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Supersymmetric σ -Models in 2-Dimensions

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Abstract

This is the text of a lecture given at the Newton Institute Euro-conference on Duality and Supersymmetric Theories, 7-18 April, 1997, Cambridge, England. To be published in the proceedings.

1 The bosonic σ -model

I have been asked to give a brief introduction to supersymmetric σ -models in two space-time dimensions.

Let us recall first the properties of the bosonic σ -models. Let $x^\mu, \mu = 0, 1$ be the two-dimensional coordinates. The fields

$$\phi^i(x) \quad i = 1, \dots, N \quad (1.1)$$

are valued in a Riemannian manifold of metric $G_{ij}(\phi)$, called the target space. The action is

$$I = -\frac{1}{2} \int d^2x G_{ij}(\phi) \partial_\mu \phi^i(x) \partial^\mu \phi^j(x). \quad (1.2)$$

A simple example can be obtained by starting from the free theory

$$I = - \int d^2x \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a \quad a = 1, \dots, N+1 \quad (1.3)$$

and imposing the constraint

$$\sum \phi^a \phi^a = 1 \quad (1.4)$$

Solving for ϕ^{N+1} we obtain an action of the form (1.2) where

$$G_{ij}(\phi) = \delta_{ij} + \frac{\phi^i \phi^j}{1 - \sum \phi^k \phi^k}, \quad i, j, k = 1, \dots, N \quad (1.5)$$

is the metric on the sphere S^N .

Another well-known example is the chiral model given by the action

$$I = -\frac{1}{2} \int d^2x \operatorname{tr} \partial_\mu U(x) \partial^\mu U^{-1}(x), \quad (1.6)$$

where U is a matrix representation of a (usually compact) Lie group and tr denotes the trace. This also can be written in the form (1.2) by introducing parameters ϕ^i on the group manifold.

A change of coordinates on the Riemannian manifold $\phi^i \rightarrow \phi'^i, G_{ij}(\phi) \rightarrow G'_{ij}(\phi')$, where

$$\phi^i = \phi'^i(\phi'), \quad (1.7)$$

$$G'_{ij}(\phi') = \frac{\partial \phi^k}{\partial \phi'^i} \frac{\partial \phi^l}{\partial \phi'^j} G_{kl}(\phi), \quad (1.8)$$

leaves the action (1.2) invariant. This reparameterization invariance gives the action a geometric meaning. Isometries of the metric of the Riemannian

manifold correspond to the internal symmetries of the model. Let us recall that an isometry is given in infinitesimal form by

$$\delta\phi^i = \xi^i(\phi), \quad (1.9)$$

$$D_j\xi_k + D_k\xi_j = 0, \quad (1.10)$$

$$D_j\xi_k = \partial_j\xi_k - \Gamma^i_{jk}(\phi)\xi_i, \quad (1.11)$$

where Γ^i_{jk} is the Levi-Civita connection

$$\Gamma^i_{jk} = \frac{1}{2}G^{il}(\partial_j G_{kl} + \partial_k G_{jl} - \partial_l G_{jk}). \quad (1.12)$$

(1.10) is Killing's equation. It should be obvious which are the isometries for the above examples of the spherical model and of the chiral model.

It is often convenient to introduce light-cone coordinates in two dimensions

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1). \quad (1.13)$$

When written in light cone coordinates the action (1.2) is invariant under the transformation $x^\pm \rightarrow x'^\pm(x^\pm)$, $x^- \rightarrow x'^-(x^-)$ (in the Euclidean setting the light-cone coordinates become two complex conjugate coordinates z and \bar{z} and the action is invariant under holomorphic transformations of these coordinates). A special case of this transformation is the scale transformation $x^\mu \rightarrow x'^\mu = \lambda x^\mu$ where λ is a constant.

