Title
Singular stochastic differential equations with elliptic and hypoelliptic diffusions

Author
Nam, Kyeongsik

Publication Date
2020

Peer reviewed|Thesis/dissertation

Singular stochastic differential equations with elliptic and hypoelliptic diffusions

by

Kyeongsik Nam

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Fraydoun Rezakhanlou, Chair
Professor Alan Hammond
Assistant Professor Shirshendu Ganguly

Spring 2020

Singular stochastic differential equations with elliptic and hypoelliptic diffusions

Abstract

Singular stochastic differential equations with elliptic and hypoelliptic diffusions

by

Kyeongsik Nam

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Fraydoun Rezakhanlou, Chair

In this thesis, the well-posedness of stochastic differential equations (SDEs) with singular coefficients is discussed. First, it is proved that SDEs with elliptic diffusion possess a unique solution when drift vector fields belong to the Orlicz-critical space. Then, it is shown that SDEs with degenerate and hypoelliptic diffusion are well-posed for a large class of singular drifts. A basic theory on Lorentz spaces and the analysis on the homogeneous Carnot group will also be introduced.

Contents

Acknowledgments

I am grateful to my advisor Professor Fraydoun Rezakhanlou for his continued support throughout PhD. I wish to thank Professor Shirshendu Ganguly for encouraging me throughout several ongoing projects. I also thank Professor Alan Hammond for helpful comments to improve this thesis.

Chapter 1

Introduction

1.1 Singular stochastic differential equations

A theory of the stochastic differential equation (SDE)

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \\ X_0 = x. \end{cases}$$

(B_t denotes a standard Brownian motion) has been central in the probability theory due to its broad applications to the analysis, in particular fluid mechanics. In fact, SDEs provide a nice Lagrangian point of view to study various types of partial differential equations in fluid mechanics such as Euler equations and Navier-Stokes equations. For instance, there is a nice stochastic Lagrangian representation of incompressible Navier-Stokes equations. It is proved in [**CI**] that for a sufficiently smooth divergence-free vector field u_0, if the pair (u, X) satisfy the following stochastic system:

$$dX = udt + \sqrt{2}dB_t,$$
$$u = \mathbb{E}\mathbf{P}\big[\nabla^T(X^{-1})(u_0 \circ X^{-1})\big],$$

($\mathbf{P}$ is the Leray-Hodge projection on divergence-free vector fields), then u satisfies the incompressible Navier-Stokes equations

$$u_t + (u \cdot \nabla)u = -\nabla p + \Delta u$$

with an initial data u_0 and some pressure p.

In addition, a theory of SDE provides the well-posedness result of the transport equations. For example, consider the following stochastically perturbed transport equation:

$$\begin{cases} d_t u(t,x) + b(t,x)\cdot Du(t,x)dt + \sum_{i=1}^d e_i \cdot Du(t,x)\circ dB_t^i = 0, \\ u(0,\cdot) = u_0 \in L^\infty \end{cases} \tag{1.1.1}$$

(e_i's are standard vectors in the Euclidean space and $\circ$ denotes the Stratonovich integral). It is proved **[FGP]** that for singular vector fields b on $\mathbb{R}^d$ satisfying

$$b \in L^\infty([0,T], C_x^\alpha), \quad \mathrm{div} b \in L^p([0,T]\times\mathbb{R}^d)$$

with $\alpha \in (0,1)$ and $p \in (2,\infty)$, the equation (1.1.1) admits a unique L^∞-weak solution. This result can be interpreted as an regularization effect by the noise since in the absence of randomness, the classical transport equation

$$\begin{cases} d_t u(t,x) + b(t,x)\cdot Du(t,x)dt = 0, \\ u(0,\cdot) = u_0 \in L^\infty \end{cases}$$

may have several weak solutions under the same condition on b.

Authors in **[FGP]** first established the fact that SDE with the additive noise and a Hölder continuous drift $b \in L^\infty([0,T], C^\alpha)$ admits a unique strong solution, and this solution possesses an improved regularity. Then, they translated this well-posedness result of SDE to the well-posedness result of the stocastically perturbed transport equation (1.1.1). The aforementioned examples show that the qualitative properties of SDEs possessing singular coefficients can be usefully applied to study various PDEs in fluid mechanics.

The primary step to study qualitative properties of SDEs is establishing the well-posedness. In the absence of randomness, SDE becomes the ordinary differential equation (ODE), and its well-posedness theory has been well-established so far. It is a classical fact that the ODE

$$\begin{aligned} x'(t) &= b(t,x(t)), \\ x(t_0) &= x_0 \end{aligned}$$

admits a unique solution $x(t)$ if the vector field $b(t,x)$ is uniformly Lipschitz continuous in x and continuous in t. This is the optimal condition in the sense that the existence or uniqueness may not hold without the Lipschitz continuity in x.

A breakthrough progress in this context was made by Diperna and Lions **[DL]**. They introduced the theory of a Lagrangian flow, which generalizes the notion of a classical flow associated with ODE. Here, $X : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is called a *regular Lagrangian flow* if it satisfies the following properties:

1. For Leb-a.e. x, $t \mapsto X(t,x)$ is an absolutely continuous integral solution.
2. There exists a constant $C > 0$ such that $X(t,\cdot)_{\#}\text{Leb} \le C\text{Leb}$.

Here, Leb denotes the Lebesgue measure on $\mathbb{R}^d$. They proved that under a suitable integrability condition on b and div$b$, which is weaker than Lipschitz continuity, it is possible to construct a Lagrangian flow. More precisely, for singular vector fields b satisfying

$$b \in L^1((0,T), W^{1,1}(\mathbb{R}^d)), \quad \text{div}b \in L^1((0,T), L^\infty(\mathbb{R}^d))$$

with a suitable growth condition, there exists a unique regular Lagrangian flow associated to the ODE with drift b. This result was extended to the bounded variation (BV) vector fields by Ambrosio [**A**]. Here, a function $u \in L^1$ is said to be bounded variation if

$$\sup\{\int u(x)\text{div}\phi(x)dx : \phi \in C_c^1,\ \|\phi\|_\infty \le 1\} < \infty.$$

Surprisingly, ODE becomes well-posed for a larger class of singular drifts b once it is perturbed by a Brownian motion. Consider the stochastic differential equation:

$$\begin{cases} dX_t = b(t,X_t)dt + dB_t, \quad 0 \le t \le T, \\ X_0 = x. \end{cases} \tag{1.1.2}$$

It is known that SDE (1.1.2) has a unique solution provided that coefficients are sufficiently regular: for instance, if b is Lipschitz continuous. There have been numerous works regarding the well-posedness for a broad class of singular coefficients. For example, Krylov and Röckner [**KR**] established the well-posedness of SDE (1.1.2) under the condition:

$$b \in L^q([0,T], L^p_x), \quad \text{for} \quad \frac{2}{q} + \frac{d}{p} < 1,\ 1 < p,q < \infty, \tag{1.1.3}$$

where d denotes the dimension of the underlying Euclidean space.

After this groundbreaking work, lots of the well-posedness results have been established for the various types of non-degenerate diffusion coefficients under the condition of type (1.1.3). For instance, Zhang [**Z1**, **Z2**] proved that SDE:

$$\begin{cases} dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad 0 \le t \le T, \\ X_0 = x. \end{cases} \tag{1.1.4}$$

admits a unique local strong solution if σ is non-degenerate and b satisfies (1.1.3). Here, σ being non-degenerate means that the infinitesimal generator of SDE (1.1.4) is elliptic.

On the other hand, at the supercritical regime:

$$b \in L^q([0,T], L^p_{\text{loc}}), \quad \text{for} \quad \frac{2}{q} + \frac{d}{p} > 1,\ 1 < p, q < \infty, \tag{1.1.5}$$

SDE (1.1.2) may not be well-posed in general. In fact, if b given by

$$b(t,x) = -\beta \frac{x}{|x|^2} \mathbb{1}_{x \neq 0}, \quad \beta > \frac{1}{2}, \tag{1.1.6}$$

the corresponding SDE (1.1.2) with the initial condition $X_0 = 0$ does not admit a solution [**BFGM**]. Since a singular drift b in (1.1.6) satisfies

$$b \in L^\infty([0,T], L^p_{\text{loc}})$$

for any $p < d$, this shows that SDE (1.1.2) is in general ill-posed at the supercritical regime (1.1.5). Therefore, the qualitative properties of SDE depend delicately on the integrability condition on the singular drift b.

However, only little is known at the critical regime:

$$b \in L^q([0,T], L^p_x), \quad \text{for} \quad \frac{2}{q} + \frac{d}{p} = 1,\ 1 < p, q < \infty. \tag{1.1.7}$$

The weak type of well-posedness result is proved at the critical regime [**BFGM**]. It is shown that for almost all realization w, there exists a stochastic Lagrangian flow associate with SDE (1.1.2). Here, $\phi : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is called a *stochastic Lagrangian flow* to (1.1.2) if it satisfies the following conditions:

1. w-almost surely, $\phi(\cdot,\cdot,w) - B_t(w)$ is a Lagrangian flow to the random ODE: $x'(t) = b^w(t, x(t))$, where $b^w(t,x) = b(t, x + B_t(w))$.

2. ϕ is weakly progressively measurable with respect to $\mathcal{F}_t$ ($\mathcal{F}_t$ by a natural filteration of the Brownian motion B_t).

However, the well-posedness in the classical sense is not known at the critical regime. In this thesis, we establish the classical well-posedness of SDE (1.1.2) at the critical regime. More precisely, we prove that SDE (1.1.2) admits a unique strong solution if the Lebesgue-type L^q integrability in a time variable is improved to the slightly stronger Lorentz-type $L^{q,1}$ integrability condition:

$$b \in L^{q,1}([0,T], L^p_x) \quad \text{for} \quad \frac{2}{q} + \frac{d}{p} = 1,\ 1 < p, q < \infty. \tag{1.1.8}$$

We refer to the condition (1.1.8) as *Orlicz-critical* condition. More precisely, Lorentz spaces are defined by:

Definition 1.1.1. (Lorentz spaces). A complex-valued function f defined on the measure space (X,μ) belongs to the *Lorentz space* $L^{p,q}(X,d\mu)$ if the quantity

$$\|f\|_{L^{p,q}(X)} := p^{\frac{1}{q}} \left\| t\mu(|f| \geq t)^{\frac{1}{p}} \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})} \tag{1.1.9}$$

is finite.

The basic properties of Lorentz spaces will be discussed in Chapter 2. The crucial fact is that $L^{p,1}$ is properly included in the standard Lebesgue space L^p for $p > 1$.

In this thesis, we show that SDE (1.1.2) admits a unique solution for singular vector fields b satisfying the Orlicz-critical condition (1.1.8).

Theorem 1.1.2. *Suppose that the drift b satisfies:*

$$b \in L^{q,1}([0,T], L^p_x) \quad \text{for} \quad \frac{2}{q} + \frac{d}{p} = 1,\ 1 < p, q < \infty. \tag{1.1.10}$$

Then, there exists a unique strong solution to SDE (1.1.2) *for any $x \in \mathbb{R}^d$.*

Using this theorem, it is possible to study some singular PDEs such as transport equations discussed before. For instance, using the strategy developed in [**FGP**], one can deduce the well-posedness of the stochastically perturbed transport equation (1.1.1) for Orlicz-critical drifts b with some additional assumptions.

In the standard Lebesgue critical spaces (1.1.7), there are some technical difficulties to show the well-posedness of SDE. For instance, the well-posedness of the associated Kolmogorov PDE may break down. However, if the time integrability is improved to the Orilcz-critical condition (1.1.10), then we obtain the well-posedness for the corresponding SDE. In particular, we establish the Orlicz-critical Sobolev embedding theorem and the well-posedness of Kolmogorov PDE under the condition (1.1.10), which will be discussed in Chapter 2.

So far, we have discussed a class of singular SDEs with additive noise for which there exists a unique solution. A natural question is to extend this well-posedness result to singular SDEs with degenerate noise. Consider the following Stratonovich SDE with hypoelliptic diffusion:

$$\begin{cases} dX_t = b(t, X_t)dt + \sum_{i=1}^m Z_i(X_t) \circ dB_t^i, \\ X_0 = x_0. \end{cases} \tag{1.1.11}$$

Here, B^i's are independent standard one dimensional Brownian motions. Stratonovich formulation has a nice control theoretical interpretation according to Stroock-Varadhan support theorem [**SV**]: if we denote x^h by a solution of the following ODE:

$$dx_t^h = b(t, x_t^h)dt + \sum_{i=1}^m Z_i(x_t^h)dh^i,$$

then the support of the law of a solution X_t of SDE (1.1.11) is the closure of a set $\{x^h \mid \frac{dh}{dt} \in L^2([0,T],\mathbb{R}^d)\}$ in C^α topology. Under Hörmander's condition [**H**]

$$\mathrm{Lie}(Z_1,\cdots,Z_m)(x) = T_x R^d,$$

the regularization effect happens in that the law of a solution possesses a smooth density. See [**H**] for an analytical approach and [**M**] for a probabilistic approach, known as Malliavin Calculus.

It is crucial to understand the qualitative properties of SDE (1.1.11) with the hypoelliptic diffusion since it appears naturally in various areas of mathematics such as sub-Riemannian geometry as well as phase space problems. For instance, several properties such as Log-Sobolev inequalities and the heat kernel estimates for the hypoelliptic diffusions have been established in [**B**, **BB**, **BBBC**, **BGM**]. Therefore, it is natural and important to develop a qualitative theory for a broad class of hypoelliptic diffusions since it provides the understanding of diffusion on sub-Riemannian manifolds. The first step to accomplish this is to establish the well-posedness result of hypoelliptic SDEs (1.1.11) for a large class of singular drifts.

In this thesis, we provide a large class of singular drifts b for which SDE (1.1.11) with hypoelliptic diffusions admits a unique solution. We consider the SDEs with singular drifts on the *homogeneous Carnot group*. We assume that vector fields $Z_1,\cdots,Z_m$ in SDE (1.1.11) are left-invariant and form a basis of the first layer of stratified Lie algebra. In the terminology of sub-Riemannian geometry, the diffusion part of SDE (1.1.11) is called the *horizontal Brownian motion*, and it is hypoelliptic. We refer to Section 3.1 for more details.

Consider the homogeneous Carnot group $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$, where $\{D(\lambda)\}_{\lambda>0}$ denotes the dilation structure. Assume that $\mathbb{G}$ has a homogeneous dimension Q, nilpotency r, and Z_i's $(1 \le i \le m)$ are left invariant vector fields that form a basis of the first layer of the Lie algebra $\mathfrak{g}$ (see Section 3.1 for details). Consider the following Stratonovich SDE on the homogeneous Carnot group $\mathbb{G}$:

$$\begin{cases} dX_t = b(t,X_t)dt + \sum_{i=1}^m Z_i(X_t)\circ dB_t^i, \\ X_0 = x_0. \end{cases} \tag{1.1.12}$$

Here, $\mathbb{G}$ is identified with $\mathbb{R}^N$ and Z_i's $(1 \le i \le m)$ are regarded as vector fields on $\mathbb{R}^N$. Also, suppose that two exponents p and q satisfying

$$\frac{2}{q} + \frac{Q}{p} < 1, \quad 1 < p,q < \infty, \tag{1.1.13}$$

are given, and that the drift b satisfies

$$b \in \mathrm{span}\{Z_1, Z_2, \cdots Z_m\}, \tag{1.1.14}$$

$$Z_I b^i \in L^q([0,T], L^p(\mathbb{G})) \quad \text{for} \quad 1 \le i \le m,\ |I| \le r-1. \tag{1.1.15}$$

Here, $Z_I f$ denotes $Z_{i_1} \cdots Z_{i_k} f$ (distribution derivatives) for a multi-index $I = (i_1, \cdots, i_k)$, $1 \le i_1, \cdots, i_k \le m$, and b^i's $(1 \le i \le m)$ are given by the expression:

$$b = \sum_{i=1}^{m} b^i Z_i.$$

Our main result Theorem 1.1.3 below claims that one can construct a (unique) solution to SDE (1.1.12) for a broad class of singular drifts b.

Theorem 1.1.3. *Let* $(\mathbb{G}, \circ, D(\lambda))$ *and* $\{Z_i | 1 \le i \le m\}$ *be as above. Assume that a singular drift* b *satisfies the conditions* (1.1.14) *and* (1.1.15) *for the exponents* p, q *satisfying* (1.1.13)*. Then, for some open set* U *containing* x_0*, there exists a unique strong solution* X_t *to SDE* (1.1.12) *before the time at which* X_t *exits* U*.*

Remarkably, Theorem 1.1.3 provides a beautiful intermediate well-posedness result between ODE case (absence of the randomness) and SDE with the additive noise case [**KR**]. Recall that as explained in the introduction, SDE with an additive noise $(\sigma = Id)$:

$$dX_t = b(t, X_t)dt + dB_t, \quad X_0 = x_0, \tag{1.1.16}$$

admits a unique strong solution if a singular drift b satisfies the condition (1.1.3). This result can be regarded as a special case of Theorem 1.1.3. In fact, if we consider the homogeneous Carnot group given by

$$(\mathbb{G}, \circ) = (\mathbb{R}^d, +), \quad Z_i = \frac{\partial}{\partial x_i}\ (1 \le i \le d),$$

with a standard dilation structure, then the homogeneous dimension Q is equal to d and nilpotency r is equal to 1. Also, SDE (1.1.12) becomes the additive noise SDE. Therefore, by comparing (1.1.3) with the conditions (1.1.13)-(1.1.15), Theorem 1.1.3 can be regarded as a considerable generalization of the well-posed result of singular SDEs from the non-degenerate diffusion case $r = 1$ to the degenerate diffusion cases $r > 1$.

In addition, one can formally check that in the limit $r \to \infty$, Theorem 1.1.3 covers the classical well-posed result in the ODE theory. Note that if we write $W^{k,p}$ for the standard Sololev spaces and $S^{k,p}$ for the Sobolev spaces with respect to vector fields $\{Z_i | 1 \le i \le m\}$:

$$S^{k,p}(\mathbb{G}) := \{f | Z_I f \in L^p(\mathbb{G}), |I| \le k\},$$

then for $1 < p < \infty$, the following relation holds:

$$W^{k,p}_{\mathrm{loc}} \subset S^{k,p}_{\mathrm{loc}} \subset W^{k/r,p}_{\mathrm{loc}} \tag{1.1.17}$$

(see [**F3**]). Also, it is obvious that under the conditions (1.1.14) and (1.1.15), each Euclidean coordinate of a singular drift b belongs to the space $L^q([0,T], S^{r-1,p}_{\mathrm{loc}})$. Thus, using this fact and the relation (1.1.17), one can conclude that

$$b \in L^q([0,T], W^{\frac{r-1}{r},p}_{\mathrm{loc}}).$$

Thus, if the noise becomes more degenerate in the sense that $r \to \infty$, it follows that $Q \to \infty$, and thus we have

$$\frac{r-1}{r} \to 1, \quad p \to \infty,$$

due to the condition (1.1.13). Since the space $L^\infty([0,T], W^{1,\infty}_x)$ is the class of drifts for which the corresponding ODE

$$x'(t) = b(t, x(t)), \quad x(0) = x_0$$

is well-posed, the formal limit $r \to \infty$ in Theorem 1.1.3 covers the classical well-posedness result in the ODE theory.

Theorem 1.1.3 provides a new perspective to study a large class of singular hypoelliptic SDEs. It is an important task to extend this theorem to more general class of hypoelliptic SDEs. Further interesting directions include the regularity of a solution and the heat kernel estimate.

Remark 1.1.4. Since Theorem 1.1.3 is a local statement, throughout this thesis, we assume that each b^i has a compact support, which is uniform in t. Note that in the case $r = 1$, we have $(\mathbb{G}, \circ) = (\mathbb{R}^N, +)$ and $Z_i = \frac{\partial}{\partial x_i}$ for $1 \leq i \leq N$, which corresponds to the additive noise case, and this case is considered in [**KR**]. Therefore, throughout this paper, we only consider the case $r > 1$ which corresponds to the degenerate diffusion case.

The condition (1.1.15) implies that $b^i(t, \cdot) \in S^{r-1,p}(\mathbb{G})$ for t-a.e. Since $(r-1)p \geq p > Q$ under the condition (1.1.13) and $r > 1$, according to the Sobolev embdding theorem (see Theorem 3.1.4), there is a version of b such that $b(t, \cdot)$ is continuous for t-a.e. In this paper, we prove Theorem 1.1.3 for such drifts.

1.2 Organization of thesis

Chapters 2 and 3 include the author's work of arXiv postings [**N1**] and [**N2**], respectively. Chapter 2 is devoted to the study of singular stochastic differential equations possessing elliptic diffusion. A basic theory of stochastic differential equations and Lorentz spaces will be discussed in Section 2.1. In Section 2.2, we obtain the existence of weak solution to SDE under the Orlicz-condition. We study the associated Kolmogorov PDE in Section 2.3. In Section 2.4, uniqueness of the strong solution is discussed. In Section 2.5, we finally prove the well-posedness of SDEs under the Orlicz-condition, and construct a stochastic flow.

The singular stochastic differential equations possessing hypoelliptic diffusion are discussed in Chapter 3. In Section 3.1, we provide a basic hypoelliptic theory. The Kolmogorov PDE associated with hypoelliptic SDE will be studied in Section 3.2. Finally, the main result will be proved in Section 3.3.

Throughout this thesis, B_t denotes the standard Brownian motion on a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with the filtration $\mathcal{F}_t = \sigma\{B_r | 0 \leq r \leq t\}$. Also, B_t^x denotes the Brownian motions starting from x. We denote ∇, Δ, and $\mathcal{M}$ by gradient, Laplacian, and the Hardy-Littlewood maximal function.

For two Banach spaces X and Y, $[X, Y]_{\theta,q}$ denotes a real interpolation of X and Y with parameters $0 < \theta < 1$ and $q \in [1, \infty]$. Also, $f \lesssim_\alpha g$ means that $f \leq Cg$ for some constant $C = C(\alpha)$. We say $f \sim_\alpha g$ provided that $f \lesssim_\alpha g$, $g \lesssim_\alpha f$. Finally, for $d \times d$ matrix A, $|A|$ denotes a Hilbert-Schmidt norm.

Chapter 2

Singular stochastic differential equations with elliptic diffusion

2.1 Preliminaries: stochastic differential equations and Lorentz spaces

In this section, we explain some key lemmas in the probability theory and some properties of Lorentz spaces.

Stochastic processes and stochastic differential equations

First, we explain notions of a weak solution and a strong solution to the SDE:

Definition 2.1.1. Consider SDE of the following form:

$$dX_t = b_t(X.)dt + \sigma_t(X.)dB_t. \tag{2.1.1}$$

Here, b and σ are progressive functions defined on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d)$ equipped with the canonical filtration $\mathcal{F}_t = \sigma\{x_s | s \leq t\}$. For a given filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\mathcal{F}_t$-Brownian motion B, and an $\mathcal{F}_0$-measurable random variable ξ, X is a *strong solution* to SDE if it is a $\mathcal{F}_t$-adapted process with $X_0 = \xi$ solving (2.1.1) almost surely. For a given initial distribution μ, a *weak solution* consists of the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $\mathcal{F}_t$-Brownian motion B, and a $\mathcal{F}_t$-adapted process X with $P \circ X_0^{-1} = \mu$ satisfying (2.1.1) almost surely.

