Title
Modified scattering for a scalar quasilinear wave equation satisfying the weak null condition

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Publication Date
2021

Peer reviewed|Thesis/dissertation

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by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Daniel Tataru, Chair
Professor Michael Christ
Professor Xin Guo

Spring 2021

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Abstract

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The objective of this dissertation is to study the long time dynamics of a scalar quasilinear wave equation

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = 0, \qquad \text{in } \mathbb{R}^{1+3}_{t,x}.$$

This equation satisfies the weak null condition introduced by Lindblad and Rodnianski [25, 24]. Lindblad [21] proved that, for small and localized initial data, this equation has a global solution. In the present work, we establish a modified scattering theory for the above equation. Such a modified scattering theory provides an accurate description of asymptotic behavior of the global solutions.

To study modified scattering, we first identify a notion of asymptotic profile and an associated notion of scattering data. One candidate for the asymptotic profile is given by the asymptotic PDE

$$2U_{sq} + G(\omega)UU_{qq} = 0$$

which was derived by Hörmander [9, 7, 8]. In Chapter 2, we derive a new reduced system, called the *geometric reduced system*, by modifying Hörmander's method. In our derivation, we make use of the optical function, i.e. a solution to the eikonal equation. In this setting, the scattering data is the initial data for our geometric reduced system, and it is chosen in a way such that the global solution to the quasilinear wave equation and the exact solution to the reduced system match at infinite time. One may infer, from this dissertation, that this new system is more accurate, in that it both describes the long time evolution and contains full information about it.

In Chapter 3, we prove the existence of the modified wave operators for the scalar quasilinear wave equation. Fixing a scattering data which is the initial data for the geometric reduced system, we can first construct an approximate solution to the model equation. Then, by studying a backward Cauchy problem, we show that there exists a global solution to the scalar quasilinear wave equation which matches the approximate solution at infinite time.

In Chapter 4, we prove the asymptotic completeness for the same equation. Given a global solution to the scalar quasilinear wave equation, we rigorously derive the geometric reduced system with error terms. These allow us to recover the scattering data, as well as to construct a matching exact solution to the reduced system.

To my parents.

Contents

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor, Professor Daniel Tataru, for his guidance and his support. The discussions between us are of great help to my research and my career development. I am very fortunate to have such a wonderful advisor.

I would like to thank Professor Sung-Jin Oh and Professor Mihaela Ifrim for many helpful discussions about my research. I would also like to thank Professor David Aldous, Professor Michael Christ and Professor Xin Guo for serving on my qualifying examination committee and/or my dissertation committee.

I could not get my PhD without the help from my fellow graduate students, staff members and other faculty members not listed above, at or outsides Berkeley. I would also like to express my gratitude to them.

Finally, I am deeply grateful to my parents for supporting me from the very beginning. With their support and help, I am fortunate to be able to focus myself on mathematics.

My research was partially supported by a James H. Simons Fellowship and by the NSF grant DMS-1800294.

Part of this dissertation was first published in *Communications in Mathematical Physics* volume 382, pages 1961-2013(2021), by Springer Nature, and it was reproduced with permission from Springer Nature.

Chapter 1

Introduction

This dissertation is devoted to the study for the long time dynamics of a scalar quasilinear wave equation in $\mathbb{R}^{1+3}_{t,x}$, of the form

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u = 0. \tag{1.1}$$

Here we use the Einstein summation convention, with the sum taken over $\alpha, \beta = 0, 1, 2, 3$ with $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$, $i = 1, 2, 3$. We assume that $g^{\alpha\beta}(u)$ are smooth functions of u, such that $g^{\alpha\beta} = g^{\beta\alpha}$ and $g^{\alpha\beta}(0)\partial_\alpha\partial_\beta = \Box = -\partial_t^2 + \Delta_x$. We also assume that $g^{00} \equiv -1$. In fact, since we expect $|u| \ll 1$, we have $g^{00}(u) < 0$, so we can replace $(g^{\alpha\beta})$ with $(g^{\alpha\beta}/(-g^{00}))$ if necessary.

This model equation is closely related to General Relativity. The vector-valued version of $g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u$ is the principal part of the Einstein equations in wave coordinates. For more physical background for the equation (1.1), we refer the readers to [21, 25, 24].

The study of global well-posededness theory of (1.1) started with Lindblad's paper [20]. Given the initial data

$$u(0) = \varepsilon u_0,\ \partial_t u(0) = \varepsilon u_1, \qquad \text{where } u_1, u_2 \in C_c^\infty(\mathbb{R}^3) \text{ and } \varepsilon > 0 \text{ is small}, \tag{1.2}$$

Lindblad conjectured that the equation (1.1) has a global solution if ε is sufficiently small. In the same paper, he proved the small data global existence for a special case

$$\partial_t^2 u - c(u)^2\Delta_x u = 0, \qquad \text{where } c(0) = 1 \tag{1.3}$$

for radially symmetric data. Later, Alinhac [1] generalized the result to general initial data for (1.3). The small data global existence result to the general case (1.1) was finally proved by Lindblad in [21].

Our main goal is to establish a modified scattering theory for (1.1).

1.1 Background

The equation (1.1) is a special case for a general scalar nonlinear wave equation in $\mathbb{R}^{1+3}_{t,x}$

$$\Box u = F(u, \partial u, \partial^2 u). \tag{1.4}$$

Here the nonlinear term is of the form

$$F(u,\partial u,\partial^2 u)=\sum a_{\alpha\beta}\partial^\alpha u\partial^\beta u+O(|u|^3+|\partial u|^3+|\partial^2 u|^3). \tag{1.5}$$

The sum in (1.5) is taken over all multiindices α,β with $|\alpha|\leq|\beta|\leq 2$, $|\beta|\geq 1$ and $|\alpha|+|\beta|\leq 3$.

Since 1980's, there have been many results on the lifespan of the solutions to the Cauchy problem (1.4) with initial data (1.2). In [11, 12], John proved that (1.4) does not necessarily have a global solution for all $t\geq 0$: any nontrivial solution to $\Box u=u_t\Delta u$ or $\Box u=u_t^2$ blows up in finite time. In contrast, (1.4) in $\mathbb{R}^{1+d}$ for $d\geq 4$ has small data global existence, proved by Hörmander [8]. For arbitrary nonlinearities in three space dimensions, the best result on the lifespan is the almost global existence: the solution exists for $t\leq e^{c/\varepsilon}$, for sufficiently small ε and some constant $c>0$. The almost global existence for (1.4) was proved by Lindblad [23]. We also refer to John and Klainerman [13], Klainerman [18], and Hörmander [9, 7] for some earlier work on almost global existence.

In contrast to the finite-time blowup in John's examples, it was proved by Klainerman [17] and by Christodoulou [3] that if the null condition is satisfied, then (1.4) has a global solution for any sufficiently small and localized initial data. The null condition was first introduced by Klainerman [16]. It states that for each $0\leq m\leq n\leq 2$ with $m+n\leq 3$, we have

$$A_{mn}(\omega):=\sum_{|\alpha|=m,|\beta|=n}a_{\alpha\beta}\widehat{\omega}^\alpha\widehat{\omega}^\beta=0,\qquad \text{for all } \widehat{\omega}=(-1,\omega)\in\mathbb{R}\times\mathbb{S}^2. \tag{1.6}$$

Equivalently, we assume $A_{mn}\equiv 0$ for all $\widehat{\omega}$ lying on the null cone $\{m^{\alpha\beta}\xi_\alpha\xi_\beta=0\}$. The null condition leads to cancellations in the nonlinear terms (1.5) so that the nonlinear effects of the equations are much weaker than the linear effects. Note that the null condition is sufficient but not necessary for the small data global existence. For example, the null condition fails for (1.1) in general, but (1.1) still has small data global existence. We also refer our readers to [32] for a general introduction on the null condition.

Later, in [25, 24], Lindblad and Rodnianski introduced the weak null condition. To state the weak null condition, we start with the asymptotic equations first introduced by Hörmander in [9, 7, 8]. We make the ansatz

$$u(t,x)\approx\frac{\varepsilon}{r}U(s,q,\omega),\qquad r=|x|,\ \omega_i=x_i/r,\ s=\varepsilon\ln(t),\ q=r-t. \tag{1.7}$$

Substituting this ansatz into (1.4), we can derive the following asymptotic PDE for $U(s,q,\omega)$

$$2\partial_s\partial_q U+\sum A_{mn}(\omega)\partial_q^m U\partial_q^n U=0. \tag{1.8}$$

Here A_{mn} is defined in (1.6) and the sum is taken over $0\leq m\leq n\leq 2$ with $m+n\leq 3$. We say that the weak null condition is satisfied if (1.8) has a global solution for all $s\geq 0$ and if the solution and all its derivatives grow at most exponentially in s, provided that the initial data decay sufficiently fast in q. In the same papers, Lindblad and Rodnianski conjectured that the weak null condition is sufficient for small data global existence. To the best of the author's knowledge, this conjecture remains open until today.

There are three remarks about the weak null condition and the corresponding conjecture. First, the weak null condition is weaker than the null condition. In fact, if the null condition is satisfied, then (1.8) becomes $\partial_s \partial_q U = 0$. Secondly, though the conjecture remains open, there are many examples of (1.4) satisfying the weak null condition and admitting small data global existence at the same time. The equation (1.1) is one of several such examples: the small data global existence for (1.1) has been proved by Lindblad [21]; meanwhile, the asymptotic equation (1.8) now becomes

$$2\partial_s \partial_q U + G(\omega) U \partial_q^2 U = 0, \tag{1.9}$$

where

$$G(\omega) := g_0^{\alpha\beta} \widehat{\omega}_\alpha \widehat{\omega}_\beta, \qquad g_0^{\alpha\beta} = \frac{d}{du} g^{\alpha\beta}(u)|_{u=0},\ \widehat{\omega} = (-1, \omega) \in \mathbb{R} \times \mathbb{S}^2,$$

whose solutions exist globally in s and satisfy the decay requirements, so (1.1) satisfies the weak null condition. There are also many examples violating the weak null condition and admitting finite-time blowup at the same time. Two such examples are $\Box u = u_t \Delta u$ and $\Box u = u_t^2$: the corresponding asymptotic equations are $(2\partial_s - U_q \partial_q) U_q = 0$ (Burger's equation) and $\partial_s U_q = U_q^2$, respectively, whose solutions are known to blow up in finite time. Thirdly, in recent years, Keir has made some further progress. In [15], he proved the small data global existence for a large class of quasilinear wave equations satisfying the weak null condition, significantly enlarging upon the class of equations for which global existence is known. His proof also applies to (1.1). In [14], he proved that if the solutions to the asymptotic system are bounded (given small initial data) and stable against rapidly decaying perturbations, then the corresponding system of nonlinear wave equations admits small data global existence.

1.2 A new reduced system

Instead of working with Hörmander's asymptotic system (1.9) directly, in this dissertation we will construct a new system of asymptotic equations. Our analysis starts as in Hörmander's derivation in [9, 7, 8], but diverges at a key point: the choice of q is different. One may contend from this work that this new system is more accurate than (1.9), in that it both describes the long time evolution and contains full information about it. In addition, if we choose the initial data appropriately, our reduced system will reduce to linear first order ODE's on μ and U_q, so it is easier to solve it than to solve (1.9).

To derive the new equations, we still make the ansatz (1.7), but now we replace $q = r - t$ with a solution $q(t, r, \omega)$ to the eikonal equation related to (1.1)

$$g^{\alpha\beta}(u) \partial_\alpha q \partial_\beta q = 0. \tag{1.10}$$

In other words, $q(t, r, \omega)$ is an optical function. There are two reasons why we choose q in this way. First, if we substitute $u = \varepsilon r^{-1} U(s, q, \omega)$ in (1.1) where $q(t, r, \omega)$ is an arbitrary function, then we obtain two terms in the expansion

$$\varepsilon r^{-1}g^{\alpha\beta}(u)q_{\alpha\beta}U_q+\varepsilon r^{-1}g^{\alpha\beta}(u)q_\alpha q_\beta U_{qq}.$$

All the other terms either decay faster than ε^2r^{-2} for $t\approx r\to\infty$, or do not contain U itself (but may contain U_q, U_{qq}, U_{sq} and etc.). If q satisfies the eikonal equation, then the second term vanishes. From the eikonal equation, we can also prove that the first term is approximately equal to a function depending on U_q but not on U. Thus, in contrast to the second order PDE (1.9) for U, we expect to get a first-order ODE for U_q which is simpler.

Secondly, the eikonal equations have been used in the previous works on the small data global existence for (1.1). In [1], Alinhac followed the method used in Christodoulou and Klainerman [4], and adapted the vector fields to the characteristic surfaces, i.e. the level surfaces of solutions to the eikonal equations. In [21], Lindblad considered the radial eikonal equations when he derived the pointwise bounds of solutions to (1.1). When they derived the energy estimates, both Alinhac and Lindblad considered a weight $w(q)$ where q is an approximate solution to the eikonal equation. Their works suggest that the eikonal equation plays an important role when we study the long time behavior of solutions to (1.1). We remark that the eikonal equations have also been used in the study of the asymptotic behavior of solutions to the Einstein vacuum equations, an analogue of (1.1); we refer our readers to [4, 22].

Since u is unknown, it is difficult to solve (1.10) directly. Instead, we introduce a new auxiliary function $\mu=\mu(s,q,\omega)$ such that $q_t-q_r=\mu$. From (1.10), we can express q_t+q_r in terms of μ and U, and then solve for all partial derivatives of q, assuming that all the angular derivatives are negligible. Then from (1.1), we can derive the following asymptotic equations for $\mu(s,q,\omega)$ and $U(s,q,\omega)$:

$$\begin{cases}\partial_s\mu=\dfrac{1}{4}G(\omega)\mu^2U_q,\\ \partial_sU_q=-\dfrac{1}{4}G(\omega)\mu U_q^2.\end{cases}\tag{1.11}$$

We call this new system of asymptotic equations the *geometric reduced system*. The derivation of (1.11) is given in Chapter 2 of this dissertation; we also refer our readers to Section 3 in [34]. In Chapter 2, we also obtain the geometric reduced system for a system of general quasilinear wave equations, which generalizes the reduced system derived in Section 3, [34]. Heuristically, one expects the solution to a system of quasilinear wave equations to correspond to an approximate solution to this geometric reduced system, and to be well approximated by an exact solution to the geometric reduced system. We then introduce the *geometric weak null condition*: for any initial data decaying sufficiently fast, the geometric reduced system has a global solution which grows at most exponentially in s. The author believes that the geometric reduced system and the geometric weak null condition might help us get a better understanding of the long time dynamics of general quasilinear wave equations.

Note that (1.11) is a system of two ODE's for (μ, U_q). Besides, we have $\partial_s(\mu U_q) = 0$ for each (s, q, ω). That is, if the initial data are given by

$$(\mu, U_q)|_{s=0}(q, \omega) = (A_1, A_2)(q, \omega),$$

then we have $\mu U_q = A_1 \cdot A_2$ at each (s, q, ω). In this dissertation, we define a function $A = A(q, \omega)$ for $(q, \omega) \in \mathbb{R} \times \mathbb{S}^2$ by

$$A(q, \omega) := -\frac{1}{2} A_1(q, \omega) \cdot A_2(q, \omega),$$

and we call the function A a *scattering data* associated to a solution u to the quasilinear wave equation (1.1). Now (1.11) reduces to a linear system of ODE's

$$\begin{cases} \partial_s \mu = -\frac{1}{2} G(\omega) A(q, \omega) \mu, \\ \partial_s U_q = \frac{1}{2} G(\omega) A(q, \omega) U_q, \end{cases}$$

whose solutions are given by

$$\begin{cases} \mu(s, q, \omega) = A_1(q, \omega) \exp(-\frac{1}{2} G(\omega) A(q, \omega) s), \\ U_q(s, q, \omega) = A_2(q, \omega) \exp(\frac{1}{2} G(\omega) A(q, \omega) s), \end{cases}$$

To solve for $U(s, q, \omega)$ uniquely, we assume that

$$\lim_{q \to -\infty} U(s, q, \omega) = 0 \qquad \text{or} \qquad \lim_{q \to \infty} U(s, q, \omega) = 0,$$

depending on which problem we are studying.

1.3 Modified scattering theory: an overview

The objective of this dissertation and [34, 33] is to study the long time dynamics, and more specifically, scattering theory for highly nonlinear dispersive equations. In other words, we would like to provide an accurate description of asymptotic behavior of the global solutions. For many nonlinear dispersive PDE's, one can establish a linear scattering theory. That is, a global solution to a nonlinear PDE scatters to a solution to the corresponding linear equation as time goes to infinity. Take the cubic defocusing NLS

$$iu_t + \Delta u = u|u|^2 \qquad \text{in } \mathbb{R}^{1+3}_{t,x}$$

as an example. Its corresponding linear equation is the linear Schrödinger equation (LS)

$$iw_t + \Delta w = 0 \qquad \text{in } \mathbb{R}^{1+3}_{t,x}.$$

One can prove that for each $u_0 \in H^1$, there exists a unique $u_+ \in H^1$ such that

$$\|u(t) - w(t)\|_{H^1} \to 0 \qquad \text{as } t \to \infty$$

where u (or w) is the global solution to NLS (or LS) with data u_0 (or u_+). This result is called the *asymptotic completeness*. One can also prove that for each $u_+ \in H^1$, there exists a unique $u_0 \in H^1$ such that the same conclusion holds. This result is called the *existence of wave operators*, where the wave operator is defined by $\Omega_+ u_+ = u_0$. We refer to Section 3.6 of [31] for this result. Some other nonlinear PDE's have modified scattering instead of linear scattering. That is, each of their global solutions scatters to a suitable modification of a linear solution. Here the modification can be made in more than one way: we can add a phase correction term, an amplitude correction term, or a velocity correction term to the linear solution. For example, in [10], when the authors study modified scattering for the cubic 1D NLS, they make use of the following asymptotic approximation:

$$\widehat{u}(t,\xi) \approx e^{-it\xi^2} W(\xi) e^{i|W(\xi)|^2 \ln t}.$$

That is, a phase shift term is introduced. For nonlinear wave equations, the modification often corresponds to a change of the geometry of the light cone foliation of the space-time. This point is reflected in the ansatz used in Section 1.2.

In general, the following steps are taken in order to study modified scattering. Given a nonlinear dispersive PDE, we hope to identify a good notion of *asymptotic profile* and an associated notion of *scattering data* for the model equation. This can be achieved by introducing some type of asymptotic equations. Like linear scattering, the two main problems in modified scattering theory are as follows:

1. *Asymptotic completeness.* Given an exact global solution to the model equation, can we find the corresponding asymptotic profile and scattering data?

2. *Existence of (modified) wave operators.* Given an asymptotic profile constructed for a scattering data, can we construct a unique exact global solution to the model equation which matches the asymptotic profile at infinite time?

There have been only a few previous results on the (modified) scattering for general quasilinear wave equations and the Einstein's equations. In [5], Dafermos, Holzegel and Rodnianski gave a scattering theory construction of nontrivial black hole solutions to the vacuum Einstein equations. That is a backward scattering problem in General Relativity. In [26], Lindblad and Schlue proved the existence of the wave operators for the semilinear models of Einstein's equations. In [6], Deng and Pusateri used the original Hörmander's asymptotic system (1.9) to prove a partial scattering result for (1.1). In their proof, they applied the spacetime resonance method; we refer to [28, 27] for some earlier applications of this method to the first order systems of wave equation.

1.4 Modified wave operators

Making use of the reduced system (1.11), we are able to prove the existence of the modified wave operators for (1.1). This result has been proved in the author's paper [34], though the assumptions made in this dissertation are weaker than those in [34]. In this dissertation, we assume that the *scattering data* $A = A(q, \omega)$, i.e. the initial data for U_q at $s = 0$, satisfies the following assumption:

$$A \in C^\infty(\mathbb{R} \times \mathbb{S}^2), \quad A \equiv 0 \text{ whenever } q \le -R, \quad (\partial_q^m \partial_\omega^n A)(q, \omega) = O_{m,n}(\langle q\rangle^{-1-\gamma-m}),\ \forall m, n. \tag{1.12}$$

Here $R \ge 1$ and $\gamma > 0$ are two fixed constants, and ∂_ω^n denotes any angular derivatives of order n. In contrast, recall that we assume $A \in C_c^\infty(\mathbb{R} \times \mathbb{S}^2)$ in [34]. As a result, the proof in this dissertation requires a more delicate analysis and substantial changes of the arguments in [34].

The first step in the proof is to construct an approximate solution to (1.1). We start by solving (1.11) explicitly with the initial data $(\mu, U_q)|_{s=0} = (-2, A)$. To get a unique solution $(\mu, U)(s, q, \omega)$, we assume that $\lim_{q\to-\infty} U(s, q, \omega) = 0$. Then, we construct an approximate solution $q(t, r, \omega)$ to the eikonal equation (1.10) by solving $q_t - q_r = \mu$ and $q(t, 0, \omega) = -t$; we can apply the method of characteristics. Both s and q are now functions of (t, r, ω), so we also obtain a function $U(t, r, \omega)$ from $U(s, q, \omega)$. Here $U(t, r, \omega)$ is our *asymptotic profile*. Next, we define u_{app} by multiplying $\varepsilon r^{-1}U$ by some cutoff functions. We expect that u_{app} is an approximate solution to (1.1), that $u_{app} = \varepsilon r^{-1}U(t, r, \omega)$ in a conic neighborhood of the light cone $\{t = r\}$ and that u_{app} is supported in a slightly larger conic neighborhood of the light cone.

The second step is to show that there exists an exact solution to (1.1) which matches u_{app} at infinite time. Fixing a large time $T > 0$, we solve a backward Cauchy problem for $v = u-u_{app}$ with zero data for $t \ge 2T$, such that $v+u_{app}$ solves (1.1) for $t \le T$. We then prove that $v = v^T$ converges to some function v^∞ as $T \to \infty$. It turns out that $u^\infty = v^\infty + u_{app}$ is a solution to (1.1) which matches the asymptotic profile at infinite time. This shows the existence of the modified wave operators.

We end this subsection with the main theorem on modified wave operators, which is Theorem 3.1. We denote by Z any of the commuting vector fields: translations ∂_α, scaling $t\partial_t + r\partial_r$, rotations $x_i\partial_j - x_j\partial_i$ and Lorentz boosts $x_i\partial_t + t\partial_i$.

Theorem 1.1. *Consider a scattering data $A = A(q, \omega)$ be a function satisfying* (1.12) *for some $R \ge 1$ and $\gamma > 0$. Fix an integer $N \ge 2$ and any sufficiently small $\varepsilon > 0$ depending on A and N. Let $q(t, r, \omega)$ and $U(t, r, \omega)$ be the associated approximate optical function and asymptotic profile. Then, there is a C^N solution u to* (1.1) *for $t \ge 0$ with the following properties:*

(i) *The solution vanishes for $|x| = r \le t - R$.*

(ii) *The solution satisfies the energy bounds: for all* $|I| \leq N-1$ *and all* $t \gg_A 1$*, we have*

$$\left\|\partial Z^I(u-\varepsilon r^{-1}U)(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ |x|\leq 5t/4\})}+\left\|\partial Z^I u(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ |x|\geq 5t/4\})}\lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}.$$

(iii) *The solution satisfies the pointwise bounds: for all* (t,r,ω) *with* $t \gg_A 1$*, we have*

$$|(\partial_t-\partial_r)u+2\varepsilon r^{-1}A(q(t,r,\omega),\omega)|\lesssim \varepsilon t^{-3/2+C\varepsilon}.$$

Moreover, for all $|I|\leq N-1$ *and all* (t,x) *with* $t\gg_A 1$*,*

$$|\partial Z^I(u-\varepsilon r^{-1}U)(t,x)|\chi_{|x|\leq 5t/4}+|\partial Z^I u(t,x)|\chi_{|x|\geq 5t/4}\lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}\langle t+r\rangle^{-1}\langle t-r\rangle^{-1/2},$$

$$|Z^I(u-\varepsilon r^{-1}U)(t,x)|\chi_{|x|\leq 5t/4}+|Z^I u(t,x)|\chi_{|x|\geq 5t/4}\lesssim_I \min\{\varepsilon t^{-1+C_I\varepsilon},\varepsilon t^{-3/2+C_I\varepsilon}\langle r-t\rangle\}.$$

For several remarks and a detailed proof, we refer our readers to Chapter 3 or [34].

1.5 Asymptotic completeness

Next we consider the asymptotic completeness question for our quasilinear wave equation (1.1). For a fixed global solution u constructed in Lindblad [21], we seek to find the corresponding asymptotic profile and scattering data.

We start the proof with the construction of a global optical function $q=q(t,x)$. In other words, we solve the eikonal equation $g^{\alpha\beta}(u)q_\alpha q_\beta=0$ in a spacetime region Ω contained in $\{2r\geq t\geq \exp(\delta/\varepsilon)\}$. Here $\delta>0$ is a fixed parameter. We apply the method of characteristics and then follow the idea in Christodoulou-Klainerman [4]. By viewing $(g_{\alpha\beta})$, the inverse of the coefficient matrix $(g^{\alpha\beta}(u))$, as a Lorentzian metric in $[0,\infty)\times\mathbb{R}^3$, we construct a null frame $\{e_k\}_{k=1}^4$ in Ω. Then, most importantly, we define the second fundamental forms χ_{ab} for $a,b=1,2$ which are related to the Levi-Civita connection and the null frame under the metric $(g_{\alpha\beta})$. By studying the Raychaudhuri equation and using a continuity argument, we can show that $\mathrm{tr}\chi>0$ everywhere. This is the key step. In addition, we can prove that $q=q(t,x)$ is smooth in some weak sense (see Section 4.2.1).

Next, we define $(\mu,U)(t,x):=(q_t-q_r,\varepsilon^{-1}ru)(t,x)$. The map

$$\Omega\to[0,\infty)\times\mathbb{R}\times\mathbb{S}^2:\qquad (t,x)\mapsto(\varepsilon\ln t-\delta,q(t,x),x/|x|):=(s,q,\omega)$$

is an invertible smooth function with a smooth inverse, so a function $(\mu,U)(s,q,\omega)$ is obtained. It can be proved that $(\mu,U)(s,q,\omega)$ is an approximate solution to the reduced system (1.11), and that there is an exact solution $(\widetilde{\mu},\widetilde{U})(s,q,\omega)$ to the geometric reduced system (1.11) which matches $(\mu,U)(s,q,\omega)$ as $s\to\infty$. A key step is to prove that $A(q,\omega):=-\frac{1}{2}\lim_{s\to\infty}(\mu U_q)(s,q,\omega)$ is well-defined for each (q,ω). The function A is called the *scattering data* in this problem. We also show a gauge independence result, which states that the scattering data is independent of the choice of q in some specific way.

Finally, we construct an approximate solution $\widetilde{u}$ to (1.1) in Ω. The construction here is similar to that in Section 4 of [34]. That is, we construct a function $\widetilde{q}$ by solving

$$\widetilde{q}_t - \widetilde{q}_r = \mu(\varepsilon \ln t - \delta, \widetilde{q}(t,x), \omega)$$

by the method of characteristics, and then define

$$\widetilde{u}(t,x) := \varepsilon r^{-1} \widetilde{U}(\varepsilon \ln t - \delta, \widetilde{q}(t,x), \omega).$$

Then, in Ω, $\widetilde{q}$ is an approximate optical function, and $\widetilde{u}$ is an approximate solution to (1.1). In addition, near the light cone $t = r$, the difference $u - \widetilde{u}$, along with its derivatives, decays much faster than $\varepsilon t^{-1+C\varepsilon}$. Since u and its derivatives is of size $O(\varepsilon t^{-1+C\varepsilon})$, we conclude that $\widetilde{u}$ offers a good approximation of u.

We end this subsection with a rough version of the main theorem. For a precise statement, we refer to Theorem 4.1.

Theorem 1.2 (Rough version)**.** *Let u be a global solution to the Cauchy problem* (1.1) *and* (1.2)*. Fix a parameter $\delta > 0$ and a sufficiently small $\varepsilon > 0$. We define a region $\Omega \subset \{2r > t > \exp(\delta/\varepsilon)\} \subset \mathbb{R}^{1+3}_{t,x}$. Then we have*

(i) *There exists a solution to the eikonal equation*

$$g^{\alpha\beta}(u)\partial_\alpha q \partial_\beta q = 0 \text{ in } \Omega; \qquad q = |x| - t \text{ on } \partial\Omega.$$

Moreover, the map

$$\Omega \to [0,\infty) \times \mathbb{R} \times \mathbb{S}^2 : \qquad (t,x) \mapsto (\varepsilon \ln t - \delta, q(t,x), x/|x|)$$

is a diffeomorphism. Thus, a smooth function $F = F(t,x)$ induces a smooth function $F = F(s,q,\omega)$ and vice versa.

(ii) *In Ω, we set $(\mu, U)(t,x) := (q_t - q_r, \varepsilon^{-1} r u)(t,x)$ which induces a smooth function $(\mu, U)(s,q,\omega)$. Then, $(\mu,U)(s,q,\omega)$ is an approximate solution to the geometric reduced system* (2.4)*. In addition, the following three limits exist for all $(q,\omega) \in \mathbb{R} \times \mathbb{S}^2$:*

$$\begin{cases} A(q,\omega) := -\frac{1}{2} \lim_{s\to\infty} (\mu U_q)(s,q,\omega), \\ A_1(q,\omega) := \lim_{s\to\infty} \exp(\frac{1}{2} G(\omega) A(q,\omega) s) \mu(s,q,\omega), \\ A_2(q,\omega) := \lim_{s\to\infty} \exp(-\frac{1}{2} G(\omega) A(q,\omega) s) U_q(s,q,\omega). \end{cases}$$

All of them are smooth functions of (q,ω) for $\varepsilon \ll 1$, and we have $A_1 A_2 \equiv -2A$. Making use of these functions, we are able to obtain an exact solution to our reduced system (2.4)*.*

(iii) *The above results are gauge independent. That is, the scattering data $A = A(q, \omega)$ is independent of the choice of the optical function $q = q(t, x)$ in some suitable sense.*

(iv) *We define $\widetilde{u} = \widetilde{u}(t, x)$ as in Section 4.1.3. Then $\widetilde{u} = \widetilde{u}(t, x)$ is an approximate solution to* (1.1). *Moreover, the difference $u - \widetilde{u}$ decays much faster than the solution u itself as $t \to \infty$.*

For several remarks and a detailed proof, we refer our readers to Chapter 4 of this dissertation. We also remark that a paper [33] including the results listed above is in preparation by the author.

1.6 Preliminaries

1.6.1 Notations

We use C to denote universal positive constants. We write $A \lesssim B$ or $A = O(B)$ if $|A| \leq CB$ for some $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We use C_v or $\lesssim_v$ if we want to emphasize that the constant depends on a parameter v. We make an additional convention that the constants C are always independent of ε; that is, we would never write C_ε or $\lesssim_\varepsilon$ in this dissertation. The values of all constants in this dissertation may vary from line to line.

In this dissertation, we always assume that $\varepsilon \ll 1$ which means $0 < \varepsilon < \varepsilon_0$ for some sufficiently small constant $\varepsilon_0 < 1$. Again, we write $\varepsilon \ll_v 1$ if we want to emphasize that ε_0 depends on a parameter v.

Unless specified otherwise, we always assume that the Latin indices i, j, l take values in $\{1, 2, 3\}$ and the Greek indices α, β take values in $\{0, 1, 2, 3\}$. In Chapter 4 we also assume $a, b \in \{1, 2\}$. We use subscript to denote partial derivatives, unless specified otherwise. For example, $u_{\alpha\beta} = \partial_\alpha \partial_\beta u$, $q_r = \partial_r q = \sum_i \omega_i \partial_i q$, $A_q = \partial_q A$ and etc. For a fixed integer $k \geq 0$, we use ∂^k to denote either a specific partial derivative of order k, or the collection of partial derivatives of order k.

To prevent confusion, we will only use ∂_ω to denote the angular derivatives under the coordinate (s, q, ω), and will never use it under the coordinate (t, r, ω). For a fixed integer $k \geq 0$, we will use ∂_ω^k to denote either a specific angular derivative of order k, or the collection of all angular derivatives of order k.

1.6.2 Commuting vector fields

Let Z be any of the following vector fields:

$$\partial_\alpha,\ \alpha = 0, 1, 2, 3;\ S = t\partial t + r\partial_r;\ \Omega_{ij} = x_i \partial_j - x_j \partial_i,\ 1 \leq i < j \leq 3;\ \Omega_{0i} = x_i \partial_t + t\partial_i,\ i = 1, 2, 3. \tag{1.13}$$

We write these vector fields as $Z_1, Z_2, \dots, Z_{11}$, respectively. For any multiindex $I = (i_1, \dots, i_m)$ with length $m = |I|$ such that $1 \leq i_* \leq 11$, we set $Z^I = Z_{i_1} Z_{i_2} \cdots Z_{i_m}$. Then we have the Leibniz's rule

$$Z^I(fg) = \sum_{|J|+|K|=|I|} C^I_{JK} Z^J f Z^K g, \qquad \text{where } C^I_{JK} \text{ are constants.} \tag{1.14}$$

We have the following commutation properties.

$$[S, \Box] = -2\Box, \qquad [Z, \Box] = 0 \text{ for other } Z; \tag{1.15}$$

$$[Z_1, Z_2] = \sum_{|I|=1} C_{Z_1,Z_2,I} Z^I, \qquad \text{where } C_{Z_1,Z_2,I} \text{ are constants;} \tag{1.16}$$

$$[Z, \partial_\alpha] = \sum_\beta C_{Z,\alpha\beta} \partial_\beta, \qquad \text{where } C_{Z,\alpha\beta} \text{ are constants.} \tag{1.17}$$

In this dissertation, we need the following lemma related to the commuting vector fields. Here we use f_0 to denote an arbitrary polynomial of $\{Z^I\omega\}$. It is then clear that $Z^I f_0 = f_0$ for each I. We also remark that while the definition of f_0 will be modified in the rest of this dissertation, an arbitrary polynomial of $\{Z^I\omega\}$ could always be denoted as f_0.

Lemma 1.3. *For each multiindex I and each function F, we have*

$$(\partial_t - \partial_r) Z^I F = Z^I(F_t - F_r) + \sum_{|J|<|I|} [f_0 Z^J (F_t - F_r) + \sum_i f_0 (\partial_i + \omega_i \partial_t) Z^J F]. \tag{1.18}$$

Besides, for each $1 \leq k < k' \leq 3$, we have

$$(\partial_t - \partial_r) Z^I \Omega_{kk'} F = Z^I \Omega_{kk'}(F_t - F_r) + \sum_{|J|<|I|} [f_0 Z^J \Omega_{kk'} (F_t - F_r) + \sum_i f_0 (\partial_i + \omega_i \partial_t) Z^J \Omega_{kk'} F]. \tag{1.19}$$

Note that in $\sum_i(\dots)$, the sum is taken over all $i = 1, 2, 3$.

Proof. First, note that $[\partial_t - \partial_r, Z] = f_0 \cdot \partial$ and $\partial = f_0(\partial_t - \partial_r) + \sum_i f_0(\partial_i + \omega_i \partial_t)$. We now prove (1.18) by induction on $|I|$. If $|I| = 0$, there is nothing to prove. Now suppose we have proved (1.18) for each $|I| < n$. Now we fix a multiindex I with $|I| = n > 0$. Then, by writing

$Z^I = ZZ^{I'}$, we have

$$\begin{aligned}
&(\partial_t - \partial_r)Z^I F = [\partial_t - \partial_r, Z]Z^{I'}F + Z((\partial_t - \partial_r)Z^{I'}F) \\
&= f_0 \cdot \partial Z^{I'}F + Z(Z^{I'}(F_t - F_r) + \sum_{|J|<n-1} [f_0 Z^J(F_t - F_r) + \sum_i f_0(\partial_i + \omega_i\partial_t)Z^J F] \\
&= f_0(f_0(\partial_t - \partial_r) + \sum_j f_0(\partial_j + \omega_j\partial_t))Z^{I'}F + Z^I(F_t - F_r) \\
&\quad + \sum_{|J|<n-1} Z[f_0 Z^J(F_t - F_r) + \sum_i f_0(\partial_i + \omega_i\partial_t)Z^J F] \\
&= f_0(\partial_t - \partial_r)Z^{I'}F + \sum_j f_0(\partial_j + \omega_j\partial_t)Z^{I'}F + Z^I(F_t - F_r) \\
&\quad + \sum_{|J|<n-1} [(Zf_0)Z^J(F_t - F_r) + \sum_i (Zf_0)(\partial_i + \omega_i\partial_t)Z^J F] \\
&\quad + \sum_{|J|<n-1} [f_0 ZZ^J(F_t - F_r) + \sum_i f_0 Z(\partial_i + \omega_i\partial_t)Z^J F].
\end{aligned}$$

In the second equality, we can apply (1.18) by the induction hypotheses. Moreover, we note that $[\partial_i + \omega_i\partial_t, Z] = f_0 \cdot \partial$, so

$$\begin{aligned}
Z(\partial_i + \omega_i\partial_t)Z^J F &= (\partial_i + \omega_i\partial_t)ZZ^J F + f_0 \cdot \partial Z^J F \\
&= (\partial_i + \omega_i\partial_t)ZZ^J F + f_0(\partial_t - \partial_r)Z^J F + \sum_j f_0(\partial_j + \omega_j\partial_t)Z^J F.
\end{aligned}$$

Now (1.18) follows from the induction hypotheses and the computations above.

To prove (1.19), we replace F with $\Omega_{kk'}F$ in (1.18) and note that

$$\begin{aligned}
[\partial_t - \partial_r, \Omega_{kk'}] &= -\partial_r(x_k)\partial_{k'} + \partial_r(x_{k'})\partial_k + \sum_i \Omega_{kk'}(\omega_i)\partial_i \\
&= -\omega_k\partial_{k'} + \omega_{k'}\partial_k + \sum_i \omega_k(\delta_{ik'} - \omega_i\omega_{k'})\partial_i - \sum_i \omega_{k'}(\delta_{ik} - \omega_i\omega_k)\partial_i = 0.
\end{aligned}$$

Now, (1.19) is obvious. □

1.6.3 Several pointwise bounds

We have the pointwise estimates for partial derivatives.

Lemma 1.4. *For any function ϕ, we have*

$$|\partial^k\phi| \leq C\langle t - r\rangle^{-k} \sum_{|I|\leq k} |Z^I\phi|, \qquad \forall k \geq 1, \tag{1.20}$$

and

$$|(\partial_t+\partial_r)\phi|+|(\partial_i-\omega_i\partial_r)\phi|\leq C\langle t+r\rangle^{-1}|Z\phi|. \tag{1.21}$$

Here, for each $x\in\mathbb{R}$, we define the Japanese bracket $\langle x\rangle:=\sqrt{1+|x|^2}$. We also define $|Z\phi|:=\sum_{|I|=1}|Z^I\phi|$.

In addition, we have the Klainerman-Sobolev inequality.

Proposition 1.5. *For $\phi\in C^\infty(\mathbb{R}^{1+3})$ which vanishes for large $|x|$, we have*

$$(1+t+|x|)(1+|t-|x||)^{1/2}|\phi(t,x)|\leq C\sum_{|I|\leq 2}\left\|Z^I\phi(t,\cdot)\right\|_{L^2(\mathbb{R}^3)}. \tag{1.22}$$

We also state the Gronwall's inequality.

Proposition 1.6. *Suppose A,E,r are bounded functions from $[a,b]$ to $[0,\infty)$. Suppose that E is increasing. If*

$$A(t)\leq E(t)+\int_a^b r(s)A(s)\ ds, \qquad \forall t\in[a,b],$$

then

$$A(t)\leq E(t)\exp(\int_a^t r(s)\ ds), \qquad \forall t\in[a,b].$$

The proofs of these results are standard. See, for example, [21, 30, 7] for the proofs.

We also need the following lemma, which can be viewed as the estimates for Taylor's series adapted to Z vector fields.

Lemma 1.7. *Fix $\varepsilon>0$, an integer $k\geq 0$ and a multiindex I. Suppose there are two functions u,v on (t,x) such that $|u|+|v|\leq 1$ for all (t,x). Suppose $f\in C^\infty(\mathbb{R})$ with $f(0)=f'(0)=0$. Then, for all (t,x), we have*

$$\begin{aligned}&|\partial^k Z^I(f(u+v)-f(u))|\\ \lesssim_{k,I}&\sum_{k_1+k_2\leq k,\ |I_1|+|I_2|\leq|I|}p_{k,I}|\partial^{k_1}Z^{I_1}v(t,x)|(|\partial^{k_2}Z^{I_2}v(t,x)|+|\partial^{k_2}Z^{I_2}u(t,x)|).\end{aligned} \tag{1.23}$$

where

$$p_{k,I}(t,x)=1+\max_{k_1+|J|\leq(k+|I|)/2}(|\partial^{k_1}Z^Ju(t,x)|+|\partial^{k_1}Z^Jv(t,x)|)^{k+|I|}.$$

Proof. By the chain rule and Leibniz's rule, $\partial^kZ^I(f(u))$ can be written as a sum of terms of the form

$$f^{(l)}(u)\partial^{k_1}Z^{I_1}u\partial^{k_2}Z^{I_2}u\cdots\partial^{k_l}Z^{I_l}u$$

where $l \leq k+|I|$, $k_i+|I_i| > 0$ for each i and $\sum_i k_i = k$, $\sum_i I_i = I$. Thus, $\partial^k Z^I(f(u+v)-f(u))$ can be written as a sum of terms of the form

$$\begin{aligned}
&f^{(l)}(u+v)\partial^{k_1}Z^{I_1}(u+v)\partial^{k_2}Z^{I_2}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v) - f^{(l)}(u)\partial^{k_1}Z^{I_1}u\partial^{k_2}Z^{I_2}u\cdots\partial^{k_l}Z^{I_l}u \\
&= (f^{(l)}(u+v)-f^{(l)}(u))\partial^{k_1}Z^{I_1}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v) \\
&\quad + \sum_{j=1}^{l} f^{(l)}(u)\partial^{k_1}Z^{I_1}u\cdots\partial^{k_{j-1}}Z^{I_{j-1}}u\cdot\partial^{k_j}Z^{I_j}v\cdot\partial^{k_{j+1}}Z^{I_{j+1}}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v)
\end{aligned}$$

where $k_i + |I_i| > 0$ for each i and $\sum_i k_i = k$, $\sum_i I_i = I$. When $l = 0$, we must have $k = |I| = 0$, so (1.23) follows from

$$|f(u+v)-f(u)| \leq \sup_{\beta\in[0,1]} |f'(u+\beta v)||v| \leq \sup_{|z|\leq 1} |f''(z)| \cdot \sup_{\beta\in[0,1]} |u+\beta v|\cdot|v| \leq C(|u|+|v|)|v|.$$

Note that now $p_{0,0} = 2$. When $l \geq 1$, since $k_i + |I_i| > (k+|I|)/2 > 0$ for at most one i and since the product of all other terms of the form $\partial^{k_i}Z^{I_i}(u+v)$ can be controlled by $p_{k,I}$, we have

$$\begin{aligned}
&|(f^{(l)}(u+v)-f^{(l)}(u))\partial^{k_1}Z^{I_1}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v)| \\
&\leq \sup_{\beta\in[0,1]} |f^{(l+1)}(u+\beta v)||v\cdot\partial^{k_1}Z^{I_1}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v)| \\
&\leq C_{k,I}p_{k,I}|v| \sum_{k_1\leq k, |J|\leq|I|} (|\partial^{k_1}Z^J u| + |\partial^{k_1}Z^J v|).
\end{aligned}$$

When $l = 1$, we have

$$|f'(u)\partial^k Z^I v| \leq C|u||\partial^k Z^I v|.$$

When $l \geq 2$, since $k_i + |I_i| > (k+|I|)/2$ for at most one i and since the product of all other terms of the form $\partial^{k_i}Z^{I_i}(u+v)$ or $\partial^{k_i}Z^{I_i}u$ can be controlled by $p_{k,I}$, we have

$$\begin{aligned}
&|f^{(l)}(u)\partial^{k_1}Z^{I_1}u\cdots\partial^{k_{j-1}}Z^{I_{j-1}}u\cdot\partial^{k_j}Z^{I_j}v\cdot\partial^{k_{j+1}}Z^{I_{j+1}}(u+v)\cdots\partial^{k_l}Z^{I_l}(u+v)| \\
&\leq C_{k,I}p_{k,I} \sum_{k_1+k_2\leq k,\ |I_1|+|I_2|\leq|I|} |\partial^{k_1}Z^{I_1}v|(|\partial^{k_2}Z^{I_2}u| + |\partial^{k_2}Z^{I_2}v|).
\end{aligned}$$

□

1.6.4 A function space

Fix a domain $\mathcal{D} \subset \mathbb{R}^{1+3}_{t,x}$ which may depend on the parameter ε. Suppose that in $\mathcal{D}$ we have $t \geq 2C$ and $r/t \in [1/C, C]$ for some constant $C > 1$ which is independent of ε. We make the following definition.

Definition 1.8. Fix $n, s, p \in \mathbb{R}$. We say that a function $F = F_\varepsilon(t,x)$ is in $\varepsilon^n S^{s,p} = \varepsilon^n S^{s,p}_{\mathcal{D}}$, if for each fixed integer $N \geq 1$, for all $\varepsilon \ll_{n,s,p,N} 1$, we have $F \in C^N(\mathcal{D})$ and

$$\sum_{|I|\leq N} |Z^I F(t,x)| \lesssim \varepsilon^n t^{s+C\varepsilon}\langle r-t\rangle^p, \qquad \forall (t,x) \in \mathcal{D}. \tag{1.24}$$

Here F is allowed to depend on ε, but all the constants in (1.24) must be independent of ε. If $n = 0$, we write $\varepsilon^0 S^{s,p}$ as $S^{s,p}$ for simplicity.

We have the following key lemma.

Lemma 1.9. *We have the following two properties.*
(a) *For any $F_1 \in \varepsilon^{n_1} S^{s_1,p_1}$ and $F_2 \in \varepsilon^{n_2} S^{s_2,p_2}$, we have*

$$F_1 + F_2 \in \varepsilon^{\min\{n_1,n_2\}} S^{\max\{s_1,s_2\},\max\{p_1,p_2\}}, \qquad F_1F_2 \in \varepsilon^{n_1+n_2} S^{s_1+s_2,p_1+p_2}.$$

(b) *For any $F \in \varepsilon^n S^{s,p}$, we have $ZF \in \varepsilon^n S^{s,p}$, $\partial F \in \varepsilon^n S^{s,p-1}$ and $(\partial_i + \omega_i \partial_t)F \in \varepsilon^n S^{s-1,p}$.*

Proof. Note that (a) follows directly from the definition and the Leibniz's rule. In (b), if $F \in \varepsilon^n S^{s,p}$, then $ZF \in \varepsilon^n S^{s,p}$ follows directly from the definition. Next, we fix an arbitrary integer $N \geq 1$. Since $F \in \varepsilon^n S^{s,p}$, for all $\varepsilon \ll_{n,s,p,N+1} 1$ we have $F \in C^{N+1}(\mathcal{D})$ and

$$\sum_{|I|\leq N+1} |Z^I F(t,x)| \lesssim \varepsilon^n t^{s+C\varepsilon} \langle r-t\rangle^p, \qquad \forall (t,x) \in \mathcal{D}.$$

Thus, $\partial F \in C^N(\mathcal{D})$. Moreover, by (1.17) and Lemma 1.4, in $\mathcal{D}$ we have

$$\sum_{|I|\leq N} |Z^I \partial F| \lesssim \sum_{|I|\leq N} |\partial Z^I F| \lesssim \sum_{|J|\leq N+1} \langle r-t\rangle^{-1} |Z^J F| \lesssim \varepsilon^n t^{s+C\varepsilon} \langle r-t\rangle^{p-1}.$$

In conclusion, $\partial F \in \varepsilon^n S^{s,p-1}$.

Next, we note that

$$(\partial_i + \omega_i\partial_t)F = r^{-1}\Omega_{0i}F + (r+t)^{-1}r^{-1}\omega_i(rSF - \sum_j t\omega_j\Omega_{0j}F) + (r-t)r^{-2}\sum_j \omega_j \Omega_{ji}F.$$

By the definition, we can easily show $t^m, r^m, (r+t)^m \in S^{m,0}$ for each $m \in \mathbb{R}$, $r-t \in S^{0,1}$ and $\partial^m \omega_i \in S^{-m,0}$ for each integer $m \geq 0$. And since $ZF \in S^{s,p}$, by part (a) we conclude that

$$(\partial_i + \omega_i\partial_t)F \in \varepsilon^n S^{s-1,p} + \varepsilon^n S^{s-2,p-1} = \varepsilon^n S^{s-1,p}.$$

Here we have $\varepsilon^n S^{s-2,p-1} \subset \varepsilon^n S^{s-1,p}$. In fact, recall that $t \geq 2C$ and $C^{-1} \leq r/t \leq C$ for some constant C independent of ε. Thus, in $\mathcal{D}$ we have

$$\langle r-t\rangle/t = \sqrt{t^{-2} + (r/t-1)^2} \leq \sqrt{1/(4C^2) + C^2} \lesssim 1.$$

In summary, in $\mathcal{D}$ we have

$$\varepsilon^n t^{s-2+C\varepsilon} \langle r-t\rangle^{p-1} \lesssim \varepsilon^n t^{s-1+C\varepsilon} \langle r-t\rangle^p,$$

so $S^{s-2,p-1} \subset S^{s-1,p}$. This finishes the proof. □

Example 1.10. We have

$$t^m, r^m, (r+t)^m \in S^{m,0},\ \forall m \in \mathbb{R}; \qquad r-t \in S^{0,1}; \qquad \partial^m \omega_i \in S^{-m,0}\ \forall m \geq 0,\ m \in \mathbb{Z}.$$

It also follows from $\frac{d}{ds}\langle s\rangle = s/\langle s\rangle$, the chain rule and Lemma 1.9 that

$$(r-t)^m, \langle r-t\rangle^m \in S^{0,m},\ \forall m \in \mathbb{R}.$$

These estimates would be very useful in the rest of this dissertation.

In addition, we have the following lemma which is relevant to the Taylor's expansion of a function.

Lemma 1.11. *Suppose $f \in C^\infty(\mathbb{R})$ and let $u \in \varepsilon^n S^{s,p}$ for some $n > 0$, $s < 0$ and $p \leq 0$. Then, we have $f(u) - f(0) - f'(0)u \in \varepsilon^{2n} S^{2s,2p}$.*

Proof. Since $u \in \varepsilon^n S^{s,p}$, $t \gtrsim 1$, $s < 0$ and $p \leq 0$, by choosing $\varepsilon \ll_{n,s} 1$, we have

$$|u| \leq C\varepsilon^n t^{s+C\varepsilon}\langle r-t\rangle^p \leq C\varepsilon^n \leq 1.$$

In this estimate, we can choose $\varepsilon \ll_{n,m,s,p} 1$ so that $s + C\varepsilon < 0$. Then,

$$\begin{aligned}|f(u) - f(0) - f'(0)u| &= |\int_0^u f'(v) - f'(u)\ dv| \leq \int_{|v|\leq|u|} |\int_u^v f''(w)\ dw|\ dv \\ &\leq \int_{|v|\leq|u|}\int_{|w|\leq|u|} |f''(w)|\ dwdv \leq \|f\|_{C^2([-1,1])}|u|^2 \lesssim_f \varepsilon^{2n} t^{2s+C\varepsilon}\langle r-t\rangle^{2p}.\end{aligned}$$

In general, we fix a multiindex I with $|I| =: m > 0$. Suppose we have proved that for $\varepsilon \ll_{n,s,p,m} 1$, the function $f(u) - f(0) - f'(0)u$ is in $C^{m-1}(\mathcal{D})$, such that (1.24) holds with $N = m-1$ and $F = f(u) - f(0) - f'(0)u$. By the Leibinz's rule and the chain rule, we can write $Z^I(f(u) - f(0) - f'(0)u)$ as a sum of $(f'(u) - f'(0))Z^I u$ and a linear combination of terms of the form

$$f^{(l)}(u) \cdot \prod_{j=1}^l Z^{I_j} u, \qquad \text{where } 2 \leq l \leq m,\ \sum |I_j| = m,\ |I_j| > 0 \text{ for each } j.$$

Since $u \in \varepsilon^n S^{s,p}$, we can choose $\varepsilon \ll_{m,n,s,p} 1$ such that $u \in C^m(\mathcal{D})$ such that in $\mathcal{D}$

$$\sum_{|J|\leq m} |Z^J u| \lesssim \varepsilon^n t^{s+C\varepsilon}\langle r-t\rangle^p.$$

Since $\sum_{l\leq m} |f^{(l)}(u)| \leq \|f\|_{C^m([-1,1])}$ and $|f'(u) - f'(0)| \leq \|f\|_{C^2([-1,1])} \cdot |u|$, we have

$$|(f'(u) - f'(0))Z^I u| \lesssim_f |u||Zu| \lesssim \varepsilon^{2n} t^{2s+C\varepsilon}\langle r-t\rangle^{2p},$$

$$|f^{(l)}(u) \cdot \prod_{j=1}^l Z^{I_j} u| \lesssim_f (\varepsilon^n t^{s+C\varepsilon}\langle r-t\rangle^p)^m \lesssim \varepsilon^{2n} t^{2s+C\varepsilon}\langle r-t\rangle^{2p}.$$

In conclusion, as long as $\varepsilon \ll_{m,s,p,n} 1$, in $\mathcal{D}$ we have

$$|Z^I(f(u) - f(0) - f'(0)u)| \lesssim \varepsilon^{2n} t^{2s+C\varepsilon} \langle r - t \rangle^{2p}.$$

We thus conclude that $f(u) - f(0) - f'(0)u \in \varepsilon^{2n} S^{2s,2p}$. □

Chapter 2

A New Reduced System

2.1 The asymptotic equations for the quasilinear wave equation (1.1)

Let $u = u(t,x)$ be a global solution to (1.1). Let $q = q(t,x)$ be a solution of the eikonal equation (1.10) related to (1.1), and let $\mu = q_t - q_r$. Suppose u has the form

$$u(t,x) \approx \varepsilon r^{-1} U(s,q,\omega) \tag{2.1}$$

where $\omega_i = x_i/r$, $s = \varepsilon \ln(t)$ and $q = q(t,x)$. Our goal in this section is to derive the asymptotic equations for (μ, U).

We make the following assumptions:

1. Every function is smooth.
2. There is a diffeomorphism between two coordinates (t,r,ω) and (s,q,ω), so any function F can be written as $F(t,r,\omega)$ and $F(s,q,\omega)$ at the same time.
3. $\varepsilon > 0$ is sufficiently small, $t, r > 0$ are both sufficiently large with $t \approx r$.
4. All the angular derivatives are negligible. In particular, $\partial_i \approx \omega_i \partial_r$.
5. $\mu, U \sim 1$ and $\nu \lesssim \varepsilon t^{-1}$, where $\nu := q_t + q_r$. The same estimates hold if we apply Z^I or $\partial_s^a \partial_q^b \partial_\omega^c$ to the left hand sides.

Here are two useful remarks. First, the solutions (μ, U) to the reduced system may not exactly satisfy the assumptions listed above. They only satisfy some weaker versions of those assumptions. For example, instead of $\mu \sim 1$, we may only get $t^{-C\varepsilon} \lesssim |\mu| \lesssim t^{C\varepsilon}$; by solving $q_t - q_r = \mu$, instead of an exact optical function, i.e. a solution to (1.10), we may only get an approximate optical function q in the sense that $g^{\alpha\beta}(u) q_\alpha q_\beta = O(t^{-2+C\varepsilon})$. However, such differences are usually negligible in the derivation of a reduced system. Thus, our assumptions above are very reasonable.

Secondly, it may seem strange that we ignore the angular derivatives of q which is $\lesssim t^{-1}$ but keep $\nu \lesssim \varepsilon t^{-1}$. This, however, is reasonable according to the form of (1.1) and (1.10). For example, if we expand the eikonal equation, we get (2.3) below. The angular derivatives are either squared or multiplied by $\varepsilon r^{-1}U$, while the major terms in (2.3) are of the order εt^{-1}. On the other hand, ν is not negligible since there is a term $\mu\nu$ in the expansion.

Recall that

$$\Box u = r^{-1}((-\partial_t + \partial_r)(\partial_t + \partial_r) + r^{-2}\Delta_\omega)ru$$

where $\Delta_\omega = \sum_{i<j}\Omega_{ij}^2$ is the Laplacian on the sphere $\mathbb{S}^2$. By the chain rule we have

$$\partial_t = \varepsilon t^{-1}\partial_s + q_t\partial_q, \qquad \partial_r = q_r\partial_q.$$

By the assumptions, we have

$$\begin{aligned}\Box u &\approx \varepsilon r^{-1}(-\partial_t + \partial_r)(\partial_t + \partial_r)U \approx -\varepsilon r^{-1}\mu\partial_q(\varepsilon t^{-1}U_s + \nu U_q)\\ &\approx -\varepsilon^2(tr)^{-1}\mu U_{sq} - \varepsilon r^{-1}\mu\nu_q U_q - \varepsilon r^{-1}\mu\nu U_{qq}.\end{aligned}$$

Since

$$\begin{aligned}
q_t &= \frac{1}{2}(\mu+\nu) \approx \frac{1}{2}\mu, & q_i &\approx \omega_i q_r \approx \frac{\omega_i}{2}(\nu-\mu) \approx -\frac{1}{2}\omega_i\mu,\\
q_{tt} &\approx \frac{1}{2}\mu_t \approx \frac{1}{2}\mu_q q_t \approx \frac{1}{4}\mu\mu_q, & q_{it} &\approx \frac{1}{2}\mu_i \approx \frac{1}{2}\mu_q q_i \approx -\frac{1}{4}\omega_i\mu\mu_q,\\
q_{ij} &\approx -\frac{1}{2}\omega_i\mu_j \approx -\frac{1}{2}\omega_i\mu_q q_j \approx \frac{1}{4}\omega_i\omega_j\mu_q\mu,
\end{aligned}$$

we have

$$g_0^{\alpha\beta}q_\alpha q_\beta \approx \frac{1}{4}G(\omega)\mu^2, \qquad g^{\alpha\beta}q_{\alpha\beta} \approx \frac{1}{4}G(\omega)\mu\mu_q,$$

where

$$G(\omega) = g_0^{\alpha\beta}\widehat{\omega}_\alpha\widehat{\omega}_\beta, \qquad g_0^{\alpha\beta} = \frac{d}{du}g^{\alpha\beta}(u)|_{u=0},\ \widehat{\omega} = (-1,\omega) \in \mathbb{R}\times\mathbb{S}^2.$$

And since

$$U_{tt} \approx U_{qq}q_{tt} + U_q q_t^2, \qquad U_{it} \approx U_{qq}q_i q_t + U_q q_{it}, \qquad U_{ij} \approx U_{qq}q_i q_j + U_q q_{ij},$$

we have from (1.1)

$$\begin{aligned}
0 &= \widetilde{g}^{\alpha\beta}(u)\partial_\alpha\partial_\beta u \approx \Box u + g_0^{\alpha\beta}u\partial_\alpha\partial_\beta u\\
&\approx -\varepsilon^2(tr)^{-1}\mu U_{sq} - \varepsilon r^{-1}\mu\nu_q U_q - \varepsilon r^{-1}\mu\nu U_{qq} + \varepsilon^2 r^{-2}g_0^{\alpha\beta}U(U_q q_{\alpha\beta} + U_{qq}q_\alpha q_\beta)\\
&\approx -\varepsilon^2(tr)^{-1}\mu U_{sq} - \varepsilon r^{-1}\mu\nu_q U_q - \varepsilon r^{-1}\mu\nu U_{qq} + \frac{1}{4}G(\omega)\varepsilon^2 r^{-2}(\mu\mu_q UU_q + \mu^2 UU_{qq}).
\end{aligned} \tag{2.2}$$

By the eikonal equation, we have

$$0 = g^{\alpha\beta}(u)q_\alpha q_\beta \approx -q_t^2 + \sum_i q_i^2 + \varepsilon r^{-1}g_0^{\alpha\beta}Uq_\alpha q_\beta \approx -\mu\nu + \frac{1}{4}\varepsilon r^{-1}G(\omega)\mu^2 U, \tag{2.3}$$

so we conclude that

$$\nu \approx \frac{1}{4}\varepsilon r^{-1}G(\omega)\mu U, \qquad \nu_q \approx \frac{1}{4}\varepsilon r^{-1}G(\omega)(\mu_q U + \mu U_q).$$

Plug everything back in (2.2). We thus have

$$\begin{aligned} 0 &\approx -\varepsilon^2(tr)^{-1}\mu U_{sq} - \frac{1}{4}\varepsilon^2 r^{-2}G(\omega)(\mu_q U + \mu U_q)\mu U_q \\ &\quad - \frac{1}{4}\varepsilon^2 r^{-2}G(\omega)\mu^2 U U_{qq} + \frac{1}{4}G(\omega)\varepsilon^2 r^{-2}(\mu\mu_q U U_q + \mu^2 U U_{qq}) \\ &= -\varepsilon^2(tr)^{-1}\mu U_{sq} - \frac{1}{4}\varepsilon^2 r^{-2}G(\omega)\mu^2 U_q^2. \end{aligned}$$

Assuming that $t = r$, we get the first asymptotic equation

$$U_{sq} = -\frac{1}{4}G(\omega)\mu U_q^2.$$

Meanwhile, note that from $(\partial_t - \partial_r)\nu = (\partial_t + \partial_r)\mu$, we have

$$\nu_q\mu \approx \nu_q\mu + \varepsilon t^{-1}\nu_s = \mu_q\nu + \varepsilon t^{-1}\mu_s$$

and thus

$$\begin{aligned} \mu_s &\approx t\varepsilon^{-1}(\nu_q\mu - \mu_q\nu) \approx t\varepsilon^{-1}(\frac{1}{4}\varepsilon r^{-1}G(\omega)(\mu_q U + \mu U_q)\mu - \frac{1}{4}\varepsilon r^{-1}G(\omega)\mu U\mu_q) \\ &\approx \frac{t}{4r}G(\omega)\mu^2 U_q. \end{aligned}$$

Again, assuming that $t = r$, we get the second asymptotic equation

$$\mu_s = \frac{1}{4}G(\omega)\mu^2 U_q.$$

In conclusion, our system of asymptotic equations is

$$\begin{cases} \partial_s\mu = \frac{1}{4}G(\omega)\mu^2 U_q, \\ \partial_s U_q = -\frac{1}{4}G(\omega)\mu U_q^2. \end{cases} \tag{2.4}$$

We call (2.4) a *geometric reduced system* for (1.1), since it is related to the geometry of the null cone with respect to the metric $g_{\alpha\beta} = (g^{\alpha\beta}(u))^{-1}$ instead of the Minkowski metric.

If the initial data is given by $(\mu, U_q)|_{s=0}(q,\omega) = (A_1, A_2)(q,\omega)$, and if we set $A := -(A_1A_2)/2$, then (2.4) has an explicit solution

$$\begin{cases} \mu(s,q,\omega) = A_1(q,\omega)\exp(-\frac{1}{2}G(\omega)A(q,\omega)s), \\ U_q(s,q,\omega) = A_2(q,\omega)\exp(\frac{1}{2}G(\omega)A(q,\omega)s), \end{cases} \tag{2.5}$$

To solve for $U(s,q,\omega)$ uniquely, we assume $\lim_{q\to-\infty} U(s,q,\omega) = 0$ in the modified wave operator problem, or $\lim_{q\to\infty} U(s,q,\omega) = 0$ in the asymptotic completeness problem.

2.2 The asymptotic equations for the general case

Though the derivation of the asymptotic system (2.4) is already sufficient for this dissertation, let us also do the corresponding computations in a more general case. In this section, we study a system of general quasilinear wave equations

$$g^{\alpha\beta}(u,\partial u)\partial_\alpha\partial_\beta u^j = f^j(u,\partial u), \qquad j=1,\dots,N. \tag{2.6}$$

Here our unknown function u is a vector-valued function. That is, we have $u=(u^1,\dots,u^N):\mathbb{R}^{1+3}_{t,x}\to\mathbb{R}^N$ for some positive integer N. In addition, we assume that $(g^{\alpha\beta})$ are smooth, symmetric and independent of j and that $g^{\alpha\beta}(0,0)=m^{\alpha\beta}$. Moreover, we assume that f^j are all smooth functions such that $f^j(0,0)=0$ and $df^j(0,0)=0$.

Assume that we have Taylor expansions

$$\begin{aligned} g^{\alpha\beta}(u,\partial u) &= m^{\alpha\beta}+g_k^{\alpha\beta}u^k+g_k^{\alpha\beta\lambda}\partial_\lambda u^k+O(|u|^2+|\partial u|^2),\\ f^j(u,\partial u) &= f^j_{kk'}u^ku^{k'}+f^{j,\alpha}_{kk'}u^k\partial_\alpha u^{k'}+f^{j,\alpha\beta}_{kk'}\partial_\alpha u^k\partial_\beta u^{k'}+O(|u|^3+|\partial u|^3).\end{aligned}$$

Here $m^{\alpha\beta}, g^*_*, f^*_*$ are all real constants. In addition, we use the Einstein summation convention and we take sum over all $1\le k,k'\le N$.

We make the ansatz

$$u^j(t,x)\approx \varepsilon r^{-1}U^j(s,q,\omega), \qquad j=1,2,\dots,N$$

with the same s,ω,r. We now assume that q is the solution to the eikonal equation

$$g^{\alpha\beta}(u,\partial u)\partial_\alpha q\partial_\beta q=0. \tag{2.7}$$

Set $\mu=q_t-q_r$ and $\nu=q_t+q_r$. Again, we assume that all the assumptions made in Section 2.1 remain valid.

