2\tau}{|\Gamma|} + \sqrt{1 + \frac{4\tau^2}{|\Gamma|^2}}.$$

Lemma 7.3. *Let v be a function in $C^0_\sigma(\mathbb{R}; V)$. We suppose that v admits an analytic extension in the strip of the complex plane $\sum(\mathbb{R}; \tau) = \{z \in \mathbb{C}, \operatorname{dist}(z, \mathbb{R}) \leq \tau\}$ for some $\tau > 0$, and that*

$$\forall z = (y + iw) \in \sum(\mathbb{R}; \tau), \quad \sigma(y)\|v(z)\|_V \leq C_v(\tau).$$

Then, for any $\delta > 0$, there exist a constant C, independent of p, and a function $\Theta(p) = O(\sqrt{p})$ such that

$$\min_{w\in\mathcal{P}_p\otimes V}\max_{y\in\mathbb{R}}|\|v(y)-w(y)\|_V e^{-\frac{(\delta y)^2}{4}}| \le C\Theta(p)e^{-\tau\delta\sqrt{p}}.$$

The proof of these two lemmas can be found in Lemma 4.4 and 4.6 of [3] which is an immediate extension of result in [8]. Based on these lemmas and regularity results, the following results for different problems are able to be derived.

7.1 Linear Stochastic Poisson Problem (Babuska's work)

The theoretical error analysis of Stochastic Collocation approach is introduced and proved by Thm 4.1 in [3], which indicates the error goes to zero exponentially as the order of p_n increasing, where p_ns are the orders of polynomials in random space $\otimes_1^N\mathcal{P}_{p_n}(\Gamma_n) = \mathcal{P}_p(\Gamma)$. The result of error analysis with respect to Linear Stochastic Poisson problem is as following,

Theorem 7.4. Error Estimates of Collocation Approach. *Under the assumption 6.1, 6.3, 6.6 and assumptions in (3.2.2), there exist $r_n > 0$, $n = 1, 2, \ldots, N$, and constant C_1 and C_2 independent of h and p, such that,*

$$\|u-u_{h,p}\|_{V_\rho^2} \le C_1 \inf_{\forall v_h\in L_\rho^{\alpha+1}\otimes H_h}\|u-v_h\|_{V_\rho^{\alpha+1}} + C_2\sum_{n=1}^N\beta_n(p_n)\exp^{-r_n p_n^{\theta_n}}, \qquad (7.1.1)$$

where

- *if Γ_n is bounded,* $\begin{cases}\theta_n = \beta_n = 1\\ r_n = \log[\frac{2\tau_n}{|\Gamma_n|}(1+\sqrt{1+\frac{|\Gamma_n|^2}{4\tau_n^2}})],\end{cases}$

- *if Γ_n is unbounded,* $\begin{cases}\theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n})\\ r_n = \tau_n\delta_n,\end{cases}$

where τ_n is defined in Lemma 6.7 and δ_n is defined in Assumption 6.1.

Later on, following the same path of the result above with several further assumptions and regularity conditions, one can show numerical solutions of nonlinear problem (4.2.2) and (4.2.4) by Stochastic Collocation method go to zero exponentially too as the increasing of order p_n.

7.2 Nonlinear Stochastic Diffusion-Reaction Elliptic Equation

In this section, we prove the results of convergence for nonlinear stochastic diffusion-reaction elliptic equations. Since the monotone increasing is required for many proofs in this section, α is always assumed as odd integer. In addition, in order to take advantage of Sobolev Imbedding theorem A.2, we assume the value of α must satisfy $H_0^1(D) \hookrightarrow L^{\alpha+1}(D)$, in particular $3 \leq \alpha < \infty$ if $D \in \mathbb{R}^2$ or $3 \leq \alpha \leq 6$ if $D \in \mathbb{R}^3$.

Before showing the results of error estimates, we here review the notations of different solutions in collocation method. $u \in V_\rho^{\alpha+1}$ is the weak solution in (4.2.2), u_h is the semidiscrete projection of u onto subspace $L_\rho^{\alpha+1}(\Gamma) \otimes H_h(D)$ in (5.3.1), and $u_{h,p} \in \mathcal{P}_p(\Gamma) \otimes H_h(D)$ is the discrete solution obtained from numerical approach in (5.1.2). The error we are interested in is $\|u - u_{h,p}\|_{V_\rho^{\alpha+1}}$, however, in order to keep the exponential convergent rate, the proof need to take advantage of orthogonality of polynomial basis, a Hilbert space is required. Hence the error convergent rate discussed in this section is $\|u - u_{h,p}\|_{V_\rho^2} \leq C\|u - u_{h,p}\|_{V_\rho^{\alpha+1}}$ where C is the imbedding constant in Theorem 2.4.

Lemma 7.5. Interpolation Error. *Under the assumption 6.1, 6.3 and 6.6, there exist $r_n > 0$, $n = 1, 2, \ldots, N$, and constant C independent of h and p, such that the interpolation error*

$$\|u_h - u_{h,p}\|_{V_\rho^2} \leq C \sum_{n=1}^{N} \beta_n(p_n) \exp^{-r_n p_n^{\theta_n}}, \tag{7.2.1}$$

where

- *if* Γ_n *is bounded,* $\begin{cases} \theta_n = \beta_n = 1 \\ r_n = \log[\frac{2\tau_n}{|\Gamma_n|}(1+\sqrt{1+\frac{|\Gamma_n|^2}{4\tau_n^2}})], \end{cases}$

- *if* Γ_n *is unbounded,* $\begin{cases} \theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n}) \\ r_n = \tau_n \delta_n, \end{cases}$

where τ_n *is defined in Lemma 6.7 and* δ_n *is defined in Assumption 6.1.*

Proof. This lemma is a result of two regularity theorems 6.5 and 6.7, and the previous lemmas 7.1, 7.2 and 7.3. The tricks of proof are introduced in Thm 4.1 of [3] as following. This error analyzed here is the interpolation error. Recall that u_h has the same regularity as the exact solution u with respect to y, and $u_{h,p} = \mathcal{I}_p u_h$.

Firstly, since the zeros of auxiliary pdf are employed as collocation points, we need to pass the error from the space L^2_ρ to $L^2_{\hat{\rho}}$ by the inclusion

$$\|u_h - \mathcal{I}_p u_h\|_{L^2_\rho \otimes H^1_0} \;\leq\; \|\frac{\rho}{\hat{\rho}}\|_{L^\infty(\infty)} \|u_h - \mathcal{I}_p u_h\|_{L^2_{\hat{\rho}} \otimes H^1_0},$$

where $\hat{\rho}$ chosen satisfies (5.1.1).

In this prove, we introduce a notation that is $\bullet_n$ as a quantity relative to the direction y_n and $\bullet^*_n$ as the analogous quantity relative to all other directions y_j with $j \neq n$. We focus on the first direction y_1 and define an interpolation operator $\mathcal{I}_1$: $C^0_{\sigma_1}(\Gamma_1; L^2_{\hat{\rho}^*_1} \otimes H^1_0) \to L^2_{\hat{\rho}_1}(\Gamma_1; L^2_{\hat{\rho}^*_1} \otimes H^1_0)$ as

$$\mathcal{I}_{p_1} v(y_1, y^*_1, x) \;=\; \sum_{k=1}^{p_1+1} v(y^k_1, y^*_1, x) l_k(y_1).$$

Then, the global interpolation operator $\mathcal{I}_p$ can be written as a composition of two operators $\mathcal{I}_p = \mathcal{I}_1 \otimes \mathcal{I}^*_1$, where $\mathcal{I}^*_1$ is the interpolation operator in all directions $y_2, \ldots, y_{N_p}$ except y_1 as $\mathcal{I}^*_1$: $C^0_{\sigma^*_1}(\Gamma^*_1; L^2_{\hat{\rho}_1} \otimes H^1_0) \to L^2_{\hat{\rho}^*_1}(\Gamma^*_1; L^2_{\hat{\rho}_1} \otimes H^1_0)$. Now, we can split the error into two terms as

$$\|u_h - \mathcal{I}_p u_h\|_{L^2_{\hat{\rho}} \otimes H^1_0} \;\leq\; \underbrace{\|u_h - \mathcal{I}_1 u_h\|_{L^2_{\hat{\rho}} \otimes H^1_0}}_{I} + \underbrace{\|\mathcal{I}_1(u_h - \mathcal{I}^*_1 u_h)\|_{L^2_{\hat{\rho}} \otimes H^1_0}}_{II}.$$