We say that *weak existence* holds for the initial distribution μ if there exists a weak solution $(\Omega, \mathcal{F}, \mathcal{F}_t, P, B, X)$ satisfying (2.1.1). *Strong existence* is said to hold for the initial distribution μ if there exists a strong solution X for every $(\Omega, \mathcal{F}, \mathcal{F}_t, P, B, \xi)$ satisfying $P \circ \xi^{-1} = \mu$.

We say that *strong uniqueness* holds for the initial distribution μ provided that for any solutions X and Y to (2.1.1) on the common filtered probability space with a given Brownian motion such that $X_0 = Y_0$ a.s. with a distribution μ, $X = Y$ almost surely. Finally, *weak uniqueness* is said to hold for the initial distribution μ if each weak solution X has the same distribution.

The following theorem proved by Watanabe and Yamada [**YW1**, **YW2**] is crucial to prove the existence of a strong solution to SDE.

Theorem 2.1.2 (Yamada-Watanabe Principle, [**YW1**, **YW2**]). *Consider the following SDE:*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{2.1.2}$$

with a given initial condition. Suppose that a weak solution to (2.1.2) *exists and a strong solution to* (2.1.2) *is unique. Then, the strong existence and weak uniqueness hold as well.*

The celebrated Itô formula states how the diffusion process is changed under the transformation: if the one-dimensional process X_t satisfy

$$dX_t = \mu_t dt + \sigma_t dB_t,$$

then

$$df(t, X_t) = (f_t + \mu_t f_x + \frac{1}{2}\sigma_t^2 f_{xx})dt + \sigma_t f_x dB_t.$$

The similar result holds in the high dimension as well.

The following lemma is crucially used throughout this thesis.

Lemma 2.1.3 ([**P**]). *Let X_t $(0 \le t \le T)$ be a nonnegative stochastic process adapted to $\mathcal{F}_t$. Assume that for any $0 \le s \le t \le T$,*

$$\mathbb{E}\left[\int_s^t X_r dr \Big| \mathcal{F}_s\right] \le f(s,t)$$

holds for some deterministic function $f(s,t)$ satisfying

- $f(s_1, t_1) \le f(s_2, t_2)$ *for* $[s_1, t_1] \subset [s_2, t_2]$,
- $\lim_{h\to 0^+} \sup_{0\le s\le t\le T, |t-s|\le h} f(s,t) = \alpha \ge 0$.

Then, for arbitrary $c < \alpha^{-1}$ (when $\alpha = 0$, α^{-1} is defined by $\alpha^{-1} := \infty$),

$$\mathbb{E} \exp\left[c\int_0^T X_r dr\right] < \infty. \tag{2.1.3}$$

Lorentz spaces

In this section, we study some useful properties about the Lorentz spaces. The concept of Lorentz spaces is introduced in [**L**]. These spaces can be regarded as generalizations of the standard Lebesgue $L^p(X, d\mu)$ spaces. In the case when $q = p$, $L^{p,p}$ coincides with the standard L^p spaces, and when $q = \infty$, $L^{p,\infty}$ coincides with the weak L^p spaces. Lorentz spaces are quasi-Banach spaces in the sense that for some constant $c = c(p, q) > 1$,

$$\|f + g\|_{L^{p,q}} \leq c(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}}) \tag{2.1.4}$$

for any $f, g \in L^{p,q}$, and it is complete with respect to $\|\cdot\|_{L^{p,q}}$. Also, Lorentz spaces are real interpolation spaces between two L^p spaces:

$$(L^1, L^\infty)_{\theta,q} = L^{p,q}$$

with $\frac{1}{p} = 1 - \theta$.

Remark 2.1.4. From the definition of Lorentz spaces, we can easily check that the following property holds: if $p < \infty$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|f\|_{L^{p,q}(A)} < \epsilon$$

for all measurable set $A \subseteq X$ satisfying $\mu(A) < \delta$. Also, one can check that for any two disjoint measurable sets $A, B \subseteq X$ and $f \in L^{p,q}(X)$,

$$\|f\|_{L^{p,q}(A)} + \|f\|_{L^{p,q}(B)} \sim_{p,q} \|f\|_{L^{p,q}(A\cup B)} .$$

The following lemma is useful throughout this thesis, which follows from the direct computation.

Lemma 2.1.5. *Let us denote $P(t, x)$ by the standard heat kernel. Then, $\nabla P \in L^{q,\infty}(\mathbb{R}, L^p_x)$ for any exponents $p, q \in (1, \infty)$ satisfying $\frac{2}{q} + \frac{d}{p} = d + 1$.*

There are counterparts of Hölder's and Young's inequalities for the Lorentz spaces. Hölder's inequality for the Lorentz spaces claims that for $1 \leq p_1, p_2, p < \infty$, $0 < q_1, q_2, q \leq \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

we have

$$\|fg\|_{L^{p,q}(X,d\mu)} \leq C(p, q, p_1, q_1, p_2, q_2) \|f\|_{L^{p_1,q_1}(X,d\mu)} \|g\|_{L^{p_2,q_2}(X,d\mu)} .$$

O'Neil's convolution inequality claims that for $1 < p_1, p_2 < \infty$, $0 < q_1, q_2 < \infty$ satisfying

$$1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

we have

$$\|f * g\|_{L^{p,q}(\mathbb{R}^d,dx)} \le C(p,q,p_1,q_1,p_2,q_2) \|f\|_{L^{p_1,q_1}(\mathbb{R}^d,dx)} \|g\|_{L^{p_2,q_2}(\mathbb{R}^d,dx)}.$$

One can extend O'Neil's convolution inequality to the mixed-norm Lorentz spaces. We in particular consider the case $p = q = \infty$ for our purposes.

Proposition 2.1.6. *Suppose that $p_1, p_2, q_1, q_2 \in (1,\infty)$ and $r_1, r_2, s_1, s_2 \in [1,\infty]$ satisfy*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2} = 1.$$

Then, for any $f \in L^{q_1,r_1}(\mathbb{R}, L^{p_1,s_1}(\mathbb{R}^d))$ and $g \in L^{q_2,r_2}(\mathbb{R}, L^{p_2,s_2}(\mathbb{R}^d))$,

$$\|f * g\|_{L^\infty_{t,x}} \le C(p_1,p_2,q_1,q_2,r_1,r_2,s_1,s_2) \|f\|_{L_t^{q_1,r_1}(L_x^{p_1,s_1})} \|g\|_{L_t^{q_2,r_2}(L_x^{p_2,s_2})}.$$

Proof. Note that

$$\begin{aligned} |f * g|(t,x) &\le \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f(s,y)g(t-s,x-y)| dyds \\ &= \|f(\cdot,\cdot)g(t-\cdot,x-\cdot)\|_{L_t^1(L_x^1)}. \end{aligned}$$

Since $\|g\|_{L_t^{q_2,r_2}(L_x^{p_2,s_2})}$ is invariant under the operations $g(\cdot) \mapsto g(c+\cdot)$ and $g(\cdot) \mapsto g(-\cdot)$, it suffices to prove that

$$\|fg\|_{L^1_{t,x}} \le C \|f\|_{L_t^{q_1,r_1}(L_x^{p_1,s_1})} \|g\|_{L_t^{q_2,r_2}(L_x^{p_2,s_2})}.$$

Using Hölder's inequality for the Lorentz spaces, we obtain

$$\begin{aligned} \|fg\|_{L^1_{t,x}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} |f|(t,x)|g|(t,x) dxdt \\ &\le C \int_{\mathbb{R}} \|f(t,\cdot)\|_{L_x^{p_1,s_1}} \|g(t,\cdot)\|_{L_x^{p_2,s_2}} dt \\ &\le C \|f\|_{L_t^{q_1,r_1}(L_x^{p_1,s_1})} \|g\|_{L_t^{q_2,r_2}(L_x^{p_2,s_2})}. \end{aligned}$$

□

Finally, we state the fixed point theorem for the quasi-Banach space.

Proposition 2.1.7. *Suppose that X is a quasi-Banach space, and for some $c > 1$,*

$$\|x + y\| \leq c(\|x\| + \|y\|)$$

hold for any $x, y \in X$. Also, assume that for some $\theta > 0$ satisfying $c\theta < 1$, a map $T : X \to X$ satisfy that for any $x, y \in X$,

$$|T(x) - T(y)| \leq \theta |x - y|.$$

Then, T has a unique fixed point.

Proof. Choose an arbitrary $x_0 \in X$ and let us define $x_n := T(x_{n-1})$ inductively for $n \geq 1$. It is obvious that

$$d(x_{n+1}, x_n) \leq \theta^n d(x_1, x_0).$$

Using a quasi-norm property of X, for any $m > n$,

$$\begin{aligned}
d(x_m, x_n) &\leq cd(x_m, x_{n+1}) + cd(x_{n+1}, x_n) \\
&\leq c^2 d(x_m, x_{n+2}) + c^2 d(x_{n+2}, x_{n+1}) + cd(x_{n+1}, x_n) \\
&\leq \cdots \\
&\leq c^{m-(n+1)} d(x_m, x_{m-1}) + \sum_{k=1}^{m-(n+1)} c^k d(x_{n+k}, x_{n+k-1}) \\
&\leq \left[c^{m-(n+1)} \theta^{m-1} + \sum_{k=1}^{m-(n+1)} c^k \theta^{n+k-1} \right] d(x_1, x_0) \\
&< ((c\theta)^{m-1} c^{-n} + (1 - c\theta)^{-1} c^{-(n-1)}) d(x_1, x_0).
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence, thus it converges to a limit x^* in X since (X, d) is complete. Since T is continuous, we can readily check that x^* is a fixed point.

Uniqueness follows immediately. □

Some useful lemmas

In this section, we introduce some useful lemmas frequently used in the thesis. First, we introduce the notion of Hardy-Littlewood maximal function.

Definition 2.1.8. For the locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$, the Hardy-Littlewood maximal function is defined as follows.

$$\mathcal{M}f(x) = \sup_r \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

Here, $B(x,r)$ is the ball of radius r centered at x, and $|B(x,r)|$ denotes the d-dimensional Lebesgue measure of $B(x,r)$.

The celebrated Hardy-Littlewood maximal inequality states that M is bounded as a sublinear operator from the L^p for any $p > 1$. In other words,

$$\|\mathcal{M}f\|_p \leq C_{p,d} \|f\|_p$$

with some constant $C_{p,d} > 0$.

Also, the following weak (1,1)-type estimate holds as well:

$$|\{Mf > \lambda\}| < \frac{C_d}{\lambda} \|f\|_1 .$$

The aforementioned Hardy-Littlewood maximal inequality can be used to derive Lebesgue differentiation theorem and Rademacher differentiation theorem.

In the next proposition, we state a useful inequality involving the Hardy-Littlewood maximal operator $\mathcal{M}$. It provides a way to control the difference $|u(x) - u(y)|$.

Proposition 2.1.9. *There exists a constant $N = N(d)$ such that the following property holds: for any $u \in C^\infty(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,*

$$|u(x) - u(y)| \leq N|x - y|(\mathcal{M}|\nabla u|(x) + \mathcal{M}|\nabla u|(y)).$$

The last proposition is a useful criteria to derive a global bijectivity of the map, which is called the *Hadamard lemma*.

Proposition 2.1.10. *Suppose that a $C^k (k \geq 1)$ map $F : \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following properties:*
(i) $\nabla F(x)$ is non-singular for every $x \in \mathbb{R}^d$,
(ii) $\lim_{|x|\to\infty} |F(x)| = \infty$.
Then, F is a C^k diffeomorphism from $\mathbb{R}^d$ to itself.

2.2 Existence of a weak solution to SDE

From now on, we construct a unique strong solution to SDE (1.1.2) under the Orlicz-critical condition (1.1.10). According to the Yamada-Watanabe principle

[**YW1**, **YW2**], it reduces to establish the existence of a weak solution and the uniqueness of a strong solution to SDE. We prove both of them separately under the Orlicz-critical condition (1.1.10).

In this section, we construct a weak solution to SDE under the Orlicz-critical condition (1.1.10). First, we recall the following key lemma by Khasminskii [**K1**]:

Lemma 2.2.1. *Suppose that a nonnegative function f satisfies*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E} \int_0^T f(s, B_s^x) ds = M < 1.$$

Then, we have

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\, e^{\int_0^T f(s,B_s^x)ds} \leq \frac{1}{1-M}.$$

The quantity $\sup_{x\in\mathbb{R}^d} \mathbb{E} \int_0^T f(s, B_s^x) ds$ can be controlled for a large class of functions:

Proposition 2.2.2. *Suppose that two exponents $p, q \in (1, \infty)$ satisfying $\frac{2}{q} + \frac{d}{p} = 2$ are given. Then, for any $f \in L^{q,1}([0,T], L_x^p)$,*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E} \int_0^T f(s, B_s^x) ds < C \|f\|_{L^{q,1}([0,T],L_x^p)}$$

holds for some constant $C = C(p, q)$ independent of f and T.

Proof. Let p', q' be the conjugate exponents of p, q, respectively. Then,

$$\begin{aligned}
\mathbb{E} \int_0^T f(s, B_s^x) ds &= \int_0^T \int_{\mathbb{R}^d} (2\pi s)^{-\frac{d}{2}} f(s, x+y) e^{-\frac{|y|^2}{2s}} dy ds \\
&\leq \int_0^T (2\pi s)^{-\frac{d}{2}} \|f(s,\cdot)\|_{L_x^p} \left\| e^{-\frac{|\cdot|^2}{2s}} \right\|_{L_x^{p'}} ds \\
&= K \int_0^T \|f(s,\cdot)\|_{L_x^p} s^{d/2p' - d/2} ds \\
&\leq C \|f\|_{L^{q,1}([0,T],L_x^p)} \left\| s^{-\frac{d}{2}(1-\frac{1}{p'})} \right\|_{L^{q',\infty}([0,T])} \\
&= C \|f\|_{L^{q,1}([0,T],L_x^p)} .
\end{aligned}$$

Here, we used the fact that

$$\left\| e^{-\frac{|\cdot|^2}{2s}} \right\|_{L_x^{p'}} = K \cdot s^{\frac{d}{2p'}}$$

for some universal constant K. Also, we applied Hölder's inequality for the Lorentz spaces in the fourth line. In addition, we used the fact $\frac{d}{2}(1-\frac{1}{p'})=\frac{1}{q'}$ to conclude that

$$\left\| s^{-\frac{d}{2}(1-\frac{1}{p'})} \right\|_{L^{q',\infty}([0,T])} = 1.$$

□

The preceding proposition, combined with the Markov property and Lemma 2.2.1, implies the following proposition.

Proposition 2.2.3. *Suppose that $f \in L^{q,1}([0,T], L^p_x)$ for $p, q \in (1,\infty)$ satisfying $\frac{2}{q}+\frac{d}{p}=2$. Then, the following quantity is finite:*

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\, e^{\int_0^T f(s,B_s^x)ds}. \tag{2.2.1}$$

Proof. Without loss of generality, we assume that $f \geq 0$. In order to apply Lemma 2.2.1, let us divide the interval $[0,T]$ into several intervals $[T_{i-1},T_i]$, $0=T_0<T_1<\cdots<T_k<T_{k+1}=T$, such that

$$\sup_{x\in\mathbb{R}^d} \mathbb{E}\int_0^{T_i-T_{i-1}} f(T_{i-1}+s, B_s^x)ds \leq \alpha$$

holds for some $\alpha<1$. Applying Lemma 2.2.1, we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{R}^d} \mathbb{E}\, e^{\int_0^T f(s,B_s^x)ds} &= \sup_{x\in\mathbb{R}^d} \mathbb{E}\, e^{\int_0^{T_1} f(s,B_s^x)ds}\cdots e^{\int_{T_k}^T f(s,B_s^x)ds} \\
&= \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[e^{\int_0^{T_1} f(s,B_s^x)ds}\cdots e^{\int_{T_{k-1}}^{T_k} f(s,B_s^x)ds}\, \mathbb{E}(e^{\int_{T_k}^T f(s,B_s^x)ds}|\mathcal{F}_{T_k})\right] \\
&= \sup_{x\in\mathbb{R}^d} \mathbb{E}\left[e^{\int_0^{T_1} f(s,B_s^x)ds}\cdots e^{\int_{T_{k-1}}^{T_k} f(s,B_s^x)ds}\, \mathbb{E}\, e^{\int_0^{T-T_k} f(T_k+s,B_s^y)ds}|_{y=B_{T_k}^x}\right] \\
&\leq \frac{1}{1-\alpha} \sup_{x\in\mathbb{R}^d} \mathbb{E}\, e^{\int_0^{T_1} f(s,B_s^x)ds}\cdots e^{\int_{T_{k-1}}^{T_k} f(s,B_s^x)ds} \\
&\leq \cdots \\
&\leq (\frac{1}{1-\alpha})^{k+1}.
\end{aligned}$$

□

Now, as an application of Girsanov theorem, one can derive the existence of a weak solution. We first briefly recall Girsanov theorem, which describes how the stochastic processes are transformed under the change of underlying probability measure. Suppose that B_t is a Brownian motion with respect to the measure $\mathbb{P}$, and Z_t is a measurable process adapted to the natural filtration of the Brownian motion B_t. Then,

$$B_t - [B, Z]_t$$

is a Brownian motion under the new measure $\mathbb{Q}$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp(Z_t - \frac{1}{2}[Z]_t).$$

Now, we briefly state Novikov condition, which provide a sufficient condition for a process, which is the Radon-Nikodym derivative in Girsanov theorem, to be a martingale. Under the same assumption as above, assume that

$$\mathbb{E}e^{\frac{1}{2}\int_0^T |Z_t|^2 dt} < \infty$$

is satisfied. Then, the process

$$t \mapsto e^{\int_0^t Z_s dB_s - \frac{1}{2}\int_0^t Z_s^2 ds}$$

is a martingale under the measure $\mathbb{P}$.

We are now ready to prove the existence of weak solution.

Theorem 2.2.4. *Suppose that b satisfies the condition* (1.1.10)*. Then, SDE* (1.1.12) *admits a weak solution. More precisely, we can construct processes X_t and B_t for $0 \le t \le T$ on some filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ such that B_t is a standard $\mathcal{F}_t$-Brownian motion and almost surely,*

$$X_t = x + \int_0^t b(s, X_s)ds + B_t \tag{2.2.2}$$

holds for all $0 \le t \le T$.

Proof. Let X_t be a Brownian motion starting from x on the probability space $(\Omega, \mathcal{G}, Q)$, equipped with a natural filtration $\mathcal{F}_t$. Then, using Proposition 2.2.3, one can conclude that

$$\alpha_t = \exp\Big[\int_0^t b(s, X_s)dX_s - \frac{1}{2}\int_0^t |b(s, X_s)|^2 ds\Big]$$

is a Q-martingale since Novikov condition is satisfied. Thus, a process defined by

$$B_t = X_t - \int_0^t b(s, X_s)ds - x$$

is a $\mathcal{F}_t$-Brownian motion starting from the origin with respect to the new probability measure $dP(w) = \alpha_T(w)dQ(w)$ on $\mathcal{F}_T$ due to Girsanov theorem. $\square$

2.3 Kolmogorov elliptic PDE

In this section, we study the following Kolmogorov PDE:

$$\begin{cases} u_t - \frac{1}{2}\Delta u + b \cdot \nabla u + f = 0, & 0 \le t \le T, \\ u(0,x) = 0, \end{cases} \tag{2.3.1}$$

for singular functions b and f that belong to the Orlicz-critical space (1.1.10). This PDE plays a crucial role to prove the strong uniqueness of SDE since the infinitesimal generator of SDE (1.1.2) is given by

$$Lu = \frac{1}{2}\Delta u + b \cdot \nabla u.$$

The PDE (2.3.1) is well-understood when singular coefficients b and f belong to the subcritical Lebesgue space (1.1.3). On the other hand, PDE (2.3.1) is not well-understood if b and f belong to the Orlicz-critical space due to the lack of nice embedding properties of the mixed-norm parabolic Sobolev spaces at the critical regime.

In this section, we obtain the parabolic Sobolev embedding properties when a slightly stronger Lorentz integrability condition is imposed on the time variable. Also, we establish the well-posedness result of PDE (2.3.1) with Orlicz-critical coefficients, and then obtain a priori estimate.

For $1 < p, q < \infty$, and $S \le T$, let us define a function space $X^{q,p}([S,T])$ to be a collection of functions satisfying

$$u, u_t, \nabla u, \nabla^2 u \in L^{q,1}([S,T], L^p_x).$$

Note that derivatives are interpreted as a distribution sense. Its norm is defined by

$$\begin{aligned} &\|u\|_{X^{q,p}([S,T])} \\ &\quad := \|u\|_{L^{q,1}([S,T],L^p_x)} + \|u_t\|_{L^{q,1}([S,T],L^p_x)} + \|\nabla u\|_{L^{q,1}([S,T],L^p_x)} + \left\|\nabla^2 u\right\|_{L^{q,1}([S,T],L^p_x)}. \end{aligned}$$

One can easily check that $X^{q,p}([S,T])$ is a quasi-Banach space. The following theorem establishes the well-posedness of PDE (2.3.1).

Theorem 2.3.1. *Assume that b satisfies* (1.1.10)*. Then, there exists $T_0 \le T$ satisfying the following properties: for any $f \in L^{q,1}([0,T_0], L^p_x)$, there exists a unique solution $u \in X^{q,p}([0,T_0])$ to* (2.3.1) *for $0 \le t \le T_0$, and the estimate*

$$\|u\|_{X^{q,p}([0,T_0])} \le C \|f\|_{L^{q,1}([0,T_0],L^p_x)} \tag{2.3.2}$$

holds for some constant C depending only on $\|b\|_{L^{q,1}([0,T_0],L^p_x)}$.

The first step to establish this theorem is to obtain an a priori estimate for the $L^{q,1}([0,T],L^p_x)$-norm of the following heat equation:

$$\begin{cases} u_t - \frac{1}{2}\Delta u = f, \quad 0 \le t \le T, \\ u_0 = 0. \end{cases} \tag{2.3.3}$$

Proposition 2.3.2. *For any $p,q \in (1,\infty)$ and $f \in L^{q,1}([0,T],L^p_x)$, there exists a unique solution $u \in X^{q,p}([0,T])$ to PDE (2.3.3). Also, there exists some constant $C = C(p,q)$ independent of T such that for any $f \in L^{q,1}([0,T],L^p_x)$,*

$$\left\|\nabla^2 u\right\|_{L^{q,1}([0,T],L^p_x)} \le C \left\|f\right\|_{L^{q,1}([0,T],L^p_x)}, \tag{2.3.4}$$

$$\left\|u\right\|_{X^{q,p}([0,T])} \le C \max\{1,T\} \left\|f\right\|_{L^{q,1}([0,T],L^p_x)}. \tag{2.3.5}$$

Proof. Let us first prove the estimate (2.3.4). For $f \in C_c^\infty([0,T]\times\mathbb{R}^d)$, let us define

$$u(t) = \int_0^t T_{t-s} f(s) ds,$$

where T_t denotes the semigroup generated by $\frac{1}{2}\Delta$. Obviously, u is a classical solution to the heat equation. According to **[K2]**, for any $p,q \in (1,\infty)$, there exists some constant $C = C(p,q)$ independent of T such that for any $f \in L^q([0,T],L^p_x)$,

$$\left\|\nabla^2 u\right\|_{L^q([0,T],L^p_x)} \le C \left\|f\right\|_{L^q([0,T],L^p_x)}.$$

Note that $L^{q,1}_t(L^p_x)$ can be realized as a real interpolation space of two mixed-norm Lebesgue spaces: for $0 < \theta < 1$ satisfying

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

we have

$$[L^{q_1}([0,T],L^p_x), L^{q_2}([0,T],L^p_x)]_{\theta,1} = L^{q,1}([0,T],L^p_x),$$

Thus, we obtain the estimate (2.3.4). Also, using the estimate (2.3.4), for some constant $C_1 = C_1(p,q)$ independent of T,

$$\left\|u_t\right\|_{L^{q,1}([0,T],L^p_x)} \le C_1 \left\|f\right\|_{L^{q,1}([0,T],L^p_x)}.$$

Using Minkowski's integral inequality, Hölder's inequality and the trivial inequality

$$u(t,x) \le \int_0^T |u_t(s,x)| ds,$$

it follows that for some constant $C_2 = C_2(p,q)$ independent of T,

$$\|u\|_{L^{q,1}([0,T],L^p_x)} \leq C_2 T \|f\|_{L^{q,1}([0,T],L^p_x)}.$$

Furthermore, using the interpolation inequality

$$\|\nabla u\|_{L^p_x} \lesssim \|u\|_{L^p_x} + \left\|\nabla^2 u\right\|_{L^p_x}$$

and the aforementioned results, we readily obtain (2.3.5).