Following the computations in Section 2.1, we have

$$\Box u^j\approx -\varepsilon^2(tr)^{-1}\mu U^j_{sq}-\varepsilon r^{-1}\mu\nu_qU^j_q-\varepsilon r^{-1}\mu\nu U^j_{qq};$$

$$q_\alpha\approx-\frac12\mu\widehat{\omega}_\alpha,\ q_{\alpha\beta}=\frac14\mu\mu_q\widehat{\omega}_\alpha\widehat{\omega}_\beta, \qquad \text{where } \widehat{\omega}:=(-1,\omega)\in\mathbb{R}\times\mathbb{S}^2;$$

$$\partial_\alpha u^j\approx\varepsilon r^{-1}U^j_qq_\alpha\approx-\frac{\varepsilon}{2r}U^j_q\mu\widehat{\omega}_\alpha,\ \partial_\alpha\partial_\beta u^j\approx\varepsilon r^{-1}(U^j_{qq}q_\alpha q_\beta+U^j_qq_{\alpha\beta})\approx\frac{\varepsilon}{4r}(U^j_{qq}\mu+U^j_q\mu_q)\mu\widehat{\omega}_\alpha\widehat{\omega}_\beta.$$

It then follows that

$$\begin{aligned} g^{\alpha\beta}(u,\partial u)&\approx m^{\alpha\beta}+g_k^{\alpha\beta}u^k+g_k^{\alpha\beta\lambda}\partial_\lambda u^k\approx m^{\alpha\beta}+\frac{\varepsilon}{r}g_k^{\alpha\beta}U^k-\frac{\varepsilon}{2r}g_k^{\alpha\beta\lambda}\widehat{\omega}_\lambda\mu U^k_q\\ g^{\alpha\beta}(u,\partial u)q_\alpha q_\beta&\approx-\mu\nu+\frac14\mu^2\widehat{\omega}_\alpha\widehat{\omega}_\beta(\frac{\varepsilon}{r}g_k^{\alpha\beta}U^k-\frac{\varepsilon}{2r}g_k^{\alpha\beta\lambda}\widehat{\omega}_\lambda\mu U^k_q),\\ g^{\alpha\beta}(u,\partial u)\partial_\alpha\partial_\beta u^j&\approx\Box u^j+(\frac{\varepsilon}{r}g_k^{\alpha\beta}U^k-\frac{\varepsilon}{2r}g_k^{\alpha\beta\lambda}\widehat{\omega}_\lambda\mu U^k_q)\cdot\frac{\varepsilon}{4r}(U^j_{qq}\mu+U^j_q\mu_q)\mu\widehat{\omega}_\alpha\widehat{\omega}_\beta\end{aligned}$$

and that

$$\begin{aligned} f^j(u,\partial u) &\approx f^j_{kk'}u^k u^{k'} + f^{j,\alpha}_{kk'}u^k\partial_\alpha u^{k'} + f^{j,\alpha\beta}_{kk'}\partial_\alpha u^k\partial_\beta u^{k'} \\ &\approx \frac{\varepsilon^2}{r^2}f^j_{kk'}U^kU^{k'} - \frac{\varepsilon^2}{2r^2}f^{j,\alpha}_{kk'}\widehat{\omega}_\alpha U^k\mu U^{k'}_q + \frac{\varepsilon^2}{4r^2}f^{j,\alpha\beta}_{kk'}\widehat{\omega}_\alpha\widehat{\omega}_\beta\mu^2U^k_qU^{k'}_q. \end{aligned}$$

For simplicity, we set

$$G_{2,k}(\omega) := g^{\alpha\beta}_k\widehat{\omega}_\alpha\widehat{\omega}_\beta, \qquad G_{3,k}(\omega) := g^{\alpha\beta\lambda}_k\widehat{\omega}_\alpha\widehat{\omega}_\beta\widehat{\omega}_\lambda;$$

$$F^j_{0,kk'}(\omega) := f^j_{kk'}, \qquad F^j_{1,kk'}(\omega) := f^{j,\alpha}_{kk'}\widehat{\omega}_\alpha, \qquad F^j_{2,kk'}(\omega) := f^{j,\alpha\beta}_{kk'}\widehat{\omega}_\alpha\widehat{\omega}_\beta.$$

Then, by the eikonal equation, we have

$$0 \approx -\mu\nu + \frac{\varepsilon}{4r}\mu^2G_{2,k}(\omega)U^k - \frac{\varepsilon}{8r}G_{3,k}(\omega)\mu^3U^k_q,$$

and thus

$$\begin{aligned} \nu &\approx \frac{\varepsilon}{4r}G_{2,k}(\omega)\mu U^k - \frac{\varepsilon}{8r}G_{3,k}(\omega)\mu^2U^k_q, \\ \nu_q &\approx \frac{\varepsilon}{4r}G_{2,k}(\omega)\partial_q(\mu U^k) - \frac{\varepsilon}{8r}G_{3,k}(\omega)\partial_q(\mu^2U^k_q). \end{aligned}$$

It follows that

$$\begin{aligned} \Box u^j \approx& -\frac{\varepsilon^2}{tr}\mu U^j_{sq} - \frac{\varepsilon^2}{8r^2}\mu(2G_{2,k}(\omega)\partial_q(\mu U^k) - G_{3,k}(\omega)\partial_q(\mu^2U^k_q))U^j_q \\ &- \frac{\varepsilon^2}{8r^2}\mu(2G_{2,k}(\omega)\mu U^k - G_{3,k}(\omega)\mu^2U^k_q)U^j_{qq}. \end{aligned}$$

Besides, by (2.1), we have

$$\begin{aligned} 0 \approx& \Box u^j + \frac{\varepsilon^2}{8r^2}(2G_{2,k}(\omega)U^k - G_{3,k}(\omega)\mu U^k_q)(U^j_{qq}\mu + U^j_q\mu_q)\mu \\ &- \frac{\varepsilon^2}{r^2}F^j_{0,kk'}(\omega)U^kU^{k'} + \frac{\varepsilon^2}{2r^2}F^j_{1,kk'}(\omega)U^k\mu U^{k'}_q - \frac{\varepsilon^2}{4r^2}F^j_{2,kk'}(\omega)\mu^2U^k_qU^{k'}_q \\ \approx& -\frac{\varepsilon^2}{tr}\mu U^j_{sq} - \frac{\varepsilon^2}{4r^2}G_{2,k}(\omega)\mu^2U^k_qU^j_q + \frac{\varepsilon^2}{8r^2}G_{3,k}(\omega)\mu^2U^j_q(\mu_qU^k_q + \mu U^k_{qq}) \\ &- \frac{\varepsilon^2}{r^2}F^j_{0,kk'}(\omega)U^kU^{k'} + \frac{\varepsilon^2}{2r^2}F^j_{1,kk'}(\omega)U^k\mu U^{k'}_q - \frac{\varepsilon^2}{4r^2}F^j_{2,kk'}(\omega)\mu^2U^k_qU^{k'}_q. \end{aligned}$$

By setting $t = r$, we obtain the first asymptotic equation

$$\begin{aligned} \mu U^j_{sq} =& -\frac{1}{4}G_{2,k}(\omega)\mu^2U^k_qU^j_q + \frac{1}{8}G_{3,k}(\omega)\mu^2U^j_q(\mu_qU^k_q + \mu U^k_{qq}) \\ &- F^j_{0,kk'}(\omega)U^kU^{k'} + \frac{1}{2}F^j_{1,kk'}(\omega)U^k\mu U^{k'}_q - \frac{1}{4}F^j_{2,kk'}(\omega)\mu^2U^k_qU^{k'}_q. \end{aligned}$$

In addition, since $(\partial_t + \partial_r)\mu = (\partial_t - \partial_r)\nu$, we have $\varepsilon t^{-1}\mu_s + \nu\mu_q \approx \mu\nu_q$. This implies

$$\begin{aligned}\varepsilon t^{-1}\mu_s &\approx \mu\nu_q - \mu_q\nu \\ &\approx \mu(\frac{\varepsilon}{4r}G_{2,k}(\omega)\partial_q(\mu U^k) - \frac{\varepsilon}{8r}G_{3,k}(\omega)\partial_q(\mu^2 U^k_q)) - \mu_q(\frac{\varepsilon}{4r}G_{2,k}(\omega)\mu U^k - \frac{\varepsilon}{8r}G_{3,k}(\omega)\mu^2 U^k_q) \\ &\approx \frac{\varepsilon}{4r}G_{2,k}(\omega)\mu^2 U^k_q - \frac{\varepsilon}{8r}G_{3,k}(\omega)(\mu^3 U^k_{qq} + \mu^2\mu_q U^k_q).\end{aligned}$$

By setting $t = r$, we obtain the second asymptotic equation

$$\mu_s = \frac{1}{4}G_{2,k}(\omega)\mu^2 U^k_q - \frac{1}{8}G_{3,k}(\omega)(\mu^3 U^k_{qq} + \mu^2\mu_q U^k_q).$$

Finally, we note that

$$\partial_s(\mu U^j_q) = -F^j_{0,kk'}(\omega)U^k U^{k'} + \frac{1}{2}F^j_{1,kk'}(\omega)U^k\mu U^{k'}_q - \frac{1}{4}F^j_{2,kk'}(\omega)\mu^2 U^k_q U^{k'}_q.$$

In summary, we make the following definition.

Definition 2.1. The system of differential equations

$$\begin{cases} \partial_s(\mu U^j_q) = -F^j_{0,kk'}(\omega)U^k U^{k'} + \frac{1}{2}F^j_{1,kk'}(\omega)U^k\mu U^{k'}_q - \frac{1}{4}F^j_{2,kk'}(\omega)\mu^2 U^k_q U^{k'}_q, \ j = 1, \dots, N \\ \partial_s\mu = \frac{1}{4}G_{2,k}(\omega)\mu^2 U^k_q - \frac{1}{8}G_{3,k}(\omega)(\mu^3 U^k_{qq} + \mu^2\mu_q U^k_q) \end{cases} \tag{2.8}$$

is called a *geometric reduced system* for (2.1).

We can then define a variant of the weak null condition.

Definition 2.2. We say that a system (2.1) of quasilinear wave equations satisfies the *geometric weak null condition*, if for any initial data at $s = 0$ decaying sufficiently fast in q, we have a global solution to the corresponding new reduced system for all $s \geq 0$, and if the solution and all the derivatives grow at most exponentially in s.

Two questions arise naturally from these definitions.

1. To what extent is the geometric weak null condition equivalent to the weak null condition?

2. Is the geometric weak null condition sufficient for the global existence of general quasilinear wave equations with small and localized initial data?

The answers to these two questions are still unclear, and the author believes that answering them might give us a better understanding of the long time dynamics of general quasilinear wave equations.

We end this section with two examples.

Example 2.3. Suppose $N = 1$, $f \equiv 0$ and $g^{\alpha\beta}(u, \partial u) = g^{\alpha\beta}(\partial u)$. In this case, (2.8) becomes

$$\begin{cases} \partial_s(\mu U_q) = 0, \\ \partial_s \mu = -\frac{1}{8}G_3(\omega)(\mu^3 U_{qq} + \mu^2 \mu_q U_q) = -\frac{1}{8}G_3(\omega)\mu^2\partial_q(\mu U_q). \end{cases}$$

Here we are working in the scalar case $N = 1$, so we can write $U = U^1$ and $G_3(\omega) = G_{3,1}(\omega)$ for simplicity.

We claim that here the geometric weak null condition is satisfied if and only if $G_3(\omega) \equiv 0$ on $\mathbb{S}^2$, i.e. the null condition is satisfied. In fact, by the first equation, for all s we have

$$\mu U_q(s, q, \omega) = \mu U_q(0, q, \omega).$$

We set $B(q, \omega) = \mu U_q(0, q, \omega)$. Then, then second equation now becomes

$$\partial_s \mu = -\frac{1}{8}G_3(\omega)\mu^2 B_q \Longrightarrow \partial_s(1/\mu) = \frac{1}{8}G_3(\omega)B_q(q, \omega).$$

This equation has a solution for all

$$0 \leq s < \inf_{(q,\omega)\in\mathbb{R}\times\mathbb{S}^2} \frac{8}{\max\{0, -\mu(0, q, \omega)G_3(\omega)B_q(q, \omega)\}}.$$

Here we use the convention that $8/0 = \infty$. If $G_3(\omega) \equiv 0$, it is obvious that the geometric weak null condition is satisfied. Otherwise, by choosing $(\mu|_{s=0}, B)(q, \omega)$ appropriately, we can make $\mu(0, q, \omega)G_3(\omega)B_q(q, \omega) < 0$ for some $(q, \omega) \in \mathbb{R} \times \mathbb{S}^2$.

Meanwhile, the Hörmander asymptotc PDE now becomes

$$2U_{sq} + G_3(\omega)U_q U_{qq} = 0.$$

We recall from Section 6.5 in [7] that this asymptotic PDE blows up in finite time unless $G_3(\omega) \equiv 0$. We thus conclude that in this example, the geometric weak null condition is equivalent to the weak null condition.

Example 2.4. In wave coordinates, the Einstein vacuum equations become a system of quasilinear wave equations with unknown functions $h_{\alpha\beta} := g_{\alpha\beta} - m_{\alpha\beta}$ for $\alpha, \beta = 0, 1, 2, 3$:

$$g^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = P(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h, \partial h) + G_{\mu\nu}(h)(\partial h, \partial h). \tag{2.9}$$

Here $(g^{\alpha\beta})$ is the inverse of $(g_{\alpha\beta}) = (m_{\alpha\beta} + h_{\alpha\beta})$, and the bilinear form P is given by

$$P(\partial_\mu h, \partial_\nu h) = \frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'}\partial_\mu h_{\alpha\alpha'}h_{\beta\beta'} - \frac{1}{2}m^{\alpha\alpha'}m^{\beta\beta'}\partial_\mu h_{\alpha\beta}h_{\alpha'\beta'},$$

$Q_{\alpha\beta}(\partial h, \partial h)$ is a null form and $G(h)(\partial h, \partial h)$ is a quadratic form in ∂h with coefficients smoothly dependent on h and $G(0)(\partial h, \partial h) = 0$. In addition, from the wave coordinate condition, for $\gamma = 0, 1, 2, 3$, we have the constraint equation

$$m^{\alpha\beta}\partial_\alpha h_{\beta\gamma} = \frac{1}{2}m^{\alpha\beta}\partial_\gamma h_{\alpha\beta} + G_\gamma(h)(\partial h). \tag{2.10}$$

Here $G(h)(\partial h)$ is a linear function of ∂h with coefficients smoothly dependent on h and $G(0)(\partial h) = 0$. We refer our readers to Lemma 3.1 in Lindblad-Rodnianski [25], or Lemma 3.2 in Lindblad-Rodnianski [24]. It is known that this system of quasilinear wave equations satisfies the weak null condition. We claim that it also satisfies the geometric weak null condition.

Using the ansatz $h_{\alpha\beta} \approx \varepsilon r^{-1} U_{(\alpha\beta)}$, we obtain a geometric reduced system

$$\begin{cases} \partial_s(\mu\partial_q U_{(\gamma\sigma)}) = -\dfrac{1}{4}\widehat{\omega}_\gamma\widehat{\omega}_\sigma(\dfrac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \dfrac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})\mu^2\partial_q U_{(\alpha\alpha')}\cdot\partial_q U_{(\beta\beta')}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma,\sigma = 0,1,2,3; \\ \partial_s\mu = -\dfrac{1}{4}m^{\alpha\beta}m^{\alpha'\beta'}\widehat{\omega}_\beta\widehat{\omega}_{\beta'}\mu^2\partial_q U_{(\alpha\alpha')}. \end{cases} \tag{2.11}$$

We remark that in order to get the equation for $\partial_s\mu$, we use the following identity:

$$g^{\alpha\beta} = m^{\alpha\beta} - m^{\alpha\alpha'}m^{\beta\beta'}h_{\alpha'\beta'} + O(|h|^2).$$

Recall that $(g^{\alpha\beta})$ is the inverse of $(m_{\alpha\beta} + h_{\alpha\beta})$. In addition, from the constraint equation (2.10), we have an additional constraint equation

$$m^{\alpha\beta}\mu\widehat{\omega}_\alpha\partial_q U_{(\beta\gamma)} = \frac{1}{2}m^{\alpha\beta}\mu\widehat{\omega}_\gamma\partial_q U_{(\alpha\beta)}, \qquad \gamma = 0,1,2,3. \tag{2.12}$$

To solve (2.11) with a constraint (2.12), we set $\overline{L} = -\partial_t + \partial_r$, $L = \partial_t + \partial_r$ and

$$Q(X,Y) := X^\alpha Y^\beta \mu\partial_q U_{(\alpha\beta)}, \qquad \text{for any vector fields } X = X^\alpha\partial_\alpha,\ Y = Y^\beta\partial_\beta.$$

In this example, we always assume that X^α and Y^α are functions of ω which are independent of t and r, so we have

$$\partial_s(Q(X,Y)) = X^\alpha Y^\beta\partial_s(\mu\partial_q U_{(\alpha\beta)}).$$

As a result, if $X(r-t) = 0$ or $Y(r-t) = 0$ everywhere, we have

$$\partial_s(Q(X,Y)) = X^\alpha Y^\beta\widehat{\omega}_\alpha\widehat{\omega}_\beta(\dots) = 0$$

and thus

$$Q(X,Y) = Q(X,Y)|_{s=0} = X^\alpha Y^\beta(\mu\partial_q U_{(\alpha\beta)})|_{s=0}, \qquad \forall s \geq 0.$$

Note that the map $(X,Y) \mapsto Q(X,Y)$ is a bilinear form. By setting $T_\alpha := \partial_\alpha - \frac{1}{2}\widehat{\omega}_\alpha\overline{L}$, we have

$$\begin{aligned} \mu\partial_q U_{(\alpha\beta)} = Q(\partial_\alpha,\partial_\beta) &= \frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_\beta Q(\overline{L},\overline{L}) + \frac{1}{2}\widehat{\omega}_\alpha Q(\overline{L},T_\beta) + \frac{1}{2}\widehat{\omega}_\beta Q(T_\alpha,\overline{L}) + Q(T_\alpha,T_\beta) \\ &=: \frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_\beta Q(\overline{L},\overline{L}) + K_{\alpha,\beta}(q,\omega). \end{aligned} \tag{2.13}$$

As explained above, $K_{\alpha,\beta}$ is independent of s since $T_\alpha(r-t)=0$ everywhere. It follows from (2.12) that for each fixed $\gamma = 0,1,2,3$,

$$m^{\alpha\beta}\widehat{\omega}_\alpha(\frac{1}{4}\widehat{\omega}_\beta\widehat{\omega}_\gamma Q(\overline{L},\overline{L}) + K_{\beta,\gamma}) = \frac{1}{2}m^{\alpha\beta}\widehat{\omega}_\gamma(\frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_\beta Q(\overline{L},\overline{L}) + K_{\alpha,\beta})$$

and thus

$$m^{\alpha\beta}\widehat{\omega}_\alpha K_{\beta,\gamma} = \frac{1}{2}m^{\alpha\beta}\widehat{\omega}_\gamma K_{\alpha,\beta}. \tag{2.14}$$

Next, we note that

$$\begin{aligned}\partial_s(Q(\overline{L},\overline{L})) &= -(\frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \frac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})\mu^2\partial_q U_{(\alpha\alpha')}\cdot\partial_q U_{(\beta\beta')}\\ &= -(\frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \frac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})(\frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_{\alpha'}Q(\overline{L},\overline{L}) + K_{\alpha,\alpha'})(\frac{1}{4}\widehat{\omega}_\beta\widehat{\omega}_{\beta'}Q(\overline{L},\overline{L}) + K_{\beta,\beta'}).\end{aligned}$$

A key observation is that there is no term involving $[Q(\overline{L},\overline{L})]^2$ on the right hand side. Moreover, if we compute the coefficient of $Q(\overline{L},\overline{L})$, we have

$$\begin{aligned}&-(\frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \frac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})(\frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_{\alpha'}K_{\beta,\beta'} + \frac{1}{4}\widehat{\omega}_\beta\widehat{\omega}_{\beta'}K_{\alpha,\alpha'})\\ =&\ \frac{1}{8}m^{\alpha\beta}m^{\alpha'\beta'}(\widehat{\omega}_\alpha\widehat{\omega}_{\alpha'}K_{\beta,\beta'} + \widehat{\omega}_\beta\widehat{\omega}_{\beta'}K_{\alpha,\alpha'}) = \frac{1}{16}m^{\alpha\beta}m^{\alpha'\beta'}(\widehat{\omega}_{\beta'}\widehat{\omega}_{\alpha'}K_{\alpha,\beta} + \widehat{\omega}_{\alpha'}\widehat{\omega}_{\beta'}K_{\beta,\alpha}) = 0.\end{aligned}$$

Here in the second identity we make use of (2.14). We thus obtain

$$\partial_s(Q(\overline{L},\overline{L})) = -(\frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \frac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})K_{\alpha,\alpha'}K_{\beta,\beta'}. \tag{2.15}$$

Since $K_{*,*}$'s are functions independent of s and determined by the initial data $(\mu,\partial_q U_{(**)})|_{s=0}$, the solution to the ODE (2.15) is of the form

$$\begin{aligned}Q(\overline{L},\overline{L}) &= -(\frac{1}{4}m^{\alpha\alpha'}m^{\beta\beta'} - \frac{1}{2}m^{\alpha\beta}m^{\alpha'\beta'})K_{\alpha,\alpha'}K_{\beta,\beta'}s + Q(\overline{L},\overline{L})|_{s=0}\\ &=: K_1(q,\omega)s + K_2(q,\omega),\end{aligned} \tag{2.16}$$

where K_1, K_2 are functions independent of s and determined by the initial data $(\mu,\partial_q U_{(**)})|_{s=0}$. Making use of (2.13), we obtain $\mu\partial_q U_{(**)}(s,q,\omega)$ for all $s\geq 0$.

By (2.13) and (2.14), we can rewrite the second equation in (2.11) as

$$\begin{aligned}\partial_s\mu &= -\frac{1}{4}m^{\alpha\beta}m^{\alpha'\beta'}\widehat{\omega}_\beta\widehat{\omega}_{\beta'}(\frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_{\alpha'}Q(\overline{L},\overline{L}) + K_{\alpha,\alpha'})\cdot\mu\\ &= -\frac{1}{4}m^{\alpha\beta}m^{\alpha'\beta'}\widehat{\omega}_\beta\widehat{\omega}_{\beta'}K_{\alpha,\alpha'}\cdot\mu = -\frac{1}{8}m^{\alpha\beta}m^{\alpha'\beta'}\widehat{\omega}_{\alpha'}\widehat{\omega}_{\beta'}K_{\beta,\alpha}\cdot\mu = 0.\end{aligned} \tag{2.17}$$

Thus, $\mu(s,q,\omega) = \mu(0,q,\omega)$ for all $s\geq 0$. We conclude by (2.13) that our reduced system (2.11) has a solution

$$\begin{cases}\partial_q U_{(\alpha\beta)}(s,q,\omega) = K_{1,(\alpha\beta)}(q,\omega)s + K_{2,(\alpha\beta)}(q,\omega),\\ \mu(s,q,\omega) = K_0(q,\omega).\end{cases} \tag{2.18}$$

Here $K_{*,(**)}$ and K_0 are all are functions independent of s and determined by the initial data $(\mu, \partial_q U_{(**)})|_{s=0}$. Explicitly, we have $K_0(q,\omega) = \mu(0,q,\omega)$,

$$K_{1,(\alpha\beta)} = \frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_\beta(K_1/K_0), \qquad K_{2,(\alpha\beta)}(q,\omega) = \frac{1}{4}\widehat{\omega}_\alpha\widehat{\omega}_\beta(K_2/K_0) + (K_{\alpha,\beta}/K_0). \tag{2.19}$$

Note that

$$(Q(X,Y)/\mu)|_{s=0} = X^\alpha Y^\beta(\partial_q U_{(\alpha\beta)})|_{s=0}$$

and that

$$\begin{aligned} K_2/K_0 &= (Q(\overline{L},\overline{L})/\mu)|_{s=0}, \\ K_{\alpha,\beta}/K_0 &= \frac{1}{2}\widehat{\omega}_\alpha(Q(\overline{L},T_\beta)/\mu)|_{s=0} + \frac{1}{2}\widehat{\omega}_\beta(Q(T_\alpha,\overline{L})/\mu)|_{s=0} + (Q(T_\alpha,T_\beta)/\mu)|_{s=0}, \\ K_1/K_0 &= C\cdot K_{*,*}\cdot(K_{*,*}/K_0). \end{aligned}$$

Thus, the solution (2.18), along with all its derivatives, grows linearly in s. We conclude that the geometric weak null condition is satisfied. We also remark that the linear growth in the solution (2.18) is consistent with the results in Lindblad's paper [22].

Chapter 3

Existence of Modified Wave Operators

3.1 Introduction

In this chapter, our main goal is to prove the existence of the modified wave operators for our model equation. This is accomplished in two steps.

The first step is to construct an approximate solution to the quasilinear wave equation (1.1). We start with solving the asymptotic system (1.11) explicitly with the initial data $(\mu, U_q)|_{s=0} = (-2, A)$. Here $A = A(q, \omega)$ is the *scattering data* associated to a solution u to the quasilinear wave equation (1.1). Then, we construct an approximate solution $q(t, r, \omega)$ to the eikonal equation (1.10) by solving $q_t - q_r = \mu$ and $q(t, 0, \omega) = -t$; this equation is an ODE along each characteristic line. Both s and q are now functions of (t, r, ω), so we also obtain a function $U(t, r, \omega)$ from $U(s, q, \omega)$. Here $U(t, r, \omega)$ is the *asymptotic profile* associated to a solution u to the quasilinear wave equation (1.1). Thirdly, we define u_{app}. We expect that u_{app} is an approximate solution to (1.1), that $u_{app} = \varepsilon r^{-1} U(t, r, \omega)$ in a conic neighborhood of the light cone $\{t = r\}$ and that u_{app} is supported in a slightly larger conic neighborhood of the light cone.

The second step is to show that there is an exact solution to (1.1) which matches u_{app} at infinite time. Fixing a large time $T > 0$, we solve a backward Cauchy problem for $v = u - u_{app}$ with zero data for $t \geq 2T$, such that $v + u_{app}$ solves (1.1) for $t \leq T$. We then prove that $v = v^T$ converges to some function v^∞ as $T \to \infty$. It turns out that $u^\infty = v^\infty + u_{app}$ is a solution to (1.1) which matches the asymptotic profile at infinite time. This shows the existence of the modified wave operators.

A more detailed discussion is given below.

3.1.1 Approximate solution

To construct an approximate solution to (1.1), we start by solving our reduced system (1.11). This requires us to assign the initial data at $s = 0$. To choose $\mu|_{s=0}$, we use the gauge freedom. Note that if $q_t - q_r = \mu$ and if $\widetilde{q} = F(q, \omega)$, then we have $\widetilde{q}_t - \widetilde{q}_r = (\partial_q F)\mu$. Thus, by choosing the function F appropriately, we can prescribe $\mu|_{s=0}$ freely. We now set $\mu|_{s=0} \equiv -2$ since we expect $q \approx r - t$. The initial data of U_q can be chosen arbitrarily, so we set $U_q|_{s=0} = A$ for an arbitrary function $A = A(q,\omega)$, which is called the *scattering data* in this chapter. An explicit solution $(\mu, U_q)(s,q,\omega)$ is given by (2.5) with (A_1, A_2) replaced by $(-2, A)$. To solve for U uniquely, in this chapter we add an assumption that $\lim_{q\to-\infty} U(s,q,\omega) = 0$.

In the author's previous paper [34], it was assumed that the scattering data A belongs to $C_c^\infty(\mathbb{R}\times\mathbb{S}^2)$. As commented in that paper, this assumption can be relaxed. In this dissertation, we assume that $A \in C^\infty(\mathbb{R}\times\mathbb{S}^2)$ and that

$$A(q,\omega) = 0, \text{ whenever } q \le -R; \tag{3.1}$$

$$\partial_q^m \partial_\omega^n A = O_{m,n}(\langle q\rangle^{-1-\gamma-m}) \text{ in } \mathbb{R}\times\mathbb{S}^2, \text{ for all } m,n \ge 0. \tag{3.2}$$

Here $R \ge 1$ and $\gamma > 0$ are two fixed constants, and ∂_ω^n denotes any angular derivatives of order n.

Next we make a change of coordinates. For a small $\varepsilon > 0$, we set $s = \varepsilon\ln(t) - \delta$, where $\delta > 0$ is a sufficiently small constant to be chosen. We remark that this choice of s is related to the almost global existence, since now $s = 0$ if and only if $t = e^{\delta/\varepsilon}$. In fact, when $t \le e^{\delta/\varepsilon}$, we expect the solution to (1.1) behaves as a solution to $\Box u = 0$, so our asymptotic equations play a role only when $t \ge e^{\delta/\varepsilon}$. Let $q(t,r,\omega)$ be the solution to

$$q_t - q_r = \mu(\varepsilon\ln(t) - \delta, q(t,r,\omega),\omega), \qquad q(t,0,\omega) = -t.$$

We can use the method of characteristics to solve this equation. Then, any function of (s,q,ω) induces a new function of (t,r,ω). With an abuse of notation, we set

$$U(t,r,\omega) = U(\varepsilon\ln(t) - \delta, q(t,r,\omega),\omega).$$

The function $U(t,r,\omega)$ is called the *asymptotic profile* in this chapter. We will prove that, near the light cone $\{t = r\}$, $\varepsilon r^{-1}U(t,r,\omega)$ is an approximate solution to (1.1), and $q(t,r,\omega)$ is an approximate optical function, i.e. an approximate solution to the eikonal equation corresponding to the metric $g^{\alpha\beta}(\varepsilon r^{-1}U)$.

3.1.2 The main theorem

We denote by Z any of the commuting vector fields: translations ∂_α, scaling $t\partial_t + r\partial_r$, rotations $x_i\partial_j - x_j\partial_i$ and Lorentz boosts $x_i\partial_t + t\partial_i$. Our main theorem in this chapter is the following.

Theorem 3.1. *Consider a scattering data $A(q,\omega)$ be a function in $C^\infty(\mathbb{R}\times\mathbb{S}^2)$ satisfying the support assumption* (3.1) *and the decay assumption* (3.2) *for some $R \ge 1$ and $\gamma > 0$. Fix an integer $N \ge 2$ and any sufficiently small $\varepsilon > 0$ depending on A and N. Let $q(t,r,\omega)$ and $U(t,r,\omega)$ be the associated approximate optical function and asymptotic profile. Then, there exists a C^N solution u to* (1.1) *for $t \ge 0$ with the following properties:*

(i) *The solution vanishes for $|x| = r \le t - R$.*

(ii) *The solution satisfies good energy bounds: for all $|I| \leq N-1$ and all $t \gg_A 1$, we have*

$$\left\|\partial Z^I(u-\varepsilon r^{-1}U)(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ |x|\leq 5t/4\})}+\left\|\partial Z^I u(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ |x|\geq 5t/4\})}\lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}.$$

(iii) *The solution satisfies good pointwise bounds: for all (t,r,ω) with $t \gg_A 1$, we have*

$$|(\partial_t-\partial_r)u+2\varepsilon r^{-1}A(q(t,r,\omega),\omega)|\lesssim \varepsilon t^{-3/2+C\varepsilon}.$$

Moreover, for all $|I|\leq N-1$ and all (t,x) with $t\gg_A 1$,

$$|\partial Z^I(u-\varepsilon r^{-1}U)(t,x)|\chi_{|x|\leq 5t/4}+|\partial Z^I u(t,x)|\chi_{|x|\geq 5t/4}\lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}\langle t+r\rangle^{-1}\langle t-r\rangle^{-1/2},$$

$$|Z^I(u-\varepsilon r^{-1}U)(t,x)|\chi_{|x|\leq 5t/4}+|Z^I u(t,x)|\chi_{|x|\geq 5t/4}\lesssim_I \min\{\varepsilon t^{-1+C_I\varepsilon},\varepsilon t^{-3/2+C_I\varepsilon}\langle r-t\rangle\}.$$

Remark 3.1.1. In [34], the author has proved Theorem 3.1 with a stronger assumption $A\in C_c^\infty(\mathbb{R}\times\mathbb{S}^2)$. The proof in this dissertation requires a more delicate analysis and substantial changes corresponding to the arguments in [34].

Remark 3.1.2. The solution in the main theorem is unique in the following sense. Suppose $N\geq 7$. Suppose u_1,u_2 are two C^N solutions to (1.1), such that they correspond to the same scattering data and that they satisfy the energy bounds and pointwise bounds in the main theorem. Then, we have $u_1=u_2$, assuming $\varepsilon\ll 1$. We also remark that u does not depend on the value 5/4 in the estimates: for each fixed $\kappa>1$, if u_κ is a solution satisfying all the estimates above with 5/4 replaced by κ, then $u=u_\kappa$ for $\varepsilon\ll_\kappa 1$, where u is the unique solution from the main theorem. We will prove these statements after the proof of the main theorem.

Remark 3.1.3. By the main theorem, we have the following pointwise bound near the light cone (e.g. when $|t-r|\lesssim t^{C\varepsilon}$):

$$|\partial Z^I(u-\varepsilon r^{-1}U)(t,x)|+|Z^I(u-\varepsilon r^{-1}U)(t,x)|\lesssim_I \varepsilon t^{-3/2+C_I\varepsilon}. \tag{3.3}$$

Note that, for the free constant coefficient linear wave equation, we can prove a stronger pointwise estimate with $t^{-3/2+C_I\varepsilon}$ replaced by t^{-2} on the right hand side. This is suggested by the fact that the solution to the forward Cauchy problem $\Box w=0$ with compactly supported initial data satisfies such a stronger pointwise estimate (see Theorem 6.2.1 in [7]). In our construction, we can achieve this stronger estimate if we add an additional assumption $\int_{-\infty}^{\infty}A(q,\omega)\,dq=0$ on the scattering data. We refer our readers to Remark (2) after Theorem 1 in [34].

Remark 3.1.4. Here we make three remarks about the scattering data A. First, the assumption that $A\in C^\infty(\mathbb{R}\times\mathbb{S}^2)$ can be relaxed. Instead, we can assume that A is $C^{N'}$ for some large integer $N'\gg_N 1$. Secondly, the support assumption (3.1) is necessary. In fact, it guarantees that the asymptotic profile $U(t,r,\omega)$ vanishes whenever $r-t\leq -R$, which is important in our proof. Thirdly, the decay assumption (3.2) is motivated by Lindblad-Schlue [26]. There the authors assumed that $(\langle q\rangle\partial_q)^m\partial_\omega^n F_0=O(\langle q\rangle^{-\gamma})$ for some $\gamma\in(1/2,1)$, where F_0 is their radiation field. For a linear wave equation, in our setting we have $A=U_q=\partial_q F_0$, so we expect $\partial_q^m\partial_\omega^n A=\partial_q^{m+1}\partial_\omega^n F_0=O(\langle q\rangle^{-m-1-\gamma})$.

3.1.3 Idea of the proof

Here we outline the main idea of the construction of u in Theorem 3.1. Roughly speaking, our starting point begins from the ideas from both Lindblad [21] and Lindblad-Schlue [26]. To construct a matching global solution, we follow the idea in Lindblad-Schlue [26]: we solve a backward Cauchy problem with some initial data at $t = T$ and then send T to infinity. However, the backward Cauchy problems in [26] are of simpler form, and their solutions can be constructed by Duhamel's formula explicitly. Here, our backward Cauchy problem is quasilinear, and it is necessary to prove that the solution does exist for all $0 \leq t \leq T$. We follow the proof of the small data global existence in [21]: we use a continuity argument with the help of the adapted energy estimates and Poincaré's lemma.

We now provide more detailed descriptions of the proof. First, we construct an approximate solution to (1.1). Let $q(t, r, \omega)$ and $U(t, r, \omega)$ be the approximate optical function and asymptotic profile associated to some scattering data $A(q, \omega)$. We set

$$u_{app}(t, x) = \varepsilon r^{-1}\eta(t)\psi(r/t)U(\varepsilon \ln(t) - \delta, q(t, r, \omega), \omega) \tag{3.4}$$

for all $t \geq 0$ and $x \in \mathbb{R}^3$. Here $\psi \equiv 1$ when $|r - t| \leq t/4$ and $\psi \equiv 0$ when $|r - t| \geq t/2$, which is used to localize $\varepsilon r^{-1}U$ near the light cone $\{r = t\}$; η is a cutoff function such that $\eta \equiv 0$ for $t \leq 2R$, which is used to remove the singularity at $|x| = 0$ and $t = 0$. We can check that u_{app} is a good approximate solution to (1.1) in the sense that

$$g^{\alpha\beta}(u_{app})\partial_\alpha\partial_\beta u_{app} = O(\varepsilon t^{-3+C\varepsilon}), \qquad t \gg_A 1.$$

Next we seek to construct an exact solution matching u_{app} at infinite time. Fixing a large time T, we consider the following equation

$$g^{\alpha\beta}(u_{app} + v)\partial_\alpha\partial_\beta v = -\chi(t/T)g^{\alpha\beta}(u_{app} + v)\partial_\alpha\partial_\beta u_{app}, \ t > 0; \qquad v \equiv 0, \ t \geq 2T. \tag{3.5}$$

Here $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi(t/T) = 1$ for $t \leq T$ and $\chi(t/T) = 0$ for $t \geq 2T$. Note that $u_{app}+v$ is now an exact solution to (1.1) for $t \leq T$. In Section 3.4 we prove that, if ε is sufficiently small, then (3.5) has a solution $v = v^T$ for all $t \geq 0$ which satisfies some decay in energy as $t \to \infty$. To prove this, we use a continuity argument. The proof relies on the energy estimates and Poincaré's lemma, which are established in Section 3.3. Note that the small constant $\delta > 0$ is not chosen until the proof of the Poincaré's lemma, and we remark that δ depends only on the scattering data $A(q, \omega)$. We also remark that the energy estimates and Poincaré's lemma in this dissertation are closely related to those in [21, 1].

Finally we prove in Section 3.5 that v^T does converge to some v^∞ in suitable function spaces, as $T \to \infty$. Thus we obtain a global solution $u_{app} + v^\infty$ to (1.1) for $t \geq 0$, such that it "agrees with" u_{app} at infinite time, in the sense that the energy of v^∞ tends to 0 as $t \to \infty$. By the Klainerman-Sobolev inequality, we can derive the pointwise bounds in the main theorem from the estimates for the energy of v^∞.

Note that to obtain a candidate for v^∞, we have a more natural choice of PDE than (3.5). We may consider the Cauchy problem (1.1) for $t \leq T$ with initial data $(u_{app}(T), \partial_t u_{app}(T))$.

The problem with such a choice is that for u_{app} constructed above, $Z^I(u-u_{app})(T)$ does not seem to have a good decay in T if Z^I only contains the scaling $S=t\partial_t+r\partial_r$ and Lorentz boosts $\Omega_{0i}=t\partial_i+x_i\partial_t$. For example, we can consider the linear wave equation $\Box u=0$. We set $v=u-u_{app}$, then $v=v_t=0$ at $t=T$. Then, at $t=T$ we have $S^2v=t^2v_{tt}=-t^2\Box u_{app}$. However, in the linear case, $u_{app}=\varepsilon r^{-1}F_0(r-t,\omega)$ for $t\approx r$ and thus $\Box u_{app}=O(\varepsilon r^{-3})$. The power -3 cannot be improved, so we can only get $S^2v=O(\varepsilon r^{-1})$ for $t\approx r$, while we expect $S^2v=O(\varepsilon r^{-3/2+C\varepsilon})$ for $t\approx r$ from Theorem 3.1. Similarly, the same applies for S^kv if $k\geq 3$. In the linear case, one possible way to deal with this difficulty is to consider more terms in the asymptotic expansion of the solutions, say take

$$u_{app}=\sum_{n=0}^{N}\frac{\varepsilon}{r^{n+1}}F_n(r-t,\omega)$$

where F_0 is the usual Friedlander radiation field, and F_n satisfies some PDE based on F_{n-1}. This method was used by Lindblad and Schlue in their construction. However, it does not seem to work in the quasilinear case, since we do not have such a good asymptotic expansion for a solution to (1.1). In this dissertation, we avoid such a difficulty by considering a variant (3.5) of (1.1). Such a difficulty does not appear in (3.5), since $v\equiv 0$ for all $t\geq 2T$.

3.2 The Asymptotic Profile and the Approximate Solution

Our main goal in this section is to construct an approximate solution u_{app} to (1.1). Fix a scattering data $A=A(q,\omega)\in C^\infty(\mathbb{R}\times\mathbb{S}^2)$ such that

$$A(q,\omega)=0,\ \text{whenever } q\leq -R; \tag{3.6}$$

$$\partial_q^m\partial_\omega^n A=O_{m,n}(\langle q\rangle^{-1-\gamma-m})\text{ in }\mathbb{R}\times\mathbb{S}^2,\ \text{for all } m,n\geq 0. \tag{3.7}$$

Here $R\geq 1$ and $\gamma>0$ are two fixed constants, and ∂_ω^n denotes any angular derivatives of order n. Fix a sufficiently large $T_A>0$ and a sufficiently small $\varepsilon>0$, both depending on $A(q,\omega)$. Let $(\mu,U)(s,q,\omega)$ be the solution to (2.4) with $(\mu,U_q)|_{s=0}=(-2,A)$ and $\lim_{q\to-\infty}U(s,q,\omega)=0$. Let $q(t,r,\omega)$ be the solution to the PDE

$$(\partial_t-\partial_r)q(t,r,\omega)=\mu(\varepsilon\ln(t)-\delta,q(t,r,\omega),\omega),\qquad q(t,0,\omega)=-t$$

and set

$$U(t,r,\omega)=U(\varepsilon\ln(t)-\delta,q(t,r,\omega),\omega).$$

Here $\delta>0$ is a sufficiently small constant depending only on the scattering data. In this section, we will show that near the light cone $\{t=r+R\}$, $\varepsilon r^{-1}U(t,r,\omega)$ and $q(t,r,\omega)$ are the approximate solution to (1.1) and the approximate optical function, respectively, in the sense that for all (t,r,ω) with $t\geq T_A$ and $-R\leq r-t\lesssim t^{C\varepsilon}$, we have

$$g^{\alpha\beta}(\varepsilon r^{-1}U)\partial_\alpha\partial_\beta(\varepsilon r^{-1}U)=O(\varepsilon t^{-3+C\varepsilon}),$$

$$g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta = O(t^{-2+C\varepsilon}).$$

For all $t \geq 0$ and $x \in \mathbb{R}^3$, we set

$$u_{app}(t,x) = \varepsilon r^{-1}\eta(t)\psi(r/t)U(\varepsilon\ln(t)-\delta, q(t,r,\omega),\omega).$$

Here $\psi \equiv 1$ when $|r-t| \leq t/4$ and $\psi \equiv 0$ when $|r-t| \geq t/2$, which is used to localize $\varepsilon r^{-1}U$ near the light cone $\{r=t\}$; η is a cutoff function such that $\eta \equiv 0$ when $t \leq 2R$. The definitions of ψ and η will be given later.

Our main proposition in this section is the following:

Proposition 3.2. *Fix a scattering data $A \in C^\infty(\mathbb{R}\times\mathbb{S}^2)$ satisfying the support assumption* (3.6) *and the decay assumption* (3.7)*. Fix a sufficiently small $\varepsilon > 0$ depending on A. Let u_{app} be the function defined as above. Then, for all (t,x) with $t \geq T_A$, we have*

$$|\partial u_{app}(t,x)| \lesssim \varepsilon(1+t)^{-1}.$$

Moreover, for all multiindices I and for all (t,x) with $t \geq 0$, we have

$$|Z^I u_{app}(t,x)| \lesssim_I \varepsilon(1+t)^{-1+C_I\varepsilon},$$

$$|Z^I(g^{\alpha\beta}(u_{app})\partial_\alpha\partial_\beta u_{app})(t,x)| \lesssim_I \varepsilon(1+t)^{-3+C_I\varepsilon}.$$

Remark 3.2.1. If we have $0 < \delta < 1$, then all the constants involved in this section are uniform in δ. Thus, it would not impact any result in this section if we do not choose the value of δ until the proof of the Poincaré's lemma in the next section.

This proposition is proved in three steps. First, in Section 3.2.1, we construct $q(t,r,\omega)$ and $U(t,r,\omega)$ for all (t,x) with $t > 0$, by solving the reduced system (2.4) and $q_t - q_r = \mu$ explicitly. Next, in Section 3.2.2, we prove that $\varepsilon r^{-1}U(t,r,\omega)$ is an approximate solution to (1.1) near the light cone $\{t = r+R\}$ when t is sufficiently large. To achieve this goal we prove several estimates for q and U in the region $t \sim r$. Finally, in Section 3.2.3, we define u_{app} and prove the pointwise bounds for large t. To define u_{app}, we use cutoff functions to restrict $\varepsilon r^{-1}U$ in a conical neighborhood of $\{t=r\}$ and remove the singularities at $|x|=0$ or $t=0$.

3.2.1 Construction of q and U

Fix $\varepsilon \ll 1$. Fix a scattering data A as in the statement of Proposition 3.2. Also fix $0<\delta<1$ depending on $A(q,\omega)$ but not on ε. Its value will be chosen in Section 3.3. We define $q(t,r,\omega)$ by solving

$$(\partial_t - \partial_r)q(t,r,\omega) = \mu(\varepsilon\ln(t)-\delta, q(t,r,\omega),\omega), \qquad q(t,0,\omega) = -t, \tag{3.8}$$

where

$$\mu(s,q,\omega) := -2\exp(-\frac{1}{2}G(\omega)A(q,\omega)s). \tag{3.9}$$

Here we set

$$G(\omega) := g_0^{\alpha\beta}\widehat{\omega}_\alpha\widehat{\omega}_\beta, \qquad g_0^{\alpha\beta} := \frac{d}{dv}g^{\alpha\beta}(v)|_{v=0}, \qquad \widehat{\omega} := (-1,\omega) \in \mathbb{R}\times\mathbb{S}^2.$$

By the chain rule and the estimates for A, we have

$$|\mu| \lesssim \exp(C|s|); \qquad |\partial_s^b\partial_q^a\partial_\omega^c\mu| \lesssim \langle q\rangle^{-1-\gamma-a}\exp(C|s|), \qquad \forall a+b+c>0. \tag{3.10}$$

Note that (3.8) has a solution $q(t,r,\omega)$ for all $t>0$. In fact, if we apply method of characteristics, for $z(\tau) = q(\tau, r+t-\tau,\omega)$ and $s(\tau) = \ln(\tau)$ we have an autonomous system of ODE's

$$\begin{cases} \dot{z}(\tau) = \mu(\varepsilon s(\tau)-\delta, z(\tau),\omega) \\ \dot{s}(\tau) = \exp(-s(\tau)) \end{cases}$$

with initial data $(z,s)(r+t) = (-r-t, \ln(r+t))$. Whenever $0<\tau<r+t$, we have

$$|\dot{z}(\tau)| \lesssim \exp(C\langle z(\tau)\rangle^{-1-\gamma}(|\varepsilon\ln(\tau)|+1)) \lesssim \max\{\tau^{C\varepsilon},\tau^{-C\varepsilon}\},$$

and then

$$\begin{aligned} |z(\tau)| &\lesssim r+t+\int_\tau^{r+t}\max\{(\tau')^{C\varepsilon},(\tau')^{-C\varepsilon}\}\ d\tau' \\ &\lesssim r+t+(\tau')^{1+C\varepsilon}\Big|_0^{r+t} + (\tau')^{1-C\varepsilon}\Big|_0^{r+t} \lesssim (r+t+1)^{1+C\varepsilon}. \end{aligned}$$

Here we choose $\varepsilon \ll 1$ so that $C\varepsilon < 1$. Thus, $|z(\tau)|$ cannot blow up when $\tau > 0$. Neither can $|s(\tau)|$ since $s(\tau) = \ln(\tau)$. We are thus able to solve this system of ODE's for all $\tau > 0$ by Picard's theorem.

We have

$$q(t,r,\omega) = -(r+t) - \int_t^{r+t}\mu(\varepsilon\ln(\tau)-\delta, q(\tau, r+t-\tau,\omega),\omega)\ d\tau. \tag{3.11}$$

Note that if $G(\omega)\equiv 0$, we have $\mu\equiv -2$ and thus $q = r-t$, which coincides with the choice of q in Hörmander's setting.

We also define $U(s,q,\omega)$ by solving the following equation

$$(\partial_q U)(s,q,\omega) = A(q,\omega)\exp(\frac{1}{2}G(\omega)A(q,\omega)s), \qquad \lim_{q\to-\infty}U(s,q,\omega) = 0. \tag{3.12}$$

The equation (3.12) has a solution $U(s,q,\omega)$ for all s, which comes from taking the following integral:

$$U(s,q,\omega) = \int_{-\infty}^q A(p,\omega)\exp(\frac{1}{2}G(\omega)A(p,\omega)s)\ dp. \tag{3.13}$$

Since $A\equiv 0$ for $q\le -R$, we have $U(s,q,\omega) = 0$ unless $q \ge -R$. In addition, by the decay assumption (3.7), we have

$$|U| \lesssim \int_{\mathbb{R}}\langle p\rangle^{-1-\gamma}\exp(C|s|)\ dp \lesssim \exp(C|s|).$$

In general, it is easy to check that for all $s \geq -1$ and all $(q, \omega) \in \mathbb{R} \times \mathbb{S}^2$,

$$\begin{aligned} &|\partial_s^b \partial_\omega^c U| \lesssim \exp(C|s|), \qquad \forall b, c \geq 0; \\ &|\partial_q^a \partial_s^b \partial_\omega^c U| \lesssim \exp(C|s|)\langle q\rangle^{-a-\gamma}, \qquad \forall a > 0,\ b, c \geq 0. \end{aligned} \tag{3.14}$$

Here the constants depend on a, b, c, but they are uniform for all $(s, q, \omega) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^2$.

From now on, we use U to denote the function on (t, r, ω):

$$U = U(t, r, \omega) = U(\varepsilon \ln(t) - \delta, q(t, r, \omega), \omega). \tag{3.15}$$

Such a U is called the *asymptotic profile* in this chapter. Note that

$$(\partial_t - \partial_r)U = \mu U_q + \varepsilon t^{-1} U_s = -2A + O(\varepsilon t^{-1+C\varepsilon}).$$

This explains the meaning of the scattering data A in our construction.

3.2.2 Estimates for q and U

Fix $T_A \gg 1$. We then choose $\varepsilon \ll 1$, so ε can depend on T_A but not vice versa. We set

$$\mathcal{D} := \{(t, x) :\ t \geq T_A,\ t/2 \leq r \leq 2t\} \tag{3.16}$$

and recall Definition 1.8 in Section 1.6.4. Our main goal now is to prove that $\varepsilon r^{-1} U(t, r, \omega) \in \varepsilon S^{-1}$ and $g^{\alpha\beta}(\varepsilon r^{-1} U)\partial_\alpha \partial_\beta(\varepsilon r^{-1} U) \in \varepsilon S^{-3}$. In other words, $\varepsilon r^{-1} U$ has some good pointwise bounds and is an approximate solution whenever $t \geq T_A$ and $t \sim r$.

We start with a lemma for $q(t, r, \omega)$.

Lemma 3.3. *Fix (t, r, ω) with $t \geq T_A \gg 1$ and we set $t_1 = (t + r + R)/2$. Then we have*

$$q(\tau, r + t - \tau, \omega) = r + t - 2\tau, \qquad \forall \tau \geq t_1. \tag{3.17}$$

Thus, when $t_1 \leq t$, i.e. $r \leq t - R$, we have $q = r - t$.

If $1 \ll T_A \leq t \leq t_1$, we have

$$(t + r)^{-C\varepsilon}(q + R) \lesssim (r - t + R) \lesssim (t + r)^{C\varepsilon}(q + R), \tag{3.18}$$

$$|q(t, r, \omega) - (r - t)| \lesssim (t + r)^{C\varepsilon}\langle q\rangle. \tag{3.19}$$

As a result, whenever $r - t \geq -R$ and $t \geq T_A \gg 1$, we have

$$(t + r)^{-C\varepsilon} \lesssim \langle r - t\rangle/\langle q\rangle \lesssim (t + r)^{C\varepsilon}.$$

Moreover, if $|q(t, r, \omega)| \lesssim t^\kappa$ for some $0 \leq \kappa < 1$, then $(t, x) \in \mathcal{D}$ as long as $T_A \gg_\kappa 1$ and $\varepsilon \ll_\kappa 1$.

Proof. Note that $\mu \equiv -2$ for $q \le -R$. Then the first part of this lemma follows directly from (3.11). Now we assume $1 \ll T_A \le t \le t_1$, i.e. $r - t \ge -R$. Since $-2e^{C|s|} \le \mu \le -2e^{-C|s|}$ and since

$$t \ge T_A \ge 1,\ 0 < \delta < 1 \Longrightarrow |\varepsilon \ln(t) - \delta| \le |\varepsilon \ln(t)| + 1 = \varepsilon \ln(t) + 1,$$

we have

$$\begin{aligned} -R - q(t,r,\omega) &= \int_t^{t_1} \mu(\tau, r+t-\tau, \omega)\ d\tau \le -2e^C \int_t^{t_1} \tau^{-C\varepsilon}\ d\tau \le -C(t_1 - t)t_1^{-C\varepsilon}, \\ -R - q(t,r,\omega) &= \int_t^{t_1} \mu(\tau, r+t-\tau, \omega)\ d\tau \ge -2e^{-C} \int_t^{t_1} \tau^{C\varepsilon}\ d\tau \ge -C^{-1}(t_1 - t)t_1^{C\varepsilon}. \end{aligned}$$

It follows that

$$\begin{aligned} t_1 - t &\le t_1^{C\varepsilon}(R + q(t,r,\omega)) \lesssim (q+R)(t+r)^{C\varepsilon}, \\ t_1 - t &\ge t_1^{-C\varepsilon}(R + q(t,r,\omega)) \gtrsim (q+R)(t+r)^{-C\varepsilon}. \end{aligned}$$

Since $t_1 - t = (r - t + R)/2$, we have $r - t = 2(t_1 - t) - R$ and thus

$$\begin{aligned} \langle r-t \rangle &\sim (t_1 - t + R) \lesssim (q+R)(t+r)^{C\varepsilon} + R(t+r)^{C\varepsilon} \lesssim \langle q \rangle (t+r)^{C\varepsilon}, \\ \langle r-t \rangle &\sim (t_1 - t + R) \gtrsim (q+R)(t+r)^{-C\varepsilon} + R(t+r)^{-C\varepsilon} \gtrsim \langle q \rangle (t+r)^{-C\varepsilon}. \end{aligned}$$

Moreover, we have

$$|q(t,r,\omega) - (r-t)| \le |q(t,r,\omega) + R| + 2|t_1 - t| \lesssim \langle q \rangle + \langle q \rangle (t+r)^{C\varepsilon} \lesssim \langle q \rangle (t+r)^{C\varepsilon}.$$

Finally, if $t \ge T_A \gg 1$, $r - t \ge -R$ and $|q| \lesssim t^\kappa$, we obtain an inequality

$$|r - t| \lesssim (t+r)^{C\varepsilon} \langle q \rangle \lesssim (t+r)^{C\varepsilon} t^\kappa.$$

If $(t,x) \notin \mathcal{D}$, then we must have $r > 2t > 1$, so

$$r/2 = r - r/2 < r - t \lesssim (r/2 + r)^{C\varepsilon} t^\kappa \lesssim r^{C\varepsilon} t^\kappa.$$

However, if we choose $\varepsilon \ll_\kappa 1$, we have $1 - C\varepsilon > (\kappa+1)/2$. We thus obtain

$$t^\kappa \gtrsim r^{1-C\varepsilon} \gtrsim t^{1-C\varepsilon} \gtrsim t^{(\kappa+1)/2} \Longrightarrow t^{(\kappa-1)/2} \gtrsim 1.$$

This estimate clearly fails for $t \gg_\kappa 1$ as $\kappa < 1$. Thus, by choosing $T_A \gg_\kappa 1$, we conclude that $(t,x) \in \mathcal{D}$. ☐

We now move on to estimates for ∂q. In Lemma 3.4, we give the pointwise bounds for $\nu = q_t + q_r$, $\partial_r \nu$ and $\lambda_i = q_i - \omega_i q_r$. In Lemma 3.5, we find the first terms in the asymptotic expansion of ν and ν_q when $t \sim r$ and $t \gg 1$.

Lemma 3.4. *For $t \geq T_A$,*

$$\nu(t,x) := (\partial_t + \partial_r)q = O(\varepsilon t^{-1+C\varepsilon}), \tag{3.20}$$

$$\lambda_i(t,x) := (\partial_i - \omega_i \partial_r)q = O((t+r)^{-1+C\varepsilon}). \tag{3.21}$$

Note that we do not need to assume that $(t,x) \in \mathcal{D}$ in this lemma.

Proof. Fix (t,r,ω). Since $\nu = \nu_r = \lambda_i = 0$ when $r - t \leq -R$, we now assume $r - t \geq -R$. Then

$$(\partial_t - \partial_r)\nu = (\partial_t + \partial_r)\mu = (\partial_q \mu)\nu - \frac{\varepsilon}{2t} G(\omega) A\mu. \tag{3.22}$$

By Lemma 3.3, for all $t > T_A$, we have

$$\begin{aligned}\int_t^{r+t} |\partial_q \mu| \, d\tau &= \int_t^{r+t} \frac{1}{2} |G(\omega)\partial_q A| \cdot |\varepsilon \ln(\tau) - \delta| \cdot |\mu| \, d\tau \\ &\lesssim (\varepsilon \ln(t+r) + 1) \int_t^{t_1} (-\dot{z}(\tau)) \langle z(\tau) \rangle^{-2-\gamma} \, d\tau \\ &\lesssim \varepsilon \ln(t+r) + 1.\end{aligned}$$

Here the integral is taken along the characteristic $(\tau, r+t-\tau, \omega)$ for $\tau \geq T_A$, as in (3.11). Also recall that $z(\tau) = q(\tau, r+t-\tau, \omega)$. From now on, $\int (\dots) \, d\tau$ would always denote an integral along a characteristic. Similarly, we have

$$\int_t^{r+t} |G(\omega)A\mu \frac{\varepsilon}{2\tau}| \, d\tau \lesssim \frac{\varepsilon}{t} \int_t^{t_1} |\mu| \langle q \rangle^{-1-\gamma} \, d\tau \lesssim \frac{\varepsilon}{t} \int_t^{t_1} (-\dot{z}(\tau)) \langle z(\tau) \rangle^{-1-\gamma} \, d\tau \lesssim \varepsilon t^{-1}.$$

Now, we integrate (3.22) along the characteristic and then apply the Gronwall's inequality. Note that the initial value of $(\partial_t + \partial_r)q$ is 0 as $q = r - t$ for $r \leq t - R$ by Lemma 3.3. Then we have $\nu = O(\varepsilon t^{-1}(t+r)^{C\varepsilon})$. This finishes the proof of (3.20) when $r \lesssim t$. If $r > 2t$, for $t \leq \tau \leq (r+t)/3$, by Lemma 3.3 we have

$$\langle z(\tau) \rangle \gtrsim (r+t)^{-C\varepsilon} \langle (r+t-\tau) - \tau \rangle \gtrsim (r+t)^{-C\varepsilon} \langle r+t \rangle,$$

$$|\mu_q(\tau, r+t-\tau, \omega)| \lesssim |A_q(\varepsilon \ln \tau - \delta)\mu| \lesssim \langle z(\tau) \rangle^{-2-\gamma} \cdot \tau^{C\varepsilon} \lesssim \langle r+t \rangle^{-2-\gamma+C\varepsilon}.$$

We integrate (3.22) along the characteristic for $t \leq \tau \leq (r+t)/3$. It follows that for each $t \leq t' \leq (t+r)/3$ we have

$$\begin{aligned}|\nu|_{\tau=t'}| &\lesssim \left|\nu|_{\tau=(r+t)/3}\right| + \int_{t'}^{(r+t)/3} |\mu_q \nu|(\tau) \, d\tau + \varepsilon(t')^{-1} \\ &\lesssim \int_{t'}^{(r+t)/3} \langle r+t \rangle^{-2} |\nu|(\tau) \, d\tau + \varepsilon(t')^{-1} + \varepsilon(r+t)^{-1+C\varepsilon}.\end{aligned}$$

Note that at $\tau = (r+t)/3$ we have $(r+t-\tau) \sim \tau$, so $\nu|_{\tau=(r+t)/3} = O(\varepsilon(t+r)^{-1+C\varepsilon})$. By the Gronwall's inequality we conclude that $\nu(t,r,\omega) = O(\varepsilon t^{-1+C\varepsilon})$ for $r > 2t$. This finishes the proof of (3.20).

To prove (3.21), we note that

$$\begin{aligned}(\partial_t - \partial_r)\lambda_i &= (\partial_i - \omega_i\partial_r)\mu + r^{-1}\lambda_i \\ &= (\mu_q + r^{-1})\lambda_i - \frac{1}{2}(\varepsilon\ln(t) - \delta)\sum_l \partial_{\omega_l}(GA)\cdot\frac{\delta_{il} - \omega_i\omega_l}{r}\mu \\ &= (\mu_q + r^{-1})\lambda_i + O(r^{-1}|\varepsilon\ln(t) - \delta|\cdot|\mu|\langle q\rangle^{-1-\gamma}).\end{aligned} \tag{3.23}$$

Note that $\lambda_i \equiv 0$ when $r < t - R$ and that for $0 < t - R \leq r$, we have

$$0 \leq \int_t^{t_1} (r+t-\tau)^{-1}\, d\tau = \ln\frac{2r}{r+t-R} \leq \ln 2,$$

$$\begin{aligned}\int_t^{t_1} (r+t-\tau)^{-1}|\varepsilon\ln(\tau) - \delta||\mu|\langle q\rangle^{-1-\gamma}\, d\tau &\leq \int_t^{t_1} (r+t-\tau)^{-1}(\varepsilon\ln(\tau)+1)\cdot|\mu|\langle q\rangle^{-1-\gamma}\, d\tau \\ &\leq \frac{\varepsilon\ln(t+r)+1}{r+t-t_1}\int_t^{t_1} |\mu|\langle q\rangle^{-1-\gamma}\, d\tau \\ &\lesssim (t+r)^{-1+C\varepsilon}.\end{aligned}$$

Apply the Gronwall's inequality again and we obtain (3.21). □

Remark 3.4.1. Since $|\mu| = -\mu \geq 2C^{-1}t^{-C\varepsilon}$, we conclude that $q_t, q_r \neq 0$ for all $t \geq T_A$ if ε is small enough. In particular, for $\varepsilon \ll 1$ and $t \gg 1$,

$$\begin{aligned}q_r &= \frac{-\mu+\nu}{2} \geq C^{-1}t^{-C\varepsilon} - C\varepsilon t^{-1+C\varepsilon} \geq (2C)^{-1}t^{-C\varepsilon}, \\ q_t &= \frac{\mu+\nu}{2} \leq -C^{-1}t^{-C\varepsilon} + C\varepsilon t^{-1+C\varepsilon} \leq -(2C)^{-1}t^{-C\varepsilon}.\end{aligned}$$

So for each fixed $t \geq T_A$ and $\omega \in \mathbb{S}^2$, the map $r \mapsto q(t,r,\omega)$ is strictly increasing and continuous for each fixed t. Moreover, $\lim_{r\to\infty} q(t,r,\omega) = \infty$. This implies that for each $t \geq T_A$ and $q^0 \geq -t$, there exists a unique r such that $q(t,r,\omega) = q^0$. So $\{t \geq T_A,\ r \lesssim t\} \ni (t,r,\omega) \mapsto (\varepsilon\ln(t) - \delta, q(t,r,\omega), \omega)$ has an inverse map $(s,q,\omega) \mapsto (e^{(s+\delta)/\varepsilon}, r(s,q,\omega), \omega)$. By the inverse function theorem, the map $(t,r,\omega) \mapsto (s,q,\omega)$ is a diffeomorphism.

From now on, any function V can be written as both $V(t,r,\omega)$ and $V(s,q,\omega)$ at the same time. Thus, for any function V on (t,r,ω), we can define $\partial_s^a\partial_q^b\partial_\omega^c V$ using the chain rule and Leibniz's rule. Note that in this paper, ∂_ω will only be used under the coordinate (s,q,ω) and will never be used under the coordinate (t,r,ω).

Lemma 3.5. *For $t \geq T_A$ and $r \lesssim t$,*

$$\nu - \frac{\varepsilon G(\omega)}{4t}\mu U = O(\varepsilon t^{-2+C\varepsilon}\langle q\rangle), \tag{3.24}$$

$$\nu_q - \frac{\varepsilon G(\omega)}{4t}\mu_q U - \frac{\varepsilon G(\omega)}{4t}\mu U_q = O(\varepsilon t^{-2+C\varepsilon}). \tag{3.25}$$

Proof. Again, we may assume $r \geq t - R$. We have

$$\begin{aligned}(\partial_t - \partial_r)(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) &= (\partial_t + \partial_r)\mu - \frac{\varepsilon G(\omega)}{4t}(\partial_t - \partial_r)(\mu U) + \frac{\varepsilon G(\omega)}{4t^2}\mu U \\ &= \mu_q \nu + \mu_s \frac{\varepsilon}{t} - \frac{\varepsilon G(\omega)}{4t}(\partial_q(\mu U)\mu + \partial_s(\mu U)\frac{\varepsilon}{t}) + \frac{\varepsilon G(\omega)}{4t^2}\mu U \\ &= \mu_q(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) + \frac{\varepsilon^2 G(\omega)}{4t^2}(-U_s + \frac{1}{2}G(\omega)AU)\mu + \frac{\varepsilon G(\omega)}{4t^2}\mu U.\end{aligned} \tag{3.26}$$

In particular, note that $\mu_s = \varepsilon G(\omega)\mu^2 U_q/4$.

Fix (t, r, ω). Integrate this equation along the characteristic $(\tau, r + t - \tau, \omega)$. Note that U vanishes if $\tau \geq t_1$ and $U, U_s = O(t^{C\varepsilon})$. We have

$$\int_t^{r+t} \frac{\varepsilon|G(\omega)|}{4\tau^2}|\mu U|\ d\tau \leq \frac{C\varepsilon(t+r)^{C\varepsilon}}{4t^2}\int_t^{t_1}|\mu|\ d\tau \lesssim \varepsilon t^{-2+C\varepsilon}\langle q\rangle$$

and

$$\int_t^{r+t}\left|\frac{\varepsilon^2 G(\omega)}{4\tau^2}(-U_s + \frac{1}{2}G(\omega)AU)\mu\right|\ d\tau \leq C\varepsilon^2\frac{(t+r)^{C\varepsilon}}{t^2}\int_t^{t_1}|\mu|\ d\tau \lesssim \varepsilon^2 t^{-2+C\varepsilon}\langle q\rangle.$$

Finally, since $\int_t^{r+t}|\mu_q|\ d\tau \lesssim \varepsilon\ln(t+r) + 1 \lesssim \varepsilon\ln(t) + 1$ and since $\nu = U = 0$ at $\tau > t_1$, by Gronwall's inequality we conclude (3.24).