One can add to the action (1.2) the interaction

$$I_{WZW} = -\frac{1}{2} \int d^2x \varepsilon^{\mu\nu} B_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j, \quad (1.14)$$

where

$$\varepsilon_{01} = -\varepsilon_{10} = 1, \quad \varepsilon_{00} = \varepsilon_{11} = 0, \quad (1.15)$$

$$B_{ij}(\phi) = -B_{ji}(\phi). \quad (1.16)$$

This interaction is usually called the Wess-Zumino-Witten (WZW) term [1] [2]. The entire action is invariant under reparameterization of the field manifold, which can be written in infinitesimal form as

$$\begin{aligned} \delta B_{ij}(\phi) &= D_i V_j(\phi) - D_j V_i(\phi) \\ &= \partial_i V_j(\phi) - \partial_j V_i(\phi) \end{aligned} \quad (1.17)$$

$$\delta\phi^i = -V^i(\phi) \quad (1.18)$$

$$\delta G_{ij}(\phi) = D_i V_j(\phi) + D_j V_i(\phi). \quad (1.19)$$

The antisymmetric quantity B_{ij} may not transform like a tensor when we go from one coordinate patch to another on the field manifold, but may change by a gauge transformation (1.17). If the manifold has a nontrivial topology

the WZW term may become multivalued and need to be multiplied by a quantized coefficient so that the exponentiated action in the functional path integral is single valued.

Let us write the total action in light-cone coordinates

$$I = \int dx^+ dx^- (G_{ij} - B_{ij}) \partial_+ \phi^i \partial_- \phi^j. \quad (1.20)$$

It is easy to see that the corresponding equations of motion are

$$\partial_+ \partial_- \phi^i + (\Gamma^i_{jk} + H^i_{jk}) \partial_+ \phi^j \partial_- \phi^k = 0, \quad (1.21)$$

$$H_{ijk}(\phi) = \frac{1}{2} (\partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij}), \quad (1.22)$$

i.e.

$$\hat{D}_+ \partial_- \phi^i = 0 \quad \text{or} \quad \hat{D}_- \partial_+ \phi^i = 0, \quad (1.23)$$

where \hat{D}_\pm are defined with the connection

$$\hat{\Gamma}_\pm^i{}_{jk} = \Gamma^i_{jk} \pm H^i_{jk}, \quad (1.24)$$

which is not symmetric in j and k . The antisymmetric part of the connection is the torsion tensor (which in this case is totally antisymmetric in all three indices, when the upper index is lowered). One can say that the WZW term introduces torsion on the field manifold, the torsion being given by the curl of B_{ij} as in (1.22).

2 Supersymmetry in two dimensions

In Minkowski two-dimensional space-time, Majorana-Weyl spinors transform as

$$\psi_+^i \rightarrow e^{\frac{1}{2}l} \psi_+^i, \quad (2.1)$$

$$\psi_-^i \rightarrow e^{-\frac{1}{2}l} \psi_-^i \quad (2.2)$$

under Lorentz transformations of parameter l . Here $+$ and $-$ denote the two different chiralities. Vectors, e.g. $v_\pm^i = \partial_\pm \phi^i$, transform as

$$v_+^i \rightarrow e^l v_+^i, \quad (2.3)$$

$$v_-^i \rightarrow e^{-l} v_-^i. \quad (2.4)$$

The (1,1) supersymmetry algebra has two fermionic generators Q_+ and Q_- which satisfy

$$Q_+^2 = P_+ = -i\partial_+, \quad Q_-^2 = P_- = -i\partial_-, \quad Q_+ Q_- + Q_- Q_+ = 0. \quad (2.5)$$

Other brackets vanish, *e.g.* $[Q_{\pm}, P_{\pm}] = 0$, *etc.* (In general, the notation (a, b) means that there are a fermionic generators of positive and b of negative chirality.) A representation of the algebra in terms of fields can be obtained by introducing a superspace of bosonic coordinates x^+ , x^- and fermionic (Grassmannian) coordinates θ_+ , θ_- . A superfield can be expanded

$$\Phi(x^+, x^-, \theta_+, \theta_-) = \phi(x) + i\theta_- \psi_+(x) + i\theta_+ \psi_-(x) + i\theta_+ \theta_- F(x). \quad (2.6)$$

where the coefficient (component) fields have obvious statistics and Lorentz properties. $F(x)$ is a nonpropagating auxiliary field, whose equations of motion are algebraic, *i.e.* contain no derivatives of $F(x)$. On superfields the supersymmetry generators are represented as

$$Q_+ = i \frac{\partial}{\partial \theta_-} - \theta_- \partial_+, \quad Q_- = i \frac{\partial}{\partial \theta_+} - \theta_+ \partial_-. \quad (2.7)$$

The “supercovariant” derivatives

$$\mathcal{D}_+ = i \frac{\partial}{\partial \theta_-} + \theta_- \partial_+, \quad \mathcal{D}_- = i \frac{\partial}{\partial \theta_+} + \theta_+ \partial_-. \quad (2.8)$$

anticommute with both Q 's and satisfy

$$\mathcal{D}_+^2 = i\partial_+, \quad \mathcal{D}_-^2 = i\partial_-, \quad \mathcal{D}_+ \mathcal{D}_- + \mathcal{D}_- \mathcal{D}_+ = 0. \quad (2.9)$$