To bound the term I, by Thm 6.2, assumption 6.1 (ii) and (6.0.1), the following inclusion holds:

$$C^0_{\sigma_1}(\Gamma_1; V) \;\subset\; C^0_{G_1}(\Gamma_1; V) \;\subset\; L^2_{\hat{\rho}_1}(\Gamma_1; V),$$

with a Hilbert space $V = L^2_{\hat{\rho}_1^*}(\Gamma_1^*) \otimes H^1_0(D)$, $\sigma_1 = G_1 = 1$ if Γ_1 is bounded and $\sigma_1 = e^{-\alpha_1|y_1|}$, $G_1 = e^{-\frac{(\delta_1 y_1)^2}{4}}$ if Γ_1 is unbounded. Here we are actually analyzing the error in a larger space with the choice of G_1 instand of σ_1. With the result of Lemma 7.1 and the inclusions above, the operator $\mathcal{I}_1$ is also continuous from $C^0_{G_1}(\Gamma_1; V)$ in $L^2_{\hat{\rho}_1}(\Gamma_1; V)$, then we can estimate

$$I \;=\; \|u_h - w + \mathcal{I}_1(w - u_h)\|_{L^2_{\hat{\rho}_1}(\Gamma_1;V)} \;\leq\; C \inf_{\forall w \in \mathcal{P}_{p_1} \otimes V} \|u_h - w\|_{C^0_{G_1}(\Gamma_1;V)},$$

by noticing $\mathcal{I}_1(w) = w$ for any $w \in \mathcal{P}_{p_1} \otimes V$. To bound the term of best approximation, we apply Lemma 7.2 for the case of Γ_1 is bounded, Lemma 7.3 for the case if Γ_1 is unbounded and the fact that $u \in C^0_{\sigma_1}(\Gamma_1; V)$. Putting everything together, we have

$$I \;\leq\; \begin{cases} Ce^{\tau_1 p_1}, & \Gamma_1 \text{ bounded} \\ C\beta(p_1)e^{-\tau_1\sqrt{p_1}}, & \Gamma_1 \text{ unbounded.} \end{cases}$$

To bound the term II, use Lemma 7.1 directly:

$$II \;\leq\; \hat{C}\|u_h - \mathcal{I}_1^* u\|_{C^0_{\sigma_1}(\Gamma_1;V)}.$$

Thus, the second bound becomes another interpolation error which could be bounded by the same approach with respect to rest $N-1$ directions. □

Next, we start to look at the error from finite element approach $\|u - u_h\|_{V^2}$. First, introduce a theorem for nonlinear finite element approach,

Theorem 7.6. *Under assumptions in Theorem 3.9, if the nonlinear operator* $b(u) \triangleq c\hat{b}(u) = cu^\alpha$ *is monotone, and* $\langle b(u), v\rangle$ *is Lipschitz in weak sense, i.e*

$$\langle b(u) - b(u_h), u - v_h\rangle \leq K\|u - u_h\|_{V_\rho^{\alpha+1}}\|u - v_h\|_{V_\rho^{\alpha+1}}, \quad \forall v_h \in L_\rho^{\alpha+1}(\Gamma) \otimes H_h(D). \tag{7.2.2}$$

Then, u_h *satisfied the following,*

$$\|u - u_h\|_{V_\rho^2} \;\leq\; C \inf_{\forall v_h \in L_\rho^{\alpha+1} \otimes H_h} \|u - v_h\|_{V_\rho^{\alpha+1}}. \tag{7.2.3}$$

Proof. This proof is slightly changed from the proof in Thm 10.2.13 of [22]. Let $X_h = L_\rho^{\alpha+1} \otimes H_h$ and $a(u, v)$ be the bilinear form, consider

$$a(u - u_h, w_h) + \langle b(u) - b(u_h), w_h\rangle \;=\; 0, \quad \forall w_h \in X_h,$$

where $w_h = v_h - u_h \in X_h$. This implies,

$$\begin{aligned} a_{min}\|u-u_h\|^2_{V^2_\rho} \;&\le\; a(u-u_h,u-u_h) \\ &\le\; a(u-u_h,u-v_h)+a(u-u_h,v_h-u_h) \\ &\le\; a(u-u_h,u-v_h)+\langle b(u_h)-b(u),v_h-u\rangle-\langle b(u_h)-b(u),u_h-u\rangle \\ &\le\; a(u-u_h,u-v_h)+\langle b(u_h)-b(u),v_h-u\rangle \\ &\le\; \hat{C}\|u-u_h\|_{V^{\alpha+1}\rho}\|u-v_h\|_{V^{\alpha+1}\rho}+K\|u-u_h\|_{V^{\alpha+1}_\rho}\|u-v_h\|_{V^{\alpha+1}_\rho}. \end{aligned}$$

Since this v_h is arbitrary in X_h, we have

$$\|u-u_h\|_{V^2_\rho} \;\le\; C \inf_{\forall v_h\in X_h} \|u-v_h\|_{V^{\alpha+1}_\rho},$$

where $C = \frac{\hat{C}+K}{a_{min}}$. □

Then, the following lemma is able to prove,

Lemma 7.7. Error of Galerkin approximation. *Under assumptions in Theorem 3.9, the Galerkin approximation has the following error estimates*

$$\|u-u_h\|_{V^2_\rho} \;\le\; C \inf_{\forall v_h\in L^{\alpha+1}_\rho\otimes H_h} \|u-v_h\|_{V^{\alpha+1}_\rho}. \tag{7.2.4}$$

Proof. In order to use Theorem 7.6, we only need to show the nonlinear operator $b(u) = c\hat{b}(u)$ is Lipschitz in weak sense. In this proof, $\|w\|_{L^\alpha(\Gamma\times D)}$ denotes as $\|w\|_\alpha$. Since

$$\langle b(u)-b(u_h),u-v_h\rangle \le \|b(u)-b(u_h)\|_{\frac{\alpha+1}{\alpha}}\|u-v_h\|_{\alpha+1}, \tag{7.2.5}$$

consider the Taylor expansion

$$b(u)-b(u_h) \;=\; c\hat{b}^{(1)}(u)(u-u_h)+\cdots+c\frac{1}{\alpha!}\hat{b}^{(\alpha)}(\xi)(u-u_h)^\alpha,$$

where $\hat{b}^{(\alpha)}(\xi) = \alpha!$, thus

$$\|b(u)-b(u_h)\|_{\frac{\alpha+1}{\alpha}} \;\le\; \sum_{i=1}^{\alpha}\|\frac{c}{i!}\hat{b}^{(i)}(u-u_h)^i\|_{\frac{\alpha+1}{\alpha}}.$$

Let $\sum_{i=1}^{\alpha} \|\frac{c}{i!}\hat{b}^{(i)}(u-u_h)^i\|_{\frac{\alpha+1}{\alpha}} = \sum_{i=1}^{\alpha} I_i$, one has when $i = 1, \ldots, \alpha - 1$,

$$\begin{aligned}
I_i &= \left(\int_\Gamma \rho \int_D |\frac{C}{i!} u^{\alpha-i}(u-u_h)^i|^{\frac{\alpha+1}{\alpha}}\right)^{\frac{\alpha}{\alpha+1}} \\
&\leq \|c\|_\infty \left(\int_\Gamma \rho \int_D u^{\frac{(\alpha-i)(\alpha+1)}{\alpha}} (u-u_h)^{\frac{(\alpha+1)i}{\alpha}}\right)^{\frac{\alpha}{\alpha+1}} \\
&\leq \|c\|_\infty \left(\int_\Gamma \rho \int_D u^{\alpha+1}\right)^{\frac{\alpha-i}{\alpha+1}} \left(\int_\Gamma \rho \int_D |u-u_h|^{\alpha+1}\right)^{\frac{i}{\alpha+1}} \\
&\leq \|c\|_\infty (\overline{C}_{f_y}+1)^\alpha \|u-u_h\|_{\alpha+1}^i \qquad \text{where } \overline{C}_{f_y} \text{ is the constant in Lemma 6.4} \\
&= \hat{C}\|u-u_h\|_{\alpha+1}^i;
\end{aligned}$$

When $i = \alpha$,

$$I_\alpha \leq \|c\|_\infty \|u-u_h\|_{\alpha+1}^\alpha.$$

Thus,

$$\sum_{i=1}^{\alpha} I_i \leq \alpha\hat{C}\left(\frac{1-\|u-u_h\|_{\alpha+1}^\alpha}{1-\|u-u_h\|_{\alpha+1}}\right)\|u-u_h\|_{\alpha+1}.$$

Since $\|u-u_h\|_{\alpha+1} \leq \overline{C}_{f_y}$, thus $\left(\frac{1-\|u-u_h\|_{\alpha+1}^\alpha}{1-\|u-u_h\|_{\alpha+1}}\right) \leq \widetilde{C}$, and let $K = \alpha\hat{C}\widetilde{C}$, the expected result is derived by substituting $\sum_{i=1}^{\alpha} I_i \leq K\|u-u_h\|_{\alpha+1}$ back to (7.2.5). □

Finally, the main result of this section is as following,

Theorem 7.8. Error Estimates of Collocation Approach. *Under the assumption 6.1, 6.3, 6.6 and assumptions in Theorem 3.9, there exist $r_n > 0$, $n = 1, 2, \ldots, N$, and constant C_1 and C_2 independent of h and p, such that,*

$$\|u-u_{h,p}\|_{V_\rho^2} \leq C_1 \inf_{\forall v_h \in L_\rho^{\alpha+1}\otimes H_h} \|u-v_h\|_{V_\rho^{\alpha+1}} + C_2 \sum_{n=1}^{N} \beta_n(p_n)\exp^{-r_n p_n^{\theta_n}}, \qquad (7.2.6)$$

where

- *if Γ_n is bounded,* $\begin{cases} \theta_n = \beta_n = 1 \\ r_n = \log[\frac{2\tau_n}{|\Gamma_n|}(1+\sqrt{1+\frac{|\Gamma_n|^2}{4\tau_n^2}})], \end{cases}$

- *if Γ_n is unbounded,* $\begin{cases} \theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n}) \\ \tau_n = \tau_n\delta_n, \end{cases}$

where τ_n is defined in Lemma 6.7 and δ_n is defined in Assumption 6.1.