To prove (3.25), we first prove it with ∂_q replaced by ∂_r. By (3.26), we have

$$\begin{aligned}(\partial_t - \partial_r)\partial_r(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) &= \partial_r(\partial_t - \partial_r)(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) \\ &= \mu_q \partial_r(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) + q_r\mu_{qq}(\nu - \frac{\varepsilon G(\omega)}{4t}\mu U) \\ &\quad + \frac{\varepsilon G(\omega)}{4t^2}(\mu U)_q q_r + \frac{\varepsilon^2 G(\omega)}{4t^2}\partial_q((-U_s + \frac{1}{2}G(\omega)AU)\mu)q_r.\end{aligned} \tag{3.27}$$

Note that $\mu_q = -\frac{1}{2}GA_q s\mu$ and $\mu_{qq} = -\frac{1}{2}GA_{qq}s\mu + (\frac{1}{2}GA_q s)^2\mu$. Integrate along the characteristic and we have

$$\begin{aligned}\int_t^{r+t}|q_r\mu_{qq}(\nu - \frac{\varepsilon G(\omega)}{4\tau}\mu U)|\ d\tau &\lesssim \int_t^{t_1}(|\nu| + |\mu|)|\mu|\langle q\rangle^{-3-\gamma}\cdot\varepsilon\tau^{-2+C\varepsilon}\langle q\rangle\ d\tau \\ &\lesssim \varepsilon t^{-2+C\varepsilon}\int_t^{t_1}|\mu|\langle q\rangle^{-2-\gamma}\ d\tau \lesssim \varepsilon t^{-2+C\varepsilon},\end{aligned}$$

$$\begin{aligned}\int_t^{r+t} |\frac{\varepsilon G(\omega)}{4\tau^2}(\mu U)_q q_r|\ d\tau &\lesssim \int_t^{t_1} \varepsilon\tau^{-2}(|A|+|\mu_q U|)(|\mu|+|\nu|)\ d\tau \\ &\lesssim \int_t^{t_1} \varepsilon\tau^{-2+C\varepsilon}\langle q\rangle^{-1-\gamma}|\mu|\ d\tau + \int_t^{t_1} \varepsilon^2\tau^{-3+C\varepsilon}\ d\tau \\ &\lesssim \varepsilon t^{-2+C\varepsilon}\end{aligned}$$

and

$$\int_t^{r+t} |\frac{\varepsilon^2 G(\omega)}{4\tau^2}\partial_q((-U_s+\frac{1}{2}G(\omega)AU)\mu)q_r|\ d\tau \lesssim \int_t^{t_1} \varepsilon^2\tau^{-2+C\varepsilon}\langle q\rangle^{-2-\gamma}|\mu|\ d\tau \lesssim \varepsilon^2 t^{-2+C\varepsilon}.$$

In the last estimate, we note that

$$\partial_q((-U_s+\frac{1}{2}G(\omega)AU)\mu = \frac{1}{2}G(\omega)A_qU\mu + (-U_s+\frac{1}{2}G(\omega)AU)\mu_q = O(t^{C\varepsilon}\langle q\rangle^{-2-\gamma}).$$

Recall that $\int_t^{r+t}|\mu_q|\ d\tau \lesssim \varepsilon\ln(t+r)+1$, $\nu=U=0$ at $\tau=r+t$, $\partial_r=q_r\partial_q$ and $q_r \gtrsim t^{-C\varepsilon}$. Apply the Gronwall's inequality and we conclude (3.25). □

Remark 3.5.1. We now prove some estimates for ν_q which will be used in the proof of the Poincaré's lemma (i.e. Lemma 3.13).

It follows from (3.25) that

$$|\nu_q| \lesssim \varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma} + \varepsilon t^{-2+C\varepsilon}$$

whenever $t \geq T_A \gg 1$ and $r \lesssim t$.

Now fix (t,x) such that $t \geq T_A$ and $r > 2t$. We seek to prove an estimate for ν_q. By differentiating (3.22), we have

$$\begin{aligned}(\partial_t-\partial_r)\partial_r\nu &= \mu_q\nu_r + \partial_r(\mu_q)\nu - \frac{\varepsilon}{2t}G\partial_r(A\mu) \\ &= \mu_q\nu_r - \frac{1}{2}\nu Gs\partial_r(A_q\mu) - \frac{\varepsilon}{2t}G\partial_r(A\mu) \\ &= \mu_q\nu_r + O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-2-\gamma}|\mu|).\end{aligned}$$

Besides, whenever $r > 2t$, we have $r \sim (r+t)$ and thus

$$\begin{aligned}\langle q\rangle &\lesssim (t+r)^{C\varepsilon}\langle r-t\rangle \lesssim (t+r)^{1+C\varepsilon}, \\ \langle q\rangle &\gtrsim (t+r)^{-C\varepsilon}\langle r-t\rangle \gtrsim (t+r)^{1-C\varepsilon}.\end{aligned}$$

Here we apply Lemma 3.3. It follows that

$$|\mu_q| \lesssim \langle q\rangle^{-2-\gamma}|s\mu| \lesssim (r+t)^{-2-\gamma+C\varepsilon}\cdot t^{C\varepsilon} \lesssim (r+t)^{-2}.$$

By setting $\ell(\tau) := \nu_r(\tau, r+t-\tau, \omega)$, we have for $t \leq \tau \leq (r+t)/3$,

$$\begin{aligned}|\dot{\ell}(\tau)| &\lesssim (r+t)^{-2}|\ell(\tau)| + \varepsilon\tau^{-1+C\varepsilon}\langle z(\tau)\rangle^{-2-\gamma}\cdot\tau^{C\varepsilon} \\ &\lesssim (r+t)^{-2}|\ell(\tau)| + \varepsilon\tau^{-1+C\varepsilon}(r+t)^{-2-\gamma+C\varepsilon}.\end{aligned}$$

Moreover, we have

$$|\ell((r+t)/3)| \lesssim \varepsilon(r+t)^{-1+C\varepsilon}\langle z((r+t)/3)\rangle^{-2-\gamma} + \varepsilon(r+t)^{-2+C\varepsilon} \lesssim \varepsilon(r+t)^{-2+C\varepsilon},$$

$$\begin{aligned}\int_t^{(r+t)/3} \varepsilon\tau^{-1+C\varepsilon}(r+t)^{-2-\gamma+C\varepsilon}\,d\tau &\lesssim \varepsilon(r+t)^{-2-\gamma+C\varepsilon}t^{-1+C\varepsilon}\cdot((r+t)/3-t)\\ &\lesssim \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon},\end{aligned}$$

$$\int_t^{(r+t)/3} (r+t)^{-2}\,d\tau \lesssim (r+t)^{-2}\cdot((r+t)/3-t) \lesssim (r+t)^{-1} \lesssim 1.$$

It follows from the Gronwall's inequality that

$$|\ell(t)| \lesssim \varepsilon(r+t)^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon}.$$

Then, for $r>2t$, we have

$$\begin{aligned}|\nu_q| &\lesssim q_r^{-1}|\nu_r| \lesssim t^{C\varepsilon}(\varepsilon(t+r)^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon})\\ &\lesssim \varepsilon(t+r)^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon}.\end{aligned}$$

Most of the estimates in the previous three lemmas will still hold, if Z^I is applied to the left hand sides for each multiindex I. We recall (3.16) and Definition 1.8.

Lemma 3.6. (a) *We have $q \in S^{0,1}$ and $\Omega_{kk'}q \in S^{0,0}$ for each $1 \le k < k' \le 3$. That is, for all $(t,x) \in \mathcal{D}$ and for all I, we have*

$$|Z^I q(t,r,\omega)| \lesssim_I \langle r-t\rangle t^{C_I\varepsilon}, \tag{3.28}$$

$$|Z^I \Omega_{kk'}q(t,r,\omega)| \lesssim_I t^{C_I\varepsilon}. \tag{3.29}$$

(b) *We have $\partial_q^a\partial_\omega^c A \in S^{0,-1-a-\gamma}$; $\mu \in S^{0,0}$ and $\partial_s^b\partial_a^q\partial_\omega^c\mu \in S^{0,-1-a-\gamma}$ for $a+b+c>0$; $\partial_s^b\partial_\omega^c U \in S^{0,0}$ for all $b,c \ge 0$, and $\partial_q^a\partial_s^b\partial_\omega^c U \in S^{0,-a-\gamma}$ for all $a>0$ and $b,c\ge 0$. Here all the functions are of $(s,q,\omega) = (\varepsilon\ln t-\delta, q(t,x),\omega)$ defined in $\mathcal{D}$.*

(c) *We have $\nu \in \varepsilon S^{-1,0}$, $\nu_q \in \varepsilon S^{-1,-1}$, $\lambda_i \in S^{-1,0}$, and*

$$\nu - \frac{\varepsilon}{4t}G(\omega)\mu U \in \varepsilon S^{-2,1}, \qquad \nu_q - \frac{\varepsilon}{4t}G(\omega)\partial_q(\mu U) \in \varepsilon S^{-2,0}.$$

Here all the functions are of $(s,q,\omega) = (\varepsilon\ln t-\delta, q(t,x),\omega)$ defined in $\mathcal{D}$.

Proof. (a) We first prove (3.28) by induction on $|I|$. The case $|I|=0$ has been proved in Lemma 3.3. In general, we fix an integer $k\ge 0$ and suppose (3.28) holds for all $|I|\le k$. Now fix a multiindex I with $|I|=k+1$. By the chain rule and Leibniz's rule, we express $Z^I\mu$ as a linear combination of terms of the form

$$(\partial_s^b\partial_q^a\partial_\omega^c\mu)\cdot Z^{I_1}q\cdots Z^{I_a}q\cdot Z^{J_1}(\varepsilon\ln t-\delta)\cdots Z^{J_b}(\varepsilon\ln t-\delta)\cdot Z^{K_1}\omega\cdots Z^{K_c}\omega \tag{3.30}$$

where $a+b+c>0$, $|I_*|,|J_*|,|K_{*,*}|$ are nonzero, and the sum of all these multiindices is $k+1$. The only term with some $|I_*|>k$ is $\mu_q Z^I q$. By (3.10) and our induction hypotheses, the remaining terms of the form (3.30) are controlled by

$$\langle q\rangle^{-1-a-\gamma}\exp(C|\varepsilon\ln t-\delta|)\cdot\langle q\rangle^a t^{C\varepsilon}\cdot\varepsilon^b\lesssim\langle q\rangle^{-1-\gamma}t^{C\varepsilon}.$$

Here we recall that $t^{-C\varepsilon}\lesssim\langle r-t\rangle/\langle q\rangle\lesssim t^{C\varepsilon}$ by Lemma 3.3. In summary, we have

$$Z^I\mu=\mu_q Z^I q+O(\langle q\rangle^{-1-\gamma}t^{C\varepsilon}).$$

Following the same proof, we can also show that

$$\sum_{0<|J|\leq k}|Z^J\mu|\lesssim\langle q\rangle^{-1-\gamma}t^{C\varepsilon},\qquad\sum_{|J|\leq k}|Z^J\mu|\ \ \lesssim|\mu|+\langle q\rangle^{-1-\gamma}t^{C\varepsilon}\lesssim t^{C\varepsilon}.$$

By (1.18), we have

$$(\partial_t-\partial_r)Z^I q=Z^I\mu+\sum_{|J|<|I|}[f_0Z^J\mu+\sum_i f_0(\partial_i+\omega_i\partial_t)Z^J q]$$

where f_0 denotes an arbitrary polynomial of $\{Z^I\omega\}$. Thus, we have

$$\begin{aligned}|(\partial_t-\partial_r)Z^I q|&\lesssim|\mu_q Z^I q|+t^{C\varepsilon}+\sum_{|J|\leq k}\sum_i|(\partial_i+\omega_i\partial_t)Z^J q|\\&\lesssim|\mu_q Z^I q|+t^{C\varepsilon}+(t+r)^{-1}\sum_{|J|\leq k+1}|Z^J q|\\&\lesssim|\mu_q Z^I q|+(t+r)^{-1}\sum_{|J|=k+1}|Z^J q|+t^{C\varepsilon}.\end{aligned}$$

In the second inequality, we apply Lemma 1.4; in the last one, we apply the induction hypotheses to control $|Z^J q|$ for $|J|\leq k$.

Now we fix $(t,x)\in\mathcal{D}$. Since (3.28) clearly holds for $q=r-t$, we can assume $r-t>-R$. By integrating along the characteristic $(\tau,r+t-\tau,\omega)$ for $t\leq\tau\leq t_1$ and taking sum over all multiindices I with $|I|=k+1$, we have

$$\begin{aligned}&\sum_{|I|=k+1}|Z^I q(t_1,r+t-t_1,\omega)-Z^I q(t,r,\omega)|\\&\lesssim\int_t^{t_1}(|\mu_q|+(t+r)^{-1})\sum_{|I|=k+1}|Z^I q|+\tau^{C\varepsilon}\ d\tau\\&\lesssim\int_t^{t_1}(|\mu_q|+(t+r)^{-1})\sum_{|I|=k+1}|Z^I q|\ d\tau+t_1^{C\varepsilon}|t_1-t|\\&\lesssim\int_t^{t_1}(|\mu_q|+(t+r)^{-1})\sum_{|I|=k+1}|Z^I q|\ d\tau+t^{C\varepsilon}\langle r-t\rangle.\end{aligned}$$

In the last inequality, we recall that $t \sim r$ in $\mathcal{D}$, so $t_1 = (r+t+R)/2 \sim t$; we also recall that $t_1 - t = (r-t+R)/2 \sim \langle r-t\rangle$. Also note that $Z^I q(t_1, t+r-t_1, \omega) = O(1)$, since $Z(r-t) = O(1)$ when $r = t-R$ and $t \gg 1$. Finally, recall that

$$\int_t^{t_1} |\mu_q(\tau)| + (r+t)^{-1}\, d\tau \lesssim \varepsilon \ln(t+r) + 1 \lesssim \varepsilon \ln(t) + 1$$

as proved in Lemma 3.4. By applying the Gronwall's inequality, we conclude that

$$\sum_{|I|=k+1} |Z^I q| \lesssim t^{C\varepsilon} \langle r-t\rangle.$$

As a result, by (3.10), we also have for each I with $|I| > 0$,

$$|Z^I \mu| \lesssim |\mu_q||Z^I q| + \langle q\rangle^{-1-\gamma} t^{C\varepsilon} \lesssim \langle q\rangle^{-2-\gamma} t^{C\varepsilon}\langle r-t\rangle + \langle q\rangle^{-1-\gamma} t^{C\varepsilon} \lesssim \langle q\rangle^{-1-\gamma} t^{C\varepsilon}. \tag{3.31}$$

Next we prove (3.29) by induction on $|I|$. By Lemma 3.4 we have

$$\Omega_{kk'} q = x_k \lambda_{k'} - x_{k'} \lambda_k = O(r \cdot (t+r)^{-1+C\varepsilon}) = O(t^{C\varepsilon}).$$

So the base case $|I| = 0$ is proved. In general we fix I with $|I| > 0$. By the induction hypotheses $\sum_{|J|<|I|} |Z^J \Omega_{kk'} q| \lesssim t^{C\varepsilon}$, (1.19) and (3.31), we have

$$\begin{aligned}
|(\partial_t - \partial_r) Z^I \Omega_{kk'} q| &= |Z^I \Omega_{kk'} \mu + \sum_{|J|<|I|} [f_0 Z^I \Omega_{kk'} \mu + \sum_i f_0 (\partial_i + \omega_i \partial_t) Z^J \Omega_{kk'} q]| \\
&\lesssim \sum_{0<|J|\le|I|+1} |Z^J \mu| + \sum_{|J|\le|I|} (t+r)^{-1} |Z^J \Omega_{kk'} q| \\
&\lesssim \langle q\rangle^{-1-\gamma} t^{C\varepsilon} + (t+r)^{-1} \sum_{|J|=|I|} |Z^J \Omega_{kk'} q| + t^{-1+C\varepsilon}.
\end{aligned}$$

Fix $(t,x) \in \mathcal{D}$. Since $\Omega_{kk'}(r-t) = 0$, we can assume $r - t \ge -R$. Then,

$$\begin{aligned}
&\int_t^{t_1} \tau^{-1+C\varepsilon} + \langle q(\tau, r+t-\tau, \omega)\rangle^{-1-\gamma} \tau^{C\varepsilon}\, d\tau \\
&\lesssim t^{-1+C\varepsilon}|t_1 - t| + t^{C\varepsilon} \int_t^{t_1} \langle z(\tau)\rangle^{-1-\gamma} (-\dot z(\tau))\, d\tau \lesssim t^{C\varepsilon}.
\end{aligned}$$

In the first inequality, we note that $|\mu| t^{C\varepsilon} \gtrsim 1$ and $-\mu = |\mu|$. Similar to the previous proof, we conclude by the Gronwall's inequality that $Z^I \Omega_{kk'} q = O(t^{C\varepsilon})$.

(b) Let $Q = Q(s,q,\omega)$ be a function of (s,q,ω). By the chain rule and Leibniz's rule, we can again write $Z^I Q$ as a sum of terms of the form (3.30) with μ replaced by Q. In conclusion, we have

$$|Z^I Q| \lesssim \sum_{a+b+c\le|I|} |\partial_s^b \partial_q^a \partial_\omega^c Q| \cdot \varepsilon^b \langle r-t\rangle^a t^{C\varepsilon}.$$

Combine this estimate with (3.7), (3.10) and (3.14) and recall that $t^{-C\varepsilon} \lesssim \langle r-t\rangle/\langle q\rangle \lesssim t^{C\varepsilon}$ in $\mathcal{D}$. We thus conclude that $\partial_q^a\partial_\omega^c A \in S^{0,-1-a-\gamma}$; $\mu \in S^{0,0}$ and $\partial_s^b\partial_a^q\partial_\omega^c\mu \in S^{0,-1-a-\gamma}$ for $a+b+c>0$; $\partial_s^b\partial_\omega^c U \in S^{0,0}$ for all $b,c\geq 0$, and $\partial_q^a\partial_s^b\partial_\omega^c U \in S^{0,-a-\gamma}$ for all $a>0$ and $b,c\geq 0$.

(c) In (a) we have proved that $\Omega_{kk'}q \in S^{0,0}$ for each $1\leq k<k'\leq 3$. Thus,

$$\lambda_i = r^{-1}\sum_j \omega_j\Omega_{ji}q \in S^{-1,0}\cdot S^{0,0}\cdot S^{0,0}\subset S^{-1,0}.$$

Here we recall from Example 1.10 that $r^{-1}\in S^{-1,0}$ and $\omega\in S^{0,0}$.

Next we set $Q=\nu-\varepsilon G(\omega)\mu U/(4t)$. By Lemma 3.5 we have $Q=O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle)$. Moreover, as computed in the proof of Lemma 3.5, we have

$$Q_t-Q_r=\mu_q Q+\frac{\varepsilon^2 G(\omega)}{4t^2}(-U_s+\frac{1}{2}G(\omega)AU)\mu+\frac{\varepsilon G(\omega)}{4t^2}\mu U=\mu_q Q+\varepsilon S^{-2,0}.$$

Fix a multiindex I with $|I|>0$, and suppose $|Z^JQ|\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ for all $|J|<|I|$. By (1.18), we have

$$\begin{aligned}|(\partial_t-\partial_r)Z^IQ| &\lesssim \sum_{|J|\leq|I|}|\mu_q Z^JQ|+\varepsilon t^{-2+C\varepsilon}+(t+r)^{-1}\sum_{|J|\leq|I|}|Z^JQ|\\ &\lesssim (|\mu_q|+(t+r)^{-1})\sum_{|J|=|I|}|Z^JQ|+\varepsilon t^{-2+C\varepsilon}.\end{aligned}$$

Note that $Q\equiv 0$ in the region $r-t<-R$, and note that when $r-t\geq -R$ we have

$$\int_t^{t_1}\varepsilon\tau^{-2+C\varepsilon}\,d\tau\lesssim \varepsilon t^{-2+C\varepsilon}(t_1-t)\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle.$$

By the Gronwall's inequality, we conclude that $\sum_{|J|=|I|}|Z^IQ|\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$. Thus, we have $Q\in\varepsilon S^{-2,1}$. By Lemma 1.9, we have $\partial Q\in \varepsilon S^{-2,0}$ and thus $Q_r\in\varepsilon S^{-2,0}$. This implies that

$$q_r\nu_q-\frac{\varepsilon}{4t}G(\omega)q_r\partial_q(\mu U)\in\varepsilon S^{-2,0}.$$

Since $q_r\geq C^{-1}t^{-C\varepsilon}$ and $q_r=\partial q\in S^{0,0}$, we can show that $q_r^{-1}\in S^{0,0}$. This easily follows from the fact that $Z^I(q_r^{-1})$ can be written as a linear combination of terms of the form

$$q_r^{-1-m}Z^{I_1}q_r\cdots Z^{I_m}q_r,\qquad \sum|I_*|=|I|.$$

We thus conclude that

$$\nu_q-\frac{\varepsilon}{4t}G(\omega)\partial_q(\mu U)\in\varepsilon S^{-2,0}.$$

Finally, note that

$$\frac{\varepsilon}{4t}G(\omega)\mu U\in\varepsilon S^{-1,0},\qquad \frac{\varepsilon}{4t}G(\omega)\partial_q(\mu U)=\frac{\varepsilon}{4t}G(\omega)(\mu_q U-2A)\in\varepsilon S^{-1,-1-\gamma}.$$

And since $\langle r-t\rangle \lesssim t$ in $\mathcal{D}$, we conclude that

$$\nu = Q + \frac{\varepsilon}{4t}G(\omega)\mu U \in \varepsilon S^{-1,0} + \varepsilon S^{-2,1} \subset \varepsilon S^{1,0},$$
$$\nu_q = \nu_q - \frac{\varepsilon}{4t}G(\omega)\partial_q(\mu U) + \frac{\varepsilon}{4t}G(\omega)\partial_q(\mu U) \in \varepsilon S^{-2,0} + \varepsilon S^{-1,-1-\gamma} \subset \varepsilon S^{-1,-1}.$$

□

The following proposition states that q is an approximate optical function.

Proposition 3.7. *We have $g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta \in S^{-2,1}$.*

Proof. Since $\nu \in \varepsilon S^{-1,0}$ and $\lambda_i \in S^{-1,0}$, we have

$$\begin{aligned}
&g_0^{\alpha\beta}q_\alpha q_\beta \\
&= g_0^{00}(\frac{\mu+\nu}{2})^2 + 2g_0^{0i}(\frac{\mu+\nu}{2})(\lambda_i + \frac{\omega_i(\nu-\mu)}{2}) + g_0^{ij}(\lambda_i + \frac{\omega_i(\nu-\mu)}{2})(\lambda_j + \frac{\omega_j(\nu-\mu)}{2}) \\
&= \frac{1}{4}G(\omega)\mu^2 + \frac{1}{2}(g_0^{00} - g_0^{ij}\omega_i\omega_j)\mu\nu + (g_0^{0i} - g_0^{ij}\omega_j)\mu\lambda_i + \frac{1}{4}g^{00}\nu^2 + \frac{1}{2}g_0^{0i}\nu(2\lambda_i + \omega_i\nu) \\
&\quad + \frac{1}{4}g_0^{ij}(2\lambda_i + \omega_i\nu)(2\lambda_j + \omega_j\nu) \\
&= \frac{1}{4}G(\omega)\mu^2 + \frac{1}{2}(g_0^{00} - g_0^{ij}\omega_i\omega_j)\mu\nu + (g_0^{0i} - g_0^{ij}\omega_j)\mu\lambda_i \mod S^{-2,0} \\
&= \frac{1}{4}G(\omega)\mu^2 \mod S^{-1,0}.
\end{aligned}$$

If we replace $(g_0^{\alpha\beta})$ with $(m^{\alpha\beta})$ in the computations above, we have

$$m^{\alpha\beta}q_\alpha q_\beta = -\mu\nu - m^{ij}\omega_j\mu\lambda_i \bmod S^{-2,0} = -\mu\nu \bmod S^{-2,0}.$$

Here note that $m^{ij}\omega_j\lambda_i = \sum_i \omega_i(q_i - \omega_i q_r) = 0$. Moreover, since $\varepsilon r^{-1}U \in \varepsilon S^{-1,0}$, by Lemma 1.11 we have

$$g^{\alpha\beta}(\varepsilon r^{-1}U) - m^{\alpha\beta} - g_0^{\alpha\beta}\varepsilon r^{-1}U \in \varepsilon^2 S^{-2,0}.$$

Moreover, by Lemma 1.9 and $q \in S^{0,1}$, we have $\partial q \in S^{0,0}$. We thus conclude that

$$\begin{aligned}
g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta &= m^{\alpha\beta}q_\alpha q_\beta + g_0^{\alpha\beta}\varepsilon r^{-1}Uq_\alpha q_\beta + (g^{\alpha\beta}(\varepsilon r^{-1}U) - m^{\alpha\beta} - g_0^{\alpha\beta}\varepsilon r^{-1}U)q_\alpha q_\beta \\
&= -\mu\nu + \frac{\varepsilon}{4r}G(\omega)\mu^2 U \mod S^{-2,0} \\
&= -\mu(\nu - \frac{\varepsilon}{4t}G(\omega)\mu U) + \frac{\varepsilon(t-r)}{4rt}G(\omega)\mu^2 U \mod S^{-2,0} \\
&\in \varepsilon S^{-2,1} + S^{-2,0} \subset S^{-2,1}.
\end{aligned}$$

In the last line we apply Lemma 3.6. □

In order to prove that $\varepsilon r^{-1}U$ is an approximate solution to (1.1), we need the following lemma.

Lemma 3.8. *We have* $g^{\alpha\beta}(\varepsilon r^{-1}U)q_{\alpha\beta} = \frac{\varepsilon}{2t}GA\mu - r^{-1}\mu \mod S^{-2,0}$.

Proof. We first note that

$$\varepsilon t^{-1}\nu_s = \nu_t - q_t\nu_q = (\nu_t + \nu_r) - \nu\nu_q,$$
$$\sum_l \partial_i\omega_l\nu_{\omega_l} = \nu_i - q_i\nu_q = (\nu_i - \omega_i\nu_r) - \lambda_i\nu_q.$$

By Lemma 1.9 and since $\nu \in \varepsilon S^{-1,0}$, we conclude that $\nu_t + \nu_r, \nu_i - \omega_i\nu_r \in \varepsilon S^{-2,0}$. By Lemma 3.6, we have $\nu\nu_q \in \varepsilon^2 S^{-2,-1}$ and $\lambda_i\nu_q \in \varepsilon S^{-2,-1}$. Thus, we have $\varepsilon t^{-1}\nu_s, \sum_l \partial_i\omega_l\nu_{\omega_l} \in \varepsilon S^{-2,0}$. Moreover, we have $\partial\lambda_i \in S^{-1,-1}$ by Lemma 1.9 and Lemma 3.6. It follows that

$$\begin{aligned}
q_{tt} &= \partial_t(\frac{\mu+\nu}{2}) = \frac{1}{4}\mu_q(\mu+\nu) + \frac{\varepsilon}{2t}\mu_s + \frac{1}{4}\nu_q(\mu+\nu) + \frac{\varepsilon}{2t}\nu_s \\
&= \frac{1}{4}\mu_q\mu + \frac{1}{4}\mu_q\nu + \frac{\varepsilon}{2t}\mu_s + \frac{1}{4}\nu_q\mu \mod \varepsilon S^{-2,0} = \frac{1}{4}\mu_q\mu \mod S^{-1,-1}, \\
q_{ti} &= \partial_i(\frac{\mu+\nu}{2}) = \frac{1}{2}(\mu_q+\nu_q)(\lambda_i + \frac{\omega_i(\nu-\mu)}{2}) + \frac{1}{2}\sum_l(\mu_{\omega_l}+\nu_{\omega_l})\partial_i\omega_l \\
&= -\frac{1}{4}\omega_i\mu_q\mu \mod S^{-1,-1},
\end{aligned}$$

$$\begin{aligned}
q_{ij} &= \partial_i(\lambda_j + \frac{\omega_j(\nu-\mu)}{2}) \\
&= \partial_i\lambda_j + \frac{1}{2}\partial_i\omega_j(\nu-\mu) + \frac{1}{2}\omega_j(\nu_q-\mu_q)(\lambda_i + \frac{\omega_i(\nu-\mu)}{2}) + \frac{1}{2}\omega_j\sum_l(\mu_{\omega_l}+\nu_{\omega_l})\partial_i\omega_l \\
&= \frac{1}{4}\omega_i\omega_j\mu\mu_q + \partial_i\lambda_j - \frac{1}{2}\mu\partial_i\omega_j - \frac{1}{4}\omega_j\mu_q(2\lambda_i + \omega_i\nu) \\
&\quad - \frac{1}{4}\omega_j\nu_q\omega_i\mu + \frac{1}{2}\omega_j\sum_l \mu_{\omega_l}\partial_i\omega_l \mod \varepsilon S^{-2,0} \\
&= \frac{1}{4}\omega_i\omega_j\mu\mu_q \mod S^{-1,0}.
\end{aligned}$$

Thus, we have $\partial^2 q \in S^{0,-1}$ and

$$g_0^{\alpha\beta}q_{\alpha\beta} = \frac{1}{4}G(\omega)\mu_q\mu \mod S^{-1,0}$$

and

$$\begin{aligned}
\Box q &= -(\frac{1}{4}\mu_q\mu + \frac{1}{4}\mu_q\nu + \frac{\varepsilon}{2t}\mu_s + \frac{1}{4}\nu_q\mu) + \frac{1}{4}\mu\mu_q - \frac{1}{4}\mu\nu_q - \frac{1}{4}\mu_q\nu \\
&\quad + \sum_i[\partial_i\lambda_i - \frac{1}{2}\mu\partial_i\omega_i - \frac{1}{2}\omega_i\mu_q\lambda_i + \frac{1}{2}\omega_i\sum_l \mu_{\omega_l}\partial_i\omega_l] \mod \varepsilon S^{-2,0} \\
&= \frac{\varepsilon}{4t}GA\mu - \frac{1}{2}\mu\nu_q - \frac{1}{2}\mu_q\nu - r^{-1}\mu + \sum_i \partial_i\lambda_i \mod \varepsilon S^{-2,0}.
\end{aligned}$$

Here we note that $\sum_i \partial_i\omega_i = 2/r$, $\sum_i \omega_i\partial_i\omega_l = 0$ and $\sum_i \omega_i\lambda_i = 0$. Moreover, we have $\sum_i \omega_i\partial_r\lambda_i = \partial_r(\sum_i \omega_i\lambda_i) = 0$ and $(\partial_i - \omega_i\partial_r)\lambda_i \in S^{-2,0}$, so

$$\sum_i \partial_i\lambda_i = \sum_i \omega_i\partial_r\lambda_i + \sum_i(\partial_i - \omega_i\partial_r)\lambda_i \in S^{-2,0}.$$

By Lemma 3.6, we conclude that

$$\begin{aligned}\Box q &= \frac{\varepsilon}{4t}GA\mu - \frac{1}{2}\mu\cdot\frac{\varepsilon}{4t}G\partial_q(\mu U) - \frac{1}{2}\mu_q\cdot\frac{\varepsilon}{4t}G\mu U - r^{-1}\mu \mod S^{-2,0}\\ &= \frac{\varepsilon}{2t}GA\mu - \frac{\varepsilon}{4t}G\mu\mu_q U - r^{-1}\mu \mod S^{-2,0}.\end{aligned}$$

In conclusion, we have

$$\begin{aligned}g^{\alpha\beta}(\varepsilon r^{-1}U)q_{\alpha\beta} &= \Box q + g_0^{\alpha\beta}\varepsilon r^{-1}Uq_{\alpha\beta} + (g^{\alpha\beta}(\varepsilon r^{-1}U) - m^{\alpha\beta} - g_0^{\alpha\beta}\varepsilon r^{-1}U)q_{\alpha\beta}\\ &= \frac{\varepsilon}{2t}GA\mu - \frac{\varepsilon}{4t}G\mu\mu_q U - r^{-1}\mu + \frac{\varepsilon}{4r}G\mu_q\mu U \mod S^{-2,0}\\ &= \frac{\varepsilon}{2t}GA\mu + \frac{\varepsilon(t-r)}{4tr}G\mu\mu_q U - r^{-1}\mu \mod S^{-2,0}\\ &= \frac{\varepsilon}{2t}GA\mu - r^{-1}\mu \mod S^{-2,0}.\end{aligned}$$

□

Finally we prove that $\varepsilon r^{-1}U$ has good pointwise bounds and is an approximate solution to (1.1) in $\mathcal{D}$.

Proposition 3.9. *We have*

$$\varepsilon r^{-1}U \in \varepsilon S^{-1,0}, \qquad g^{\alpha\beta}(\varepsilon r^{-1}U)\partial_\alpha\partial_\beta(\varepsilon r^{-1}U) \in \varepsilon S^{-3,0}.$$

In other word, for $(t,r,\omega) \in \mathcal{D}$,

$$|Z^I(\varepsilon r^{-1}U)| \lesssim_I \varepsilon t^{-1+C_I\varepsilon},$$

$$|Z^I(g^{\alpha\beta}(\varepsilon r^{-1}U)\partial_\alpha\partial_\beta(\varepsilon r^{-1}U))| \lesssim_I \varepsilon t^{-3+C_I\varepsilon}.$$

Note that we have a better bound for $\partial(\varepsilon r^{-1}U)$*: for all* $(t,r,\omega) \in \mathcal{D}$,

$$|\partial(\varepsilon r^{-1}U)| \lesssim \varepsilon t^{-1}.$$

Proof. We have proved $U \in S^{0,0}$ in Lemma 3.6, so it is clear that $\varepsilon r^{-1}U \in \varepsilon S^{-1,0}$. In addition,

$$\begin{aligned}\partial_t(\varepsilon r^{-1}U) &= \varepsilon r^{-1}(U_q q_t + U_s\varepsilon t^{-1}) = \varepsilon r^{-1}(\frac{1}{2}(\mu+\nu)U_q + U_s\varepsilon t^{-1})\\ &= -\varepsilon r^{-1}A \mod \varepsilon S^{-2,0},\end{aligned}$$

$$\partial_i(\varepsilon r^{-1}U) = \varepsilon r^{-2}\omega_i U + \varepsilon r^{-1}(U_q q_i + \sum_j U_{\omega_j}\partial_i\omega_j)$$
$$= \varepsilon r^{-1}U_q(\frac{1}{2}(\nu-\mu)\omega_i + \lambda_i) \mod \varepsilon S^{-2,0} = \varepsilon r^{-1}A\omega_i \mod \varepsilon S^{-2,0}.$$

Since $|A| \lesssim 1$, we conclude that $|\partial(\varepsilon r^{-1}U)| \lesssim \varepsilon t^{-1}$ in $\mathcal{D}$.

We have

$$\begin{aligned}(\varepsilon r^{-1}U)_{tt} &= \varepsilon r^{-1}(-U_s\varepsilon t^{-2} + 2U_{sq}q_t\varepsilon t^{-1} + U_{ss}\varepsilon^2 t^{-2} + q_{tt}U_q + q_t^2 U_{qq})\\ &= \varepsilon r^{-1}(2U_{sq}q_t\varepsilon t^{-1} + q_{tt}U_q + q_t^2U_{qq}) \mod \varepsilon S^{-3,0}\\ &= \varepsilon r^{-1}(q_{tt}U_q + q_t^2U_{qq}) \mod \varepsilon S^{-2,-1},\end{aligned}$$

$$\begin{aligned}(\varepsilon r^{-1}U)_{ti} &= \varepsilon r^{-1}(U_{qq}q_tq_i + \sum_l U_{\omega_l q}q_t\partial_i\omega_l + U_q q_{it} + U_{sq}q_i\varepsilon t^{-1} + \sum_l U_{s\omega_l}\partial_i\omega_l\varepsilon t^{-1})\\ &\quad - \varepsilon r^{-2}\omega_i(U_q q_t + U_s\varepsilon t^{-1})\\ &= \varepsilon r^{-1}(U_{qq}q_tq_i + U_q q_{it}) \mod \varepsilon S^{-2,-1},\end{aligned}$$

$$\begin{aligned}(\varepsilon r^{-1}U)_{ij} &= \varepsilon r^{-1}(U_{qq}q_iq_j + \sum_l U_{q\omega_l}(q_i\partial_j\omega_l + q_j\partial_i\omega_l) + U_q q_{ij} + \sum_{l,l'} U_{\omega_l\omega_{l'}}\partial_i\omega_l\partial_j\omega_{l'})\\ &\quad - \varepsilon r^{-2}\omega_i(U_q q_j + \sum_l U_{\omega_l}\partial_j\omega_l) - \varepsilon r^{-2}\omega_j(U_q q_i + \sum_l U_{\omega_l}\partial_i\omega_l) + \varepsilon\partial_j(r^{-2}\omega_i)\\ &= \varepsilon r^{-1}(U_{qq}q_iq_j + \sum_l U_{q\omega_l}(q_i\partial_j\omega_l + q_j\partial_i\omega_l) + U_q q_{ij}) - \varepsilon r^{-2}U_q(\omega_i q_j + \omega_j q_i) \mod \varepsilon S^{-3,0}\\ &= \varepsilon r^{-1}(U_{qq}q_iq_j + U_q q_{ij}) \mod \varepsilon S^{-2,-1}\end{aligned}$$

Here we note that $\varepsilon S^{-2,-1-\gamma} + \varepsilon S^{-3,0} \subset \varepsilon S^{-2,-1}$. Besides, we have $g^{\alpha\beta}(\varepsilon r^{-1}U) - m^{\alpha\beta} \in \varepsilon S^{-1,0}$. In summary, we have

$$\begin{aligned}&g^{\alpha\beta}(\varepsilon r^{-1}U)\partial_\alpha\partial_\beta(\varepsilon r^{-1}U)\\ &= g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta\varepsilon r^{-1}U_{qq} + g^{\alpha\beta}(\varepsilon r^{-1}U)q_{\alpha\beta}\varepsilon r^{-1}U_q + g^{00}(\varepsilon r^{-1}U)\varepsilon r^{-1}(2U_{sq}q_t\varepsilon t^{-1})\\ &\quad + g^{ij}(\varepsilon r^{-1}U)[\varepsilon r^{-1}\sum_l U_{q\omega_l}(q_i\partial_j\omega_l + q_j\partial_i\omega_l) - \varepsilon r^{-2}U_q(\omega_i q_j + \omega_j q_i)] \mod \varepsilon S^{-3,0}\\ &= g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta\varepsilon r^{-1}U_{qq} + g^{\alpha\beta}(\varepsilon r^{-1}U)q_{\alpha\beta}\varepsilon r^{-1}U_q - \varepsilon r^{-1}(2U_{sq}q_t\varepsilon t^{-1})\\ &\quad + \sum_i[2\varepsilon r^{-1}\sum_l U_{q\omega_l}q_i\partial_i\omega_l - 2\varepsilon r^{-2}U_q\omega_i q_i] \mod \varepsilon S^{-3,0}.\end{aligned}$$

Here we have

$$\sum_i q_i\partial_i\omega_l = \sum_i \lambda_i\partial_i\omega_l + \sum_i \omega_i q_r\partial_i\omega_l = \sum_i \lambda_i\partial_i\omega_l + 0 \in S^{-2,0}.$$

By Proposition 3.7 and Lemma 3.8, we have

$$\begin{aligned}g^{\alpha\beta}(\varepsilon r^{-1}U)\partial_\alpha\partial_\beta(\varepsilon r^{-1}U) &= (\frac{\varepsilon}{2t}GA\mu - r^{-1}\mu)\varepsilon r^{-1}U_q - \frac{\varepsilon^2}{rt}GAU_q q_t - 2\varepsilon r^{-2}U_q q_r \mod \varepsilon S^{-3,0}\\ &= -\frac{\varepsilon^2}{2tr}GA\nu U_q - \varepsilon r^{-2}\nu U_q \mod \varepsilon S^{-3,0} \in \varepsilon S^{-3,0}.\end{aligned}$$

This finishes the proof. □

3.2.3 Approximate solution u_{app}

Let $T_A \gg 1$ be a large constant such that all the estimates in Section 3.2.2 hold for $t \geq T_A$. Choose $\eta \in C^\infty(\mathbb{R})$ such that $\eta \equiv 1$ on $[2T_A, \infty)$ and $\eta \equiv 0$ on $(-\infty, T_A]$. In addition, choose $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on $[3/4, 5/4]$ and $\psi \equiv 0$ outsides $[1/2, 3/2]$.

We now define the approximate solution u_{app} by

$$u_{app}(t,x) := \varepsilon r^{-1}\eta(t)\psi(r/t)U(\varepsilon\ln(t) - \delta, q(t,r,\omega), \omega), \qquad r = |x|,\ \omega_i = x_i/r. \tag{3.32}$$

Note that $u_{app}(t,x)$ is defined for all $(t,x) \in [0,\infty) \times \mathbb{R}^3$. If $t \leq T_A$, then we have $u_{app} \equiv 0$. If $t \geq T_A \geq 2R$, since $U \equiv 0$ for $r \leq t - R$, u_{app} has no singularity at $|x| = 0$. Moreover, since $\psi \equiv 0$ when $|t-r| > t/2$, we have $u_{app} \equiv 0$ unless $(t,x) \in \mathcal{D}$; since $\psi \equiv 1$ when $|t-r| \leq t/4$, we have $u_{app} = \varepsilon r^{-1}U$ whenever $|t-r| \leq t/4$ and $t \geq 2T_A$.

We now prove the estimates for u_{app} in Proposition 3.2. The estimates are in fact the same as those in Proposition 3.9. However, note that in Proposition 3.9 we assume that $(t,x) \in \mathcal{D}$ while here we only assume $t \geq 0$.

Proof of Proposition 3.2. When $t \leq T_A$, we have $u_{app} \equiv 0$. When $T_A \leq t \leq 2T_A$, we have $Z^I u_{app} = O_R(\varepsilon)$. This is because the support of u_{app} lies in $|x| \sim_A 1$, and because U, η, ψ and all their derivatives are $O(1)$. Also note that $\varepsilon \leq (2T_A)^M \varepsilon t^{-M}$ for each M and all $t \leq 2T_A$.

Suppose $t \geq 2T_A$. Now η plays no role since $\eta(t) = 1$ for all $t \geq 2T_A \gg 1$. For $|r-t| \leq t/4$, all the estimates follow directly from Proposition 3.9. If $q(t,r,\omega) \leq -R$ i.e. $r - t \leq -R$, or if $r > 3t/2$, then $u_{app} \equiv 0$ so there is nothing to prove. So now we can assume $t \geq 2T_A$, $5t/4 \leq r \leq 3t/2$. Note that now we have $|r-t| \sim t$ and $(t,x) \in \mathcal{D}$, so

$$|\partial^k Z^I(\varepsilon r^{-1}U)| \lesssim \langle t-r\rangle^{-k} \sum_{|J|\leq|I|+k} |Z^J(\varepsilon r^{-1}U)| \lesssim \varepsilon t^{-1-k+C\varepsilon}.$$

Since $\partial^k Z^I(r/t) = O(t^{-k})$ for $t \sim r$, we have $\partial^k Z^I(\psi(r/t)) = O(t^{-k})$ for all $t \geq T_A$. In particular, we have $\partial(\psi(r/t)) = \psi'\partial(r/t) = O(t^{-1})$. It follows that for each I,

$$|\partial u_{app}| \lesssim |\partial(\psi(r/t))| \cdot |\varepsilon r^{-1}U| + |\psi(r/t)| \cdot |\partial(\varepsilon r^{-1}U)| \lesssim \varepsilon t^{-2+C\varepsilon} \lesssim \varepsilon t^{-1}, \tag{3.33}$$

$$|Z^I u_{app}| \lesssim \sum_{|J|+|K|=|I|} |Z^J(\psi(r/t))| \cdot |Z^K(\varepsilon r^{-1}U)| \lesssim \varepsilon t^{-1+C\varepsilon}, \tag{3.34}$$

$$|Z^I\partial^2 u_{app}| \lesssim \sum_{|J|\leq|I|} |\partial^2 Z^J u_{app}| \lesssim \langle r-t\rangle^{-2} \sum_{|J|\leq|I|+2} |Z^J u_{app}| \lesssim \varepsilon t^{-3+C\varepsilon}. \tag{3.35}$$

And since $Z^I(g^{\alpha\beta}(u_{app})) = O(1)$ for each I, we conclude that

$$|Z^I(g^{\alpha\beta}(u_{app})\partial_\alpha\partial_\beta u_{app})(t,x)| \lesssim \varepsilon(1+t)^{-3+C\varepsilon}.$$

□

3.3 Energy estimates and Poincare's lemma

We now derive the energy estimates and Poincaré's lemma, which are the main tools in the proof of our main theorem. The results in this section are similar to those in [21, 1].

3.3.1 Setup

Suppose $t \geq T_A \gg 1$ and $\varepsilon \ll 1$. Assume that u is a solution of (1.1) vanishing for $r \leq t-R$ and satisfying the pointwise estimates: for all $t \geq T_A \gg 1$ we have

$$|u| \lesssim \varepsilon t^{-1+C\varepsilon}, \qquad |\partial u(t,x)| \lesssim \varepsilon t^{-1}; \tag{3.36}$$

if $q(t,r,\omega) \leq t^{1/4}$ and $t \geq T_A$, we have

$$|u - \varepsilon r^{-1} U| \lesssim \varepsilon t^{-5/4+C\varepsilon}. \tag{3.37}$$

Recall that $U = U(t,r,\omega)$ is the asymptotic profile defined in (3.15). In Section 3.4 we will check these estimates when we apply the energy estimates.

3.3.2 Energy estimates

Fix a smooth function $\phi(t,x)$ with $\phi(t) \in C_c^\infty(\mathbb{R}^3)$ for each $t \geq T_A$ and ϕ is supported in $r \geq t-R$. We define the energy

$$\begin{aligned} E_u(\phi)(t) &= \int_{\mathbb{R}^3} w(t,x)(-2g^{0\alpha}(u)\phi_t\phi_\alpha + g^{\alpha\beta}(u)\phi_\alpha\phi_\beta)(t,x)\;dx \\ &= \int_{\mathbb{R}^3} w(t,x)(|\partial\phi|^2 - 2(g^{0\alpha}(u) - m^{0\alpha})\phi_t\phi_\alpha - (g^{\alpha\beta}(u) - m^{\alpha\beta})\phi_\alpha\phi_\beta)(t,x)\;dx. \end{aligned} \tag{3.38}$$

The weight function w is defined by

$$w(t,x) = \exp(c_0\varepsilon \ln(t) \cdot \sigma(q(t,r,\omega))) \tag{3.39}$$

with

$$\sigma(q) = (R+q+1)^{-\lambda}.$$

Here $q(t,r,\omega)$ is defined in Section 3.2; $c_0 \gg_A 1$ is a large constant to be chosen; $0 < \lambda < \gamma$ where γ comes from the decay assumption (3.7) of A. Note that $\phi \equiv 0$ unless $r \geq t-R$, and $q(t,r,\omega) \geq -R$ when $r \geq t-R$. So $w(t,x)$ is well-defined in the support of ϕ.

We remark that this type of the weight w was already used in the previous work on small data global existence by Lindblad [21] and Alinhac [1]. It can be viewed as an extended version of the method of ghost weight introduced by Alinhac; see [2].

Our goal is to prove the following energy estimates.

Proposition 3.10. *For $1 \ll T_A \leq t \leq T$, we have*

$$E_u(\phi)(t) \leq E_u(\phi)(T) + \int_t^T 2 \left\| g^{\alpha\beta}(u)\partial_\alpha\partial_\beta\phi(\tau) \right\|_{L^2(w)} \|\partial\phi(\tau)\|_{L^2(w)} + C\varepsilon\tau^{-1} \|\partial\phi\|^2_{L^2(w)} \ d\tau. \tag{3.40}$$

Here $\|f\|^2_{L^2(w)} := \int_{\mathbb{R}^3} |f|^2 w \ dx$ and $C > 0$ is a constant (which could depend on $u, \partial u$ and the weight w).

The proof starts with a computation of $\frac{d}{dt}E_u(\phi)(t)$. For simplicity, we write $g^{\alpha\beta} = g^{\alpha\beta}(u)$. Then, by applying integration by parts, we have

$$\begin{aligned}
&\frac{d}{dt}E_u(\phi)(t) \\
&= \int_{\mathbb{R}^3} w_t(-2g^{0\alpha}\phi_t\phi_\alpha + g^{\alpha\beta}\phi_\alpha\phi_\beta) \\
&\qquad + w(-2g^{0\alpha}\phi_{tt}\phi_\alpha - 2g^{0\alpha}\phi_t\phi_{\alpha t} - 2\partial_t g^{0\alpha}\phi_t\phi_\alpha + 2g^{\alpha\beta}\phi_{\alpha t}\phi_\beta + \partial_t g^{\alpha\beta}\phi_\alpha\phi_\beta) \ dx \\
&= \int_{\mathbb{R}^3} w_t(-2g^{0\alpha}\phi_t\phi_\alpha + g^{\alpha\beta}\phi_\alpha\phi_\beta) + w(-2g^{0\alpha}\phi_{\alpha t}\phi_t + 2g^{i\beta}\phi_{it}\phi_\beta - 2\partial_t g^{0\alpha}\phi_t\phi_\alpha + \partial_t g^{\alpha\beta}\phi_\alpha\phi_\beta) \ dx \\
&= \int_{\mathbb{R}^3} w_t(-2g^{0\alpha}\phi_t\phi_\alpha + g^{\alpha\beta}\phi_\alpha\phi_\beta) - 2w_i g^{i\beta}\phi_t\phi_\beta \\
&\qquad + w(-2g^{0\alpha}\phi_{\alpha t}\phi_t - 2g^{i\beta}\phi_t\phi_{i\beta} - 2\partial_t g^{0\alpha}\phi_t\phi_\alpha - 2\partial_i g^{i\beta}\phi_t\phi_\beta + \partial_t g^{\alpha\beta}\phi_\alpha\phi_\beta) \ dx \\
&= \int_{\mathbb{R}^3} w_t g^{\alpha\beta}\phi_\alpha\phi_\beta + w(-2g^{\alpha\beta}\phi_{\alpha\beta}\phi_t - 2\partial_\alpha g^{\alpha\beta}\phi_t\phi_\beta + \partial_t g^{\alpha\beta}\phi_\alpha\phi_\beta) - 2w_\alpha g^{\alpha\beta}\phi_t\phi_\beta \ dx.
\end{aligned}$$

By setting $T_\alpha := q_t\partial_\alpha - q_\alpha\partial_t$, we have $\phi_\alpha = q_t^{-1}(T_\alpha\phi + q_\alpha\phi_t)$. Note that

$$w_t = c_0(\varepsilon t^{-1}\sigma(q) + \varepsilon \ln(t)\sigma'(q)q_t)w, \qquad w_i = c_0\varepsilon\ln(t)\sigma'(q)q_i w.$$

Thus,

$$\begin{aligned}
g^{\alpha\beta}\phi_\alpha\phi_\beta q_t - 2g^{\alpha\beta}\phi_t\phi_\beta q_\alpha &= g^{\alpha\beta}q_t^{-1}(T_\alpha\phi + q_\alpha\phi_t)(T_\beta\phi + q_\beta\phi_t) - 2g^{\alpha\beta}q_\alpha\phi_t q_t^{-1}(T_\beta\phi + q_\beta\phi_t) \\
&= g^{\alpha\beta}q_t^{-1}T_\alpha\phi T_\beta\phi - g^{\alpha\beta}q_t^{-1}q_\alpha q_\beta\phi_t^2
\end{aligned}$$

and

$$\begin{aligned}
w_t g^{\alpha\beta}\phi_\alpha\phi_\beta - 2w_\alpha g^{\alpha\beta}\phi_t\phi_\beta &= c_0\varepsilon\ln(t)\sigma'(q)w(g^{\alpha\beta}q_t^{-1}T_\alpha\phi T_\beta\phi - g^{\alpha\beta}q_t^{-1}q_\alpha q_\beta\phi_t^2) \\
&\quad + c_0\varepsilon t^{-1}\sigma(q)w(-g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j).
\end{aligned}$$

Note that $T_0 = 0$, $(g^{ij}) = (\delta_{ij} + O(\varepsilon t^{-1+C\varepsilon}))$ is positive definite for $\varepsilon \ll 1$ and $t \geq T_A \gg 1$; $\sigma'(q) = -\lambda(R + q + 1)^{-1-\lambda} < 0$; by Lemma 3.4 we have

$$q_t = (\mu + \nu)/2 \leq -ct^{-C\varepsilon} + C\varepsilon(t + r)^{-1+C\varepsilon} < 0.$$

We conclude that

$$c_0\varepsilon\ln(t)\sigma'(q)wg^{\alpha\beta}q_t^{-1}T_\alpha\phi T_\beta\phi \geq 0.$$

In addition, we have the following lemma.

Lemma 3.11. *For all $t \geq T_A$, we have*

$$|c_0\varepsilon \ln(t)\sigma'(q)wg^{\alpha\beta}q_t^{-1}q_\alpha q_\beta \phi_t^2| \leq Cc_0\varepsilon t^{-1}\sigma(q)w\phi_t^2.$$

Note that we do not need to assume that $(t,x) \in \mathcal{D}$ in this lemma.

Proof. First, we suppose $q(t,r,\omega) \leq t^{1/4}$. By Proposition 3.7 and (3.37), we have

$$\begin{aligned} |g^{\alpha\beta}(u)q_\alpha q_\beta| &\leq |(g^{\alpha\beta}(u) - g^{\alpha\beta}(\varepsilon r^{-1}U))q_\alpha q_\beta| + |g^{\alpha\beta}(\varepsilon r^{-1}U)q_\alpha q_\beta| \\ &\lesssim |u - \varepsilon r^{-1}U| \cdot |\partial q|^2 + t^{-2+C\varepsilon}\langle r-t\rangle \lesssim t^{-5/4+C\varepsilon}. \end{aligned}$$

Here we note that $\partial q = O(t^{C\varepsilon})$ since $|\mu| \lesssim t^{C\varepsilon}$ and $|\nu| + \sum_i |\lambda_i| \lesssim t^{-1+C\varepsilon}$ by Lemma 3.4. We also note that $\langle r-t\rangle \lesssim \langle q\rangle t^{C\varepsilon} \lesssim t^{1/4+C\varepsilon}$ by Lemma 3.3. Thus,

$$\begin{aligned} |c_0\varepsilon \ln(t)\sigma'(q)wg^{\alpha\beta}q_t^{-1}q_\alpha q_\beta \phi_t^2| &\lesssim c_0\varepsilon \ln(t) \cdot \lambda(q+R+1)^{-1}\sigma(q)w \cdot t^{C\varepsilon} \cdot t^{-5/4+C\varepsilon} \cdot \phi_t^2 \\ &\lesssim c_0\varepsilon t^{-5/4+1/8+C\varepsilon}\sigma(q)w\phi_t^2 \\ &\lesssim c_0\varepsilon t^{-1}\sigma(q)w\phi_t^2. \end{aligned}$$

Here we note that $\ln(t) \lesssim t^{1/8}$ and that $|q_t| \gtrsim t^{-C\varepsilon}$ by Remark 3.4.1.

Next we suppose $q(t,r,\omega) \geq t^{1/4}$. Since $\mu = O(t^{C\varepsilon})$ and $\nu, \lambda_i = O(t^{-1+C\varepsilon})$ for all $t \geq T_A$ (we do not need to assume $(t,x) \in \mathcal{D}$; see Lemma 3.4), we have $\partial q = O(t^{C\varepsilon})$ and $|q_t| \gtrsim t^{-C\varepsilon}$. Thus,

$$|g^{\alpha\beta}q_\alpha q_\beta| \lesssim |m^{\alpha\beta}q_\alpha q_\beta| + |u||\partial q|^2 \lesssim |\mu\nu| + \sum_i |\lambda_i|^2 + \varepsilon t^{-1+C\varepsilon} \lesssim t^{-1+C\varepsilon}.$$

It follows that

$$\begin{aligned} |c_0\varepsilon \ln(t)\sigma'(q)wg^{\alpha\beta}q_t^{-1}q_\alpha q_\beta \phi_t^2| &\lesssim c_0\varepsilon \ln t \cdot \lambda(q+R+1)^{-1}\sigma(q)|q_t|^{-1}|g^{\alpha\beta}q_\alpha q_\beta| \cdot w\phi_t^2 \\ &\lesssim c_0\varepsilon(\ln t)t^{-1/4}\sigma(q) \cdot \varepsilon t^{-1+C\varepsilon} \cdot w\phi_t^2 \\ &\lesssim c_0\varepsilon t^{-9/8+C\varepsilon}\sigma(q)w\phi_t^2 \lesssim c_0\varepsilon t^{-1}\sigma(q)w\phi_t^2. \end{aligned}$$

□

We now finish the proof of Proposition 3.10. Since

$$-g^{00}\phi_t^2 + g^{ij}\phi_i\phi_j = |\partial\phi|^2 + O(|u||\partial\phi|^2) \sim |\partial\phi|^2,$$

we have

$$g^{\alpha\beta}\phi_\alpha\phi_\beta w_t - 2g^{\alpha\beta}\phi_t\phi_\beta w_\alpha \geq -Cc_0\varepsilon t^{-1}\sigma(q)w|\partial\phi|^2.$$

In conclusion,

$$\begin{aligned}\frac{d}{dt}E_u(\phi)(t) &\geq \int_{\mathbb{R}^3} w(-2g^{\alpha\beta}\phi_{\alpha\beta}\phi_t - 2\partial_\alpha g^{\alpha\beta}\phi_t\phi_\beta + \partial_t g^{\alpha\beta}\phi_\alpha\phi_\beta) - Cc_0\varepsilon t^{-1}\sigma(q)w|\partial\phi|^2\ dx \\ &\geq \int_{\mathbb{R}^3} -2w|g^{\alpha\beta}\phi_{\alpha\beta}||\phi_t| - C\varepsilon t^{-1}w|\partial\phi|^2\ dx \\ &\geq -2\left\|g^{\alpha\beta}\phi_{\alpha\beta}\right\|_{L^2(w)}\left\|\phi_t\right\|_{L^2(w)} - C\varepsilon t^{-1}\left\|\partial\phi\right\|^2_{L^2(w)}.\end{aligned}$$

Here we note that $\partial g^{**} = O(\varepsilon t^{-1})$ because of (3.36). We also note that our constant C depends on c_0 from (3.39). Integrate this inequality with respect to t on $[t, T]$ and we conclude (3.40).

3.3.3 Poincare's lemma

Fix a smooth function $\phi(t,x)$ with $\phi(t) \in C_c^\infty(\mathbb{R}^3)$ for each $t \geq T_A$ and ϕ is supported in $r \geq t - R$. As in the previous sections, we shall assume that $t \geq T_A \gg 1$ and $\varepsilon \ll 1$.

Lemma 3.12. *For ϕ as above, we have*

$$\int_{\mathbb{R}^3} \langle t-r\rangle^{-2}|\phi|^2\ dx \lesssim \int_{\mathbb{R}^3} |\partial\phi|^2\ dx. \tag{3.41}$$

Proof. Note that $\langle r-t\rangle \sim (r-t+R+1)$ if $r-t \geq -R$. Then we have

$$\begin{aligned}\int \langle t-r\rangle^{-2}|\phi|^2\ dx &\lesssim_R \int_{\mathbb{S}^2}\int_0^\infty (r-t+R+1)^{-2}|\phi|^2\ r^2drdS_\omega \\ &= \int_{\mathbb{S}^2}\int_0^\infty |\phi|^2\ r^2\partial_r(-(r-t+R+1)^{-1})\ drdS_\omega \\ &= \int_{\mathbb{S}^2}\int_0^\infty \partial_r(|\phi|^2r^2)(r-t+R+1)^{-1}\ drdS_\omega \\ &= \int_{\mathbb{S}^2}\int_0^\infty (2|\phi|^2r + 2\phi\phi_r r^2)(r-t+R+1)^{-1}\ drdS_\omega \\ &\lesssim_R \int_{\mathbb{S}^2}\int_0^\infty 2|\phi r^{-1} + \phi_r|\cdot|\phi|\langle t-r\rangle^{-1}\ r^2drdS_\omega \\ &\lesssim \left(\int \langle t-r\rangle^{-2}|\phi|^2\ dx\right)^{1/2}\left(\int |\phi r^{-1} + \phi_r|^2\ dx\right)^{1/2}.\end{aligned}$$

Since

$$\int 2\phi\phi_r r^{-1}\ dx = \int_{\mathbb{S}^2}\int_0^\infty \partial_r(\phi^2)r\ drdS_\omega = \int_{\mathbb{S}^2}\int_0^\infty -\phi^2\ drdS_\omega = -\int \phi^2 r^{-2}\ dx,$$

we have

$$\int |\phi r^{-1} + \phi_r|^2 \, dx = \int \phi_r^2 \, dx.$$

We then conclude (3.41). □

We can also prove a weighted version of Poincaré's lemma. Note that the value of δ in $s = \varepsilon \ln(t) - \delta$ is chosen in the proof of the next lemma.

Lemma 3.13. *For ϕ as above, we have*

$$\int \phi^2 q_r^2 \langle q\rangle^{-2} w \, dx \lesssim \int |\partial\phi|^2 w \, dx. \tag{3.42}$$

Proof. Note that $\langle q\rangle \sim (q + R + 1)$ since ϕ is supported in $q \geq -R$. We first claim that whenever $r - t \geq -R$ and $t \geq T_A$, we have

$$\partial_q(q_r)w + q_r w_q \leq C\delta\langle q\rangle^{-1} w q_r. \tag{3.43}$$

Since $q_r \sim |\mu|$ whenever $t \geq T_A$, it suffices to prove (3.43) with q_r replaced by $|\mu|$ on the right hand side.

Note that

$$\begin{aligned}
\partial_q(q_r)w + q_r w_q &= w(\partial_q(q_r) - q_r c_0 \varepsilon \ln(t) \cdot \lambda(q + R + 1)^{-1-\lambda}) \\
&= \frac{1}{2} w(\nu_q - \mu_q - c_0\varepsilon \ln(t) \cdot \lambda(q + R + 1)^{-1-\lambda}(\nu - \mu)) \\
&= \frac{1}{2} w(\frac{1}{2} G A_q(\varepsilon \ln(t) - \delta) + c_0 \varepsilon \ln(t) \cdot \lambda(q + R + 1)^{-1-\lambda})\mu \\
&\quad + O(w(|\nu_q| + c_0\lambda\varepsilon \ln(t) \cdot (q + R + 1)^{-1-\lambda}|\nu|)).
\end{aligned}$$

First we suppose $r \lesssim t$. In this case, recall from Remark 3.5.1 and Lemma 3.4 that

$$\begin{aligned}
&|\nu_q| + c_0\lambda\varepsilon \ln(t) \cdot (q + R + 1)^{-1-\lambda}|\nu| \\
&\lesssim \varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma} + \varepsilon t^{-2+C\varepsilon} + c_0\lambda\varepsilon \ln(t)(q + R + 1)^{-1-\lambda} \cdot \varepsilon t^{-1+C\varepsilon} \\
&\lesssim (\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma} + \varepsilon t^{-2+C\varepsilon} + c_0\lambda\varepsilon \ln(t)(q + R + 1)^{-1-\lambda} \cdot \varepsilon t^{-1+C\varepsilon})|\mu|.
\end{aligned}$$

In the last estimate, we note that $|\mu| \gtrsim t^{-C\varepsilon}$. It follows that

$$\begin{aligned}
\partial_q(q_r)w + q_r w_q &\leq \frac{1}{2} w\varepsilon \ln(t) \cdot (\frac{1}{2} G A_q - c_0 \cdot \lambda(q + R + 1)^{-1-\lambda})|\mu| - \frac{1}{4} w G A_q \delta\mu \\
&\quad + Cw(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma} + \varepsilon t^{-2+C\varepsilon} + c_0\lambda\varepsilon \ln(t)(q + R + 1)^{-1-\lambda} \cdot \varepsilon t^{-1+C\varepsilon})|\mu| \\
&\leq w\varepsilon \ln(t) \cdot (C(q + R + 1)^{-2-\gamma} + (-\frac{1}{2} + C\varepsilon t^{-1+C\varepsilon})c_0\lambda(q + R + 1)^{-1-\lambda})|\mu| \\
&\quad + C\delta\langle q\rangle^{-2-\gamma}|\mu| + C\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma}|\mu| + C\varepsilon t^{-2+C\varepsilon}|\mu|.
\end{aligned}$$

We choose $\varepsilon \ll 1$ and $T_A \gg 1$ so that $C\varepsilon T_A^{-1+C\varepsilon} \leq 1/6$. We also choose $c_0 \gg_\lambda 1$ so that $c_0\lambda > 6C$. It follows that
$$C + (-\frac{1}{2} + C\varepsilon t^{-1+C\varepsilon})c_0\lambda < 0.$$
Also note that $\langle q\rangle \lesssim (t+r)^{C\varepsilon}\langle r-t\rangle \lesssim t^{1+C\varepsilon}$ whenever $t \sim r$. Thus, for $\varepsilon < \delta$ we have
$$\partial_q(q_r)w + q_r w_q \leq 0 + C\delta\langle q\rangle^{-2-\gamma}|\mu| + C\varepsilon\langle q\rangle^{-2-\gamma+C\varepsilon}|\mu| + C\varepsilon\langle q\rangle^{-2+C\varepsilon}|\mu| \leq C\delta\langle q\rangle^{-1}|\mu|.$$

Next we suppose $r > 2t$. As proved in Remark 3.5.1, we have $(r+t)^{1-C\varepsilon} \lesssim \langle q\rangle \lesssim (r+t)^{1+C\varepsilon}$ and $|\nu_q| \lesssim \varepsilon(t+r)^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon}$. It follows that
$$\begin{aligned}\partial_q(q_r)w + q_r w_q &\leq w\varepsilon\ln(t)\cdot(C\langle q\rangle^{-2-\gamma} + (-\frac{1}{2} + C\varepsilon t^{-1+C\varepsilon})c_0\lambda(q+R+1)^{-1-\lambda})|\mu| \\ &\quad + C\delta w\langle q\rangle^{-2-\gamma}|\mu| + Cw|\mu|(\varepsilon(t+r)^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}(r+t)^{-1-\gamma+C\varepsilon}) \\ &\leq w\varepsilon\ln(t)\cdot(C\langle q\rangle^{-2-\gamma} + (-\frac{1}{2} + C\varepsilon t^{-1+C\varepsilon})c_0\lambda(q+R+1)^{-1-\lambda})|\mu| \\ &\quad + C\delta w\langle q\rangle^{-2-\gamma}|\mu| + C\varepsilon w|\mu|(\langle q\rangle^{-2+C\varepsilon} + t^{-1+C\varepsilon}\langle q\rangle^{-1-\gamma+C\varepsilon}).\end{aligned}$$

Again, by choosing $\varepsilon \ll_\delta 1$ and $c_0 \gg_\lambda 1$, we have
$$\partial_q(q_r)w + q_r w_q \leq 0 + C\delta w\langle q\rangle^{-2-\gamma}|\mu| + C\varepsilon w|\mu|\langle q\rangle^{-1} \leq C\delta w\langle q\rangle^{-1}|\mu|.$$

This finishes the proof of (3.43).