The action can be written as a superspace integral

$$I = \int dx^+ dx^- d\theta_- d\theta_+ G_{ij}(\Phi) \mathcal{D}_+ \Phi^i \mathcal{D}_- \Phi^j, \quad (2.10)$$

where the Grassmannian integrals can be defined, according to Berezin, as

$$\int d\theta_+ \equiv \frac{\partial}{\partial \theta_+}, \quad \int d\theta_- \equiv \frac{\partial}{\partial \theta_-}. \quad (2.11)$$

In order to find the ordinary space-time Lagrangian one must perform the integration over the Grassmannian variables and eliminate the auxiliary fields by using their equations of motion.

A convenient way to evaluate the action (2.10) in component form is to observe that the θ integrations, in the form of θ derivatives, can be replaced by supercovariant derivatives, because the additional x derivatives one introduces in this way integrate to zero upon x integration. So one can compute the Lagrangian as

$$L = [\mathcal{D}_+ \mathcal{D}_- (G_{ij} \mathcal{D}_+ \Phi^i \mathcal{D}_- \Phi^j)]_{\theta=0}. \quad (2.12)$$

In doing the computation one must use the algebra (2.9) and the relations

$$[\Phi^i]_{\theta=0} = \phi^i, \quad [\mathcal{D}_{\pm} \Phi^i]_{\theta=0} = -\psi_{\pm}^i, \quad [\mathcal{D}_+ \mathcal{D}_- \Phi^i]_{\theta=0} = -iF^i. \quad (2.13)$$

This method is especially convenient when the superfield satisfies supercovariant constraints, which is not the case here, but will be the case below (see Section 4, Eq. (4.10)).

The result for the Lagrangian (2.12) is

$$L = G_{ij}\partial_+\phi^i\partial_-\phi^j + iG_{ij}(\psi_+^i D_-\psi_+^j + \psi_-^i D_+\psi_-^j) + 2iF_k\Gamma_{ij}^k\psi_+^i\psi_-^j + G_{ij}F^iF^j + \partial_k\partial_l G_{ij}\psi_+^k\psi_-^l\psi_+^i\psi_-^j, \quad (2.14)$$

where Γ is the Levi-Civita connection and the covariant derivatives on the spinors are defined by

$$D_-\psi_+^j = \partial_-\psi_+^j + \Gamma_{kl}^j\partial_-\phi^k\psi_+^l, \quad D_+\psi_-^j = \partial_+\psi_-^j + \Gamma_{kl}^j\partial_+\phi^k\psi_-^l. \quad (2.15)$$

Using standard formulas of Riemannian geometry the last three terms in (2.14) can be rearranged to give

$$L = G_{ij}\partial_+\phi^i\partial_-\phi^j + iG_{ij}(\psi_+^i D_-\psi_+^j + \psi_-^i D_+\psi_-^j) + \frac{1}{2}R_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l + G_{ij}(F^i + i\Gamma_{kl}^i\psi_+^k\psi_-^l)(F^j + i\Gamma_{mn}^j\psi_+^m\psi_-^n). \quad (2.16)$$

In either form (2.14) or (2.16) we see that the equations of motion for the auxiliary fields are

$$F^i + i\Gamma_{kl}^i\psi_+^k\psi_-^l = 0. \quad (2.17)$$

Using these equations the Lagrangian reduces to the first three terms in (2.16), which have an obvious geometric meaning, and the action is given by [3]

$$I = \int dx^+ dx^- [G_{ij}(\partial_+\phi^i\partial_-\phi^j + i\psi_+^i D_-\psi_+^j + i\psi_-^i D_+\psi_-^j) + \frac{1}{2}R_{ijkl}\psi_+^i\psi_+^j\psi_-^k\psi_-^l]. \quad (2.18)$$