Proof. Since $\|u - u_{h,p}\| \leq \|u - u_h\| + \|u_h - u_{h,p}\|$, and this theorem is direct result of Lemma 7.5 and 7.7. □

Remark 7.9. *This result shows the error converges exponentially as the the increasing of amount of collocation points. Or in another word, the interpolation error with respect to the direction y_n goes to zero exponentially as the dimension p_n of polynomial space $\mathcal{P}_{p_n}$ increasing.*

Remark 7.10. *This estimator is analyzed in a larger space $V_\rho^2 \supset V_\rho^{\alpha+1}$, it is enough to analyze the error with respect to mean and variance, however the error estimates of the exact solution space need further analysis.*

7.2.1 Convergence of Moments

According to the Theorem 7.8, the error estimates of first two moments are straightforward.

Theorem 7.11. Approximation of 1st Moment.

$$\|E[u - u_{h,p}]\|_{L^1(D)} \leq C\|u - u_{h,p}\|_{V_\rho^2}. \tag{7.2.7}$$

Proof. Simply apply Holder's inequality, we have

$$\|E[u - u_{h,p}]\|_{L^1(D)} \leq C_1\|u - u_{h,p}\|_{L^2(\Gamma\times D)} \leq C_2\|u - u_{h,p}\|_{V_\rho^2}.$$

□

Theorem 7.12. Approximation of 2nd Moment.

$$\|E[u^2 - u_{h,p}^2]\|_{L^1(D)} \leq C_1\|u - u_{h,p}\|_{V_\rho^2} + C_2\|u - u_{h,p}\|_{V_\rho^2}^2. \tag{7.2.8}$$

Proof. Consider,

$$\begin{aligned}
\|E[u^2 - u_{h,p}^2]\|_{L^1(D)} &= \int_D \int_\Gamma \rho(u - u_{h,p})(u + u_{h,p}) \\
&\leq \|u - u_{h,p}\|_{L^2(\Gamma\times D)}\|u + u_{h,p}\|_{L^2(\Gamma\times D)} \\
&\leq \|u - u_{h,p}\|_{L^2(\Gamma\times D)}(2\|u\|_{L^2(\Gamma\times D)} + \|u - u_{h,p}\|_{L^2(\Gamma\times D)}) \\
&\leq 2C_1(C_{f_y})\|u - u_{h,p}\|_{V_\rho^2} + C_2\|u - u_{h,p}\|_{V_\rho^2}^2.
\end{aligned}$$

□

Remark 7.13. *To analyze convergent property of higher order moments, more regularity assumptions are required.*

7.3 Linearized Elastostatic Problem with Random Media

In this section, we prove results of convergence for linearized Elastostatic problems with random media. Before showing the results of error estimates, we here review the notations of different solutions in collocation method. $u \in \mathbf{V}_\rho^2$ is the weak solution in (4.2.4), u_h is the semidiscrete projection of u onto subspace $\mathbf{L}_\rho^2(\Gamma) \otimes \mathbf{H}_h(\mathscr{D})$ in (5.4.1), and $\mathbf{u}_{h,p} \in \mathbf{P}_p(\Gamma) \otimes \mathbf{H}_h(\mathscr{D})$ is the discrete solution obtained from numerical approach in (5.1.2). The error we are interested in is $\|u - u_{h,p}\|_{\mathbf{V}_\rho^2}$.

Lemma 7.14. Interpolation Error. *Under the assumption 6.1, 6.8 and 6.11, there exist $r_n > 0$, $n = 1, 2, \ldots, N$, and constant C independent of h and p, such that the interpolation error*

$$\|u_h - u_{h,p}\|_{\mathbf{V}_\rho^2} \leq C \sum_{n=1}^{N} \beta_n(p_n) \exp^{-r_n p_n^{\theta_n}}, \tag{7.3.1}$$

where

- *if Γ_n is bounded,* $\begin{cases} \theta_n = \beta_n = 1 \\ r_n = \log[\frac{2\tau_n}{|\Gamma_n|}(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2}})], \end{cases}$

- *if Γ_n is unbounded,* $\begin{cases} \theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n}) \\ r_n = \tau_n \delta_n, \end{cases}$

where τ_n is defined in Lemma 6.12 and δ_n is defined in Assumption 6.1.

Proof. This lemma is a result of two regularity theorems 6.10 and 6.12, and the previous lemmas 7.1, 7.2 and 7.3. The trick of proof is introduced in Lemma 7.5. □

Next, we start to look at the error from finite element approach $\|u - u_h\|_{\mathbf{V}^2}$. First, introduce the generalized Cea's Lemma,

Theorem 7.15. Generalized Cea's Lemma *Let X be a Banach space, let $X_h \subset X$, let $u \in X$ be the solution to a bilinear form satisfied three conditions listed in Thm 3.19, and let $u_h \in X_h$ be the Glaerkin approximation satisfying*

$$a(u_h, v_h) = f(v_h), \quad \forall v_h \in X_h \subset X.$$

Then,

$$\|u - u_h\|_X \;\leq\; C \inf_{\forall v_h \in X_h} \|u - v_h\|_X.$$

The following lemma is straightforward based on Cea's lemma.

Lemma 7.16. Error of Galerkin approximation. *Under assumptions in Theorem 3.21, the Galerkin approximation has the following error estimates*

$$\|u - u_h\|_{\mathbf{V}^2_\rho} \;\leq\; C \inf_{\forall v_h \in \mathbf{L}^2_\rho \otimes \mathbf{H}_h} \|u - v_h\|_{\mathbf{V}^2_\rho}. \tag{7.3.2}$$

Proof. This theorem is a direct result by Thm 7.15. □

Finally, the main result of this section is as following,

Theorem 7.17. Error Estimates of Collocation Approach. *Under the assumption 6.1, 6.8, 6.11 and assumptions in Theorem 3.21, there exist $r_n > 0$, $n = 1, 2, \ldots, N$, and constant C_1 and C_2 independent of h and p, such that,*

$$\|u - u_{h,p}\|_{\mathbf{V}^2_\rho} \;\leq\; C_1 \inf_{\forall v_h \in \mathbf{L}^2_\rho \otimes \mathbf{H}_h} \|u - v_h\|_{\mathbf{V}^2_\rho} + C_2 \sum_{n=1}^{N} \beta_n(p_n) \exp^{-r_n p_n^{\theta_n}}, \tag{7.3.3}$$

where

- *if Γ_n is bounded,* $\begin{cases} \theta_n = \beta_n = 1 \\ r_n = \log[\frac{2\tau_n}{|\Gamma_n|}(1 + \sqrt{1 + \frac{|\Gamma_n|^2}{4\tau_n^2}})], \end{cases}$

- *if Γ_n is unbounded,* $\begin{cases} \theta_n = \frac{1}{2}, \quad \beta_n = O(\sqrt{p_n}) \\ \tau_n = \tau_n \delta_n, \end{cases}$

where τ_n is defined in Lemma 6.7 and δ_n is defined in Assumption 6.1.

Proof. Since $\|u - u_{h,p}\| \leq \|u - u_h\| + \|u_h - u_{h,p}\|$, and this theorem is direct result of Lemma 7.14 and 7.16. □

Remark 7.18. *This result shows the error converges exponentially as the the increasing of amount of collocation points. Or in another word, the interpolation error with respect to the direction y_n goes to zero exponentially as the dimension p_n of polynomial space $\mathcal{P}_{p_n}$ increasing.*

7.3.1 Convergence of Moments

According to the Theorem 7.17, the error estimates of first two moments are straightforward.