Now we have
$$\begin{aligned}&\int |\phi|^2 q_r^2\langle q\rangle^{-2} w\ dx \\ &\leq C\int_{\mathbb{S}^2}\int_0^\infty |\phi(t,r\omega)|^2 r^2 q_r^2 (q+R+1)^{-2} w\ drdS_\omega \\ &= C\int_{\mathbb{S}^2}\int_0^\infty (q+R+1)^{-1}\partial_r(\phi^2 r^2 q_r w)\ drdS_\omega \\ &= C\int_{\mathbb{S}^2}\int_0^\infty (q+R+1)^{-1}[2\phi\phi_r r^2 w + 2\phi^2 rw + \phi^2 r^2\partial_q(q_r)w + \phi^2 r^2 q_r w_q]q_r\ drdS_\omega \\ &\leq C\int_{\mathbb{S}^2}\int_0^\infty (q+R+1)^{-1}(2\phi\phi_r + 2\phi^2 r^{-1})r^2 q_r w\ drdS_\omega \\ &\quad + C\int_{\mathbb{S}^2}\int_0^\infty (q+R+1)^{-1}\phi^2 r^2\cdot C\delta\langle q\rangle^{-1}q_r w\cdot q_r\ drdS_\omega \\ &\leq C\left(\int (|\phi_r|^2 + r^{-2}|\phi|^2)w\ dx\right)^{1/2}\cdot\left(\int \phi^2\langle q\rangle^{-2}q_r^2 w\ dx\right)^{1/2} \\ &\quad + C_A\delta\int \langle q\rangle^{-2}\phi^2 q_r^2 w\ dx.\end{aligned}$$

Here the constant C_A in the second term only depends on the scattering data, and in particular it does not depend on ε, t or T_A. Thus, by choosing $\delta := \frac{1}{4C_A}$, we conclude that

$$\int |\phi|^2 q_r^2 \langle q\rangle^{-2} w\ dx \lesssim \int (|\phi_r|^2 + r^{-2}|\phi|^2) w\ dx.$$

Now recall that $r \geq t - R$ when $\phi \neq 0$. If $q \leq t^{1/2}$ we have $\langle q\rangle^2 \leq Ct$ and $q_r \geq C^{-1}t^{-C\varepsilon}$, as proved before. Thus, if $t \geq T_A \gg 1$,

$$\begin{aligned}\int_{q\leq t^{1/2}} r^{-2}\phi^2 w\ dx &\lesssim (t-R)^{-2} \cdot Ct^{C\varepsilon} \cdot Ct \int \phi^2 q_r^2 \langle q\rangle^{-2} w\ dx \\ &\lesssim t^{-1+C\varepsilon} \int \phi^2 q_r^2 \langle q\rangle^{-2} w\ dx.\end{aligned}$$

If $q \geq t^{1/2}$, we have $w(q) \leq \exp(Cc_0\varepsilon \ln(t) \cdot t^{-\lambda/2}) \leq C$ for $t \gg_A 1$ and $\varepsilon \ll 1$. Besides, we also have $w \geq 1$. Thus, by Hardy's inequality,

$$\int_{q\geq t^{1/2}} r^{-2}\phi^2 w\ dx \lesssim \int r^{-2}\phi^2\ dx \lesssim \int |\partial\phi|^2\ dx \lesssim \int |\partial\phi|^2 w\ dx.$$

By choosing $T_A \gg 1$ and $\varepsilon \ll 1$, we have

$$\begin{aligned}\int |\phi|^2 q_r^2 \langle q\rangle^{-2} w\ dx &\leq C\int |\phi_r|^2 w\ dx + C\int_{q\geq t^{1/2}} r^{-2}|\phi|^2 w\ dx + C\int_{q<t^{1/2}} r^{-2}|\phi|^2 w\ dx \\ &\leq C\int |\partial\phi|^2 w\ dx + Ct^{-1+C\varepsilon}\int \phi^2 q_r^2 \langle q\rangle^{-2} w\ dx \\ &\leq C\int |\partial\phi|^2 w\ dx + \frac{1}{2}\int \phi^2 q_r^2 \langle q\rangle^{-2} w\ dx.\end{aligned}$$

This finishes the proof. □

We end this section with the following key lemma. It is crucial that we get a factor εt^{-1} instead of $\varepsilon t^{-1+C\varepsilon}$ in the estimate below.

Lemma 3.14. *Suppose ϕ is supported in $|x| - t \geq -R$ and $\phi(t) \in C_c^\infty(\mathbb{R}^3)$ for each t. Let $F := g_0^{\alpha\beta}\partial_\alpha\partial_\beta u_{app}$ where u_{app} is defined in (3.32). Then for $t \geq T_A \gg 1$, we have*

$$\|\phi F\|_{L^2(w)} \lesssim \varepsilon t^{-1} \|\partial\phi\|_{L^2(w)}.$$

Proof. Write $F = \frac{\varepsilon}{4r}G(\omega)q_r A_q + F_2$. By the weighted Poincaré's lemma, i.e. Lemma 3.13, we have

$$\left\|\varepsilon r^{-1}G(\omega)q_r A_q\phi\right\|_{L^2(w)}^2 \lesssim \varepsilon^2(t-R)^{-2}\int q_r^2\langle q\rangle^{-2}\phi^2 w\ dx \lesssim \varepsilon^2 t^{-2}\|\partial\phi\|_{L^2(w)}^2.$$

We next estimate F_2. If $-R \le r-t \le t/4$, we have $u_{app} = \varepsilon r^{-1}U$ and $(t,x) \in \mathcal{D}$. As computed in the proofs of Proposition 3.7, Lemma 3.8 and Proposition 3.9, we have

$$\begin{aligned}
F &= \varepsilon r^{-1}(g_0^{\alpha\beta}q_{\alpha\beta}U_q + g_0^{\alpha\beta}q_\alpha q_\beta U_{qq}) \mod \varepsilon S^{-2,-1} \\
&= \varepsilon r^{-1}(\frac{1}{4}G(\omega)\mu_q\mu U_q + \frac{1}{4}G(\omega)\mu^2 U_{qq}) \mod \varepsilon S^{-2,-1} \\
&= -\frac{\varepsilon}{2r}G(\omega)\mu A_q \mod \varepsilon S^{-2,-1} \\
&= \frac{\varepsilon}{4r}G(\omega)q_r A_q \mod \varepsilon S^{-2,-1}.
\end{aligned}$$

Here we also apply Lemma 3.6 to control the remainder terms. Thus, $F_2 = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-1})$ whenever $-R \le r-R \le t/4$. If $r-t < -R$, we have $A \equiv 0$ and $u_{app} \equiv 0$. Thus $F_2 \equiv 0$. If $r-t > t/4$, we have $u_{app} \equiv 0$ if $r-t > t/2$, or $\partial^2 u_{app} = O(\varepsilon t^{-3+C\varepsilon})$ if $t/4 \le r-t \le t/2$ by (3.35). In both cases, we have $F = O(\varepsilon r^{-3+C\varepsilon})$. Moreover, whenever $r-t \ge t/4$, by Lemma 3.3 we have $\langle q\rangle \gtrsim \langle r-t\rangle(t+r)^{-C\varepsilon} \gtrsim r^{1-C\varepsilon}$; by Lemma 3.4 we have $q_r = (\nu-\mu)/2 = O(t^{C\varepsilon})$. Thus,

$$|\frac{\varepsilon}{4r}G(\omega)q_r A_q| \lesssim \varepsilon r^{-1}\cdot t^{C\varepsilon}\cdot\langle q\rangle^{-2-\gamma} \lesssim \varepsilon r^{-3-\gamma+C\varepsilon}.$$

It follows that $F_2 = O(\varepsilon r^{-3+C\varepsilon})$ whenever $r-t \ge t/4$.

Since $1 \le w \le Ct^{C\varepsilon}$, we have

$$\begin{aligned}
\|\phi F_2\|^2_{L^2(w)} &= \left\|\phi F_2\chi_{r-t\le t/4}\right\|^2_{L^2(w)} + \left\|\phi F_2\chi_{r-t\ge t/4}\right\|^2_{L^2(w)} \\
&\lesssim \int_{r-t\le t/4}\varepsilon^2 t^{-4+C\varepsilon}\langle r-t\rangle^{-2}|\phi|^2\,dx + \int_{r-t\ge t/4}\varepsilon^2 r^{-6+C\varepsilon}|\phi|^2\,dx \\
&\lesssim \int \varepsilon^2 t^{-2}\langle t-r\rangle^{-2}\phi^2\,dx \lesssim \varepsilon^2 t^{-2}\|\partial\phi\|^2_{L^2(\mathbb{R}^3)} \lesssim \varepsilon^2 t^{-2}\|\partial\phi\|^2_{L^2(w)}.
\end{aligned}$$

Here we use the Poincaré's lemma, i.e. Lemma 3.12. We are done. □

3.4 Continuity Argument

3.4.1 Setup

Fix $\chi(s) \in C_c^\infty(\mathbb{R})$ such that $\chi \in [0,1]$ for all s, $\chi \equiv 1$ for $|s| \le 1$ and $\chi \equiv 0$ for $|s| \ge 2$. Also fix a large time $T > 0$. Consider the equation of $v = v^T(t,x)$

$$g^{\alpha\beta}(u_{app}+v)\partial_\alpha\partial_\beta v = -\chi(t/T)g^{\alpha\beta}(u_{app}+v)\partial_\alpha\partial_\beta u_{app},\ t>0; \qquad v \equiv 0,\ t \ge 2T. \quad (3.44)$$

We have the following results.

(a) By the local existence theory of quasilinear wave equations, we can find a local smooth solution to (3.44) near $t = 2T$.

(b) The solution on $[T_1,\infty)$ can be extended to $[T_1-\epsilon,\infty)$ for some small $\epsilon>0$ if

$$\left\|\partial^k v\right\|_{L^\infty([T_1,\infty)\times\mathbb{R}^3)}<\infty,\qquad \text{for all } k\leq 4.$$

(c) The solution to (3.44) has a finite speed of propagation: $v^T(t,x)=0$ if $r+t>6T$ or $r<t-R$, so $Z^I(t/T)=O(1)$ when $T/2\leq t\leq 2T$.

(d) If the solution exists for $t\leq T$, we have $g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u=0$ for $t\leq T$ where $u=u_{app}+v$.

The proofs of these statements are standard. We refer to [30] for the proofs of (a) and (b). In this section, our goal is to prove the following proposition.

Proposition 3.15. *Fix an integer $N\geq 6$. Then there exist constants $\varepsilon_N>0$ which depend on N and R, such that for any $0<\varepsilon<\varepsilon_N$,* (3.44) *has a solution $v=v^T(t,x)$ for all $t\geq 0$. In addition, $v\equiv 0$ if $r<t-R$; for all $|I|\leq N$, we have*

$$\left\|\partial Z^I v(t)\right\|_{L^2(\mathbb{R}^3)}\lesssim_I \varepsilon(1+t)^{-1/2+C_I\varepsilon},\qquad \forall t\geq 0. \tag{3.45}$$

Recall that we choose R based on the support assumption (3.6) *of our scattering data A.*

It should be pointed out that the N in this proposition is different from the N in Theorem 3.1.

We use a continuity argument to prove this proposition. From now on we assume $\varepsilon\ll 1$, which means ε is arbitrary in $(0,\varepsilon_N)$ for some fixed small constant ε_N depending on N. First we prove the result for $t\geq T_{N,A}$, where $T_{N,A}\gg_{N,A}1$ is a sufficiently large constant depending on N. We start with a solution $v(t,x)$ for $t\geq T_1$ such that for all $t\geq T_1\geq T_{N,A}$ and $k+i\leq N$,

$$E_{k,i}(t):=\sum_{l\leq k,|I|\leq i}E_u(\partial^l Z^I v)(t)\leq B_{k,i}\varepsilon^2 t^{-1+C_{k,i}\varepsilon}, \tag{3.46}$$

$$|u|\leq B_0\varepsilon t^{-1+C_{0,2}\varepsilon/2},\ |\partial u|\leq B_1\varepsilon t^{-1}. \tag{3.47}$$

Here $u:=v+u_{app}$ and E_u is defined in (3.38). We remark that $C_{k,i},B_{k,i}$ depend on k,i but not on N. Our goal is to prove that (3.46) and (3.47) hold with $B_{k,i},B_0,B_1$ replaced by smaller constants $B'_{k,i},B'_0,B'_1$, and with $C_{k,i}$ unchanged, assuming that $\varepsilon\ll 1$ and $T_{N,A}\gg 1$. To achieve this goal, we first induct on i, and then we induct on k for each fixed i. For each (k,i), we want to prove the following inequality

$$\begin{aligned}\sum_{l\leq k,|I|\leq i}\left\|g^{\alpha\beta}(u)\partial_\alpha\partial_\beta\partial^l Z^I v\right\|_{L^2(w)}\leq{}& C_N\varepsilon t^{-1}E_{k,i}(t)^{1/2}\\ &+C_N\varepsilon t^{-1+C\varepsilon}(E_{k-1,i}(t)^{1/2}+E_{k+1,i-1}(t)^{1/2})\\ &+C\varepsilon t^{-3/2+C\varepsilon}.\end{aligned} \tag{3.48}$$

Here $E_{-1,\cdot}=E_{\cdot,-1}=0$, and C,C_N are constants whose meanings will be explained later. We then combine (3.48) with the energy estimates (3.40) to derive an inequality on $E_{k,i}(t)$.

We remark that the proof in this section is closely related to that of the energy estimates in Section 9 of Lindblad [21].

In the following computation, let C denote a universal constant or a constant from the previous estimates for q and u_{app} (e.g. from Proposition 3.2). Here C is allowed to depend on (k,i) or N, but we will never write it as $C_{k,i}$ or C_N. We will choose the constants in the following order:

$$\begin{aligned} &C \to C_{0,0}, B_{0,0} \to C_{1,0}, B_{1,0} \to \cdots \to C_{N,0}, B_{N,0} \\ &\to C_{0,1}, B_{0,1} \to \cdots \to C_{N-1,1}, B_{N-1,1} \\ &\to C_{0,2}, B_{0,2} \to \cdots \to C_{N-2,2}, B_{N-2,2} \\ &\dots \\ &\to C_{0,N}, B_{0,N} \\ &\to B_0, B_1 \to C_N \to T_{N,A} \to \varepsilon. \end{aligned}$$

We emphasize that if a constant B appears before a constant B', then B cannot depend on B'.

In addition, since $T_{N,A} \gg 1$ and $\varepsilon \ll 1$ are chosen at the end, we can control terms like $C_N\varepsilon$ and $C_N T_{N,A}^{-\gamma+C_N\varepsilon}$ for $\gamma > 0$ for any (k,i) by a universal constant, e.g. 1.

To end the setup, we derive a differential equation for $Z^I v$ from (3.44). If we commute (3.44) with Z^I, we have

$$\begin{aligned} &g^{\alpha\beta}(u)\partial_\alpha\partial_\beta Z^I v \\ &= [\Box, Z^I]v + [g^{\alpha\beta}(u) - m^{\alpha\beta}, Z^I]\partial_\alpha\partial_\beta v + (g^{\alpha\beta}(u) - m^{\alpha\beta})[\partial_\alpha\partial_\beta, Z^I]v \\ &\quad - Z^I(\chi(t/T)(g^{\alpha\beta}(u) - g^{\alpha\beta}(u_{app}))\partial_\alpha\partial_\beta u_{app}) - Z^I(\chi(t/T)g^{\alpha\beta}(u_{app})\partial_\alpha\partial_\beta u_{app}) \\ &=: R_1 + R_2 + R_3 + R_4 + R_5 \end{aligned} \tag{3.49}$$

with $Z^I v \equiv 0$ for $t \geq 2T$.

3.4.2 Pointwise bounds (3.47)

In the next few subsections, we always assume $t \geq T_{N,A} \gg 1$. Since $1 \leq w \leq Ct^{C\varepsilon}$, by (3.47) and (3.38) we have

$$C^{-1}\|\partial\phi\|_{L^2(\mathbb{R}^3)} \leq \|\partial\phi\|_{L^2(w)} \sim E_u(\phi)^{1/2} \leq Ct^{C\varepsilon}\|\partial\phi\|_{L^2(\mathbb{R}^3)}. \tag{3.50}$$

Here we can choose $\varepsilon \ll 1$ and $T_{N,A} \gg 1$ so that all constants in this inequality are universal. If we combine this inequality with (3.46), we have

$$\left\|\partial Z^I v(t)\right\|^2_{L^2(\mathbb{R}^3)} \leq CE_u(Z^I v)(t) \leq CB_{0,i}\varepsilon^2 t^{-1+C_{0,i}\varepsilon}, \qquad |I| = i \leq N,$$

so by the Klainerman-Sobolev inequality, we have

$$|\partial Z^I v(t)| \leq CB_{0,i+2}^{1/2}\varepsilon t^{-1/2+C_{0,i+2}\varepsilon/2}(1+t+r)^{-1}\langle t-r\rangle^{-1/2}, \qquad |I| = i \leq N-2. \tag{3.51}$$

Note that

$$\begin{aligned}\int_0^{2t}(1+t+\rho)^{-1}\langle t-\rho\rangle^{-1/2}\ d\rho &\leq (1+t)^{-1}\int_0^{2t}\langle t-\rho\rangle^{-1/2}\ d\rho\\ &\leq 2(1+t)^{-1}\int_0^t(1+\rho)^{-1/2}\ d\rho\\ &\lesssim (1+t)^{-1/2},\\ \int_{2t}^{\infty}(1+t+\rho)^{-1}\langle t-\rho\rangle^{-1/2}\ d\rho &\lesssim \int_{2t}^{\infty}(1+\rho)^{-3/2}\ d\rho \lesssim (1+t)^{-1/2}.\end{aligned}$$

Thus, by integrating $\partial_r Z^I v(t,\rho\omega)$ from $\rho=t-R$ to $\rho=r$, we conclude that

$$|Z^I v(t)| \leq CB_{0,i+2}^{1/2}\varepsilon t^{-1+C_{0,i+2}\varepsilon/2}, \qquad |I|=i\leq N-2. \tag{3.52}$$

If we let $I=0$ in (3.51) and (3.52), we have

$$|\partial v| \leq CB_{0,2}^{1/2}\varepsilon t^{-3/2+C_{0,2}\varepsilon/2}, \qquad |v| \leq CB_{0,2}^{1/2}\varepsilon t^{-1+C_{0,2}\varepsilon/2}.$$

Note that $|u_{app}|\leq C\varepsilon t^{-1+C\varepsilon}$ and $|\partial u_{app}|\leq C\varepsilon t^{-1}$. This allows us to replace B_0, B_1 with $B_0/2, B_1/2$ in (3.47) as long as we choose $T_{N,A}, B_0, B_1$ sufficiently large and ε sufficiently small (e.g. $CB_{0,2}^{1/2}<B_0/4$, $C<B_0/4$; same for B_1; $T_{N,A}>10$; $C_{0,2}\varepsilon<1/4$).

In the following computation, we will use (3.51) and (3.52) directly instead of (3.47) for the pointwise bounds, so the choice of $C_{k,i}, B_{k,i}$ will be independent of B_0, B_1.

We remark that if $N\geq 6$, (3.51) and (3.52) allow us to extend the solution $v(t,x)$ of (3.44) below $t=T_1$, by the local existence theory of quasilinear wave equations. Moreover, these two pointwise bounds, together with $Z^I u_{app}=O(\varepsilon(1+t)^{-1+C\varepsilon})$, allow us to use Lemma 1.7 freely, as long as $\varepsilon\ll 1$ and $T_{N,A}\gg 1$.

3.4.3 Energy estimate (3.46) with $k=i=0$

Let $k=i=0$ and fix $T_1\leq t\leq 2T$. Now $R_1=R_2=R_3=0$ in (3.49).

For R_4, since $|\chi(t/T)|\leq 1$, we have

$$\begin{aligned}\|R_4\|_{L^2(w)} &\leq \left\|g_0^{\alpha\beta}v\partial_\alpha\partial_\beta u_{app}\right\|_{L^2(w)} + C\left\||v|(|u_{app}|+|v|)|\partial^2 u_{app}|\right\|_{L^2(w)}\\ &\leq C\varepsilon t^{-1}\|\partial v\|_{L^2(w)} + C_N\varepsilon^2 t^{-2+C_N\varepsilon}\left\||v|\langle t-r\rangle^{-1}\right\|_{L^2(w)}\\ &\leq C\varepsilon t^{-1}E_u(v)(t)^{1/2}.\end{aligned}$$

Here we apply Lemma 1.7 in the first inequality, Lemma 3.14 in the second inequality, Lemma 3.12 and (3.50) in the third inequality.

For R_5, since u_{app} is supported in the ball centered at origin with radius $2t$, by Proposition 3.2 we have

$$\|R_5\|_{L^2(w)} \leq C\varepsilon t^{-3/2+C\varepsilon}.$$

Thus, by (3.40), we conclude that

$$\begin{aligned} E_u(v)(t) &\leq \int_t^{2T} C_N\varepsilon\tau^{-1}E_u(v)(\tau) + C\varepsilon\tau^{-3/2+C\varepsilon}E_u(v)(\tau)^{1/2}\ d\tau \\ &\leq \int_t^{2T} C_N B_{0,0}\varepsilon^3\tau^{-2+C_{0,0}\varepsilon} + CB_{0,0}^{1/2}\varepsilon^2\tau^{-2+(C+C_{0,0}/2)\varepsilon}\ d\tau \\ &\leq CC_N B_{0,0}\varepsilon^3 t^{-1+C_{0,0}\varepsilon} + CB_{0,0}^{1/2}\varepsilon^2 t^{-1+(C+C_{0,0}/2)\varepsilon}. \end{aligned}$$

In particular, the constants C do not depend on C_N or $C_{k,i}, B_{k,i}$ in (3.46). If $\varepsilon \ll 1$ (say $CC_N\varepsilon \leq 1/4$) and $C_{0,0}, B_{0,0}$ are large enough (say $C_{0,0}/2 + C < C_{0,0}, C\sqrt{B_{0,0}} < B_{0,0}/4$), we obtain (3.46) with $B_{0,0}$ replaced by $B_{0,0}/2$.

3.4.4 Energy estimate (3.46) with $i = 0$ and $k > 0$

Let $i = 0$ and $k > 0$ and fix $T_1 \leq t \leq 2T$. Now $R_1 = R_3 = 0$.

For R_2, we have

$$\begin{aligned} \|R_2\|_{L^2(w)} &\leq \left\|[g_0^{\alpha\beta}u, \partial^k]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} + \left\|[g^{\alpha\beta}(u) - m^{\alpha\beta} - g_0^{\alpha\beta}u, \partial^k]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} \\ &\leq C \sum_{k_1+k_2\leq k, k_1>0} \left\||\partial^{k_1}u||\partial^{k_2+2}v|\right\|_{L^2(w)} \\ &\quad + C \sum_{k_1+k_2+k_3\leq k, k_3<k} \left\||\partial^{k_1}u||\partial^{k_2}u||\partial^{k_3+2}v|\right\|_{L^2(w)}. \end{aligned}$$

The second sum comes from Lemma 1.7. By writing $u = v + u_{app}$, we have the following terms in the sums:

$$\begin{aligned} &\left\||\partial u_{app}||\partial^{k_2+2}v|\right\|_{L^2(w)} \leq C\varepsilon t^{-1}E_{k,0}(t)^{1/2}, && k_2 < k; \\ &\left\||\partial^{k_1}u_{app}||\partial^{k_2+2}v|\right\|_{L^2(w)} \leq C\varepsilon t^{-1+C\varepsilon}E_{k-1,0}(t)^{1/2}, && k_1 + k_2 \leq k, k_1 > 1; \\ &\left\||\partial^{k_1}v||\partial^{k_2+2}v|\right\|_{L^2(w)} \leq C_N\varepsilon t^{-3/2+C_N\varepsilon}E_{k,0}(t)^{1/2}, && k_1 + k_2 \leq k, k_1 > 0; \\ &\left\||\partial^{k_1}u_{app}||\partial^{k_2}u_{app}||\partial^{k_3+2}v|\right\|_{L^2(w)} \leq C\varepsilon^2 t^{-2+C\varepsilon}E_{k,0}(t)^{1/2}, && k_1 + k_2 + k_3 \leq k, k_3 < k; \\ &\left\||\partial^{k_1}u_{app}||\partial^{k_2}v||\partial^{k_3+2}v|\right\|_{L^2(w)} \leq C_N\varepsilon^2 t^{-2+C_N\varepsilon}E_{k,0}(t)^{1/2}, && k_1 + k_2 + k_3 \leq k, k_3 < k; \\ &\left\||\partial^{k_1}v||\partial^{k_2}v||\partial^{k_3+2}v|\right\|_{L^2(w)} \leq C_N\varepsilon^2 t^{-2+C_N\varepsilon}E_{k,0}(t)^{1/2}, && k_1 + k_2 + k_3 \leq k, k_3 < k. \end{aligned}$$

Here we use Proposition 3.2, (3.51) and (3.52). We take $L^2(w)$ norm on the derivative of v with the highest order, and apply pointwise bounds on the derivatives of u_{app} or derivatives of v with lower orders. Here we need $N/2 + 1 \leq N - 2$, i.e. $N \geq 6$, to apply the pointwise bounds. Thus, we have

$$\|R_2\|_{L^2(w)} \leq C\varepsilon t^{-1}E_{k,0}(t)^{1/2} + C\varepsilon t^{-1+C\varepsilon}E_{k-1,0}(t)^{1/2}.$$

The constants here are universal, as long as we choose $\varepsilon \ll 1$ (say $C_N\varepsilon < 1$) and $T_{N,A}$ sufficiently large (say $C_N/\sqrt{T_{N,A}} \le 1$).

For R_4, since $\partial^l(\chi(t/T)) = O(1)$ for all l, by Lemma 1.7 we have

$$\begin{aligned}\|R_4\|_{L^2(w)} \le{}& C\sum_{k_1\le k}\left\|g_0^{\alpha\beta}\partial^{k_1}v\partial_\alpha\partial_\beta u_{app}\right\|_{L^2(w)} + C\sum_{k_1+k_2\le k,\ k_2>0}\left\||\partial^{k_1}v||\partial^{k_2+2}u_{app}|\right\|_{L^2(w)}\\ &+ C\sum_{k_1+k_2+k_3\le k}\left\||\partial^{k_1}v|(|\partial^{k_2}u_{app}|+|\partial^{k_2}v|)|\partial^{k_3+2}u_{app}|\right\|_{L^2(w)}.\end{aligned}$$

By Lemma 3.14, the first sum has an upper bound

$$C\varepsilon t^{-1}\sum_{k_1\le k}\left\|\partial\partial^{k_1}v\right\|_{L^2(w)} \le C\varepsilon t^{-1}E_{k,0}(t)^{1/2}.$$

By Lemma 3.12, the second sum has an upper bound

$$\begin{aligned}C\varepsilon t^{-1+C\varepsilon}\sum_{k_1<k}\left\|\partial^{k_1}v\langle t-r\rangle^{-2}\right\|_{L^2(w)} &\le C\varepsilon t^{-1+C\varepsilon}\sum_{k_1<k}\left\|\partial\partial^{k_1}v\right\|_{L^2(\mathbb{R}^3)}\\ &\le C\varepsilon t^{-1+C\varepsilon}E_{k-1,0}(t)^{1/2}.\end{aligned}$$

The third sum is controlled by the second one, because $|\partial^{k_2}u_{app}| \le C\varepsilon t^{-1+C\varepsilon} \le 1$, and at least one of $|\partial^{k_1}v|$ and $|\partial^{k_2}v|$ are controlled by $C_N\varepsilon t^{-1+C_N\varepsilon} \le 1$ (since $\min\{k_1,k_2\} \le k/2 \le N-2$). In conclusion,

$$\|R_4\|_{L^2(w)} \le C\varepsilon t^{-1}E_{k,0}(t)^{1/2} + C\varepsilon t^{-1+C\varepsilon}E_{k-1,0}(t)^{1/2}.$$

The constants here are again universal.

For R_5, we have

$$\|R_5\|_{L^2(w)} \le C\varepsilon t^{-3/2+C\varepsilon}.$$

Thus, by (3.40), we have

$$\begin{aligned}E_{k,0}(t) \le{}& \int_t^{2T} C_N\varepsilon(1+\tau)^{-1}E_{k,0}(\tau) + C_N\varepsilon\tau^{-1+C\varepsilon}E_{k-1,0}(\tau)^{1/2}E_{k,0}(\tau)^{1/2}\\ &+ C\varepsilon\tau^{-3/2+C\varepsilon}E_{k,0}(\tau)^{1/2}d\tau\\ \le{}& \int_t^{2T} C_NB_{k,0}\varepsilon^3\tau^{-2+C_{k,0}\varepsilon} + C_NB_{k,0}\varepsilon^3\tau^{-2+(C+C_{k,0}/2+C_{k-1,0}/2)\varepsilon}\\ &+ CB_{k,0}^{1/2}\varepsilon^2\tau^{-2+(C+C_{k,0}/2)\varepsilon}\ d\tau\\ \le{}& CC_NB_{k,0}\varepsilon^3t^{-1+C_{k,0}\varepsilon} + CC_NB_{k,0}\varepsilon^3t^{-1+(C+C_{k,0}/2+C_{k-1,0}/2)\varepsilon}\\ &+ CB_{k,0}^{1/2}\varepsilon^2t^{-1+(C+C_{k,0}/2)\varepsilon}.\end{aligned}$$

Similarly we can prove (3.46) with $B_{k,0}$ replaced by $B_{k,0}/2$, if we assume that $B_{k,0}, C_{k,0}$ are large enough and $\varepsilon \ll 1$ (say $CC_N\varepsilon < 1/8$, $C_{k,0} \ge C_{k-1,0}$, $C\sqrt{B_{k,0}} \le B_{k,0}/8$).

3.4.5 Energy estimate (3.46) with $k = 0$ and $i > 0$

Let $k = 0$ and $i > 0$ and fix $T_1 \le t \le 2T$. Also fix Z^I with $|I| = i$.

For R_2, we have

$$\begin{aligned}\|R_2\|_{L^2(w)} &\le \left\|[g_0^{\alpha\beta}u, Z^I]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} + \left\|[g^{\alpha\beta}(u) - m^{\alpha\beta} - g_0^{\alpha\beta}u, Z^I]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} \\ &\le C \sum_{|J_1|+|J_2|\le i,\ |J_1|>0} \left\||Z^{J_1}u||\partial^2 Z^{J_2}v|\right\|_{L^2(w)} \\ &\quad + C \sum_{|J_1|+|J_2|+|J_3|\le i,\ |J_3|<i} \left\|Z^{J_1}uZ^{J_2}u\partial^2 Z^{J_3}v\right\|_{L^2(w)}.\end{aligned}$$

The second sum comes from Lemma 1.7. Note that the second sum is controlled by the first sum. In fact, since $|J_1|, |J_2|$ cannot be greater than $i/2$ at the same time, without loss of generality we assume $|J_1| \le i/2 \le N - 2$. Thus $|Z^{J_1}u| \le C_N\varepsilon t^{-1+C_N\varepsilon} \le 1$ by (3.52) if we choose $\varepsilon \ll 1$. For the first sum, by writing $u = v + u_{app}$, we have the following terms in the sum:

$$\begin{aligned}&\left\||Z^{J_1}u_{app}||\partial^2 Z^{J_2}v|\right\|_{L^2(w)}, \qquad |J_1| + |J_2| \le i, |J_1| > 0; \\ &\left\||Z^{J_1}v||\partial^2 Z^{J_2}v|\right\|_{L^2(w)}, \qquad |J_1| + |J_2| \le i, |J_1| > 0.\end{aligned}$$

The first term has an upper bound

$$C\varepsilon t^{-1+C\varepsilon}E_{1,i-1}(t)^{1/2}.$$

By Lemma 1.4, we can see that the second term is controlled by

$$C\left\||\langle t-r\rangle^{-1}Z^{J_1}v||\partial ZZ^{J_2}v|\right\|_{L^2(w)}, \qquad |J_2| < i.$$

If $|J_1| \le N - 2$, then by (3.51) we have

$$|\langle t-r\rangle^{-1}Z^{J_1}v| \le \langle t-r\rangle^{-1}\int_{t-R}^{r} |\partial_\rho Z^{J_1}(t,\rho\omega)|\ d\rho \le C\left\|\partial Z^{J_1}v(t)\right\|_{L^\infty(\mathbb{R}^3)} \le C_N\varepsilon t^{-3/2+C_N\varepsilon},$$

which implies that

$$C\left\||\langle t-r\rangle^{-1}Z^{J_1}v||\partial ZZ^{J_2}v|\right\|_{L^2(w)} \le CC_N t^{-3/2+C_N\varepsilon}E_{0,i}(t)^{1/2}.$$

If $|J_1| \ge N - 1$, then $|J_2| \le 1$. In this case, by (3.51), (3.50) and Lemma 3.12, we have

$$|\partial ZZ^{J_2}v| \le C_N\varepsilon t^{-3/2+C_N\varepsilon},$$

$$\begin{aligned}\left\|\langle t-r\rangle^{-1}Z^{J_1}v\right\|_{L^2(w)} &\le Ct^{C\varepsilon}\left\|\langle t-r\rangle^{-1}Z^{J_1}v\right\|_{L^2(\mathbb{R}^3)} \le Ct^{C\varepsilon}\left\|\partial Z^{J_1}v\right\|_{L^2(\mathbb{R}^3)} \\ &\le Ct^{C\varepsilon}E_{0,i}(t)^{1/2}.\end{aligned}$$

Thus, the term above is controlled by

$$CC_N\varepsilon t^{-3/2+(C_N+C)\varepsilon}E_{0,i}(t)^{1/2}.$$

For R_3, following the same discussion as above, we have

$$\begin{aligned}\|R_3\|_{L^2(w)} &\leq C\sum_{|J|<i}\left\||u||\partial^2 Z^J v|\right\|_{L^2(w)}\\ &\leq C\varepsilon t^{-1+C\varepsilon}\sum_{|J|<i}\left\|\partial^2 Z^J v\right\|_{L^2(w)} + C\sum_{|J|<i}\left\||v||\partial^2 Z^J v|\right\|_{L^2(w)}\\ &\leq C\varepsilon t^{-1+C\varepsilon}E_{1,i-1}(t)^{1/2} + CC_N\varepsilon t^{-3/2+(C_N+C)\varepsilon}E_{0,i}(t)^{1/2}.\end{aligned}$$

For R_4, since $Z^J(\chi(t/T)) = O(1)$ for all J by finite speed of propagation, we have

$$\begin{aligned}\|R_4\|_{L^2(w)} \leq{}& C\sum_{|J|\leq i}\left\|g_0^{\alpha\beta}Z^J v\partial_\alpha\partial_\beta u_{app}\right\|_{L^2(w)} + C\sum_{|J_1|+|J_2|\leq i,|J_2|>0}\left\||Z^{J_1}v||\partial^2 Z^{J_2}u_{app}|\right\|_{L^2(w)}\\ &+ C\sum_{|J_1|+|J_2|+|J_3|\leq i}\left\||Z^{J_1}v|(|Z^{J_2}v| + |Z^{J_2}u_{app}|)|\partial^2 Z^{J_3}u_{app}|\right\|_{L^2(w)}\\ \leq{}& C\varepsilon t^{-1}E_{0,i}(t)^{1/2} + C\varepsilon t^{-1+C\varepsilon}E_{0,i-1}(t)^{1/2}.\end{aligned}$$

The proof is very similar to the proof on estiamte of R_4 in the case $i=0$ and $k>0$.

For R_5, again we have

$$\|R_5\|_{L^2(w)} \leq C\varepsilon t^{-3/2+C\varepsilon}.$$

For R_1, we have

$$\begin{aligned}|[\Box, Z^I]v| &\lesssim \sum_{|J_1|+|J_2|<i}|Z^{J_1}\Box Z^{J_2}v| \lesssim \sum_{|J|<i}|Z^J\Box v|\\ &\lesssim \sum_{|J|<i}|Z^J(g^{\alpha\beta}(u)\partial_\alpha\partial_\beta v)| + |Z^J((g^{\alpha\beta}(u) - m^{\alpha\beta})\partial_\alpha\partial_\beta v)|\\ &\lesssim \sum_{|J|<i}|Z^J(\chi(t/T)g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u_{app})| + |Z^J((g^{\alpha\beta}(u) - m^{\alpha\beta})\partial_\alpha\partial_\beta v)|.\end{aligned}$$

Here all the constants are universal which depend only on i, N. The first term is simply $R_4 + R_5$ with a lower order I. The second term can be controlled in the same way as we control R_2, R_3. In conclusion,

$$\begin{aligned}
E_{0,i}(t) &\leq \int_t^{2T} CC_N\varepsilon\tau^{-1}E_{0,i}(\tau) + CC_N\varepsilon\tau^{-1+C\varepsilon}E_{1,i-1}(\tau)^{1/2}E_{0,i}(\tau)^{1/2} \\
&\quad + C\varepsilon\tau^{-3/2+C\varepsilon}E_{0,i}(\tau)^{1/2}\ d\tau \\
&\leq \int_t^{2T} CC_N B_{0,i}\varepsilon^3\tau^{-2+C_{0,i}\varepsilon} + CC_N B_{0,i}\varepsilon^3\tau^{-2+(C+C_{1,i-1}/2+C_{0,i}/2)\varepsilon} \\
&\quad + CB_{0,i}^{1/2}\varepsilon^2\tau^{-2+(C+C_{0,i}/2)\varepsilon}\ d\tau \\
&\leq CC_N B_{0,i}\varepsilon^3 t^{-1+C_{0,i}\varepsilon} + CC_N B_{0,i}\varepsilon^3 t^{-1+(C+C_{0,i}/2+C_{1,i-1}/2)\varepsilon} \\
&\quad + CB_{0,i}^{1/2}\varepsilon^2 t^{-1+(C+C_{0,i}/2)\varepsilon}.
\end{aligned}$$

Again, we can choose $B_{0,i}, C_{0,i}$ sufficiently large such that (3.46) holds with $B_{0,i}$ replaced by $B_{0,i}/2$. Note that $B_{1,i-1}, C_{1,i-1}$ are already chosen when we consider the case $k = 0, i > 0$.

3.4.6 Energy estimate (3.46) with $k, i > 0$

Let $k, i > 0$ and fix $T_1 \leq t \leq 2T$. Also fix Z^I with $|I| = i$. This case can be viewed as a combination of the case $k = 0, i > 0$ and the case $i = 0, k > 0$.

For R_2, we have

$$\begin{aligned}
\|R_2\|_{L^2(w)} &\leq \left\|[g_0^{\alpha\beta}u, \partial^k Z^I]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} + \left\|[g^{\alpha\beta}(u) - m^{\alpha\beta} - g_0^{\alpha\beta}u, \partial^k Z^I]\partial_\alpha\partial_\beta v\right\|_{L^2(w)} \\
&\leq C \sum_{k_1+k_2\leq k, |J_1|+|J_2|\leq i, k_1+|J_1|>0} \left\||\partial^{k_1}Z^{J_1}u||\partial^{2+k_2}Z^{J_2}v|\right\|_{L^2(w)} \\
&\quad + C \sum_{\substack{k_1+k_2+k_3\leq k \\ |J_1|+|J_2|+|J_3|\leq i \\ k_3+|J_3|<k+i}} \left\|\partial^{k_1}Z^{J_1}u\partial^{k_2}Z^{J_2}u\partial^{2+k_3}Z^{J_3}v\right\|_{L^2(w)}.
\end{aligned}$$

The second sum is again easy to handle. For the first sum, we consider the following three cases: $k_1 = 0$ and $|J_1| > 0$; $k_1 = 1$ and $|J_1| = 0$; all the remaining choices of (k, J_1). For the first case, we apply Proposition 3.2 and Lemma 1.4 to obtain a factor $\langle t - r\rangle^{-1}$ with one ∂ replaced by Z; for the second case, we use $|\partial u| \leq C_N\varepsilon t^{-1}$; for the third, we use (3.51) directly. The proof here is very similar to the proof in the previous cases. We thus have

$$\|R_2\|_{L^2(w)} \leq C_N\varepsilon t^{-1}E_{k,i}(t)^{1/2} + C_N\varepsilon t^{-1+C\varepsilon}(E_{k-1,i}(t)^{1/2} + E_{k+1,i-1}(t)^{1/2}).$$

For R_3, we have

$$\|R_3\|_{L^2(w)} \leq C \sum_{k_1\leq k, |J|<i} \left\||u||\partial^{k_1+2}Z^J v|\right\|_{L^2(w)}.$$

We can use Proposition 3.2 and Lemma 1.4 to obtain

$$\|R_3\|_{L^2(w)} \leq C_N\varepsilon t^{-1}E_{k,i}(t)^{1/2} + C_N\varepsilon t^{-1+C\varepsilon}(E_{k-1,i}(t)^{1/2} + E_{k+1,i-1}(t)^{1/2}).$$

For R_4, we have

$$\begin{aligned}\|R_4\|_{L^2(w)} \leq{}& C\sum_{k_1\leq k,|J|\leq i}\left\|g_0^{\alpha\beta}\partial^{k_1}Z^Jv\partial_\alpha\partial_\beta u_{app}\right\|_{L^2(w)}\\ &+C\sum_{k_1+k_2\leq k,|J_1|+|J_2|\leq i,k_2+|J_2|>0}\left\||\partial^{k_1}Z^{J_1}v||\partial^{k_2+2}Z^{J_2}u_{app}|\right\|_{L^2(w)}\\ &+C\sum_{\substack{k_1+k_2+k_3\leq k\\ |J_1|+|J_2|+|J_3|\leq i\\ k_3+|J_3|<k+i}}\left\||\partial^{k_1}Z^{J_1}v|(|\partial^{k_2}Z^{J_2}v|+|\partial^{k_2}Z^{J_2}u_{app}|)|\partial^{k_3+2}Z^{J_3}u_{app}|\right\|_{L^2(w)}\\ \leq{}& C_N\varepsilon t^{-1}E_{k,i}(t)^{1/2} + C_N\varepsilon t^{-1+C\varepsilon}(E_{k-1,i}(t)^{1/2} + E_{k+1,i-1}(t)^{1/2}).\end{aligned}$$

This can be handled in the same way as we handle R_4 in the case $k=0, i>0$ or $i=0, k>0$.

For R_5, again we have

$$\|R_5\|_{L^2(w)} \leq C\varepsilon t^{-3/2+C\varepsilon}.$$

For R_1, since $[\Box,\partial^kZ^I] = \partial^k[\Box, Z^I]$, we can conclude that the $L^2(w)$ norm of R_1 can be controlled by the bounds of the $L^2(w)$ norms of all other R_i.

In conclusion, we have

$$\begin{aligned}E_{k,i}(t) \leq{}& \int_t^{2T} CC_N\varepsilon\tau^{-1}E_{k,i}(\tau) + CC_N\varepsilon\tau^{-1+C\varepsilon}(E_{k-1,i}(\tau)^{1/2} + E_{k+1,i-1}(\tau)^{1/2})E_{k,i}(\tau)^{1/2}\\ &+ C\varepsilon\tau^{-3/2+C\varepsilon}E_{k,i}(\tau)^{1/2}\ d\tau\\ \leq{}& \int_t^{2T} CC_NB_{k,i}\varepsilon^3\tau^{-2+C_{k,i}\varepsilon} + CC_NB_{k,i}\varepsilon^3\tau^{-2+(C+C_{k+1,i-1}/2+C_{k-1,i}/2+C_{k,i}/2)\varepsilon}\\ &+ CB_{k,i}^{1/2}\varepsilon^2\tau^{-2+(C+C_{k,i}/2)\varepsilon}\ d\tau\\ \leq{}& CC_NB_{k,i}\varepsilon^3t^{-1+C_{k,i}\varepsilon} + CC_NB_{k,i}\varepsilon^3t^{-1+(C+C_{k+1,i-1}/2+C_{k-1,i}/2+C_{k,i}/2)\varepsilon}\\ &+ CB_{k,i}^{1/2}\varepsilon^2t^{-1+(C+C_{k,i}/2)}.\end{aligned}$$

Again, we can choose $B_{k,i}, C_{k,i}$ sufficiently large such that (3.46) holds with $B_{k,i}$ replaced by $B_{k,i}/2$. Note that $B_{k+1,i-1}, C_{k+1,i-1}, B_{k-1,i}, C_{k-1,i}$ are already chosen when we consider the case $k, i > 0$.

3.4.7 Existence for $0 \leq t \leq T_{N,A}$

In the previous subsections, we prove that there exists a solution v to (3.44) for all $t \geq T_{N,A}$ with (3.45) hold for all $|I| \leq N$ and $t \geq T_{N,A}$. Now we finish the proof of Proposition 3.15

by extending the solution to all $t \geq 0$. At a small time, u_{app} does not approximate u well, but u_{app} and all its derivatives stay bounded for all (t,x) with $0 \leq t \leq T_{N,A}$. See Proposition 3.2. So, it is better to use (1.1) to control u directly instead of using (3.44).

Fix $N \geq 6$. By using the pointwise bounds in Proposition 3.2 and the support of u_{app}, we have

$$\left\|Z^I u_{app}(t)\right\|_{L^2(\mathbb{R}^3)} \lesssim_{I,N,R} \varepsilon, \qquad 0 \leq t \leq T_{N,A}.$$

Thus, it suffices to prove that the solution u to (1.1) with $u = v + u_{app}$ for $t \geq T_{N,A}$ exists for $0 \leq t \leq T_{N,A}$, with

$$\left\|\partial Z^I u(t)\right\|_{L^2(\mathbb{R}^3)} \lesssim_{I,N,R} \varepsilon, \qquad 0 \leq t \leq T_{N,A},\ |I| \leq N.$$

If we apply Z^I to (1.1), we have

$$g^{\alpha\beta}(u)\partial_\alpha\partial_\beta Z^I u = [\Box, Z^I]u + [g^{\alpha\beta}(u) - m^{\alpha\beta}, Z^I]\partial_\alpha\partial_\beta u + (g^{\alpha\beta}(u) - m^{\alpha\beta})[\partial_\alpha\partial_\beta, Z^I]u. \tag{3.53}$$

We can now set up the continuity argument. Suppose that we have a solution u to (1.1) for $T_1 \leq t \leq T_{N,A}$ for some $0 \leq T_1 \leq T_{N,A}$, such that

$$\left\|\partial Z^I u(t)\right\|_{L^2(\mathbb{R}^3)} \leq B\varepsilon, \qquad |I| \leq N, T_1 \leq t \leq T_{N,A}. \tag{3.54}$$

Here $B = B_N$ depends on N. We remark that (3.54) implies (3.45) for $t \leq T_{N,A}$, where the power is the same but the constant in $\lesssim_I$ now depends on N. This is because $1 \lesssim_N t^{-1/2+C_I\varepsilon}$ for $t \leq T_{N,A}$, assuming $\varepsilon \ll 1$.

By the Klainerman-Sobolev inequality, we conclude that for $t \geq T_1$

$$|\partial Z^I u(t,x)| \leq CB\varepsilon(1+t+r)^{-1}\langle t-r\rangle^{-1/2}, \qquad |I| \leq N-2$$

and

$$|Z^I u(t,x)| \leq CB\varepsilon(1+t)^{-1/2}, \qquad |I| \leq N-2.$$

The proof of the second estimate is similar to that of (3.52). Thus, assuming $\varepsilon \ll 1$, from (3.53) we have for $|I| \leq N$

$$\begin{aligned} |g^{\alpha\beta}(u)\partial_\alpha\partial_\beta Z^I u| &\leq C \sum_{|J|+|K|\leq|I|,|K|<|I|} |Z^J u||\partial^2 Z^K u| \\ &\leq C \sum_{|J|+|K|\leq|I|,|K|<|I|} \langle t-r\rangle^{-1}|Z^J u||\partial Z Z^K u| \\ &\leq C_N\varepsilon \sum_{|J|\leq|I|} (|\partial Z^J u| + \langle t-r\rangle^{-1}|Z^J u|). \end{aligned}$$

Here we apply Lemma 1.7 in the first inequality and the pointwise bounds in the third one. Note that if $|J| + |K| \leq |I|$ and $|K| < |I|$, then $\min\{|J|, |K|+1\} \leq N/2+1 \leq N-2$ when $N \geq 6$.

Now we can use the standard energy estimates, say Proposition 2.1 in Chapter I in Sogge [30] or Proposition 6.3.2 in Hörmander [7]. We apply the Poincaré's lemma, i.e. Lemma 3.12, to $\langle t-r\rangle^{-1}|Z^J u|$, so its $L^2(\mathbb{R}^3)$ norm is controlled by the that of $|\partial Z^J u|$. By setting

$$E_N(t) = \sum_{|I|\leq N} \left\|\partial Z^I v(t)\right\|^2_{L^2(\mathbb{R}^3)},$$

for small $\varepsilon \ll 1$, we have

$$\begin{aligned} E_N(t)^{1/2} &\leq 2(E_N(T_{N,A})^{1/2} + C_N\varepsilon \int_t^{T_{N,A}} E_N(\tau)^{1/2}\, d\tau) \exp(\int_t^{T_{N,A}} C_N\varepsilon\, d\tau) \\ &\leq C_N\varepsilon + C_N B^{1/2}\varepsilon^2. \end{aligned}$$

Then by choosing ε small enough and B large enough, both depending on N, we can replace B with $B/2$ in (3.54). We are done.

Finally, we remark that for each $|I| \leq N$ and $\varepsilon \ll 1$, we can apply Proposition 3.15 with N replaced by $N' = \max\{6, |I|\} \leq N$. Note that when $\varepsilon < \varepsilon_N \leq \varepsilon_{N'}$ and $T > T_{N,A} \geq T_{N',R}$, the solution for N and the solution for N' are exactly the same. But the constants in (3.45) now depend on $\max\{6, |I|\}$ instead of N. This allows us to remove the dependence of N in the coefficients of (3.45).

3.5 Limit as $T \to \infty$

Our goal for this section is to prove the following proposition.

Proposition 3.16. *Fix $N \geq 6$. Then for the same ε_N in Proposition 3.15 and for $0 < \varepsilon < \varepsilon_N$, there is a solution u to (1.1) in C^{N-4} for all $t \geq 0$, such that for all $|I| \leq N-5$*

$$\left\|\partial Z^I (u - u_{app})(t)\right\|_{L^2(\mathbb{R}^3)} \lesssim_I \varepsilon(1+t)^{-1/2+C_I\varepsilon}, \qquad t \geq 0. \tag{3.55}$$

Besides, for all $|I| \leq N-5$ and $t \gg_A 1$,

$$|\partial Z^I(u - u_{app})(t,x)| \lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}\langle r+t\rangle^{-1}\langle t-r\rangle^{-1/2}, \tag{3.56}$$

$$|Z^I(u - u_{app})(t,x)| \lesssim_I \min\{\varepsilon t^{-1+C_I\varepsilon}, \varepsilon t^{-3/2+C_I\varepsilon}\langle r-t\rangle\}. \tag{3.57}$$

It should be pointed out that the value of "N" in Theorem 3.1 is equal to $N-4$ for the N in this proposition.

From now on, the constant C is allowed to depend on all the constants in the previous sections (say $C_{k,i}, B_{k,i}, N$), but it must be independent of ε and T.

3.5.1 Existence of the limit

Fix $N \geq 6$ and $T_2 > T_1 \gg 1$. By Proposition 3.15, for each $0 < \varepsilon < \varepsilon_N$, we get two corresponding solutions $v_1 = v^{T_1}$ and $v_2 = v^{T_2}$ which exist for all $t \geq 0$. Our goal now is to prove that $v_1 - v_2$ tends to 0 in some Banach space as $T_2 > T_1 \to \infty$.

Recall that $\varepsilon_N, T_{N,A}$ are independent of the choice of T, as long as $T > T_{N,A}$. In addition, v_1 and v_2 satisfy (3.45), (3.46), (3.47), (3.51) and (3.52), as shown in the continuity argument, for $t \geq T_{N,A}$, and they satisfy (3.54) along with the pointwise bounds for $0 \leq t \leq T_{N,A}$. All the constants involved in these estimates are independent of T. We define $u_1 = v^{T_1} + u_{app}$, $u_2 = v^{T_2} + u_{app}$ and $\widetilde{v} = v^{T_2} - v^{T_1}$. Then, for $t \geq T_1$ and $|I| \leq N$, by (3.45), we have

$$\left\|\partial Z^I \widetilde{v}(t)\right\|_{L^2(\mathbb{R}^3)} \leq \left\|\partial Z^I v_1(t)\right\|_{L^2(\mathbb{R}^3)} + \left\|\partial Z^I v_2(t)\right\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^2 T_1^{-1+C\varepsilon}.$$

In addition, for $t \leq T_1$ (now $\chi(t/T_1) = \chi(t/T_2) = 1$) and for each $|I| \leq N$, we have

$$\begin{aligned} g^{\alpha\beta}(u_1)\partial_\alpha\partial_\beta Z^I \widetilde{v} = {} & [\Box, Z^I]\widetilde{v} + [g^{\alpha\beta}(u_1) - m^{\alpha\beta}, Z^I]\partial_\alpha\partial_\beta\widetilde{v} + [g^{\alpha\beta}(u_2) - g^{\alpha\beta}(u_1), Z^I]\partial_\alpha\partial_\beta v_2 \\ & + (g^{\alpha\beta}(u_1) - m^{\alpha\beta})[\partial_\alpha\partial_\beta, Z^I]\widetilde{v} + (g^{\alpha\beta}(u_2) - g^{\alpha\beta}(u_1))[\partial_\alpha\partial_\beta, Z^I]v_2 \\ & - Z^I((g^{\alpha\beta}(u_2) - g^{\alpha\beta}(u_1))\partial_\alpha\partial_\beta u_{app}) - (g^{\alpha\beta}(u_2) - g^{\alpha\beta}(u_1))\partial_\alpha\partial_\beta Z^I v_2. \end{aligned} \tag{3.58}$$

Define a new energy

$$\widetilde{E}_{k,i}(t) := \sum_{l\leq k, |I|\leq i} E_{u_1}(\partial^l Z^I \widetilde{v})(t).$$

Here E_{u_1} is defined in (3.38) with u replaced by u_1. For $k + i \leq N - 3$ with $|I| = i$, and for $t \geq T_{N,A}$ we have

$$\left\|g^{\alpha\beta}(u_1)\partial_\alpha\partial_\beta\partial^k Z^I\widetilde{v}\right\|_{L^2(w)} \leq C\varepsilon t^{-1}\widetilde{E}_{k,i}(t)^{1/2} + C\varepsilon t^{-1+C\varepsilon}(\widetilde{E}_{k-1,i}(t)^{1/2} + \widetilde{E}_{k+1,i-1}(t)^{1/2}) \tag{3.59}$$

with $\widetilde{E}_{-1,\cdot} = \widetilde{E}_{\cdot,-1} = 0$. This is a simple application of Lemma 1.4, Lemma 1.7 and the estimates for u_1, v_1, u_2, v_2. We skip the detail of the proof here, since it is very similar to the proof of (3.48) on $E_{k,i}$. However, we should always put $L^2(w)$ norm on the terms involving $\widetilde{v}$ and put L^∞ norm on terms involving u_1, u_2, v_1, v_2. The pointwise bounds only holds for $|I| \leq N - 2$, as seen in (3.51) and (3.52), so we need to assume $k + i \leq N - 3$ instead of $k + i \leq N$ above. Besides, there is no term like R_5 in the previous section, so we expect $\widetilde{E}_{k,i}$ to have a better decay than $E_{k,i}$.

Since (3.51) and (3.52) hold for v_1, we can apply energy estimate (3.40) for E_{u_1}. Thus, for all $T_{N,A} \leq t \leq T_1$ and for $k + i \leq N - 3$,

$$\begin{aligned} \widetilde{E}_{k,i}(t) \leq {} & C\varepsilon^2 T_1^{-1+C\varepsilon} + B\int_t^{T_1} \varepsilon\tau^{-1}\widetilde{E}_{k,i}(\tau)\ d\tau \\ & + C\varepsilon\int_t^{T_1} \tau^{-1+C\varepsilon}(\widetilde{E}_{k-1,i}(\tau)^{1/2} + \widetilde{E}_{k+1,i-1}(\tau)^{1/2})\widetilde{E}_{k,i}(\tau)^{1/2}\ d\tau. \end{aligned}$$

Using this estimate, we claim that $\widetilde{E}_{k,i}(t) \leq C\varepsilon^2 T_1^{-1+C\varepsilon}$ for all $k+i \leq N-3$. Here C may depend on k,i. To prove this claim, we first induct on $i=0,1,\dots,N$ and then on $k=0,\dots,N-3-i$ for each fixed i. If we fix (k,i) and let $V(t)=V_{k,i}(t)$ be the right hand side, then we have

$$\begin{aligned} dV/dt &= -B\varepsilon t^{-1}\widetilde{E}_{k,i}(t) - C\varepsilon t^{-1+C\varepsilon}(\widetilde{E}_{k-1,i}(t)^{1/2} + \widetilde{E}_{k+1,i-1}(t)^{1/2})\widetilde{E}_{k,i}(t)^{1/2} \\ &\geq -B\varepsilon t^{-1}V(t) - C\varepsilon t^{-1+C\varepsilon}(\widetilde{E}_{k-1,i}(t)^{1/2} + \widetilde{E}_{k+1,i-1}(t)^{1/2})V(t)^{1/2}. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d}{dt}(t^{B\varepsilon/2}\sqrt{V}) &= \frac{1}{2}B\varepsilon t^{-1+B\varepsilon/2}\sqrt{V} + t^{B\varepsilon/2}\frac{dV/dt}{2\sqrt{V}} \\ &= \frac{1}{2\sqrt{V}}t^{B\varepsilon/2}(B\varepsilon t^{-1}V + dV/dt) \\ &\geq -C\varepsilon t^{-1+(C+B/2)\varepsilon}(\widetilde{E}_{k-1,i}(t)^{1/2} + \widetilde{E}_{k+1,i-1}(t)^{1/2}) \\ &\geq -C\varepsilon^2 t^{-1+C\varepsilon}T_1^{-1+C\varepsilon}. \end{aligned}$$

The last line holds by induction hypothesis. We then have

$$\begin{aligned} t^{B\varepsilon/2}\sqrt{V(t)} &\leq T_1^{B\varepsilon/2}\sqrt{V(T_1)} + \int_t^{T_1} C\varepsilon^2\tau^{-1+C\varepsilon}T_1^{-1/2+C\varepsilon}\ d\tau \\ &\leq C\varepsilon T_1^{-1/2+C\varepsilon}, \end{aligned}$$

and thus for all $t \geq T_{N,A}$, we have

$$\widetilde{E}_{k,i}(t) \leq V(t) \leq t^{B\varepsilon}V(t) \leq C\varepsilon^2 T_1^{-1+C\varepsilon}.$$

Here C in different places may denote different values.

For $0 \leq t \leq T_{N,A}$, we can also prove that

$$\left\|\partial Z^I(v_2-v_1)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C_N\varepsilon T_1^{-1/2+C_N\varepsilon}.$$

The proof is very similar to the proof in Section 3.4. We can use the equation

$$g^{\alpha\beta}(u_1)\partial_\alpha\partial_\beta(u_2-u_1) = -(g^{\alpha\beta}(u_2)-g^{\alpha\beta}(u_1))\partial_\alpha\partial_\beta u_2$$

and apply the standard energy estimates to establish the continuity argument. Again, we can remove the dependence of N in the constants, using the same argument in Section 3.4.

By (3.50), for each $|I| \leq N-3$ and $\varepsilon \ll 1$, we have

$$\sup_{t\geq 0}\left\|\partial Z^I(v_2-v_1)(t)\right\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon T_1^{-1/2+C\varepsilon} \to 0$$

as $T_2 > T_1 \to \infty$. By the Klainerman-Sobolev inequality and

$$\int_0^\infty (1+t+\rho)^{-1}\langle t-\rho\rangle^{-1/2}\ d\rho \lesssim (1+t)^{-1/2},$$

for all $|I| \leq N-5$, we have

$$\sup_{t\geq 0,\ x\in\mathbb{R}^3} |\partial Z^I(v_2-v_1)(t,x)| \leq C\varepsilon T_1^{-1/2+C\varepsilon} \to 0$$
$$\sup_{t\geq 0,\ x\in\mathbb{R}^3} |Z^I(v_2-v_1)(t,x)| \leq C\varepsilon T_1^{-1/2+C\varepsilon} \to 0,$$

as $T_2 > T_1 \to \infty$. Then, there is $v^\infty \in C^{N-4}(\{(t,x):\ t\geq 0\})$, such that $\partial Z^I v^T \to \partial Z^I v^\infty$ and $Z^I v^T \to Z^I v^\infty$ pointwisely for $t \geq 0$ as $T \to \infty$, for each $|I| \leq N-5$. It is clear that the pointwise bounds (3.51) and (3.52) also hold for v^∞ for $|I| \leq N-5$. By Fatou's lemma, for each $|I| \leq N-5$ we have

$$\left\|\partial Z^I v^\infty(t)\right\|_{L^2(\mathbb{R}^3)} \leq \liminf_{T\to\infty} \left\|\partial Z^I v^T(t)\right\|_{L^2(\mathbb{R}^3)} \leq C_I\varepsilon(1+t)^{-1/2+C_I\varepsilon}. \tag{3.60}$$

Meanwhile, if $N \geq 6$, then by taking $T \to \infty$ in

$$g^{\alpha\beta}(u_{app}+v^T)\partial_\alpha\partial_\beta v^T = -\chi(t/T)g^{\alpha\beta}(u_{app}+v^T)\partial_\alpha\partial_\beta u_{app},$$

we conclude that $u^\infty := v^\infty + u_{app}$ is a solution to (1.1) for $t \geq 0$.

3.5.2 End of the proof of Theorem 3.1

For $t \geq T_A \gg 1$ and $t \leq 3t/2$, we have $u_{app} = \varepsilon r^{-1}U$ if $r \leq 5t/4$, and $\partial^k Z^I(u_{app}, \varepsilon r^{-1}U) = O(\varepsilon t^{-k-1+C\varepsilon})$ if $t/4 \leq r-t \leq t/2$. See the proof of Proposition 3.2. Thus,

$$|\partial Z^I(u_{app}-\varepsilon r^{-1}U)(t,x)|\chi_{|x|\leq 3t/2} \lesssim_I \varepsilon t^{-2+C_I\varepsilon}$$

and

$$\begin{aligned}
&\left\|\partial Z^I(u_{app}-\varepsilon r^{-1}U)(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ |x|\leq 3t/2\})}\\
&= \left\|\partial Z^I((1-\psi(r/t))\varepsilon r^{-1}U)(t)\right\|_{L^2(\{x\in\mathbb{R}^3:\ 5t/4\leq|x|\leq 3t/2\})}\\
&\lesssim_I \varepsilon t^{-2+C_I\varepsilon}\cdot|\{x\in\mathbb{R}^3:\ 5t/4\leq|x|\leq 3t/2\}|^{1/2}\\
&\lesssim_I \varepsilon t^{-1/2+C_I\varepsilon}.
\end{aligned}$$

These two bounds allows us to get the estimates in Theorem 3.1 from (3.55), (3.56) and (3.57), since

$$u-u_{app} = (u-\varepsilon r^{-1}U)\chi_{|x|\leq 3t/2} - (u_{app}-\varepsilon r^{-1}U)\chi_{|x|\leq 3t/2} + u\chi_{|x|>3t/2}.$$

We also remark that starting from the estimates in Theorem 3.1, we can also derive (3.55), (3.56) and (3.57), using the essentially same derivation here.

By (3.60), for $t \gg_A 1$, we have

$$|(\partial_t-\partial_r)(u^\infty-u_{app})(t,x)| \lesssim \varepsilon t^{-1/2+C_I\varepsilon}(1+t+r)^{-1}.$$

Since $\psi(r/t) = 0$ unless $t \sim r$, for $t \gg_A 1$ we have

$$\begin{aligned}(\partial_t - \partial_r)u_{app} &= (\partial_t - \partial_r)(\varepsilon r^{-1}\psi(r/t)U)\\ &= -\varepsilon r^{-2}\psi U + \varepsilon r^{-1}\psi\mu U_q + \varepsilon^2 r^{-1}t^{-1}\psi U_s\\ &\quad + \varepsilon r^{-1}t^{-2}(t-r)\psi' U\\ &= -2\varepsilon r^{-1}\psi A + O(\varepsilon t^{-2+C\varepsilon}).\end{aligned}$$

When $r \leq 5t/4$, we have $\psi = 1$ or $A = 0$, so here we have $\psi A = A$. When $r > 5t/4$, we have

$$|(1-\psi(r/t))A(q(t,r,\omega),\omega)| \lesssim \langle q(t,r,\omega)\rangle^{-1-\gamma} \lesssim (t+r)^{C\varepsilon}\langle r-t\rangle^{-1-\gamma} \lesssim (t+r)^{-1-\gamma+C\varepsilon}.$$

Here we apply Lemma 3.3 and we note that $\langle r-t\rangle \sim r \sim (t+r)$ if $r \geq 5t/4$. In summary, for all $t \gg_A 1$, we have

$$|(\partial_t - \partial_r)u^\infty + \frac{2\varepsilon}{r}A(q(t,r,\omega),\omega)| \lesssim \varepsilon t^{-3/2+C\varepsilon}. \tag{3.61}$$

This finishes the proof of part (iii) in Theorem 3.1.