This model is invariant under supersymmetry transformations which, in superspace, are given by

$$\delta\Phi^i = -i(\varepsilon_- Q_+ + \varepsilon_+ Q_-)\Phi^i, \quad (2.19)$$

where ε_- and ε_+ are Grassmannian infinitesimal parameters. The action (2.10) was constructed with supercovariant derivatives so that this be true. In terms of component fields the transformations become

$$\delta\phi^i = i\varepsilon_- \psi_+^i + i\varepsilon_+ \psi_-^i, \quad (2.20)$$

$$\delta\psi_+^i = -\varepsilon_- \partial_+ \phi^i - \varepsilon_+ F^i, \quad \delta\psi_-^i = -\varepsilon_+ \partial_- \phi^i + \varepsilon_- F^i, \quad (2.21)$$

$$\delta F^i = -i\varepsilon_- \partial_+ \psi_-^i + i\varepsilon_+ \partial_- \psi_+^i. \quad (2.22)$$

The algebra of these transformations closes: the commutator of two transformations δ and δ' of parameters ε_{\pm} and ε'_{\pm} is a two-dimensional translation of parameter $2i\varepsilon'_-\varepsilon_-$, $2i\varepsilon'_+\varepsilon_+$. For instance,

$$(\delta'\delta - \delta\delta')\phi^i = 2i\varepsilon'_-\varepsilon_-\partial_+\phi^i + 2i\varepsilon'_+\varepsilon_+\partial_-\phi^i. \quad (2.23)$$

One can shorten (2.20) to (2.21) by replacing the auxiliary fields by their value from the equations of motion, i.e. taking

$$\delta\psi_{\pm}^i = -\varepsilon_{\mp}\partial_{\pm}\phi^i \pm \varepsilon_{\pm}i\Gamma_{kl}^i\psi_{\pm}^k\psi_{\pm}^l. \quad (2.24)$$

The action is still invariant under (2.20) with (2.24). However now the algebra closes only on the mass shell, i.e. by use of the spinor fields equations of motion.

For certain geometries of the target space, the (1, 1) supersymmetry can be enlarged to a (2, 2) supersymmetry (when the Riemannian manifold is a complex Kähler manifold) or even to a (4, 4) supersymmetry (when it is a hyper-Kähler manifold). This is discussed in the next two sections.

3 Complex manifolds

In this section I discuss briefly those properties of complex manifolds which will be needed later [4]. An almost complex structure on a real $2n$ -dimensional differentiable orientable manifold is a tensor field $J_j^i(\phi)$ such that

$$J_k^i J_j^k = -\delta_j^i. \quad (3.1)$$

A manifold endowed with an almost complex structure is called an almost complex manifold. One defines the Nijenhuis torsion of J to be the tensor field

$$N_{jk}^i = 2(J_j^h \partial_h J_k^i - J_k^h \partial_h J_j^i - J_h^i \partial_j J_k^h + J_h^i \partial_k J_j^h) = -N_{kj}^i, \quad (3.2)$$

where ∂_h is the ordinary partial derivative. It is remarkable that N is actually a tensor. When N vanishes, J is called a complex structure and the manifold a complex manifold. In this case it is possible to define in local patches n complex coordinates ϕ^α and their complex conjugates $\bar{\phi}^\alpha$ such that $J\phi = -i\phi$, $J\bar{\phi} = i\bar{\phi}$. The transition functions between different coordinate patches are holomorphic functions of the coordinates ϕ (anti-holomorphic functions of $\bar{\phi}$).

If the $2n$ -dimensional manifold is a Riemannian manifold one can require the complex structure to satisfy

$$G_{ij} J_k^i J_l^j = G_{kl}, \quad (3.3)$$

(invariance of the metric) and

$$D_k J^i_j \equiv \partial_k J^i_j + \Gamma^i_{lk} J^l_j - J^i_l \Gamma^l_{jk} = 0, \quad (3.4)$$

(the tensor J is covariantly constant) where Γ is the Levi-Civita connection. With these conditions the complex manifold is called a Kähler manifold. Notice that the vanishing of the Nijenhuis torsion (3.2) follows from (3.4) and the symmetry of Γ , i.e. the absence of torsion on the Riemannian manifold.