Theorem 7.19. Approximation of 1st Moment.

$$\|E[u - u_{h,p}]\|_{\mathbf{L}^1(\mathscr{D})} \leq C\|u - u_{h,p}\|_{\mathbf{V}^2_\rho}. \tag{7.3.4}$$

Proof. Simply apply Holder's inequality, we have

$$\|E[u - u_{h,p}]\|_{\mathbf{L}^1(\mathscr{D})} \leq C_1\|u - u_{h,p}\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})} \leq C_2\|u - u_{h,p}\|_{\mathbf{V}^2_\rho}.$$

□

Theorem 7.20. Approximation of 2nd Moment.

$$\|E[u^2 - u^2_{h,p}]\|_{\mathbf{L}^1(\mathscr{D})} \leq C_1\|u - u_{h,p}\|_{\mathbf{V}^2_\rho} + C_2\|u - u_{h,p}\|^2_{\mathbf{V}^2_\rho}. \tag{7.3.5}$$

Proof. Consider,

$$\begin{aligned}
\|E[u^2 - u^2_{h,p}]\|_{\mathbf{L}^1(D)} &= \sum \int_{\mathscr{D}} \int_\Gamma |\rho(u - u_{h,p})(u + u_{h,p})| \\
&\leq \|u - u_{h,p}\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})}\|u + u_{h,p}\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})} \\
&\leq \|u - u_{h,p}\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})}(2\|u\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})} + \|u - u_{h,p}\|_{\mathbf{L}^2(\Gamma\times\mathscr{D})}) \\
&\leq 2C_1(C_{\bar{f}_y})\|u - u_{h,p}\|_{\mathbf{V}^2_\rho} + C_2\|u - u_{h,p}\|^2_{\mathbf{V}^2_\rho}.
\end{aligned}$$

□

Remark 7.21. *To analyze convergent property of higher order moments, more regularity assumptions are required.*

Chapter 8

Numerical Results

8.1 Error indicators

Based on the error indicators employed in [3], we generalize them as an muti-dimension extrapolation type error indicator, and further apply it for all the targeting problems. In the following numerical results, the computational results are in accordance with the convergence rate shown in the chapter 7. To study the computational error, we estimate it along each direction i corresponding to a multi-index $p = (p_1, \dots, p_i, \dots, p_N)$ by calculating the L_1 norm of the mean of computational error $\|E[e]\|_{L^1} = \|E[u_{h,p} - u_{h,\overline{p}}]\| \triangleq EE_i$ where $\overline{p} = (p_1, \dots, p_i + 1, \dots, p_N)$.

Here we use an extrapolation type estimator with the help of *a priori* error estimates in Thm 7.4, Lemma 7.5 and 7.14. Assume the interpolation error $\|E[u_h - E_{h,p}]\| = \|E[u_h - \mathcal{I}_p u_h]\| \approx \sum_n C_n e^{-p_n}$, i.e. $\|u_h - \mathcal{I}_{p_n} u_h\| \approx C_n e^{-p_n}$ for each direction y_n with $n = 1, \dots, N$. Then, we have

$$\begin{aligned} EE_i = \|E[u_{h,p} - u_{h,\overline{p}}]\| &\geq \|E[u_{h,p} - u_h]\| - \|E[u_h - u_{h,\overline{p}}]\| \\ &\approx \sum_{n=1}^{N} C_n e^{-p_n} - \sum_{n=1,n\neq i}^{N} C_n e^{-p_n} - C_i e^{-p_i} \\ &\approx C_i e^{-p_i}(1 - e^{-1}) \\ &\approx (1 - e^{-1})\|E[u_h - \mathcal{I}_{p_i} u_h]\|. \end{aligned} \tag{8.1.1}$$

Thus, if the computational error $\|E[u_{h,p} - u_{h,\overline{p}}]\|$ goes to zero exponentially as expected, it implies the interpolation error along direction y_i is decreasing exponentially.

Similarly, the second order moment is approximated by $\|E[u_{h,p}^2 - u_{h,\overline{p}}^2]\|_{L_1}$. In order to verify the exponential convergence rate with respect to p_i, we stay in the same semidiscrete space H_h while increasing p_i only. The L_1 norms are approximated by applying quadrature for the inner integral and trapezoidal method for the outer integral,

$$\begin{aligned} EE_i \;=\; \|E[u_{h,p} - u_{h,\overline{p}}]\|_{L^1} \;&=\; \int_D |\int_\Gamma (u - u_{h,p})| \\ &\approx\; \sum_i a_i |\sum_j w_j u_{h,p}(y_j, x) - \sum_k \overline{w}_k u_{h,\overline{p}}(y_k, x)|, \end{aligned} \tag{8.1.2}$$

with the quadrature weights w_j, $\overline{w}_k$ and weights of trapezoidal method a_i.

As desired, the error decreases exponentially as the polynomial order p_i increasing for both mean and second order moment.

8.2 Adaptive Algorithm

Recall the upper bound of the error and consider the result obtained from (8.1.1),

$$\begin{aligned} \|E[u - u_{h,p]}\| \;&\leq\; \|E[u - u_h]\| + \|E[u_h - u_{h,p}]\| \\ &\leq\; \|E[u - u_h]\| + \sum_{n=1}^{N} \|E[u_h - \mathcal{I}_{p_n} u_h]\| \\ &\leq\; C_1 \inf_{\forall v_h \in L_\rho^{\alpha+1} \otimes H_h} \|u - v_h\|_{V_\rho^{\alpha+1}} + \sum_{n=1}^{N} \frac{EE_n}{(1 - e^{-1})}, \end{aligned}$$

the first term of error is determined by the semidiscrete space H_h only and the second term is a sum of errors from each direction $n = 1, \ldots, N$. Thus, in real computing, by incresing the degree of one direction i only, the convergence rate will be dominated by the error induced by the accuracy of deterministic solver and other direction along y_j, with $j \neq i$. To avoid this, an adaptive algorithm with anisotropic strategy is described briefly as following:

1. Set up a semidiscrete space H_h.

2. Increase the order of polynomial along ith direction as much as possible.

3. Increase the other directions $j \neq i$ as much as possible one by one.

4. Refine the semidistrete space and continue step 2 and 3.

The deterministic solver employed in this section is MCLite developed by Prof.Michael Holst introduced in [12, 13], and is overwritten by Mr.Shi(Fox) Cheng for stochastic collocation approach. MCLite is a FEM deterministic solver employed piecewise linear element and Newton's method for nonlinearity, and is extended to be a stochastic collocation solver. The extended routines to generate stochastic collocation points are introduced in chapter A.5.

8.3 Numerical Examples

In the following numerical examples, three types of problems setting are tested, and for each setting, Gaussian probability density $\Gamma = [-\infty, \infty]^N$ with Hermite zeros and uniform probability density $\Gamma = [-1, 1]^N$ with Legendre zeros are applied. For more types of density function ρ, please refer to works [25, 27].

Setting 1. Consider a stochastic problem on domain $D = [0, 1] \times [0, 1]$ and $(Y_1, Y_2) \in \Gamma$ is 2 dimensional probability space with collocation points as tensor product of Legendre or Hermite polynomial zeros. The coefficient of diffusion is

$$a = 1 + \exp((Y_1 + Y_2)\exp(-1/8)).$$

And exact solution is

$$u = \sin(\pi x)\sin(\pi y)(\exp(Y_1) - \exp(1)/2 + \exp(-1)/2)(\cos(Y_2) - \sin(1)),$$

with zero boundary condition. The corresponding RHS f can be calculated by PDE itself then.

Setting 2. Consider a stochastic problem on domain $D = [0, 1] \times [0, 1]$ and $(Y_1, Y_2) \in \Gamma$ is 2 dimensional probability space with collocation points as tensor product of Legendre or Hermite zeros. The coefficient of diffusion is

$$a = 1 + \exp((Y_1 + Y_2)\exp(-1/8)).$$

And exact solution is

$$u = \frac{1}{8}\sin(\pi x + Y1)\sin(\pi y + Y2)(-2Y_1 + \frac{1}{4})(-2Y_2 + \frac{1}{3})\exp(Y_1/4 + Y_2/3),$$

with nontrival boundary condition. The corresponding RHS f can be calculated by PDE itself then.

Setting 3. Consider a stochastic linearized Elasticity problem on a piece of 2D ground surface section $\mathscr{D} = [0, 10] \times [0, 1]$ formed by material with random properties described by $(Y_1, Y_2) \in \Gamma$ which is a 2 dimensional probability space with collocation points as tensor product of Legendre or Hermite zeros. The index four random tensor field A_{ijpq} is isotropic in the constitutive law with the property shown in Assumption 3.16. The exact solution is not explicit.