3.5.3 Uniqueness

Now we give a brief proof of the uniqueness statement given in the remark of Theorem 3.1. It suffices to prove the uniqueness of Proposition 3.16, assuming $N \geq 11$ and $\varepsilon \ll 1$. This is because (3.55), (3.56) and (3.57) are equivalent to the estimates in the main theorem, even if we replace 5/4 with a fixed constant $\kappa > 1$. We refer to Section 3.5.2 for the proof.

Now, suppose we have two C^{N-4} solutions u_1, u_2 constructed in Proposition 3.16. Fix $T \gg 1$. We can prove that $\|\partial Z^I(u_1 - u_2)(t)\| \lesssim \varepsilon T^{-1/2+C\varepsilon}$ for all $t \geq 0$ and $|I| \leq N - 10$. Here the constants are independent of T. The proof is essentially the same as that in Section 3.5.1. Let $T \to \infty$ and we get $u_1 \equiv u_2$.

Chapter 4

Asymptotic Completeness

4.1 Introduction

In this chapter, our main goal is to prove the asymptotic completeness for our model equation. For a fixed global solution u constructed in Lindblad [21], we seek to find the corresponding asymptotic profile and scattering data.

We start the proof with construction of a global optical function $q = q(t,x)$. In other words, we solve the eikonal equation $g^{\alpha\beta}(u)q_\alpha q_\beta = 0$ in a spacetime region Ω contained in $\{2r \geq t \geq \exp(\delta/\varepsilon)\}$. Here $\delta > 0$ is a fixed parameter. We apply the method of characteristics and then follow the idea in Christodoulou-Klainerman [4]. By viewing $(g_{\alpha\beta})$, the inverse of the coefficient matrix $(g^{\alpha\beta}(u))$, as a Lorentzian metric in $[0,\infty) \times \mathbb{R}^3$, we construct a null frame $\{e_k\}_{k=1}^4$ in Ω. Then, most importantly, we define χ_{ab} for $a, b = 1, 2$ which are related to the Levi-Civita connection and the null frame under the metric $(g_{\alpha\beta})$. By studying the Raychaudhuri equation and using a continuity argument, we can show that the $\mathrm{tr}\chi > 0$ everywhere. This is the key step. In addition, we can prove that $q = q(t,x)$ is smooth in some weak sense (see Section 4.2.1). We refer our readers to Section 4.3 and Section 4.4 for more details in the proof.

Next, we define $(\mu, U)(t,x) := (q_t - q_r, \varepsilon^{-1} r u)(t,x)$. The map

$$\Omega \to [0,\infty) \times \mathbb{R} \times \mathbb{S}^2 : \qquad (t,x) \mapsto (\varepsilon \ln t - \delta, q(t,x), x/|x|) := (s, q, \omega)$$

is an invertible smooth function with a smooth inverse, so a function $(\mu, U)(s,q,\omega)$ is obtained. It can be proved that $(\mu, U)(s,q,\omega)$ is an approximate solution to the reduced system (2.4), and that there is an exact solution $(\widetilde{\mu}, \widetilde{U})(s,q,\omega)$ to(2.4) which matches $(\mu, U)(s,q,\omega)$ as $s \to \infty$. A key step is to prove that $A(q,\omega) := -\frac{1}{2}\lim_{s\to\infty}(\mu U_q)(s,q,\omega)$ is well-defined for each (q,ω). The function A is called the *scattering data* in this chapter. We also show a gauge independence result, which states that the scattering data is independent of the choice of the optical function q in a suitable sense. We refer our readers to Section 4.5 and Section 4.6.

Finally, we construct an approximate solution $\widetilde{u}$ to (1.1) in Ω. The construction here is similar to that in Section 4 of [34], or in Section 3.2 in this dissertation. That is, we construct a function $\widetilde{q}$ by solving

$$\widetilde{q}_t - \widetilde{q}_r = \mu(\varepsilon \ln t - \delta, \widetilde{q}(t,x), \omega)$$

by the method of characteristics, and then define

$$\widetilde{u}(t,x) := \varepsilon r^{-1} \widetilde{U}(\varepsilon \ln t - \delta, \widetilde{q}(t,x), \omega).$$

Then, in Ω, $\widetilde{q}$ is an approximate optical function, and $\widetilde{u}$ is an approximate solution to (1.1). In addition, near the light cone $t = r$, the difference $u - \widetilde{u}$, along with its derivatives, decays much faster than $\varepsilon t^{-1+C\varepsilon}$. Since u and its derivatives is of size $O(\varepsilon t^{-1+C\varepsilon})$, we conclude that $\widetilde{u}$ offers a good approximation of u.

A more detailed discussion is given below.

4.1.1 Construction of an optical function

Let $u = u(t,x)$ be a global solution to (1.1) and (1.2) constructed in Lindblad [21]. Here we fix a constant $R > 0$ such that supp $(u_0, u_1) \subset \{|x| \leq R\}$, so we have $u \equiv 0$ for $|x| \geq t + R$ by the finite speed of propagation. Our goal in this section is to construct an optical function, i.e. a solution to the eikonal equation

$$g^{\alpha\beta}(u) q_\alpha q_\beta = 0. \tag{4.1}$$

Here we do not expect to solve (4.1) for all $(t,x) \in \mathbb{R}^{1+3}_{t,x}$. Instead, we solve it in a region $\Omega \subset \mathbb{R}^{1+3}_{t,x}$ which is defined by

$$\Omega := \{(t,x) : \ t > T_0, \ |x| > (t + T_0)/2 + 2R\}.$$

Here $T_0 = \exp(\delta/\varepsilon)$ and $\delta > 0$ is a fixed constant independent of ε. We also assign the initial data by setting $q = r - t$ on $\partial\Omega$. It is then clear that $q = r - t$ in $\Omega \cap \{r - t > R\}$, so from now on we focus on the region $\Omega \cap \{r - t < 2R\}$.

To construct an optical function, we apply the method of characteristics. In fact, the characteristics for (4.1) are the geodesics with respect to the Lorentzian metric $(g_{\alpha\beta})$ which is the inverse of the matrix $(g^{\alpha\beta}(u))$. Moreover, we only need to study those geodesics emanating from the cone

$$H := \partial\Omega \cap \{t > T_0\} = \{(t,x) : \ t > T_0, \ |x| = (t + T_0)/2 + 2R\}.$$

Now we follow the idea in Christodoulou-Klainerman [4]. Fix $T > T_0$ and suppose that the optical function exists in $\Omega_T := \Omega \cap \{t \leq T, r - t \leq 2R\}$. Then, every point in Ω_T can be reached by a unique characteristic emanating from H. We first define a null frame $\{e_k\}_{k=1}^4$ in Ω_T, such that e_4 is tangent to the unique characteristic passing through that point. We then define the second fundamental form of the time slices of the null cones:

$$\chi_{ab} := \langle D_{e_a} e_4, e_b \rangle, \qquad a, b \in \{1, 2\}.$$

Here D is the Levi-Civita connection associated to the Lorentzian metric $(g_{\alpha\beta})$, and $\langle \cdot, \cdot \rangle$ is the bilinear form associated to the metric $(g_{\alpha\beta})$. We now use a continuity argument. Suppose that in Ω_T we have

$$\max_{a,b=1,2} |\chi_{ab} - \delta_{ab} r^{-1}| \leq At^{-2+B\varepsilon}. \tag{4.2}$$

The positive constants A and B are both independent of ε and T. Our goal is to prove that (4.2) holds with A replaced by $A/2$. It follows that $\mathrm{tr}\chi := \chi_{11} + \chi_{22}$, sometimes called the *null mean curvature* of the level sets of q, is positive everywhere, and that the characteristics emanating from H will not intersect with each other. This allows us to extend the optical function to $\Omega_{T+\epsilon}$ for a small $\epsilon > 0$, such that (4.2) holds everywhere in $\Omega_{T+\epsilon}$. We conclude from this continuity argument that the optical function exists everywhere in Ω.

In order to prove that (4.2) holds with A replaced by $A/2$, we make use of the Raychaudhuri equation

$$e_4(\chi_{ab}) = -\sum_{c=1,2} \chi_{ac}\chi_{cb} + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{ab} + \langle R(e_4, e_a)e_4, e_b \rangle,$$

which describes the evolution of χ along the null geodesics foliating the light cones. In this equation, Γ^*_{**}'s are the Christoffel symbols, and $\langle R(X,Y)Z, W \rangle$ is the curvature tensor, both with respect to the Lorentzian metric $(g_{\alpha\beta})$. Note that we have a decomposition

$$\langle R(e_4, e_a)e_4, e_b \rangle = e_4(f_1) + f_2$$

where $f_1 = O(\varepsilon t^{-2+C\varepsilon})$ and $f_2 = O(\varepsilon t^{-3+C\varepsilon})$; see Lemma 4.17 for a more accurate statement. We also refer our readers to Corollary 5.9 in [29] for a similar decomposition of curvature tensors. Moreover, it follows from (1.1) that

$$|e_4(e_3(u)) + r^{-1} e_3(u)| \lesssim \varepsilon A t^{-3+B\varepsilon}, \qquad |e_4(e_3(u))| \lesssim \varepsilon t^{-2}.$$

Combining all these estimates and the Gronwall's inequality, we are able to prove (4.2) with A replaced by $A/2$.

So far, we have constructed a global optical function $q = q(t, x)$ in Ω which is C^2 by the method of characteristics. In fact, the optical function $q = q(t, x)$ is smooth[1] in Ω in the followings sense: for each integer $N \geq 2$, there exists $\varepsilon_N > 0$ such that q is a C^N function in Ω for each $0 < \varepsilon < \varepsilon_N$. Moreover, if Z is one of the commuting vector fields: translations ∂_α, scaling $t\partial_t + r\partial_r$, rotations $x_i\partial_j - x_j\partial_i$ and Lorentz boosts $x_i\partial_t + t\partial_i$, then in Ω we have $Z^I q = O(\langle q \rangle t^{C\varepsilon})$ and $Z^I \Omega_{ij} q = O(t^{C\varepsilon})$ for each multiindex I and $\varepsilon \ll_I 1$. To prove these estimates, we introduce the commutator coefficients $\{\xi^l_{k_1k_2}\}_{1 \leq k_1, k_2, l \leq 4}$ for which we have $[e_{k_1}, e_{k_2}] = \xi^l_{k_1k_2} e_l$. We also introduce a weighted null frame

$$(V_1, V_2, V_3, V_4) := (re_1, re_2, (3R - r + t)e_3, te_4)$$

[1]See Section 4.2.1. In particular, a smooth function may not be C^∞.

which combines the advantages of a usual null frame $\{e_k\}$ and the commuting vector fields Z's. By computing $e_4(V^I\xi^l_{k_1k_2})$ for each multiindex I and applying the Gronwall's inequality, we are able to obtain several estimates for $V^I(\xi^l_{k_1k_2})$; see Proposition 4.31. These estiamtes for ξ then imply the estimates for q, so we finish the proof.

We finally remark that the map

$$\Omega \to [0,\infty)\times\mathbb{R}\times\mathbb{S}^2: \qquad (t,x)\mapsto(\varepsilon\ln t-\delta, q(t,x), x/|x|) := (s,q,\omega)$$

is an invertible smooth function with a smooth inverse. This is because $q_r > 0$ everywhere in Ω. Thus, a smooth function $F = F(t,x)$ induces a smooth function $F = F(s,q,\omega)$ and vice versa.

4.1.2 The asymptotic equations and the scattering data

For each $(t,x)\in\Omega$, we define

$$\mu(t,x) := (q_t - q_r)(t,x), \qquad U(t,x) := \varepsilon^{-1}ru(t,x).$$

We then obtain two smooth functions $\mu(s,q,\omega)$ and $U(s,q,\omega)$ as discussed at the end of Section 4.1.1.

To state the results in this subsection, we introduce a new notation $\mathfrak{R}_{s,p}$ for each $s,p\in\mathbb{R}$. For a function $F = F(t,x)$ defined in $\Omega\cap\{r-t<2R\}$, we write $F = \mathfrak{R}_{s,p}$ if for each integer $N\geq 1$ and for each $\varepsilon \ll_N 1$, we have

$$\sum_{|I|\leq N}|V^I(F)| \lesssim t^{s+C\varepsilon}\langle q\rangle^p, \qquad \forall (t,x)\in\Omega\cap\{r-t<2R\}.$$

Here recall that $\{V_*\}$ is the weighted null frame.

By the chain rule, we have

$$\partial_s = \varepsilon^{-1}t(\partial_t - q_tq_r^{-1}\partial_r), \qquad \partial_q = q_r^{-1}\partial_r, \qquad \partial_{\omega_i} = r(\partial_i - q_iq_r^{-1}\partial_r).$$

Then we can express $(\partial_s,\partial_q,\partial_\omega)$ in terms of the weighted null frame $\{V_*\}$. In fact, we have

$$\partial_s = \sum_a \varepsilon^{-1}\mathfrak{R}_{-1,0}V_a + (\varepsilon^{-1}+\mathfrak{R}_{-1,0})V_4, \qquad \partial_q = \sum_k \mathfrak{R}_{0,-1}V_k,$$
$$\partial_{\omega_i} = \sum_{k\neq 3}\mathfrak{R}_{-1,0}V_k + \sum_a e^i_aV_a = \sum_{k\neq 3}\mathfrak{R}_{0,0}V_k.$$

Meanwhile, from (1.1) and $e_4(e_3(q)) = -\Gamma^0_{\alpha\beta}e^\alpha_4e^\beta_4e_3(q)$, we can show that

$$e_4(e_3(u)) + r^{-1}e_3(u) = \varepsilon\mathfrak{R}_{-3,0}, \qquad e_4(e_3(q)) = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}.$$

Combine these estimates, and we obtain that

$$\begin{cases} \partial_s \mu = \dfrac{1}{4} G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}, \\ \partial_s U_q = -\dfrac{1}{4} G(\omega)\mu U_q^2 + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{cases} \tag{4.3}$$

That is, $(\mu, U)(s, q, \omega)$ is an approximate solution to the geometric reduced system (2.4).

Next, we note from (4.3) that $\partial_s(\mu U_q) = O(\varepsilon^{-1} t^{-1+C\varepsilon})$. By integrating the remainder term $\varepsilon^{-1} t^{-1+C\varepsilon}$ (viewed as a function of s) with respect to s, we can show that $\{(\mu U_q)(s, q, \omega)\}_{s\gg 1}$ is uniformly Cauchy for each $(q, \omega) \in \mathbb{R} \times \mathbb{S}^2$. Thus, the limit

$$A(q, \omega) := -\frac{1}{2} \lim_{s\to\infty} (\mu U_q)(s, q, \omega)$$

exists and the convergence is uniform in (q, ω). This function A is then the *scattering data* in the asymptotic completeness problem.

Similarly, we can show that for each m and n, the limit

$$A_{m,n}(q, \omega) := -\frac{1}{2} \lim_{s\to\infty} (\langle q\rangle \partial_q)^m \partial_\omega^n (\mu U_q)(s, q, \omega)$$

exists and the convergence is uniform in (q, ω). The uniform convergences of these limits imply that

$$(\langle q\rangle \partial_q)^m \partial_\omega^n A(q, \omega) = A_{m,n}(q, \omega).$$

Following the same method, we can define

$$A_1(q, \omega) := \lim_{s\to\infty} \exp\Big(\frac{1}{2} G(\omega) A(q, \omega) s\Big) \mu(s, q, \omega),$$
$$A_2(q, \omega) := \lim_{s\to\infty} \exp\Big(-\frac{1}{2} G(\omega) A(q, \omega) s\Big) U_q(s, q, \omega).$$

Both of these limits exist and have derivatives of any order with respect to q and ω, as long as ε is sufficiently small. It is clear that $A_1 A_2 \equiv -2A$, so we obtain an exact solution to the reduced system (2.4):

$$\begin{cases} \widetilde{\mu}(s, q, \omega) = A_1(q, \omega) \exp\Big(-\dfrac{1}{2} G(\omega) A(q, \omega) s\Big), \\ \widetilde{U}_q(s, q, \omega) = A_2(q, \omega) \exp\Big(\dfrac{1}{2} G(\omega) A(q, \omega) s\Big), \end{cases} \tag{4.4}$$

By assuming $\lim_{q\to\infty} \widetilde{U}(s, q, \omega) = 0$, we obtain a unique function $\widetilde{U} = \widetilde{U}(s, q, \omega)$. By the definition of (A, A_1, A_2), we expect the $(\mu - \widetilde{\mu}, U - \widetilde{U})$, along with their derivatives with respect to (s, q, ω) of any order, decays faster than μ and U.

We refer our readers to Proposition 4.49 for a complete list of estimates.

4.1.3 Approximation

We now show that the exact solution (4.4) gives a good approximation of the exact solution u to (1.1).

We first solve

$$\widetilde{q}_t - \widetilde{q}_r = \widetilde{\mu}(\varepsilon \ln(t) - \delta, \widetilde{q}(t,x), \omega) \quad \text{in } \Omega \cap \{r - t < 2R\}; \qquad \widetilde{q} = r - t \quad \text{when } r - t \geq 2R$$

and set

$$\widetilde{u}(t,x) = \varepsilon r^{-1} \widetilde{U}(\varepsilon \ln(t) - \delta, \widetilde{q}(t,x), \omega) \quad \text{in } \Omega \cap \{r - t < 2R\}.$$

Then, we can prove that $\widetilde{u}$ is an approximate solution to (1.1) in the following sense: for each integer $N \geq 1$ and $\varepsilon \ll_N 1$, we have

$$\sum_{|I| \leq N} |Z^I (g^{\alpha\beta}(\widetilde{u}) \partial_\alpha \partial_\beta \widetilde{u})| \lesssim \varepsilon t^{-3+C\varepsilon}, \qquad \text{in } \Omega \cap \{r - t < 2R\}. \tag{4.5}$$

To make our proof simpler, we introduce a new function $F = F(q,\omega)$ such that $F_q = -2/A_1$. It can be shown that $q \mapsto F(q,\omega)$ has an inverse $q \mapsto \hat{F}(q,\omega)$. Now we define $\hat{A}(q,\omega) := A(\hat{F}(q,\omega),\omega)$ and define $(\hat{\mu}, \hat{U}_q)(s,q,\omega)$ by replacing (A_1, A_2, A) in (4.4) with $(-2, \hat{A}, \hat{A})$. Then, $\hat{q}(t,x) := F(\widetilde{q}(t,x),\omega)$ is a solution to

$$\hat{q}_t - \hat{q}_r = \hat{\mu}(\varepsilon \ln t - \delta, \hat{q}(t,x), \omega) \quad \text{in } \Omega \cap \{r - t < 2R\}; \qquad \hat{q} = r - t \quad \text{when } r - t \geq 2R.$$

In addition, we have

$$\widetilde{U}(\varepsilon \ln(t) - \delta, \widetilde{q}(t,x), \omega) = \hat{U}(\varepsilon \ln(t) - \delta, \hat{q}(t,x), \omega).$$

We can now follow the proof in Section 4 of [34] to prove (4.5).

In order to estimate $u - \widetilde{u}$, we set $p(t,x) := F(q(t,x),\omega) - \hat{q}(t,x)$ in Ω. We claim that, for each fixed $\gamma \in (0,1/2)$, an integer $N \geq 1$, and for each $\varepsilon \ll_{\gamma,N} 1$, whenever $(t,x) \in \Omega$ such that $|r - t| \lesssim t^\gamma$, we have $|Z^I p(t,x)| \lesssim t^{-1+C\varepsilon} \langle r - t \rangle$ for each $|I| \leq N$. To show this claim, we compute $p_t - p_r$ and apply a continuity argument. This claim then implies that, under the same assumptions on γ, N and ε, whenever $(t,x) \in \Omega$ such that $|r - t| \lesssim t^\gamma$, we have $|Z^I(u - \widetilde{u})(t,x)| \lesssim \varepsilon t^{-2+C\varepsilon} \langle r - t \rangle$ for each $|I| \leq N$. Recall from Lindblad [21] that we only have $Z^I u = O(\varepsilon t^{-1+C\varepsilon})$, so $\widetilde{u}$ provides a good approximation of u.

4.1.4 The main theorem

We now state the main theorem which is a summary of the previous subsections. In this theorem, we say that a function $f = f(t,x)$ is smooth if for each large integer N, f is C^N whenever $\varepsilon \ll_N 1$. See Section 4.2.1 for details.

Theorem 4.1. *Let u be a smooth solution to the Cauchy problem* (1.1) *and* (1.2). *Fix a constant $R > 0$ such that supp $(u_0, u_1) \subset \{|x| \leq R\}$, so $u \equiv 0$ for $|x| \geq t + R$ by the finite speed of propagation. Set $T_0 := \exp(\delta/\varepsilon)$ for a fixed constant $\delta > 0$. Then we have*

a) *There exists a smooth solution to the eikonal equation*

$$g^{\alpha\beta}(u)\partial_\alpha q\partial_\beta q = 0 \ \textit{in}\ \Omega; \qquad q = |x| - t \ \textit{on}\ \partial\Omega.$$

Here the region $\Omega \subset \mathbb{R}^{1+3}_{t,x}$ *is defined by*

$$\Omega := \{(t,x):\ t > T_0,\ |x| > (t+T_0)/2 + 2R\}.$$

In Ω*, for each* I *we have*

$$|Z^I q| \lesssim \langle q\rangle t^{C\varepsilon}, \qquad \sum_{1\le i,j\le 3} |Z^I\Omega_{ij}q| \lesssim t^{C\varepsilon}.$$

Moreover, the map

$$\Omega \to [0,\infty)\times\mathbb{R}\times\mathbb{S}^2: \qquad (t,x)\mapsto (\varepsilon\ln t - \delta, q(t,x), x/|x|)$$

is an invertible smooth function with a smooth inverse. Thus, a smooth function $F = F(t,x)$ *induces a smooth function* $F = F(s,q,\omega)$ *and vice versa.*

b) *In* Ω*, we set* $(\mu, U)(t,x) := (q_t - q_r, \varepsilon^{-1} r u)(t,x)$ *which induces a smooth function* $(\mu,U)(s,q,\omega)$*. Then,* $(\mu,U)(s,q,\omega)$ *is an approximate solution to the geometric reduced system* (2.4) *in the sense that*

$$\begin{cases} \partial_s\mu = \dfrac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}, \\[2ex] \partial_s U_q = -\dfrac{1}{4}G(\omega)\mu U_q^2 + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{cases}$$

Here the notation $\mathfrak{R}_{*,*}$ *has been defined in Section 4.1.1. In addition, the following three limits exist for all* $(q,\omega)\in\mathbb{R}\times\mathbb{S}^2$*:*

$$\begin{cases} A(q,\omega) := -\dfrac{1}{2}\lim\limits_{s\to\infty}(\mu U_q)(s,q,\omega), \\ A_1(q,\omega) := \lim\limits_{s\to\infty}\exp(\dfrac{1}{2}G(\omega)A(q,\omega)s)\mu(s,q,\omega), \\ A_2(q,\omega) := \lim\limits_{s\to\infty}\exp(-\dfrac{1}{2}G(\omega)A(q,\omega)s)U_q(s,q,\omega). \end{cases}$$

All of them are smooth functions of (q,ω) *for* $\varepsilon \ll 1$*, and we have* $A_1A_2 \equiv -2A$*. By setting*

$$\begin{cases} \widetilde{\mu}(s,q,\omega) := A_1\exp(-\dfrac{1}{2}GAs), \\[2ex] \widetilde{U}_q(s,q,\omega) := A_2\exp(\dfrac{1}{2}GAs). \end{cases}$$

we obtain an exact solution to our reduced system (2.4)*.*

c) *We define* $\widetilde{u} = \widetilde{u}(t, x)$ *as in Section 4.1.3. The function* $\widetilde{u} = \widetilde{u}(t, x)$ *is an approximate solution to* (1.1) *in the following sense:*

$$|Z^I(g^{\alpha\beta}(\widetilde{u})\partial_\alpha\partial_\beta\widetilde{u})(t,x)| \lesssim \varepsilon t^{-3+C\varepsilon}, \qquad \forall (t,x) \in \Omega,\ \forall I.$$

Moreover, if we fix a constant $0 < \gamma < 1$ *and a large integer* N*. Then, for* $\varepsilon \ll_{\gamma,N} 1$*, at each* $(t, x) \in \Omega$ *such that* $|r - t| \lesssim t^\gamma$*, we have*

$$|Z^I(u - \widetilde{u})| \lesssim_\gamma \varepsilon t^{-2+C\varepsilon}\langle r - t\rangle, \qquad \forall |I| \le N.$$

Remark 4.1.1. We choose the region Ω in a way that $t \sim r$ in $\Omega \cap \{r - t < 2R\}$, that $t \ge T_0 = \exp(\delta/\varepsilon)$ in $\overline{\Omega}$, and that $u \equiv 0$ in $\partial\Omega \cap \{t = T_0\}$. The proof in this chapter is expected to work if we start with a different region Ω with these three properties hold. For example, we can replace the definition of Ω with

$$\Omega = \Omega_{\kappa,\delta} := \{(t, x) :\ t > \exp(\delta/\varepsilon),\ |x| - \exp(\delta/\varepsilon) - 2R > \kappa(t - \exp(\delta/\varepsilon))\}$$

for some fixed constants $\delta > 0$ and $0 < \kappa < 1$. For different pairs of (κ, δ), we do not expect to get the same scattering data. However, Proposition 4.57 states that the scattering data associated to different regions $\Omega_{\kappa,\delta}$ are in fact related to each other in some sense. This is a result on *gauge independence*.

Remark 4.1.2. In our construction, we fix a parameter $\delta > 0$ and solve the eikonal equation in a region contained in $\{t > \exp(\delta/\varepsilon)\}$. In fact, the proof in this chapter is expected to work for each fixed $\delta > 0$. However, we do not simply set $\delta = 1$ here. Instead, we choose a sufficiently small $\delta > 0$ which depends on the pair (u_0, u_1), such that the nonlinear effects of (1.1) are negligible until we reach the time $\exp(\delta/\varepsilon)$. For example, we can set δ to be the small constant c in the almost global existence result.

Remark 4.1.3. We compare the results in this work with those in Deng-Pusateri [6]. First, the approximation result (i.e. part c) in Theorem 4.1) is better than that in [6] (i.e. Theorem 2.3). This suggests that the geometric reduced system (2.4) gives a more accurate descriptions of the global solutions to (1.1) than the Hörmander's asymptotic PDE (1.9) does. Further, the proof in this chapter relies on the null geometry while the authors in [6] made use of the spacetime resonance method.

4.2 Preliminaries for this chapter

In addition to Section 1.6, we need to introduce some notations and lemmas which are only used in this chapter.

4.2.1 A key theorem and a convention

This chapter is based on the following global existence result.

Theorem 4.2 (Lindblad [21])**.** *Fix a large integer $N \gg 1$. Then, for $\varepsilon \ll_N 1$, the Cauchy problem* (1.1) *with the initial data* (1.2) *has a global C^N solution $u = u(t,x)$ for all $t \geq 0$. Moreover, we have pointwise decays: $Z^I u = O_I(\varepsilon \langle t \rangle^{-1+C_I\varepsilon})$ for each multiindex I such that $|I| \leq N$. Moreover, we have $\partial u = O(\varepsilon \langle t \rangle^{-1})$.*

Most of the functions in this chapter have similar properties. That is, they depend on a small parameter ε, and they are C^N for any large integer N as long as $\varepsilon \ll_N 1$. For convenience, we make the following definition.

Definition 4.3. Fix a function $f = f_\varepsilon(t,x)$ which depends on a small parameter ε. In this chapter, we say that f is *smooth*, if for each large integer N, f is C^N whenever $\varepsilon \ll_N 1$.

Following the same spirits, we say that all derivatives of a function satisfy some properties, if for each large integer N, all its derivatives of order $\leq N$ exist and satisfy such properties whenever $\varepsilon \ll_N 1$.

We remark that under this definition, a smooth function does not need to be a C^∞ function. It would be more convenient to work with this seemingly strange definition.

Under such a convention, we can state Theorem 4.2 as follows: For $\varepsilon \ll 1$, the Cauchy problem (1.1) with the initial data (1.2) has a global smooth solution $u = u(t,x)$ for all $t \geq 0$. Moreover, we have pointwise decays: $Z^I u = O_I(\varepsilon \langle t \rangle^{-1+C_I\varepsilon})$ for each multiindex I and $\partial u = O(\varepsilon \langle t \rangle^{-1})$.

4.2.2 The null condition of a matrix

The definition and lemmas in this subsection will be used in Section 4.4.2. In this subsection, we assume that every matrix is in $\mathbb{R}^{4\times 4}$ and is a symmetric constant matrix.

Definition 4.4. A matrix $g = (g^{\alpha\beta})_{\alpha,\beta=0,1,2,3}$ satisfies the *null condition* if

$$g^{\alpha\beta}\xi_\alpha\xi_\beta = 0, \qquad \text{whenever } \xi \in \mathbb{R}^{1+3} \text{ and } |\xi_0|^2 = |\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2.$$

We remark that a real symmetric constant matrix g satisfies the null condition if and only if $g^{\alpha\beta}\xi_\alpha\eta_\beta$ is a linear combination of $-\xi_0\eta_0 + \sum_{j=1}^3 \xi_j\eta_j$ and $\xi_\alpha\eta_\beta - \xi_\beta\eta_\alpha$.

We start with the following useful lemma.

Lemma 4.5. *Suppose g is a constant matrix satisfying the null condition. Then, for any two functions $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$, we have*

$$Z(g^{\alpha\beta}\phi_\alpha\psi_\beta) = g^{\alpha\beta}(\partial_\alpha Z\phi)\psi_\beta + g^{\alpha\beta}\phi_\alpha(\partial_\beta Z\psi) + g_1^{\alpha\beta}\phi_\alpha\psi_\beta.$$

Here g_1 is another symmetric constant matrix satisfying the null condition. Moreover, if $Z = \Omega_{ij}$ for $1 \leq i,j \leq 3$ and if $(g^{\alpha\beta}) = (m^{\alpha\beta})$ is the usual Minkowski metric, then $g_1 = 0$.

We refer our readers to Lemma 6.6.5 in [7] for the proof.

In addition, we have the following pointwise estimates related to the null condition.

Lemma 4.6. *Suppose g is a matrix satisfying the null condition. Then, for any two functions $\phi = \phi(t,x)$ and $\psi = \psi(t,x)$, if $t \sim r \gg 1$, we have*

$$|g^{\alpha\beta}\phi_\alpha\psi_\beta| \lesssim \langle t\rangle^{-1}(|Z\phi||\partial\psi| + |Z\psi||\partial\phi|).$$

Here $|Zf| = \sum_{|J|=1} |Z^J f|$ for a function $f = f(t,x)$.

We refer our readers to Lemma I.5.4 in [30] for the proof.

4.3 Construction of the optical function

Let $u = u(t,x)$ be a global solution to (1.1) and (1.2) constructed in Theorem 4.2. If we fix a constant $R > 0$ such that supp $(u_0, u_1) \subset \{|x| \leq R\}$, then $u \equiv 0$ for $|x| \geq t + R$ by the finite speed of propagation. Our goal in this section is to construct an optical function, i.e. a solution to the eikonal equation

$$g^{\alpha\beta}(u)\partial_\alpha q \partial_\beta q = 0 \text{ in } \Omega; \qquad q = |x| - t \text{ on } \partial\Omega. \tag{4.6}$$

The region $\Omega \subset \mathbb{R}^{1+3}_{t,x}$ is defined by

$$\Omega := \{(t,x) : \ t > T_0, \ |x| > (t + T_0)/2 + 2R\}. \tag{4.7}$$

Here $T_0 := \exp(\delta/\varepsilon)$ for a fixed constant $\delta > 0$.

Our main result of this section is the following proposition.

Proposition 4.7. *The eikonal equation (4.6) has a global C^2 solution in the region Ω.*

In Section 4.4, we will show that this C^2 solution is in fact smooth (in the sense defined in Section 4.2.1).

Here we briefly explain how the optical function is constructed. In Section 4.3.1, we apply the method of characteristics and solve the characteristic ODE's. Here the characteristics are in fact the geodesics with respect to the Lorentzian metric $(g_{\alpha\beta})$ which is the inverse of the coefficients $(g^{\alpha\beta}(u))$ in (4.6). In Section 4.3.2, assuming that the optical function q exists in some region, we prove several preliminary estimates for q by studying the characteristic ODE's.

To finish the proof, we need to show that the characteristics, i.e. the geodesics, do not intersect with each other. This is related to the null geometry of the level sets of the optical function. In Section 4.3.3 and 4.3.4, we construct a null frame $\{e_k\}_{k=1}^4$ and then define several connection coefficients under the Lorentzian metric $(g_{\alpha\beta})$. Most importantly, we define

$$\chi_{ab} := \langle D_{e_a} e_4, e_b\rangle, \qquad a, b = 1, 2.$$

Here D is the Levi-Civita connection and $\langle\cdot,\cdot\rangle$ is the bilinear form, both with respect to $(g_{\alpha\beta})$. It suffices to prove that the trace of χ, sometimes called the null mean curvature, is positive everywhere.

We now follow the idea in Christodoulou-Klainerman [4]. In Section 4.3.5, we derive an equation for χ, called the Raychaudhuri equation. In Section 4.3.6, we use a continuity argument and the Raychaudhuri equation to prove that in the region where the optical function exists, we have

$$\max_{a,b=1,2}|\chi_{ab}-\delta_{ab}r^{-1}|\lesssim t^{-2+C\varepsilon}.$$

We conclude that $\mathrm{tr}\chi>0$ everywhere, and thus end the proof.

4.3.1 The method of characteristics

Now we use the method of characteristics to solve (4.6). We have the characteristic ODE's

$$\begin{cases}\dot{x}^\alpha(s)=2g^{\alpha\beta}(x(s))p_\beta(s),\\ \dot{z}(s)=2g^{\alpha\beta}(x(s))p_\beta(s)p_\alpha(s)=0,\\ \dot{p}_\alpha(s)=-(\partial_\alpha g^{\mu\nu})(x(s))p_\mu(s)p_\nu(s).\end{cases}\tag{4.8}$$

Here we write $g^{\alpha\beta}(t,x)=g^{\alpha\beta}(u(t,x))$ with an abuse of notation. We expect that $z(s)=q(x(s))$ and $p(s)=(\partial q)(x(s))$ for some optical function $q(t,x)$. By differentiating the first equation, we obtain the geodesic equation

$$\ddot{x}^\alpha(s)+\Gamma^\alpha_{\mu\nu}\dot{x}^\mu(s)\dot{x}^\nu(s)=0.\tag{4.9}$$

Here Γ is the Christoffel symbol of the Levi-Civita connection D of the Lorentzian metric $(g_{\alpha\beta})$. Thus, in this chapter, the curve $x(s)$ is either called a *characteristic curve*, or a *geodesic*.

To solve the eikonal equation (4.6), we only need to consider the geodesics emanating from the surface

$$H:=\{(t,x):\ t\geq T_0,\ r=(t+T_0)/2+2R\}\subset\partial\Omega.\tag{4.10}$$

From these geodesics, later we will construct a solution $q(t,x)$ in the region $\Omega\cap\{r-t<2R\}$ such that $q=r-t$ in $\Omega\cap\{R<r-t<2R\}$. Since $u\equiv 0$ in the region $r-t>R$, we can then extend our solution to the whole region Ω by defining $q=r-t$ when $r>t+R$.

To solve the characteristic ODE's (4.8) and the geodesic equation (4.9), we need to first determine $(\partial q)|_H$. Fix $(t,x)\in H$ and recall that $q=r-t$ on H. Since $X_i:=\partial_i+2\omega_i\partial_t$ is tangent to H, we have $X_iq=X_i(r-t)=-\omega_i$ on H. Thus, for $(t,x)\in H$ we have $q_i=X_iq-2\omega_iq_t=-\omega_i-2\omega_iq_t$ and

$$\begin{aligned}0&=-q_t^2+2g^{0i}q_t(-\omega_i-2\omega_iq_t)+g^{ij}(-\omega_i-2\omega_iq_t)(-\omega_j-2\omega_jq_t)\\&=(-1-4g^{0i}\omega_i+4g^{ij}\omega_i\omega_j)q_t^2+(4g^{ij}\omega_i\omega_j-2g^{0i}\omega_i)q_t+g^{ij}\omega_i\omega_j.\end{aligned}$$

Since $g^{\alpha\beta}(u) = m^{\alpha\beta} + O(|u|)$, we have

$$\begin{aligned} 0 &= (-1 + 4m^{ij}\omega_i\omega_j + O(|u|))q_t^2 + (4m^{ij}\omega_i\omega_j + O(|u|))q_t + (m^{ij}\omega_i\omega_j + O(|u|)) \\ &= (3 + O(|u|))q_t^2 + (4 + O(|u|))q_t + (1 + O(|u|)). \end{aligned}$$

Since $|u| \ll 1$, by the root formula we can uniquely determine $q_t = -1 + O(|u|)$ at (t, x) (the other root $q_t = -1/3 + O(|u|)$ is discarded since we expect q to behave like $r - t$). We also have $q_i = -\omega_i - 2\omega_i q_t = \omega_i + O(|u|)$ and $q_r = \omega_i q_r$. If moreover $t < T_0 + 2R$, then $r = (t + T_0)/2 + 2R > t + R$ and thus $g^{\alpha\beta} \equiv m^{\alpha\beta}$. Thus, we have $q_t = -1$ and $q_i = \omega_i$ for $(t, x) \in H$ such that $t < T_0 + 2R$.

Now fix $x(0) \in H$. We set

$$z(0) = r(x(0)) - x^0(0), \qquad p_\alpha(0) = (\partial_\alpha q)(x(0))$$

where we set

$$r(V) := \Big(\sum_{i=1}^{3}(V^i)^2\Big)^{1/2}, \qquad \text{for a vector } V = (V^\alpha)_{\alpha=0}^3.$$

We have the following lemma.

Lemma 4.8. *Fix $x(0) \in H$ and construct $z(0), p(0)$ as above. Then the system* (4.8) *along with the initial data $(x(0), z(0), p(0))$ has a unique solution $(x(s), z(s), p(s))$ on* $[0, \infty)$. *In addition, we have $\dot{x}^0(s) > 0$ for all $s \geq 0$, and $x^0(s) \to \infty$ as $s \to \infty$.*

If moreover we have $x(0) \in H \cap \{t < T_0 + 2R\}$, then $x(s) = (2s, 2s\omega) + x(0)$. In other words, the geodesics emanating from $H \cap \{t < T_0 + 2R\}$ are straight lines. Thus $q = r - t$ whenever $r > t + R$.

Proof. We apply the Picard existence and uniqueness theorem, e.g. Theorem 1.17 in [31], to (4.8). From the theorem, we obtain a unique solution $(x(s), z(s), p(s))$ for all $0 \leq s < s_{\max}$. By the blowup criterion in the theorem, either we have $s_{\max} < \infty$ and $|x(s)|+|z(s)|+|p(s)| \to \infty$ as $s \to s_{\max}$, or we have $s_{\max} = \infty$. Here $|x(s)| + |z(s)| + |p(s)| \to \infty$ is equivalent to $|x(s)| + |\dot{x}(s)| \to \infty$ due to $z(s) = z(0)$ and the first equation in (4.8).

We claim that, along each geodesic, for all $s \geq 0$ we have

$$4g^{\alpha\beta}(x(s))p_\alpha(s)p_\beta(s) = 2\dot{x}^\alpha(s)p_\alpha(s) = g_{\alpha\beta}(x(s))\dot{x}^\alpha(s)\dot{x}^\beta(s) = 0. \tag{4.11}$$

In other words, the geodesics $x(s)$ are null curves. The first two equations follow from the first equation in (4.8), so here we only prove the last one. Note that the equality holds for $s = 0$ by the construction of $(\partial q)|_H$. In addition,

$$\begin{aligned} \frac{d}{ds}(g^{\alpha\beta}(x(s))p_\alpha(s)p_\beta(s)) &= 2g^{\alpha\beta}(x(s))\dot{p}_\alpha(s)p_\beta(s) + (\partial_\mu g^{\alpha\beta})(x(s))\dot{x}^\mu(s)p_\alpha(s)p_\beta(s) \\ &= \dot{x}^\alpha(s)\dot{p}_\alpha(s) - \dot{p}_\mu(s)\dot{x}^\mu(s) = 0. \end{aligned}$$

In the last line we use the third equation in (4.8). This ends the proof of (4.11).

Next we claim that $\dot{x}^0(s) > 0$ for all s. Since $g^{\alpha\beta}(u) = m^{\alpha\beta} + O(|u|)$ for $|u| \ll 1$, its inverse $(g_{\alpha\beta}(u))$ is also a small pertubation of the Minkowski metric, i.e. $g_{\alpha\beta} = m_{\alpha\beta}+O(|u|)$. Thus, (4.11) implies

$$0 = g_{00}(\dot{x}^0)^2 + 2g_{0i}\dot{x}^0\dot{x}^i + g_{ij}\dot{x}^i\dot{x}^j = -(\dot{x}^0(s))^2 + \sum_i(\dot{x}^i(s))^2 + O(|u(x(s))||\dot{x}|^2).$$

We first show that $\dot{x}^0(s) \neq 0$ for all s. If $\dot{x}^0(s_0) = 0$ for some $s_0 > 0$, then we have $g_{ij}\dot{x}^i\dot{x}^j = 0$ at $s = s_0$. Since $g_{ij} = \delta_{ij} + O(|u|)$, the symmetric matrix (g_{ij}) is positive definite. Then $\dot{x}(s_0) = 0$. However, recall that $x(s)$ is a geodesic, and the only geodesic passing through $x(s_0)$ with $\dot{x}(s_0) = 0$ is the constant curve $x(s) = x(s_0)$. This leads to a contradiction. In addition, since $q_t = -1 + O(|u|)$ on H and $\dot{x}^0(0) = 2g^{0\beta}p_\beta(0)$, we have $\dot{x}^0(0) = 2 + O(|u|)$. Thus $\dot{x}^0(s) > 0$ for all s.

Moreover, since $u = O(\varepsilon\langle t\rangle^{-1+C\varepsilon})$, we have

$$|-(\dot{x}^0(s))^2 + \sum_i(\dot{x}^i(s))^2| \leq C\varepsilon\langle x^0(s)\rangle^{-1+C\varepsilon}(|\dot{x}^0(s)|^2 + \sum_i(\dot{x}^i(s))^2).$$

By choosing $\varepsilon \ll 1$, we can make $C\varepsilon \leq 1/2$. Thus, for $\varepsilon \ll 1$, we have

$$\sum_i(\dot{x}^i(s))^2 \leq (\dot{x}^0(s))^2 + \frac{1}{2}(|\dot{x}^0(s)|^2 + \sum_i(\dot{x}^i(s))^2) \Longrightarrow \sum_i(\dot{x}^i(s))^2 \lesssim (\dot{x}^0(s))^2.$$

Thus, for each i we have

$$|x^i(s)| = |x^i(0) + \int_0^s \dot{x}^i(\tau)\ d\tau| \leq |x^i(0)| + C\int_0^s \dot{x}^0(\tau)\ d\tau = |x^i(0)| + Cx^0(s).$$

In conclusion, if $|x(s)| + |\dot{x}(s)| \to \infty$, then we must have $x^0(s) + \dot{x}^0(s) \to \infty$.

If we differentiate the first equation in (4.8) and use the third one, we obtain

$$|\ddot{x}^0(s)| \leq |2g^{0\beta}\dot{p}_\beta| + |2(\partial_\mu g^{0\beta})\dot{x}^\mu p_\beta| \lesssim |\partial u(x(s))||\dot{x}(s)|^2 \lesssim \varepsilon\langle x^0(s)\rangle^{-1}(\dot{x}^0(s))^2.$$

The last inequality follows since $|\dot{x}^i(s)| \lesssim \dot{x}^0(s)$ and since $\partial u = O(\varepsilon\langle t\rangle^{-1})$. Since $\dot{x}^0 > 0$, we then have

$$|\frac{d}{ds}\ln\dot{x}^0| = \frac{|\ddot{x}^0|}{\dot{x}^0} \leq C\varepsilon\frac{\dot{x}^0}{x^0} = C\varepsilon\frac{d}{ds}\ln x^0,$$

which implies that

$$|\ln\dot{x}^0(s) - \ln\dot{x}^0(0)| \lesssim \varepsilon(\ln x^0(s) - \ln x^0(0)).$$

The last inequality is equivalent to

$$\dot{x}^0(0)(\frac{x^0(s)}{x^0(0)})^{-C\varepsilon} \leq \dot{x}^0(s) \leq \dot{x}^0(0)(\frac{x^0(s)}{x^0(0)})^{C\varepsilon}.$$

It follows that

$$\frac{d}{ds}((x^0(s))^{1-C\varepsilon}) = (1 - C\varepsilon)(x^0(s))^{-C\varepsilon}\dot{x}^0(s) \leq \dot{x}^0(0)(x^0(0))^{-C\varepsilon},$$
$$\frac{d}{ds}((x^0(s))^{1+C\varepsilon}) = (1 + C\varepsilon)(x^0(s))^{C\varepsilon}\dot{x}^0(s) \geq \dot{x}^0(0)(x^0(0))^{C\varepsilon} > 0,$$

and thus

$$(x^0(s))^{1-C\varepsilon} \leq (x^0(0))^{1-C\varepsilon} + \dot{x}^0(0)s(x^0(0))^{-C\varepsilon}, \tag{4.12}$$

$$(x^0(s))^{1+C\varepsilon} \geq (x^0(0))^{1+C\varepsilon} + \dot{x}^0(0)s(x^0(0))^{C\varepsilon}. \tag{4.13}$$

If $s_{\max} < \infty$, then $x^0(s) \to \infty$ as $s \to s_{\max}$ fails because of (4.12). On the other hand, if $s_{\max} < \infty$, then $x^0(s)+\dot{x}^0(s) \to \infty$ as discussed above. But since $\dot{x}^0(s) \leq \dot{x}^0(0)(x^0(s)/x^0(0))^{C\varepsilon}$, we must have $x^0(s) \to \infty$ as $s \to s_{\max}$. A contradiction. Thus, $s_{\max} = \infty$. We thus conclude $x^0(s) \to \infty$ as $s \to \infty$ by (4.13).

The proof of the second half of this lemma is easy. We simply use the fact that $g^{\alpha\beta}(u) = m^{\alpha\beta}$ when $r \geq t + R$. □

Remark 4.8.1. We let $\mathcal{A}$ denote the set of all the geodesics constructed in this lemma.

4.3.2 Estimates for the optical function

Fix a time $T > T_0 = \exp(\delta/\varepsilon)$ and we set $\Omega_T = \Omega \cap \{t \leq T,\ r - t \leq 2R\}$. Note that $r \sim t$ in Ω_T. From now on, we assume that the optical function $q = q(t, x)$ exists in Ω_T, that q is C^2 and that $q_t < 0$ everywhere. We remark that the assumptions are true for $T = T_0 + 2R$ since $g^{\alpha\beta} \equiv m^{\alpha\beta}$ in Ω_{T_0+2R}. Our goal is to derive some estimates which allow us to extend the optical function to $\Omega_{T+\epsilon}$ for some $\epsilon > 0$.

First of all, we claim that each point in Ω_T lies on exactly one geodesic in $\mathcal{A}$ (which is defined in Remark 4.8.1). A direct corollary is that to define a function $F(t, x)$ in Ω_T, we can define $F(x(s))$ along each geodesic in $\mathcal{A}$. To prove this claim, we define a vector field $L = L^\alpha\partial_\alpha$ by $L^\alpha := 2g^{\alpha\beta}q_\beta$. Note that $L^0 > 0$ everywhere. In fact, we have

$$g_{\alpha\beta}L^\alpha L^\beta = 4g_{\alpha\beta}g^{\alpha\alpha'}g^{\beta\beta'}q_{\alpha'}q_{\beta'} = 4g^{\alpha'\beta'}q_{\alpha'}q_{\beta'} = 0.$$

If $L^0 = 0$, then $g_{ij}L^iL^j = 0$. But $g_{ij} = \delta_{ij} + O(|u|)$, so (g_{ij}) is positive definite for $\varepsilon \ll 1$. Thus, $L^\alpha = 0$ and $q_t = \frac{1}{2}g_{0\beta}L^\beta = 0$. This contradicts with the assumption that $q_t < 0$. And since $L^0 = -2q_t + O(|u\partial q|) = 2 + O(|u|) > 0$ on $\partial\Omega$, we have $L^0 > 0$ in Ω_T. Moreover, because of the characteristic ODE's (4.8), a curve in Ω_T is a geodesic in $\mathcal{A}$ if and only if it is an integral curve of L emanating from H. By the existence and uniqueness of integral curves, we finish the proof of the claim.

We also claim that each geodesic emanating from $H \cap \partial\Omega_T$ must stay in Ω_T until it intersects with $\{t = T\}$. This claim simply follows from the fact that the optical function remains constant along each geodesic and that the optical function is injective when restricted to $(\partial\Omega_T) \setminus \{t = T\}$.

Here a useful lemma which follows directly from the chain rule and the pointwise estimates in Theorem 4.2 (also see Proposition 6.1 in Lindblad [21]).

Lemma 4.9. *For each $k \geq 0$ and $\varepsilon \ll_k 1$, we have*

$$\sum_{|I|\leq k}(|Z^I(g^{\alpha\beta}-m^{\alpha\beta})|+|Z^I(g_{\alpha\beta}-m_{\alpha\beta})|)\lesssim_k \sum_{|I|\leq k}|Z^I u|\lesssim_k \varepsilon\langle t\rangle^{-1+C_k\varepsilon}.$$

Moreover,

$$|\partial g^{\alpha\beta}|+|\partial g_{\alpha\beta}|+|\Gamma^\alpha_{\mu\nu}|\lesssim|\partial u|\lesssim\varepsilon\langle t\rangle^{-1}.$$

Now we can prove several useful estimates for q in Ω_T.

Lemma 4.10. *In Ω_T, we have $|Sq|+\sum_i|\Omega_{0i}q|\lesssim|q|+t^{C\varepsilon}$, $|\partial q|+\sum_{i,j}|\Omega_{ij}q|\lesssim t^{C\varepsilon}$ and $\sum_i|q_i-\omega_iq_r|\lesssim t^{-1+C\varepsilon}$.*

Proof. If we apply a vector field Z defined by (1.13) to the eikonal equation, we obtain

$$0=(Zg^{\alpha\beta})q_\alpha q_\beta+2g^{\alpha\beta}q_\alpha Zq_\beta=(Zg^{\alpha\beta})q_\alpha q_\beta+2g^{\alpha\beta}q_\alpha\partial_\beta Zq+2g^{\alpha\beta}q_\alpha[Z,\partial_\beta]q.$$

It is easy to check that $2m^{\alpha\beta}q_\alpha[Z,\partial_\beta]q=0$ if $Z\neq S$ and $[S,\partial_\beta]=-\partial_\beta$. Thus, for some geodesic $x(s)$, we have

$$|\frac{d}{ds}(Zq(x(s)))|\lesssim(|Zg^{\alpha\beta}|+|g^{\alpha\beta}-m^{\alpha\beta}|)|p(s)|^2\lesssim\varepsilon(x^0(s))^{-1+C\varepsilon}|\dot{x}(s)|^2\lesssim\varepsilon(x^0(s))^{-1+C\varepsilon}\dot{x}^0(s).$$

Recall that $p(s)=(\partial q)(x(s))$ and that we have $|\dot{x}^i(s)|\lesssim\dot{x}^0(s)\lesssim(x^0(s))^{C\varepsilon}$ from the proof of Lemma 4.8. Since $\partial q=(-1,\omega)+O(|u|)$ on H, we have $|Sq|+|\Omega_{0j}q|=O(|q|+\varepsilon t^{C\varepsilon})$ and $|\Omega_{ij}q|=O(\varepsilon t^{C\varepsilon})$ on H. By integrating the inequality, we have

$$|Zq(x(s))-Zq(x(0))|\lesssim\int_0^s\varepsilon(x^0(\tau))^{-1+C\varepsilon}\dot{x}^0(\tau)\ d\tau\lesssim(x^0(s))^{C\varepsilon},$$

so we have

$$|Zq(x(s))|\lesssim|Zq(x(0))|+(x^0(s))^{C\varepsilon}\lesssim1+|q(x(0))|+(x^0(s))^{C\varepsilon}=1+|q(x(s))|+(x^0(s))^{C\varepsilon}.$$

In conclusion, we have $|Zq|=O(|q|+t^{C\varepsilon})$ in Ω_T. For $Z=\partial_\alpha$ or Ω_{ij} we have better bounds $|\Omega_{ij}q|+|\partial q|=O(t^{C\varepsilon})$, since the estimates for $\partial q|_H$ and $\Omega_{ij}q|_H$ are better. In addition, we have $|q_i-\omega_iq_r|=r^{-1}|\sum_j\omega_j\Omega_{ij}q|\lesssim t^{-1+C\varepsilon}$. □

Lemma 4.11. *For each $(t,x)\in\Omega_T$, we have $q_r\geq C^{-1}t^{-C\varepsilon}$, $-q_t\geq C^{-1}t^{-C\varepsilon}$ and $|q_t+q_r|\lesssim\varepsilon t^{-1+C\varepsilon}$.*

Proof. Recall that from the proof of Lemma (4.8), we have $|\dot{x}^i(s)| \lesssim \dot{x}^0(s)$ and

$$(x^0(s))^{-C\varepsilon} \leq \dot{x}^0(0)(\frac{x^0(s)}{x^0(0)})^{-C\varepsilon} \leq \dot{x}^0(s) \leq \dot{x}^0(0)(\frac{x^0(s)}{x^0(0)})^{C\varepsilon} \leq (x^0(s))^{C\varepsilon}$$

along each geodesic $x(s)$ in $\mathcal{A}$. At $(t_0, x_0) = x(s_0)$ for some geodesic $x(s)$ in $\mathcal{A}$, we have

$$q_t = \frac{1}{2}g_{0\alpha}\dot{x}^\alpha(s_0) = -\frac{1}{2}\dot{x}^0(s_0) + O(|u(x(s_0))||\dot{x}(s_0)|) \leq -\frac{1}{2}t_0^{-C\varepsilon} + C\varepsilon t_0^{-1+C\varepsilon} \leq -\frac{1}{4}t_0^{-C\varepsilon}. \tag{4.14}$$

Here we take $\varepsilon \ll 1$ as usual.

To prove the estimate for q_r, we first prove that $q_r > 0$ in Ω_T. Assume $q_r = 0$ at some $(t_0, x_0) \in \Omega_T$. By the eikonal equation (4.6) and the previous lemma, at (t_0, x_0) we have

$$\begin{aligned} 0 &= g^{00}q_t^2 + 2g^{0i}q_t(q_i - q_r\omega_i) + g^{ij}(q_i - \omega_i q_r)(q_j - \omega_j q_r) \\ &= -q_t^2 + O(|u||q_t|\sum_i |q_i - q_r\omega_i|) + O((\sum_i |q_i - \omega_i q_r|)^2) \\ &= -q_t^2 + O(t_0^{-2+C\varepsilon}). \end{aligned} \tag{4.15}$$

Plug (4.14) into (4.15), and we conclude that $t_0^{-2C\varepsilon} \lesssim q_t^2 \lesssim t_0^{-2+C\varepsilon}$ and $t_0^{2-3C\varepsilon} \lesssim 1$. This is impossible, since $t_0^{2-3C\varepsilon} \geq t_0 \geq T_0 = \exp(\delta/\varepsilon) \gg 1$ for $\varepsilon \ll 1$. So we have $q_r \neq 0$ everywhere in Ω_T. Since $q_r = 1 + O(|u|) > 0$ on H, we have $q_r > 0$ everywhere in Ω_T. By (4.14), we have $-q_t + q_r \geq -q_t \geq \frac{1}{4}t^{-C\varepsilon}$. Then since

$$\begin{aligned} 0 &= -q_t^2 + \sum_i q_i^2 + O(|u||\partial q|^2) = (q_t + q_r)(-q_t + q_r) + \sum_i (q_i - q_r\omega_i)^2 + O(\varepsilon t^{-1+C\varepsilon}|\partial q|^2) \\ &= (q_t + q_r)(-q_t + q_r) + O(t^{-2+2C\varepsilon} + \varepsilon t^{-1+C\varepsilon}) \end{aligned}$$

and since $t^{-1} \leq T_0^{-1} \ll \varepsilon$, we have

$$|q_t + q_r| = (-q_t + q_r)^{-1}O(\varepsilon t^{-1+C\varepsilon}) \lesssim t^{C\varepsilon} \cdot \varepsilon t^{-1+C\varepsilon} \lesssim \varepsilon t^{-1+C\varepsilon}.$$

Then we have $q_r = -q_t + (q_t + q_r) \geq C^{-1}t^{-C\varepsilon} - C\varepsilon t^{-1+C\varepsilon} \geq C^{-1}t^{-C\varepsilon}$. □

4.3.3 A null frame

We construct a null frame $\{e_1, e_2, e_3, e_4\}$ in Ω_T as follows. Define two vector fields e_3, e_4 by

$$e_4 := (L^0)^{-1}L, \qquad e_3 := e_4 + 2g^{0\alpha}\partial_\alpha.$$

Since $g^{00} \equiv -1$, we have $e_4^0 \equiv 1$ and $e_3^0 \equiv -1$. Moreover, we have

$$\begin{aligned} \langle e_4, e_4\rangle &= (L^0)^{-2}\langle L, L\rangle = (L^0)^{-2}g_{\alpha\beta}L^\alpha L^\beta = 0, \\ \langle e_4, e_3\rangle = \langle e_3, e_4\rangle &= \langle 2g^{0\alpha}\partial_\alpha, e_4\rangle = 2g_{\alpha\beta}g^{0\alpha}e_4^\beta = 2e_4^0 = 2, \\ \langle e_3, e_3\rangle &= \langle e_4, e_3\rangle + \langle 2g^{0\alpha}\partial_\alpha, e_3\rangle = 2 + 2g_{\alpha\beta}g^{0\alpha}e_3^\beta = 2 + 2e_3^0 = 0. \end{aligned} \tag{4.16}$$

Here $\langle\cdot,\cdot\rangle$ is the bilinear form defined by the Lorentzian metric $(g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$.

Next we define $\{e_a\}_{a=1,2}$. When restricted to the 2-sphere $H \cap \{t = T'\}$ for some $T' \geq T_0$, the metric $(g_{\alpha\beta})$ is positive definite. Thus, we can choose a smooth orthonormal basis $\{E_a\}_{a=1,2}$ locally on this 2-sphere. Here we make our choice such that $E_a|_H$ depends only on ω and not on t. Note that E_a is tangent to $H \cap \{t = T'\}$, that $E_a^0 = 0$ and that $\langle E_a, E_b\rangle = \delta_{ab}$. Then we take the parallel transport of E_a along the geodesics. That is, we consider the equations $D_4 E_a = 0$ for $a = 1, 2$. Here D is the Levi-Civita connection of the Lorentzian metric, and $D_4 := D_{e_4}$. Since e_4 is tangent to the geodesic, equivalently we need to solve the ODE's

$$\frac{d}{ds}E_a^\alpha(x(s)) + \dot{x}^\mu(s)E_a^\nu(x(s))\Gamma^\alpha_{\mu\nu}(x(s)) = 0. \tag{4.17}$$

By the existence and uniqueness for linear ODE's (e.g. Theorem 4.12 in [19]), these ODE's admit a unique solution for all $0 \leq s \leq s_0$. Finally, we define

$$e_a := E_a - E_a^0 e_4, \qquad a = 1, 2.$$

Thus $e_a^0 = 0$. Unlike e_3, e_4, the vector fields e_1, e_2 cannot be defined globally in Ω_T. This is because there is no global orthonormal basis on a 2-sphere. In the rest of this chapter, when we state a property of e_a on Ω_T, we mean that any locally defined e_a satisfies this property.

We conclude that $\{e_k\}_{k=1,2,3,4}$ is a null frame by (4.16) and the following lemma.

Lemma 4.12. *In Ω_T we have $\langle e_a, e_b\rangle = \delta_{ab}$ and $\langle e_4, e_a\rangle = \langle e_3, e_a\rangle = 0$ for each $a, b = 1, 2$.*

Proof. We first prove that $\langle E_a, E_b\rangle = \delta_{ab}$ and $\langle e_4, E_a\rangle = 0$ on H. The first equality follows directly from the construction of $\{E_a\}$. To prove the second one, we recall that $q_i = q_r\omega_i$ on H; see the computations right above Lemma 4.8. Moreover, note that $\sum_i x^i(0)E_a^i = 0$ since E_a is tangent to the sphere on H. Thus, on H, we have

$$\langle L, E_a\rangle = g_{\alpha\beta}L^\alpha E_a^\beta = 2q_\beta E_a^\beta = 2q_i E_a^i = 2q_r\omega_i E_a^i = 0.$$

And since $e_4 = (L^0)^{-1}L$, we have $\langle e_4, E_a\rangle = 0$ at $x(0)$.

Along each geodesic $x(s)$ in $\mathcal{A}$, we have

$$\begin{aligned} e_4\langle E_a, E_b\rangle &= \langle D_4 E_a, E_b\rangle + \langle E_a, D_4 E_b\rangle = 0,\\ e_4\langle L, E_a\rangle &= \langle D_4 L, E_a\rangle + \langle L, D_4 E_a\rangle = 0.\end{aligned}$$

Because of the equalities at $s = 0$, we conclude that $\langle E_a, E_b\rangle = \delta_{ab}$ and $\langle L, E_a\rangle = 0$ (and thus $\langle e_4, E_a\rangle = 0$) along each geodesic.

Finally, note that

$$\begin{aligned} &\langle e_a, e_b\rangle = \langle E_a, E_b\rangle - E_a^0\langle e_4, E_b\rangle - E_b^0\langle E_a, e_4\rangle + E_a^0E_b^0\langle e_4, e_4\rangle = \delta_{ab},\\ &\langle e_4, e_a\rangle = \langle e_4, E_a\rangle - E_a^0\langle e_4, e_4\rangle = 0,\\ &\langle e_3, e_a\rangle = \langle 2g^{0\alpha}\partial_\alpha, e_a\rangle + \langle e_4, e_a\rangle = 2g_{\alpha\beta}g^{0\alpha}e_a^\beta = 2e_a^0 = 0.\end{aligned}$$

This finishes the proof. ∎

Before we move on to the next lemma, we summarize some important properties of a null frame. First, any vector field X can be uniquely expressed as a linear combination of the null frame:

$$X = \sum_{a=1,2} \langle X, e_a\rangle e_a + \frac{1}{2}\langle X, e_4\rangle e_3 + \frac{1}{2}\langle X, e_3\rangle e_4. \tag{4.18}$$

In addition, for each $k = 1, 2, 3, 4$ we have

$$\langle g^{\alpha\beta}\partial_\beta, e_k\rangle = g^{\alpha\beta} g_{\beta\mu} e_k^\mu = e_k^\alpha,$$

so we obtain

$$g^{\alpha\beta}\partial_\beta = \sum_{a=1,2} e_a^\alpha e_a + \frac{1}{2}e_4^\alpha e_3 + \frac{1}{2}e_3^\alpha e_4 \Longrightarrow g^{\alpha\beta} = \sum_{a=1,2} e_a^\alpha e_a^\beta + \frac{1}{2}e_4^\alpha e_3^\beta + \frac{1}{2}e_3^\alpha e_4^\beta. \tag{4.19}$$

Finally, we have $e_1(q) = e_2(q) = e_4(q) = 0$ and $e_3(q) = L^0$ in Ω_T. In fact, since $q_\alpha = \frac{1}{2}g_{\alpha\beta}L^\beta$, we have $Xq = \frac{1}{2}\langle X, L\rangle = \frac{1}{2}L^0\langle e_4, X\rangle$ for each vector field X. Then we use the properties of a null frame. The equality $e_1(q) = e_2(q) = e_4(q) = 0$ implies that e_1, e_2, e_4 are tangent to the level set of q, so e_1, e_2, e_4 are sometimes called the tangential derivatives.

The next lemma shows several better estimates for the tangential derivatives.

Lemma 4.13. *In Ω_T, we have $e_4 = \partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial$, $e_3 = e_4 + 2g^{0\alpha}\partial_\alpha = -\partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial$ and $e_a = O(1)\partial$. Then, for all I, s, l, we have*

$$\sum_{k=1,2,4} (|e_k(\partial^s Z^I u)| + |e_k(\partial^s Z^I g^{\alpha\beta})| + |e_k(\partial^s Z^I g_{\alpha\beta})|) \lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-s}.$$

Here we use the convention given in Section 4.2.1. Moreover, we have

$$|e_1(\partial_\alpha g_{\mu\nu})e_2^\alpha| + |e_2(\partial_\alpha g_{\mu\nu})e_1^\alpha| + |e_1(\partial_\alpha g_{\mu\nu})e_1^\alpha - e_2(\partial_\alpha g_{\mu\nu})e_2^\alpha| \lesssim \varepsilon t^{-3+C\varepsilon}.$$

Proof. By the lemmas in Section 4.3.2, we have

$$e_4^i - \omega_i = \frac{L^i - L^0\omega_i}{L^0} = \frac{2q_i + 2q_t\omega_i + O(|u||\partial q|)}{-2q_t + O(|u||\partial q|)} = \frac{2(q_i - q_r\omega_i) + 2(q_r + q_t)\omega_i + O(|u||\partial q|)}{-2q_t + O(|u||\partial q|)}.$$

By Lemma 4.10 and Lemma 4.11, the denominator has a lower bound $C^{-1}t^{-C\varepsilon} - C\varepsilon t^{-1+C\varepsilon} \geq (2C)^{-1}t^{-C\varepsilon}$ and the numerator is $O(t^{-1+C\varepsilon})$. In conclusion, $e_4 = \partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial$. It follows that for each I,

$$\begin{aligned}
|e_4(\partial^s Z^I u)| &\lesssim |(\partial_t + \partial_r)\partial^s Z^I u| + t^{-1+C\varepsilon}|\partial\partial^s Z^I u| \\
&\lesssim \langle t+r\rangle^{-1}\sum_{|J|=1}|Z^J\partial^s Z^I u| + t^{-1+C\varepsilon}\langle r-t\rangle^{-s-1}\sum_{|J|\leq s+1}|Z^J Z^I u| \\
&\lesssim \langle t+r\rangle^{-1}\sum_{|J|\leq 1}|\partial^s Z^J Z^I u| + t^{-1+C\varepsilon}\langle r-t\rangle^{-s-1}\cdot\varepsilon t^{-1+C\varepsilon} \\
&\lesssim \langle t+r\rangle^{-1}\cdot\varepsilon t^{-1+C\varepsilon}\langle r-t\rangle^{-s} + \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-s-1} \\
&\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-s}.
\end{aligned}$$

Here we apply Lemma 1.4, the pointwise decays in Theorem 4.2, and (1.17). By the chain rule and Leibniz's rule, we can express $e_4(\partial^s Z^I(g^{\alpha\beta}, g_{\alpha\beta}))$ as a linear combination of terms of the form

$$\frac{d^m}{du^m}(g^{\alpha\beta}, g_{\alpha\beta})(u) \cdot (\partial^{s_1} Z^{I_1} u) \cdots (\partial^{s_{m-1}} Z^{I_{m-1}} u) \cdot e_4(\partial^{s_m} Z^{I_m} u)$$

where $\sum s_* = s$, $\sum |I_*| = |I|$ and $m > 0$. These terms have an upper bound

$$\varepsilon t^{-1+C\varepsilon} \langle r-t\rangle^{-s_1} \cdots \varepsilon t^{-1+C\varepsilon} \langle r-t\rangle^{-s_{m-1}} \cdot \varepsilon t^{-2+C\varepsilon} \langle r-t\rangle^{-s_m} \lesssim \varepsilon t^{-2+C\varepsilon} \langle r-t\rangle^{-s}.$$

We thus have $e_4(\partial^s Z^I(g^{\alpha\beta}, g_{\alpha\beta})) = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-s})$.