The existence of a solution $u \in X^{q,p}([0,T])$ to the heat equation (2.3.3) can be established via a standard approximation argument and the estimate (2.3.5). Uniqueness immediately follows from the estimate (2.3.5). □

In order to obtain (2.3.2) for the PDE (2.3.1), we need to control the first order term $\|b \cdot \nabla u\|_{L^{q,1}([0,T],L^p_x)}$. In order to accomplish, we need the embedding theorem for the mixed-norm parabolic Sobolev spaces. This type of Sobolev embedding theorem was obtained in **[K1]**: ∇u is bounded and Hölder continuous in (t,x) provided that

$$u_t, \nabla^2 u \in L^q([0,T], L^p_x)$$

for $1 < p, q < \infty$ satisfying the subcritical condition $\frac{2}{q} + \frac{d}{p} < 1$.

However, in general, ∇u may not be bounded under the critical condition $\frac{2}{q} + \frac{d}{p} = 1$. It turns out that when a slightly stronger Lorentz integrability condition is imposed on the time variable, the boundedness of ∇u can be established at the critical regime $\frac{2}{q} + \frac{d}{p} = 1$:

Proposition 2.3.3. *Suppose that $u \in X^{q,p}([0,T])$ with $u(0) = 0$, and the exponents $1 < p, q < \infty$ satisfy the condition:*

$$\frac{2}{q} + \frac{d}{p} = 1.$$

Then, we have $\nabla u \in L^\infty([0,T] \times \mathbb{R}^d)$. Also, there exists some constant $C = C(p,q)$ independent of T such that for any $u \in X^{q,p}([0,T])$,

$$\|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C(\|u_t\|_{L^{q,1}([0,T],L^p_x)} + \left\|\nabla^2 u\right\|_{L^{q,1}([0,T],L^p_x)}). \tag{2.3.6}$$

Proof. Let us define $f := u_t - \Delta u$. One can represent ∇u in terms of the heat kernel:

$$\nabla u(t,x) = \int_0^t \int_{\mathbb{R}^d} \nabla\left(\frac{1}{s^{d/2}} e^{-|y|^2/4s}\right) \cdot f(t-s, x-y) dy ds.$$

If we denote p', q' by the conjugate exponents of p, q, respectively, then according to Lemma 2.1.5, we have

$$\nabla(\frac{1}{t^{d/2}}e^{-|x|^2/4t}) \in L^{q',\infty}([0,T], L_x^{p'}).$$

Thus, using Proposition 2.1.6,

$$\begin{aligned}\|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^d)} &\le C \left\|\nabla(\frac{1}{t^{d/2}}e^{-|x|^2/4t})\right\|_{L^{q',\infty}([0,T],L_x^{p'})} \|f\|_{L^{q,1}([0,T],L_x^p)} \\ &\le C(p,q)(\|u_t\|_{L^{q,1}([0,T],L_x^p)} + \left\|\nabla^2 u\right\|_{L^{q,1}([0,T],L_x^p)}).\end{aligned}$$

Note that the estimate Proposition 2.1.6 is global in time, whereas the above inequality is integrated only over $[0,T]$. This subtle problem can be easily overcome by extending two functions $g(s,y) = \nabla(\frac{1}{s^{d/2}}e^{-|y|^2/4s})$ and $f(s,y)$ to the whole real line by setting $f, g = 0$ outside $[0,T]$. □

Now, we are ready to study the Kolmogorov PDE (2.3.1).

Proof of Theorem 2.3.1. We use a fixed point theorem for the quasi-Banach spaces to prove the existence of a solution. For $u \in X^{q,p}([0,T])$, we have $\nabla u \in L^\infty([0,T] \times \mathbb{R}^d)$ according to Proposition 2.3.3. Therefore, for $b, f \in L^{q,1}([0,T], L_x^p)$, we have $f + b \cdot \nabla u \in L^{q,1}([0,T], L_x^p)$. Using Proposition 2.3.2, define $w = F(u) \in X^{q,p}([0,T])$ to be a unique solution of the following PDE:

$$\begin{cases} w_t - \frac{1}{2}\Delta w = -(f + b\cdot\nabla u), & 0 \le t \le T, \\ w(0,x) = 0. \end{cases}$$

Using the estimates (2.3.5) and (2.3.6), for some constants C, C_1 independent of T,

$$\begin{aligned}\|F(u_1) - F(u_2)\|_{X^{q,p}([0,T])} &\le C\max\{1,T\} \|b\cdot\nabla(u_1-u_2)\|_{L^{q,1}([0,T],L_x^p)} \\ &\le C\max\{1,T\} \|b\|_{L^{q,1}([0,T],L_x^p)} \cdot \|\nabla(u_1-u_2)\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\ &\le C_1\max\{1,T\} \|b\|_{L^{q,1}([0,T],L_x^p)} \cdot \|(u_1-u_2)\|_{X^{q,p}([0,T])} \,.\end{aligned}$$

Let us denote $c = c(q,1) > 1$ by a constant from (2.1.4), and choose a sufficiently small T_0 satisfying

$$\|b\|_{L^{q,1}([0,T_0],L_x^p)} < \frac{1}{2cC_1\max\{1,T_0\}}.$$

Then, a map $F : X^{q,p}([0,T_0]) \to X^{q,p}([0,T_0])$ satisfies

$$|F(x) - F(y)| < \frac{1}{2c}|x-y|.$$

Therefore, applying a fixed point theorem for the quasi-Banach spaces, there exists $u \in X^{q,p}([0,T_0])$ satisfying PDE for $0 \le t \le T_0$.

Now, let us prove the estimate (2.3.2). Using (2.3.5) and (2.1.4), for some constants C, C_1,

$$\begin{aligned}\|u\|_{X^{q,p}([0,T_0])} &\le C \max\{1,T_0\} \|f + b\cdot\nabla u\|_{L^{q,1}([0,T_0],L^p_x)} \\ &\le C_1 \max\{1,T_0\}(\|f\|_{L^{q,1}([0,T_0],L^p_x)} + \|b\|_{L^{q,1}([0,T_0],L^p_x)} \|u\|_{X^{q,p}([0,T_0])}).\end{aligned}$$

Therefore, for sufficiently small T_0 satisfying

$$\|b\|_{L^{q,1}([0,T_0],L^p_x)} < \frac{1}{C_1 \max\{1,T_0\}}, \tag{2.3.7}$$

we obtain the estimate (2.3.2). Note that a constant C in (2.3.2) can be chosen depending only on $\|b\|_{L^{q,1}([0,T_0],L^p_x)}$. □

Remark 2.3.4. From the proof of Theorem 2.3.1, one can check that for any b with sufficiently small $\|b\|_{L^{q,1}([0,T],L^p_x)}$, there exists a unique solution u to PDE (2.3.1) for $0 \le t \le T$ satisfying:

$$\|u\|_{X^{q,p}([0,T])} \le C(\|b\|_{L^{q,1}([0,T],L^p_x)}, p, q) \|f\|_{L^{q,1}([0,T],L^p_x)}.$$

For these b's, one can easily derive a stability property of PDE (2.3.1). More precisely, there exist a constant C_0 depending on T satisfying the following statement: for any b_i and f_i, $i = 1,2$, satisfying

$$\|f_i\|_{L^{q,1}([0,T],L^p_x)}, \|b_i\|_{L^{q,1}([0,T],L^p_x)} < C_0,$$

define u_i to be a solution to PDE (2.3.1) with b_i and f_i in place of b and f, respectively. Then, for some constant $\bar{C} > 1$ depending on C_0,

$$\begin{aligned}&\|u_1 - u_2\|_{X^{q,p}([0,T])}, \|u_1 - u_2\|_{L^\infty([0,T]\times\mathbb{R}^d)}, \|\nabla(u_1 - u_2)\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\ &\qquad \le \frac{\bar{C}}{2}(\|b_1 - b_2\|_{L^{q,1}([0,T],L^p_x)} + \|f_1 - f_2\|_{L^{q,1}([0,T],L^p_x)}).\end{aligned} \tag{2.3.8}$$

In particular, when $f_i = b_i$, the RHS of (2.3.8) can be written as $\bar{C}\|b_1 - b_2\|_{L^{q,1}([0,T],L^p_x)}$.

Assume that b satisfies (1.1.10), and T_0 is from Theorem 2.3.1. According to Theorem 2.3.1, there exists a unique solution $\tilde{u} \in X^{q,p}([0,T_0])$ to the following PDE:

$$\begin{cases} u_t + \frac{1}{2}\Delta u + b\cdot\nabla u + b = 0, & 0 \le t \le T_0, \\ u(T_0,x) = 0. \end{cases} \tag{2.3.9}$$

The following proposition plays an essential role in Section 2.4.

Proposition 2.3.5. *There exists a sufficiently small T_1 such that the following holds: if $\tilde{u}$ is a solution to* (2.3.9) *with T_1 in place of T_0, then there exists a version u of $\tilde{u}$, which is continuous in (t,x), such that $\Phi(t,x) := x + u(t,x)$ satisfies the following conditions:*
(i) $\Phi(t,\cdot)$ is a C^1 diffeomorphism from $\mathbb{R}^d$ to itself for each $0 \le t \le T_1$.
(ii) For each $0 \le t \le T_1$,

$$\frac{1}{2} \le \|\nabla\Phi(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le 2, \quad \frac{1}{2} \le \left\|\nabla\Phi^{-1}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^d)} \le 2.$$

Here, we say u_1 is a version of u_2 if $u_1 = u_2$ for (t,x)-a.e.

Proof. Let us first prove that there exist a version u of $\tilde{u}$ which is C^1 in x. Choose a smooth approximation u_n of $\tilde{u}$ in $X^{q,p}([0,T_0])$ norm. Using Proposition 2.3.3,

$$\|\nabla(u_n - u_m)\|_{L^\infty_{t,x}([0,T_0]\times\mathbb{R}^d)} \le C\,\|u_n - u_m\|_{X^{q,p}([0,T_0])}.$$

Therefore, ∇u_n converge uniformly to some continuous function w. Since u_n converge uniformly to some continuous function u which is a version of $\tilde{u}$, u is differentiable in x and its spatial derivative is w. Since w is continuous, u is C^1 in x.

Now, let us show that for sufficiently small T_1, $\nabla\Phi(t,x)$ is non-singular for each $0 \le t \le T_1$. Note that using the estimates (2.3.5) and (2.3.6), for some constants C, C_1, C_2 independent of T,

$$\begin{aligned}\|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^d)} &\le C\,\|u\|_{X^{q,p}([0,T])}\\ &\le C_1\max\{1,T\}\,\|b\cdot\nabla u + b\|_{L^{q,1}([0,T],L^p_x)}\\ &\le C_2\max\{1,T\}(\|b\|_{L^{q,1}([0,T],L^p_x)}\,\|\nabla u\|_{L^\infty([0,T]\times\mathbb{R}^d)} + \|b\|_{L^{q,1}([0,T],L^p_x)}).\end{aligned}$$

Therefore, if we choose sufficiently small T_1 so that $\|b\|_{L^{q,1}([0,T_1],L^p_x)}$ is small enough, then

$$\|\nabla u\|_{L^\infty([0,T_1]\times\mathbb{R}^d)} \le \frac{1}{2}.$$

This immediately implies the first inequality in the condition (ii). From this, we obtain the non-singularity of $\nabla\Phi(t,\cdot)$, and $\lim_{|x|\to\infty}|\Phi(t,x)| = \infty$ for each $t \in [0,T_1]$. Therefore, according to the Hadamard's Lemma, $\Phi(t,\cdot)$ is a global diffeomorphism for each $t \in [0,T_1]$, which concludes the proof of the first property.

The second inequality in (ii) follows from the identity

$$\nabla\Phi^{-1}(t,x) = [\nabla\Phi(t,\Phi^{-1}(t,x))]^{-1} = [I + \nabla u(t,\Phi^{-1}(t,x))]^{-1},$$

and the fact

$$\sup_{t\in[0,T_1]} \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{2}.$$

□

Throughout this thesis, we use the notations $u(t,x)$, $\Phi(t,x)$, and T_1 from the above Proposition.

2.4 Uniqueness

In this section, we prove the uniqueness of a strong solution to SDE (1.1.2) up to time T_1. The following proposition claims that a strong solution to (1.1.2) yields a new strong solution to the auxiliary SDE which contains no drift terms. It is called the Zvonkin's transformation method [**Z3**].

Proposition 2.4.1. *Suppose that b satisfies* (1.1.10)*, and X_t is a strong solution to SDE* (1.1.2) *up to time T_1. Then, Y_t defined by $Y_t = \Phi(t, X_t)$ is a strong solution to the following SDE:*

$$\begin{cases} dY_t = \tilde{\sigma}(t, Y_t)dB_t, \quad 0 \leq t \leq T_1, \\ Y_0 = \Phi(0, x) = y, \end{cases} \tag{2.4.1}$$

for $\tilde{\sigma}$ defined by

$$\tilde{\sigma}(t, x) = I + \nabla u(t, \Phi^{-1}(t, x)). \tag{2.4.2}$$

Proof. One can check that the standard Itô's formula

$$f(t, X_t) - f(0, X_0) = \int_0^t (f_t + b\nabla f + \frac{1}{2}\Delta f)(s, X_s)ds + \int_0^t \nabla f(s, X_s)dB_s$$

holds for any functions $f \in X^{q,p}([0,T])$ with p, q satisfying $\frac{2}{q} + \frac{d}{p} = 1$. Thus, applying Itô's formula to a function u, we have

$$\begin{aligned} u(t, X_t) &= u(0, X_0) + \int_0^t (u_t + b \cdot \nabla u + \frac{1}{2}\Delta u)(s, X_s)ds + \int_0^t \nabla u(s, X_s)dB_s \\ &= u(0, X_0) - \int_0^t b(s, X_s)ds + \int_0^t \nabla u(s, X_s)dB_s \\ &= u(0, X_0) - X_t + X_0 + B_t + \int_0^t \nabla u(s, X_s)dB_s. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} Y_t - Y_0 &= \Phi(t, X_t) - \Phi(t, X_0) \\ &= \int_0^t \nabla u(s, X_s) dB_s + B_t \\ &= \int_0^t \nabla u(s, \Phi^{-1}(s, Y_s)) dB_s + B_t. \end{aligned}$$

□

Let us call SDE (2.4.1) by a *conjugated SDE*. Before proving the strong uniqueness of SDE (1.1.2), we prove the following two lemmas which will be used frequently.

Lemma 2.4.2. *For any* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* b_1, b_2 *satisfying the condition*

$$b_1, b_2 \in L^{q,1}([0,T], L^p_x) \quad \textit{for} \quad \frac{2}{q} + \frac{d}{p} = 1,\ 1 < p, q < \infty,$$

we have

$$\sup_x \mathbb{E} \exp \Big[\lambda_1 \int_0^T b_1(s, B^x_s) dB^x_s + \lambda_2 \int_0^T b_2^2(s, B^x_s) ds \Big] < \infty. \tag{2.4.3}$$

Proof. Let us first briefly recall the basic property of Doléans-Dade exponential. Recall that Doléans-Dade exponential $\mathscr{E}(M)_t$ of a semimartingale M_t is defined to be the solution Z_t to the stochastic differential equation

$$dZ_t = Z_t dM_t$$

with initial condition $Z_0 = 1$. Applying Itô formula to the function $f(Z) = \log Z$, we have

$$d \log Z_t = \frac{1}{Z_t} dZ_t - \frac{1}{2Z_t^2} d[Z]_t = dM_t - \frac{1}{2} d[M]_t.$$

Taking exponential, we obtain

$$Z_t = \exp(M_t - M_0 - \frac{1}{2}[M]_t).$$

In particular, when M_t is given by

$$M_t = \int_0^t N_t dB_t,$$

Doléans-Dade exponential of M_t is given by

$$\mathscr{E}(M)_t = \exp(\int_0^t N_s dB_s - \frac{1}{2}\int_0^t N_s^2 ds).$$

The crucial property of Doléans-Dade exponential is that $\mathscr{E}(M)_t$ is a local martingale if M_t is a local martingale.

Now, let us prove the lemma. By Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\exp\Big[\lambda_1 \int_0^T b_1(s, B_s^x)dB_s^x + \lambda_2 \int_0^T b_2^2(s, B_s^x)ds\Big] \\
&\leq \Big[\mathbb{E}\mathscr{E}\Big[\int_0^T 2\lambda_1 b_1(s, B_s^x)dB_s^x\Big]\Big]^{1/2}\Big[\mathbb{E}\exp\Big[\int_0^T (2\lambda_1^2 b_1^2 + 2\lambda_2 b_2^2)(s, B_s^x)ds\Big]\Big]^{1/2}.
\end{aligned} \tag{2.4.4}$$

Since $b_1, b_2 \in L^{q,1}([0,T], L_x^p)$, it follows that $b_1^2, b_2^2 \in L^{q/2,1/2}([0,T], L_x^{p/2})$. Letting $\tilde{q} = \frac{q}{2}$ and $\tilde{p} = \frac{p}{2}$, we have $b_1^2, b_2^2 \in L^{\tilde{q},1}([0,T], L_x^{\tilde{p}})$ with $\frac{2}{\tilde{q}} + \frac{d}{\tilde{p}} = 2$. Therefore, the second term of (2.4.4) is finite according to Proposition 2.2.3. The first term of (2.4.4) is equal to 1 since Novikov's condition is satisfied. □

Lemma 2.4.3. *Let X_t be a solution to SDE* (1.1.12) *with b satisfying the condition* (1.1.10)*. Then, for arbitrary $\lambda \in \mathbb{R}$ and $f \in L^{q,1}([0,T], L_x^p)$,*

$$\sup_x \mathbb{E}\exp\Big[\lambda \int_0^T f^2(s, X_s)ds\Big] < \infty. \tag{2.4.5}$$

Proof. By Girsanov formula, LHS of (2.4.5) equals to

$$\sup_x \mathbb{E}\Big[\exp\Big[\lambda\int_0^T f^2(s, B_s^x)ds\Big]\cdot \exp\Big[\int_0^T b(s, B_s^x)dB_s^x - \frac{1}{2}\int_0^T b^2(s, B_s^x)ds\Big]\Big].$$

Since both b and f belong to $L^{q,1}([0,T], L_x^p)$ with $\frac{2}{q} + \frac{d}{p} = 1$, Hölder's inequality and Lemma 2.4.2 conclude the proof. In fact, by (2.4.4),

$$\mathbb{E}\exp\Big[\lambda\int_0^T f^2(s, X_s)ds\Big] \leq \Big[\mathbb{E}\exp\Big[\int_0^T (b^2 + 2\lambda f^2)(s, B_s^x)ds\Big]\Big]^{1/2}. \tag{2.4.6}$$

□

Remark 2.4.4. It is proved in [**FF3**, **F**] that under the subcritical condition (1.1.3), quantities (2.4.3) and (2.4.5) can be controlled by $\|b\|_{L^q([0,T],L_x^p)}$. At the Orlicz-critical regime (1.1.10), these quantities can be controlled by $\|b\|_{L^{q,1}([0,T],L_x^p)}$ in some weak sense. In fact, one can show that there exists a constant $K = K(p, q, \lambda_1, \lambda_2)$ and functions $C_1, C_2 : \mathbb{R} \to \mathbb{R}$ such that the following holds: for any f and b satisfying

$$\|f\|_{L^{q,1}([0,T],L^p_x)}, \|b\|_{L^{q,1}([0,T],L^p_x)} < K,$$

we have

$$\sup_x \mathbb{E} \exp\Big[\lambda_1 \int_0^T b(s, B^x_s) dB^x_s + \lambda_2 \int_0^T b^2(s, B^x_s) ds\Big] \le C_1(K),$$

and

$$\sup_x \mathbb{E} \exp\Big[\lambda_1 \int_0^T f(s, X_s) dB^x_s + \lambda_2 \int_0^T f^2(s, X_s) ds\Big] \le C_2(K). \tag{2.4.7}$$

If we denote X^μ_t by a solution to SDE (1.1.2) with the initial distribution μ, then (2.4.7) implies that

$$\sup_\mu \mathbb{E} \exp\Big[\lambda_1 \int_0^T f(s, X^\mu_s) dB^x_s + \lambda_2 \int_0^T f^2(s, X^\mu_s) ds\Big] \le C_2(K) \tag{2.4.8}$$

(sup takes over all of the probability measures on $\mathbb{R}^d$). This is because if we denote P_x by a law of $\{X_t \mid 0 \le t \le T\}$ which is a solution of (1.1.2) starting from x, then $P_\mu = \int P_x d\mu(x)$ is a law of $\{X^\mu_t \mid 0 \le t \le T\}$.

Also, by letting $\lambda_1 = 0$ and $\lambda_2 = 1$ in Lemma 3.1.7 and using the inequality $1 + x \le e^x$, one can conclude that there exists a function $C : \mathbb{R} \to \mathbb{R}$ such that for any f and b satisfying

$$\|f\|_{L^{q,1}([0,T],L^p_x)}, \|b\|_{L^{q,1}([0,T],L^p_x)} < K(p, q, 0, 1),$$

we have

$$\sup_x \mathbb{E} \int_0^T f^2(s, X_s) ds < C(K).$$

Now, we are ready to prove the strong uniqueness of SDE (1.1.2) under the condition (1.1.10). Proof follows the argument in **[FF3]**.

Proposition 2.4.5. *A strong solution to SDE* (1.1.2) *is unique up to* T_1.