8.3.1 Linear Stochastic Poisson Problem.

Consider the following linear problem, find $u(y, x) : D \times \Gamma \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla(a(y,x)\cdot\nabla u(y,x)) &= f(y,x) \quad \text{on } D\times\Gamma, \\ u(y,x) &= \partial u \quad \text{on } \partial D. \end{aligned} \tag{8.3.1}$$

Denote the order of polynomials used in Γ as $[p_1, p_2]$. In order to observe the exponential decaying proved in Thm 7.4, only one of p_1 or p_2 will increase. The numerical results based on Setting 1 and Setting 2 are shown as following,

Table 8.1: Linear Problem with **Setting 1** and Legendre zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\bar{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\bar{p}}]\|_{L_1}$
$[2, 2]$	$2.11251707e - 05$	$7.04736282e - 07$
$[3, 2]$	$1.80162053e - 07$	$2.64285614e - 08$
$[4, 2]$	$8.13689743e - 10$	$4.84046708e - 10$
$[5, 2]$	$2.27600348e - 12$	$5.40067729e - 12$

Table 8.2: Linear Problem with **Setting 1** and Hermite zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$2.20656612e - 03$	$1.89202414e - 03$
$[3, 2]$	$1.11852927e - 04$	$4.41652912e - 04$
$[4, 2]$	$4.02916383e - 06$	$6.68041141e - 05$
$[5, 2]$	$1.12606911e - 07$	$7.64250708e - 06$

Table 8.3: Linear Problem with **Setting 2** and Legendre zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$4.16896969e - 03$	$7.02491798e - 03$
$[2, 3]$	$5.94854479e - 05$	$7.54632251e - 04$
$[2, 4]$	$4.11014569e - 07$	$3.03637355e - 05$
$[2, 5]$	$1.62821741e - 09$	$6.40541102e - 07$

Table 8.4: Linear Problem with **Setting 2** and Hermite zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$2.55732820e - 02$	$3.27609015e - 02$
$[2, 3]$	$2.08748127e - 03$	$2.01452560e - 02$
$[2, 4]$	$1.12479700e - 04$	$6.20045682e - 03$
$[2, 5]$	$4.38754653e - 06$	$1.28904335e - 03$

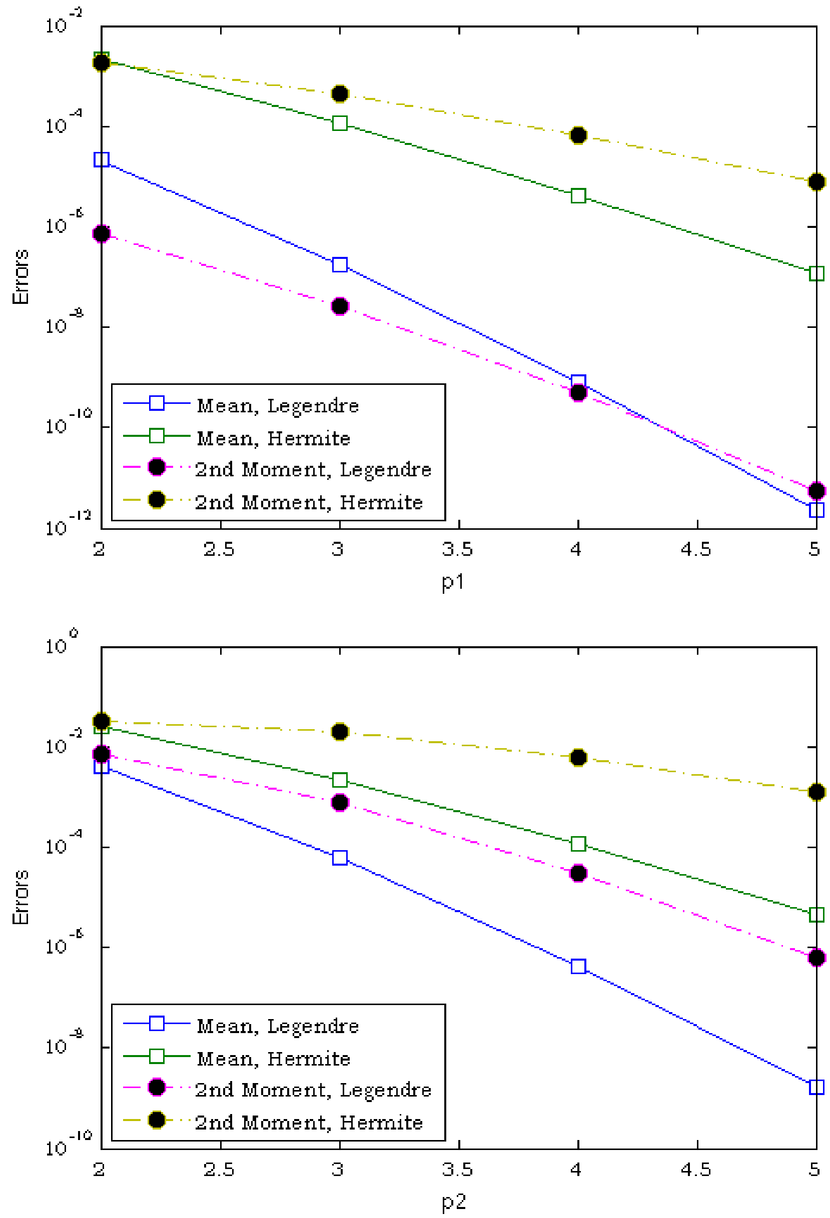

Figure 8.1: Error convergence with respect to linear problem. Left: **Setting 1**. Right: **Setting 2**.

Obviously, for above linear problem, the mean and 2nd order moment goes to zero exponentially as p_1 or p_2 increasing, which accords with the result from [3].

8.3.2 Nonlinear Stochastic Problem with Nonlinearity $\mathbf{u^3}$.

Consider the following nonlinear problem, find $u(y,x) : D \times \Gamma \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla(a(y,x)\cdot\nabla u(y,x)) + \lambda u(y,x)^3 &= f(y,x) \quad \text{on } D\times\Gamma, \\ u(y,x) &= \partial u \quad \text{on } \partial D. \end{aligned} \tag{8.3.2}$$

Where $\lambda = \exp(Y_1 + Y_2)$. Numerical results are as following

Table 8.5: Nonlinear Problem with nonlinearity u^3, **Setting 1** and Legendre zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2,2]$	$2.14866983e-05$	$7.16250061e-03$
$[2,3]$	$1.75710454e-07$	$3.07709305e-04$
$[2,4]$	$3.91734984e-09$	$5.93994847e-06$
$[2,5]$	$2.39519408e-08$	$6.76712198e-08$

Table 8.6: Nonlinear Problem with nonlinearity u^3, **Setting 1** and Hermite zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2,2]$	$2.02914348e-03$	$4.23150905e-02$
$[2,3]$	$1.00737768e-04$	$1.00782435e-02$
$[2,4]$	$3.99244913e-06$	$1.46724560e-03$
$[2,5]$	$2.65266585e-07$	$1.61882410e-04$

Table 8.7: Nonlinear Problem with u^3, **Setting 2** and Legendre zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2,2]$	$3.89331194e-03$	$6.61852538e-03$
$[3,2]$	$5.32283183e-05$	$6.84436527e-04$
$[4,2]$	$3.44940534e-07$	$2.57631839e-05$
$[5,2]$	$1.47536345e-08$	$5.16144546e-07$

Table 8.8: Nonlinear Problem with u^3, **Setting 2** and Hermite zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2,2]$	$2.37577431e-02$	$2.98363043e-02$
$[3,2]$	$1.86305410e-03$	$1.78092770e-02$
$[4,2]$	$9.32958060e-05$	$5.24491294e-03$
$[5,2]$	$3.49452921e-06$	$1.01535255e-03$

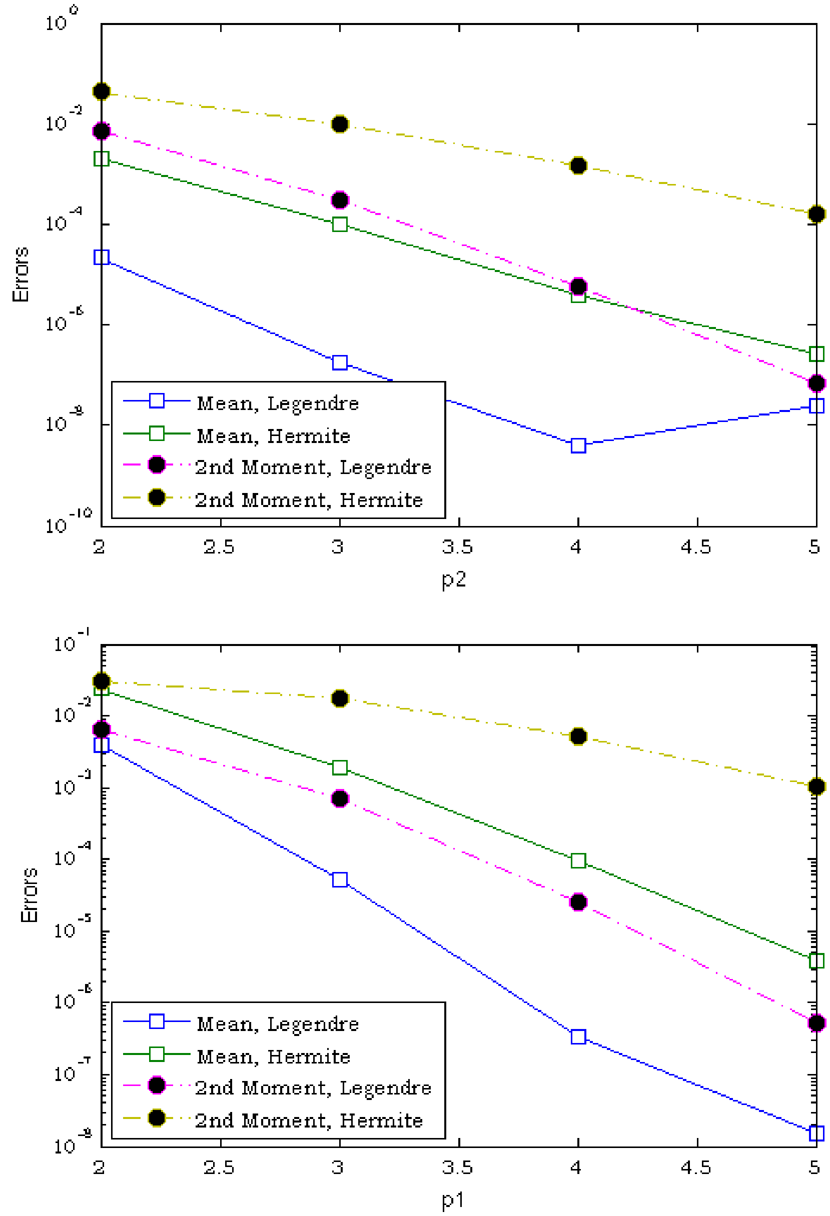

Figure 8.2: Error convergence with respect to problem with u^3. Left: **Setting 1**. Right: **Setting 2**.