Next we fix $(t_0, x_0) \in \Omega_T$. Without loss of generality, we assume $|q_3| = \max\{|q_j| : j = 1,2,3\}$ at (t_0, x_0). For $i = 1,2$, we define

$$Y_i := q_i\partial_3 - q_3\partial_i = r^{-1}q_r\Omega_{i3} + (q_i - \omega_i q_r)\partial_3 - (q_3 - \omega_3 q_r)\partial_i = r^{-1}q_r\Omega_{i3} + O(t^{-1+C\varepsilon})\partial.$$

Here $\{Y_1, Y_2\}$ is a basis of the tangent space of the 2-sphere $\Sigma_{(t_0,x_0)} = \{t = t_0, q = q(t_0, x_0)\}$ at (t_0, x_0). Since e_a lies in the tangent space (as $e_a^0 = 0$ and $e_a(q) = 0$), we can write $e_a = \sum_{i=1,2} c_{ai} Y_i$ in a unique way. Since

$$\langle Y_i, Y_j\rangle = q_iq_jg_{33} + q_3^2 g_{ij} - q_iq_3g_{3j} - q_jq_3g_{3i} = q_iq_j + q_3^2\delta_{ij} + O(|u|q_3^2), \qquad i,j = 1,2,$$

we have

$$1 = \langle e_a, e_a\rangle = \sum_{i,j} c_{ai}c_{aj}\langle Y_i, Y_j\rangle = (\sum_i c_{ai}q_i)^2 + (1 + O(|u|))q_3^2 \sum_i c_{ai}^2.$$

Then, for $\varepsilon \ll 1$ we have

$$1 \geq 0 + (1 + O(\varepsilon t^{-1+C\varepsilon}))q_3^2 \sum_i c_{ai}^2 \geq \frac{1}{2} q_3^2 \sum_i c_{ai}^2.$$

Thus, we have $|q_3 c_{ai}| \lesssim 1$ for each a, i and thus $e_a^\alpha = \sum_i c_{ai} Y_i^\alpha = O(|c_{ai}q_3|) = O(1)$. And since $C^{-1}t^{-C\varepsilon} \leq |q_r| = |\sum_i \omega_i q_i| \leq \sum_i |q_i| \leq 3|q_3|$, for each multiindex I, we have

$$\begin{aligned}|e_a(\partial^s Z^I u)| &\leq \sum_i |c_{ai} Y_i(\partial^s Z^I u)| \lesssim \sum_i |c_{ai}|(r^{-1}|q_r||\Omega\partial^s Z^I u| + t^{-1+C\varepsilon}|\partial\partial^s Z^I u|) \\ &\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-s}.\end{aligned}$$

By the chain rule and Leibniz's rule, we finish the proof of the first estimate.

In addition,

$$\begin{aligned}0 = \langle e_1, e_1\rangle - \langle e_2, e_2\rangle &= (\sum_i c_{1i}q_i)^2 - (\sum_i c_{2i}q_i)^2 + q_3^2 \sum_i (c_{1i}^2 - c_{2i}^2) + O(|u|q_3^2 \sum_{a,i} c_{ai}^2) \\ &= (\sum_i c_{1i}q_i)^2 - (\sum_i c_{2i}q_i)^2 + q_3^2 \sum_i (c_{1i}^2 - c_{2i}^2) + O(|u|) \\ &= \sum_{i,j}(c_{1i}c_{1j} - c_{2i}c_{2j})q_iq_j - (\sum_i c_{2i}q_i)^2 + q_3^2 \sum_i (c_{1i}^2 - c_{2i}^2) + O(|u|),\end{aligned} \tag{4.20}$$

$$\begin{aligned} 0=\langle e_1,e_2\rangle &= \sum_{i,j} c_{1i}c_{2j}\langle Y_i,Y_j\rangle = \sum_{i,j} c_{1i}c_{2j}q_iq_j + \sum_i c_{1i}c_{2i}q_3^2 + O(|u|q_3^2\sum_{i,j}|c_{1i}c_{2j}|) \\ &= \sum_{i,j} c_{1i}c_{2j}q_iq_j + \sum_i c_{1i}c_{2i}q_3^2 + O(|u|). \end{aligned} \tag{4.21}$$

Then, we have

$$Y_i(Zg) = r^{-1}q_r\Omega_{i3}g + O(t^{-1+C\varepsilon}|\partial g|) = O(\varepsilon t^{-2+C\varepsilon}),$$

$$\begin{aligned} Y_i(\partial_\alpha g)Y_j^\alpha &= (r^{-1}q_r\Omega_{i3}(\partial_\alpha g) + (q_i-\omega_iq_r)\partial_3\partial_\alpha g - (q_3-\omega_3q_r)\partial_i\partial_\alpha g)Y_j^\alpha \\ &= r^{-1}q_r(Y_j^\alpha[\Omega_{i3},\partial_\alpha]g + Y_j\Omega_{i3}g) + (q_i-\omega_iq_r)Y_j(\partial_3 g) - (q_3-\omega_3q_r)Y_j(\partial_i g) \\ &= r^{-1}q_r(-Y_j^i\partial_3 g + Y_j^3\partial_i g) + r^{-1}q_rY_j\Omega_{i3}g + O(t^{-1+C\varepsilon}|Y_j(\partial g)|) \\ &= r^{-1}q_r(\delta_{ij}q_3\partial_3 g + q_j\partial_i g) + O(\varepsilon t^{-3+C\varepsilon}), \end{aligned}$$

$$\begin{aligned} e_a(\partial_\alpha g)e_b^\alpha &= \sum_{i,j} c_{ai}Y_i(\partial_\alpha g)c_{bj}Y_j^\alpha = \sum_{i,j} c_{ai}c_{bj}(r^{-1}q_r(\delta_{ij}q_3\partial_3 g + q_j\partial_i g) + O(\varepsilon t^{-3+C\varepsilon})) \\ &= \sum_i r^{-1}c_{ai}c_{bi}q_rq_3\partial_3 g + \sum_{i,j} r^{-1}c_{ai}c_{bj}q_rq_j\partial_i g + O(\sum_{i,j}|c_{ai}c_{bj}||q_3|\varepsilon t^{-3+C\varepsilon}) \\ &= \sum_i r^{-1}c_{ai}c_{bi}q_rq_3\partial_3 g + \sum_{i,j} r^{-1}c_{ai}c_{bj}q_rq_j\partial_i g + O(\varepsilon t^{-3+C\varepsilon}). \end{aligned}$$

When $a\neq b$, by (4.21) we have

$$\begin{aligned} e_a(\partial_\alpha g)e_b^\alpha &= r^{-1}q_rq_3^{-1}(-\sum_{i,j} c_{ai}c_{bj}q_iq_j + O(|u|))\partial_3 g + \sum_{i,j} r^{-1}c_{ai}c_{bj}q_rq_j\partial_i g + O(\varepsilon t^{-3+C\varepsilon}) \\ &= r^{-1}q_rq_3^{-1}\sum_{i,j} c_{ai}c_{bj}q_j(-q_i\partial_3 g + q_3\partial_i g) + O(r^{-1}|q_rq_3^{-1}||u||\partial g|) + O(\varepsilon t^{-3+C\varepsilon}) \\ &= r^{-1}q_rq_3^{-1}\sum_{i,j} c_{ai}c_{bj}q_j(-Y_i g) + O(\varepsilon t^{-3+C\varepsilon}) = O(\varepsilon t^{-3+C\varepsilon}). \end{aligned}$$

By (4.20) we have

$$\begin{aligned} &e_1(\partial_\alpha g)e_1^\alpha - e_2(\partial_\alpha g)e_2^\alpha \\ &= \sum_i r^{-1}(c_{1i}^2 - c_{2i}^2)q_rq_3\partial_3 g + \sum_{i,j} r^{-1}(c_{1i}c_{1j} - c_{2i}c_{2j})q_rq_j\partial_i g + O(\varepsilon t^{-3+C\varepsilon}) \\ &= r^{-1}q_rq_3^{-1}(-\sum_{i,j}(c_{1i}c_{1j} - c_{2i}c_{2j})q_iq_j)\partial_3 g + \sum_{i,j} r^{-1}(c_{1i}c_{1j} - c_{2i}c_{2j})q_rq_j\partial_i g + O(\varepsilon t^{-3+C\varepsilon}) \\ &= \sum_{i,j} r^{-1}q_rq_3^{-1}q_j(c_{1i}c_{1j} - c_{2i}c_{2j})(-Y_i g) + O(\varepsilon t^{-3+C\varepsilon}) = O(\varepsilon t^{-3+C\varepsilon}). \end{aligned}$$

It is clear that our proof would still work if we assume $|q_1| = \max\{|q_j| : j=1,2,3\}$ or $|q_2| = \max\{|q_j| : j=1,2,3\}$. This ends the proof.

□

Lemma 4.14. *In Ω_T, we have $|q-(r-t)| \lesssim t^{C\varepsilon}$.*

Proof. By the previous lemma and Lemma 4.11, we have

$$e_4^i - \omega_i = \frac{2(q_i - q_r\omega_i) + 2(q_r+q_t)\omega_i + O(|u||\partial q|)}{L^0} = 2(L^0)^{-1}(q_i - q_r\omega_i) + O(\varepsilon t^{-1+C\varepsilon}).$$

Thus,

$$e_4(q-r+t) = (\partial_t + \partial_r)(-r+t) - 2(L^0)^{-1}\sum_i (q_i - q_r\omega_i)\omega_i + O(\varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1+C\varepsilon}).$$

Suppose $(t,x) \in \Omega_T$ lies on a geodesic $x(s)$ in Ω_T. Since $q-r+t=0$ on H, by integrating $e_4(q-r+t)$ along this geodesic, we have

$$|q-r+t| \lesssim \int_{x^0(0)}^t \varepsilon\tau^{-1+C\varepsilon}\,d\tau \lesssim t^{C\varepsilon}.$$

□

4.3.4 Connection coefficients

From now on, we write $D_k = D_{e_k}$ for $k=1,2,3,4$ for simplicity.

Lemma 4.15. *In Ω_T, we have*

$$D_4 e_k = (\Gamma^0_{\alpha\beta}e_4^\alpha e_k^\beta)e_4, \qquad k=1,2,4.$$

As a result, we have $e_4(e_k^\alpha) = O(\varepsilon t^{-2+C\varepsilon})$ for each $k=1,2,3,4$.

Proof. Since a geodesic in $\mathcal{A}$ is an integral curve of L, we have $L^\alpha = \dot{x}^\alpha(s)$ at $x(s)$. Then, the geodesic equation (4.9) implies

$$L(L^0) = \dot{x}^\alpha(s)(\partial_\alpha L^0) = \frac{d}{ds}L^0(x(s)) = \ddot{x}^0(s) = -\Gamma^0_{\mu\nu}L^\mu L^\nu, \qquad \text{at } x(s).$$

Divide both sides by L^0, and we conclude $e_4(L^0) = -\Gamma^0_{\mu\nu}e_4^\mu L^\nu$ in Ω_T and thus $e_4(\ln L^0) = -\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu$. Similarly, from (4.17) we obtain $e_4(E_a^0) = -\Gamma^0_{\mu\nu}e_4^\mu E_a^\nu$. Thus, we have

$$D_4e_4 = D_4((L^0)^{-1}L) = -(L^0)^{-2}e_4(L^0)L + (L^0)^{-1}D_4L = -(L^0)^{-1}e_4(L^0)e_4 = (\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu)e_4.$$

For $a=1,2$, since $D_4E_a = 0$, we have

$$\begin{aligned} D_4e_a &= D_4(E_a - E_a^0 e_4) = -D_4(E_a^0 e_4) = -e_4(E_a^0)e_4 - E_a^0 D_4 e_4 \\ &= (\Gamma^0_{\mu\nu}e_4^\mu E_a^\nu)e_4 - (E_a^0\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu)e_4 = \Gamma^0_{\mu\nu}e_4^\mu(E_a^\nu - E_a^0 e_4^\nu)e_4 \\ &= (\Gamma^0_{\mu\nu}e_4^\mu e_a^\nu)e_4. \end{aligned}$$

In addition, $D_4 e_k = e_4(e_k^\alpha)\partial_\alpha + \Gamma^\alpha_{\mu\nu} e_4^\mu e_k^\nu \partial_\alpha$. If we consider the coefficients of ∂_α in $D_4 e_k$ for $k = 1, 2, 4$, we have $e_4(e_k^\alpha) = \Gamma^0_{\mu\nu} e_4^\mu e_k^\nu e_4^\alpha - \Gamma^\alpha_{\mu\nu} e_4^\mu e_k^\nu$. By Lemma 4.13, we have

$$\begin{aligned}\Gamma^\alpha_{\mu\nu} &= \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}) \\ &= \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{2}(\sum_a e_a^\alpha e_a(g_{\mu\nu}) + \frac{1}{2}(e_3^\alpha e_4(g_{\mu\nu}) + e_4^\alpha e_3(g_{\mu\nu}))) \\ &= \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{4} e_4^\alpha e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon}).\end{aligned} \tag{4.22}$$

Then, since $e_4^0 = 1$, for $k = 1, 2, 4$ we have

$$\begin{aligned}e_4(e_k^\alpha) &= (\frac{1}{2} g^{0\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{4} e_4^0 e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon})) e_4^\mu e_k^\nu e_4^\alpha \\ &\quad - (\frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta}) - \frac{1}{4} e_4^\alpha e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon})) e_4^\mu e_k^\nu \\ &= \frac{1}{2} g^{0\beta}(e_4(g_{\nu\beta}) e_k^\nu e_4^\alpha + e_k(g_{\mu\beta}) e_4^\mu e_4^\alpha) + \frac{1}{2} g^{\alpha\beta}(e_4(g_{\nu\beta}) e_k^\nu + e_k(g_{\mu\beta}) e_4^\mu) \\ &\quad - \frac{1}{4} e_3(g_{\mu\nu})(e_4^\mu e_k^\nu e_4^\alpha e_4^0 - e_4^\mu e_k^\nu e_4^\alpha) + O(\varepsilon t^{-2+C\varepsilon}) \\ &= O(\varepsilon t^{-2+C\varepsilon}).\end{aligned}$$

It follows that $e_4(e_3^\alpha) = e_4(e_4^\alpha) + e_4(2g^{0\alpha}) = O(\varepsilon t^{-2+C\varepsilon})$. This finishes the proof. □

Remark 4.15.1. Since $e_3(q) = L^0$, we have

$$e_4(e_3(q)) = e_4(L^0) = -\Gamma^0_{\alpha\beta} e_4^\alpha L^\beta = -\Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta e_3(q).$$

This equality is useful in the rest of this chapter.

Next, we set $\chi_{ab} := \langle D_a e_4, e_b\rangle$ for $a, b = 1, 2$.

Lemma 4.16. *In Ω_T, we have*

(a) $\chi_{12} = \chi_{21}$.

(b) $\mathrm{tr}\chi := \chi_{11} + \chi_{22}$ *is independent of the choice of* e_1 *and* e_2.

(c)

$$[e_4, e_a] = -\sum_b \chi_{ab} e_b,\ \ D_a e_4 = \sum_b \chi_{ab} e_b + (e_4^\mu e_a^\nu \Gamma^0_{\mu\nu}) e_4,\ \ e_a(e_4^\alpha) \ = \sum_b \chi_{ab} e_b^\alpha + O(\varepsilon t^{-2+C\varepsilon}).$$

Proof. (a) Since $e_a(q) = 0$, we have

$$\langle e_4, [e_1, e_2]\rangle = (L^0)^{-1}\langle L, [e_1, e_2]\rangle = 2(L^0)^{-1}[e_1, e_2]q = 2(L^0)^{-1}(e_1(e_2(q)) - e_2(e_1(q))) = 0.$$

And since

$$\langle D_k e_l, e_m\rangle = e_k(\langle e_l, e_m\rangle) - \langle e_l, D_k e_m\rangle = -\langle e_l, D_k e_m\rangle, \qquad k,l,m = 1,2,3,4,$$

we have

$$\chi_{12} - \chi_{21} = \langle D_1 e_4, e_2\rangle - \langle D_2 e_4, e_1\rangle = \langle e_4, -D_1 e_2 + D_2 e_1\rangle = -\langle e_4, [e_1, e_2]\rangle = 0.$$

(b) Suppose that $\{e'_k\}$ is another null frame with $e_3 = e'_3$ and $e_4 = e'_4$. Then we have $e'_a = \sum_b \langle e'_a, e_b\rangle e_b$, $e_a = \sum_b \langle e_a, e'_b\rangle e'_b$ and thus

$$e_a = \sum_b \langle e_a, e'_b\rangle e'_b = \sum_{b,c} \langle e_a, e'_b\rangle\langle e'_b, e_c\rangle e_c \Longrightarrow \sum_{b,c}\langle e_a, e'_b\rangle\langle e'_b, e_c\rangle = \delta_{ac}.$$

Then,

$$\begin{aligned}\chi'_{11} + \chi'_{22} &= \sum_a \langle D_{e'_a} e_4, e'_a\rangle = \sum_a \sum_{b,c} \langle e'_a, e_b\rangle\langle e'_a, e_c\rangle\langle D_b e_4, e_c\rangle \\ &= \sum_{b,c}\sum_a \langle e'_a, e_b\rangle\langle e'_a, e_c\rangle \chi_{bc} = \sum_{b,c} \delta_{bc}\chi_{bc} = \chi_{11} + \chi_{22}.\end{aligned}$$

(c) Since $D_4 e_k = (\Gamma^0_{\alpha\beta} e_4^\alpha e_k^\beta) e_4$ for $k = 1,2,4$, we have $\langle D_4 e_k, e_a\rangle = 0$ for $k = 1,2,4$ and thus

$$\begin{aligned}\langle e_4, [e_4, e_a]\rangle &= \langle e_4, D_4 e_a - D_a e_4\rangle = -\langle D_4 e_4, e_a\rangle - \frac{1}{2} e_a \langle e_4, e_4\rangle = 0, \\ \langle e_b, [e_4, e_a]\rangle &= \langle e_b, D_4 e_a - D_a e_4\rangle = \langle e_b, D_4 e_a\rangle - \chi_{ab} = -\chi_{ab}.\end{aligned}$$

Since $e_4^0 = 1$ and $e_a^0 = 0$, we have $[e_4, e_a]^0 = 0$ (where $[e_4, e_a] = [e_4, e_a]^\alpha \partial_\alpha$) and thus

$$\langle e_3, [e_4, e_a]\rangle = \langle e_4, [e_4, e_a]\rangle + 2g^{0\alpha} g_{\alpha\beta} [e_4, e_a]^\beta = 0 + 2[e_4, e_a]^0 = 0.$$

By (4.18) we conclude that $[e_4, e_a] = -\sum_{b=1,2} \chi_{ab} e_b$. The second equality follows from $D_a e_4 = [e_a, e_4] + D_4 e_a$. The third one follows from $e_a(e_4^\alpha) - e_4(e_a^\alpha) = [e_a, e_4]^\alpha$ and the previous lemma. □

4.3.5 The Raychaudhuri equation

It turns out the estimates for χ_{ab} are crucial in the proof of the global existence of the optical function. To obtain such estimates, we need the Raychaudhuri equation

$$e_4(\chi_{ab}) = -\sum_c \chi_{ac}\chi_{cb} + \Gamma^0_{\alpha\beta} e_4^\alpha e_4^\beta \chi_{ab} + \langle R(e_4, e_a) e_4, e_b\rangle. \tag{4.23}$$

Here $\langle R(X,Y)Z,W\rangle := \langle D_XD_YZ - D_YD_XZ - D_{[X,Y]}Z,W\rangle$ is the curvature tensor. In fact, since $2\langle D_ae_4,e_4\rangle = e_a\langle e_4,e_4\rangle = 0$, we have

$$\begin{aligned}
e_4(\chi_{ab}) &= e_4\langle D_ae_4,e_b\rangle = \langle D_4D_ae_4,e_b\rangle + \langle D_ae_4,D_4e_b\rangle \\
&= \langle D_aD_4e_4,e_b\rangle + \langle D_{[e_4,e_a]}e_4,e_b\rangle + \langle R(e_4,e_a)e_4,e_b\rangle + \Gamma^0_{\alpha\beta}e_4^\alpha e_b^\beta\langle D_ae_4,e_4\rangle \\
&= \langle D_a(\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta e_4),e_b\rangle - \sum_c \chi_{ac}\langle D_ce_4,e_b\rangle + \langle R(e_4,e_a)e_4,e_b\rangle \\
&= e_a(\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta)\langle e_4,e_b\rangle + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{ab} - \sum_c \chi_{ac}\chi_{cb} + \langle R(e_4,e_a)e_4,e_b\rangle \\
&= \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{ab} - \sum_c \chi_{ac}\chi_{cb} + \langle R(e_4,e_a)e_4,e_b\rangle.
\end{aligned}$$

From (4.23), we can compute $e_4(\chi_{11}-\chi_{22})$, $e_4(\chi_{12})$ and $e_4(\mathrm{tr}\chi)$. Note that

$$\sum_c \chi_{1c}\chi_{c1} - \sum_c \chi_{2c}\chi_{c2} = \chi_{11}^2 - \chi_{22}^2 = \mathrm{tr}\chi(\chi_{11}-\chi_{22}),$$

$$\sum_c \chi_{1c}\chi_{c2} = \sum_c \chi_{2c}\chi_{c1} = \chi_{11}\chi_{12} + \chi_{12}\chi_{22} = \chi_{12}\mathrm{tr}\chi,$$

$$\sum_c \chi_{1c}\chi_{c1} + \sum_c \chi_{2c}\chi_{c2} = \chi_{11}^2 + \chi_{22}^2 + 2\chi_{12}^2 = \frac{1}{2}(\mathrm{tr}\chi)^2 + \frac{1}{2}(\chi_{11}-\chi_{22})^2 + 2\chi_{12}^2.$$

As for the curvature tensor, we have the following lemma.

Lemma 4.17. *In Ω_T, we have*

$$\langle R(e_4,e_a)e_4,e_b\rangle = e_4(f_{ab}) + \frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu} + O(\varepsilon^2 t^{-3+C\varepsilon})$$

where

$$f_{ab} := \frac{1}{2}(e_a^\beta e_b^\nu e_4(g_{\beta\nu}) - e_a^\beta e_4^\mu e_b(g_{\beta\mu})) - \frac{1}{2}e_4^\alpha e_a(g_{\alpha\nu})e_b^\nu = O(\varepsilon t^{-2+C\varepsilon}).$$

Moreover,

$$\langle R(e_4,e_1)e_4,e_1\rangle - \langle R(e_4,e_2)e_4,e_2\rangle = e_4(f_{11}-f_{22}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$\langle R(e_4,e_1)e_4,e_2\rangle = e_4(f_{12}) + O(\varepsilon t^{-3+C\varepsilon}),$$

$$\langle R(e_4,e_1)e_4,e_1\rangle + \langle R(e_4,e_2)e_4,e_2\rangle = e_4(\mathrm{tr}f - \frac{1}{2}e_4^\alpha e_4^\mu e_3(g_{\alpha\mu})) + O(\varepsilon^2 t^{-3+C\varepsilon}).$$

Proof. We have $\langle R(e_4, e_a)e_4, e_b\rangle = e_4^\alpha e_a^\beta e_4^\mu e_b^\nu R_{\alpha\beta\mu\nu}$ where $R_{\alpha\beta\mu\nu}$ is given by

$$\begin{aligned}
R_{\alpha\beta\mu\nu} &:= \langle R(\partial_\alpha, \partial_\beta)\partial_\mu, \partial_\nu\rangle = g_{\sigma\nu}(\partial_\alpha \Gamma^\sigma_{\beta\mu} - \partial_\beta \Gamma^\sigma_{\alpha\mu} + \Gamma^\delta_{\beta\mu}\Gamma^\sigma_{\alpha\delta} - \Gamma^\delta_{\alpha\mu}\Gamma^\sigma_{\beta\delta}) \\
&= \partial_\alpha \Gamma_{\nu\beta\mu} - \partial_\beta \Gamma_{\nu\alpha\mu} - \Gamma^\sigma_{\beta\mu}\partial_\alpha g_{\sigma\nu} + \Gamma^\sigma_{\alpha\mu}\partial_\beta g_{\sigma\nu} + \Gamma^\delta_{\beta\mu}\Gamma_{\nu\alpha\delta} - \Gamma^\delta_{\alpha\mu}\Gamma_{\nu\beta\delta} \\
&= \partial_\alpha \Gamma_{\nu\beta\mu} - \partial_\beta \Gamma_{\nu\alpha\mu} - \Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha} + \Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta} \\
&= \frac{1}{2}(\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\beta\partial_\nu g_{\alpha\mu}) - \Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha} + \Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta}.
\end{aligned}$$

Here for simplicity we set $\Gamma_{\alpha\mu\nu} := g_{\alpha\beta}\Gamma^\beta_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu})$. Then

$$\begin{aligned}
&\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu(\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\beta\partial_\nu g_{\alpha\mu}) \\
&= \frac{1}{2}e_4(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})e_a^\beta e_4^\mu e_b^\nu - \frac{1}{2}e_4^\alpha e_a^\beta e_4(\partial_\beta g_{\alpha\nu})e_b^\nu + \frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu} \\
&= e_4(\frac{1}{2}(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})e_a^\beta e_4^\mu e_b^\nu - \frac{1}{2}e_4^\alpha e_a^\beta(\partial_\beta g_{\alpha\nu})e_b^\nu) + \frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu} \\
&\quad + O(|\partial g|\sum_{k=1,2,4}|e_4(e_k^\alpha)|) \\
&= e_4(f_{ab}) + \frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu} + O(\varepsilon^2 t^{-3+C\varepsilon}).
\end{aligned}$$

To finish the proof of the first part, we note that

$$\Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha} = g^{\sigma\delta}\Gamma_{\sigma\beta\mu}\Gamma_{\delta\nu\alpha} = \frac{1}{4}g^{\sigma\delta}(\partial_\beta g_{\sigma\mu} + \partial_\mu g_{\beta\sigma} - \partial_\sigma g_{\beta\mu})(\partial_\alpha g_{\delta\nu} + \partial_\nu g_{\alpha\delta} - \partial_\delta g_{\alpha\nu}).$$

By (4.19), we have

$$\begin{aligned}
e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha} &= \frac{1}{4}g^{\sigma\delta}\partial_\sigma g\partial_\delta g + \sum_{k=1,2,4}O(1)e_k(g)\partial g \\
&= \frac{1}{4}\sum_{c=1,2}e_c(g)e_c(g) + \frac{1}{8}e_3(g)e_4(g) + \frac{1}{8}e_4(g)e_3(g) + O(\sum_{k=1,2,4}|e_k(g)||\partial g|) \\
&= O(\varepsilon t^{-2+C\varepsilon}\cdot\varepsilon t^{-1+C\varepsilon}) = O(\varepsilon^2 t^{-3+C\varepsilon}).
\end{aligned}$$

Similarly, we have $e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta} = O(\varepsilon^2 t^{-3+C\varepsilon})$.

To prove the second half, we only need to consider the term $\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu}$. By Lemma 4.13, we have

$$\frac{1}{2}e_4^\alpha e_1^\beta e_4^\mu e_2^\nu \partial_\beta\partial_\nu g_{\alpha\mu} = \frac{1}{2}e_4^\alpha e_4^\mu e_1^\beta e_2(\partial_\beta g_{\alpha\mu}) = O(\varepsilon t^{-3+C\varepsilon}),$$

$$\frac{1}{2}e_4^\alpha e_1^\beta e_4^\mu e_1^\nu \partial_\beta\partial_\nu g_{\alpha\mu} - \frac{1}{2}e_4^\alpha e_2^\beta e_4^\mu e_2^\nu \partial_\beta\partial_\nu g_{\alpha\mu} = \frac{1}{2}e_4^\alpha e_4^\mu(e_1^\beta e_1(\partial_\beta g_{\alpha\mu}) - e_2^\beta e_2(\partial_\beta g_{\alpha\mu})) = O(\varepsilon t^{-3+C\varepsilon}).$$

Finally, note that

$$\begin{aligned}\sum_a e_4^\alpha e_a^\beta e_4^\mu e_a^\nu \partial_\beta\partial_\nu g_{\alpha\mu} &= \frac{1}{2}e_4^\alpha e_4^\mu(g^{\beta\nu} - \frac{1}{2}e_3^\beta e_4^\nu - \frac{1}{2}e_4^\beta e_3^\nu)\partial_\beta\partial_\nu g_{\alpha\mu} \\ &= \frac{1}{2}e_4^\alpha e_4^\mu g^{\beta\nu}\partial_\beta\partial_\nu g_{\alpha\mu} - \frac{1}{2}e_4^\alpha e_4^\mu e_3^\beta e_4(\partial_\beta g_{\alpha\mu}) \\ &= -e_4(\frac{1}{2}e_4^\alpha e_4^\mu e_3^\beta \partial_\beta g_{\alpha\mu}) + O(\varepsilon^2 t^{-3+C\varepsilon}).\end{aligned}$$

We briefly explain how we obtain the third estimate here. If $F = F(u)$ is a function of u which is a solution to (1.1), then by (4.19)

$$\begin{aligned}g^{\beta\nu}\partial_\beta\partial_\nu(F(u)) &= F'(u)g^{\beta\nu}u_{\beta\nu} + F''(u)g^{\beta\nu}u_\beta u_\nu = 0 + F''(u)(\sum_c e_c(u)e_c(u) + e_3(u)e_4(u)) \\ &= O(\varepsilon t^{-3+C\varepsilon}).\end{aligned}$$

We thus have $e_4^\alpha e_4^\mu g^{\beta\nu}\partial_\beta\partial_\nu g_{\alpha\mu} = O(\varepsilon t^{-3+C\varepsilon})$. To handle the other term, we note that

$$e_4(\frac{1}{2}e_4^\alpha e_4^\mu e_3^\beta\partial_\beta g_{\alpha\mu}) - \frac{1}{2}e_4^\alpha e_4^\mu e_3^\beta e_4(\partial_\beta g_{\alpha\mu}) = \frac{1}{2}e_4(e_4^\alpha e_4^\mu e_3^\beta)\partial_\beta g_{\alpha\mu} = O(\varepsilon^2 t^{-3+C\varepsilon}).$$

□

Thus, it follows from (4.23) that

$$\begin{cases} e_4(\chi_{11}-\chi_{22}) = -\mathrm{tr}\chi(\chi_{11}-\chi_{22}) + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta(\chi_{11}-\chi_{22}) + e_4(f_{11}-f_{22}) + O(\varepsilon t^{-3+C\varepsilon}), \\ e_4(\chi_{12}) = -\chi_{12}\mathrm{tr}\chi + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{12} + e_4(f_{12}) + O(\varepsilon t^{-3+C\varepsilon}), \\ e_4(\mathrm{tr}\chi) = -\frac{1}{2}(\mathrm{tr}\chi)^2 - \frac{1}{2}(\chi_{11}-\chi_{22})^2 - 2\chi_{12}^2 + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\mathrm{tr}\chi \\ \qquad +e_4(\mathrm{tr}f - \frac{1}{2}e_4^\alpha e_4^\mu e_3(g_{\alpha\mu})) + O(\varepsilon^2 t^{-3+C\varepsilon}). \end{cases} \tag{4.24}$$

It turns out to be more convenient to work with (4.24) instead of (4.23).

4.3.6 Continuity argument

Fix a geodesic $x(s)$ in $\mathcal{A}$ with $x^0(0) \in H \cap \{t < T\}$. Since $\dot{x}^0(s) > 0$ for all $s \geq 0$ and $\lim_{s\to\infty} x^0(s) = \infty$, there exists a unique $0 < s_0 < \infty$ such that $x^0(s_0) = T$. Also fix some $s_1 \in [0, s_0]$. Our assumption is that for all $s \in [0, s_1]$, at $(t,x) = x(s) \in \Omega_T$ we have

$$\max_{a,b=1,2}|\chi_{ab} - \delta_{ab}r^{-1}| \leq At^{-2+B\varepsilon}. \tag{4.25}$$

Here A and B are large constants which are independent of T, ε, s_1, s_0 and the geodesic $x(s)$. In the derivation below, we always assume that the constants C in the inequalities are given before we choose A, B, and that the constants C are also independent of T, ε, s_1, s_0 and $x(s)$. Note that for $A, B \gg 1$, we have (4.25) for $s_1 = 0$ by the next lemma.

Lemma 4.18. *On H, we have $|\partial^2 q| \lesssim t^{-1}$ and $\max_{a,b=1,2} |\chi_{ab} - \delta_{ab} r^{-1}| \lesssim t^{-2+C\varepsilon}$.*

Proof. Recall from Section 4.3.1 that on H we have

$$(-1 - 4g^{0i}\omega_i + 4g^{ij}\omega_i\omega_j)q_t^2 + (4g^{ij}\omega_i\omega_j - 2g^{0i}\omega_i)q_t + g^{ij}\omega_i\omega_j = 0.$$

To compute $X_i q_t$ where $X_i = \partial_i + 2\omega_i\partial_t$, we apply X_i to the equation and then solve for $X_i q_t$. Then,

$$X_i q_t = -\frac{q_t^2 X_i(-1 - 4g^{0i}\omega_i + 4g^{ij}\omega_i\omega_j) + q_t X_i(4g^{ij}\omega_i\omega_j - 2g^{0i}\omega_i) + X_i(g^{ij}\omega_i\omega_j)}{2q_t(-1 - 4g^{0i}\omega_i + 4g^{ij}\omega_i\omega_j) + 4g^{ij}\omega_i\omega_j - 2g^{0i}\omega_i}.$$

Note that every term on the right hand side is known. The denominator is equal to $-2 + O(|u|)$ on H, so it is nonzero for $\varepsilon \ll 1$. In addition, we have $X_i\omega_j = O(r^{-1}) = O(t^{-1})$ and $X_i u = O(|\partial u|) = O(\varepsilon t^{-1})$, so $X_i q_t = O(t^{-1})$. Next, we have

$$X_i q_j = X_i(-\omega_j - 2\omega_j q_t) = -(\partial_i\omega_j)(1 + 2q_t) - \omega_i X_i q_t = O(t^{-1}).$$

By applying ∂_t to the eikonal equation, we have

$$0 = 2g^{\alpha\beta}q_\beta q_{t\alpha} + (\partial_t g^{\alpha\beta})q_\alpha q_\beta = 2g^{0\beta}q_\beta q_{tt} + 2g^{i\beta}q_\beta(X_i q_t - 2\omega_i q_{tt}) + (\partial_t g^{\alpha\beta})q_\alpha q_\beta.$$

And since $(q_t, q_i) = (-1, \omega) + O(|u|)$ on H, we have

$$q_{tt} = -\frac{2g^{i\beta}q_\beta X_i q_t + (\partial_t g^{\alpha\beta})q_\alpha q_\beta}{2g^{0\beta}q_\beta - 4g^{i\beta}\omega_i q_\beta} = -\frac{O(|\partial q|t^{-1} + \varepsilon t^{-1}|\partial q|^2)}{-2q_t - 4q_r + O(|u||\partial q|)} = O(t^{-1}).$$

Finally we note that $q_{it} = X_i q_t - 2\omega_i q_{tt} = O(t^{-1})$ and $q_{ij} = X_i q_j - 2\omega_i q_{jt} = O(t^{-1})$.

We move on to the estimates for χ. By definition, we have

$$\chi_{ab} = \langle D_a e_4, e_b\rangle = (e_a(e_4^\alpha) + e_a^\mu e_4^\nu \Gamma_{\mu\nu}^\alpha)e_b^\beta g_{\alpha\beta}.$$

As computed in Lemma 4.13, we have

$$\begin{aligned} e_a^\mu e_4^\nu \Gamma_{\mu\nu}^\alpha e_b^\beta g_{\alpha\beta} &= (\frac{1}{2}g^{\alpha\gamma}(\partial_\mu g_{\nu\gamma} + \partial_\nu g_{\mu\gamma}) - \frac{1}{4}e_4^\alpha e_3(g_{\mu\nu}) + O(\varepsilon t^{-2+C\varepsilon}))e_a^\mu e_4^\nu e_b^\beta g_{\alpha\beta} \\ &= \frac{1}{2}(e_a(g_{\nu\beta})e_4^\nu e_b^\beta g_{\alpha\beta} + e_4(g_{\mu\beta})e_a^\mu e_b^\beta g_{\alpha\beta}) - \frac{1}{4}e_3(g_{\mu\nu})e_a^\mu e_4^\nu\langle e_4, e_b\rangle + O(\varepsilon t^{-2+C\varepsilon}) \\ &= O(\varepsilon t^{-2+C\varepsilon}). \end{aligned}$$

In addition, recall from Section 4.3.1 that $q_i = \omega_i q_r$ on H. Since e_a is tangent to H, on H we have

$$e_a(q_i) = e_a(\omega_i q_r) = e_a^j r^{-1}(\delta_{ij} - \omega_i\omega_j)q_r + \omega_i e_a(q_r) = e_a^i r^{-1} - \omega_i q_r r^{-1} e_a(r) + \omega_i e_a(q_r).$$

Since e_a is tangent to the 2-sphere $\{t = t_0, q = q(t_0, x_0)\} = \{t = t_0, |x| = |x_0|\}$ at $(t_0, x_0) \in H$, we have $e_a(r) = e_a^i \omega_i = 0$ on H. Thus, on H we have

$$\begin{aligned} e_b^\gamma e_a(q_\gamma) &= e_b^i e_a(q_i) = \sum_i e_b^i (e_a^i r^{-1} - 0 + \omega_i e_a(q_r)) \\ &= r^{-1} g_{ij} e_a^i e_b^j - r^{-1}(g_{ij} - \delta_{ij}) e_a^i e_b^j + 0 = r^{-1}\delta_{ab} + O(\varepsilon t^{-2+C\varepsilon}). \end{aligned}$$

It follows that

$$e_a(e_4^\alpha) = e_a(\frac{L^\alpha}{L^0}) = \frac{L^0 e_a(2g^{\alpha\gamma}q_\gamma) - L^\alpha e_a(2g^{0\gamma}q_\gamma)}{(L^0)^2} = \frac{2(g^{\alpha\gamma} - e_4^\alpha g^{0\gamma})e_a(q_\gamma)}{L^0} + O(\varepsilon t^{-2+C\varepsilon}),$$

$$\begin{aligned} e_a(e_4^\alpha)e_b^\beta g_{\alpha\beta} &= \frac{2(e_b^\gamma - \langle e_4, e_b\rangle g^{0\gamma})e_a(q_\gamma)}{-2q_t + O(|u||\partial q|)} + O(\varepsilon t^{-2+C\varepsilon}) = \frac{2e_b^\gamma e_a(q_\gamma)}{2 + O(|u|)} + O(\varepsilon t^{-2+C\varepsilon}) \\ &= r^{-1}\delta_{ab} + O(\varepsilon t^{-2+C\varepsilon}). \end{aligned}$$

This finishes the proof. □

To complete the continuity argument, we need to prove (4.25) with A replaced by $A/2$. We start with χ_{12} and $\chi_{11} - \chi_{22}$. By (4.24), we have

$$\begin{aligned} e_4(r^2(\chi_{12} - f_{12})) &= 2re_4(r)(\chi_{12} - f_{12}) + r^2 e_4(\chi_{12} - f_{12}) \\ &= 2re_4(r)(\chi_{12} - f_{12}) + r^2((-\mathrm{tr}\chi + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta)\chi_{12} + O(\varepsilon t^{-3+C\varepsilon})) \\ &= r(2e_4(r) - r\mathrm{tr}\chi + r\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta)\chi_{12} - 2re_4(r)f_{12} + O(\varepsilon t^{-1+C\varepsilon}). \end{aligned}$$

Recall that $e_4(r) = 1 + O(t^{-1+C\varepsilon})$, $f_{12} = O(\varepsilon t^{-2+C\varepsilon})$ and $r\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta = O(r|\partial g|) = O(\varepsilon)$. By (4.25), we have $|2 - r\mathrm{tr}\chi| \leq 2Art^{-2+B\varepsilon}$. In conclusion,

$$\begin{aligned} |e_4(r^2(\chi_{12} - f_{12}))| &\leq r(2Art^{-2+B\varepsilon} + C\varepsilon + Ct^{-1+C\varepsilon}) \cdot At^{-2+B\varepsilon} + C\varepsilon t^{-1+C\varepsilon} \\ &\leq CA^2 t^{-2+2B\varepsilon} + CA\varepsilon t^{-1+B\varepsilon} + CAt^{-2+(B+C)\varepsilon} + C\varepsilon t^{-1+C\varepsilon} \\ &\leq CA^2 t^{-2+2B\varepsilon} + CA\varepsilon t^{-1+B\varepsilon}. \end{aligned}$$

By choosing $A, B \gg C$, we obtain the last inequality. On H, we have $|r^2(\chi_{12} - f_{12})| \leq Ct^{C\varepsilon}$ by the previous lemma. Thus, by integrating $e_4(r^2(\chi_{12} - f_{12}))$ along the geodesic, we have

$$\begin{aligned} |r^2(\chi_{12} - f_{12})| &\leq C(x^0(0))^{C\varepsilon} + CA^2(x^0(0))^{-1+2B\varepsilon} + CAB^{-1}t^{B\varepsilon} \\ &\leq Ct^{C\varepsilon} + CA^2 T_0^{-1+2B\varepsilon} + CAB^{-1}t^{B\varepsilon}. \end{aligned}$$

Since $T_0 \gg \varepsilon^{-1}$, we have $A^2T_0^{-1+2B\varepsilon} \leq 1$ for $\varepsilon \ll 1$. In addition, by choosing $B \geq A$, we have

$$|\chi_{12}| \leq r^{-2}(|f_{12}| + Ct^{C\varepsilon} + C + Ct^{B\varepsilon}) \leq Ct^{-2+B\varepsilon}.$$

Here C is independent of A and B, so if we choose $A \geq 4C$, we obtain with $|\chi_{12}| \leq \frac{1}{4}At^{-2+B\varepsilon}$. The proof for $|\chi_{11} - \chi_{22}| \leq \frac{1}{4}At^{-2+B\varepsilon}$ is essentially the same.

To finish the continuity argument, we need to prove that $|\text{tr}\chi - 2r^{-1}| \leq \frac{1}{4}At^{-2+B\varepsilon}$. For $h = \text{tr}\chi - \text{tr}f = \text{tr}\chi + O(\varepsilon t^{-2+C\varepsilon})$, by (4.25) we have $h = 2r^{-1} + O(At^{-2+B\varepsilon}) \sim 2r^{-1}$. Then, for $\varepsilon \ll 1$, by the last equation in (4.24) we have

$$\begin{aligned}
&e_4(h^{-1}) = -h^{-2}e_4(h)\\
&= -h^{-2}(-\frac{1}{2}(\text{tr}\chi)^2 + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta \text{tr}\chi - \frac{1}{2}e_4(e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})) + O(\varepsilon^2 t^{-3+C\varepsilon} + (\chi_{11} - \chi_{22})^2 + \chi_{12}^2))\\
&= -h^{-2}(-\frac{1}{2}h^2 + \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta h - \frac{1}{2}e_4(e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})) + O(\varepsilon t^{-3+C\varepsilon} + \varepsilon^2 t^{-3+C\varepsilon} + A^2t^{-4+2B\varepsilon}))\\
&= \frac{1}{2} - \Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta h^{-1} + \frac{1}{2}h^{-2}e_4^\alpha e_4^\beta e_4(e_3(g_{\alpha\beta})) + O(\varepsilon t^{-1+C\varepsilon}).
\end{aligned}$$

In the last line we use the product rule and the estimate $e_4(e_4^\alpha) = O(\varepsilon t^{-2+C\varepsilon})$. In addition, we have

$$|h^{-1} - r/2| = \frac{|2 - r(\text{tr}\chi - \text{tr}f)|}{2h} \lesssim r(|2 - r\text{tr}\chi| + |r\text{tr}f|) \lesssim At^{B\varepsilon};$$

by (4.22), we have

$$\begin{aligned}
\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta &= \frac{1}{2}g^{0\gamma}(e_4^\beta e_4(g_{\beta\gamma}) + e_4^\alpha e_4(g_{\alpha\gamma})) - \frac{1}{4}e_4^0 e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta + O(\varepsilon t^{-2+C\varepsilon})\\
&= -\frac{1}{4}e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta + O(\varepsilon t^{-2+C\varepsilon}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
e_4(h^{-1}) &= \frac{1}{2} + \frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})h^{-1} + \frac{1}{4}rh^{-1}e_4^\alpha e_4^\beta e_4(e_3(g_{\alpha\beta}))\\
&\quad + O(\varepsilon t^{-1+C\varepsilon} + h^{-1}\varepsilon t^{-2+C\varepsilon} + At^{B\varepsilon}h^{-1}|e_4(e_3(g))|)\\
&= \frac{1}{2} + \frac{1}{4}h^{-1}e_4^\alpha e_4^\beta(e_3(g_{\alpha\beta}) + re_4(e_3(g_{\alpha\beta}))) + O(At^{1+B\varepsilon}|e_4(e_3(g_{\alpha\beta}))| + \varepsilon t^{-1+C\varepsilon}).
\end{aligned} \tag{4.26}$$

The next three lemmas are necessary for us to control $e_3(g_{\alpha\beta}) + re_4(e_3(g_{\alpha\beta}))$ and $e_4(e_3(g_{\alpha\beta}))$.

Lemma 4.19. *Under the assumption* (4.25)*, in* Ω_T *we have* $|e_a(e_3(q))| + |e_a(\partial q)| \lesssim t^{-1+C\varepsilon}$, $|e_a(\Omega_{ij}q)| \lesssim At^{-1+B\varepsilon}|e_3(q)| + t^{-1+C\varepsilon}$ *and* $|\partial^2 q| \lesssim t^{C\varepsilon}$.

Proof. We have (assuming $\{a, a'\} = \{1, 2\}$)

$$\begin{aligned} e_4(e_a(e_3(q))) &= [e_4, e_a]e_3(q) + e_a(e_4(e_3(q))) = -\sum_b \chi_{ab} e_b(e_3(q)) - e_a(\Gamma^0_{\mu\nu} e_4^\mu e_4^\nu e_3(q)) \\ &= -\sum_b \chi_{ab} e_b(e_3(q)) - 2\Gamma^0_{\mu\nu}(\sum_b \chi_{ab} e_b^\mu + O(\varepsilon t^{-2+C\varepsilon}))e_4^\nu e_3(q) \\ &\quad - \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu e_a(e_3(q)) - e_a(\Gamma^0_{\mu\nu}) e_4^\mu e_4^\nu e_3(q) \\ &= -(\chi_{aa} + \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu) e_a(e_3(q)) - \chi_{12} e_{a'}(e_3(q)) \\ &\quad - (2\Gamma^0_{\mu\nu} \sum_b \chi_{ab} e_b^\mu e_4^\nu + e_a(\Gamma^0_{\mu\nu}) e_4^\mu e_4^\nu + O(\varepsilon t^{-2+C\varepsilon}|\Gamma|))e_3(q). \end{aligned}$$

Since $\chi_{ab} = r^{-1}\delta_{ab} + O(At^{-2+B\varepsilon}) \sim r^{-1}$ for $\varepsilon \ll_{A,B} 1$, the last term is $O(\varepsilon t^{-2+C\varepsilon}|e_3(q)|) = O(\varepsilon t^{-2+C\varepsilon})$. Then,

$$\begin{aligned} &|e_4(re_a(e_3(q)))| = |e_4(r)e_a(e_3(q)) + re_4(e_a(e_3(q)))| \\ &\leq |(1 + O(t^{-1+C\varepsilon}))e_a(e_3(q)) - r(\chi_{aa} + \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu)e_a(e_3(q)) - r\chi_{12} e_{a'}(e_3(q))| + C\varepsilon t^{-1+C\varepsilon} \\ &\leq (|r^{-1} - \chi_{aa}| + |\Gamma^0_{\mu\nu} e_4^\mu e_4^\nu| + O(t^{-2+C\varepsilon}))|re_a(e_3(q))| + |r\chi_{12} e_{a'}(e_3(q))| + C\varepsilon t^{-1+C\varepsilon} \\ &\leq (At^{-2+B\varepsilon} + C\varepsilon t^{-1} + Ct^{-2+C\varepsilon})|re_a(e_3(q))| + CAt^{-2+B\varepsilon}|re_{a'}(e_3(q))| + C\varepsilon t^{-1+C\varepsilon} \\ &\leq C\varepsilon t^{-1} \sum_b |re_b(e_3(q))| + C\varepsilon t^{-1+C\varepsilon}. \end{aligned}$$

In the last line, we choose $\varepsilon \ll 1$ so that $C\varepsilon t^{-1} \geq At^{-2+B\varepsilon} + t^{-2+C\varepsilon}$ for $t \geq T_0 = \exp(\delta/\varepsilon)$. Since e_a is tangent to H, on H we have $e_a(e_3(q)) = e_a(2g^{0\alpha}q_\alpha) = O(|\partial^2 q| + |e_a(g)\partial q|) = O(t^{-1})$ by Lemma 4.18. In conclusion, if $(t, x) \in \Omega_T$ lies on a geodesic $x(s)$ in $\mathcal{A}$, at (t, x) we have

$$\begin{aligned} \sum_a |re_a(e_3(q))| &\leq \sum_a |re_a(e_3(q))|(x(0)) + \int_{x^0(0)}^t C\varepsilon\tau^{-1} \sum_a |re_a(e_3(q))|(\tau, \widetilde{x}(\tau))\, d\tau + Ct^{C\varepsilon} \\ &\leq C + Ct^{C\varepsilon} + \int_{x^0(0)}^t C\varepsilon\tau^{-1} \sum_a |re_a(e_3(q))|(\tau, \widetilde{x}(\tau))\, d\tau. \end{aligned}$$

Here $(\tau, \widetilde{x}(\tau))$ is a reparametrization of the geodesic $x(s)$. We conclude that $\sum_a |re_a(e_3(q))| \lesssim Ct^{C\varepsilon}$ by the Gronwall's inequality. In addition, in Ω_T we have

$$\begin{aligned} e_a(q_\alpha) &= e_a(\frac{1}{2}\langle\partial_\alpha, e_4\rangle e_3(q)) = e_a(\frac{1}{2}e_4^\beta g_{\alpha\beta} e_3(q)) \\ &= \frac{1}{2}e_a(e_4^\beta)g_{\alpha\beta}e_3(q) + \frac{1}{2}e_4^\beta e_a(g_{\alpha\beta})e_3(q) + \frac{1}{2}e_4^\beta g_{\alpha\beta} e_a(e_3(q)) = O(t^{-1+C\varepsilon}). \end{aligned}$$

Next we compute $e_a(\Omega_{ij}q)$. Note that

$$\Omega_{ij}q = \frac{1}{2}\langle\Omega_{ij}, e_4\rangle e_3(q) = \frac{1}{2}(x_i g_{j\beta} - x_j g_{i\beta})e_4^\beta e_3(q) = \frac{1}{2}r(\omega_i g_{j\beta} e_4^\beta - \omega_j g_{i\beta} e_4^\beta)e_3(q).$$

We have

$$\omega_i g_{j\beta} e_4^\beta - \omega_j g_{i\beta} e_4^\beta = \omega_i e_4^j - \omega_j e_4^i + O(|u|) = O(\sum_j |e_4^j - \omega_j|) + O(|u|) = O(t^{-1+C\varepsilon}),$$

so $r(\omega_i g_{j\beta} e_4^\beta - \omega_j g_{i\beta} e_4^\beta) e_a(e_3(q)) = O(t^{-1+C\varepsilon})$. In addition,

$$\begin{aligned}
&e_a((x_i g_{j\beta} - x_j g_{i\beta}) e_4^\beta) \\
&= (e_a^i g_{j\beta} - e_a^j g_{i\beta}) e_4^\beta + (x_i g_{j\beta} - x_j g_{i\beta}) e_a(e_4^\beta) + O(|e_a(g)|) \\
&= e_a^i e_4^j - e_a^j e_4^i + (x_i g_{j\beta} - x_j g_{i\beta}) \sum_b (\chi_{ab} e_b^\beta + O(\varepsilon t^{-2+C\varepsilon})) + O(|e_a(g)| + |u|) \\
&= e_a^i e_4^j - e_a^j e_4^i + \sum_b \chi_{ab}(x_i e_b^j - x_j e_b^i + O(r|u|)) + O(\varepsilon t^{-1+C\varepsilon}) \\
&= e_a^i e_4^j - e_a^j e_4^i + r^{-1}(x_i e_a^j - x_j e_a^i) + O(r(|\chi_{aa} - r^{-1}| + |\chi_{12}|)) + O(\varepsilon t^{-1+C\varepsilon}) \\
&= e_a^i (e_4^j - \omega_j) - e_a^j (e_4^i - \omega_i) + O(At^{-1+B\varepsilon}) + O(\varepsilon t^{-1+C\varepsilon}) = O(At^{-1+B\varepsilon}).
\end{aligned}$$

By the product rule we obtain the second estimate.

Finally, we consider $\partial^2 q$. Recall that $e_4^\alpha = L^\alpha / L^0$ and that $|\partial q| \sim |q_r| \sim |q_t| \sim e_3(q)$. By the characteristic ODE's, we have

$$e_4(q_\alpha) = \frac{-(\partial_\alpha g^{\mu\nu}) q_\mu q_\nu}{e_3(q)} = O(\varepsilon t^{-1}) e_3(q)$$

and thus

$$\begin{aligned}
\partial_\alpha(e_4(q_\beta)) &= \frac{-\partial_\alpha((\partial_\beta g^{\mu\nu}) q_\mu q_\nu) e_3(q) + (\partial_\beta g^{\mu\nu}) q_\mu q_\nu \cdot 2\partial_\alpha(g^{0\gamma} q_\gamma)}{(e_3(q))^2} \\
&= \frac{-2(\partial_\beta g^{\mu\nu}) q_\mu q_{\alpha\nu} e_3(q) + (\partial_\beta g^{\mu\nu}) q_\mu q_\nu \cdot 2 g^{0\gamma} q_{\alpha\gamma}}{(e_3(q))^2} + O(\varepsilon t^{-1+C\varepsilon}) \\
&= O(|\partial g||\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1} |\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon}).
\end{aligned}$$

In the second line, we take out those terms without $\partial^2 q$ and control them using the estimates for g and ∂q. In the last line, we use the estimate $|\partial q| \sim e_3(q)$. Besides, we have

$$\begin{aligned}
\partial_\alpha e_4^\beta &= \frac{\partial_\alpha(L^\beta) L^0 - L^\beta \partial_\alpha(L^0)}{(L^0)^2} = \frac{2\partial_\alpha(g^{\beta\nu} q_\nu) - 2 e_4^\beta \partial_\alpha(g^{0\nu} q_\nu)}{e_3(q)} \\
&= \frac{2(g^{\beta\nu} - e_4^\beta g^{0\nu}) q_{\alpha\nu}}{e_3(q)} + O(|\partial g||\partial q|(e_3(q))^{-1}) \\
&= \frac{(\sum_a 2 e_a^\beta e_a^\nu + e_3^\beta e_4^\nu + e_4^\beta e_4^\nu) q_{\alpha\nu}}{e_3(q)} + O(\varepsilon t^{-1}) \\
&= \frac{2\sum_a e_a^\beta e_a(q_\alpha) + (e_3^\beta + e_4^\beta) e_4(q_\alpha)}{e_3(q)} + O(\varepsilon t^{-1}) = \frac{2\sum_a e_a^\beta e_a(q_\alpha)}{e_3(q)} + O(\varepsilon t^{-1}).
\end{aligned}$$

Thus, we have

$$\begin{aligned}e_4(q_{\alpha\beta}) &= [e_4, \partial_\alpha]q_\beta + \partial_\alpha(e_4(q_\beta)) = -\partial_\alpha(e_4^\mu)\partial_\mu(q_\beta) + \partial_\alpha(e_4(q_\beta))\\ &= O((e_3(q))^{-1}\sum_a |e_a(q_\beta)e_a(q_\alpha)|) + O(\varepsilon t^{-1}|\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon})\\ &= O(\varepsilon t^{-1}|\partial^2 q|) + O(\varepsilon t^{-1+C\varepsilon} + t^{-2+C\varepsilon}).\end{aligned}$$

In the last line we use the estimate $e_3(q) \geq C^{-1}t^{-C\varepsilon}$. Since $\partial^2 q = O(t^{-1})$ on H, we conclude $\partial^2 q = O(t^{C\varepsilon})$ by the Gronwall's inequality.

□

Lemma 4.20. *Set $h_i := r(\partial_i(ru) - q_i q_r^{-1}\partial_r(ru))$. Under the assumption (4.25), in Ω_T we have $|h_i| \lesssim \varepsilon t^{C\varepsilon}$, $|e_a(h_i)| \lesssim A\varepsilon t^{-1+B\varepsilon}$ and $e_a(ru) = \sum_i e_a(\omega_i)h_i$.*

Proof. We have

$$\begin{aligned}h_i &= r(\omega_i u + r u_i - q_i q_r^{-1} u - q_i q_r^{-1} r u_r) = r u q_r^{-1}(q_r\omega_i - q_i) + r^2(u_i - q_i q_r^{-1} u_r)\\ &= (ru + r^2 u_r)q_r^{-1}(q_r\omega_i - q_i) + r^2(u_i - \omega_i u_r) = (u + r u_r)q_r^{-1}\sum_j \omega_j\Omega_{ij}q + \sum_j x_j\Omega_{ji}u.\end{aligned}$$

Since $|u|+|u_r| \lesssim \varepsilon t^{-1+C\varepsilon}$, $|q_i - \omega_i q_r| \lesssim t^{-1+C\varepsilon}$ and $|u_i - \omega_i u_r| \lesssim \varepsilon t^{-2+C\varepsilon}$, we obtain $|h_i| \lesssim \varepsilon t^{C\varepsilon}$. Moreover,

$$e_a(x_j\Omega_{ij}u) = e_a^j\Omega_{ij}u + x_j e_a(\Omega_{ij}u) = O(\varepsilon t^{-1+C\varepsilon}),$$

$$\begin{aligned}e_a((u+ru_r)q_r^{-1}\omega_j\Omega_{ij}q) &= e_a(u+ru_r)q_r^{-1}\omega_j\Omega_{ij}q - (u+ru_r)q_r^{-2}e_a(q_r)\omega_j\Omega_{ij}q\\ &\quad + (u+ru_r)q_r^{-1}e_a(\omega_j)\Omega_{ij}q + (u+ru_r)q_r^{-1}\omega_j e_a(\Omega_{ij}q)\\ &= O(\varepsilon t^{-1+C\varepsilon}) + O(\varepsilon|q_r|^{-1}|e_a(\Omega q)|)\\ &= O(\varepsilon t^{-1+C\varepsilon}) + O(A\varepsilon t^{-1+B\varepsilon}\frac{e_3(q)}{q_r}) = O(A\varepsilon t^{-1+B\varepsilon}).\end{aligned}$$

Here we apply many estimates such as $e_a(r) = O(1)$, $e_a(\omega_i) = O(r^{-1})$, $\Omega q = O(t^{C\varepsilon})$, $q_r \geq C^{-1}t^{-C\varepsilon}$ and etc. In particular, we apply $e_a(\Omega q) = O(At^{-1+B\varepsilon}e_3(q) + t^{-1+C\varepsilon})$ from the previous lemma. Thus, we have $e_a(h_i) = O(A\varepsilon t^{-1+B\varepsilon})$.

Finally, we have

$$\begin{aligned}\sum_i e_a(\omega_i)h_i &= \sum_{i,j} e_a^j r^{-1}(\delta_{ij} - \omega_i\omega_j)h_i\\ &= \sum_i e_a^i(\partial_i(ru) - q_i q_r^{-1}\partial_r(ru)) - \sum_{i,j} e_a^j\omega_i\omega_j(\partial_i(ru) - q_i q_r^{-1}\partial_r(ru))\\ &= e_a(ru) - e_a(q)q_r^{-1}\partial_r(ru) - \sum_j e_a^j\omega_j\sum_i(\omega_i\partial_i(ru) - \omega_i q_i q_r^{-1}\partial_r(ru))\\ &= e_a(ru).\end{aligned}$$

□

Lemma 4.21. *Under the assumption* (4.25)*, in* Ω_T *we have* $|r^{-1}e_3(u)+e_4(e_3(u))| \lesssim \varepsilon A t^{-3+B\varepsilon}$ *and* $|e_4(e_3(u))| \lesssim \varepsilon t^{-2}$.

Proof. The second inequality follows directly from the first one. To prove the first one, we note that for each function $F = F(t,x)$, we have

$$\begin{aligned} g^{\alpha\beta}\partial_\alpha\partial_\beta F &= (\sum_a e_a^\alpha e_a^\beta + \frac{1}{2}e_4^\alpha e_3^\beta + \frac{1}{2}e_3^\alpha e_4^\beta)\partial_\alpha\partial_\beta F \\ &= \sum_a (e_a(e_a(F)) - e_a(e_a^\alpha)F_\alpha) + e_4(e_3(F)) - e_4(e_3^\alpha)F_\alpha \\ &= \sum_a (e_a(e_a(F)) - (D_a e_a)F + e_a^\mu e_a^\nu \Gamma_{\mu\nu}^\alpha F_\alpha) + e_4(e_3(F)) - (D_4 e_3)F + e_4^\mu e_3^\nu \Gamma_{\mu\nu}^\alpha F_\alpha. \end{aligned}$$

By (4.22), we have

$$\begin{aligned} e_a^\mu e_a^\nu \Gamma_{\mu\nu}^\alpha F_\alpha &= \frac{1}{2} g^{\alpha\beta} F_\alpha (e_a^\nu e_a(g_{\nu\beta}) + e_a^\mu e_a(g_{\mu\beta})) - \frac{1}{4} e_3(g_{\mu\nu}) e_a^\mu e_a^\nu e_4(F) + O(\varepsilon t^{-2+C\varepsilon}|\partial F|) \\ &= O(\varepsilon t^{-2+C\varepsilon}|\partial F| + \varepsilon t^{-1}|e_4(F)|), \end{aligned}$$

$$\begin{aligned} e_4^\mu e_3^\nu \Gamma_{\mu\nu}^\alpha F_\alpha &= \frac{1}{2} g^{\alpha\beta} F_\alpha (e_3^\nu e_4(g_{\nu\beta}) + e_4^\mu e_3(g_{\mu\beta})) - \frac{1}{4} e_4^\mu e_3^\nu e_3(g_{\mu\nu}) e_4(F) + O(\varepsilon t^{-2+C\varepsilon}|\partial F|) \\ &= \frac{1}{2}(\sum_a e_a^\beta e_a(F) + \frac{1}{2} e_3^\beta e_4(F) + \frac{1}{2} e_4^\beta e_3(F)) e_4^\mu e_3(g_{\mu\beta}) + O(\varepsilon t^{-2+C\varepsilon}|\partial F| + \varepsilon t^{-1}|e_4(F)|) \\ &= \frac{1}{4} e_3(F) e_4^\beta e_4^\mu e_3(g_{\mu\beta}) + O(\varepsilon t^{-2+C\varepsilon}|\partial F| + \varepsilon t^{-1} \sum_{k=1,2,4} |e_k(F)|). \end{aligned}$$

Moreover, since

$$\begin{aligned} D_a e_a &= \langle D_a e_a, e_{a'}\rangle e_{a'} + \frac{1}{2}\langle D_a e_a, e_4\rangle e_3 + \frac{1}{2}\langle D_a e_a, e_3\rangle e_4 \\ &= \langle D_a e_a, e_{a'}\rangle e_{a'} + \frac{1}{2}(-\chi_{aa}) e_3 + (-\frac{1}{2}\chi_{aa} + e_a^\mu e_a^\nu \Gamma_{\mu\nu}^0) e_4, \qquad a \neq a' \\ D_4 e_3 &= \sum_b \langle D_4 e_3, e_b\rangle e_b + \frac{1}{2}\langle D_4 e_3, e_4\rangle e_3 + \frac{1}{2}\langle D_4 e_3, e_3\rangle e_4 \\ &= -2\sum_b \Gamma_{\mu\nu}^0 e_4^\mu e_b^\nu e_b - \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu e_3, \end{aligned}$$

we have

$$\begin{aligned}\sum_a (D_a e_a)F &= \langle D_1 e_1, e_2\rangle e_2(F) + \langle D_2 e_2, e_1\rangle e_1(F) - \frac{1}{2}(\mathrm{tr}\chi)(e_3(F)+e_4(F)) + \sum_a e_a^\mu e_a^\nu \Gamma^0_{\mu\nu} e_4(F)\\ &= \langle D_1 e_1, e_2\rangle e_2(F) + \langle D_2 e_2, e_1\rangle e_1(F) - \frac{1}{2}(\mathrm{tr}\chi)e_3(F) + O(t^{-1}|e_4(F)|)\\ &= \langle D_1 e_1, e_2\rangle e_2(F) + \langle D_2 e_2, e_1\rangle e_1(F) - r^{-1}e_3(F) + O(t^{-1}|e_4(F)| + At^{-2+B\varepsilon}|e_3(F)|),\\ (D_4 e_3)F &= -2\sum_b \Gamma^0_{\mu\nu} e_4^\mu e_b^\nu e_b(F) - \Gamma^0_{\mu\nu} e_4^\mu e_4^\nu e_3(F)\\ &= \frac{1}{4} e_3(g_{\alpha\beta}) e_4^\alpha e_4^\beta e_3(F) + O(\varepsilon t^{-1}\sum_b |e_b(F)| + \varepsilon t^{-2+C\varepsilon}|e_3(F)|).\end{aligned}$$

Here we use the assumption (4.25) and $|e_3(u)| \lesssim |\partial u| \lesssim \varepsilon t^{-1}$. In conclusion, we have

$$\begin{aligned}g^{\alpha\beta}\partial_\alpha\partial_\beta F &= \sum_a e_a(e_a(F)) - \langle D_1 e_1, e_2\rangle e_2(F) - \langle D_2 e_2, e_1\rangle e_1(F) + e_4(e_3(F)) + r^{-1}e_3(F)\\ &\quad + O(t^{-1}|e_4(F)| + At^{-2+B\varepsilon}|e_3(F)|) + O(\varepsilon t^{-2+C\varepsilon}|\partial F| + \varepsilon t^{-1}\sum_{k=1,2,4}|e_k(F)|).\end{aligned}$$

By taking $F = u$, we obtain

$$\begin{aligned}0 = g^{\alpha\beta}\partial_\alpha\partial_\beta u &= \sum_a e_a(e_a(u)) - \langle D_1 e_1, e_2\rangle e_2(u) - \langle D_2 e_2, e_1\rangle e_1(u)\\ &\quad + r^{-1}e_3(u) + e_4(e_3(u)) + O(A\varepsilon t^{-3+B\varepsilon}).\end{aligned} \tag{4.27}$$

In addition, note that

$$\begin{aligned}e_4(e_3(F)) + r^{-1}e_3(F) &= e_4(2g^{0\alpha} + e_4^\alpha)F_\alpha + (2g^{0\alpha} + e_4^\alpha)e_4(F_\alpha) + r^{-1}e_3(F)\\ &= O((|e_4(g^{0\alpha})| + |e_4(e_4^\alpha)|)|\partial F| + |e_4(F_\alpha)| + r^{-1}|e_3(F)|)\\ &= O(\varepsilon t^{-2+C\varepsilon}|\partial F| + |e_4(\partial F)| + r^{-1}|e_3(F)|).\end{aligned}$$

Thus, we have

$$\begin{aligned}&|\sum_a e_a(e_a(F)) - \langle D_1 e_1, e_2\rangle e_2(F) - \langle D_2 e_2, e_1\rangle e_1(F)|\\ &\lesssim |\partial^2 F| + \varepsilon t^{-2+C\varepsilon}|\partial F| + r^{-1}|e_3(F)| + t^{-1}|e_4(F)| + At^{-2+B\varepsilon}|e_3(F)|) + \varepsilon t^{-1}\sum_{k=1,2,4}|e_k(F)|.\end{aligned}$$

When $F = r^{-1}$, the right hand side has an upper bound $Ct^{-3+C\varepsilon}$. When $F = \omega_i$, the right hand side has an upper bound $Ct^{-2+C\varepsilon}$. Here we choose $\varepsilon \ll_{A,B} 1$ so that $At^{-2+B\varepsilon}|e_3(r^{-1})| \lesssim At^{-4+B\varepsilon} \lesssim t^{-3}$ and $At^{-2+B\varepsilon}|e_3(\omega_i)| \lesssim At^{-3+B\varepsilon} \lesssim t^{-2}$.