The Riemann tensor can be defined by considering the action of the commutator of two covariant derivatives on tensors. Using this definition and (3.4), one can easily show that the complex structure satisfies

$$R_{ijkl} J^k_m J^l_n = R_{ijmn} = R_{klmn} J^k_i J^l_j. \quad (3.5)$$

In terms of the complex coordinates (3.3) implies that the metric satisfies

$$\begin{aligned} G_{\alpha\beta}(\phi, \bar{\phi}) &= G_{\bar{\alpha}\bar{\beta}}(\phi, \bar{\phi}) = 0, \\ G_{\alpha\bar{\beta}}(\phi, \bar{\phi}) &= G_{\bar{\beta}\alpha}(\phi, \bar{\phi}), \end{aligned} \quad (3.6)$$

while from (3.4) one can derive that the two-form

$$\Omega = -2i G_{\alpha\bar{\beta}} d\phi^\alpha \wedge d\bar{\phi}^\beta \quad (3.7)$$

is closed, i.e.

$$\partial_\alpha G_{\beta,\bar{\gamma}} = \partial_\beta G_{\alpha,\bar{\gamma}}, \quad \partial_{\bar{\alpha}} G_{\beta,\bar{\gamma}} = \partial_{\bar{\gamma}} G_{\beta\bar{\alpha}}, \quad (3.8)$$

where $\partial_\alpha = \partial/\partial\phi^\alpha$, $\partial_{\bar{\alpha}} = \partial/\partial\bar{\phi}^\alpha$. These equations imply that one can find locally a real function K such that

$$G_{\alpha\bar{\beta}}(\phi, \bar{\phi}) = \partial_\alpha \partial_{\bar{\beta}} K(\phi, \bar{\phi}). \quad (3.9)$$

Ω is called the Kähler form, and K is called the Kähler potential. K does not transform like a scalar from one patch to another and is defined only up to so-called Kähler transformations

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + f(\phi) + \overline{f(\phi)} \quad (3.10)$$

which leave the metric invariant. The Kähler potentials in two patches will in general be connected by a suitable Kähler transformation.

If a Kähler manifold admits two complex structures J^1 and J^2 which anticommute

$$(J^1)^i_k (J^2)^k_j + (J^2)^i_k (J^1)^k_j = 0, \quad (3.11)$$

then the product $J^3 = J^1 J^2$ satisfies all conditions (3.1), (3.3) and (3.4) and is also a complex structure. For instance

$$(J^3)^2 = J^1 J^2 J^1 J^2 = -J^1 J^1 J^2 J^2 = -1. \quad (3.12)$$

Furthermore

$$\begin{aligned} J^2 J^3 &= J^2 J^1 J^2 = -J^1 J^2 J^2 = J^1 = -J^3 J^2 \\ J^3 J^1 &= J^1 J^2 J^1 = -J^2 J^1 J^1 = J^2 = -J^1 J^3. \end{aligned} \quad (3.13)$$

So the tensors J^1, J^2 and J^3 satisfy (by matrix multiplication) the hyper-complex algebra of the quaternions. The manifold is called a hyper-Kähler manifold; its real dimension is necessarily a multiple $4n$ of four.

4 The Kähler and hyper-Kähler case

If the scalar fields of a supersymmetric σ -model are valued in a Kähler manifold, the Lagrangian of (2.18) is invariant under the transformation

$$\phi \rightarrow \phi' = \phi, \quad \psi_+ \rightarrow \psi'_+ = J\psi_+, \quad \psi_- \rightarrow \psi'_- = -J\psi_-, \quad (4.1)$$

where we have omitted the target space indices. This is easy to see using the equations (3.3) to (3.5). As a consequence one can define a second supersymmetry transformation

$$\delta' \phi = i\varepsilon'_- \psi'_+ + i\varepsilon'_+ \psi'_-, \quad \delta' \psi'_\pm = -\partial_\pm \phi \varepsilon'_\mp \mp F' \varepsilon'_\pm, \quad (4.2)$$

where the value of the new auxiliary field is taken to be

$$F'^k = -i\Gamma^k_{lm} \psi'_+{}^l \psi'_-{}^m. \quad (4.3)$$

With some algebra one can check that the two supersymmetry transformations (2.20) with (2.24) and (4.2) commute (on the mass shell of the auxiliary fields). Thus the (1, 1) supersymmetry is enlarged to a (2, 2) supersymmetry [5].

If one does not want to use the equation of motion of the auxiliary fields one can complete the transformation (4.1) by

$$F^k \rightarrow F'^k = F^k + iM^k_{lm} \psi_+{}^l \psi_-{}^m, \quad (4.4)$$

where

$$M^k_{lm}[J] = (\partial_l J^k_n - \partial_n J^k_l) J^n_m, \quad (4.5)$$

and (4.2) by

$$\delta' F' = -i\varepsilon'_- \partial_+ \psi'_- + i\varepsilon'_+ \partial_- \psi'_+. \quad (4.6)$$

Using (4.1) and (4.4), the two equations of motion (2.17) and (4.3) go into each other.