Proof. Let X^1_t and X^2_t be strong solutions to SDE (1.1.2) starting from x^1 and x^2, respectively. According to Proposition 2.4.1, if we define $Y^i_t = \Phi(t, X^i_t)$, then Y^i_t is a solution to the conjugated SDE (2.4.1) starting from $y^i = \Phi(0, x^i)$, respectively. Thus, we have

$$d(Y^1_s - Y^2_s) = [\tilde{\sigma}(s, Y^1_s) - \tilde{\sigma}(s, Y^2_s)] dB_s. \tag{2.4.9}$$

For any $r \in (1,\infty)$, using Itô's formula,

$$
\begin{aligned}
&d|Y_s^1 - Y_s^2|^r \\
&= \frac{r(r-1)}{2}\mathrm{Trace}\big([\tilde\sigma(s,Y_s^1) - \tilde\sigma(s,Y_s^2)][\tilde\sigma(s,Y_s^1) - \tilde\sigma(s,Y_s^2)]^T\big)|Y_s^1 - Y_s^2|^{r-2}ds + dM_s \\
&\le \frac{r(r-1)}{2}|\tilde\sigma(s,Y_s^1) - \tilde\sigma(s,Y_s^2)|^2|Y_s^1 - Y_s^2|^{r-2}ds + dM_s \\
&= |Y_s^1 - Y_s^2|^r dA_s + dM_s
\end{aligned}
$$

for some martingale M_s with zero mean (the martingale property can be checked as in [**F1**]). Here, we introduced an auxiliary process A_t ($0 \le t \le T_1$) satisfying

$$
\frac{r(r-1)}{2}\int_0^t |\tilde\sigma(s,Y_s^1) - \tilde\sigma(s,Y_s^2)|^2 ds = \int_0^t |Y_s^1 - Y_s^2|^2 dA_s, \tag{2.4.10}
$$

and for any $c > 0$,

$$
\mathbb{E}\, e^{cA_t} < \infty \tag{2.4.11}
$$

Thus, applying the product rule,

$$
d(e^{-A_s}|Y_s^1 - Y_s^2|^r) = -e^{-A_s}|Y_s^1 - Y_s^2|^r dA_s + e^{-A_s}d|Y_s^1 - Y_s^2|^r \le e^{-A_s}dM_s.
$$

Integrating this inequality in time and then taking the expectation, we have

$$
\mathbb{E}[e^{-A_t}|Y_t^1 - Y_t^2|^r] \le |y^1 - y^2|^r.
$$

Therefore, using Hölder's inequality,

$$
\begin{aligned}
\mathbb{E}\,|Y_t^1 - Y_t^2|^{r/2} &= \mathbb{E}\, e^{\frac{-A_t}{2}}|Y_t^1 - Y_t^2|^{r/2}e^{\frac{A_t}{2}} \\
&\le [\mathbb{E}\, e^{-A_t}|Y_t^1 - Y_t^2|^r]^{1/2}[\mathbb{E}\, e^{A_t}]^{1/2} \\
&\le |y^1 - y^2|^{r/2}[\mathbb{E}\, e^{A_t}]^{1/2},
\end{aligned}
$$

which implies that for each $t \in [0, T_1]$,

$$
\mathbb{E}\,|Y_t^1 - Y_t^2|^{r/2} \le C|y^1 - y^2|^{r/2}. \tag{2.4.12}
$$

In particular, when $x^1 = x^2$, we have $\mathbb{E}\,|Y_t^1 - Y_t^2|^{r/2} = 0$. Since trajectories are continuous and $\Phi(t,\cdot)$ is bijective, we obtain the strong uniqueness of SDE (1.1.2). □

Theorem 2.4.6. *Existence and uniqueness of a strong solution to SDE* (1.1.2) *holds up to time* T_1.

Proof. We have already proved the weak existence and the strong uniqueness. Therefore, according to the Yamabe-Watanabe principle [**YW1**, **YW2**], we obtain the existence and uniqueness of a strong solution to SDE (1.1.2) up to time T_1. □

In the next section, we construct a strong solution to SDE (1.1.2) up to time T.

2.5 Construction of stochastic flow

In this section, we conclude the proof of main theorem, and construct a stochastic flow associated with SDE. Let us first define a stochastic flow.

Definition 2.5.1. (Stochastic flow). A map $(s,t,x,w) \to \phi(s,t,x)(w)$, $0 \le s \le t \le T$ is called a *stochastic flow* associated to the stochastic differential equation (1.1.2) on the filtered space with a Brownian motion $(\Omega, \mathcal{F}, \mathcal{F}_t, P, B_t)$ provided that it satisfies:
(i) For any $x \in \mathbb{R}^d$ and $0 \le s \le T$, the process $X^s_{t,x} = \phi(s,t,x)$ for $s \le t \le T$ is a $\mathcal{F}_{s,t}$-adapted solution to SDE (1.1.2). Here,

$$\mathcal{F}_{s,t} := \sigma(B_u - B_r | s \le r \le u \le t).$$

(ii) w-almost surely, $\phi(s,t,x) = \phi(u,t,\phi(s,u,x))$ holds for any $0 \le s \le u \le t \le T$ and $x \in \mathbb{R}^d$.

We refer to **[K3]** for the classical theory of stochastic flows. This classical theory has been extended to a large class of SDEs with singular coefficients. For instance, Flandoli et al. **[FGP]** constructed a regular stochastic flow when the SDE with additive noise possess a low Hölder regularity of drift.

In this section, we prove that a stochastic flow associated with SDE (1.1.2) exists under the Orlicz-critical condition (1.1.10).

Theorem 2.5.2. *There exists a stochastic flow ϕ to* (1.1.2) *up to time T.*

The main ingredient to prove Theorem 2.5.2 is the Kolmogorov regularity theorem. Thanks to Proposition 2.4.1, there exists a strong solution Y^y_t, $0 \le t \le T_1$, to (2.4.1). We first prove the Hölder regularity of Y^y_t.

Proposition 2.5.3. *There exists some constant C such that for any $1 \le r < \infty$, $0 \le t < s \le T_1$, and $x, y \in \mathbb{R}^d$,*

$$\mathbb{E}\,|Y^x_t - Y^x_s|^r \le C|t-s|^{\frac{r}{2}}, \quad \mathbb{E}\,|Y^x_t - Y^y_t|^r \le C|x-y|^r.$$

Proof. The proof heavily uses Burkholder-Davis-Gundy inequality, so we briefly recall it. Burkholder-Davis-Gundy inequality provides the bound for the maximum of a martingale in terms of the quadratic variation. More precisely, if M_t is a local martingale with $M_0 = 0$, then for any $p \ge 1$,

$$\mathbb{E}(\sup_{0 \le t \le T} M_t)^p \approx \mathbb{E}[M]^{p/2}_T.$$

Now, using Burkholder-Davis-Gundy inequality, let us prove the first inequality. Applying Burkholder-Davis-Gundy inequality and using the fact that $\|\nabla u\|_{L^\infty([0,T_1]\times\mathbb{R}^d)}$ is finite, one can conclude that

$$\begin{aligned}\mathbb{E}\,|Y_t^y - Y_s^y|^r &= \mathbb{E}\,|\int_s^t (I + \nabla u(\sigma, \Phi^{-1}(r, Y_\sigma^x)))dB_\sigma|^r \\ &\le C\,\mathbb{E}\,|\int_s^t |I + \nabla u(\sigma, \Phi^{-1}(\sigma, Y_\sigma^x))|^2 d\sigma|^{\frac{r}{2}} \\ &\le C|t-s|^{\frac{r}{2}}.\end{aligned}$$

We have already obtained the second inequality in (2.4.12). □

Now, one can prove Theorem 2.5.2 by applying Kolmogorov regularity theorem. We briefly review Kolmogorov regularity theorem, which states that the stochastic process for which the moments of its increments are well-controlled possesses a certain regularity. More precisely, if a stochastic process Z_t satisfies

$$\mathbb{E}|Z_s - Z_t|^\alpha \le C|s-t|^{1+\beta}$$

for some $\alpha, \beta > 0$, then there exists a modification $\tilde{Z}$ of Z which is γ-Hölder continuous for every $0 < \gamma < \frac{\beta}{\alpha}$.

Proof of Theorem 2.5.2. Since both Φ and Φ^{-1} are continuous in (t,x), we first prove the same statement for the conjugated SDE (2.4.1). Thanks to Kolmogorov regularity theorem, one can construct a stochastic flow ψ associated with SDE (2.4.1) up to time T_1, which is a version of Y_t^y, satisfying the following property: almost surely, $\psi(s,\cdot,\cdot)$ is (α,β)-Hölder continuous for each $0 \le s \le T_1$ and any $0 < \alpha < \frac{1}{2}$, $0 < \beta < 1$.

In order to construct a stochastic flow of SDE (1.1.2), let us define

$$\phi(s,t,x) := \Phi^{-1}(t, \psi(s,t,\Phi(s,x)))$$

for $0 \le s \le t \le T_1$. It is obvious that ϕ is a stochastic flow associated with (1.1.2) up to time T_1, and almost surely, $\phi(s,\cdot,\cdot)$ is continuous for each $0 \le s \le T_1$.

Now, we extend this construction globally up to time T. Divide $[0,T]$ into the finite number of intervals $[T_{k-1}, T_k]$, $1 \le k \le N$, such that the stochastic flow ϕ of SDE (1.1.2) on each $[T_{k-1}, T_k]$ can be constructed. More precisely, we take a sufficiently small interval $[T_{k-1}, T_k]$ such that the following property holds: if u^k is a solution to PDE

$$\begin{cases} u_t^k + \frac{1}{2}\Delta u^k + b\cdot\nabla u^k + b = 0, & T_{k-1} \le t \le T_k, \\ u^k(T_k, x) = 0, \end{cases} \tag{2.5.1}$$

then u^k satisfies the conditions in Proposition 3.1.4. In other words, $\Phi^k(t,x) = x + u^k(t,x)$ is a global diffeomorphism for each $T_{k-1} \leq t \leq T_k$ and

$$\frac{1}{2} < \left\|\nabla\Phi^k(t,x)\right\|_{L^\infty([T_{k-1},T_k]\times\mathbb{R}^d)}, \left\|\nabla^{-1}\Phi^k(t,x)\right\|_{L^\infty([T_{k-1},T_k]\times\mathbb{R}^d)} < 2. \tag{2.5.2}$$

Repeating the arguments mentioned before, one can construct a stochastic flow $\phi(s,t,x)$ associated with SDE (1.1.2) for $T_{k-1} \leq s \leq t \leq T_k$. Then, we can glue them together as follows: for each $0 \leq s \leq t \leq T$, choose the indices i and j satisfying

$$T_{i-1} \leq s < T_i < \cdots < T_j < t \leq T_{j+1},$$

and then define

$$\phi(s,t,\cdot) = \phi(T_j,t,\cdot) \circ \phi(T_{j-1},T_j,\cdot) \circ \cdots \circ \phi(s,T_i,\cdot). \tag{2.5.3}$$

Here, composition happens in the spatial variable. It is obvious that ϕ satisfies the properties of the stochastic flow. □

Chapter 3

Singular stochastic differential equations with hypoelliptic diffusion

3.1 Analysis on the nilpotent Lie group

In this section, we briefly overview the theory of analysis on the homogeneous Carnot group.

Preliminaries : homogeneous Carnot group

Definition 3.1.1. We say that $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$, endowed with a Lie group structure by the composition law $\circ$, is called a *homogeneous group* if it is equipped with a one parameter family $\{D(\lambda)\}_{\lambda>0}$ of automorphisms of the following form

$$D(\lambda) : (u_1, u_2, \cdots, u_N) \mapsto (\lambda^{\alpha_1} u_1, \lambda^{\alpha_2} u_2, \cdots, \lambda^{\alpha_N} u_N)$$

for some exponents $0 < \alpha_1 \leq \cdots \leq \alpha_N$. *Homogeneous dimension* Q of $\mathbb{G}$ is defined by

$$Q = \alpha_1 + \cdots + \alpha_N.$$

For $1 \leq i \leq N$, let Z_i be a left-invariant vector field which coincides with $\frac{\partial}{\partial x_i}$ at the origin. If Lie algebra generated by $Z_1, \cdots, Z_m$, which are 1-homogeneous left-invariant vector fields, is the whole Lie algebra $\mathfrak{g}$ of $\mathbb{G}$, then $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$ is called a *homogeneous Carnot group*.

If $\mathbb{G}$ is a homogeneous Carnot group, then its Lie algebra $\mathfrak{g}$ has a natural stratification. In fact, if $\alpha_j = 1$ for $j \leq m$ and

$$V_1 = \text{span}(Z_1, Z_2, \cdots, Z_m),$$

$$V_{i+1} = [V_i, V_1], \ i > 1,$$

then there exists r, which is called a *nilpotency* of $\mathbb{G}$, satisfying

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r.$$

We assume for a moment that $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$ is a homogeneous group with a homogeneous dimension Q. One can associate the *homogeneous norm* $\|\cdot\| : \mathbb{G} \to \mathbb{R}$ to $\mathbb{G}$, smooth away from the origin, satisfying

$$\|u\| \geq 0, \quad \|u\| = 0 \Longleftrightarrow u = 0, \quad \|D(\lambda)u\| = \lambda \|u\|.$$

If we denote $|\cdot|$ by a Euclidean norm, then it satisfies

$$|x| = \mathcal{O}(\|x\|)$$

as $x \to 0$. The Lebesgue measure on $\mathbb{G} = \mathbb{R}^N$ is a bi-invariant haar measure, and once we make a change of coordinate

$$x = D(\lambda)y,$$

we have

$$dx = \lambda^Q dy.$$

This implies that the homogeneous group $\mathbb{G}$ can be regarded as a homogeneous space in the sense of Coifman and Weiss [**CW**]. This fact plays an important rule in developing a singular integral theory on the homogeneous group.

Function spaces on the homogeneous Carnot group

In this section, we introduce function spaces on the homogeneous Carnot group. First, let us define the kernels of type α and the operators of type α. A function f is said to be *homogeneous of degree* α provided that for $\lambda > 0$,

$$f(D(\lambda)x) = \lambda^\alpha f(x).$$

Definition 3.1.2. [**F3**] K is called a *kernel of type* α ($\alpha > 0$) if it is smooth away from the origin and homogeneous of degree $\alpha - Q$. Also, K is called a *singular integral kernel* if it is smooth away from the origin, homogeneous of degree $-Q$, and satisfies

$$\int_{a<|x|<b} K(x)dx = 0$$

for any $0 < a < b < \infty$. T is called the *operator of type* α $(0 \leq \alpha < Q)$ if T is given by

$$T : f \to f * K$$

for some kernel K of type α. In the case $\alpha = 0$, convolution is understood as a principal value sense.

From now on, we assume that $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$ is a homogeneous Carnot group with a homogeneous dimension Q and nilpotency r. Recall that $Z_1, \cdots, Z_m$ is a (linear) basis of V_1, and define the *sub-Laplacian* L by

$$L = Z_1^2 + \cdots + Z_m^2.$$

According to the result by Folland **[F2]**, there exists a fundamental solution of $\frac{\partial}{\partial t} - L$, which is called the heat kernel. It turns out that the heat kernel p possesses a nice Gaussian upper bound (see **[J]**).

Theorem 3.1.3. *For any* $k \geq 0$ *and indices* $I = (i_1, \cdots, i_s)$ *with* $|I| = s \geq 0$,

$$|\partial_t^k Z_I p(t,x)| \leq C(k,I) t^{-k-\frac{s+Q}{2}} e^{-c\|x\|^2/t} \tag{3.1.1}$$

holds for some constant c *independent of* $k, s, i_1, \cdots, i_s$.

Let us now define Sobolev spaces $S^{k,p}(\mathbb{G})$ associated with vector fields $\{Z_i | 1 \leq i \leq m\}$:

$$S^{k,p}(\mathbb{G}) := \{f | Z_I f \in L^p(\mathbb{G}), |I| \leq k\},$$

and the associated norm $\|\cdot\|_{S^{k,p}}$ by

$$\|f\|_{S^{k,p}} = \sum_{|I| \leq k} \|Z_I f\|_{L^p}.$$

Note that $Z_I f$ is understood as a distributional sense. Like the standard Sobolev embedding theorems in the Euclidean spaces, $S^{k,p}(\mathbb{G})$ enjoy the embedding theorems as well. Let us define Lipschitz spaces $\Gamma^\alpha(\mathbb{G})$ as follows: for $0 < \alpha < 1$,

$$\Gamma^\alpha(\mathbb{G}) := \Big\{ f \in C_b(\mathbb{G}) \Big| \sup_{x,y \in \mathbb{G}} \frac{|f(x \circ y) - f(x)|}{\|y\|^\alpha} < \infty \Big\},$$

$$\Gamma^1(\mathbb{G}) := \Big\{ f \in C_b(\mathbb{G}) \Big| \sup_{x,y \in \mathbb{G}} \frac{|f(x \circ y) + f(x \circ y^{-1}) - 2f(x)|}{\|y\|} < \infty \Big\}.$$

For $\alpha = n + \alpha'$ with a nonnegative integer n and $0 < \alpha' \leq 1$, define $\Gamma^\alpha(\mathbb{G})$ by

$$\Gamma^\alpha(\mathbb{G}) := \{f \in \Gamma^{\alpha'}(\mathbb{G}) \mid X_I f \in \Gamma^{\alpha'}(\mathbb{G}) \text{ for } |I| \leq n\}.$$

We state the Sobolev embedding theorem for the spaces $S^{k,p}(\mathbb{G})$:

Theorem 3.1.4. *[**F3**] Suppose that $l \le k$. Then, the space $S^{k,p}(\mathbb{G})$ is continuously embedded into $S^{l,q}(\mathbb{G})$ for $1 < p < q < \infty$ satisfying*

$$k - l = Q(\frac{1}{p} - \frac{1}{q}).$$

Also, the space $S^{k,p}(\mathbb{G})$ is continuously embedded into $\Gamma^{\alpha}(\mathbb{G})$ for

$$\alpha = k - \frac{Q}{p} > 0.$$

Remark 3.1.5. Let us define a different version of Lipschitz spaces $\tilde{\Gamma}^{\alpha}(\mathbb{G})$ $(0 < \alpha < 1)$:

$$\tilde{\Gamma}^{\alpha}(\mathbb{G}) := \Big\{ f \in C_b(\mathbb{G}) \Big| \sup_{x,y \in \mathbb{G}} \frac{|f(y \circ x) - f(x)|}{\|y\|^{\alpha}} < \infty \Big\}.$$

Then, it is not hard to check that $\Gamma^{\alpha}_{\text{loc}}(\mathbb{G}) \subset \tilde{\Gamma}^{\alpha/r}_{\text{loc}}(\mathbb{G})$. In fact, if we define the Euclidean Lipschitz spaces $\Lambda^{\alpha}(\mathbb{G})$ $(0 < \alpha < 1)$:

$$\Lambda^{\alpha}(\mathbb{G}) := \Big\{ f \in C_b(\mathbb{G}) \Big| \sup_{x,y \in \mathbb{G}} \frac{|f(x+y) - f(x)|}{|y|^{\alpha}} < \infty \Big\},$$

then according to [**F3**],

$$\Gamma^{\alpha}_{\text{loc}}(\mathbb{G}) \subset \Lambda^{\alpha/r}_{\text{loc}}(\mathbb{G}).$$

Also, from the fact $|x| = \mathcal{O}(\|x\|)$, we can deduce that

$$\Lambda^{\alpha/r}_{\text{loc}}(\mathbb{G}) \subset \tilde{\Gamma}^{\alpha/r}_{\text{loc}}(\mathbb{G}).$$

Thus, we obtain

$$\Gamma^{\alpha}_{\text{loc}}(\mathbb{G}) \subset \tilde{\Gamma}^{\alpha/r}_{\text{loc}}(\mathbb{G}).$$

Analysis on the homogeneous Carnot group

In this section, a basic theory on the analysis on the homogeneous Carnot group will be introduced. Recall that $Z_1, \cdots, Z_N$ are left-invariant vector fields. Now, let us denote Z_i^R $(1 \le i \le N)$ by a right-invariant vector field which coincides with $\frac{\partial}{\partial x_i}$ at the origin. Then, we have

$$(Z_i f) * g = f * (Z_i^R g).$$

Using the fact that the lie algebra of $Z_1, \cdots, Z_m$ generates the whole tangent space, there exist homogeneous functions β_{ji} of degree $\alpha_j - 1$ $(1 \leq i \leq m, 1 \leq j \leq N)$ such that the following holds: for each $1 \leq i \leq m$ and any test functions u,

$$Z_i u = \sum_{j=1}^{N} Z_j^R(\beta_{ji} u). \tag{3.1.2}$$

It is also obvious that for any $1 \leq i, j \leq N$,

$$[Z_i^R, Z_j^R] = -[Z_i, Z_j]^R. \tag{3.1.3}$$

The properties (3.1.2) and (3.1.3) will be crucially used to obtain the higher order estimate for PDE on the homogeneous Carnot group.

Calderón-Zygmund theory

In this section, we briefly review the Calderón-Zygmund theory. A singular integral of convolution type is an operator T defined by convolution with a kernel K that is locally integrable on $\mathbb{R}^n \backslash \{0\}$:

$$T(f)(x) = \lim_{\epsilon \to 0} \int_{|y-x|>\epsilon} K(x-y) f(y) dy$$

Suppose that K satisfies the following conditions:

- The Fourier transform of K is essentially bounded.
- For some constants $c, C > 1$,

$$\sup_y \int_{|x|>c|y|} |K(x-y) - K(x)| dx \leq C$$

Then, T is bounded on L^p for any $p \in (1, \infty)$.

In practical, lots of kernels K satisfy the following cancellation property: for any $r < R$,

$$\int_{r<|x|<R} K(x) = 0.$$

It is known that if the second condition above and the cancellation property as well as the size condition

$$\sup_R \int_{R<|x|<2R} |K(x)| dx < C$$

are satisfied, then the first condition above holds.

There is a useful criteria to verify the second condition above. In fact, K satisfies the above second condition if K satisfies the following two conditions:

- $K \in C^1(\mathbb{R}^n \backslash \{0\})$
- $|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}$.

Examples that enjoy the above property include the Hilbert and Riesz transforms.

Finally, it is known that if T is already bounded on L^p for some $p \in (1, \infty)$ and K satisfies the second condition above, then T is bounded on L^q for any $q \in (1, \infty)$. Note that similar results also hold if K is vector valued.

Mixed-norm parabolic Sobolev spaces

In this section, we define the mixed-norm parabolic Sobolev spaces with respect to vector fields $\{Z_i | 1 \leq i \leq m\}$ and study their properties. Note that if $(\mathbb{G}, \circ, D(\lambda))$ is a homogeneous group with a homogeneous dimension Q and a homogeneous norm $\|\cdot\|$, then the homogeneous group structure on $\mathbb{R} \times \mathbb{G}$ can be endowed as follows:

$$(s, x)\tilde{\circ}(t, y) = (s + t, x \circ y)$$

and

$$\tilde{D}(\lambda)(t, x) = (\lambda^2 t, D(\lambda)x).$$

Also, the homogeneous dimension of $(\mathbb{R} \times \mathbb{G}, \tilde{\circ}, \tilde{D}(\lambda))$ is equal to $Q + 2$. In addition, the norm

$$\|(t, x)\| := \sqrt{|t| + \|x\|^2}$$

defines a homogeneous norm on $\mathbb{R} \times \mathbb{G}$.

Definition 3.1.6. For $1 \leq p, q \leq \infty$ and the integer $k \geq 0$, let us define (inhomogeneous) mixed-norm Sobolev spaces $S^{k,(q,p)}([0, T] \times \mathbb{G})$ with respect to vector fields $\{Z_i | 1 \leq i \leq m\}$:

$$S^{k,(q,p)}([0, T] \times \mathbb{G}) := \{f \in L^q([0, T], L^p(\mathbb{G})) \mid Z_I f \in L^q([0, T], L^p(\mathbb{G})), |I| \leq k\}.$$

The corresponding norm is defined by

$$\|f\|_{S^{k,(q,p)}([0,T] \times \mathbb{G})} := \sum_{|I| \leq k} \|Z_I f\|_{L^q([0,T], L^p(\mathbb{G}))}.$$

One can also define the homogeneous mixed-norm Sobolev spaces $\dot{S}^{k,(q,p)}([0,T]\times\mathbb{G})$ and the corresponding norm

$$\|f\|_{\dot{S}^{k,(q,p)}([0,T]\times\mathbb{G})} := \sum_{|I|=k} \|Z_I f\|_{L^q([0,T],L^p(\mathbb{G}))}$$

similarly.