We set $U(t,x) = ru(t,x)$. Then, by the previous lemma,

$$e_a(u) = e_a(r^{-1}U) = e_a(r^{-1})U + r^{-1}e_a(U) = e_a(r^{-1})U + r^{-1}\sum_i e_a(\omega_i)h_i,$$

$$\begin{aligned}e_a(e_a(u)) &= e_a(e_a(r^{-1}))U + 2e_a(r^{-1})\sum_i e_a(\omega_i)h_i + r^{-1}\sum_i e_a(e_a(\omega_i))h_i + r^{-1}\sum_i e_a(\omega_i)e_a(h_i)\\ &= e_a(e_a(r^{-1}))U + r^{-1}\sum_i e_a(e_a(\omega_i))h_i + O(A\varepsilon t^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon}).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\sum_a e_a(e_a(u)) - \langle D_1e_1, e_2\rangle e_2(u) - \langle D_2e_2, e_1\rangle e_1(u)\\ &= (\sum_a e_a(e_a(r^{-1})) - \langle D_1e_1, e_2\rangle e_2(r^{-1}) - \langle D_2e_2, e_1\rangle e_1(r^{-1}))U\\ &\quad + r^{-1}\sum_i(\sum_a e_a(e_a(\omega_i)) - \langle D_1e_1, e_2\rangle e_2(\omega_i) - \langle D_2e_2, e_1\rangle e_1(\omega_i))h_i + O(A\varepsilon t^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon})\\ &= O(t^{-3+C\varepsilon}|ru| + t^{-2+C\varepsilon}r^{-1}|h_i| + A\varepsilon t^{-3+B\varepsilon} + \varepsilon t^{-3+C\varepsilon}) = O(A\varepsilon t^{-3+B\varepsilon}).\end{aligned}$$

We finish the proof by this estimate and (4.27). □

We now finish the continuity argument. By writing $g'_{\alpha\beta} := \frac{d}{du}|_{u=0}g^{\alpha\beta}(u)$, we have

$$\begin{aligned}e_3(g_{\alpha\beta}) &= g'_{\alpha\beta}(u)e_3(u),\\ e_4(e_3(g_{\alpha\beta})) &= g'_{\alpha\beta}(u)e_4(e_3(u)) + g''_{\alpha\beta}(u)e_4(u)e_3(u)\\ &= O(\varepsilon t^{-2} + \varepsilon t^{-2+C\varepsilon}\cdot\varepsilon t^{-1}) = O(\varepsilon t^{-2}),\end{aligned}$$

and thus

$$\begin{aligned}e_3(g_{\alpha\beta}) + re_4(e_3(g_{\alpha\beta})) &= g'_{\alpha\beta}(u)(e_3(u) + re_4(e_3(u))) + g''_{\alpha\beta}(u)e_4(u)e_3(u)\\ &= O(rA\varepsilon t^{-3+B\varepsilon} + r\varepsilon t^{-2+C\varepsilon}\cdot\varepsilon t^{-1}) = O(A\varepsilon t^{-2+B\varepsilon}).\end{aligned}$$

Thus, by (4.26),

$$|e_4(h^{-1}) - \frac{1}{2}| \lesssim t\cdot A\varepsilon t^{-2+B\varepsilon} + At^{1+B\varepsilon}\cdot\varepsilon t^{-2} + \varepsilon t^{-1+C\varepsilon} \lesssim A\varepsilon t^{-1+B\varepsilon}.$$

By the initial condition, on H we have

$$|h^{-1} - r/2| = \frac{|2 - r(\mathrm{tr}\chi - \mathrm{tr}f)|}{2h} \lesssim r(|2 - r\mathrm{tr}\chi| + |r\mathrm{tr}f|) \lesssim t^{C\varepsilon}$$

where the constants are known before we choose A, B. Now, suppose that $(t,x) \in \Omega_T$ lies on a geodesic $x(s)$ in $\mathcal{A}$. At $x(0)$, we have $h^{-1}|_{x(0)} = r(x(0))/2 + O((x^0(0))^{C\varepsilon})$. Thus,

$$\begin{aligned}|h^{-1}|_{(t,x)} - \frac{1}{2}r(x(0)) - \frac{1}{2}(t - x^0(0))| &\le |h^{-1}|_{(t,x)} - h^{-1}|_{x(0)} - \frac{1}{2}(t - x^0(0))| + Ct^{C\varepsilon}\\ &\lesssim \int_{x^0(0)}^t A\varepsilon\tau^{-1+B\varepsilon}\,d\tau + t^{C\varepsilon} \lesssim B^{-1}At^{B\varepsilon} + t^{C\varepsilon}.\end{aligned}$$

Also note that $r(x(0)) - x^0(0) + t = q(t,x) + t = r + O(t^{C\varepsilon})$ by Lemma 4.14. In conclusion, $|h^{-1} - r/2| \lesssim t^{C\varepsilon} + B^{-1}At^{B\varepsilon}$ at (t,x). This implies that $h^{-1} \sim r$ and

$$\begin{aligned}|\mathrm{tr}\chi - \frac{2}{r}| &\leq |h - \frac{2}{r}| + C\varepsilon t^{-2+C\varepsilon} \lesssim |\frac{r - 2h^{-1}}{rh^{-1}}| + C\varepsilon t^{-2+C\varepsilon} \\ &\leq Cr^{-2}(Ct^{C\varepsilon} + CB^{-1}At^{B\varepsilon}) + C\varepsilon t^{-2+C\varepsilon} \leq Ct^{-2+C\varepsilon} + CB^{-1}At^{-2+B\varepsilon}.\end{aligned}$$

By choosing $B \geq A \gg_C 1$, we conclude that $|\mathrm{tr}\chi - 2/r| \leq \frac{1}{4}At^{-2+B\varepsilon}$. This finishes the continuity argument as we have proved that (4.25) holds with A replaced by $A/4$.

4.4 Derivatives of the optical function

In this section, we aim to prove that q is smooth in Ω, where smoothness is defined in Section 4.2.1. Our main result is the following proposition.

Proposition 4.22. *The optical function $q = q(t,x)$ constructed in Proposition 4.7 is a smooth function in Ω. Moreover, in Ω, we have $Z^I q = O(\langle q\rangle t^{C\varepsilon})$ and $Z^I \Omega_{ij} q = O(t^{C\varepsilon})$ for each multiindex I and $1 \leq i < j \leq 3$.*

In Section 4.4.1, we define the commutator coefficients ξ_{**}^{*} with respect to the null frame $\{e_k\}$, and derive several differential equations for ξ and their derivatives. Note that the estimates for these ξ would imply the estimates for q in Proposition 4.22. We also define a weighted null frame $\{V_k\}$ which will be used in the rest of this chapter. In Section 4.4.2, we focus on the estimates for q on the surface H where the initial data of q are assigned. In Section 4.4.3, we prove Proposition 4.31 which gives several important estimates for ξ. Here we make use of the differential equations and the estimates on H proved in the first two subsections. Finally, in Section 4.4.4, we conclude the proof of Proposition 4.22 by applying Proposition 4.31.

To end this section, in Section 4.4.5 we derive two equations (4.58) and (4.59) for $e_3(u)$ and $e_3(q)$, respectively. In these two equations, we have estimates for all derivatives of the remainder terms. While they are not related to the proof of Proposition 4.22, they will be very useful in the next section.

4.4.1 Setup

As a convention, we use k,l to denote a number in $\{1,2,3,4\}$, and we use a,b,c to denote a number in $\{1,2\}$. For a finite sequence of indices $K = (k_1,\ldots,k_m)$, we set $|K| = m$, $n_{K,k} = \{j : \ k_j = k\}$ and $e_K = e_{k_1}e_{k_2}\cdots e_{k_m}$.

4.4.1.1 Commutator coefficients

We define

$$\xi_{kl}^a = \langle [e_k,e_l], e_a\rangle, \ a = 1,2; \ \xi_{kl}^3 = \frac{1}{2}\langle [e_k,e_l], e_4\rangle, \ \xi_{kl}^4 = \frac{1}{2}\langle [e_k,e_l], e_3\rangle.$$

By (4.18) we have $[e_{k_1}, e_{k_2}] = \xi^l_{k_1k_2} e_l$. Thus these ξ^*_{**}'s are also called *commutator coefficients* in this chapter.

We now derive several equations for ξ. Note that $\xi^l_{k_1k_2} = -\xi^l_{k_2k_1}$ (so $\xi^l_{kk} = 0$) and that $\xi^3_{kl} = \xi^4_{kl}$ since $[e_k, e_l]$ never contains ∂_t. Thus, we only need to study those $\xi^l_{k_1k_2}$'s with $k_1 < k_2$ and $l \leq 3$.

We start with $[e_3, e_4]$. By Lemma 4.15 we have

$$\langle [e_3, e_4], e_4\rangle = \langle D_3 e_4 - D_4 e_3, e_4\rangle = -\langle D_4 e_3, e_4\rangle = \langle e_3, D_4 e_4\rangle = 2\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_4,$$

so $\xi^3_{34} = \Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_4$. For ξ^a_{34}, we have the following equation

$$\begin{aligned}
e_4(\xi^a_{34}) &= e_4(\langle D_3 e_4 - D_4 e_3, e_a\rangle) = e_4(\langle D_3 e_4, e_a\rangle) + e_4(\langle e_3, D_4 e_a\rangle)\\
&= \langle D_4 D_3 e_4, e_a\rangle + \langle D_3 e_4, D_4 e_a\rangle + 2e_4(\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_a)\\
&= \langle D_3 D_4 e_4, e_a\rangle + \langle D_{[e_4,e_3]} e_4, e_a\rangle + \langle R(e_4, e_3)e_4, e_a\rangle + \langle D_3 e_4, (\dots)e_4\rangle + 2e_4(\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_a)\\
&= \langle D_3((\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_4) e_4), e_a\rangle - \xi^l_{34}\langle D_l e_4, e_a\rangle + \langle R(e_4, e_3)e_4, e_a\rangle + 2e_4(\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_a)\\
&= -\chi_{ba}\xi^b_{34} + \langle R(e_4, e_3)e_4, e_a\rangle + 2e_4(\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_a).
\end{aligned}$$

Next we consider $[e_a, e_4]$. From Lemma 4.16, we have $\xi^b_{a4} = \chi_{ab}$ and $\xi^3_{a4} = 0$. Thus we have the Raychaudhuri equation

$$e_4(\chi_{ab}) = \Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_4 \chi_{ab} - \sum_c \chi_{ac}\chi_{cb} + \langle R(e_4, e_a)e_4, e_b\rangle.$$

Next we consider $[e_1, e_2]$. Note that $\xi^3_{12} = 0$ as $\langle [e_1, e_2], e_4\rangle = 0$. For ξ^a_{12}, we have $\xi^1_{12} = \langle D_1 e_2 - D_2 e_1, e_1\rangle = \langle D_1 e_2, e_1\rangle$ and $\xi^2_{12} = \langle D_1 e_2 - D_2 e_1, e_2\rangle = -\langle D_2 e_1, e_2\rangle = \langle D_2 e_2, e_1\rangle$. So, $\xi^a_{12} = \langle D_a e_2, e_1\rangle$ and

$$\begin{aligned}
e_4(\xi^a_{12}) &= e_4(\langle D_a e_2, e_1\rangle) = \langle D_4 D_a e_2, e_1\rangle + \langle D_a e_2, D_4 e_1\rangle\\
&= \langle D_a D_4 e_2, e_1\rangle + \langle D_{[e_4,e_a]} e_2, e_1\rangle + \langle R(e_4, e_a)e_2, e_1\rangle + \Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_1 \langle D_a e_2, e_4\rangle\\
&= \Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_2 \chi_{a1} - \Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_1 \chi_{a2} - \chi_{ac}\xi^c_{12} + \langle R(e_4, e_a)e_2, e_1\rangle.
\end{aligned}$$

We end with $[e_a, e_3]$. Note that

$$\begin{aligned}
\xi^3_{a3} &= \frac{1}{2}\langle D_a e_3 - D_3 e_a, e_4\rangle = -\frac{1}{2}\langle e_3, D_a e_4\rangle + \frac{1}{2}\langle e_a, D_3 e_4\rangle\\
&= -\frac{1}{2}\xi^4_{a4} - \frac{1}{2}\langle e_3, D_4 e_a\rangle + \frac{1}{2}\xi^a_{34} + \frac{1}{2}\langle e_a, D_4 e_3\rangle = -\langle e_3, D_4 e_a\rangle + \frac{1}{2}\xi^a_{34}\\
&= -2\Gamma^0_{\alpha\beta} e^\alpha_4 e^\beta_a + \frac{1}{2}\xi^a_{34},
\end{aligned}$$

$$\begin{aligned}
\xi^a_{a3} &= \langle D_a e_3 - D_3 e_a, e_a\rangle = \langle D_a e_3, e_a\rangle = \chi_{aa} + \langle D_a(2g^{0\alpha}\partial_\alpha), e_a\rangle\\
&= \chi_{aa} + 2e_a(g^{0\alpha}) g_{\alpha\beta} e^\beta_a + 2g^{0\alpha} e^\beta_a \Gamma^\mu_{\beta\alpha} g_{\mu\nu} e^\nu_a.
\end{aligned}$$

For ξ_{a3}^b where $a \neq b$, we have

$$\begin{aligned}
e_4(\xi_{a3}^b) &= e_4(\langle D_a e_3 - D_3 e_a, e_b\rangle) = e_4(\chi_{ab} + \langle D_a(2g^{0\alpha}\partial_\alpha), e_b\rangle - \langle D_3 e_a, e_b\rangle) \\
&= e_4(\chi_{ab} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_b^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_b^\nu) - \langle D_4 D_3 e_a, e_b\rangle - \langle D_3 e_a, D_4 e_b\rangle \\
&= e_4(\chi_{ab} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_b^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_b^\nu) - \langle D_3 D_4 e_a, e_b\rangle - \langle D_{[e_4,e_3]}e_a, e_b\rangle \\
&\quad - \langle R(e_4,e_3)e_a, e_b\rangle - \Gamma_{\alpha\beta}^0 e_4^\alpha e_b^\beta\langle D_3 e_a, e_4\rangle \\
&= (e_4 + \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu)(\chi_{ab} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_b^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_b^\nu) - \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu\xi_{a3}^b - \sum_c \xi_{34}^c\xi_{ab}^c \\
&\quad - \langle R(e_4,e_3)e_a, e_b\rangle - \Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta\xi_{34}^b + \Gamma_{\alpha\beta}^0 e_4^\alpha e_b^\beta\xi_{34}^a.
\end{aligned}$$

Given ξ, we can express $e_{k_1}(e_{k_2}^\alpha)$ in terms of e_*^* and ξ_{**}^*. In fact, the formulas for $e_4(e_k^\alpha)$ follow from Lemma 4.15. Besides,

$$\begin{aligned}
e_k(e_4^\alpha) &= [e_k, e_4]^\alpha + e_4(e_k^\alpha) = \xi_{k4}^l e_l^\alpha + e_4(e_k^\alpha), \\
e_k(e_3^\alpha) &= e_k(e_4^\alpha) + 2e_k(g^{0\alpha}), \\
e_3(e_k^\alpha) &= [e_3, e_k]^\alpha + e_k(e_3^\alpha) = \xi_{3k}^l e_l^\alpha + e_k(e_3^\alpha), \\
e_a(e_b^\alpha) &= (D_a e_b)^\alpha - e_a^\mu e_b^\nu\Gamma_{\mu\nu}^\alpha \\
&= \sum_c\langle D_a e_b, e_c\rangle e_c^\alpha + \frac{1}{2}\langle D_a e_b, e_3\rangle e_4^\alpha + \frac{1}{2}\langle D_a e_b, e_4\rangle e_3^\alpha - e_a^\mu e_b^\nu\Gamma_{\mu\nu}^\alpha \\
&= -\sum_c \xi_{bc}^a e_c^\alpha - \frac{1}{2}\chi_{ab}(e_4^\alpha + e_3^\alpha) - \langle e_b, D_a(g^{0\beta}\partial_\beta)\rangle e_4^\alpha - e_a^\mu e_b^\nu\Gamma_{\mu\nu}^\alpha \\
&= -\sum_c \xi_{bc}^a e_c^\alpha - \frac{1}{2}\chi_{ab}(e_4^\alpha + e_3^\alpha) - (e_b^\mu g_{\mu\beta}e_a(g^{0\beta}) + e_b^\mu g_{\mu\nu}g^{0\beta}e_a^\sigma\Gamma_{\sigma\beta}^\nu)e_4^\alpha - e_a^\mu e_b^\nu\Gamma_{\mu\nu}^\alpha.
\end{aligned}$$

4.4.1.2 A weighted null frame

A new frame $\{V_k\}$ defined below turns out to be very useful in this section.

Definition 4.23. We define a new frame $\{V_k\}_{k=1}^4$ by $V_a = re_a$ for $a = 1,2$ and $V_3 = (3R - r + t)e_3$ and $V_4 = te_4$. We call $\{V_k\}_{k=1}^4$ a *weighted null frame*, since V_k is a multiple of e_k for each k.

As usual, for each multiindex $K = (k_1, \dots, k_m)$ with $k_* \in \{1,2,3,4\}$, we define $V^I = V_{k_1}\cdots V_{k_m}$ as the product of $|I|$ vector fields.

It is easy to see that

$$\begin{cases} V_4 = t(t+r)^{-1}S + (t+r)^{-1}t\omega_j\Omega_{0j} + t(e_4^i - \omega_i)\partial_i, \\ V_3 = (3R - r + t)r^{-1}V_4 + 2g^{0\alpha}(3R - r + t)\partial_\alpha, \\ V_a = V_a(r)\omega_i\partial_i + e_a^i\omega_j\Omega_{ji}; \end{cases} \tag{4.28}$$

$$Z = r^{-1}\sum_a \langle Z, e_a\rangle V_a + \frac{1}{2}t^{-1}\langle Z, e_3\rangle V_4 + \frac{1}{2}(3R - r + t)^{-1}\langle Z, e_4\rangle V_3. \tag{4.29}$$

These formulas illustrate the connection between the weighted null frame and the commuting vector fields.

Here we briefly explain why we work with $\{V_k\}$. First, we note that

$$Z \approx \sum_{k\neq 3} O(t)e_k + O(\langle r-t\rangle)e_3 \approx \sum_k O(1)V_k.$$

If we work with a usual null frame, then in order to prove $Z^I q = O(\langle q\rangle t^{C\varepsilon})$, we might need to prove

$$|e_I(q)| \lesssim \langle r-t\rangle^{1-n_{I,3}} t^{-n_{I,1}-n_{I,2}-n_{I,4}+C\varepsilon} \tag{4.30}$$

where e_I and $n_{I,*}$ are defined at the beginning of Section 4.4.1. In contrast, if we work with a weighted null frame, then we can prove

$$|V^I q| \lesssim \langle r-t\rangle t^{C\varepsilon}. \tag{4.31}$$

Since (4.30) is much more complicated than (4.31), we expect the proof to be much simpler if we choose to work with the new weighted null frame.

Next, to prove an estimate for $V^I q$, we need to compute

$$e_4(V^I q) = t^{-1}\sum_{I=(J,j,J')} V^J[V_4, V_j]V^{J'}q.$$

Since V_k is a multiple of e_k for each k, we expect $[V_4, V_k]$ to be relatively simple. If we choose to work with the commuting vector fields defined in (1.13), then we need to compute either $[e_4, Z]$ or $[V_4, Z]$. Neither of these two terms has a simple form.

4.4.2 Estimates on H

We start with the estimates on the surface H. Recall that the vector fields $X_i = \partial_i + 2\omega_i\partial_t$ are tangent to H for $i = 1, 2, 3$. For a multiindex $I = (i_1, \dots, i_m)$ where $i_j \in \{1, 2, 3\}$, we write $X^I = X_{i_1}\cdots X_{i_m}$ and $|I| = m$.

In this subsection, we keep using the convention stated in Section 4.2.1.

We have the following pointwise estimate. We ask our readers to compare this lemma with Lemma 1.4.

Lemma 4.24. *Suppose that $F = F(t, x)$ is a smooth function whose domain is contained in $\{(t, x) \in \mathbb{R}^{1+3} :\ r \sim t \gtrsim 1\}$. Then, for nonnegative integers m, n, we have*

$$\sum_{|I|=m,\ |J|=n} |Z^I X^J F| \lesssim \langle r-t\rangle^{-n} \sum_{|I|\leq m+n} |Z^I F|.$$

Proof. We induct first on $m+n$ and then on n. There is nothing to prove when $n=0$. If $m=0$ and $n=1$, we simply apply Lemma 1.4. In general, we fix multiindicies I,J such that $|I|=m$ and $|J|=n$, such that $m+n>1$ and $n>0$. We can write $X^J=X^{J'}X_j$. Then, by our induction hypotheses, we have

$$\begin{aligned}|Z^IX^JF| &\le |Z^IX^{J'}\partial_jF|+|Z^IX^{J'}(\omega_j\partial_tF)|\\ &\lesssim \langle r-t\rangle^{1-n}\sum_{|K|\le n+m-1}(|Z^K\partial F|+|Z^K(\omega_j\partial_tF)|).\end{aligned}$$

Since $Z^K\omega=O(1)$ for each $|K|\ge 0$, by the Leibniz's rule we have

$$\begin{aligned}|Z^IX^JF| &\lesssim \langle r-t\rangle^{1-n}\sum_{|K|\le n+m-1}|Z^K\partial F|\lesssim \langle r-t\rangle^{1-n}\sum_{|K|\le n+m-1}|\partial Z^KF|\\ &\lesssim \langle r-t\rangle^{-n}\sum_{|K|\le n+m}|Z^KF|.\end{aligned}$$

In the second inequality here we use the commutation property $[Z,\partial]=C\partial$. □

The next lemma is a variant of Lemma 4.5 with Z replaced by X. Note that we do not need to assume that $(m_0^{\alpha\beta})$ satisfies the null condition defined in Section 4.2.

Lemma 4.25. *Fix two functions $\phi(t,x)$ and $\psi(t,x)$. Let $(m_0^{\alpha\beta})$ be a constant matrix. Then,*

$$X_i(m_0^{\alpha\beta}\phi_\alpha\psi_\beta)=m_0^{\alpha\beta}(\partial_\alpha X_i\phi)\psi_\beta+m_0^{\alpha\beta}\phi_\alpha(\partial_\beta X_i\psi)+r^{-1}\sum_{\alpha,\beta}f_0\phi_\alpha\psi_\beta.$$

Here f_0 denotes a polynomial of ω; we allow f_0 to vary from line to line.

Proof. We have $[X_i,\partial_\alpha]=-2(\partial_\alpha\omega_i)\partial_t$. By the Leibniz's rule, we have

$$\begin{aligned}X_i(m_0^{\alpha\beta}\phi_\alpha\psi_\beta) &= m_0^{\alpha\beta}(\partial_\alpha X_i\phi)\psi_\beta+m_0^{\alpha\beta}\phi_\alpha(\partial_\beta X_i\psi)-2m_0^{\alpha\beta}(\partial_\alpha\omega_i)\phi_t\psi_\beta-2m_0^{\alpha\beta}(\partial_\beta\omega_i)\psi_t\phi_\alpha\\ &= m_0^{\alpha\beta}(\partial_\alpha X_i\phi)\psi_\beta+m_0^{\alpha\beta}\phi_\alpha(\partial_\beta X_i\psi)\\ &\quad -2r^{-1}[m_0^{j\beta}(\delta_{ji}-\omega_j\omega_i)\phi_t\psi_\beta+m_0^{\alpha j}(\delta_{ji}-\omega_j\omega_i)\psi_t\phi_\alpha]\\ &= m_0^{\alpha\beta}(\partial_\alpha X_i\phi)\psi_\beta+m_0^{\alpha\beta}\phi_\alpha(\partial_\beta X_i\psi)+r^{-1}\sum_{\alpha,\beta}f_0\phi_\alpha\psi_\beta.\end{aligned}$$

□

Using the previous two lemmas, we can now prove the estimates for Z^Iq on H. In the next two lemmas, Ω^I denotes the product of $|I|$ vector fields in $\{\Omega_{12},\Omega_{23},\Omega_{13}\}$. In the rest of Section 4.4.2, we would use Ω to denote any vector field in $\{\Omega_{12},\Omega_{23},\Omega_{13}\}$ instead of the region. There should be no confusion as we focus on estimates on H.

Lemma 4.26. *On H, for all multiindices I, we have $Z^Iq=O(\langle q\rangle t^{C\varepsilon})$ and $Z^I\Omega q=O(t^{C\varepsilon})$.*

Proof. For convenience, we set

$$\mathcal{O}_{m,n,p} = \mathcal{O}_{m,n,p}(t,x) := \sum_{|I|=m,\ |J|=n,\ |K|=p} |Z^I X^J \Omega^K q|.$$

On H, we claim that

$$\mathcal{O}_{m,n,0} \lesssim \langle q\rangle^{1-n} t^{C\varepsilon},\ \forall m,n \geq 0; \qquad \mathcal{O}_{m,n,p} \lesssim \langle q\rangle^{-n} t^{C\varepsilon},\ \forall m,n \geq 0,\ p>0.$$

We first assume $m=0$. Since Ω and X are tangent to H and since $q|_H = r-t$, we have $X^J\Omega^K q = X^J\Omega^K(r-t)$ for all multiindices J,K. If $|K|>0$, we have $X^J\Omega^K(r-t)=0$; if $|J|>0$, we have $X^J(r-t) = O(r^{1-|J|}) = O(\langle q\rangle^{1-|J|})$. Then, on H we have $\mathcal{O}_{0,0,0} = |q|$, $\mathcal{O}_{0,n,p}=0$ for $p>0$, and $\mathcal{O}_{0,n,0} = O(\langle q\rangle^{1-n})$ for $n>0$. So the claim is true for $m=0$.

In general, we fix (m,n,p) with $m>0$. Suppose we have proved

$$\begin{aligned} \mathcal{O}_{m',n',0} &\lesssim \langle q\rangle^{1-n'} t^{C\varepsilon}, && \forall m',n' \geq 0 \text{ such that } m'+n' < m+n+p \\ & && \text{or } m'+n' = m+n+p,\ m'<m; \\ \mathcal{O}_{m',n',p'} &\lesssim \langle q\rangle^{-n'} t^{C\varepsilon}, && \forall m',n' \geq 0,\ p'>0 \text{ such that } m'+n'+p' < m+n+p \\ & && \text{or } m'+n'+p' = m+n+p,\ m'<m. \end{aligned} \tag{4.32}$$

From now on, we fix three multiindices I,J,K such that $|I|=m$, $|J|=n$, and $|K|=p$.

We write $Z^I = ZZ^{I'}$ and apply $Z^{I'}X^J\Omega^K$ to the eikonal equation. We have

$$0 = 2g^{\alpha\beta} q_\beta (\partial_\alpha Z^{I'} X^J \Omega^K q) + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3$$

where the remainders are given by

$$\begin{aligned} \mathcal{R}_1 &= Z^{I'}X^J\Omega^K(m^{\alpha\beta}q_\alpha q_\beta) - 2m^{\alpha\beta}(\partial_\alpha Z^{I'}X^J\Omega^K q) q_\beta, \\ \mathcal{R}_2 &= Z^{I'}X^J\Omega^K((g^{\alpha\beta}-m^{\alpha\beta})q_\alpha q_\beta) - 2(g^{\alpha\beta}-m^{\alpha\beta})q_\beta(Z^{I'}X^J\Omega^K q_\alpha), \\ \mathcal{R}_3 &= 2(g^{\alpha\beta}-m^{\alpha\beta})q_\beta(Z^{I'}X^J\Omega^K q_\alpha - \partial_\alpha Z^{I'}X^J\Omega^K q) \end{aligned}$$

We start with $\mathcal{R}_3$. Recall that $g-m = O(\varepsilon t^{-1+C\varepsilon})$ and $q_\beta = O(1)$ on H. Besides, $Z^{I'}X^J\Omega^K q_\alpha - \partial_\alpha Z^{I'}X^J\Omega^K q$ is a linear combination of terms of the following forms

$$\begin{aligned} Z^{I_1}[Z,\partial_\alpha]Z^{I_2}X^J\Omega^K q &= CZ^{I_1}\partial Z^{I_2}X^J\Omega^K q, && Z^{I_1}ZZ^{I_2} = Z^{I'}; \\ Z^{I'}X^{J_1}[X,\partial_\alpha]X^{J_2}\Omega^K q &= CZ^{I'}X^{J_1}((\partial_\alpha\omega)\partial_t X^{J_2}\Omega^K q), && X^{J_1}XX^{J_2} = X^J; \\ Z^{I'}X^J\Omega^{K_1}[\Omega,\partial_\alpha]\Omega^{K_2} q &= CZ^{I'}X^J\Omega^{K_1}\partial\Omega^{K_2} q, && \Omega^{K_1}\Omega\Omega^{K_2} = \Omega^K. \end{aligned}$$

The first row has an upper bound

$$\begin{aligned} \sum_{|K'|\leq|I_1|+|I_2|} |\partial Z^{K'}X^J\Omega^K q| &\lesssim \langle r-t\rangle^{-1} \sum_{|K'|\leq m-1} |Z^{K'}X^J\Omega^K q| = \langle q\rangle^{-1}\sum_{m'\leq m-1}\mathcal{O}_{m',n,p} \\ &\lesssim \langle q\rangle^{-1}\cdot\langle q\rangle^{1-n}t^{C\varepsilon} \lesssim \langle q\rangle^{-n}t^{C\varepsilon}. \end{aligned}$$

We can use the induction hypotheses (4.32) to control the sum $\sum_{m'\leq m-1}\mathcal{O}_{m',n,p}$, since $m'+n+p\leq m-1+n+p<m+n+p$. The second row has an upper bound

$$\begin{aligned}&\sum_{\substack{|I_1|+|I_2|=m-1\\|J_1'|+|J_2'|=|J_1|}}|Z^{I_1}X^{J_1'}\partial\omega|\cdot|Z^{I_2}X^{J_2'}\partial X^{J_2}\Omega^Kq|\\&\lesssim\sum_{|J_1'|+|J_2'|=|J_1|}r^{-1-|J_1'|}\cdot\langle r-t\rangle^{-|J_2'|-1-|J_2|}\sum_{|K'|\leq|I_2|+|J_2'|+1+|J_2|}|Z^{K'}\Omega^Kq|\\&\lesssim\langle q\rangle^{-1-n}\sum_{m'\leq m-1+n}\mathcal{O}_{m',0,p}\lesssim\langle q\rangle^{-n}t^{C\varepsilon}.\end{aligned}$$

In the first inequality we apply Lemma 1.4 and Lemma 4.24. In the second line, we apply (4.32). The third row has an upper bound

$$\begin{aligned}\langle r-t\rangle^{-n}\sum_{|K'|\leq m-1+n}|Z^{K'}\Omega^{K_1}\partial\Omega^{K_2}q|&\lesssim\langle r-t\rangle^{-1-n}\sum_{|K'|\leq m-1+n+|K_1|+1}|Z^{K'}\Omega^{K_2}q|\\&\lesssim\langle q\rangle^{-1-n}\sum_{m'\leq m-1+n+p}\mathcal{O}_{m',0,0}\lesssim\langle q\rangle^{-n}t^{C\varepsilon}.\end{aligned}$$

In conclusion, $\mathcal{R}_3=O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-n})$.

We move on to $\mathcal{R}_2$. By the Leibniz's rule, we can express $\mathcal{R}_2$ as a linear combination of terms of the form

$$Z^{I_1}X^{J_1}\Omega^{K_1}(g^{\alpha\beta}-m^{\alpha\beta})\cdot Z^{I_2}X^{J_2}\Omega^{K_2}q_\alpha\cdot Z^{I_3}X^{J_3}\Omega^{K_3}q_\beta,$$

where $\sum|I_*|=m-1$, $\sum|J_*|=n$, $\sum|K_*|=p$, $\max_{l=2,3}\{|I_l|+|J_l|+|K_l|\}<m+n+p-1$. On H, by Lemma 4.24 and (4.32) we have

$$|Z^{I_2}X^{J_2}\Omega^{K_2}q_\alpha|\lesssim\langle q\rangle^{-|J_2|}\sum_{|K'|\leq|I_2|+|J_2|+|K_2|}|Z^{K'}q_\alpha|\lesssim\langle q\rangle^{-|J_2|-1}\sum_{|K'|<m+n+p}|Z^{K'}q|\lesssim\langle q\rangle^{-|J_2|}t^{C\varepsilon}.$$

We can estimate $Z^{I_3}X^{J_3}\Omega^{K_3}q_\beta$ in the same way. And since $Z^{I_1}X^{J_1}\Omega^{K_1}(g^{\alpha\beta}-m^{\alpha\beta})=O(\varepsilon\langle q\rangle^{-|J_1|}t^{-1+C\varepsilon})$ by Lemma 4.24, we conclude that $\mathcal{R}_2=O(\varepsilon\langle q\rangle^{-n}t^{-1+C\varepsilon})$ on H.

We move on to $\mathcal{R}_1$. By Lemma 4.5, we can write $\Omega^K(m^{\alpha\beta}q_\alpha q_\beta)$ as a linear combination (with real constant coefficients) of terms of the form

$$m^{\alpha\beta}(\partial_\alpha\Omega^{K_1}q)(\partial_\beta\Omega^{K_2}q),\qquad\min\{1,p\}\leq|K_1|+|K_2|\leq p.\tag{4.33}$$

Here $(m^{\alpha\beta})$ is the usual Minkowski metric. In fact, if $p=0$, then (4.33) is $m^{\alpha\beta}q_\alpha q_\beta$ so there is nothing to prove; if $p>0$, then we guarantee that $|K_1|+|K_2|>0$ in (4.33) since

$$\Omega^K(m^{\alpha\beta}q_\alpha q_\beta)=\Omega^{K'}(m^{\alpha\beta}(\partial_\alpha\Omega q)q_\beta+m^{\alpha\beta}q_\alpha(\partial_\beta\Omega q)),\qquad\Omega^K=\Omega^{K'}\Omega.$$

Next we consider $X^J\Omega^K(m^{\alpha\beta}q_\alpha q_\beta)$, so we apply X^J to (4.33). By Lemma 4.25, we can write $X^J\Omega^K(m^{\alpha\beta}q_\alpha q_\beta)$ as a linear combination (with real constant coefficients) of terms of the form

$$\begin{cases} m^{\alpha\beta}(\partial_\alpha X^{J_1}\Omega^{K_1}q)(\partial_\beta X^{J_2}\Omega^{K_2}q), & |J_1|+|J_2|=n, \\ & \min\{1,p\}\le|K_1|+|K_2|\le p; \\ X^{J_1}(r^{-1}f_0)\cdot(X^{J_2}\partial X^{J_2'}\Omega^{K_1}q)(X^{J_3}\partial X^{J_3'}\Omega^{K_2}q), & \sum|J_*|+|J_*'|=n-1, \\ & \min\{1,p\}\le|K_1|+|K_2|\le p. \end{cases}$$

Again $(m^{\alpha\beta})$ is the Minkowski metric. We finally apply $Z^{I'}$ to each of these terms. By Lemma 4.5 and the Leibniz's rule, we can write $\mathcal{R}_1$ as a linear combination (with real constant coefficients) of terms of the form

$$\begin{cases} m_0^{\alpha\beta}(\partial_\alpha Z^{I_1}X^{J_1}\Omega^{K_1}q)(\partial_\beta Z^{I_2}X^{J_2}\Omega^{K_2}q), \\ \qquad |I_1|+|I_2|\le m-1,\ |J_1|+|J_2|=n,\ \min\{1,p\}\le|K_1|+|K_2|\le p \\ \qquad |I_1|+|J_1|+|K_1|,\ |I_2|+|J_2|+|K_2|<m-1+n+p; \\ Z^{I_3}X^{J_3}(r^{-1}f_0)\cdot(Z^{I_1}X^{J_1}\partial X^{J_1'}\Omega^{K_1}q)(Z^{I_2}X^{J_2}\partial X^{J_2'}\Omega^{K_2}q), \\ \qquad \sum|I_*|=m-1,\ \sum|J_*|+|J_*'|=n-1,\ \min\{1,p\}\le|K_1|+|K_2|\le p. \end{cases} \tag{4.34}$$

Here $(m_0^{\alpha\beta})$ is some constant matrix satisfying the null condition defined in Section 4.2. It follows from Lemma 4.6 that on H the terms of the first type in (4.34) has an upper bound

$$\begin{aligned} &\langle t\rangle^{-1}\sum_{|L|=1}(|Z^LZ^{I_1}X^{J_1}\Omega^{K_1}q||\partial Z^{I_2}X^{J_2}\Omega^{K_2}q|+|\partial Z^{I_1}X^{J_1}\Omega^{K_1}q||Z^LZ^{I_2}X^{J_2}\Omega^{K_2}q|) \\ &\lesssim t^{-1}\langle q\rangle^{-1}\sum_{|L_1|=|L_2|=1}|Z^{L_1}Z^{I_1}X^{J_1}\Omega^{K_1}q||Z^{L_2}Z^{I_2}X^{J_2}\Omega^{K_2}q|\lesssim t^{-1}\langle q\rangle^{-1}\mathcal{O}_{1+|I_1|,|J_1|,|K_1|}\mathcal{O}_{1+|I_2|,|J_2|,|K_2|}. \end{aligned}$$

Since $\min_{l=1,2}\{|I_l|+|J_l|+|K_l|+1\}<m+n+p$ and since $|J_1|+|J_2|=n$, we can apply (4.32) to conclude that on H

$$\begin{aligned} |m_0^{\alpha\beta}(\partial_\alpha Z^{I_1}X^{J_1}\Omega^{K_1}q)(\partial_\beta Z^{I_2}X^{J_2}\Omega^{K_2}q)|&\lesssim t^{-1+C\varepsilon}\langle q\rangle^{1-n}, \qquad \text{if } p=0; \\ |m_0^{\alpha\beta}(\partial_\alpha Z^{I_1}X^{J_1}\Omega^{K_1}q)(\partial_\beta Z^{I_2}X^{J_2}\Omega^{K_2}q)|&\lesssim t^{-1+C\varepsilon}\langle q\rangle^{-n}, \qquad \text{if } p>0. \end{aligned}$$

Meanwhile, by Lemma 4.24 and (4.32), on H we have

$$\begin{aligned} |Z^{I_3}X^{J_3}(r^{-1}f_0)|&\lesssim t^{-1+C\varepsilon}\langle q\rangle^{-|J_3|}, \\ |Z^{I_1}X^{J_1}\partial X^{J_1'}\Omega^{K_1}q|&\lesssim\langle q\rangle^{-1-|J_1|-|J_1'|}\sum_{m'\le|I_1|+1+|J_1|+|J_1'|}\mathcal{O}_{m',0,|K_1|}, \\ |Z^{I_2}X^{J_2}\partial X^{J_2'}\Omega^{K_2}q|&\lesssim\langle q\rangle^{-1-|J_2|-|J_2'|}\sum_{m'\le|I_2|+1+|J_2|+|J_2'|}\mathcal{O}_{m',0,|K_2|}. \end{aligned}$$

Here we can apply (4.32) as $\max_{l=1,2}\{|I_l|+|J_l|+|J_l'|+|K_l|+1\}<m+n+p$. Thus, the product of these terms is $O(t^{-1+C\varepsilon}\langle q\rangle^{1-n})$ if $p=0$, or $O(t^{-1+C\varepsilon}\langle q\rangle^{-n})$ if $p>0$. Thus, on H we have $\mathcal{R}_1=O(t^{-1+C\varepsilon}\langle q\rangle^{1-n})$ if $p=0$, and $\mathcal{R}_1=O(t^{-1+C\varepsilon}\langle q\rangle^{-n})$ if $p>0$. In conclusion, we have

$$\begin{aligned} 2g^{\alpha\beta}q_\beta(\partial_\alpha Z^{I'}X^J\Omega^K q)&=O(t^{-1+C\varepsilon}\langle q\rangle^{1-n}), &&\text{if } p=0;\\ 2g^{\alpha\beta}q_\beta(\partial_\alpha Z^{I'}X^J\Omega^K q)&=O(t^{-1+C\varepsilon}\langle q\rangle^{-n}), &&\text{if } p>0.\end{aligned}$$

Next, we note that

$$\begin{aligned} X_jZ^{I'}X^J\Omega^Kq&=Z^{I'}X_jX^J\Omega^Kq+\sum_{I'=(I_1,i,I_2)}Z^{I_1}[X_j,Z_i]Z^{I_2}X^J\Omega^Kq,\\ \Omega_{kk'}Z^{I'}X^J\Omega^Kq&=Z^{I'}X^J\Omega_{kk'}\Omega^Kq+\sum_{I'=(I_1,i,I_2)}Z^{I_1}[\Omega_{kk'},Z_i]Z^{I_2}X^J\Omega^Kq\\ &\quad+\sum_{J=(J_1,j,J_2)}Z^{I'}X^{J_1}[\Omega_{kk'},X_j]X^{J_2}\Omega^Kq.\end{aligned}$$

Recall that $[\Omega,Z]=\sum f_0Z$ and $[X,Z]=\sum f_0\partial$ where f_0 denotes any function such that $Z^{K'}f_0=O(1)$ for all K'. By Lemma 1.4 we have

$$\begin{aligned} |X_jZ^{I'}X^J\Omega^Kq|&\lesssim\mathcal{O}_{m-1,n+1,p}+\sum_{I'=(I_1,i,I_2)}|Z^{I_1}(f_0\partial Z^{I_2}X^J\Omega^Kq)|\\ &\lesssim\mathcal{O}_{m-1,n+1,p}+\langle q\rangle^{-1}\sum_{m'\le m-1}\mathcal{O}_{m',n,p},\end{aligned}$$

$$\begin{aligned} &|\Omega_{kk'}Z^{I'}X^J\Omega^Kq|\\ &\lesssim\mathcal{O}_{m-1,n,p+1}+\sum_{I'=(I_1,i,I_2)}|Z^{I_1}(f_0ZZ^{I_2}X^J\Omega^Kq)|+\sum_{J=(J_1,j,J_2)}|Z^{I'}X^{J_1}(f_0\partial X^{J_2}\Omega^Kq)|\\ &\lesssim\mathcal{O}_{m-1,n,p+1}+\sum_{m'\le m-1}\mathcal{O}_{m',n,p}+\sum_{|J_1|+|J_2|=n-1}\langle q\rangle^{-|J_1|}|Z^{I'}Z^{J_1}(f_0\partial X^{J_2}\Omega^Kq)|\\ &\lesssim\mathcal{O}_{m-1,n,p+1}+\sum_{m'\le m-1}\mathcal{O}_{m',n,p}+\langle q\rangle^{-n}\sum_{m'\le m+n-1}\mathcal{O}_{m',0,p}.\end{aligned}$$

In conclusion, on H we have

$$\begin{aligned} &|XZ^{I'}X^J\Omega^Kq|\lesssim\langle q\rangle^{-n}t^{C\varepsilon}, &&\text{if } p=0; &&|XZ^{I'}X^J\Omega^Kq|\lesssim\langle q\rangle^{-1-n}t^{C\varepsilon}, &&\text{if } p>0;\\ &|\Omega Z^{I'}X^J\Omega^Kq|\lesssim\langle q\rangle^{1-n}t^{C\varepsilon}, &&\text{if } p=0; &&|\Omega Z^{I'}X^J\Omega^Kq|\lesssim\langle q\rangle^{-n}t^{C\varepsilon}, &&\text{if } p>0.\end{aligned}$$

We now end the proof. By setting $L^\alpha=2g^{\alpha\beta}q_\beta$ and $L=L^\alpha\partial_\alpha$, we have

$$\begin{aligned} \partial_t&=\frac{L-L^iX_i}{L^0-2\omega_iL^i}=-\frac{1}{2}L+\sum_i\omega_iX_i+O(|u|)L+\sum_iO(|u|)X_i,\\ \partial_j&=X_j-2\omega_j\partial_t=\omega_jL+X_j-2\omega_j\sum_i\omega_iX_i+O(|u|)L+\sum_iO(|u|)X_i.\end{aligned}$$

Note that $L^0 = 2 + O(|u|)$ and $L^i = 2\omega_i + O(|u|)$ on H. Then, we have

$$\begin{aligned} S &= (-\frac{1}{2}t + r)L + (t-r)\sum_i \omega_i X_i + O((r+t)|u|)L + \sum_i O((r+t)|u|)X_i \\ &= O(t + \varepsilon t^{C\varepsilon})L + \sum_i O(\langle q\rangle + \varepsilon t^{C\varepsilon})X_i. \end{aligned}$$

And since $\Omega_{kk'} = x_k X_{k'} - x_{k'} X_k$, we have $\sum_k r^{-1}\omega_k \Omega_{kk'} = X_{k'} - \sum_k \omega_{k'}\omega_k X_k$. Thus,

$$\begin{aligned} \Omega_{0j} &= (-\frac{1}{2}x_j + t\omega_j)L + tX_j + (x_j - 2t\omega_j)\sum_i \omega_i X_i + O((r+t)|u|)L + \sum_i O((r+t)|u|)X_i \\ &= t(X_j - \omega_j\omega_i X_i) + O(t + \varepsilon t^{C\varepsilon})L + \sum_i O(\langle q\rangle + \varepsilon t^{C\varepsilon})X_i \\ &= tr^{-1}\sum_i \omega_i \Omega_{ij} + O(t + \varepsilon t^{C\varepsilon})L + \sum_i O(\langle q\rangle + \varepsilon t^{C\varepsilon})X_i. \end{aligned}$$

In conclusion, for each $Z \in \{\partial_\alpha, S, \Omega_{0j}\}$, we have

$$|ZZ^{I'}X^J\Omega^K q| \lesssim \sum_{1\le i<j\le 3} |\Omega_{ij}Z^{I'}X^J\Omega^K q| + t|LZ^{I'}X^J\Omega^K q| + (\langle q\rangle + t^{C\varepsilon})\sum_i |X_i Z^{I'}X^J\Omega^K q|.$$

If $p = 0$, the right hand side has an upper bound $\langle q\rangle^{1-n}t^{C\varepsilon}$; if $p > 0$, the right hand side has an upper bound $\langle q\rangle^{-n}t^{C\varepsilon}$. We finish the proof by induction. □

Lemma 4.27. *On H, we have $Z^I(q_i - \omega_i q_r) = O(t^{-1+C\varepsilon})$ and $Z^I(q_t + q_r) = O(\varepsilon t^{-1+C\varepsilon})$ for each I. As a result, $Z^I(q_i + \omega_i q_t) = O(t^{-1+C\varepsilon})$.*

Proof. Recall that $q_i - \omega_i q_r = \sum_j r^{-1}\omega_j \Omega_{ji} q$. By Lemma 4.26 and the Leibniz's rule, for each I we have

$$|Z^I(r^{-1}\omega_j\Omega_{ji}q)| \lesssim \sum_{|I_1|+|I_2|=|I|} |Z^{I_1}(r^{-1}\omega_j)| \cdot |Z^{I_2}\Omega q| \lesssim t^{-1+C\varepsilon}.$$

So $Z^I(q_i - \omega_i q_r) = O(t^{-1+C\varepsilon})$. Moreover, by the eikonal equation we have

$$-(q_t + q_r)(q_t - q_r) + \sum_i (q_i - \omega_i q_r)^2 + (g^\alpha(u) - m^{\alpha\beta})q_\alpha q_\beta = 0,$$

so

$$q_t + q_r = \frac{\sum_i (q_i - \omega_i q_r)^2 + (g^\alpha(u) - m^{\alpha\beta})q_\alpha q_\beta}{q_t - q_r}.$$

Thus, $Z^I(q_t + q_r)$ is a linear combination of terms of the form

$$(q_t - q_r)^{-1-s} \cdot Z^{I_1}(q_t - q_r) \cdots Z^{I_s}(q_t - q_r) \cdot Z^{I_0}(\sum_i (q_i - \omega_i q_r)^2 + (g^\alpha(u) - m^{\alpha\beta})q_\alpha q_\beta)$$

where $\sum |I_*| = |I|$. It is clear that $Z^{I_*}(q_t - q_r) = O(t^{C\varepsilon})$ and that $q_t - q_r = -2 + O(\varepsilon t^{-1+C\varepsilon}) \leq -1$ on H. Moreover, since $Z^I(r^{-1}\Omega q) = O(t^{-1+C\varepsilon})$ for each I, we have $Z^{I_0}((q_i - \omega_i q_r)^2) = O(t^{-2+C\varepsilon})$. Finally, for each I we have

$$|Z^I((g^{\alpha\beta} - m^{\alpha\beta})q_\alpha q_\beta)| \lesssim \sum_{|I_1|+|I_2|+|I_3|=|I|} |Z^{I_1}(g-m)||Z^{I_2}\partial q||Z^{I_3}\partial q| \lesssim \varepsilon t^{-1+C\varepsilon}.$$

In conclusion, $Z^I(q_t + q_r) = O(t^{-2+C\varepsilon} + \varepsilon t^{-1+C\varepsilon}) = O(\varepsilon t^{-1+C\varepsilon})$, as $t \geq T_0 = \exp(\delta/\varepsilon)$. Since $q_i + \omega_i q_t = q_i - \omega_i q_r + \omega_i(q_t + q_r)$, we can easily show $Z^I(q_i + \omega_i q_t) = O(t^{-1+C\varepsilon})$ by the Leibniz's rule. □

We move on to estimates for e_*^* and ξ_{**}^* on H.

Lemma 4.28. *On H, we have $Z^I e_k^\alpha = O(t^{C\varepsilon})$ and $Z^I(e_3^i - \omega_i, e_4^i - \omega_i) = O(t^{-1+C\varepsilon})$ for each I.*

Proof. Since $e_4^0 = 1$, $e_3^0 = -1$ and $e_a^0 = 0$, we can ignore the case $\alpha = 0$. We write

$$\begin{aligned} e_4^i - \omega_i &= (g^{0\mu}q_\mu)^{-1}(g^{i\beta}q_\beta - \omega_i g^{0\beta}q_\beta) \\ &= (g^{0\mu}q_\mu)^{-1}(q_i + \omega_i q_t + (g^{i\beta} - m^{i\beta})q_\beta - \omega_i(g^{0\beta} - m^{0\beta})q_\beta) \\ &=: (g^{0\mu}q_\mu)^{-1}\mathcal{Q}. \end{aligned}$$

By Lemma 4.26, Lemma 4.27 and the Leibniz's rule, we have

$$Z^I\mathcal{Q} = O(t^{-1+C\varepsilon}), \qquad Z^I(g^{0\mu}q_\mu) = O(t^{C\varepsilon}), \qquad g^{0\mu}q_\mu = 1 + O(\varepsilon t^{-1+C\varepsilon}) \geq 1/2.$$

Besides, $Z^I(e_4^i - \omega_i)$ is a linear combination of terms of the form

$$(g^{0\mu}q_\mu)^{-1-s}Z^{I_1}(g^{0\mu}q_\mu)\cdots Z^{I_s}(g^{0\mu}q_\mu)Z^{I_0}\mathcal{Q}, \qquad \sum |I_*| = |I|,\ |I_j| > 0 \text{ for } j \neq 0.$$

We conclude that $Z^I(e_4^i - \omega_i) = O(t^{-1+C\varepsilon})$. Since $Z^I\omega = O(1)$ on H, we conclude that $Z^I e_4^i = O(t^{C\varepsilon})$. And since $Z^I(e_3^i - e_4^i) = 2Z^I g^{0i} = O(\varepsilon t^{-1+C\varepsilon})$, we conclude that $Z^I(e_3^i - \omega_i) = O(t^{-1+C\varepsilon})$ and $Z^I e_3^i = O(t^{C\varepsilon})$ on H for each I. The proofs of these estimates do not rely on the estimates for $Z^I e_a^*$, so we can use them freely in the following proof.

Next, we claim that $Z^I X^J \Omega^K e_a^i = O(\langle q\rangle^{-|J|}t^{C\varepsilon})$ on H for all I, J, K and $a = 1, 2$. Recall that Ω^K is the product of $|K|$ vector fields in $\{\Omega_{12}, \Omega_{23}, \Omega_{13}\}$. We induct first on $|I|+|J|+|K|$ and then on $|I|$. When $|I| + |J| + |K| = 0$, there is nothing to prove. When $|I| = 0$ and $|J| + |K| > 0$, we have $X^J\Omega^K e_a^i = O(r^{-|K|})$ on H, since $e_a^i|_H$ is a locally defined function of ω and it is independent of t.

In general, we fix I, J, K such that $|I| > 0$. Suppose we have proved the claim for all (I', J', K') such that $|I'| + |J'| + |K'| < |I| + |J| + |K|$, or $|I'| + |J'| + |K'| = |I| + |J| + |K|$ and $|I'| < |I|$. We write $Z^I = ZZ^{I'}$. For $a = 1, 2$ we have

$$Z^{I'}X^J\Omega^K e_4(e_a^i) = Z^{I'}X^J\Omega^K(e_4^\alpha e_a^\beta \Gamma_{\alpha\beta}^0 e_4^i - e_4^\alpha e_a^\beta \Gamma_{\alpha\beta}^i).$$

Since we can write $\Gamma = g\cdot\partial g$, for each K', we have $Z^{K'}\Gamma = O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1})$ on H. By induction hypotheses, Lemma 4.24 and the Leibniz's rule, we conclude that

$$Z^{I'}X^J\Omega^K e_4(e_a^i) = O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1-|J|}).$$

Moreover, $Z^{I'}X^J\Omega^K e_4(e_a^i)$ is equal to the sum of $e_4(Z^{I'}X^J\Omega^K e_a^i)$ and a linear combination of terms of the form

$$\begin{aligned}
&Z^{I_1}[e_4,Z^{I_2}]Z^{I_3}X^J\Omega^K e_a^i, &&(I_1,I_2,I_3)=I',\ |I_2|=1;\\
&Z^{I'}X^{J_1}[e_4,X^{J_2}]X^{J_3}\Omega^K e_a^i, &&(J_1,J_2,J_3)=J,\ |J_2|=1;\\
&Z^{I'}X^J\Omega^{K_1}[e_4,\Omega^{K_2}]\Omega^{K_3} e_a^i, &&(K_1,K_2,K_3)=K,\ |K_2|=1.
\end{aligned} \tag{4.35}$$

Note that

$$[e_4,Z] = e_4(z^\nu)\partial_\nu - Z(e_4^\nu)\partial_\nu = e_4(z^\nu)\partial_\nu - Z(\omega_j)\partial_j - Z(e_4^j-\omega_j)\partial_j,$$
$$[e_4,X_l] = e_4(2\omega_l)\partial_t - X_l(e_4^j)\partial_j = 2r^{-1}(e_4^l-\omega_l-(\omega_j-e_4^j)\omega_j\omega_l)\partial_t - (\partial_l\omega_j)\partial_j - X_l(e_4^j-\omega_j)\partial_j$$

where we write $Z = z^\nu(t,x)\partial_\nu$. We have

$$e_4(z^\nu)\partial_\nu - Z(\omega_j)\partial_j = \begin{cases} -\partial(\omega_j)\partial_j, & Z=\partial;\\ (r+t)^{-1}S + (r+t)^{-1}\omega_l\Omega_{0l} + (e_4^j-\omega_j)\partial_j, & Z=S;\\ r^{-1}\Omega_{ij} + (e_4^i-\omega_i)\partial_j - (e_4^j-\omega_j)\partial_i - r^{-1}\Omega_{ij}, & Z=\Omega_{ij};\\ r^{-1}\Omega_{0i} + r^{-1}(t-r)\partial_i + (e_4^i-\omega_i)\partial_t - tr^{-2}\omega_l\Omega_{li}, & Z=\Omega_{0i}. \end{cases}$$

In conclusion,

$$[e_4,Z] = f_1\cdot Z, \qquad [e_4,X] = f_1\cdot\partial$$

where f_1 denotes any function satisfying $Z^{J'}f_1 = O(t^{-1+C\varepsilon})$ for each J' on H. Thus, the first row in (4.35) has an upper bound

$$|Z^{I_1}(f_1ZZ^{I_3}X^J\Omega^K e_a^i)| \lesssim \sum_{|J'|\le|I_1|} t^{-1+C\varepsilon}|Z^{J'}ZZ^{I_3}X^J\Omega^K e_a^i| \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-|J|}.$$

We can use the induction hypotheses here as

$$|J'|+1+|I_3|+|J|+|K| \le |I_1|+1+|I_3|+|J|+|K| = |I'|+|J|+|K| < |I|+|J|+|K|.$$

The second row in (4.35) has an upper bound

$$\begin{aligned}
|Z^{I'}X^{J_1}(f_1\partial X^{J_3}\Omega^K e_a^i)| &\lesssim \sum_{|J'|\le|I'|+|J_1|} \langle q\rangle^{-|J_1|}|Z^{J'}(f_1\partial X^{J_3}\Omega^K e_a^i)|\\
&\lesssim \sum_{|J'|\le|I'|+|J_1|} \langle q\rangle^{-|J_1|}t^{-1+C\varepsilon}|Z^{J'}\partial X^{J_3}\Omega^K e_a^i|\\
&\lesssim \sum_{|J'|\le|I'|+|J_1|+1} \langle q\rangle^{-|J_1|-1}t^{-1+C\varepsilon}|Z^{J'}X^{J_3}\Omega^K e_a^i| \lesssim \langle q\rangle^{-|J|}t^{-1+C\varepsilon}.
\end{aligned}$$

We can use the induction hypotheses here as

$$|J'|+|J_3|+|K|\leq |I'|+|J_1|+1+|J_3|+|K|=|I'|+|J|+|K|<|I|+|J|+|K|.$$

The third row in (4.35) has an upper bound

$$\begin{aligned}|Z^{I'}X^J\Omega^{K_1}(f_1Z\Omega^{K_3}e_a^i)|&\lesssim \sum_{|J'|\leq|I'|+|J|}\langle q\rangle^{-|J|}|Z^{J'}\Omega^{K_1}(f_1Z\Omega^{K_3}e_a^i)|\\&\lesssim\sum_{|J'|\leq|I'|+|J|+|K_1|}\langle q\rangle^{-|J|}t^{-1+C\varepsilon}|Z^{J'}Z\Omega^{K_3}e_a^i|\lesssim\langle q\rangle^{-|J|}t^{-1+C\varepsilon}.\end{aligned}$$

We can use the induction hypotheses here as

$$|J'|+|K_3|+1\leq|I'|+|J|+|K_1|+1+|K_3|=|I'|+|J|+|K|<|I|+|J|+|K|.$$

In conclusion, on H we have

$$e_4(Z^{I'}X^J\Omega^Ke_a^i)=Z^{I'}X^J\Omega^Ke_4(e_a^i)+O(t^{-1+C\varepsilon}\langle q\rangle^{-|J|})=O(t^{-1+C\varepsilon}\langle q\rangle^{-|J|}).$$

We recall from the proof of Lemma 4.24 that $[Z,\Omega]=C\cdot Z$ and $[Z,X]=f_0\cdot\partial$ where f_0 denotes any function such that $Z^{K'}f_0=O(t^{C\varepsilon})$ on H for each K'. If we keep commuting Ω with each vector field in $Z^{I'}X^J$ and applying the Leibniz's rule, we get $\Omega Z^{I'}X^J\Omega^Ke_a^i=O(t^{C\varepsilon}\langle q\rangle^{-|J|})$. If we keep commuting X_l with each vector field in $Z^{I'}$ and applying the Leibniz's rule, we get $X_lZ^{I'}X^J\Omega^Ke_a^i=O(t^{C\varepsilon}\langle q\rangle^{-1-|J|})$. Finally, we recall from the proof of Lemma 4.24 that we can write

$$(\partial,S,\Omega_{0j})=O(t)L+O(1)\cdot\Omega+O(\langle q\rangle+\varepsilon t^{C\varepsilon})\cdot X$$

where $L=2g^{\alpha\beta}q_\beta\partial_\alpha=O(1)e_4$ on H. In conclusion, when $Z=\partial,S,\Omega_{0j}$, we have

$$|ZZ^{I'}X^J\Omega^Ke_a^i|\lesssim t|e_4(Z^{I'}X^J\Omega^Ke_a^i)|+|\Omega Z^{I'}X^J\Omega^Ke_a^i|+\langle q\rangle t^{C\varepsilon}|XZ^{I'}X^J\Omega^Ke_a^i|\lesssim t^{C\varepsilon}\langle q\rangle^{-|J|}.$$

We finish the proof by induction. □

We now prove the following lemma which illustrates the connection between the weighted null frame and the commuting vector fields.

Lemma 4.29. *Let $F=F(t,x)$ be a smooth function defined near H. Then, on H we have*

$$|V^IF|\lesssim\sum_{|J|\leq|I|}t^{C\varepsilon}|Z^IF|.$$

Proof. We induct on $|I|$. When $|I| = 0$, there is nothing to prove. Suppose we have proved the estimate for each function F and for each multiindex I' such that $|I'| < |I|$. Then, by writing $V^I = V^{I'}V_k$ and applying the induction hypotheses, we have

$$|V^I F| \lesssim \sum_{|J|\le|I|-1} t^{C\varepsilon}|Z^J(V_kF)|.$$

We then apply (4.28). When $k = 4$, we have $V_4F = f_0 \cdot ZF$. Here f_0 denotes any function such that $Z^{J'} f_0 = O(t^{C\varepsilon})$ on H for each J'. In particular, since $Z^{J'}(e^i_4 - \omega_i) = O(t^{-1+C\varepsilon})$ for each J' by Lemma 4.28, we have $Z^{J'}(t(e^i_4 - \omega_i)) = O(t^{-1+C\varepsilon})$ and thus $t(e^i_4 - \omega_i) = f_0$. By the Leibniz's rule, we have

$$|V^I F| \lesssim \sum_{|J|\le|I|-1} t^{C\varepsilon}|Z^J(f_0 \cdot ZF)| \lesssim \sum_{|J|\le|I|-1} t^{C\varepsilon}|Z^J ZF| \lesssim \sum_{|J|\le|I|} t^{C\varepsilon}|Z^J F|.$$

The proof for $k = 3$ follows from the case $k = 4$ and the estimate $Z^{J'}(r - t) = O(\langle r - t\rangle)$ for all J'. Finally, when $k = a \in \{1, 2\}$, we note that

$$V_a(r) = re^j_a\omega_j = re^\alpha_a(-g^{\alpha\beta} + m^{\alpha\beta})e^\beta_4 + re^j_a m^{jl}(-e^l_4 + \omega_l).$$

By Lemma 4.28, we have $Z^{J'}(\omega_*, e^*_*) = O(t^{C\varepsilon})$ and thus $Z^{J'}(V_a(r)) = O(t^{C\varepsilon})$ on H for each $|J'|$. Thus, for all $|J| \le |I| - 1$, we have

$$\begin{aligned}|Z^J(V_aF)| &\lesssim |Z^J(V_a(r)\omega_i\partial_iF)| + |Z^J(e^i_a\omega_j\Omega_{ji}F)|\\ &\lesssim t^{C\varepsilon}\sum_{|K|\le|J|}|Z^K\partial F| + t^{C\varepsilon}\sum_{|K|\le|J|}|Z^KF| \lesssim t^{C\varepsilon}\sum_{|K|\le|I|}|Z^KF|.\end{aligned}$$

This finishes the proof. □

Remark 4.29.1. With the help of this lemma, we conclude immediately that

$$V^I(g - m) = O(\varepsilon t^{-1+C\varepsilon}),\quad V^I((3R - r + t)^{-1}) = O(\langle q\rangle^{-1}t^{C\varepsilon}),\quad V^I(r^{-1}, t^{-1}) = O(t^{-1+C\varepsilon}),$$

$$V^I(q) = \langle q\rangle t^{C\varepsilon},\qquad V^Ie^\alpha_k = O(t^{C\varepsilon}),\qquad V^I(e^i_3 - \omega_i, e^i_4 - \omega_i) = O(t^{-1+C\varepsilon})$$

on H for each I.