Notice that the inverse of the relation (4.4) between F and F' has the same form

$$F^k = F'^k + iM^k_{lm} \psi_+{}^l \psi_-{}^m. \quad (4.7)$$

Also, comparing with (3.2), we see that

$$M^k_{lm} - M^k_{ml} = \frac{1}{2}N^k_{lm} \quad (4.8)$$

which vanishes for a complex structure. Therefore M^k_{lm} is symmetric in l, m .

The (2, 2) model can be formulated more symmetrically by introducing from the beginning all four supersymmetry generators. This was indeed the first example of a geometric formulation of nonlinear σ -models [6]. In a superspace of coordinates $x^+, x^-, \theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-$, the generators and the supercovariant derivatives satisfy

$$\begin{aligned} [Q_{\pm}, Q_{\pm}]_+ &= [Q_{\pm}, Q_{\mp}]_+ = 0 & [Q_{\pm}, \bar{Q}_{\pm}]_+ &= 2P_{\pm}. \\ Q_{\pm} &= i\frac{\partial}{\partial\theta_{\mp}} - \theta_{\mp}\partial_{\pm}, & \bar{Q}_{\pm} &= i\frac{\partial}{\partial\bar{\theta}_{\mp}} - \bar{\theta}_{\mp}\partial_{\pm}, \\ \mathcal{D}_{\pm} &= i\frac{\partial}{\partial\theta_{\mp}} + \theta_{\mp}\partial_{\pm}, & \bar{\mathcal{D}}_{\pm} &= i\frac{\partial}{\partial\bar{\theta}_{\mp}} + \bar{\theta}_{\mp}\partial_{\pm}. \end{aligned} \quad (4.9)$$

A "chiral" superfield (necessarily complex) satisfies

$$\Phi(x^+, x^-, \theta_+, \theta_-, \bar{\theta}_+, \bar{\theta}_-), \quad \mathcal{D}_{\pm}\bar{\Phi} = 0, \quad \bar{\mathcal{D}}_{\pm}\Phi = 0. \quad (4.10)$$

These constraints can be solved by observing that the four combinations $y^{\pm} = x^{\pm} + i\theta_{\mp}\bar{\theta}_{\mp}$ and $\bar{\theta}_{\mp}$ are annihilated by $\bar{\mathcal{D}}_{\pm}$, so that we can take

$$\begin{aligned} \Phi(y^+, y^-, \bar{\theta}_+, \bar{\theta}_-) &= \phi(y^+, y^-) + i\bar{\theta}_+\psi_-(y^+, y^-) + i\bar{\theta}_-\psi_+(y^+, y^-) \\ &\quad + i\bar{\theta}_+\bar{\theta}_-F(y^+, y^-). \end{aligned} \quad (4.11)$$

The superspace action takes the extremely simple form

$$\begin{aligned} I &= \int dx^+ dx^- d\theta_+ d\theta_- d\bar{\theta}_+ d\bar{\theta}_- K(\Phi^{\alpha}, \bar{\Phi}^{\alpha}) \\ &= \int dx^+ dx^- G_{\alpha\bar{\beta}}(\partial_+\phi^{\alpha}\partial_-\bar{\phi}^{\beta} + \partial_-\phi^{\alpha}\partial_+\bar{\phi}^{\beta}) + \dots, \\ G_{\alpha\bar{\beta}} &= \partial_{\alpha}\partial_{\bar{\beta}}K, \end{aligned} \quad (4.12)$$

where the dots denote additional terms involving also fermions (see [6]). It is easy to see that the action is invariant under the Kähler transformation (3.10). It should be noticed that (4.9) to (4.12) parallel very closely the formulas for the $N = 1$ (four Majorana components) supersymmetry in four space-time dimensions. Indeed the present model can be obtained by dimensional reduction from the four-dimensional model, simply by assuming that all fields are independent of the two space coordinates x^2 and x^3 . This is especially clear if the four-dimensional theory is formulated using the van der Waerden two-component spinor notation.