From now on, we assume that $\mathbb{G} = (\mathbb{R}^N, \circ, D(\lambda))$ is a homogeneous Carnot group with a homogeneous dimension Q and nilpotency r. We first study the boundedness properties of the operators of type 0 in the mixed-norm spaces. It is the classical theory that the operators of type 0 are bounded on $L^p(\mathbb{R}\times\mathbb{G})$ for $1 < p < \infty$ (see [**F3**]). One can generalize this result to the mixed-norm spaces $L^q(\mathbb{R}, L^p(\mathbb{G}))$ using Calderón-Zygmund theory mentioned before.

Theorem 3.1.7. *Operators T of type 0 are bounded in $L^q(\mathbb{R}, L^p(\mathbb{G}))$ for any $1 < p, q < \infty$.*

Proof. Let us denote $K(t,x)$ by a singular integral kernel of T. For $t \in \mathbb{R}$, let us define the operator $P_t : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ by

$$P_t f(x) = \int_{\mathbb{G}} K(t, x\circ y^{-1}) f(y) dy.$$

Then, Tf can be written as

$$Tf(t,x) = \int_{\mathbb{R}} P_s f(t-s, \cdot)(x) ds.$$

Since the operators of type 0 are bounded in L^p for $1 < p < \infty$ according to [**F3**], T is bounded in $L^p(\mathbb{R}, L^p(\mathbb{G}))$. In order to extend this to the general cases $q \neq p$, it suffices to prove the following inequality: for some constant $c > 0$ independent of s,

$$\int_{|t|\geq c|s|} \|P_t - P_{t-s}\|_{L^p\to L^p}\, dt \leq C < \infty, \tag{3.1.4}$$

according to the aforementioned Calderón-Zygmund theory. One can represent $P_t - P_{t-s}$ in terms of the singular integral kernel K:

$$(P_t - P_{t-s})f(x) = \int_{\mathbb{G}} [K(t, x\circ y^{-1}) - K(t-s, x\circ y^{-1})] f(y) dy. \tag{3.1.5}$$

Let us define a homogeneous norm $\|\cdot\|$ on $\mathbb{R}\times\mathbb{G}$ by $\|(t,x)\| := \sqrt{|t|+\|x\|^2}$. Since the singular integral kernel $K(t,x)$ is homogeneous of degree $-(Q+2)$, there exist constants $C,\delta>0$ such that whenever $C\,\|(s,y)\|\le\|(t,x)\|$,

$$|K((t,x)\circ(s,y))-K(t,x)|+|K((s,y)\circ(t,x))-K(t,x)|\le C\frac{\|(s,y)\|^\delta}{\|(t,x)\|^{Q+2+\delta}}.$$

Therefore, for some constant C_1 depending on δ, whenever $|t|>C^2|s|$,

$$\begin{aligned}\|K(t,x)-K(t-s,x)\|_{L^1(\mathbb{G})} &\le \left\|\frac{|s|^{\delta/2}}{(|t|+\|x\|^2)^{\delta/2+(Q+2)/2}}\right\|_{L^1(\mathbb{G})}\\ &= |s|^{\delta/2}\int_{\mathbb{G}}\frac{|t|^{Q/2}}{|t|^{\delta/2+(Q+2)/2}(1+\|z\|^2)^{\delta/2+(Q+2)/2}}dz\\ &= C_1\frac{|s|^{\delta/2}}{|t|^{\delta/2+1}}.\end{aligned}$$

Thus, applying Young's convolution inequality to (3.1.5),

$$\|P_t-P_{t-s}\|_{L^p\to L^p}\le C_1\frac{|s|^{\delta/2}}{|t|^{\delta/2+1}}.$$

Integrating this in t, we obtain

$$\int_{|t|\ge C^2|s|}\|P_t-P_{t-s}\|_{L^p\to L^p}\,dt\le C_1\int_{|t|\ge C^2|s|}\frac{|s|^{\delta/2}}{|t|^{\delta/2+1}}dt\le\frac{4C_1}{\delta C^\delta},$$

which immediately implies (3.1.4). □

We now focus on the following parabolic equation, involving the sub-Laplacian L:

$$\begin{cases}u_t-Lu=f, & 0\le t\le T,\\ u(0,x)=0.\end{cases}\tag{3.1.6}$$

We establish the well-posedness result of the equation (3.1.6) in the mixed-norm parabolic Sobolev spaces. We introduce the auxiliary function spaces: for $1\le p,q\le\infty$ and $k\ge 2$, u belongs to the function space $\tilde{S}^{k,(q,p)}([0,T]\times\mathbb{G})$ when

$$u\in S^{k,(q,p)}([0,T]\times\mathbb{G}),\quad u_t\in S^{k-2,(q,p)}([0,T]\times\mathbb{G}).\tag{3.1.7}$$

The corresponding norm $\|\cdot\|_{\tilde{S}^{k,(q,p)}}$ is defined by

$$\|u\|_{\tilde{S}^{k,(q,p)}([0,T]\times\mathbb{G})}:=\|u\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})}+\|u_t\|_{S^{k-2,(q,p)}([0,T]\times\mathbb{G})}.$$

One can prove the well-posedness result of the PDE (3.1.6) in the class $\tilde{S}^{k,(q,p)}([0,T]\times\mathbb{G})$:

Theorem 3.1.8. *Suppose that* $1 < p, q < \infty$. *Then, for any* $f \in S^{k,(q,p)}([0,T] \times \mathbb{G})$, *there exist a unique solution* $u \in \tilde{S}^{k+2,(q,p)}([0,T] \times \mathbb{G})$ *to PDE* (3.1.6). *Also, there exists some constant* C *independent of* T *such that for any* $f \in S^{k,(q,p)}([0,T] \times \mathbb{G})$,

$$\|u\|_{\dot{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})} \le C \|f\|_{\dot{S}^{k,(q,p)}([0,T]\times\mathbb{G})}, \tag{3.1.8}$$

$$\|u\|_{\tilde{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})} \le C \max\{T, 1\} \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})}. \tag{3.1.9}$$

Proof. Throughout the proof, we use a notation P for the heat kernel in order to avoid a confusion with the exponent p. Also, P_t denotes a semigroup generated by the sub-Laplacian L. For test functions $f \in C_c^\infty(\mathbb{R} \times \mathbb{G})$, let us define

$$Qf(t,x) := \int_{-\infty}^{t} P_{t-s} f(s)(x) ds. \tag{3.1.10}$$

Note that $u := Qf$ is a classical solution to the equation

$$u_t - Lu = f.$$

Step 1. A priori estimate on $\|Qf\|_{\dot{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})}$: let us first prove that

$$\|Qf\|_{\dot{S}^{k+2,(q,p)}(\mathbb{R}\times\mathbb{G})} \le C \|f\|_{\dot{S}^{k,(q,p)}(\mathbb{R}\times\mathbb{G})} \tag{3.1.11}$$

for any test functions f. In the case of $k = 0$, Theorem 3.1.7 immediately implies (3.1.11). In fact, for any $1 \le i_1, i_2 \le m$, we have a representation formula:

$$Z_{i_1} Z_{i_2} Qf = f * Z_{i_1} Z_{i_2} P$$

(convolution acts on $\mathbb{R} \times \mathbb{G}$), and note that $Z_{i_1} Z_{i_2} P$ is a singular kernel.

Now, let us prove (3.1.11) when $k = 1$. Recalling that each Z_j $(1 \le j \le N)$ can be written as a commutator of Z_i's $(1 \le i \le m)$ with order α_j, using (3.1.3), we have

$$Z_j^R = \sum_{l,I} Z_{jl}^R Z_{jI}^R,$$

where each Z_{jl} is one of Z_i's $(1 \le i \le m)$, and each Z_{jI}^R is of the form $Z_{s_1}^R \cdots Z_{s_{\alpha_j - 1}}^R$ for $1 \le s_1, \cdots, s_{\alpha_j - 1} \le m$. Therefore, applying this to (3.1.2), for any indices $1 \le i_2, i_3 \le m$,

$$f * Z_{i_2} Z_{i_3} P = f * (\sum_{j=1}^{N} Z_j^R (\beta_{ji_2} Z_{i_3} P)) = \sum_{j=1}^{N} \sum_{l,I} Z_{jl} f * (Z_{jI}^R (\beta_{ji_2} Z_{i_3} P))$$

(convolution acts on $\mathbb{R} \times \mathbb{G}$). Differentiating this in Z_{i_1} ($1 \le i_1 \le m$) direction,

$$Z_{i_1} Z_{i_2} Z_{i_3} u = \sum_{j=1}^{N} \sum_{l,I} Z_{jl} f * (Z_{i_1} Z_{jI}^R (\beta_{ji_2} Z_{i_3} P)).$$

Recall that P is a kernel of type 2, β_{ji_2} is homogeneous of degree $\alpha_j - 1$, Z_{jI}^R is a differential operator of order $\alpha_j - 1$, and Z_{i_2}, Z_{i_3} are differential operators of order 1. From this, it follows that $Z_{i_1} Z_{jI}^R (\beta_{ji_2} Z_{i_3} P)$ is a singular integral kernel. Since the operators of type 0 are bounded in $L^q(\mathbb{R}, L^p(\mathbb{G}))$ according to Theorem 3.1.7, we have

$$\| Z_{i_1} Z_{i_2} Z_{i_3} u \|_{L^q(\mathbb{R}, L^p(\mathbb{G}))} \le C \sum_{1 \le i \le m} \| Z_i f \|_{L^q(\mathbb{R}, L^p(\mathbb{G}))} .$$

This concludes the proof when $k = 1$. Similar arguments work for general k as well.

Step 2. A priori estimate on $\|Qf\|_{\tilde{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})}$: we prove that

$$\|Qf\|_{\tilde{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})} \le C \max\{T, 1\} \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} \tag{3.1.12}$$

for any test functions f. Since $u(t) := Qf(t)$ with $0 \le t \le T$ depends only on $f(s)$ with $s \le t$, according to the estimate (3.1.11), for any $0 \le l \le k$,

$$\|u\|_{\dot{S}^{l+2,(q,p)}([0,T]\times\mathbb{G})} \le C \|f\|_{\dot{S}^{l,(q,p)}([0,T]\times\mathbb{G})} .$$

Note that the constant C can be chosen independently of T due to the existence of scaling $u(t,x) \mapsto u(\lambda^2 t, D(\lambda)x)$ for $\lambda > 0$. Summing these inequalities over $0 \le l \le k$,

$$\sum_{l=0}^{k} \|u\|_{\dot{S}^{l+2,(q,p)}([0,T]\times\mathbb{G})} \le C \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} . \tag{3.1.13}$$

From the equation $u_t - Lu = f$ and the estimate (3.1.13), we have

$$\|u_t\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} \le C \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} . \tag{3.1.14}$$

Applying this to the trivial inequality $u(t,x) \le \int_0^T |u_t(s,x)| ds$, we obtain

$$\|u\|_{L^q([0,T],L^p(\mathbb{G}))} \le T \|u_t\|_{L^q([0,T],L^p(\mathbb{G}))} \le CT \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} . \tag{3.1.15}$$

Also, the interpolation type inequality allows us to obtain

$$\|u\|_{\dot{S}^{1,(q,p)}([0,T]\times\mathbb{G})} \le C \max\{T, 1\} \|f\|_{S^{k,(q,p)}([0,T]\times\mathbb{G})} . \tag{3.1.16}$$

Thus, using (3.1.13), (3.1.14), (3.1.15), and (3.1.16), we obtain (3.1.12).

Step 3. Existence of a solution: it can be proved by a standard approximation argument thanks to the estimate (3.1.12). Also, (3.1.8) and (3.1.9) hold for any $f \in S^{k,(q,p)}([0,T] \times \mathbb{G})$.

Step 4. Uniqueness of a solution: it suffices to prove that if $u \in \tilde{S}^{k+2,(q,p)}([0,T] \times \mathbb{G})$ is a solution to the equation (3.1.6) with $f = 0$, then $u = 0$. Choose the approximation u_n, each of which is smooth and has compact support, converging to u in $\tilde{S}^{k+2,(q,p)}([0,T] \times \mathbb{G})$ norm. It follows that

$$\|(u_n)_t - Lu_n\|_{S^{k,(q,p)}} \to 0$$

as $n \to \infty$. Since $u_n = Q((u_n)_t - Lu_n)$, according to the estimate (3.1.12), we have

$$\|u_n\|_{\tilde{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})} \to 0$$

as $n \to \infty$. Thus, $\|u\|_{\tilde{S}^{k+2,(q,p)}([0,T]\times\mathbb{G})} = 0$, which concludes the proof. □

Mixed-norm parabolic Sobolev embedding theorem

In this subsection, we obtain the parabolic Sobolev embedding theorem for the spaces $S^{k,(q,p)}([0,T] \times \mathbb{G})$. This is a key ingredient to establish the well-posedness result of the Kolmogorov PDE possessing singular coefficients.

Theorem 3.1.9. *Suppose that u satisfies $u(0,x) = 0$ and*

$$u \in S^{k+2,(q,p)}([0,T] \times \mathbb{G}), \quad u_t \in S^{k,(q,p)}([0,T] \times \mathbb{G}).$$

Also, assume that for $l = k, k+1$, exponents p, q, p_1, q_1 satisfy

$$1 \le p \le p_1 \le \infty,\ 1 \le q \le q_1 \le \infty,\ \frac{2}{q} + \frac{Q}{p} < (k+2-l) + \frac{2}{q_1} + \frac{Q}{p_1}. \tag{3.1.17}$$

Then, $u \in S^{l,(q_1,p_1)}([0,T]\times\mathbb{G})$. Also, if we denote $\alpha := \frac{1}{2}\Big[(k+2-l+\frac{2}{q_1}+\frac{Q}{p_1})-(\frac{2}{q}+\frac{Q}{p})\Big]$, then for some constant C independent of T and u,

$$\|u\|_{S^{l,(q_1,p_1)}([0,T]\times\mathbb{G})} \le CT^{\alpha}(\|u\|_{S^{k+2,(q,p)}([0,T]\times\mathbb{G})} + \|u_t\|_{S^{k+2,(q,p)}([0,T]\times\mathbb{G})}). \tag{3.1.18}$$

Proof. It suffices to prove the estimate (3.1.18) for all test functions u.

Step 1. The case $l = k$: for any indices $|I| \le l$, let us define $w = Z_I u$. Then, for any indices $1 \le i, j \le m$,

$$w,\ w_t,\ Z_i,\ Z_i Z_j w \in L^q([0,T], L^p(\mathbb{G})).$$

If we denote $f := w_t + Lw$, then we have the representation formula:

$$w(t,x) = \int_0^t \int_{\mathbb{G}} p(s,y) f(t-s, x \circ y^{-1}) dyds. \tag{3.1.19}$$

From this, we prove the estimate

$$\|w\|_{L^{q_1}([0,T],L^{p_1}(\mathbb{G}))} \leq C T^{\frac{1}{2}[(2+\frac{2}{q_1}+\frac{Q}{p_1})-(\frac{2}{q}+\frac{Q}{p})]} \|f\|_{L^q([0,T],L^p(\mathbb{G}))}. \tag{3.1.20}$$

Let us define a new function $\tilde{p}(t,x)$ defined on $\mathbb{R} \times \mathbb{G}$ via

$$\begin{cases} \tilde{p}(t,x) = p(t,x) & 0 \leq t \leq T, \\ \tilde{p}(t,x) = 0 & \text{otherwise}, \end{cases}$$

and $\tilde{f}(t,x)$ similarly. Then, from (3.1.19), we have

$$|w(t,x)| \leq \int_{\mathbb{R}} \int_{\mathbb{G}} \tilde{p}(s,y) |f|(t-s, x \circ y^{-1}) dyds = (\tilde{p} * |f|)(t,x) \tag{3.1.21}$$

(convolution acts on $\mathbb{R} \times \mathbb{G}$). Note that for any $1 \leq a < \infty$,

$$\left\| e^{-c\|\cdot\|^2/t} \right\|_{L^a(\mathbb{G})} = C_0 t^{Q/2a} \left\| e^{-c\|\cdot\|^2} \right\|_{L^a(\mathbb{G})} = C t^{Q/2a}, \tag{3.1.22}$$

and $\left\| e^{-c\|\cdot\|^2/t} \right\|_{L^\infty(\mathbb{G})} = 1$. Thus, using the heat kernel estimate (3.1.1), for any $1 \leq a \leq \infty$,

$$\|p(t,\cdot)\|_{L^a(\mathbb{G})} \leq C \frac{1}{t^{Q/2}} \left\| e^{-c\|\cdot\|^2/t} \right\|_{L^a(\mathbb{G})} = C t^{-\frac{Q}{2}(1-\frac{1}{a})}. \tag{3.1.23}$$

Let us choose two exponents $1 \leq r, s \leq \infty$ such that

$$\frac{1}{q_1} + 1 = \frac{1}{q} + \frac{1}{r}, \quad \frac{1}{p_1} + 1 = \frac{1}{p} + \frac{1}{s}. \tag{3.1.24}$$

According to the condition (3.1.17), we have $\frac{Qr}{2}(1-\frac{1}{s}) < 1$. Therefore, due to (3.1.23),

$$\|\tilde{p}\|_{L^r(\mathbb{R},L^s(\mathbb{G}))} = \|p\|_{L^r([0,T],L^s(\mathbb{G}))} \leq C_0 [\int_0^T t^{-\frac{Qr}{2}(1-\frac{1}{s})} dt]^{1/r} = C T^{\frac{1}{r}-\frac{Q}{2}(1-\frac{1}{s})}.$$

Thus, applying the convolution inequality for the mixed-norm spaces to (3.1.21),

$$\|w\|_{L^{q_1}(\mathbb{R},L^{p_1}(\mathbb{G}))} \leq \|\tilde{p}\|_{L^r(\mathbb{R},L^s(\mathbb{G}))} \left\| \tilde{f} \right\|_{L^q(\mathbb{R},L^p(\mathbb{G}))}$$

$$= CT^{\frac{1}{r}-\frac{Q}{2}(1-\frac{1}{s})} \left\|\tilde{f}\right\|_{L^q(\mathbb{R},L^p(\mathbb{G}))} = CT^{\frac{1}{2}[(2+\frac{2}{q_1}+\frac{Q}{p_1})-(\frac{2}{q}+\frac{Q}{p})]} \left\|\tilde{f}\right\|_{L^q(\mathbb{R},L^p(\mathbb{G}))}.$$

Thus, we obtain (3.1.20), which immediately implies (3.1.18).

Step 2. The case $l = k+1$: proof is almost same as the previous case. Applying the heat kernel estimate (3.1.1): for each $1 \le i \le m$,

$$|Z_i p(t,x)| \le Ct^{-(1+Q)/2} e^{-c\|x\|^2/t} \tag{3.1.25}$$

to the following representation formula

$$Z_i w(t,x) = \int_0^t \int_{\mathbb{G}} Z_i p(s,y) f(t-s, x \circ y^{-1}) dy ds, \tag{3.1.26}$$

we can derive the conclusion as before. □

Heat kernel estimates

In this subsection, we provide useful estimates related to the semigroup generated by the sub-Laplacian L. Let us first derive the L^p-estimate on the derivatives of a heat kernel:

Lemma 3.1.10. *Suppose that f is a homogeneous function with degree k and $1 \le p < \infty$, $|I| = a \ge 0$. Then, there exists some constant C depending on f such that for any $t > 0$,*

$$\|fZ_I p_t\|_{L^p(\mathbb{G})} \le Ct^{\frac{Q}{2p}+\frac{k-(Q+a)}{2}}.$$

Here, for a multi-index $I = (i_1, \cdots, i_a)$ with $1 \le i_1, \cdots, i_a \le m$, Z_I denotes $Z_{i_1} \cdots Z_{i_a}$.

Proof. Recall that under the change of variable $x = D(\sqrt{t})y$, we have $dx = t^{Q/2}dy$. Using this fact and the heat kernel estimate (3.1.1), we have

$$\begin{aligned}\|fZ_I p_t\|_{L^p(\mathbb{G})} &\le C\Big[\int_{\mathbb{G}} \big(f(x)t^{-\frac{Q+a}{2}} e^{-c\|x\|^2/t}\big)^p dx\Big]^{1/p} \\ &= Ct^{\frac{k-(Q+a)}{2}} \Big[\int_{\mathbb{G}} \big(f(y)e^{-c\|y\|^2}\big)^p t^{Q/2} dy\Big]^{1/p} = Ct^{\frac{Q}{2p}+\frac{k-(Q+a)}{2}}.\end{aligned}$$

□

Using the previous lemma, we obtain the following lemma, which is a key ingredient in the proof of Proposition 3.2.2:

Lemma 3.1.11. *Suppose that* $1 < p < \infty$. *Then, for any* $|I| = a \geq 1$, $f \in S^{a-1,p}(\mathbb{G})$, *and* $t > 0$,

$$\|f * Z_I p_t\|_{L^\infty(\mathbb{G})} \leq Ct^{-(\frac{Q}{2p}+\frac{1}{2})} \|f\|_{S^{a-1,p}(\mathbb{G})} . \tag{3.1.27}$$

Proof. Using the heat kernel estimate (3.1.1) and the convolution inequality, (3.1.27) immediately follows when $a = 1$. Key idea of the proof when $a \geq 2$ is transferring the directional derivative Z_I from p_t to f. By the argument in the proof of Theorem 3.1.8, for each $1 \leq i \leq m$ and any smooth functions g,

$$f * Z_i g = f * (\sum_{j=1}^{N} Z_j^R(\beta_{ji} g)) = \sum_{j=1}^{N} \sum_{l,I} Z_{jl} f * (Z_{jI}^R(\beta_{ji} g)). \tag{3.1.28}$$

Let us first prove (3.1.27) when $a = 2$, and assume that $Z_I = Z_{i_1} Z_{i_2}$, $1 \leq i_1, i_2 \leq m$. If we denote p' by the conjugate exponent of p, then applying the convolution inequality to (3.1.28),

$$\|f * Z_I p_t\|_{L^\infty(\mathbb{G})} = \|f * Z_{i_1}(Z_{i_2} p_t)\|_{L^\infty(\mathbb{G})} \leq C \|f\|_{S^{1,p}(\mathbb{G})} \sum_{j,l,I} \left\| Z_{jI}^R(\beta_{ji} Z_{i_2} p_t) \right\|_{L^{p'}(\mathbb{G})} . \tag{3.1.29}$$

Note that each Z_i^R, $1 \leq i \leq m$, can be written as

$$Z_i^R u = \sum_{j=1}^{N} Z_j(\gamma_{ji} u)$$

for some homogeneous functions γ_{ji} of degree $\alpha_j - 1$ ($1 \leq j \leq N$). Thus, applying the product rule, $Z_{jI}^R(\beta_{ji} Z_{i_2} p_t)$ can be written as the (finite) sum of $h_k Z_{I_k} p_t$'s for some homogeneous functions h_k of degree $k - 1$ and $|I_k| = k$. Note that according to Lemma 3.1.10, each term $\|h_k Z_{I_k} p_t\|_{L^{p'}(\mathbb{G})}$ is bounded by

$$Ct^{\frac{Q}{2p'}+\frac{k-1-(Q+k)}{2}} = Ct^{-(\frac{Q}{2p}+\frac{1}{2})}.$$

Thus, combining this with (3.1.29), we obtain (3.1.27) in the case $a = 2$.