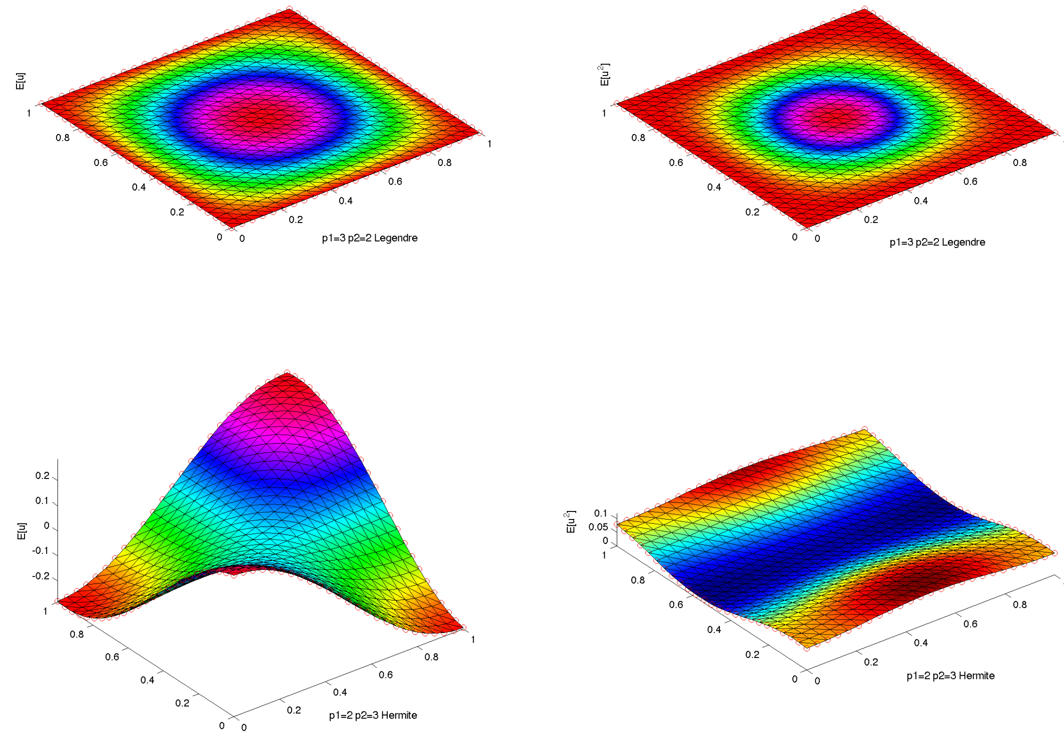

Figure 8.3: Numerical solutions with respect to problem with u^3. Top: **Setting 1**. Bottom: **Setting 2**.

Obviously, for above problem with nonlinearity, the mean and 2nd order moment goes to zero exponentially as p_1 or p_2 increasing, which accords with the result of Thm 7.8.

8.3.3 Nonlinear Stochastic Problem with Nonlinearity $\mathbf{e^u}$.

Consider the following nonlinear problem, find $u(y, x) : D \times \Gamma \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla(a(y,x) \cdot \nabla u(y,x)) + \lambda e^u &= f(y,x) \quad \text{on } D \times \Gamma, \\ u(y,x) &= \partial u \quad \text{on } \partial D. \end{aligned} \tag{8.3.3}$$

Where $\lambda = \exp(Y_1 + Y_2)$. Numerical results are as following

Table 8.9: Nonlinear Problem with e^u, **Setting 1** and Legendre zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$2.11487378e-05$	$7.04829626e-07$
$[3, 2]$	$1.75980245e-07$	$2.64196560e-08$
$[4, 2]$	$1.41498782e-08$	$5.79322270e-10$
$[5, 2]$	$9.93568951e-09$	$4.07054200e-11$

Table 8.10: Nonlinear Problem with e^u, **Setting 1** and Hermite zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$2.20668482e-03$	$1.89227538e-03$
$[3, 2]$	$1.11833481e-04$	$4.41696139e-04$
$[4, 2]$	$4.02761297e-06$	$6.68022392e-05$
$[5, 2]$	$1.12269765e-07$	$7.64084823e-06$

Table 8.11: Nonlinear Problem with e^u, **Setting 2** and Legendre zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$4.16902571e-03$	$7.02515778e-03$
$[2, 3]$	$5.94867536e-05$	$7.54667475e-04$
$[2, 4]$	$4.11142904e-07$	$3.03647307e-05$
$[2, 5]$	$1.62546605e-09$	$6.40561495e-07$

Table 8.12: Nonlinear Problem with e^u, **Setting 2** and Hermite zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$2.55735984e-02$	$3.27619389e-02$
$[2, 3]$	$2.08745399e-03$	$2.01463602e-02$
$[2, 4]$	$1.12500315e-04$	$6.20065297e-03$
$[2, 5]$	$4.39050137e-06$	$1.28905295e-03$

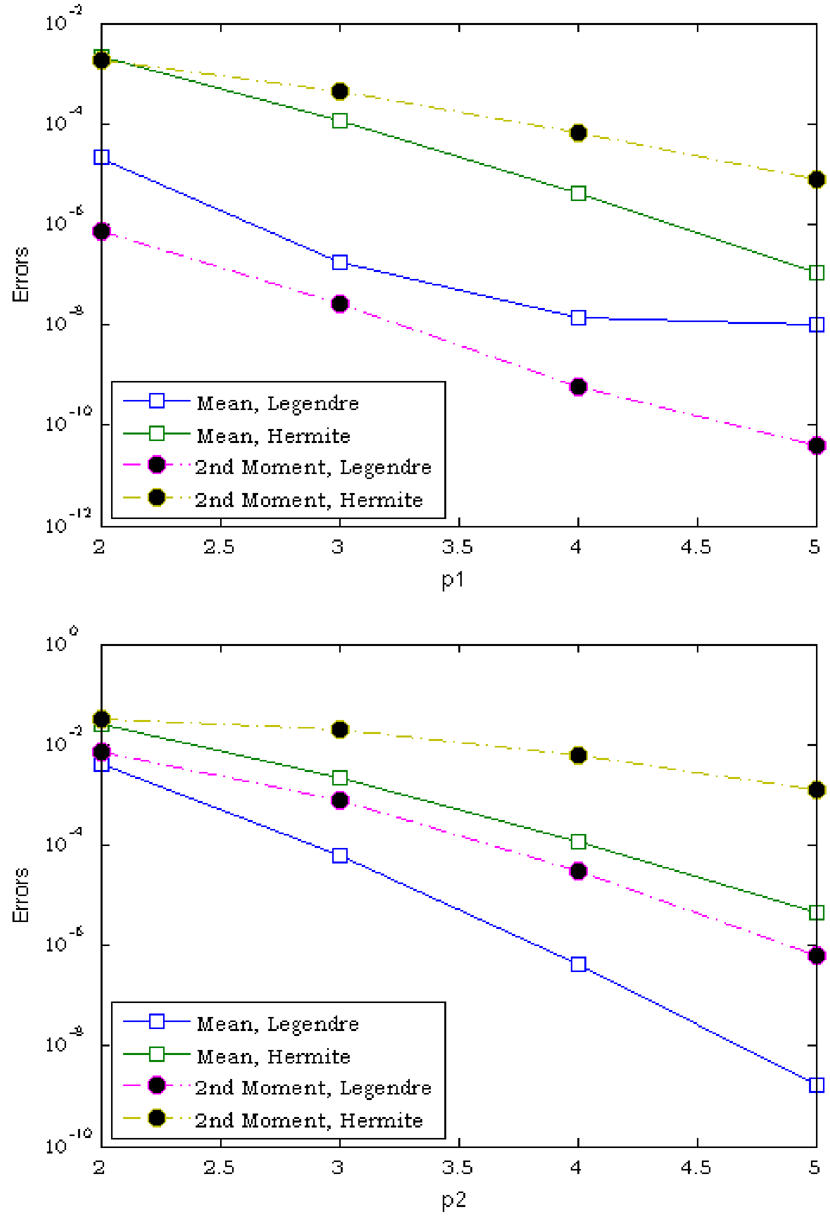

Figure 8.4: Error convergence with respect to problem with e^u. Left: **Setting 1**. Right: **Setting 2**.