If the manifold admits more than one complex structure, one can define more supersymmetries. For the hyper-Kähler case

$$\begin{aligned}\delta^a \phi &= i\varepsilon_-^a \psi_+^a + i\varepsilon_+^a \psi_-^a, \\ \delta^a \psi_\pm^a &= \mp \varepsilon_\pm^a F^a - \varepsilon_\mp^a \partial_\pm \phi, \\ \delta^a F^a &= -i\varepsilon_-^a \partial_+ \psi_-^a + i\varepsilon_+^a \partial_- \psi_+^a,\end{aligned}\tag{4.13}$$

where $a = 1, 2, 3$. Here

$$\begin{aligned}\psi_\pm^a &= \pm J^a \psi_\pm, \\ F^a &= F + iM[J^a] \psi_+ \psi_-,\end{aligned}\tag{4.14}$$

the omitted target space index structure being obvious. The three new auxiliary fields are related by

$$F^{1k} + F^{2k} = i(\partial_l J_m^{3k} - \partial_m J_l^{3k}) \psi_+^{2m} \psi_-^{1l}\tag{4.15}$$

plus the two equations obtained by rotating cyclically the indices 1, 2, 3. The three supersymmetries δ^a commute with each other and with the original δ of (2.20) to (2.22). Thus the algebra of the four super-generators is now

$$[Q^a, \bar{Q}^b]_+ = 2\delta^{ab} \mathcal{P},\tag{4.16}$$

where $a = 0, 1, 2, 3$ and Q^0 is the generator of the original supersymmetry δ . The (1, 1) supersymmetry has been enhanced to a (4, 4) supersymmetry [5].

Just as a natural way to understand the Kähler $N = 2$ supersymmetry model in two dimensions is to relate it to the $N = 1$ model in four dimensions [6], the $N = 4$ model in two dimensions can be obtained by dimensional reduction from the $N = 1$ in six, or the $N = 2$ in four. The $N = 1$ in six dimensions has been studied in [7] [8], where a more detailed description of hyper-Kähler manifolds was given than we have done here, and the six dimensional Lagrangian (on the auxiliary shell, no other is known) was written out explicitly. Various dimensional reductions can be obtained from it.

5 Chiral supersymmetries

In two dimensions one can formulate various supersymmetric models which have only supersymmetry generators of given chirality [10]. For instance, using a superspace of coordinates x^+, x^- and θ_- and using the superfield

$$\Phi(x^+, x^-, \theta_-) = \phi(x^+, x^-) + i\theta_- \psi_+(x^+, x^-),\tag{5.1}$$

one can write the action for the chiral model (1, 0) with torsion

$$\begin{aligned}I &= \int dx^+ dx^- d\theta_- (G_{ij}(\Phi) - B_{ij}(\Phi)) \mathcal{D}_+ \Phi^i \partial_- \Phi^j \\ &= \int dx^+ dx^- [(G_{ij}(\phi) - B_{ij}(\phi)) \partial_+ \phi^i \partial_- \phi^j + iG_{ij}(\phi) \psi_+^i \hat{D}_- \psi_+^j], \\ \hat{D}_- \psi_+^j &= \partial_- \psi_+^j + \hat{\Gamma}_{-ki}^j \partial_- \phi^k \psi_+^i,\end{aligned}\tag{5.2}$$

where the connection is $\hat{\Gamma}^j_{kl} = \Gamma^j_{kl} - H^j_{kl}$ as in (1.24).

In absence of torsion, if the manifold admits a complex structure and is Kähler, the (1,0) model can be elevated to a (2,0) model by a procedure analagous to that we employed above to go from (1,1) to (2,2). In presence of torsion the complex structure must satisfy

$$\hat{D}_i J^j_k(\phi) = 0, \quad (5.3)$$

where \hat{D}_i is the covariant derivative with torsion. It is easy to see that in complex coordinates this implies $\partial_\beta G_{\alpha\bar{\gamma}} - \partial_\alpha G_{\beta\bar{\gamma}} = -2H_{\alpha\beta\bar{\gamma}}$, $H_{\alpha\beta\bar{\gamma}} = 0$, so that the manifold is not Kähler (which would require $H_{\alpha\beta\bar{\gamma}} = 0$). With a little algebra one can show that the metric can be written locally as $G_{\alpha\bar{\beta}} = \partial_\alpha \bar{X}_{\bar{\beta}} + \partial_{\bar{\beta}} X_\alpha$. The (2,0) model with torsion is specified by the superspace action [11]

$$I = \frac{i}{2} \int dx^+ dx^- d\theta_- d\bar{\theta}_- (X_\alpha \partial_- \Phi^\alpha - \bar{X}_{\bar{\beta}} \partial_- \bar{\Phi}^{\bar{\beta}}) \quad (5.4)$$