The aforementioned argument works for $a > 2$ as well. In fact, differentiating (3.1.28) in Z_i directions ($1 \leq i \leq m$) and then using (3.1.2), we can deduce that

$$\|f * Z_I p_t\|_{L^\infty(\mathbb{G})} \leq C \|f\|_{S^{l-1,p}(\mathbb{G})} \|h\|_{L^{p'}(\mathbb{G})} ,$$

where h is the sum of finitely many $h_k Z_{I_k} p_t$'s for some homogeneous functions h_k of degree $k - 1$ and $|I_k| = k$. As mentioned above, $\|h\|_{L^{p'}(\mathbb{G})} \leq Ct^{-(\frac{Q}{2p}+\frac{1}{2})}$, which concludes the proof. □

3.2 Kolmogorov hypoelliptic PDE

In this section, we establish the well-posedness result of the following Kolmogorov PDE:

$$\begin{cases} u_t - \frac{1}{2}Lu + \sum_{i=1}^m b^i Z_i u + \lambda u = f, \quad 0 \le t \le T, \\ u(0,x) = 0. \end{cases} \tag{3.2.1}$$

on the homogeneous Carnot group $\mathbb{G}$ for singular functions b, f and $\lambda \in \mathbb{R}$. The solution u to Kolmogorov PDE (3.2.1) plays a crucial role in proving the uniqueness of a strong solution to SDE (1.1.12). In fact, this PDE appears when we apply the Zvonkin's transformation method [**Z3**] to obtain an auxiliary SDE. From now on, for any Banach spaces X, let us define

$$\|b\|_X := \sum_{i=1}^m \|b_i\|_X .$$

Theorem 3.2.1. *Assume that b satisfies the conditions* (1.1.14) *and* (1.1.15) *for exponents p and q satisfying* (1.1.13)*. Then, for any $f \in S^{r-1,(q,p)}([0,T]\times\mathbb{G})$, there exists a unique solution $u \in \tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$ to PDE* (3.2.1)*. Furthermore, we have the following estimate:*

$$\|u\|_{S^{r+1,(q,p)}([0,T]\times\mathbb{G})} + \|u_t\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})} \le C(b,\lambda)\,\|f\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})} \tag{3.2.2}$$

Proof. Let us first prove an a priori estimate (3.2.2). For $0 \le t \le T$, let us define

$$I(t) = \|u\|^q_{S^{r+1,(q,p)}([0,t]\times\mathbb{G})} + \|u_t\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})} .$$

Then, using the estimate (3.1.9), we have

$$I(t) \le C \left\| \sum_{i=1}^m b^i Z_i u + \lambda u + f \right\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})} . \tag{3.2.3}$$

Since p and q satisfy (1.1.13), according to the parabolic Sobolev embedding Theorem 3.1.9,

$$\|u\|^q_{S^{r,(\infty,\infty)}([0,t]\times\mathbb{G})} \le C(\|u\|^q_{S^{r+1,(q,p)}([0,t]\times\mathbb{G})} + \|u_t\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})}) = CI(t).$$

Therefore, one can deduce that for each $1 \le i \le m$,

$$\left\|b^i Z_i u\right\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})} = \int_0^t \left\|b^i Z_i u(s)\right\|^q_{S^{r-1,p}(\mathbb{G})} ds$$

$$
\begin{aligned}
&\leq \int_0^t \left\| b^i(s) \right\|^q_{S^{r-1,p}(\mathbb{G})} \left\| Z_i u(s) \right\|^q_{L^\infty(\mathbb{G})} ds \\
&\leq C \int_0^t \left\| b^i(s) \right\|^q_{S^{r-1,p}(\mathbb{G})} I(s) ds. \qquad (3.2.4)
\end{aligned}
$$

Also, using Minkowski's integral inequality,

$$
\begin{aligned}
\left\| \lambda u \right\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})} ds &\leq C\lambda^q \int_0^t \Big[\int_0^s \left\| u_t(l) \right\|^q_{S^{r-1,p}(\mathbb{G})} dl \Big] ds \\
&\leq C\lambda^q \int_0^t I(s) ds. \qquad (3.2.5)
\end{aligned}
$$

Therefore, applying (3.2.4) and (3.2.5) to (3.2.3),

$$
I(t) \leq C \int_0^t (\left\| b(s) \right\|^q_{S^{r-1,p}(\mathbb{G})} + \lambda^q) I(s) ds + C \left\| f \right\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})}.
$$

Using Grönwall's inequality, for each $0 \leq t \leq T$,

$$
I(t) \leq C \left\| f \right\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})} \exp \left[C \left\| b \right\|^q_{S^{r-1,(q,p)}([0,t]\times\mathbb{G})} + Ct\lambda^q \right].
$$

In particular, the case $t = T$ implies (3.2.2).

Once a priori estimate (3.2.2) is obtained, the existence and uniqueness of a solution to the PDE (3.2.1) immediately follows from the standard method of continuity. □

Assume that b satisfies the conditions (1.1.14), (1.1.15) for the exponents p, q satisfying (1.1.13). Also, suppose that a function $f \in S^{r-1,(q,p)}([0,T] \times \mathbb{G})$ taking values in $\mathbb{G} = \mathbb{R}^N$ is given. This means that each Euclidean coordinate of f belongs to $S^{r-1,(q,p)}([0,T] \times \mathbb{G})$. Let us now consider the following PDE:

$$
\begin{cases} u_t + \frac{1}{2} Lu + \sum_{i=1}^m b^i Z_i u - \lambda u = f, \quad 0 \leq t \leq T, \\ u(T,x) = 0. \end{cases} \qquad (3.2.6)
$$

u being a solution to PDE (3.2.6) means that (3.2.6) holds in each Euclidean coordinate. According to Theorem 3.2.1, by reversing time, one can deduce that PDE (3.2.6) has a (unique) solution $\tilde{u}^\lambda \in \tilde{S}^{r+1,(q,p)}([0,T] \times \mathbb{G})$ taking values in $\mathbb{G} = \mathbb{R}^N$. We introduce an auxiliary function Φ^λ in the next proposition, which plays a crucial role in Section 3.3.

Proposition 3.2.2. *There exist an open set Ω containing x_0, $\lambda \in \mathbb{R}$, and a version u^λ of $\tilde{u}^\lambda$ such that $\Phi^\lambda(t,x) := x + u^\lambda(t,x)$ satisfies the following properties:*

(i) Φ^λ *is continuous in* (t,x) *and* $\Phi^\lambda(t,\cdot)$ *is* C^1 *for each* $0 \le t \le T$.
(ii) $\Phi^\lambda(t,\cdot)$ *is a* C^1 *diffeomorphism from* Ω *onto its image for each* $0 \le t \le T$.
(iii) For each $0 \le t \le T$,

$$\frac{1}{2} \le \left\|\nabla \Phi^\lambda(t,\cdot)\right\|_{L^\infty(\Omega)} \le 2, \quad \frac{1}{2} \le \left\|\nabla (\Phi^\lambda)^{-1}(t,\cdot)\right\|_{L^\infty(\Phi^\lambda(t,\Omega))} \le 2.$$

Proof. Throughout the proof, in order to simplify the notation, we use $\tilde{u}$ and Φ instead of $\tilde{u}^\lambda$ and Φ^λ, respectively.

Step 1. Proof of the property (i): let us prove that there exists a continuous version u of $\tilde{u}$ such that $u(t,\cdot)$ is C^1 for each t. It suffices to show that for the arbitrary bounded and open set U in $\mathbb{R}^N$, there exists a version u of $\tilde{u}$ such that u is continuous on $[0,T] \times U$ and $u(t,\cdot) \in C^1(U)$. Choose a smooth approximation u_n converging to $\tilde{u}$ in $\tilde{S}^{r+1,(q,p)}([0,T] \times \mathbb{G})$ norm. According to Theorem 3.1.9, for any indices $|I| \le r$,

$$\|Z_I(u_n - u_m)\|_{L^\infty([0,T]\times\mathbb{G})} \le C \|u_n - u_m\|_{\tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})} . \tag{3.2.7}$$

Since each standard vector field on $\mathbb{R}^N$ can be written as commutators of Z_i's up to order r, it follows from (3.2.7) that

$$\|\nabla(u_n - u_m)\|_{L^\infty([0,T]\times U)} \le C \|u_n - u_m\|_{\tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})} .$$

for some constant $C = C(U)$. This implies that there exists $w \in C_b([0,T] \times U)$ such that

$$\|w - \nabla u_n\|_{L^\infty([0,T]\times U)} \to 0 \tag{3.2.8}$$

as $n \to 0$. Also, since the sequence $\{u_n\}$ is Cauchy in $L^\infty([0,T] \times \mathbb{G})$ norm by Theorem 3.1.9, there exists $u \in C_b([0,T] \times \mathbb{G})$, which is a version of $\tilde{u}$, such that as $n \to \infty$,

$$\|u - u_n\|_{L^\infty([0,T]\times\mathbb{G})} \to 0. \tag{3.2.9}$$

Thanks to (3.2.8) and (3.2.9), for each t, $u(t,\cdot)$ is C^1 on U and its spatial derivative is $w(t,\cdot)$.

Step 2. Estimate on $\|u\|_{S^{r,(\infty,\infty)}}$: from now on, we denote u by a function selected in the Step 1. We now claim that for arbitrary $\epsilon > 0$, there exists a sufficiently large λ such that

$$\|Z_I u\|_{L^\infty([0,T]\times\mathbb{G})} \le \epsilon \tag{3.2.10}$$

holds for all indices $|I| \le r$. We have the following representation formula for u:

$$u(t) = \int_t^T e^{-\lambda(s-t)} P_{s-t}(f + \sum_{i=1}^m b^i Z_i u)(s) ds.$$

Differentiating this in Z_I ($|I| \le r$) directions,

$$Z_I u(t) = \int_t^T e^{-\lambda(s-t)} Z_I P_{s-t}(f + \sum_{i=1}^m b^i Z_i u)(s) ds. \tag{3.2.11}$$

Note that for $g \in S^{r-1,p}(\mathbb{G})$, according to Lemma 3.1.11,

$$\|Z_I P_t g\|_{L^\infty(\mathbb{G})} = \|g * Z_I p_t\|_{L^\infty(\mathbb{G})} \le C t^{-(\frac{Q}{2p}+\frac{1}{2})} \|g\|_{S^{r-1,p}(\mathbb{G})}$$

for each $0 \le t \le T$ and any indices $|I| \le r$. Therefore, applying this to (3.2.11), we have

$$\begin{aligned}
&\sum_{|I|\le r} \|Z_I u(t)\|_{L^\infty(\mathbb{G})} \le \sum_{|I|\le r} \int_t^T e^{-\lambda(s-t)} \left\| Z_I P_{s-t}(f + \sum_{i=1}^m b^i Z_i u)(s) \right\|_{L^\infty(\mathbb{G})} ds \\
&\le C \int_t^T e^{-\lambda(s-t)} (s-t)^{-(\frac{Q}{2p}+\frac{1}{2})} \left\| (f + \sum_{i=1}^m b^i Z_i u)(s) \right\|_{S^{r-1,p}(\mathbb{G})} ds \\
&\le C \int_t^T e^{-\lambda(s-t)} (s-t)^{-(\frac{Q}{2p}+\frac{1}{2})} \Big(\|f(s)\|_{S^{r-1,p}(\mathbb{G})} + \|b(s)\|_{S^{r-1,p}(\mathbb{G})} \sum_{|I|\le r} \|Z_I u(s)\|_{L^\infty(\mathbb{G})} \Big) ds.
\end{aligned}$$

Using the modified version of Grönwall's inequality (see [**FF3**]), we obtain

$$\sum_{|I|\le r} \|Z_I u(t)\|_{L^\infty(\mathbb{G})} \le \alpha(t) + \int_t^T \alpha(s)\beta_t(s) \exp\Big[\int_t^s \beta_t(l) dl \Big] ds, \tag{3.2.12}$$

where $\alpha(s)$ and $\beta_t(s)$ are defined by

$$\begin{aligned}
\alpha(s) &= C \int_s^T e^{-\lambda(l-s)} (l-s)^{-(\frac{Q}{2p}+\frac{1}{2})} \|f(l)\|_{S^{r-1,p}(\mathbb{G})} dl, \\
\beta_t(s) &= C e^{-\lambda(s-t)} (s-t)^{-(\frac{Q}{2p}+\frac{1}{2})} \|b(s)\|_{S^{r-1,p}(\mathbb{G})}.
\end{aligned}$$

If we denote q' by a conjugate exponent of q, then by Hölder's inequality,

$$\begin{aligned}
\alpha(t) &= C \int_t^T e^{-\lambda(s-t)} (s-t)^{-(\frac{Q}{2p}+\frac{1}{2})} \|f(s)\|_{S^{r-1,p}(\mathbb{G})} ds \\
&\le C \Big[\int_0^T e^{-q'\lambda s} s^{-q'(\frac{Q}{2p}+\frac{1}{2})} ds \Big]^{1/q'} \|f\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})}.
\end{aligned}$$

Since the condition (1.1.13) implies that $q'(\frac{Q}{2p}+\frac{1}{2}) < 1$, one can easily check that

$$\lim_{\lambda\to\infty} \int_0^T e^{-q'\lambda s} s^{-q'(\frac{Q}{2p}+\frac{1}{2})} ds = 0,$$

which implies that

$$\lim_{\lambda\to\infty}\left[\sup_{0\le t\le T}\alpha(t)\right]=0. \tag{3.2.13}$$

Similarly, applying Hölder's inequality as above,

$$\int_t^T \beta_t(s)ds \le C\,\|b\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})}<\infty. \tag{3.2.14}$$

Therefore, (3.2.12), (3.2.13), and (3.2.14) imply that for sufficiently large λ, (3.2.10) holds for all indices $|I|\le r$.

Step 3. Proof of the properties (ii) and (iii): since each standard vector field on $\mathbb{R}^N$ can be written as a linear combination of commutators of Z_i's with order $\le r$, for any bounded set U in $\mathbb{R}^N$ containing x_0, there exists a constant $C=C(U)$ satisfying

$$\|\nabla u\|_{L^\infty([0,T]\times U)}\le C\,\|u\|_{S^{r,(\infty,\infty)}([0,T]\times U)}\,.$$

Therefore, thanks to the claim (3.2.10) proved in Step 2, for sufficiently large λ, we have

$$\|\nabla u\|_{L^\infty([0,T]\times U)}\le\frac{1}{2}, \tag{3.2.15}$$

which immediately implies the first inequality in the condition (iii). Since $\nabla\Phi(t,\cdot)$ is continuous and non-singular on U, there exists an open set $\Omega\subset U$ containing x_0 such that $\Phi(t,\cdot)$ is C^1 diffeomorphism from Ω onto its image according to the inverse function theorem. Also, using (3.2.15) and the identity

$$\nabla\Phi^{-1}(t,x)=[\nabla\Phi(t,\Phi^{-1}(t,x))]^{-1}=[I+\nabla u(t,\Phi^{-1}(t,x))]^{-1},$$

we obtain the second inequality in the condition (iii). This concludes the proof. □

Remark 3.2.3. Since we assumed that b^i's have compact support (see Remark 1.1.4), and Z_i's are smooth vector fields, each Euclidean coordinate of b belongs to $S^{r-1,(q,p)}([0,T]\times\mathbb{G})$. Thus, Proposition 3.2.2 is applicable for $f=b$. In fact, there exists $u\in\tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$ taking values in $\mathbb{R}^N$, which is C^1 in x, satisfying

$$\begin{cases} u_t+\frac{1}{2}Lu+\sum_{i=1}^m b^iZ_iu-\lambda u=-b, & 0\le t\le T,\\ u(T,x)=0.\end{cases} \tag{3.2.16}$$

Also, there exist $\lambda\in\mathbb{R}$ and an open set Ω in $\mathbb{R}^N$ containing x_0 such that

$$\Phi(t,x)=x+u(t,x)$$

satisfies (i), (ii), and (iii) in Proposition 3.2.2. From now on, we use these notations u, Φ, and Ω.

Finally, we choose versions of u_t, $Z_i u$ $(1 \le i \le m)$, Lu such that t-a.e., $u_t(t,\cdot)$, $Z_i u(t,\cdot)$ $(1 \le i \le m)$, $Lu(t,\cdot)$ are continuous. In fact,

$$u_t, Z_i u, Lu \in S^{r-1,(q,p)}([0,T] \times \mathbb{G}),$$

which implies that

$$u_t(t,\cdot), Z_i u(t,\cdot), Lu(t,\cdot) \in S^{r-1,p}(\mathbb{G})$$

for almost every t, and thus such versions can be obtained according to Theorem 3.1.4. Since the left hand side of (3.2.16) and $b(t,\cdot)$ are both continuous in x for t-a.e (see Remark 1.1.4), it follows that t-a.e, the equation (3.2.16) is satisfied for every $x \in \mathbb{G}$.

3.3 Proof of the main theorem

In this section, we prove the main result Theorem 1.1.3. We identify the space $\mathbb{G}$ with the Euclidean space $\mathbb{R}^N$, and then we do a stochastic calculus. Throughout this section, we add a time parameter t to the time independent vector fields Z_i's, i.e. $Z_i(t,x) = Z_i(x)$ for $0 \le t \le T$.

Itô's formula for singular functions

In order to prove the strong uniqueness of SDE (1.1.12), we use the Zvonkin's transformation method [**Z3**] to obtain an auxiliary SDE. This auxiliary SDE is more tractable than the original SDE (1.1.12) since it possesses a more regular drift coefficient. When we use the Zvonkin's transformation method, a function to which we apply Itô's formula is not as regular. In order to overcome this problem, we need to establish Itô's formula for a large class of singular functions.

The key ingredient to obtain Itô's formula for non-smooth functions is a Krylov-type estimate [**K1**]. This type of estimate has been used successfully to prove the well-posedness of a singular SDE with the non-degenerate noise. In next proposition, we establish a Krylov-type estimate for the degenerate diffusion case (1.1.12). Since we are working on the homogeneous Carnot group and the SDE (1.1.12) possesses the degenerate diffusion, the proof involves some technical difficulties.

Proposition 3.3.1. *Assume that b satisfies the conditions* (1.1.14), (1.1.15) *for the exponents p, q satisfying* (1.1.13). *Suppose that X_t is a solution to SDE* (1.1.12).

Then, for each $0 \le s \le t \le T$, the estimate

$$\mathbb{E}\left[\int_s^t f(r, X_r)dr \middle| \mathcal{F}_s\right] \le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})} \|f\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))} \tag{3.3.1}$$

holds for any $f \in L^{q/2}([0,t], L^{p/2}(\mathbb{G}))$ such that $f(r,\cdot)$ is continuous for a.e. $r \in [0,t]$. Here, a constant C is independent of s, t, and a function f.

Proof. It suffices to prove (3.3.1) for non-negative $f \in L^{q/2}([0,t], L^{p/2}(\mathbb{G}))$ such that $f(r,\cdot)$ is continuous for a.e. $r \in [0,t]$.

Step 1. The auxiliary PDE result: let us prove that for any $f \in L^{q/2}([0,t], L^{p/2}(\mathbb{G}))$, one can find a solution $w \in \tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})$ to the equation:

$$\begin{cases} w_t + \frac{1}{2}Lw + \sum_{i=1}^m b^i Z_i w = f, & \text{in } [0,t]\times\mathbb{G}, \\ w(t,x) = 0, \end{cases} \tag{3.3.2}$$

satisfying that for some constant C,

$$\|w\|_{\tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})} \le C \|f\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))}. \tag{3.3.3}$$

For $u \in \tilde{S}^{2,(q/2,p/2)}([0,T]\times\mathbb{G})$ (see (3.1.7) for definition), let us consider the following PDE:

$$\begin{cases} w_t + \frac{1}{2}Lw = f - \sum_{i=1}^m b^i Z_i u, & \text{in } [0,t]\times\mathbb{G}, \\ w(t,x) = 0. \end{cases} \tag{3.3.4}$$

Note that according to Hölder's inequality and the parabolic Sobolev embedding Theorem 3.1.9, for each $1 \le i \le m$,

$$\begin{aligned} \left\|b^i Z_i u\right\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))} &\le \left\|b^i\right\|_{L^q([0,t],L^p(\mathbb{G}))} \|Z_i u\|_{L^q([0,t],L^p(\mathbb{G}))} \\ &\le CT^{\frac{1}{2}[1-(\frac{2}{q}+\frac{Q}{p})]} \left\|b^i\right\|_{L^q([0,t],L^p(\mathbb{G}))} \|u\|_{\tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})}. \end{aligned}$$

Therefore, the right hand side of PDE (3.3.4) belongs to $L^{q/2}([0,T], L^{p/2}(\mathbb{G}))$. Applying Theorem 3.1.8, let us define

$$F(u) := w \in \tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})$$

to be a unique solution to the PDE (3.3.4). For $u_1, u_2 \in \tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})$, according to the estimate (3.1.9), we have

$$\begin{aligned} &\|F(u_1) - F(u_2)\|_{\tilde{S}^{2,(q/2,p/2)}([0,t]\times\mathbb{G})} \\ &\qquad \le C\max\{t,1\} t^{\frac{1}{2}[1-(\frac{2}{q}+\frac{Q}{p})]} \|b\|_{L^q([0,t],L^p(\mathbb{G}))} \|u_1 - u_2\|_{\tilde{S}^{2,(r,s)}([0,t]\times\mathbb{G})}. \end{aligned}$$

It follows that for a small enough t, a map $u \to F(u)$ is a strict contraction on $\tilde{S}^{2,(q/2,p/2)}([0,t] \times \mathbb{G})$. Thus, for a sufficiently small $0 < T_1 \le t$,

$$F : \tilde{S}^{2,(q/2,p/2)}([t-T_1,t] \times \mathbb{G}) \to \tilde{S}^{2,(q/2,p/2)}([t-T_1,t] \times \mathbb{G})$$

has a unique fixed point u (note that (3.3.4) is a backward PDE). For such T_1 and u, we have

$$\begin{aligned}\|u\|_{\tilde{S}^{2,(q/2,p/2)}([t-T_1,t]\times\mathbb{G})} &\le C\max\{T_1,1\}\left\| f - \sum_{i=1}^m b^i Z_i u \right\|_{L^{q/2}([t-T_1,t],L^{p/2}(\mathbb{G}))} \\ &\le C\max\{T_1,1\}(\|f\|_{L^{q/2}([t-T_1,t],L^{p/2}(\mathbb{G}))} \\ &\quad + CT_1^{\frac{1}{2}[1-(\frac{2}{q}+\frac{Q}{p})]}\|b\|_{L^q([t-T_1,t],L^p(\mathbb{G}))}\|u\|_{\tilde{S}^{2,(q/2,p/2)}([t-T_1,t]\times\mathbb{G})})\end{aligned}$$

For small enough T_1, we have the estimate (3.3.3) with u in place of w on the interval $[t-T_1,t]$. We then redefine $u(t-T_1,x) = 0$, and repeat the aforementioned argument to obtain a solution defined on the whole interval $[0,t]$ and the estimate (3.3.3).