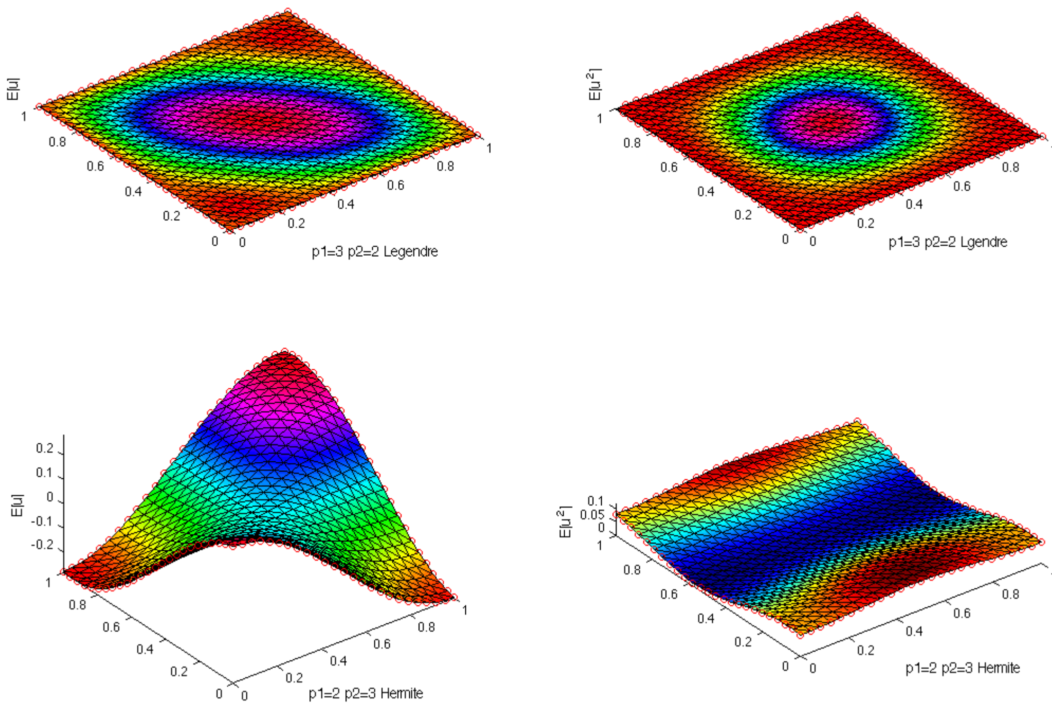

Figure 8.5: Numerical solutions with respect to problem with e^u. Top: **Setting 1**. Bottom: **Setting 2**.

Although, the above problem with nonlinearity e^u is not included in Thm 7.8, its mean and 2nd order moment goes to zero exponentially as fast as the problem with u^3.

8.3.4 Linearized Stochastic Elasticity in random media.

Consider a linearized Elasticity problem based on **Setting 3** similar as the test problem in [28], find $\mathbf{u}(y,x) : \mathscr{D} \times \Gamma \to \mathbb{R}^2$ such that

$$\begin{aligned} -\partial_j \sigma_{ij}(y,x)) &= f_i(x) \quad \text{on } \mathscr{D} \times \Gamma, \\ \sigma_{ij}(y,x) &= A_{ijpq}(y,x) e_{pq}(\mathbf{u}(y,x)) \quad \text{on } \mathscr{D} \times \Gamma \quad p,q = 1,2 \\ n_j \sigma_{ij}(y,x) &= g_i(y,x) \quad \text{on } \partial_{\mathcal{N}} \mathscr{D} \times \Gamma, \\ u_i(y,x) &= 0 \quad \text{on } \partial_{\mathcal{D}} \mathscr{D}, \end{aligned} \tag{8.3.4}$$

with isotropic material in the constitutive law. It indicates that Poisson's ratio ν and Young's modulus E are the only two variables required to fully define the index four random tensor field A shown in Assumption 3.16. The gravity is considered as the body force f, an stochastic fourier series as vertical boundary force from the bottom of the ground section and a constant horizontal boundary force is imposed by Neumann boundary condition g. For simplicity, the Poisson's ratio ν is assumed to be a constant 0.28, and the Young's modulus is described by a first order truncated KL expansion

$$E(y,x) \;=\; 21 + \sqrt{\lambda_1}\xi(y)\frac{\cos(\gamma_1 x_1)\cos(\gamma_1 x_2)}{1+\sin(2\gamma_1)/2\gamma_1},$$

where $\lambda_1 = (\frac{2b}{b^2\gamma_1^2+1})^2$ with $b = 16$, γ_1 is the first positive root of equation $\gamma\tan(\gamma) - 1/b = 0$, $\xi(y) = Y_1 Y_2$ is the product of first order Legendre and Hermite polynomial. Numerical results are as following,

Table 8.13: Linearized Elasticity with **Setting 3** and Legendre zeros, errors with increasing p_1.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$6.08150935e-03$	$4.61749358e-02$
$[3, 2]$	$5.21193888e-05$	$1.56682957e-03$
$[4, 2]$	$2.35511221e-08$	$2.85760595e-05$
$[5, 2]$	$6.56227638e-10$	$3.21547894e-07$

Table 8.14: Linearized Elasticity with **Setting 3** and Hermite zeros, errors with increasing p_2.

$[p_1, p_2]$	$\|E[u_{h,p} - u_{u,\overline{p}}]\|_{L_1}$	$\|E[u^2_{h,p} - u^2_{h,\overline{p}}]\|_{L_1}$
$[2, 2]$	$1.00755241e-02$	$4.82492099e-02$
$[2, 3]$	$4.93341702e-04$	$8.73378120e-03$
$[2, 4]$	$1.72877005e-05$	$1.19259559e-03$
$[2, 5]$	$4.73011838e-07$	$1.28182196e-04$

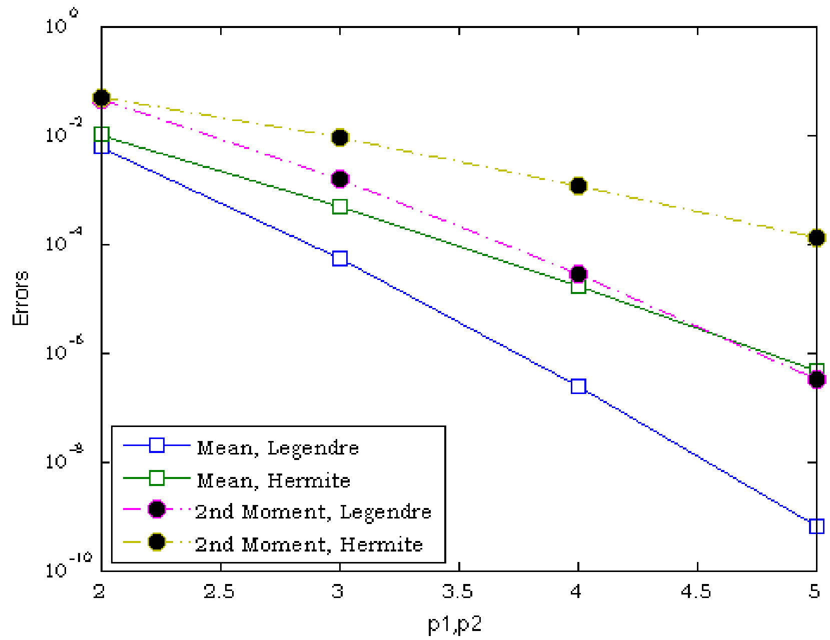

Figure 8.6: Error convergence with respect to problem of linearized Elasticity.

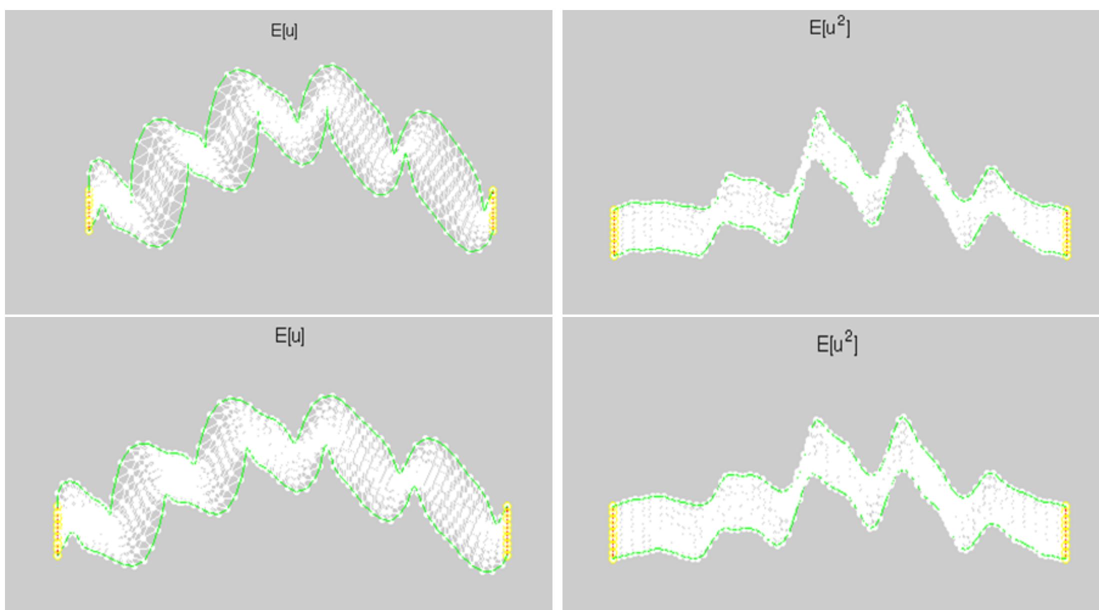

Figure 8.7: Numerical solutions with respect to linearized Elasticity problem. Top: Legendre zeros. Bottom: Hermite zeros.