(for the Kähler case, *i.e.* no torsion, $X_\alpha \propto \partial_\alpha K(\Phi, \bar{\Phi})$), and the constraints

$$\begin{aligned} \mathcal{D}_+ \bar{\Phi}^{\bar{\beta}} &= \overline{\mathcal{D}_+ \Phi^\alpha} = 0, \\ \mathcal{D}_+ &= i \frac{\partial}{\partial \theta_-} + \theta_- \partial_+, \quad \overline{\mathcal{D}_+} = i \frac{\partial}{\partial \bar{\theta}_-} + \bar{\theta}_- \partial_+, \\ \{\mathcal{D}_+, \overline{\mathcal{D}_+}\} &= 2i \partial_+. \end{aligned} \quad (5.5)$$

The (1,0) model can be coupled to a fermionic multiplet of negative chirality [10], by using the spinor superfield

$$\Lambda_-(x^+, x^-, \theta_-) = \lambda_-(x^+, x^-) + \theta_- F(x^+, x^-). \quad (5.6)$$

One can make the model more interesting by assuming that this superfield belongs to a representation V of a gauge group, $\Lambda_-^A (A = 1, \dots, P)$, with gauge field $A_i^A{}_B(\Phi)$, and introducing a metric $G_{AB}(\Phi)$. Then we can add to the action given in (5.2) the action

$$I_V = -i \int dx^+ dx^- d\theta_- G_{AB}(\Phi) \Lambda_-^A (\mathcal{D}_+ + A_+)^B \Lambda_-^C, \quad (A_+)^B{}_C = A_i^B{}_C \mathcal{D}_+ \Phi^i. \quad (5.7)$$

In component form the total action is

$$\begin{aligned} I &= \int dx^+ dx^- [(G_{ij} - B_{ij}) \partial_+ \phi^i \partial_- \phi^j + i G_{ij} \psi_+^i \hat{D}_- \psi_+^j \\ &\quad + i G_{AB} \lambda_-^A \nabla_+ \lambda_-^B + \frac{1}{2} F_{ijAB} \psi_+^i \psi_+^j \lambda_-^A \lambda_-^B], \end{aligned} \quad (5.8)$$

where the field strength and gauge covariant derivative are

$$F_{ijAB} = (\partial_i \hat{A}_j - \partial_j \hat{A}_i + [\hat{A}_i, \hat{A}_j]_-)_{AB} \quad (5.9)$$

$$\hat{A}_i^A{}_B = A_i^A{}_B + \frac{1}{2} G^{AC} G_{BC,i}, \quad \nabla_+^B{}_C = \partial_+ \delta^B{}_C + A_i^B{}_C \partial_+ \phi^i. \quad (5.10)$$

In general G_{ij} , B_{ij} and $A_i^A{}_B$ are independent external fields. In the particular case in which the gauge group is the structure group of the field manifold (tangent space group) the indices A, B, \dots becomes the tangent space indices a, b, \dots , $A_i^a{}_b \equiv \omega_i^a{}_b$ and $F_{ij}^a{}_b \equiv R_{ij}^a{}_b$, so the present model reduces to the model (1, 1) with torsion described earlier, but now in tangent space notation.

There exists also an interesting modification of the (2, 2) model described earlier, where the superfield satisfies the chirality constraints (4.10). It is the (2, 2) twisted model [14] where the superfield X satisfies the twisted chirality constraints

$$\mathcal{D}_+ X = \overline{\mathcal{D}}_- X = 0 \quad \overline{\mathcal{D}}_+ \bar{X} = \mathcal{D}_- \bar{X} = 0. \quad (5.11)$$

A general classification of (2, 2) models with torsion is given in [9], where an off shell formulation can be found.

6 Concluding Remarks

The qualitative lesson to be learned from the above discussion is that the number of supersymmetries is intimately related to the geometric structure of the target space manifold: more geometric structure corresponds to more supersymmetries.

The list of publications in the present article is very incomplete. A variety of other supersymmetric σ -models can be constructed (see, *e.g.*, [12], [13]). An excellent review up to 1986 is that of S. Mukhi [15] which contains numerous references. He also discusses the cancellations of ultraviolet divergences due to supersymmetry and some of the relevance to string theory. After 1986 the literature on the subject has exploded and no comprehensive review is known to me. I would like to mention only two relatively more recent references, [16] and [17], on the ultraviolet divergences respectively of the (4, 0) model with torsion and of the (2, 2) model without torsion. These papers contain also numerous references.

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