Step 2. Regularization processes: since w is not smooth in general, the standard Itô's formula is not applicable to a function w. In order to overcome this problem, we take a nonnegative test function $\varphi \in C_c^\infty(\mathbb{G})$, and introduce mollifiers

$$\varphi_n(x) := n^Q \varphi(D(n)x).$$

Then, define regularized functions

$$w_n(t,x) := (\varphi_n * w)(t,x) = \int_{\mathbb{G}} \varphi_n(x \circ y^{-1}) w(t,y)dy.$$

If we denote f_n by

$$f_n := (w_n)_t + \sum_{i=1}^m b^i Z_i w_n + \frac{1}{2} L w_n, \tag{3.3.5}$$

then by Itô's formula, we have

$$\begin{aligned}&w_n(t,X_t) - w_n(s,X_s) \\ &\quad= \int_s^t ((w_n)_t + \sum_{i=1}^m b^i Z_i w_n + \frac{1}{2}Lw_n)(r,X_r)dr + \int_s^t \sum_{i=1}^m Z_i w_n(r,X_r)dB_r^i \\ &\quad= \int_s^t f_n(r,X_r)dr + \int_s^t \sum_{i=1}^m Z_i w_n(r,X_r)dB_r^i.\end{aligned} \tag{3.3.6}$$

Note that using (3.3.3) and Theorem 3.1.9, one can deduce that $Z_i w \in L^q([0,t], L^p(\mathbb{G}))$ for each $1 \le i \le m$. Thus, if we denote p' by a conjugate exponent of p, then for each n,

$$\begin{aligned} \|Z_i w_n\|^q_{L^q([0,t],L^\infty(\mathbb{G}))} &= \int_0^t \|\varphi_n * Z_i w\|^q_{L^\infty(\mathbb{G})}\, dr \\ &\le \int_0^t \|\varphi_n\|^q_{L^{p'}(\mathbb{G})} \|Z_i w\|^q_{L^p(\mathbb{G})}\, dr \\ &< \|\varphi_n\|^q_{L^{p'}(\mathbb{G})} \|Z_i w\|^q_{L^q([0,t],L^p(\mathbb{G}))} < \infty \end{aligned} \tag{3.3.7}$$

(note that $\|Z_i w_n\|_{L^q([0,t],L^\infty(\mathbb{G}))}$ may not be uniformly bounded in n). This implies that for each n, a stochastic process $r \to Z_i w_n(r, X_r)$ is square-integrable on $[0,t]$ since $q > 2$ (see the condition (1.1.13)) and

$$\mathbb{E}\Big[\int_0^t |Z_i w_n(r,X_r)|^2 dr\Big] \le \int_0^t \|Z_i w_n(r,\cdot)\|^2_{L^\infty(\mathbb{G})}\, dr = \|Z_i w_n\|^2_{L^2([0,t],L^\infty(\mathbb{G}))} < \infty.$$

Therefore, one can deduce that

$$\mathbb{E}\Big[\int_s^t \sum_{i=1}^m Z_i w_n(r,X_r) dB^i_r \Big| \mathcal{F}_s\Big] = 0.$$

Using this and taking a conditional expectation with respect to $\mathcal{F}_s$ in (3.3.6), we obtain

$$\begin{aligned} \mathbb{E}\Big[\int_s^t f_n(r,X_r)dr\Big|\mathcal{F}_s\Big] &= \mathbb{E}[w_n(t,X_t) - w_n(s,X_s)|\mathcal{F}_s] \\ &\le 2 \sup_{r\in[s,t]} \|w_n(r,\cdot)\|_{L^\infty(\mathbb{G})} \\ &\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})}(\|w_n\|_{S^{2,(q/2,p/2)}([0,t]\times\mathbb{G})} + \|(w_n)_t\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))}) \\ &\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})}(\|w\|_{S^{2,(q/2,p/2)}([0,t]\times\mathbb{G})} + \|w_t\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))}) \\ &\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})} \|f\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))}. \end{aligned} \tag{3.3.8}$$

Here, we used Theorem 3.1.9 in the third line, convolution inequality in the fourth line, and (3.3.3) in the last line (note that $w_n(t,x) = 0$).

Now, we establish the commutator estimate. For a.e. $r \in [s,t]$ and any $x \in \mathbb{G}$, we have that for some $0 < \beta < 1$,

$$|\varphi_n * (b^i Z_i w) - b^i Z_i(\varphi_n * w)|(r,x)$$

$$
\begin{aligned}
&= \Big| \int_{\mathbb{G}} (b^i(r, y^{-1} \circ x) - b^i(r, x)) Z_i w(r, y^{-1} \circ x) \varphi_n(y) dy \Big| \\
&\leq C \int_{\mathbb{G}} \left\| b^i(r, \cdot) \right\|_{S^{r-1,p}(\mathbb{G})} \|y\|^{\beta} |Z_i w(r, y^{-1} \circ x)| |\varphi_n(y)| dy \\
&\leq C \left\| b^i(r, \cdot) \right\|_{S^{r-1,p}(\mathbb{G})} \|Z_i w(r, \cdot)\|_{L^\infty(\mathbb{G})} \left\| \|y\|^{\beta} \varphi_n(y) \right\|_{L^1(\mathbb{G})}.
\end{aligned} \tag{3.3.9}
$$

Here, we used Sobolev embedding Theorem 3.1.4 and Remark 3.1.5 in the third line (we have $(r-1)p > Q$: see Remark 1.1.4), and Hölder's inequality in the last line.

Therefore, integrating (3.3.9) in time and then applying Hölder's inequality,

$$
\begin{aligned}
&\left\| \varphi_n * (b^i Z_i w) - b^i Z_i (\varphi_n * w) \right\|_{L^1([s,t], L^\infty(\mathbb{G}))} \\
&\leq C \left\| \|y\|^{\beta} \varphi_n(y) \right\|_{L^1(\mathbb{G})} \left\| b^i \right\|_{L^q([s,t], S^{r-1,p}(\mathbb{G}))} \|Z_i w\|_{L^{q'}([s,t], L^\infty(\mathbb{G}))}
\end{aligned} \tag{3.3.10}
$$

(q' is the conjugate exponent of q). Note that $q' < 2$ since $q > 2$ (see the condition (1.1.13)). This implies that

$$
\frac{2}{q/2} + \frac{Q}{p/2} < 2 < 1 + \frac{2}{q'}.
$$

Thus, since $w \in \tilde{S}^{2,(q/2,p/2)}([0,t] \times \mathbb{G})$ and according to Theorem 3.1.9,

$$
\|Z_i w\|_{L^{q'}([s,t], L^\infty(\mathbb{G}))} < \infty. \tag{3.3.11}
$$

Also, it is obvious that

$$
\begin{aligned}
\left\| \|y\|^{\beta} \varphi_n(y) \right\|_{L^1(\mathbb{G})} &= n^Q \int_{\mathbb{G}} \|y\|^{\beta} \varphi(D(n)y)) dy \\
&= n^{-\beta} \int_{\mathbb{G}} \|z\|^{\beta} \varphi(z) dz,
\end{aligned}
$$

where the last identity is obtained by the change of variable $D(n)y = z$. Since $\varphi \in C_c^\infty(\mathbb{G})$,

$$
\int_{\mathbb{G}} \|z\|^{\beta} \varphi(z) dz < \infty,
$$

which implies that

$$
\lim_{n \to \infty} \left\| \|y\|^{\beta} \varphi_n(y) \right\|_{L^1(\mathbb{G})} = 0. \tag{3.3.12}
$$

Therefore, using (1.1.15), (3.3.10), (3.3.11), and (3.3.12), we have

$$\lim_{n\to\infty} \left\|\varphi_n * (b^i Z_i w) - b^i Z_i(\varphi_n * w)\right\|_{L^1([s,t],L^\infty(\mathbb{G}))} = 0. \tag{3.3.13}$$

Step 3. Proof of the estimate (3.3.1): since $f(r,\cdot)$ is continuous for r-a.e.,

$$(\varphi_n * f)(r,x) \to f(r,x)$$

everywhere in $x \in \mathbb{G}$ for r-a.e. This implies that r-a.e.,

$$(\varphi_n * f)(r,X_r) \to f(r,X_r)$$

for any realization $\omega \in \Omega$. Since we assumed that f is non-negative and $\varphi \geq 0$, it follows that $\varphi_n * f \geq 0$. Thus, according to the Fatou's lemma, for any realization $\omega \in \Omega$,

$$\int_s^t f(r,X_r)dr \leq \liminf_{n\to\infty} \int_s^t (\varphi_n * f)(r,X_r)dr. \tag{3.3.14}$$

Applying Fatou's lemma for the conditional expectation,

$$\mathbb{E}\left[\liminf_{n\to\infty} \int_s^t (\varphi_n * f)(r,X_r)dr \middle| \mathcal{F}_s\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_s^t (\varphi_n * f)(r,X_r)dr \middle| \mathcal{F}_s\right]. \tag{3.3.15}$$

From (3.3.14) and (3.3.15), we have

$$\mathbb{E}\left[\int_s^t f(r,X_r)dr \middle| \mathcal{F}_s\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[\int_s^t (\varphi_n * f)(r,X_r)dr \middle| \mathcal{F}_s\right]. \tag{3.3.16}$$

On the other hand, it is easy to check that $\varphi_n * f$ can be written as

$$\varphi_n * f = f_n + \sum_{i=1}^m (\varphi_n * (b^i Z_i w) - b^i Z_i(\varphi_n * w)).$$

Therefore, using this, (3.3.8), (3.3.13), and (3.3.16), we obtain

$$\begin{aligned}
\mathbb{E}\left[\int_s^t f(r,X_r)dr \middle| \mathcal{F}_s\right] &\leq \liminf_{n\to\infty} \mathbb{E}\left[\int_s^t (\varphi_n * f)(r,X_r)dr \middle| \mathcal{F}_s\right] \\
&\leq \limsup_{n\to\infty} \mathbb{E}\left[\int_s^t f_n(r,X_r)dr \middle| \mathcal{F}_s\right] \\
&+ \sum_{i=1}^m \mathbb{E}\left[\int_s^t (\varphi_n * (b^i Z_i w) - b^i Z_i(\varphi_n * w))(r,X_r)dr \middle| \mathcal{F}_s\right] \\
&\leq C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})} \|f\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{G}))}.
\end{aligned}$$

This concludes the proof. □

Remark 3.3.2. Let us denote X_t by a solution to SDE (1.1.12). It is a priori not clear whether or not the integral $\int_0^t f(s, X_s)ds$ depends on the version of f. In other words, it is not obvious whether or not

$$\int_0^t f(s, X_s)ds = \int_0^t g(s, X_s)ds$$

holds when $f = g$ a.e. In Proposition 3.3.1, we proved the estimate (3.3.1) for continuous functions f in order that $(\varphi_n * f)(r, X_r)$ converges to $f(r, X_r)$ for any realization, which enables us to apply the Fatou's lemma in Step 3 of the proof. Note that in general $(\varphi_n * f)(r, \cdot)$ converges to $f(r, \cdot)$ only at the Lebesgue point of $f(r, \cdot)$, and it is not a priori clear whether or not $(\varphi_n * f)(r, X_r)$ converges to $f(r, X_r)$ almost surely.

Since Theorem 1.1.3 is a local statement, we introduce the following notion of a solution, which is useful for our purpose:

Definition 3.3.3. Suppose that τ is a $\mathcal{F}_t$-stopping time. X_t is called a *τ-solution* to SDE

$$dX_t = b(s, X_s)ds + \sigma(s, X_s)dB_s, \quad 0 \le t \le T$$

if w-almost surely,

$$X_t - X_0 = \int_0^{t\wedge\tau} b(s, X_s)ds + \int_0^{t\wedge\tau} \sigma(s, X_s)dB_s$$

holds for all $0 \le t \le T$.

Note that if X_t is a solution to SDE (1.1.12) and τ is any $\mathcal{F}_t$-stopping time, then $Y_t := X_{t\wedge\tau}$ is a τ-solution to SDE (1.1.12). The notion of τ-solution is useful when we consider a stochastic process before the time at which the process exits a certain region.

Remark 3.3.4. For $\mathcal{F}_t$-stopping time τ, let us denote $\mathcal{G}_t := \mathcal{F}_{t\wedge\tau}$. Assume that b satisfies the conditions (1.1.14), (1.1.15) for the exponents p, q satisfying (1.1.13), and X_t is τ-solution to SDE (1.1.12). Following the proof of Proposition 3.3.1, one can conclude that for any $f \in L^{q/2}([0,t], L^{p/2}(\mathbb{G}))$ such that $f(r, \cdot)$ is continuous for a.e. $r \in [0, t]$,

$$\mathbb{E}\left[\int_{s\wedge\tau}^{t\wedge\tau} f(r, X_r)dr \Big| \mathcal{G}_s\right] \le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})} \|f\|_{L^{q/2}([0,t], L^{p/2}(\mathbb{G}))} . \tag{3.3.17}$$

Thus, the estimate (3.3.1) is a special case of (3.3.17) with $\tau = \infty$. The Krylov-type estimate (3.3.17) is useful to prove the strong uniqueness of SDE (1.1.12) in Section 3.3.

Similarly, one can also prove that for any $f \in S^{r-1,(q,p)}([0,t]\times\mathbb{G})$ such that $f(r,\cdot)$ is continuous for a.e. $r \in [0,t]$,

$$\begin{aligned}\mathbb{E}\Big[\int_{s\wedge\tau}^{t\wedge\tau} f(r,X_r)dr\Big|\mathcal{G}_s\Big] &\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})}\|f\|_{L^q([0,T],L^p(\mathbb{G}))} \\ &\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})}\|f\|_{S^{r-1,(q,p)}([0,T]\times\mathbb{G})}. \qquad (3.3.18)\end{aligned}$$

Using the Krylov-type estimate (3.3.18), one can derive Itô's formula for the mixed-norm parabolic Sobolev spaces $\tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$.

Theorem 3.3.5. *Suppose that assumptions in Theorem 1.1.3 are satisfied, and X_t is a τ-solution to the SDE:*

$$dX_t = b(t,X_t)dt + \sum_{i=1}^m Z_i(t,X_t)\circ dB_t^i, \quad 0\le t\le T.$$

Then, for any $f \in \tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$ satisfying

$$f \quad \textit{continuous in } (t,x),$$

$$(f_t + \sum_{i=1}^m b^i Z_i f + \frac{1}{2}\sum_{i=1}^m Z_i^2 f)(t,\cdot),\ Z_1 f(t,\cdot),\cdots,Z_m f(t,\cdot) \quad \textit{continuous in } x \textit{ for } t\textit{-a.e.},$$

a process $f(t,X_t)$ is a τ-solution to

$$df(t,X_t) = (f_t + \sum_{i=1}^m b^i Z_i f + \frac{1}{2}\sum_{i=1}^m Z_i^2 f)(t,X_t)dt + \sum_{i=1}^m Z_i f(t,X_t)dB_t^i, \quad 0\le t\le T.$$

Proof. Since $t\to f(t,X_t)$ is continuous, it suffices to check that for each t,

$$\begin{aligned}&f(t\wedge\tau, X_{t\wedge\tau}) \\ &= \int_0^{t\wedge\tau}(f_t + \sum_i b^i Z_i f + \frac{1}{2}\sum_i Z_i^2 f)(s,X_s)ds + \int_0^{t\wedge\tau}\sum_i Z_i f(s,X_s)dB_s^i \quad (3.3.19)\end{aligned}$$

holds almost surely. Let us approximate f by smooth functions f_n in $\tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$ norm. More precisely, for a mollifier $\varphi_n(x) := n^Q\varphi(D(n)x)$ with $\varphi\in C_c^\infty(\mathbb{G})$, let us define $f_n := \varphi_n * f$. Then,

$$(f_n)_t + \sum_i b^i Z_i f_n + \frac{1}{2}\sum_i Z_i^2 f_n \to f_t + \sum_i b^i Z_i f + \frac{1}{2}\sum_i Z_i^2 f \quad \text{in } S^{r-1,(q,p)}.$$

Since for t-a.e., both $\Big[(f_n)_t + \sum_i b^i Z_i f_n + \frac{1}{2}\sum_i Z_i^2 f_n\Big](t,\cdot)$ and $(f_t + \sum_i b^i Z_i f + \frac{1}{2}\sum_i Z_i^2 f)(t,\cdot)$ are continuous, using the estimate (3.3.18), one can conclude that

$$\lim_{n\to\infty} \mathbb{E}\Big|\int_0^{t\wedge\tau} ((f_n)_t + \sum_i b^i Z_i f_n + \frac{1}{2}\sum_i Z_i^2 f_n)(s, X_s)ds - \int_0^{t\wedge\tau} (f_t + \sum_i b^i Z_i f + \frac{1}{2}\sum_i Z_i^2 f)(s, X_s)ds\Big| = 0. \quad (3.3.20)$$

Also, since for t-a.e., both $Z_i f_n(t,\cdot)$ and $Z_i f(t,\cdot)$ are continuous, using the Itô's isometry and (3.3.18), we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\Big|\int_0^{t\wedge\tau} \sum_i Z_i f_n(s, X_s)dB_s^i - \int_0^{t\wedge\tau} \sum_i Z_i f(s, X_s)dB_s^i\Big|^2 &= \lim_{n\to\infty} \mathbb{E}\int_0^{t\wedge\tau} \Big[\sum_i Z_i(f_n - f)\Big]^2(s, X_s)ds \\ &\le C \lim_{n\to\infty} \Big\| \Big[\sum_i Z_i(f_n - f)\Big]^2 \Big\|_{S^{r-1,(q,p)}} \\ &\le C \lim_{n\to\infty} \|f_n - f\|^2_{S^{r,(2q,2p)}} \\ &\le C \lim_{n\to\infty} \|f_n - f\|^2_{\tilde{S}^{r+1,(q,p)}} = 0. \end{aligned} \quad (3.3.21)$$

Note that parabolic Sobolev embedding Theorem 3.1.9 is applicable in the last line since $\frac{2}{q} + \frac{Q}{p} < 1 + \frac{2}{2q} + \frac{Q}{2p}$. Furthermore, according to Theorem 3.1.9 again,

$$\|f_n - f\|_{L^\infty} \le \|f_n - f\|_{S^{r,(\infty,\infty)}} \le C\,\|f_n - f\|_{\tilde{S}^{r+1,(q,p)}}.$$

Therefore,

$$\lim_{n\to\infty} |f_n(t\wedge\tau, X_{t\wedge\tau}) - f(t\wedge\tau, X_{t\wedge\tau})| \le \lim_{n\to\infty} \|f_n - f\|_{L^\infty} = 0. \quad (3.3.22)$$

Since f_n's are smooth, Itô's formula yields that

$$f_n(t\wedge\tau, X_{t\wedge\tau}) = \int_0^{t\wedge\tau} ((f_n)_t + \sum_i b^i Z_i f_n + \frac{1}{2}\sum_i Z_i^2 f_n)(s, X_s)ds + \int_0^{t\wedge\tau} \sum_i Z_i f_n(s, X_s)dB_s^i. \quad (3.3.23)$$

Therefore, sending $n\to\infty$ along the appropriate subsequence using (3.3.20), (3.3.21), and (3.3.22), we obtain (3.3.19). □

Conjugated SDE

In this section, we derive an auxiliary SDE transformed by the original SDE (1.1.12), which is called a *conjugated SDE*. The advantage of this new SDE over the original SDE is that it possesses a more regular drift coefficient. This idea goes back to the Zvonkin's work [**Z3**], and has been successfully used to prove the well-posedness of SDEs with the additive noise. In the next proposition, as in [**FF3**, **Z1**], we obtain an auxiliary SDE using a function u. Recall that functions u, Φ, and the open set Ω are defined in Remark 3.2.3.

Proposition 3.3.6. *For $0 \le t \le T$ and $x \in \Phi(t, \Omega)$, let us define vector fields $\tilde{b}$ and $\tilde{\sigma}_i$ $(1 \le i \le m)$ via*

$$\tilde{b}(t,x) = [\lambda u + \frac{1}{2}\sum_{i=1}^{m} Z_i' Z_i](t, \Phi^{-1}(t,x)), \quad \tilde{\sigma}_i(t,x) = (Z_i + Z_i u)(t, \Phi^{-1}(t,x))$$

($Z_i' : \mathbb{R}^N \to \mathbb{R}^{N\times N}$ is a standard derivative of the map $Z_i : \mathbb{R}^N \to \mathbb{R}^N$, and $Z_i' Z_i$ is interpreted as a product of $N \times N$ matrix Z_i' and a vector $Z_i \in \mathbb{R}^N$). Suppose that X_t is a τ-solution to SDE (1.1.12) *for a $\mathcal{F}_t$-stopping time τ such that $X_t \in \Omega$ for $0 \le t \le T$. Then, $Y_t = \Phi(t, X_t)$ is a τ-solution to the following SDE:*

$$\begin{cases} dY_t = \tilde{b}(t,Y_t)dt + \sum_{i=1}^m \tilde{\sigma}_i(t,Y_t)dB_t^i, \\ Y_0 = \Phi(0,x_0). \end{cases} \tag{3.3.24}$$

Proof. In order to alleviate notations, we omit the summation symbol $\sum_i$. Recall that $u(t,x)$ is continuous, $(\partial_t u + b^i Z_i u + \frac{1}{2}Lu)(s,\cdot)$, $Z_i u(s,\cdot)$ are continuous for s-a.e. (see Remark 3.2.3), and $u \in \tilde{S}^{r+1,(q,p)}([0,T]\times\mathbb{G})$. Therefore, using Itô's formula for non-smooth functions Theorem 3.3.5, we have

$$\begin{aligned}
u(t,X_t) &= u(0,X_0) + \int_0^{t\wedge\tau} (\partial_t u + b^i Z_i u + \frac{1}{2}Lu)(s,X_s)ds + \int_0^{t\wedge\tau} Z_i u(s,X_s)dB_s^i \\
&= u(0,X_0) - \int_0^{t\wedge\tau} (b - \lambda u)(s,X_s)ds + \int_0^{t\wedge\tau} Z_i u(s,X_s)dB_s^i \\
&= u(0,X_0) - X_t + X_0 + \int_0^{t\wedge\tau} \lambda u(s,X_s)ds \\
&+ \int_0^{t\wedge\tau} Z_i(X_s)\circ dB_s^i + \int_0^{t\wedge\tau} Z_i u(s,X_s)dB_s^i \\
&= u(0,X_0) - X_t + X_0 + \int_0^{t\wedge\tau} [\lambda u + \frac{1}{2}Z_i' Z_i](s,X_s)ds + \int_0^{t\wedge\tau} (Z_i + Z_i u)(s,X_s)dB_s^i.
\end{aligned}$$

Since $X_t \in \Omega$ for $0 \le t \le T$, $Y_t \in \Phi(t, \Omega)$. Therefore,

$$\begin{aligned}
Y_t - Y_0 &= \Phi(t, X_t) - \Phi(t, X_0) \\
&= \int_0^{t\wedge\tau} [\lambda u + \frac{1}{2} Z_i' Z_i](s, X_s) ds + \int_0^{t\wedge\tau} (Z_i + Z_i u)(s, X_s) dB_s^i \\
&= \int_0^{t\wedge\tau} [\lambda u + \frac{1}{2} Z_i' Z_i](s, \Phi^{-1}(s, Y_s)) ds + \int_0^{t\wedge\tau} (Z_i + Z_i u)(s, \Phi^{-1}(s, Y_s)) dB_s^i \\
&= \int_0^{t\wedge\tau} \tilde{b}(s, Y_s) ds + \int_0^{t\wedge\tau} \tilde{\sigma}_i(s, Y_s) dB_t^i.
\end{aligned}$$

□

Strong uniqueness

Using the conjugated SDE (3.3.24), one can prove that a strong solution to SDE (1.1.12) is unique:

Theorem 3.3.7. *Suppose that X_t^1, X_t^2 are τ-solutions to (1.1.12) for a $\mathcal{F}_t$-stopping time τ such that $X_t^1, X_t^2 \in \Omega'$ for $0 \le t \le T$. Then, $X_t^1 = X_t^2$ almost surely.*

Proof. Let us define $Y_t^k = \Phi(t, X_t^k)$ for $k = 1, 2$. Then, according to Proposition 3.3.6, Y_t^k is a τ-solution to SDE (3.3.24), and $Y_t^k \in \Phi(t, \Omega)$ for each t. Thus,

$$Y_t^1 - Y_t^2 = \int_0^{t\wedge\tau} [\tilde{b}(s, Y_s^1) - \tilde{b}(s, Y_s^2)] ds + \sum_{i=1}^{m} \int_0^{t\wedge\tau} [\tilde{\sigma}_i(s, Y_s^1) - \tilde{\sigma}_i(s, Y_s^2)] dB_s^i. \quad (3.3.25)$$

Let us first check that $\tilde{b}(t, \cdot)$ is Lipschitz continuous on $\Phi(t, \Omega)$ uniformly in t. Note that $\|\nabla u\|_{L^\infty([0,T]\times\Omega)} \le \frac{1}{2}$ (see Step 3 of the proof of Proposition 3.2.2) and a map $x \to Z_i' Z_i u(x)$ is smooth on $\mathbb{R}^N$. Thus, applying a chain rule, we obtain the uniform Lipschitz continuity of $\tilde{b}(t, \cdot)$ since $\|\nabla \Phi^{-1}(t, \cdot)\|_{L^\infty(\Phi(t,\Omega))}$ is uniformly bounded in t (see Proposition 3.2.2).