Clearly, for above problem of linearized Elasticity, the mean and 2nd order moment goes to zero exponentially as p_1 or p_2 increasing, which accords with the result of Thm 7.17.

Remark 8.1. *Although, the model of linearized Elasticity is only suitable for problems with very small forces, we still apply large boundary forces in our testing problems in order to observe the behavior of numerical solutions by the front figures. For the real model with large boundary forces, the nonlinear Elasticity model should be applied.*

Chapter 9

Conclusion

In this dissertation, the wellpossedness of the nonlinear stochastic diffusion reaction problem and the general linearized elastostatic problem in random media are analyzed in a newly solution space V^p under certain assumptions.

The a priori error estimates for solving the nonlinear stochastic diffusion reaction problem and general linearized elastostatic problem in random media by stochastic collocation approach is derived. The error goes to zero exponentially as the order p_n increasing in space of polynomial chaos $\otimes_1^N \mathcal{P}_{p_n}(\Gamma_n)$, which keeps the same convergent rate as stochastic linear possion problem. Hence, we successfully generalize stochastic collocation approach to a much wider region of elliptic problems. As desired, the theoretical result is admitted by the numerical experiments verified by a posterior error estimator.

For real computing, in order to solve the stochastic equations by collocation method one must actually solve N_p decouple deterministic equations, and this can fit into a parallel strategy naturally. The demand of more efficient solver to solve stochastic problems is realistic because a number of deterministic equations need to be solved or a number of extra unknowns are introduced into the original deterministic problem. Therefore, a parallel solver designed with the adaptive strategy described in this work is a significant task to implement.

To generalize the analysis and numerical treatments to wider types of elliptic problems or even time depending equations is still ongoing research. Any numerical approach for solving stochastic problems with better performance than Monte Carlo simulation is needed, since for most practical problems, the only trusted numerical approach is still Monte Carlo simulation which is slow and computationally expensive.

Appendix A

A.1 Imbedding Theorems of L^p and Sobolev Space

Theorem A.1. L^P Imbedding. *If $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \supset L^q(\mu)$ and $\|f\|_p \leq C\|f\|_q$, where $C = \mu(X)^{1/p-1/q}$.*

Theorem A.2. Sobolev Imbedding. *Let $\Omega = \mathbb{R}^n_+$ or an open set of class C^1 with bounded boundary $\partial\Omega$. Then we have the continuous inclusions*

- *if $1 \leq p < n$, then $W^{1,p}(\Omega) \hookrightarrow L^{p\star}(\Omega)$, where $p\star = \frac{np}{n-p}$,*
- *if $p = n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [n, \infty)$*
- *if $p > n$, then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$*

and further, in the latter case, u is Holder continuous of exponent $\alpha = 1 - n/p$. In particular, $W^{1,p}(\Omega) \subset C(\overline{\Omega})$, $p > n$.

Theorem A.3. Rellich-Kondrasov. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, then the above continuous inclusions in Theorem A.2 are compact.*

A.2 Useful Inequalities

Theorem A.4. Cauchy-Schwarz Inequality. *Here introduce the compact written of Cauchy-Schwarz Inequality. For vectors $x, y \in \mathbb{R}^d$, one has*

$$|\sum_{i=1}^{d} x_i \bar{y}_i|^2 \leq \sum_{j=1}^{d} |x_j|^2 \sum_{k=1}^{d} |y_k|^2. \tag{A.2.1}$$

Theorem A.5. Holder's Inequality. *Suppose* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$*. If* f *and* g *are measurable functions on* X*, then*

$$\|fg\|_{L^1} \;=\; \|f\|_p\|g\|_q. \qquad \text{(A.2.2)}$$

Theorem A.6. Holder's Inequality for Vector-valued functions. *Suppose* $1 < p < \infty$ *and* $p^{-1} + q^{-1} = 1$*. If* $\mathbf{f}$ *and* $\mathbf{g}$ *are vector valued measurable functions on* $X \to \mathbb{R}^d$*, then*

$$\int_X \sum_{i=1}^{d} |f_i g_i| \;\leq\; (\int_X \sum_{i=1}^{d} |f_i|^p)^{1/p} (\int_X \sum_{i=1}^{d} |g_i|^q)^{1/q}. \qquad \text{(A.2.3)}$$

Theorem A.7. Minkowski's Inequality for Integrals. *Suppose that* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathcal{N}, \nu)$ *are* σ*-finite measure spaces, and let* f *be an* $(\mathcal{M} \otimes \mathcal{N})$*-measurable function on* $X \times Y$*,*

$$\left[\int \left(\int f(x,y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x,y)^p d\mu(x)\right]^{1/p} d\nu(y). \qquad \text{(A.2.4)}$$

Theorem A.8. Poincare's Inequality. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$*. Then there exists a positive constant* $C = C(\Omega, p)$ *such that for* $1 \leq p < \infty$

$$\|u\|_{L^p} \;\leq\; C|u|_{W^{1,p}} \quad \text{for every } \mathrm{u} \in \mathrm{W}_0^{1,\mathrm{p}}(\Omega). \qquad \text{(A.2.5)}$$

A.3 Reflexive Banach Space

Definition A.9. Reflexive Space. *Suppose* X *is a normed vector space, and the mapping* $J : X \to X''$ *defined by* $J(x)(\phi) = \phi(x)$ *for every* $x \in X$ *and* $\phi \in X'$ *is bijective, then the space* X *is called reflexive.*

Theorem A.10. *All Hilbert spaces are reflexive.*

The proof of above theorem is based on Riesz Representation Theorem.

Theorem A.11. *Any closed subspace of reflexive space is reflexive.*

The proof of above theorem is based on Hahn-Banach theorem.

A.4 Variational Methods in Banach Spaces

Let X be a Banach space, and U be a subset of X, consider the minimization problem of a energy functional $J : U \subset X \to \overline{\mathbb{R}}$:

$$\text{Find } \mathrm{u_0 \in U \subset X \text{ s.t } J(u_0) = \inf_{u \in U} J(u).} \tag{A.4.1}$$

Some of the simplest and most general results which guarantees the existence of a solution to the minimization problem (A.4.1) is following.

Theorem A.12. *Let X be are flexive Banach space, let U be a weakly closed subset of X, and let $J : U \subset X \to \overline{\mathbb{R}}$ be a proper, bounded below, and weakly lower semicontinuous on U. Then there exists a solution to problem (A.4.1).*

Theorem A.13. *Let X be a reflexive Banach space, let U be a closed convex subset of X, and let $J : U \subset X \to \overline{\mathbb{R}}$ be a proper, bounded below, convex, and lower semicontinuous on U. Then there exists a solution to problem (A.4.1). Moreover, if J is strictly convex, the solution is unique.*

Here two sufficient conditions of weakly lower semicontinous are provided.

Theorem A.14. *Let X be a Banach space, let $U \subset X$ be a closed convex subset, and $J : U \subset X \to \overline{\mathbb{R}}$ be convex and lower semicontinuous on U. Then J is weakly lower semicontinuous on U.*

Theorem A.15. *Let X be a Banach space, let $U \subset X$ be a closed convex subset, and $J : U \subset X \to \overline{\mathbb{R}}$ be convex on U. If J is G-differentiable on U. Then J is weakly lower semicontinuous on U.*

A.5 Collocation Points Generator of MCLite

The stochastic collocation points generator are built by three files which are `collocationpoints.m`, `HermiteZero.m` and `LegendreZero.m`.

To generate collocation points, simply call `collocationpoints.m` at the beginning of `go.m`(the driver script controller of MCLite package) as

```
Zero = collocationpoints[type,n,p1,p2]
```

where `type==1` is for Hermite zeros, `type==2` is for Legendre zeros, `n` is the dimension of random space $\mathcal{P}_p$, and `p1`, `p2` are the order of polynomials with respect to the first dimension and second dimension of space $\mathcal{P}_p$. The corresponding `HermiteZero.m` or `LegendreZero.m` will be called in by `collocationpoints.m`, and the collocation points matrix `Zero` is the output. Then, `go.m` will solve the stochastic problem by a loop of each collocation point generated.

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