Therefore, using this fact and applying Itô's formula to (3.3.25), for any $a > 2$,

$$\begin{aligned}
&d|Y_t^1 - Y_t^2|^a \\
&\le (L|Y_s^1 - Y_s^2|^a + \frac{a(a-1)}{2} |\tilde{\sigma}(s, Y_s^1) - \tilde{\sigma}(s, Y_s^2)|^2 |Y_s^1 - Y_s^2|^{a-2}) \mathbb{1}_{[0,\tau]} ds + W_s \mathbb{1}_{[0,\tau]} dB_s
\end{aligned} \quad (3.3.26)$$

for some constant $L > 0$ and the process W_s satisfying

$$|W_s| \le C|Y_s^1 - Y_s^2|^{a-1} |\tilde{\sigma}(s, Y_s^1) - \tilde{\sigma}(s, Y_s^2)|. \quad (3.3.27)$$

Here, $\tilde{\sigma}$ denotes a $N \times m$ matrix whose columns consist of $\tilde{\sigma}_i$'s. In order to deal with the right hand side of (3.3.26), we need the following lemma, motivated by **[FF3]** and **[KR]**.

Lemma 3.3.8. *There exists a continuous and $\mathcal{F}_t$-adapted process A_t satisfying*

$$\frac{a(a-1)}{2} \int_0^t |\tilde{\sigma}(s, Y_s^1) - \tilde{\sigma}(s, Y_s^2)|^2 \mathbb{1}_{[0,\tau]} ds = \int_0^t |Y_s^1 - Y_s^2|^2 dA_s \tag{3.3.28}$$

and

$$\mathbb{E}\, e^{cA_s} < \infty \tag{3.3.29}$$

for any $c > 0$.

Proof. Let us define a process A_t by

$$A_t := \frac{a(a-1)}{2} \int_0^t \mathbb{1}_{Y_s^1 \neq Y_s^2} \frac{|\tilde{\sigma}(s, Y_s^1) - \tilde{\sigma}(s, Y_s^2)|^2}{|Y_s^1 - Y_s^2|^2} \mathbb{1}_{[0,\tau]} ds.$$

Then, it is obvious that A_t satisfies (3.3.28), and it suffices to prove the estimate (3.3.29). Note that since $Y_t^k \in \Phi(t, \Omega)$, using the property (iii) in Proposition 3.2.2,

$$|X_t^1 - X_t^2| = |\Phi^{-1}(t, Y_t^1) - \Phi^{-1}(t, Y_t^2)| \leq 2|Y_t^1 - Y_t^2|.$$

Using this, we have

$$\begin{aligned} A_t &\leq C \sum_i \int_0^t \mathbb{1}_{Y_s^1 \neq Y_s^2} \frac{|(Z_i + Z_i u)(s, X_s^1) - (Z_i + Z_i u)(s, X_s^2)|^2}{|Y_s^1 - Y_s^2|^2} \mathbb{1}_{[0,\tau]} ds \\ &\leq C \sum_i \int_0^t \mathbb{1}_{X_s^1 \neq X_s^2} \frac{|(Z_i + Z_i u)(s, X_s^1) - (Z_i + Z_i u)(s, X_s^2)|^2}{|X_s^1 - X_s^2|^2} \mathbb{1}_{[0,\tau]} ds. \end{aligned} \tag{3.3.30}$$

Here, we used the Lipschitz continuity of $\Phi^{-1}(t, \cdot)$ (see Proposition 3.2.2). For mollifiers $\rho_n(x) := n^N \rho(nx)$ with $\rho \in C_c^\infty(\mathbb{R}^N)$, let us first prove that for any $c \in \mathbb{R}$,

$$\begin{aligned} &\limsup_n \mathbb{E} \exp \Bigg[c \int_0^{t\wedge\tau} \mathbb{1}_{X_s^1 \neq X_s^2} \\ &\qquad \cdot \frac{|[(Z_i + Z_i u) * \rho_n](s, X_s^1) - [(Z_i + Z_i u) * \rho_n](s, X_s^2)|^2}{|X_s^1 - X_s^2|^2} ds \Bigg] < \infty. \end{aligned} \tag{3.3.31}$$

Here, $*$ denotes the standard convolution operator on the Euclidean spaces: $(f * g)(x) = \int_{\mathbb{R}^N} f(x - y) g(y) dy$. Let us choose a cutoff function $\phi \in C_c^\infty(\mathbb{R}^N)$ such that $\phi = 1$ on Ω, and denote $K := \text{supp}(\phi)$. Also, denote $\mathcal{M}$ by a Hardy-Littlewood maximal operator with respect to the Euclidean distance and Lebesgue measure on $\mathbb{R}^N$. Since $X_t^k \in \Omega$, we have

$$\begin{aligned}
&\frac{|[(Z_i + Z_i u) * \rho_n](s, X_s^1) - [(Z_i + Z_i u) * \rho_n](s, X_s^2)|^2}{|X_s^1 - X_s^2|^2} \\
&= \frac{|\phi \cdot \{(Z_i + Z_i u) * \rho_n\}(s, X_s^1) - \phi \cdot \{(Z_i + Z_i u) * \rho_n\}(s, X_s^2)|^2}{|X_s^1 - X_s^2|^2} \\
&\leq C(|\mathcal{M}\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]|^2(s, X_s^1) + |\mathcal{M}\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]|^2(s, X_s^2)).
\end{aligned} \tag{3.3.32}$$

Here, we used the fact that for some constant $C = C(N)$, the inequality

$$|f(x) - f(y)| \leq C|x - y|(\mathcal{M}\nabla f(x) + \mathcal{M}\nabla f(y)) \tag{3.3.33}$$

holds for any $f \in C^\infty(\mathbb{R}^N)$. On the other hand, using the fact that ϕ has compact support K and $u \in S^{r+1,(q,p)}$, for any n,

$$\begin{aligned}
&\|\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]\|_{L^q([0,T], L^p(\mathbb{R}^N))} \\
&\leq \|\phi \cdot \{[\nabla(Z_i + Z_i u)] * \rho_n\}\|_{L^q([0,T], L^p(\mathbb{R}^N))} + \|\nabla\phi \cdot \{(Z_i + Z_i u) * \rho_n\}\|_{L^q([0,T], L^p(\mathbb{R}^N))} \\
&\leq C\big(\|\nabla(Z_i + Z_i u)\|_{L^q([0,T], L^p(K))} + \|Z_i + Z_i u\|_{L^q([0,T], L^p(K))} \big) \\
&\leq C(1 + \|u\|_{S^{r+1,(q,p)}([0,T] \times \mathbb{R}^N)}).
\end{aligned}$$

Here, we used the convolution inequality in the second line. Also, in the third line, we used

$$\|\nabla(Z_i u)\|_{L^q([0,T], L^p(K))} \leq C(K) \, \|u\|_{S^{r+1,(q,p)}([0,T] \times \mathbb{R}^N)}$$

(recall that each standard vector field on $\mathbb{R}^N$ can be written as a linear combination of commutators of Z_i's with order $\leq r$).

Therefore, we obtain

$$\begin{aligned}
&\limsup_n \big\| |\mathcal{M}\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]|^2 \big\|_{L^{q/2}([0,T], L^{p/2}(\mathbb{R}^N))} \\
&\qquad \leq C \limsup_n \big\| |\nabla[\phi \cdot \{(Z_i + Z_i u) * \varphi_n\}]|^2 \big\|_{L^{q/2}([0,T], L^{p/2}(\mathbb{R}^N))} < \infty
\end{aligned} \tag{3.3.34}$$

since the maximal operator $\mathcal{M}$ is bounded in $L^p(\mathbb{R}^N)$. Since $\mathcal{M}\nabla[\phi \cdot \{(Z_k + Z_k u) * \rho_n\}](s, \cdot)$ is continuous s-a.e, using (3.3.17), one can conclude that

$$\limsup_n \mathbb{E}\Big[\int_{s\wedge\tau}^{t\wedge\tau} |\mathcal{M}\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]|^2(r, X_r^k) dr \Big| \mathcal{G}_s \Big]$$

$$\le C(t-s)^{1-(\frac{2}{q}+\frac{Q}{p})} \limsup_n \left\| |\mathcal{M}\nabla[\phi \cdot \{(Z_i + Z_i u) * \rho_n\}]|^2 \right\|_{L^{q/2}([0,t],L^{p/2}(\mathbb{R}^N))} . \tag{3.3.35}$$

Therefore, using (3.3.32), (3.3.34), (3.3.35), we obtain the estimate (3.3.31).

Finally, let us check that (3.3.31) implies (3.3.29). Since $(Z_i + Z_i u)(s,\cdot)$ is continuous for s-a.e, $[(Z_i + Z_i u) * \rho_n](s,\cdot)$ converges to $(Z_i + Z_i u)(s,\cdot)$ pointwisely in $x \in \mathbb{R}^N$ for s-a.e. Thus, using (3.3.30), (3.3.31), and the Fatou's lemma, we obtain (3.3.29). □

Let us go back to the proof of Theorem 3.3.7. Applying Lemma 3.3.8 to (3.3.26), we have

$$\begin{aligned} e^{-A_t}|Y_t^1 - Y_t^2|^a &= \int_0^t -e^{-A_s}|Y_s^1 - Y_s^2|^a dA_s + \int_0^t e^{-A_s} d|Y_s^1 - Y_s^2|^a \\ &\le \int_0^t Le^{-A_s}|Y_s^1 - Y_s^2|^a \mathbb{1}_{[0,\tau]} ds + \int_0^t e^{-A_s} W_s \mathbb{1}_{[0,\tau]} dM_s. \end{aligned} \tag{3.3.36}$$

Let us define a $\mathcal{F}_t$-stopping time τ_l by

$$\tau_l = \inf\{0 \le t \le T \mid |Y_t^1| > l \text{ or } |Y_t^2| > l\},$$

and $\tau_l = T$ if the above set is empty. Then, by (3.3.36),

$$\begin{aligned} &e^{-A_{t\wedge\tau_l}}|Y_{t\wedge\tau_l}^1 - Y_{t\wedge\tau_l}^2|^a \\ &\quad \le \int_0^t Le^{-A_s}|Y_s^1 - Y_s^2|^a \mathbb{1}_{[0,\tau]}\mathbb{1}_{[0,\tau_l]} ds + \int_0^t e^{-A_s} W_s \mathbb{1}_{[0,\tau]}\mathbb{1}_{[0,\tau_l]} dM_s. \end{aligned} \tag{3.3.37}$$

Let us check that that $\tilde{\sigma}$ is bounded on $\Phi(t,\Omega)$ uniformly in t. Since $u \in \tilde{S}^{r+1,(q,p)}$ (see Remark 3.2.3), according to Theorem 3.1.9, $Z_i u \in L^\infty([0,T] \times \mathbb{R}^N)$. Also, it is obvious that $Z_i(\cdot)$ is bounded on Ω. These facts imply the uniform boundedness of $\tilde{\sigma}(t,\cdot)$ on $\Phi(t,\Omega)$.

Thus, since $|Y_s^1|, |Y_s^2| \le l$ for $s \in [0,\tau_l]$, for some constant C,

$$|Y_s^1 - Y_s^2|^{a-1}|\tilde{\sigma}(s,Y_s^1) - \tilde{\sigma}(s,Y_s^2)| < Cl^{a-1}$$

for any $s \in [0,\tau_l]$. From this and (3.3.27), it follows that $s \mapsto e^{-A_s} W_s \mathbb{1}_{[0,\tau]}\mathbb{1}_{[0,\tau_l]}$ is a square-integrable process. Therefore, taking the expectation in (3.3.37),

$$\mathbb{E}[e^{-A_{t\wedge\tau_l}}|Y_{t\wedge\tau_l}^1 - Y_{t\wedge\tau_l}^2|^a] \le L\int_0^t \mathbb{E}[e^{-A_s}|Y_s^1 - Y_s^2|^a \mathbb{1}_{[0,\tau]}\mathbb{1}_{[0,\tau_l]}] ds.$$

Sending $l \to \infty$ and applying the Fatou's lemma,

$$\mathbb{E}[e^{-A_t}|Y_t^1 - Y_t^2|^a] \leq L \int_0^t \mathbb{E}[e^{-A_s}|Y_s^1 - Y_s^2|^a \mathbb{1}_{[0,\tau]}]ds.$$

Applying Grönwall's inequality, we obtain

$$\mathbb{E}[e^{-A_t}|Y_t^1 - Y_t^2|^a] = 0.$$

Using Hölder's inequality,

$$\mathbb{E}\,|Y_t^1 - Y_t^2|^{a/2} \leq [\mathbb{E}\,e^{-A_t}|Y_t^1 - Y_t^2|^a]^{1/2}[\mathbb{E}\,e^{A_t}]^{1/2} = 0$$

since $\mathbb{E}\,e^{A_t}$ is finite (see the estimate (3.3.29)). Thus, we have

$$\mathbb{E}\,|Y_t^1 - Y_t^2|^{a/2} = 0.$$

Since trajectories are continuous in time and $\Phi(t, \cdot)$ is bijective from Ω onto $\Phi(t, \Omega)$ for each t, proof is concluded. □

Conclusion of the proof of Theorem 1.1.3

In this section, we finally complete the proof of the main result Theorem 1.1.3. We show the weak existence and strong uniqueness separately, and then apply the Yamada-Watanabe principle. Since we have already proved the uniqueness of a solution in Section 3.3, it suffices to derive the existence of a weak solution. Let us first recall the well-known fact about the existence of a weak solution:

Theorem 3.3.9. *Suppose that $b(t, \cdot)$ and $\sigma(t, \cdot)$ are continuous in x and have linear growth for each $0 \leq t \leq T$. Then, SDE*

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad 0 \leq t \leq T, \\ X_0 = x_0, \end{cases}$$

admits a weak solution.

Since Theorem 1.1.3 is a local statement, we localize coefficients of SDE (1.1.12), and then apply Theorem 3.3.9. More precisely, choose a cutoff function $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi = 1$ on Ω, and consider the following SDE:

$$\begin{cases} dX_t = (\psi b)(t, X_t)dt + \sum_{i=1}^m (\psi Z_i)(t, X_t) \circ dB_t^i, \quad 0 \leq t \leq T, \\ X_0 = x_0. \end{cases} \tag{3.3.38}$$

Corollary 3.3.10. *There exists a weak solution to SDE* (3.3.38)*.*

Proof. Recall that $b(t,\cdot)$ is continuous for t-a.e (see Remark 1.1.4) and ψ is a cutoff function. Thus, according to Theorem 3.3.9, SDE (3.3.38) has a weak solution. □

Now, we are ready to conclude the proof of the main result Theorem 1.1.3.

Proof of Theorem 1.1.3. Let us consider the following SDE:

$$\begin{cases} dX_t = b_t(X.)\mathbb{1}_{t<\tau(X.)}dt + \sum_{i=1}^m (Z_i)_t(X.)\mathbb{1}_{t<\tau(X.)} \circ dB_t^i, \quad 0 \le t \le T, \\ X_0 = x_0. \end{cases} \tag{3.3.39}$$

Here, $b_t(x.)$, $(Z_i)_t(x.)$ are $\mathbb{R}^N$-valued progressive functions on the space $[0,T] \times C([0,T],\mathbb{R}^N)$, equipped with the canonical filtration $\mathcal{F}_t = \sigma\{x_s|s \le t\}$, defined by $b_t(x.) := b(t,x_t)$, $(Z_i)_t(x.) := Z_i(t,x_t)$. Also, $\mathcal{F}_t$-stopping time τ is defined by $\tau(x.) := \inf\{t \le T \mid x_t \notin \Omega\}$ and $\tau(x.) = T$ if the set is empty. Uniqueness of a strong solution to SDE (3.3.39) follows from Theorem 3.3.7, and the existence of a weak solution follows from Corollary 3.3.10. Therefore, Yamada-Watanabe principle (see Theorem 2.1.2) concludes that a unique strong solution exists to SDE (3.3.39). This concludes the proof of Theorem 1.1.3. □

Bibliography

[A] Ambrosio, L. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.* **158** (2004), no. 2, 227-260.

[B] Baudoin, F. Stochastic analysis on sub-Riemannian manifolds with transverse symmetries. *Ann. Probab.* **45** (2017), no. 1, 56-81.

[BB] Baudoin, F., Bonnefont, M. Log-Sobolev inequalities for subelliptic operators satisfying a generalized curvature dimension inequality. *J. Funct. Anal.* **262** (2012), no. 6, 2646-2676.

[BBBC] Bakry, D., Baudoin, F., Bonnefont, M., Chafai, D. On gradient bounds for the heat kernel on the Heisenberg group. *J. Funct. Anal.* **255** (2008), no. 8, 1905-1938.

[BFGM] Beck, L., Flandoli, F., Gubinelli, M., Maurelli, M. Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness, Arxiv (2014).

[BGM] Baudoin, F., Gordina, M., Melcher, T. Quasi-invariance for heat kernel measures on sub-Riemannian infinite dimensional Heisenberg groups. *Trans. Amer. Math. Soc.* **365** (2013), no. 8, 4313-4350.

[BL] Bergh, J., Löfström, J. *Interpolation Spaces.* Springer-Verlag, Berlin Heidelberg New York, 1976.

[BLU] Bonfiglioli, A., Lanconelli, E., Uguzzoni, F. Stratified Lie groups and potential theory for their sub-Laplacians. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[CI] Constantin, P., Iyer, G. A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations. *Comm. Pure Appl. Math.* **61** (2008), no 3, 330-345.

[CW] Coifman, R. R, Weiss G. Analyse harmonique non-commutative sur certains espaces homogènes, Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.

[DL] Diperna, R.J., Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.* **98** (1989), no. 3, 511-547.

[FF1] Fedrizzi, E., Flandoli, F. Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. *Stochastics.* **83** (2011), no. 3, 241-257.

[FF2] Fedrizzi, E., Flandoli, F. Noise prevents singularities in linear transport equations. *J. Funct. Anal.* **264** (2013), no. 6, 1329-1354.

[FF3] Fedrizzi, E., Flandoli, F. Hölder flow and differentiability for SDEs with non-regular drift. *Stoch. Anal. Appl.* **31** (2013), no. 4, 708-736.

[F1] Fedrizzi, E. Uniqueness and flow theorems for solutions of SDEs with low regularity of the drift. Tesi di Laurea in Mathematica, Università di Pisa (2009).

[FGP] Flandoli, F., Gubinelli, M., Priola, E. Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (2010), no. 1, 1-53.

[F2] Folland, G. B. Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.* **13** (1975), no. 2, 161-207.

[F3] Folland, G. B. Applications of analysis on nilpotent groups to partial differential equations. *Bull. Amer. Math. Soc.* **83** (1977), no. 5, 912-930.

[G] Gopala Rao V.R., Parabolic Function Spaces With Mixed Norm, *Trans. Amer. Math. Soc.* **246** (1978), 451-461.

[H] Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **119** (1967), 147-171.

[J] Jerison, D., Sánchez-Calle, A.: Estimates for the heat kernel for a sum of squares of vector fields. *Indiana Univ. Math. J.* **35** (1986), no. 4, 835-854.

[KV] Korányi, A., Vági, S. Singular integrals on homogeneous spaces and some problems of classical analysis. *Ann. Scuola Norm. Sup. Pis.* **25** (1971), 575-648.

[K] Khasminskii, R.Z. On positive solutions of the equation $Au + Vu = 0$. *Theoret. Probab. Appl.* **4** (1959), no. 3, 332-341.

[K1] Krylov, N. V. On estimates of the maximum of a solution of a parabolic equation and estimates of the distribution of a semimartingale. *Mat. Sb.* **130(172)** (1986), 207-221, 284.

[K2] Krylov, N.V. The heat equation in $L_q((0,T), L_p)$-spaces with weights. *Siam J. Math. Anal.* **32** (2001), no. 5, 1117-1131.

[K3] Kunita, H. *Stochastic Flows and Stochastic Differential Equations.* Cambridge Univ. Press, Cambridge, 1990.

[KR] Krylov, N.V., Röckner, M. Strong solutions to stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields.* **131** (2005), no. 2, 154-196.

[L] Lorentz, G. Some new function spaces. *Ann. of Math.* **51** (1950), no. 1, 37-55.

[M] Malliavin, P. Stochastic calculus of variation and hypoelliptic operators. In Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976) Wiley, New York. 195-263.

[N1] Nam, K. Stochastic Differential Equations with Critical Drifts. (2018) https://arxiv.org/abs/1802.00074.

[N2] Nam, K. Hypoelliptic diffusions with singular drifts. (2018) https://arxiv.org/abs/1806.10172.

[P] Portenko, N. I. *Generalized diffusion processes.* Translations of Mathematical Monographs, 83. American Mathematical Society, Providence, RI, 1990.

[SV] Stroock, D. W., Varadhan, S.R.S. On the support of diffusion processes with applications to the strong maximum principle. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability.* (1970/71). vol. III. 333-359..

[YW1] Watanabe, S., Yamada, T. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.* **11** (1971), 155-167.

[YW2] Watanabe, S., Yamada, T. On the uniqueness of solutions of stochastic differential equations. II. *J. Math. Kyoto Univ.* **11** (1971), 553-563.

[Z1] Zhang, X. Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. *Electronic Journal of Probability.* **16** (2011), no. 38, 1096-1116.

[Z2] Zhang, X. Stochastic differential equations with Sobolev diffusion and singular drift and applications. *Ann. Appl. Probab.* **26** (2016), no. 5, 2697-2732.

[Z3] Zvonkin, A.K. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb.* **93** (1974), 129-149.