Lemma 4.30. *For each I, on H we have $V^I(\xi^2_{13}, \xi^1_{23}) = O(\langle q\rangle^{-1}t^{C\varepsilon})$, $V^I(\xi^a_{34}) = O(t^{-1+C\varepsilon}\langle q\rangle^{-1})$ and $V^I(\xi^a_{k_1k_2}) = O(t^{-1+C\varepsilon})$ for all other $k_1 < k_2$ and $a \in \{1, 2\}$; $V^I(\xi^3_{k_1k_2}) = O(t^{-1+C\varepsilon}\langle q\rangle^{-1})$ for all $k_1 < k_2$; $V^I(\chi_{ab} - r^{-1}\delta_{ab}) = O(t^{-2+C\varepsilon})$.*

Proof. First, for any function $F = F(t, x)$ and for each $1 \le k \le 4$, on H we have

$$|V^I(e_k(F))| \lesssim \langle q\rangle^{-1}t^{C\varepsilon}\sum_{|J|\le|I|+1}|V^J(F)|. \tag{4.36}$$

This inequality easily follows from the Leibniz's rule, Remark 4.29.1 and the estimate $\langle r-t\rangle \lesssim t$ on H.

Since $e_l(\langle e_{k_1}, e_{k_2}\rangle) = 0$ for each k_1, k_2, l, we have

$$\begin{aligned} 2\xi^3_{k_1k_2} &= \langle [e_{k_1}, e_{k_2}], e_4\rangle = e_{k_1}(e^\alpha_{k_2})g_{\alpha\beta}e^\beta_4 - e_{k_2}(e^\alpha_{k_1})g_{\alpha\beta}e^\beta_4 \\ &= -e^\alpha_{k_2}e_{k_1}(g_{\alpha\beta})e^\beta_4 - e^\alpha_{k_2}g_{\alpha\beta}e_{k_1}(e^\beta_4) + e^\alpha_{k_1}e_{k_2}(g_{\alpha\beta})e^\beta_4 + e^\alpha_{k_1}g_{\alpha\beta}e_{k_2}(e^\beta_4). \end{aligned}$$

We assume $k_1 \neq k_2$ as $\xi^*_{k_1k_1} \equiv 0$. By (4.36) and the Leibniz's rule, on H for each I we have

$$|V^I(-e^\alpha_{k_2}e_{k_1}(g_{\alpha\beta})e^\beta_4 + e^\alpha_{k_1}e_{k_2}(g_{\alpha\beta})e^\beta_4)| \lesssim \varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

Moreover, since $e^0_4 \equiv 1$, we have

$$\begin{aligned} e^\alpha_{k_2}g_{\alpha\beta}e_{k_1}(e^\beta_4) &= e^\alpha_{k_2}g_{\alpha j}e_{k_1}(e^j_4 - \omega_j) + e^\alpha_{k_2}g_{\alpha j}e_{k_1}(\omega_j) \\ &= e^\alpha_{k_2}g_{\alpha j}e_{k_1}(e^j_4 - \omega_j) + r^{-1}e^\alpha_{k_2}g_{\alpha j}(e^j_{k_1} - e^l_{k_1}\omega_l\omega_j). \end{aligned}$$

Again, by (4.36) and the Leibniz's rule, on H for each I we have

$$|V^I(e^\alpha_{k_2}g_{\alpha j}e_{k_1}(e^j_4 - \omega_j))| \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

If $k_1 = 3$ or 4, then since

$$e^j_{k_1} - e^l_{k_1}\omega_l\omega_j = e^j_{k_1} - \omega_j + (1 - e^l_{k_1}\omega_l)\omega_j = e^j_{k_1} - \omega_j + \sum_l(\omega_l - e_{k_l})\omega_l\omega_j,$$

by the Leibniz's rule and the estimate $V^I(e^i_3-\omega_i, e^i_4-\omega_i) = O(t^{-1+C\varepsilon})$ for each I, we conclude that

$$|V^I(r^{-1}e^\alpha_{k_2}g_{\alpha j}(e^j_{k_1} - \omega_j + (1 - e^l_{k_1}\omega_l)\omega_j))| \lesssim t^{-2+C\varepsilon}, \qquad k_1 \geq 3.$$

If $k_1 = 1$ or 2, then $e^0_{k_1} = 0$.

$$r^{-1}e^\alpha_{k_2}g_{\alpha j}(e^j_{k_1} - e^l_{k_1}\omega_l\omega_j) = r^{-1}\langle e_{k_2}, e_{k_1}\rangle - r^{-1}e^\alpha_{k_2}g_{\alpha j}e^l_{k_1}\omega_l\omega_j = -r^{-1}e^\alpha_{k_2}g_{\alpha j}e^l_{k_1}\omega_l\omega_j.$$

Note that

$$\begin{aligned} e^l_{k_1}\omega_l &= e^l_{k_1}\delta_{ll'}e^{l'}_4 + e^l_{k_1}\delta_{ll'}(\omega_{l'} - e^{l'}_4) = e^\mu_{k_1}g_{\mu\nu}e^\nu_4 - e^\mu_{k_1}(g_{\mu\nu} - m_{\mu\nu})e^\nu_4 + e^l_{k_1}\delta_{ll'}(\omega_{l'} - e^{l'}_4) \\ &= -e^\mu_{k_1}(g_{\mu\nu} - m_{\mu\nu})e^\nu_4 + e^l_{k_1}\delta_{ll'}(\omega_{l'} - e^{l'}_4). \end{aligned}$$

Thus, by the Leibniz's rule, we have $V^I(e^l_{k_1}\omega_l) = O(t^{-1+C\varepsilon})$ and thus

$$|V^I(r^{-1}e^\alpha_{k_2}g_{\alpha j}(e^j_{k_1} - e^l_{k_1}\omega_l\omega_j))| \lesssim t^{-2+C\varepsilon}, \qquad k_1 \leq 2.$$

In conclusion, for each I, on H we have

$$|V^I(\xi^3_{k_1k_2})| \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1} + t^{-2+C\varepsilon} \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

Next, we have

$$\xi^c_{k_1k_2} = \langle [e_{k_1}, e_{k_2}], e_c\rangle = e_{k_1}(e^\alpha_{k_2})g_{\alpha\beta}e^\beta_c - e_{k_2}(e^\alpha_{k_1})g_{\alpha\beta}e^\beta_c.$$

We first prove some estimates for $e_{k_1}(e^\alpha_{k_2})g_{\alpha\beta}e^\beta_c$ with $k_1 \neq k_2$. If $k_1 = a \in \{1,2\}$ and $k_2 = b \in \{1,2\}$, we have $e_a = r^{-1}V_a$ and thus $V^I(e_a(e^\alpha_b)g_{\alpha\beta}e^\beta_c) = O(t^{-1+C\varepsilon})$ on H. If $k_2 = 3$ and $k_1 = a \in \{1,2\}$, then

$$\begin{aligned} e_a(e^\alpha_3)g_{\alpha\beta}e^\beta_c &= e_a(\omega_i)g_{i\beta}e^\beta_c + e_a(e^i_3 - \omega_i)g_{i\beta}e^\beta_c = r^{-1}(e^i_a - e^l_a\omega_l\omega_i)g_{ij}e^j_c + e_a(e^i_3 - \omega_i)g_{i\beta}e^\beta_c \\ &= r^{-1}\delta_{ac} - r^{-1}(e^l_a\omega_l)\omega_i g_{ij}e^j_c + r^{-1}V_a(e^i_3 - \omega_i)g_{i\beta}e^\beta_c. \end{aligned}$$

Recall that $V^I(e^l_a\omega_l) = O(t^{-1+C\varepsilon})$ on H. By Remark 4.29.1, we have $V^I(e_a(e^\alpha_3)g_{\alpha\beta}e^\beta_c - r^{-1}\delta_{ac}) = O(t^{-2+C\varepsilon})$ on H. Following the same proof, we can show that $V^I(e_a(e^\alpha_4)g_{\alpha\beta}e^\beta_c - r^{-1}\delta_{ac}) = O(t^{-2+C\varepsilon})$ on H. Next, for $k \neq 3$ we have

$$\begin{aligned} e_4(e^\alpha_k)g_{\alpha\beta}e^\beta_c &= e^\mu_4 e^\nu_k(\Gamma^0_{\mu\nu}e^\alpha_4 - \Gamma^\alpha_{\mu\nu})g_{\alpha\beta}e^\beta_c = -e^\mu_4 e^\nu_k \Gamma^\alpha_{\mu\nu}g_{\alpha\beta}e^\beta_c \\ &= -\frac{1}{2}e^\mu_4 e^\nu_k e^\beta_c(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}) \\ &= -\frac{1}{2}(t^{-1}e^\nu_k e^\beta_c V_4(g_{\beta\nu}) + e^\mu_4 e^\beta_c(t^{-1}, r^{-1})V_k(g_{\beta\mu}) - r^{-1}e^\mu_4 e^\nu_k V_c(g_{\mu\nu})). \end{aligned}$$

$$e_4(e^\alpha_3)g_{\alpha\beta}e^\beta_c = e_4(2g^{0\alpha})g_{\alpha\beta}e^\beta_c + e_4(e^\alpha_4)g_{\alpha\beta}e^\beta_c = t^{-1}V_4(2g^{0\alpha})g_{\alpha\beta}e^\beta_c + e_4(e^\alpha_4)g_{\alpha\beta}e^\beta_c.$$

Then, on H we have $V^I(e_4(e^\alpha_k)g_{\alpha\beta}e^\beta_c) = O(\varepsilon t^{-2+C\varepsilon})$. Next, we have

$$\begin{aligned} e_3(e^\alpha_4)g_{\alpha\beta}e^\beta_c &= e_3(\omega_j)g_{j\beta}e^\beta_c + (3R - r + t)^{-1}V_3(e^j_4 - \omega_j)g_{j\beta}e^\beta_c \\ &= r^{-1}(e^j_3 - \omega_j + (1 - \sum_l e^l_3\omega_l)\omega_j)g_{j\beta}e^\beta_c + (3R - r + t)^{-1}V_3(e^j_4 - \omega_j)g_{j\beta}e^\beta_c. \end{aligned}$$

Then, on H we have $V^I(e_3(e^\alpha_4)g_{\alpha\beta}e^\beta_c) = O(t^{-1+C\varepsilon}\langle q\rangle^{-1})$. Besides, we have

$$e_3(e^\alpha_c)g_{\alpha\beta}e^\beta_c = -e^\alpha_c g_{\alpha\beta}e_3(e^\beta_c) - e^\alpha_c e_3(g_{\alpha\beta})e^\beta_c \Longrightarrow e_3(e^\alpha_c)g_{\alpha\beta}e^\beta_c = -\frac{1}{2}(3R - r + t)^{-1}e^\alpha_c V_3(g_{\alpha\beta})e^\beta_c,$$

so we have $V^I(e_3(e^\alpha_c)g_{\alpha\beta}e^\beta_c) = O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1})$ on H. If $c' \neq c$, then

$$e_3(e^\alpha_{c'})g_{\alpha\beta}e^\beta_c = (3R - r + t)^{-1}V_3(e^\alpha_{c'})g_{\alpha\beta}e^\beta_c,$$

so we have $V^I(e_3(e^\alpha_{c'})g_{\alpha\beta}e^\beta_c) = O(\langle q\rangle^{-1}t^{C\varepsilon})$ on H if $c \neq c'$. All these estimates imply that on H, we have

$$V^I(\xi^c_{ab}, \xi^c_{a4}, \xi^c_{c3}) = O(t^{-1+C\varepsilon}); \qquad V^I(\xi^c_{c'3}) = O(\langle q\rangle^{-1}t^{C\varepsilon}),\ c \neq c'; \qquad V^I(\xi^c_{34}) = O(t^{-1+C\varepsilon}\langle q\rangle^{-1}).$$

Moreover,

$$|V^I(\chi_{ab} - r^{-1}\delta_{ab})| \leq |V^I(e_a(e^\alpha_4)g^{\alpha\beta}e^\beta_b - r^{-1}\delta_{ab})| + |V^I(e_a(e^\alpha_4)g^{\alpha\beta}e^\beta_b)| \lesssim t^{-2+C\varepsilon}.$$

□

4.4.3 Estimates in Ω

Recall that we defined a weighted null frame $\{V_k\}_{k=1}^4$ in Section 4.4.1. Our goal in this section is to prove the following proposition. Note that the estimates here are the same as those in Lemma 4.30.

Proposition 4.31. *In $\Omega\cap\{r-t<2R\}$, for each I we have the following estimates:*

$$|V^I(\xi_{13}^2)|+|V^I(\xi_{23}^1)|\lesssim\langle q\rangle^{-1}t^{C\varepsilon};\tag{4.37}$$

$$|V^I(\xi_{34}^a)|\lesssim\langle q\rangle^{-1}t^{-1+C\varepsilon};\tag{4.38}$$

for all other (k_1,k_2,a) such that $k_1<k_2$ and $a=1,2$, we have

$$|V^I(\xi_{k_1k_2}^a)|\lesssim t^{-1+C\varepsilon};\tag{4.39}$$

for all $k_1<k_2$, we have

$$|V^I(\xi_{k_1k_2}^3)|\lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1};\tag{4.40}$$

for $\xi_{a4}^b=\chi_{ab}$, we have

$$|V^I(\chi_{ab}-r^{-1}\delta_{ab})|\lesssim t^{-2+C\varepsilon}.\tag{4.41}$$

In this proposition we use the convention given in Section 4.2.1. That is, for each fixed integer $N>0$, we can choose $\varepsilon\ll_N 1$, such that the estimates in this proposition hold for all multiindices I with $|I|\le N$.

Since it is known that $q=r-t$ for $r-t>R$, we only care about the region where $r-t<2R$ in this subsection. Recall that every point in $\Omega\cap\{r-t<2R\}$ lies on exactly one geodesic in $\mathcal{A}$ emanating from H. The following lemma would be the key lemma in the proof of Proposition 4.31.

Lemma 4.32. *Fix $0<\varepsilon\ll 1$. Let $Q_1,\dots,Q_m$ be m functions defined in $\Omega\cap\{r-t<2R\}$. For each $i=1,\dots,m$, suppose in $\Omega\cap\{r-t<2R\}$ we have*

$$e_4(Q_i)=(-n_0r^{-1}+n_1e_4(\ln(3R-r+t)))Q_i+O(\varepsilon t^{-1}\sum_j|Q_j|)+O(f(t)).\tag{4.42}$$

Here $n_0,n_1\ge 0$ are two fixed real numbers which do not depend on i. Moreover, for some fixed $s\ge 1$, we suppose that $Q_i|_H=O(h(t))$ for each i. Then, in $\Omega\cap\{r-t<2R\}$ we have

$$\sum_i|Q_i|\lesssim t^{-n_0+C\varepsilon}((x^0(0))^{n_0}h(x^0(0))+\int_{x^0(0)}^t\tau^{n_0+C\varepsilon}f(\tau)\ d\tau).\tag{4.43}$$

Here we suppose that (t,x) lies on the geodesic $x(s)$ in $\mathcal{A}$ and that the integral is taken along the geodeisc $x(s)$.

Proof. Recall that $e_4(r) = 1 + O(t^{-1+C\varepsilon})$. If we define $Q_i' = (3R - r + t)^{-n_1} r^{n_0} Q_i$, then by (4.42), we have

$$\begin{aligned} e_4(Q_i') &= -n_1(3R - r + t)^{-n_1-1} e_4(3R - r + t) r^{n_0} Q_i + n_0 (3R - r + t)^{-n_1} r^{n_0-1} e_4(r) Q_i \\ &\quad + (3R - r + t)^{-n_1} r^{n_0} e_4(Q_i) \\ &= n_0 r^{-1}(e_4(r) - 1) Q_i' + O(\varepsilon t^{-1} \sum_j |Q_j'| + (3R - r + t)^{-n_1} r^{n_0} f(t)) \\ &= O(\varepsilon t^{-1} \sum_j |Q_j'| + (3R - r + t)^{-n_1} r^{n_0} f(t)). \end{aligned}$$

To get the last equality, we note that $r^{-1}(e_4(r) - 1) = O(t^{-2+C\varepsilon}) = O(\varepsilon t^{-1})$ as $t \geq \exp(\delta/\varepsilon)$.

In addition, we have $\langle q \rangle / \langle r - t \rangle = t^{O(\varepsilon)}$. In fact, by Lemma 4.14, we have $|q - (r - t)| \lesssim t^{C\varepsilon}$ and thus

$$\begin{aligned} 1 + |q| \lesssim 1 + |r - t| + t^{C\varepsilon} \lesssim t^{C\varepsilon} \langle r - t \rangle &\Longrightarrow \langle r - t \rangle^{-1} \lesssim \langle q \rangle^{-1} t^{C\varepsilon} \\ 1 + |r - t| \lesssim 1 + |q| + t^{C\varepsilon} \lesssim t^{C\varepsilon} \langle q \rangle &\Longrightarrow \langle q \rangle^{-1} \lesssim \langle r - t \rangle^{-1} t^{C\varepsilon}. \end{aligned}$$

Thus, in $\Omega \cap \{r - t < 2R\}$ we have

$$(3R - r + t)^{-n_1} r^{n_0} f(t) \lesssim \langle q \rangle^{-n_1} t^{n_0 + C\varepsilon} f(t).$$

Fix a point (t_0, x_0) in $\Omega \cap \{r - t < 2R\}$, and let $x(s)$ be the unique geodesic in $\mathcal{A}$ passing through (t_0, x_0). Note that $t_0 \geq x^0(0) \geq T_0$ and that q remains constant along each geodesic in $\mathcal{A}$. Then by integrating $e_4(Q_i')$, we have

$$\begin{aligned} \sum_i |Q_i'(t_0, x_0)| &\lesssim \sum_i |Q_i'(x(0))| + \int_{x^0(0)}^{t_0} \varepsilon \tau^{-1} \sum_j |Q_j'(\tau, y(\tau))| + \langle q \rangle^{-n_1} \tau^{n_0 + C\varepsilon} f(\tau)\ d\tau \\ &\lesssim \langle q \rangle^{-n_1} (x^0(0))^{n_0} h(x^0(0)) + \int_{x^0(0)}^{t_0} \varepsilon \tau^{-1} \sum_j |Q_j'(\tau, y(\tau))| + \langle q \rangle^{-n_1} \tau^{n_0 + C\varepsilon} f(\tau)\ d\tau. \end{aligned}$$

Here $(\tau, y(\tau))$ is a reparameterization of $x(s)$ such that $y(t_0) = x_0$. By the Gronwall's inequality, we conclude that

$$\sum_i |Q_i'(t_0, x_0)| \lesssim t_0^{C\varepsilon} \langle q \rangle^{-n_1} ((x^0(0))^{n_0} h(x^0(0)) + \int_{x^0(0)}^{t_0} \tau^{n_0 + C\varepsilon} f(\tau)\ d\tau).$$

To end the proof, we multiply both sides by $r^{-n_0}(3R - r + t)^{n_1}$, and recall that $t \sim r$ in $\Omega \cap \{r - t < 2R\}$. □

To prove Proposition 4.31, we induct on $|I|$.

4.4.3.1 The base case $I = 0$.

From Section 4.4.1, in $\Omega \cap \{r - t < 2R\}$ we already have the following estimates: $\xi_{34}^3 = O(|\Gamma|) = O(\min\{\varepsilon t^{-1}, \varepsilon t^{-1+C\varepsilon}\langle r-t\rangle^{-1}\})$, $\xi_{a4}^b = \chi_{ab} = \delta_{ab} r^{-1} + O(t^{-2+C\varepsilon}) = O(t^{-1})$, $\xi_{a3}^a = \chi_{aa} + O(\varepsilon t^{-1}) = O(t^{-1})$, $\xi_{a4}^3 = \xi_{12}^3 = 0$. To control the rest ξ, we recall that

$$\begin{aligned} &\langle R(e_k, e_l)e_r, e_s\rangle = e_k^\alpha e_l^\beta e_r^\mu e_s^\nu R_{\alpha\beta\mu\nu} \\ =\ & e_k^\alpha e_l^\beta e_r^\mu e_s^\nu (\frac{1}{2}(\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\beta\partial_\nu g_{\alpha\mu}) - \Gamma_{\beta\mu}^\delta \Gamma_{\delta\nu\alpha} + \Gamma_{\alpha\mu}^\delta\Gamma_{\delta\nu\beta}). \end{aligned} \tag{4.44}$$

If at most one of k, l, r, s is equal to 3, then we have $\langle R(e_k, e_l)e_r, e_s\rangle = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-1})$ by Lemma 4.13. From the equations in Section 4.4.1 we have

$$\begin{aligned} |e_4(\xi_{34}^a) + r^{-1}\xi_{34}^a| &\lesssim t^{-2+C\varepsilon}\sum_b |\xi_{34}^b| + \varepsilon t^{-2+C\varepsilon}\langle q\rangle^{-1}, \\ |e_4(\xi_{12}^a) + r^{-1}\xi_{12}^a| &\lesssim t^{-2+C\varepsilon}\sum_b |\xi_{12}^b| + \varepsilon t^{-2+C\varepsilon}\langle q\rangle^{-1}. \end{aligned}$$

By Lemma 4.32 with $n_0 = 1$, $n_1 = 0$ and $f(t) = \varepsilon t^{-2+C\varepsilon}\langle q\rangle^{-1}$, we have

$$\begin{aligned} |\xi_{34}^a| &\lesssim t^{-1+C\varepsilon}(\langle q\rangle^{-1}(x^0(0))^{C\varepsilon} + t^{C\varepsilon}\langle q\rangle^{-1}) \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}, \\ |\xi_{12}^a| &\lesssim t^{-1+C\varepsilon}((x^0(0))^{C\varepsilon} + t^{C\varepsilon}\langle q\rangle^{-1}) \lesssim t^{-1+C\varepsilon}. \end{aligned}$$

Here we get different estimates for ξ_{34}^a and ξ_{12}^a because their estimates on H are different; see Lemma 4.30.

It follows from Section 4.4.1 that $\xi_{a3}^3 = \frac{1}{2}\xi_{34}^a + O(\varepsilon t^{-1}) = O(t^{-1+C\varepsilon})$. It remains to estimate $\xi_{a3}^{a'}$ where $a \neq a'$. Note that

$$\begin{aligned} e_4(\xi_{a3}^{a'}) &= (e_4 + \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu)(\chi_{aa'} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_{a'}^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_{a'}^\nu) - \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu \xi_{a3}^{a'} - \sum_c \xi_{34}^c\xi_{aa'}^c \\ &\quad - \langle R(e_4, e_3)e_a, e_{a'}\rangle - \Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta \xi_{34}^{a'} + \Gamma_{\alpha\beta}^0 e_4^\alpha e_{a'}^\beta \xi_{34}^a \\ &= e_4(\chi_{aa'}) - \Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu \xi_{a3}^{a'} - \sum_c \xi_{34}^c \xi_{aa'}^c + O(\varepsilon t^{-2+C\varepsilon}) = O(\varepsilon t^{-1}|\xi_{a3}^{a'}|) + O(t^{-2+C\varepsilon}). \end{aligned}$$

By Lemma 4.32 with $n_0 = n_1 = 0$ and $f(t) = t^{-2+C\varepsilon}$ we have $|\xi_{a3}^{a'}| \lesssim (x^0(0))^{-1+C\varepsilon}t^{C\varepsilon}$. Here note that if (t, x) lies on a geodesic $x(s)$ in $\mathcal{A}$, then

$$q(t,x) = q(x(0)) = r(x(0)) - x^0(0) = \frac{T_0 - x^0(0)}{2} + 2R \Longrightarrow x^0(0) = T_0 - 2(q - 2R).$$

And since we only care about the region where $q < 2R$, we have $t \geq x^0(0) \sim (T_0 + \langle q\rangle) \geq \langle q\rangle$. In conclusion, we prove Proposition 4.31 in the case $I = 0$.

4.4.3.2 The general case.

Fix $m > 0$. Suppose we have proved Proposition 4.31 for all $|I| < m$. Our goal is to prove Proposition 4.31 for $|I| = m$.

Under the induction hypotheses, we can prove a key lemma which is Lemma 4.34 below. For convenience, we introduce the following notation.

Definition 4.33. Let $F = F(t, x)$ be a function with domain $\Omega \cap \{r - t < 2R\}$. For any integer $m \geq 0$ and any real numbers s, p, we write $F = \mathfrak{R}^m_{s,p}$ if for $\varepsilon \ll_{s,p,m} 1$ we have

$$\sum_{|I| \leq m} |V^I(F)| \lesssim t^{s+C\varepsilon}\langle q\rangle^p \qquad \text{in } \Omega \cap \{r - t < 2R\}.$$

By the Leibniz's rule, we can easily prove that $\mathfrak{R}^{m_1}_{s_1,p_1} \cdot \mathfrak{R}^{m_2}_{s_2,p_2} = \mathfrak{R}^{\min\{m_1,m_2\}}_{s_1+s_2,p_1+p_2}$. In addition, under the induction hypotheses, we have

$$\begin{gathered}(\xi^2_{13}, \xi^1_{23}) = \mathfrak{R}^{m-1}_{0,-1};\ \xi^a_{34} = \mathfrak{R}^{m-1}_{-1,-1};\ \xi^a_{k_1k_2} = \mathfrak{R}^{m-1}_{-1,0} \text{ for all other } k_1 < k_2 \text{ and } a = 1, 2; \\ \xi^3_{k_1k_2} = \mathfrak{R}^{m-1}_{-1,-1} \text{ for all } k_1 < k_2;\ \chi_{ab} - r^{-1}\delta_{ab} = \mathfrak{R}^{m-1}_{-2,0}.\end{gathered} \tag{4.45}$$

Lemma 4.34. *For $\varepsilon \ll_m 1$, we have*

$$e^\alpha_k = \mathfrak{R}^m_{0,0}; \tag{4.46}$$

$$(e^i_4 - \omega_i, e^i_3 - \omega_i) = \mathfrak{R}^m_{-1,0}; \tag{4.47}$$

$$(g^{\alpha\beta} - m^{\alpha\beta}, g_{\alpha\beta} - m_{\alpha\beta}) = \varepsilon\mathfrak{R}^{m+1}_{-1,0},\ \Gamma^\alpha_{\mu\nu} = \varepsilon\mathfrak{R}^{m+1}_{-1,-1}; \tag{4.48}$$

for each fixed $s \in \mathbb{R}$, we have

$$\omega_i = \mathfrak{R}^{m+1}_{0,0},\ (t^s, r^s) = \mathfrak{R}^{m+1}_{s,0},\ (3R - r + t)^s = \mathfrak{R}^{m+1}_{0,s}. \tag{4.49}$$

Proof. We prove by induction. First, since $e^*_* = O(1)$, we have $e^\alpha_k = \mathfrak{R}^0_{0,0}$; by Lemma 4.13, we have $(e^i_4 - \omega_i, e^i_3 - \omega_i) = \mathfrak{R}^0_{-1,0}$. Besides, $(g_{**} - m_{**}, g^{**} - m^{**}) = O(\varepsilon t^{-1+C\varepsilon})$ and

$$|\Gamma| \lesssim |g||\partial g| \lesssim \varepsilon t^{-1+C\varepsilon}\langle r - t\rangle^{-1} \lesssim \varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

Here we use the estimate $\langle r - t\rangle/\langle q\rangle = t^{O(\varepsilon)}$. Besides,

$$\sum_k |V_k(g)| \lesssim \sum_{k\neq 3}(t + r)|e_k(g)| + \langle r - t\rangle|\partial g| \lesssim \varepsilon t^{-1+C\varepsilon}.$$

Since Γ is a linear combination of terms of the form $g \cdot \partial g$ with constant real coefficients, by Lemma 4.13 we have

$$\begin{aligned}\sum_k |V_k(\Gamma)| &\lesssim \sum_k (|V_k(g)||\partial g| + |g| \cdot |V_k(\partial g)|)\\ &\lesssim \varepsilon t^{-1+C\varepsilon} \cdot \varepsilon \langle r-t\rangle^{-1} t^{-1+C\varepsilon} + \sum_{k\neq 3} (t+r)|e_k(\partial g)| + \langle r-t\rangle |\partial^2 g|\\ &\lesssim \varepsilon \langle q\rangle^{-1} t^{-1+C\varepsilon}.\end{aligned}$$

We thus obtain (4.48) with $m = 0$. Since $3R - r + t \sim \langle r - t\rangle$ in $\Omega \cap \{r - t < 2R\}$, (4.49) with $m + 1$ replaced by 0 is obvious. In addition, by writing $Vf := (V_1 f, V_2 f, V_3 f, V_4 f)$, we have

$$\begin{cases} V(t) = (0, 0, -(3R - r + t), t);\\ V(r) = (re_1(r), re_2(r), (3R - r + t)(e_3^i \omega_i), te_4^i \omega_i);\\ V(\omega_i) = (e_1^i - \omega_i e_1(r), e_2^i - \omega_i e_2(r), r^{-1}(3R - r + t)(e_3^i - \omega_i e_3^j \omega_j), r^{-1} t(e_4^i - \omega_i e_4^j \omega_j));\\ V(3R - r + t) = (-re_1(r), -re_2(r), (3R - r + t)(-1 - e_3^i \omega_i), t(1 - e_4^i \omega_i)) \end{cases} \tag{4.50}$$

Since $e_3, e_4 = \pm\partial_t + \partial_r + O(t^{-1+C\varepsilon})\partial$, we have

$$\begin{aligned} e_a(r) = e_a^i \omega_i &= \sum_i e_a^i e_4^i + \sum_i e_a^i (\omega_i - e_4^i)\\ &= \langle e_a, e_4\rangle - (g^{\alpha\beta} - m^{\alpha\beta}) e_4^\alpha e_a^\beta + \sum_i e_a^i(\omega_i - e_4^i) = O(t^{-1+C\varepsilon}),\\ 1 - e_4^i \omega_i &= -\sum_i (e_4^i - \omega_i)\omega_i = O(t^{-1+C\varepsilon}). \end{aligned} \tag{4.51}$$

Also note that for each fixed $s \in \mathbb{R}$ and for each funtion $\phi(t, x)$, $V(\phi^s) = s\phi^{s-1} V(\phi)$. Then, we have $V(\omega) = O(t^{C\varepsilon})$, $V(t^s, r^s) = O(t^{s+C\varepsilon})$, $V((3R - r + t)^s) = O(\langle r - t\rangle^s t^{C\varepsilon})$. We thus obtain (4.49) with $m = 0$. This finishes the proof in the base case.

In general, we assume that we have proved (4.46)-(4.49) with m replaced by n where $0 \le n < m$. We first prove (4.46) with m replaced by $n + 1$. Fix a multiindex I such that $|I| = n + 1$. If $I = (I', 4)$, note that $te_4(e_k^\alpha)$ is a linear combination (with constant real coefficients) of terms of the form $t\Gamma^*_{**}(e^*_*)(e^*_*)(e^*_*)$, $-t\Gamma^*_{**}(e^*_*)(e^*_*)$ and $V_4(g^{0\alpha})$. By the induction hypotheses, we notice that

$$t\Gamma^*_{**}(e^*_*)(e^*_*)(e^*_*) = \mathfrak{R}^{n+1}_{1,0} \cdot \varepsilon\mathfrak{R}^{n+1}_{-1,-1} \cdot \mathfrak{R}^n_{0,0} \cdot \mathfrak{R}^n_{0,0} \cdot \mathfrak{R}^n_{0,0} = \varepsilon\mathfrak{R}^n_{0,-1}$$

and similarly

$$t\Gamma^*_{**}(e^*_*)(e^*_*) = \varepsilon\mathfrak{R}^n_{0,-1}.$$

Besides,

$$g^{0\alpha} - m^{0\alpha} = \varepsilon\mathfrak{R}^{n+1}_{-1,0} \Longrightarrow V_k(g^{0\alpha}) = \varepsilon\mathfrak{R}^n_{-1,0}.$$

So in conclusion,

$$V_4(e_k^\alpha)=\varepsilon\mathfrak{R}^n_{0,-1}\Longrightarrow V^I(e_k^\alpha)=O(\varepsilon\langle q\rangle^{-1}t^{C\varepsilon}).$$

If $I=(I',k')$ where $k'\neq 4$, then by the formulas at the end of Section 4.4.1, we have

$$\begin{aligned}V_{k'}(e_4^\alpha)&=r\xi^l_{a4}e_l^\alpha+rt^{-1}V_4(e_a^\alpha)\\&=\mathfrak{R}^{n+1}_{1,0}\cdot\mathfrak{R}^{m-1}_{-1,0}\cdot\mathfrak{R}^n_{0,0}+\mathfrak{R}^{n+1}_{1,0}\cdot\mathfrak{R}^{n+1}_{-1,0}\cdot\varepsilon\mathfrak{R}^n_{0,-1}=\mathfrak{R}^n_{0,0},\qquad k'=a=1,2;\\V_3(e_4^\alpha)&=(3R-r+t)\xi^l_{34}e_l^\alpha+t^{-1}(3R-r+t)V_4(e_3^\alpha)\\&=\mathfrak{R}^{n+1}_{0,1}\cdot\mathfrak{R}^{m-1}_{-1,-1}\cdot\mathfrak{R}^n_{0,0}+\mathfrak{R}^{n+1}_{-1,0}\cdot\mathfrak{R}^{n+1}_{0,1}\cdot\varepsilon\mathfrak{R}^n_{0,-1}=\mathfrak{R}^n_{-1,0}.\end{aligned}$$

In addition, note that $e_3^\alpha=e_4^\alpha+2g^{0\alpha}$, so

$$V_{k'}(e_4^\alpha,e_3^\alpha)=\mathfrak{R}^n_{0,0}\Longrightarrow V^I(e_4^\alpha,e_3^\alpha)=O(t^{C\varepsilon}).$$

If $I=(I',3)$, we have

$$\begin{aligned}V_3(e_a^\alpha)&=(3R-r+t)\xi^l_{a3}e_l^\alpha+r^{-1}(3R-r+t)V_a(e_3^\alpha)\\&=\mathfrak{R}^{n+1}_{0,1}\cdot\mathfrak{R}^{m-1}_{0,-1}\cdot\mathfrak{R}^n_{0,0}+\mathfrak{R}^{n+1}_{-1,0}\cdot\mathfrak{R}^{n+1}_{0,1}\cdot\mathfrak{R}^n_{0,0}=\mathfrak{R}^n_{0,0}.\end{aligned}$$

Here we recall that $t\gtrsim x^0(0)\sim\langle q\rangle+T_0$, so $\mathfrak{R}^n_{-s,s}=\mathfrak{R}^n_{0,0}$ for each $s>0$. Thus,

$$V^I(e_a^\alpha)=O(t^{C\varepsilon}).$$

If $I=(I',a)$, then

$$V_a(e_b^\alpha)=-\sum_c r\xi^a_{bc}e_c^\alpha-\frac{1}{2}r\chi_{ab}(e_4^\alpha+e_3^\alpha)-(e_b^\mu g_{\mu\beta}V_a(g^{0\beta})+re_b^\mu g_{\mu\nu}g^{0\beta}e_a^\sigma\Gamma^\nu_{\sigma\beta})e_4^\alpha-re_a^\mu e_b^\nu\Gamma^\alpha_{\mu\nu}.$$

Again, by our induction hypotheses, we conclude that

$$V_a(e_b^\alpha)=\mathfrak{R}^n_{0,0}\Longrightarrow V^I(e_b^\alpha)=O(t^{C\varepsilon}).$$

Summarize all the results above and we conclude that $e_*^*=\mathfrak{R}^{n+1}_{0,0}$. Note that the computations above work as long as $n\le m-1$.

Next we prove (4.47) with m replaced by $n+1$. It suffices to consider $e_4^i-\omega_i$ as $e_3^i-e_4^i=2g^{0i}=\varepsilon\mathfrak{R}^{n+1}_{-1,0}$. Fix a multiindex I with $|I|=n+1$. Note that

$$\begin{aligned}V_a(e_4^i-\omega_i)&=re_a(e_4^i-\omega_i)=r(\xi^l_{a4}e_l^i+e_4(e_a^i)-r^{-1}(e_a^i-\omega_ie_a(r)))\\&=r(\chi_{ab}-\delta_{ab}r^{-1})e_b^i+re_4(e_a^i)+r^{-1}\omega_iV_a(r)\\&=\mathfrak{R}^{n+1}_{1,0}\cdot\mathfrak{R}^{m-1}_{-2,0}\cdot\mathfrak{R}^n_{0,0}+re_4(e_a^i)+\mathfrak{R}^n_{-1,0}=re_4(e_a^i)+\mathfrak{R}^n_{-1,0},\\V_4(e_4^i-\omega_i)&=te_4(e_4^i-\omega_i)=t(e_4(e_4^i)-(e_4^j-\omega_j)\partial_j\omega_i)\\&=te_4(e_4^i)-tr^{-1}(e_4^i-\omega_i-\omega_i\omega_j(e_4^j-\omega_j))\\&=te_4(e_4^i)+\mathfrak{R}^{n+1}_{0,0}\cdot(\mathfrak{R}^n_{-1,0}+\mathfrak{R}^{n+1}_{0,0}\cdot\mathfrak{R}^n_{-1,0})=te_4(e_4^i)+\mathfrak{R}^n_{-1,0},\\V_3(e_4^i-\omega_i)&=(3R-r+t)e_3(e_4^i-\omega_i)=(3R-r+t)(\xi^l_{34}e_l^i+e_4(e_3^i)-(e_3^j-\omega_j)\partial_j\omega_i)\\&=(3R-r+t)(\xi^l_{34}e_l^i+e_4(e_4^i)+2t^{-1}V_4(g^{0i})-r^{-1}(e_3^i-\omega_i-(e_3^j-\omega_j)\omega_i\omega_j))\\&=(3R-r+t)e_4(e_4^i)+\mathfrak{R}^{n+1}_{0,1}\cdot(\mathfrak{R}^{m-1}_{-1,-1}\cdot\mathfrak{R}^n_{0,0}+\varepsilon\mathfrak{R}^n_{-2,0}+\mathfrak{R}^{n+1}_{-1,0}\cdot\mathfrak{R}^n_{-1,0})\\&=(3R-r+t)e_4(e_4^i)+\mathfrak{R}^n_{-1,0}.\end{aligned}$$

Here we use (4.45). To finish the proof, we note that for $k \neq 3$,

$$\begin{aligned}2e_4(e_k^i) &= 2e_4^\alpha e_k^\beta(\Gamma^0_{\alpha\beta}e_4^i - \Gamma^i_{\alpha\beta}) = e_4^\alpha e_k^\beta(g^{0\delta}e_4^i - g^{i\delta})(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}) \\ &= (g^{0\delta}e_4^i - g^{i\delta})(e_4(g_{\delta\beta})e_k^\beta + e_k(g_{\delta\alpha})e_4^\alpha) + e_4^\alpha e_k^\beta(-\frac{1}{2}e_4(g_{\alpha\beta})(e_4^i + e_3^i) - \sum_b e_b^i e_b(g_{\alpha\beta})) \\ &= \mathfrak{R}^{n+1}_{0,0}t^{-1}V_4(g) + \mathfrak{R}^{n+1}_{0,0}r^{-1}V_a(g) = \varepsilon\mathfrak{R}^{n+1}_{-2,0}.\end{aligned}$$

Also note that $e_4(g) = t^{-1}V_4(g) = \varepsilon\mathfrak{R}^{n+1}_{-2,0}$ and that e_k^0 is a constant, so we have $e_4(e_k^\alpha) = \varepsilon\mathfrak{R}^{n+1}_{-2,0}$ for each k, α. Thus,

$$V(e_4^i - \omega_i) = \mathfrak{R}^{n+1}_{-1,0} \Longrightarrow e_4^i - \omega_i = \mathfrak{R}^{n+1}_{-1,0}.$$

Finally, we prove (4.48) and (4.49) with $m+1$ replaced by $n+2$. Fix a multiindex I such that $|I| = n+2$. Note that

$$\begin{aligned}(3R + t - r)\partial_t &= 3R\partial_t + \frac{tS - x_i\Omega_{0i}}{r+t} = \mathfrak{R}^{n+1}_{0,0} \cdot Z, \\ (3R + t - r)\partial_r &= 3R\partial_r + \frac{t\omega_i\Omega_{0i} - rS}{r+t} = \mathfrak{R}^{n+1}_{0,0} \cdot Z, \\ (3R + t - r)\partial_i &= 3R\partial_i + (t-r)\omega_i\partial_r + (t-r)r^{-1}\omega_j\Omega_{ji} = \mathfrak{R}^{n+1}_{0,0} \cdot Z.\end{aligned}$$

Thus, $\partial = (3R + t - r)^{-1}\mathfrak{R}^{n+1}_{0,0} \cdot Z = \mathfrak{R}^{n+1}_{0,-1} \cdot Z$. Since we have just proved $e_*^* = \mathfrak{R}^{n+1}_{0,0}$ and $e_4^i - \omega_i = \mathfrak{R}^{n+1}_{-1,0}$, by (4.51) we have $e_a(r) = \mathfrak{R}^{n+1}_{-1,0}$. In conclusion, by (4.28) we have

$$\begin{aligned}V_4 &= t(t+r)^{-1}S + (t+r)^{-1}t\omega_j\Omega_{0j} + t(e_4^i - \omega_i)\partial_i = \mathfrak{R}^{n+1}_{0,0} \cdot Z, \\ V_3 &= (3R - r + t)r^{-1}V_4 + 2g^{0\alpha}(3R - r + t)\partial_\alpha = \mathfrak{R}^{n+1}_{0,0} \cdot Z, \\ V_a &= re_a(r)\omega_i\partial_i + e_a^i\omega_j\Omega_{ji} = \mathfrak{R}^{n+1}_{-1,0} \cdot \mathfrak{R}^{n+1}_{0,-1} \cdot Z + \mathfrak{R}^{n+1}_{0,0} \cdot Z = \mathfrak{R}^{n+1}_{0,0} \cdot Z.\end{aligned}$$

Now, given a function $F = F(t,x)$, if $|I| = n+2$, we can write $V^I F$ as a linear combination of terms of the form

$$V^{I_1}(\mathfrak{R}^{n+1}_{0,0}) \cdots V^{I_s}(\mathfrak{R}^{n+1}_{0,0})Z^s F, \qquad \sum |I_*| + s = n+2, \ s > 0. \tag{4.52}$$

Since $|I_j| < n+2$ for each j, we have $V^{I_j}(\mathfrak{R}^{n+1}_{0,0}) = O(t^{C\varepsilon})$. Note that for each J with $|J| > 0$, we have $Z^J g = O(\varepsilon t^{-1+C\varepsilon})$, $Z^J\omega = O(1)$, $Z^J(t^s, r^s) = O(t^s)$, $Z^J((3R - r + t)^s) = O(\langle r - t\rangle^s)$ and $Z^J(\Gamma) = O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1})$. The last one is true because $Z^J\Gamma$ is a linear combination (with constant real coefficients) of terms of the form $(Z^{J_1}g) \cdot (Z^{J_2}\partial g) = O(\varepsilon t^{-1+C\varepsilon}\langle r - t\rangle^{-1})$. By plugging these estimates into (4.52), we conclude (4.48) and (4.49) with $m+1$ replaced by $n+2$. □

Remark 4.34.1. We have $Z^I\partial^k g = \varepsilon\mathfrak{R}^{m+1}_{-1,-k}$ for each I and k, as long as $\varepsilon \ll_{I,k} 1$. This follows directly from (4.52), Lemma 1.4 and $[Z, \partial] = C \cdot \partial$.

From the proof, we note that $e_4(e_k^\alpha) = \varepsilon\mathfrak{R}^m_{-2,0}$ and $e_a(r) = \mathfrak{R}^m_{-1,0}$. These estimates are better than what we can get from (4.46) and (4.49).

By Lemma 4.34, we have $e_4^i\omega_i - 1 = (e_4^i - \omega_i)\omega_i = \mathfrak{R}^m_{-1,0}$. This result can be improved as shown in the next lemma.

Lemma 4.35. *For $\varepsilon \ll_m 1$, we have $e_4^i\omega_i - 1 = \varepsilon\mathfrak{R}^m_{-1,0}$.*

Proof. By Lemma 4.34, we have

$$e_a^j\omega_j = -(g^{\alpha\beta} - m^{\alpha\beta})e_4^\alpha e_a^\beta + \sum_i e_a^i(\omega_i - e_4^i) = \mathfrak{R}^m_{-1,0}.$$

Recall that

$$g^{\alpha\beta} = \sum_a e_a^\alpha e_a^\beta + \frac{1}{2}(e_4^\alpha e_3^\beta + e_3^\alpha e_4^\beta).$$

Then,

$$\begin{aligned} g^{\alpha\beta}(\partial_\alpha(r-t))(\partial_\beta(r-t)) &= \sum_a (e_a^i\omega_i)(e_a^j\omega_j) + (e_4^i\omega_i - 1)(e_3^j\omega_j + 1) \\ &= \mathfrak{R}^m_{-2,0} + (e_4^i\omega_i - 1)(2 + (e_3^j - \omega_j)\omega_j). \end{aligned}$$

Meanwhile, we have

$$\begin{aligned} g^{\alpha\beta}(\partial_\alpha(r-t))(\partial_\beta(r-t)) &= g^{00} - 2g^{0i}\omega_i + g^{ij}\omega_i\omega_j \\ &= -2g^{0i}\omega_i + (g^{ij} - m^{ij})\omega_i\omega_j = \varepsilon\mathfrak{R}^{m+1}_{-1,0}. \end{aligned}$$

Thus,

$$e_4^i\omega_i - 1 = (2 + (e_3^j - \omega_j)\omega_j)^{-1}(\varepsilon\mathfrak{R}^m_{-1,0} + \mathfrak{R}^m_{-2,0}) = (2 + (e_3^j - \omega_j)\omega_j)^{-1}\cdot\varepsilon\mathfrak{R}^m_{-1,0}.$$

Here we note that $\mathfrak{R}^m_{-2,0} = \varepsilon\mathfrak{R}^m_{-1,0}$ as $t \geq \exp(\delta/\varepsilon)$.

Fix a multiindx I with $|I| \leq m$. Then, $V^I(e_4^i\omega_i - 1)$ is a linear combination of terms of the form

$$(2 + (e_3^j - \omega_j)\omega_j)^{-s-1}V^{I_0}(\varepsilon\mathfrak{R}^m_{-1,0})V^{I_2}(2 + (e_3^j - \omega_j)\omega_j)\cdots V^{I_s}(2 + (e_3^j - \omega_j)\omega_j)$$

where $\sum|I_*| = |I| \leq m$ such that $|I_k| > 0$ for each $k > 0$. Thus, we can replace $V^{I_*}(2 + (e_3^j - \omega_j)\omega_j)$ with $V^{I_*}((e_3^j - \omega_j)\omega_j)$ in the product. By Lemma 4.34 we have $(e_3^j - \omega_j)\omega_j = \mathfrak{R}^m_{-1,0}$. Since $e_3^j - \omega_j = O(t^{-1+C\varepsilon})$, we have $2 + (e_3^j - \omega_j)\omega_j \geq 1$ for $\varepsilon \ll 1$. In conclusion, we have

$$|V^I(e_4^i\omega_i - 1)| \lesssim \varepsilon t^{-1+C\varepsilon}\cdot \max_{0\leq s\leq m}\{(t^{-1+C\varepsilon})^s\} \lesssim \varepsilon t^{-1+C\varepsilon}.$$

Thus, $e_4^i\omega_i - 1 = \varepsilon\mathfrak{R}^m_{-1,0}$. □

We can now control the curvature tensor terms.

Lemma 4.36. *We have* $\langle R(e_4, e_k)e_l, e_p\rangle) = \varepsilon\mathfrak{R}^m_{-2,-1}$ *if* $l, p \neq 3$.

Proof. By (4.44), we can express $e_4^\alpha e_k^\beta e_l^\mu e_p^\nu R_{\alpha\beta\mu\nu}$ as a linear combination of terms of the form

$$e_4(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})e_k^\beta e_l^\mu e_p^\nu,\ e_l(\partial_\beta g_{\alpha\nu})e_4^\alpha e_k^\beta e_p^\nu,\ e_p(\partial_\beta g_{\alpha\mu})e_4^\alpha e_k^\beta e_l^\mu,\ e_4^\alpha e_k^\beta e_l^\mu e_p^\nu \cdot \Gamma \cdot (g \cdot \Gamma).$$

By Lemma 4.34 and Remark 4.34.1, we have

$$e_4(\partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})e_k^\beta e_l^\mu e_p^\nu = t^{-1}V_4(\partial g) \cdot \mathfrak{R}^m_{0,0} = \mathfrak{R}^m_{-1,0} \cdot Z(\partial g) = \varepsilon\mathfrak{R}^m_{-2,-1}.$$

Since $l \neq 3$, we either have $e_l = t^{-1}V_l$ or $e_l = r^{-1}V_l$. In both cases, we can follow the same proof as above to conclude that

$$e_l(\partial_\beta g_{\alpha\nu})e_4^\alpha e_k^\beta e_p^\nu = \varepsilon\mathfrak{R}^m_{-2,-1}.$$

Similarly, we also have

$$e_p(\partial_\beta g_{\alpha\mu})e_4^\alpha e_k^\beta e_l^\mu = \varepsilon\mathfrak{R}^m_{-2,-1}.$$

Finally, note that

$$e_4^\alpha e_k^\beta e_l^\mu e_p^\nu \cdot \Gamma \cdot (g \cdot \Gamma) = (\varepsilon\mathfrak{R}^{m+1}_{-1,-1})^2 \cdot \mathfrak{R}^m_{0,0} = \varepsilon^2\mathfrak{R}^m_{-2,-2}.$$

Thus, $\langle R(e_4, e_k)e_l, e_p\rangle = \varepsilon\mathfrak{R}^m_{-2,-1}$. □

Lemma 4.36 can be improved in a special case.

Lemma 4.37. (a) *We have*

$$\langle R(e_4, e_a)e_4, e_b\rangle = e_4(f_{ab}) + \frac{1}{4}e_4^\alpha e_4^\mu r^{-1}\delta_{ab}e_3(g_{\alpha\mu}) + \varepsilon\mathfrak{R}^m_{-3,0}.$$

Here we set

$$f_{ab} = \frac{1}{2}(e_a^\beta e_b^\nu e_4(g_{\beta\nu}) - e_a^\beta e_4^\mu e_b(g_{\beta\mu})) - \frac{1}{2}e_4^\alpha e_a(g_{\alpha\nu})e_b^\nu = \varepsilon\mathfrak{R}^m_{-2,0}.$$

(b) *Assume that* $\chi_{ab} = \mathfrak{R}^m_{-1,0}$. *Then we have*

$$\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta \chi_{ab} + \frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})\chi_{ab} = \varepsilon\mathfrak{R}^m_{-3,0}.$$

Proof. (a) Recall that $\langle R(e_4, e_a)e_4, e_b\rangle = e_4^\alpha e_a^\beta e_4^\mu e_b^\nu R_{\alpha\beta\mu\nu}$ where $R_{\alpha\beta\mu\nu}$ is given by

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(\partial_\alpha\partial_\mu g_{\beta\nu} - \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\beta\partial_\nu g_{\alpha\mu}) - \Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha} + \Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta}.$$

Note that (for simplicity we take the sum over all the indices without writing the summation)

$$
\begin{aligned}
&\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu} \\
=&\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^i(\omega_i\partial_r)(\partial_\beta g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^i(\partial_i-\omega_i\partial_r)(\partial_\beta g_{\alpha\mu}) \\
=&\frac{1}{2}e_4^\alpha e_4^\mu e_b(r)e_a^\beta\partial_r(\partial_\beta g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^i r^{-1}\omega_j\Omega_{ji}(\partial_\beta g_{\alpha\mu}) \\
=&\frac{1}{2}e_4^\alpha e_4^\mu e_b(r)\omega_j e_a(\partial_j g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^i r^{-1}\omega_j[\Omega_{ji},\partial_\beta](g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_4^\mu e_b^i r^{-1}\omega_j e_a(\Omega_{ji}g_{\alpha\mu}) \\
=&\frac{1}{2}e_4^\alpha e_4^\mu e_b(r)\omega_j e_a(\partial_j g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}(-e_a(r)e_b(g_{\alpha\mu})+e_a^ie_b^i\partial_r(g_{\alpha\mu})) \\
&+\frac{1}{2}e_4^\alpha e_4^\mu e_b^i r^{-1}\omega_j e_a(\Omega_{ji}g_{\alpha\mu}) \\
=&\frac{1}{2}e_4^\alpha e_4^\mu e_b(r)\omega_j e_a(\partial_j g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}(-e_a(r)e_b(g_{\alpha\mu})+(\delta_{ab}-e_a^\beta(g_{\beta\nu}-m_{\beta\nu})e_b^\nu)\partial_r(g_{\alpha\mu})) \\
&+\frac{1}{2}e_4^\alpha e_4^\mu e_b^i r^{-1}\omega_j e_a(\Omega_{ji}g_{\alpha\mu}) \\
=&\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}e_b(r)\omega_j V_a(\partial_j g_{\alpha\mu})+\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}(-r^{-1}e_a(r)V_b(g_{\alpha\mu})+(\delta_{ab}-e_a^\beta(g_{\beta\nu}-m_{\beta\nu})e_b^\nu)\partial_r(g_{\alpha\mu})) \\
&+\frac{1}{2}e_4^\alpha e_4^\mu e_b^i r^{-2}\omega_j V_a(\Omega_{ji}g_{\alpha\mu}).
\end{aligned}
$$

Recall that in Lemma 4.34, we have proved that $e_a(r)=\mathfrak{R}^m_{-1,0}$. Thus, we have

$$
\begin{aligned}
&\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu \partial_\beta\partial_\nu g_{\alpha\mu}=\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}\delta_{ab}(\partial_r g_{\alpha\mu})+\varepsilon\mathfrak{R}^m_{-3,0} \\
=&\frac{1}{2}e_4^\alpha e_4^\mu r^{-1}\delta_{ab}(\omega_j-\frac{1}{2}e_3^j-\frac{1}{2}e_4^j)\partial_j g_{\alpha\mu}+\frac{1}{4}e_4^\alpha e_4^\mu r^{-1}\delta_{ab}(e_3(g_{\alpha\mu})+e_4(g_{\alpha\mu}))+\varepsilon\mathfrak{R}^m_{-3,0} \\
=&\frac{1}{4}e_4^\alpha e_4^\mu r^{-1}\delta_{ab}e_3(g_{\alpha\mu})+\varepsilon\mathfrak{R}^m_{-3,0}.
\end{aligned}
$$

Next, we note that

$$
\begin{aligned}
&\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu(\partial_\alpha\partial_\mu g_{\beta\nu}-\partial_\alpha\partial_\nu g_{\beta\mu}-\partial_\beta\partial_\mu g_{\alpha\nu}) \\
=&\frac{1}{2}e_a^\beta e_4^\mu e_b^\nu e_4(\partial_\mu g_{\beta\nu}-\partial_\nu g_{\beta\mu})-\frac{1}{2}e_4^\alpha e_a^\beta e_b^\nu e_4(\partial_\beta g_{\alpha\nu}) \\
=&e_4(f_{ab})-\frac{1}{2}e_4(e_a^\beta e_4^\mu e_b^\nu)(\partial_\mu g_{\beta\nu}-\partial_\nu g_{\beta\mu})-\frac{1}{2}e_4(e_4^\alpha e_a^\beta e_b^\nu)(\partial_\beta g_{\alpha\nu}).
\end{aligned}
$$

In Lemma 4.34, we have proved that $e_4(e_k^\alpha)=\varepsilon\mathfrak{R}^m_{-2,0}$. By Lemma 4.34, we can easily prove that $f_{ab}=\varepsilon\mathfrak{R}^m_{-2,0}$. This implies that

$$
\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu(\partial_\alpha\partial_\mu g_{\beta\nu}-\partial_\alpha\partial_\nu g_{\beta\mu}-\partial_\beta\partial_\mu g_{\alpha\nu})=e_4(f_{ab})+\varepsilon\mathfrak{R}^m_{-3,0}.
$$

Finally, we note that

$$\begin{aligned}
&e_4^\alpha e_a^\beta e_4^\mu e_b^\nu(-\Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha}+\Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta})\\
=&-\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu\Gamma^\delta_{\beta\mu}\Gamma_{\delta\nu\alpha}+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu\Gamma^\delta_{\alpha\mu}\Gamma_{\delta\nu\beta}\\
=&-\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu g^{\delta\sigma}(\partial_\beta g_{\mu\sigma}+\partial_\mu g_{\beta\sigma}-\partial_\sigma g_{\beta\mu})(\partial_\alpha g_{\nu\delta}+\partial_\nu g_{\alpha\delta}-\partial_\delta g_{\alpha\nu})\\
&+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu g^{\delta\sigma}(\partial_\alpha g_{\mu\sigma}+\partial_\mu g_{\alpha\sigma}-\partial_\sigma g_{\alpha\mu})(\partial_\beta g_{\nu\delta}+\partial_\nu g_{\beta\delta}-\partial_\delta g_{\beta\nu}).
\end{aligned}$$

Note that in the expansion of the right hand side, each term contains a product $e_k(g)\cdot e_l(g)$ where $l\neq 3$, except

$$I:=-\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu g^{\delta\sigma}\partial_\sigma g_{\beta\mu}\partial_\delta g_{\alpha\nu}+\frac{1}{2}e_4^\alpha e_a^\beta e_4^\mu e_b^\nu g^{\delta\sigma}\partial_\sigma g_{\alpha\mu}\partial_\delta g_{\beta\nu}.$$

Now we apply $g^{\delta\sigma}=\sum_a e_a^\delta e_a^\sigma+\frac{1}{2}(e_3^\delta e_4^\sigma+e_3^\sigma e_4^\delta)$. Then, we can also write I as a sum of several terms containing $e_k(g)\cdot e_l(g)$ where $l\neq 3$. Since $e_l(g)=V_l(g)\cdot\mathfrak{R}^{m+1}_{-1,0}$, the whole sum is $\varepsilon^2\mathfrak{R}^m_{-3,0}$. Combine all the disccussion above and we finish the proof.

(b) We have

$$\begin{aligned}
\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{ab}&=\frac{1}{2}g^{0\mu}(e_4^\beta e_4(g_{\beta\mu})+e_4^\alpha e_4(g_{\alpha\mu})-e_4^\alpha e_4^\beta\partial_\mu g_{\alpha\beta})\chi_{ab}\\
&=-\frac{1}{2}g^{0\mu}e_4^\alpha e_4^\beta\partial_\mu g_{\alpha\beta}\chi_{ab}+\mathcal{R}=-\frac{1}{4}e_4^\alpha e_4^\beta(e_3(g_{\alpha\beta})-e_4(g_{\alpha\beta}))\chi_{ab}+\mathcal{R}\\
&=-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})\chi_{ab}+\mathcal{R}.
\end{aligned}$$

Here the remainder $\mathcal{R}$ is a linear combination of $g\cdot(e_*^*)\cdot e_4(g)\cdot\chi$ or $(e_*^*)\cdot(e_*^*)\cdot e_4(g)\cdot\chi$. Since $e_4(g)=t^{-1}V_4(g)=\varepsilon\mathfrak{R}^m_{-2,0}$ and $(g,e_*^*)=\mathfrak{R}^m_{0,0}$, under our assumption on χ, it follows from the Leibniz's rule that $\mathcal{R}=\varepsilon\mathfrak{R}^m_{-3,0}$. □

Remark 4.37.1. Note we only have $\chi=\mathfrak{R}^{m-1}_{-1,0}$ from our induction hypotheses, so we cannot apply (b) directly assuming (4.45) only.

We now prove Proposition 4.31 for $|I|=m$. Fix a multiindex I such that $|I|=m$. We have

$$\begin{aligned}
&[V_4,V_4]=0,\\
&[V_4,V_a]=t(e_4^i-\omega_i)\omega_i e_a-t(r\chi_{ab}-\delta_{ab})e_b,\\
&[V_4,V_3]=-t(e_4^i-\omega_i)\omega_i e_3+(3R-r+t)e_4-t(3R-r+t)\xi^l_{34}e_l.
\end{aligned}$$

We write $[V_4, V_k] := \eta_k^l V_l$. Then by Lemma 4.34, Lemma 4.35 and the inudction hypotheses (4.45), we have

$$\begin{cases} \eta_a^a = (e_4^i - \omega_i)\omega_i t r^{-1} - t(\chi_{aa} - r^{-1}) = \mathfrak{R}^{m-1}_{-1,0}; \\ \eta_a^{a'} = -t\chi_{12} = \mathfrak{R}^{m-1}_{-1,0}, \quad a \neq a' \\ \eta_3^3 = -t(e_4^i - \omega_i)\omega_i(3R - r + t)^{-1} - t\xi_{34}^3 = \varepsilon\mathfrak{R}^m_{0,-1}; \\ \eta_3^4 = (3R - r + t)t^{-1} - (3R - r + t)\xi_{34}^4 = \mathfrak{R}^m_{-1,1} \\ \eta_3^a = -(3R - r + t)\xi_{34}^a t r^{-1} = \mathfrak{R}^{m-1}_{-1,0}; \\ \eta_*^* \equiv 0 \text{ in all other cases.} \end{cases} \tag{4.53}$$

In summary we have $\eta_*^* = \mathfrak{R}^{m-1}_{-1,1}$. Here we briefly explain why $\eta_3^3 = \varepsilon\mathfrak{R}^m_{0,-1}$, since all other estimates are clear. Note that $(e_4^i - \omega_i)\omega_i = \varepsilon\mathfrak{R}^m_{-1,0}$ by Lemma 4.35. Also note that $\xi_{34}^4 = \xi_{34}^3 = e_4^\alpha e_4^\beta \Gamma^0_{\alpha\beta} = \varepsilon\mathfrak{R}^m_{-1,-1}$. Thus,

$$\eta_3^3 = -t(e_4^i - \omega_i)\omega_i(3R - r + t)^{-1} - t\xi_{34}^3 = \mathfrak{R}^{m+1}_{1,0} \cdot \varepsilon\mathfrak{R}^m_{-1,0} \cdot \mathfrak{R}^{m+1}_{0,-1} + \mathfrak{R}^{m+1}_{1,0} \cdot \varepsilon\mathfrak{R}^m_{-1,-1} = \varepsilon\mathfrak{R}^m_{0,-1}.$$

In addition, since $\Gamma = O(\varepsilon t^{-1})$, we have

$$\eta_3^3 = (3R - r + t)^{-1} t e_4(3R - r + t) - t\xi_{34}^3 = V_4(\ln(3R - r + t)) + O(\varepsilon).$$

Next, we note that

$$\begin{aligned} &V_4(V^I(\xi^{l_1}_{k_1k_2})) \\ =& \sum_{(J,k,J')=I} V^J[V_4, V_k]V^{J'}(\xi^{l_1}_{k_1k_2}) + V^I(V_4(\xi^{l_1}_{k_1k_2})) \\ =& \sum_{(J,k,J')=I} V^J(\eta_k^l V_l(V^{J'}(\xi^{l_1}_{k_1k_2}))) + V^I(V_4(\xi^{l_1}_{k_1k_2})) \\ =& \sum_{(J,k,J')=I} \eta_k^l V^{(J,l,J')}(\xi^{l_1}_{k_1k_2}) + \sum_{\substack{|J_1|+|J_2|=m \\ 0<|J_1|<m}} C_{J_1,J_2} V^{J_1}(\eta_k^l) V^{J_2}(\xi^{l_1}_{k_1k_2}) + V^I(V_4(\xi^{l_1}_{k_1k_2})) \\ =:& \, Q_1 + Q_2 + Q_3. \end{aligned} \tag{4.54}$$

In Q_1, we note that if $\eta_k^l \not\equiv 0$, then we must have $n_{(J,l,J'),3} \leq n_{(J,k,J'),3}$. Recall that $n_{J,3}$ denotes the number of V_3 in the product V^J. This is because $\eta_k^3 \equiv 0$ for $k \neq 3$. In addition, we note that $n_{(J,l,J'),3} < n_{(J,k,J'),3}$ if $k=3$ and $l \neq 3$. Then,

$$\begin{aligned} Q_1 &= (n_{I,3}\eta_3^3 - \sum_a n_{I,a}\eta_a^a)V^I(\xi_{k_1k_2}^{l_1}) + O((|\eta_1^2| + |\eta_2^1|) \sum_{\substack{|J|=m \\ n_{J,3}=n_{I,3}}} |V^J(\xi_{k_1k_2}^{l_1})|) \\ &\quad + O(\sum_{l\neq 3}|\eta_3^l| \sum_{(J_1,3,J_2)=I} |V^{(J_1,l,J_2)}(\xi_{k_1k_2}^{l_1})|) \\ &= n_{I,3}V_4(\ln(3R-r+t))V^I(\xi_{k_1k_2}^{l_1}) + O((\varepsilon + t^{-1+C\varepsilon}) \sum_{\substack{|J|=m, \\ n_{J,3}=n_{I,3}}} |V^J(\xi_{k_1k_2}^{l_1})|) \\ &\quad + O(\langle q\rangle t^{-1+C\varepsilon} \sum_{\substack{|J|=m, \\ n_{J,3}<n_{I,3}}} |V^J(\xi_{k_1k_2}^{l_1})|). \end{aligned} \tag{4.55}$$

In Q_2, we have $|J_1|, |J_2| < m$. Since $\eta_*^* = \mathfrak{R}_{-1,1}^{m-1}$, we have

$$|Q_2| \lesssim \sum_{\substack{|J_1|+|J_2|=m \\ 0<|J_1|<m}} |V^{J_1}(\mathfrak{R}_{-1,1}^{m-1})V^{J_2}(\xi_{k_1k_2}^{l_1})| \lesssim t^{-1+C\varepsilon}\langle q\rangle \sum_{0<|J|<m} |V^J(\xi_{k_1k_2}^{l_1})|. \tag{4.56}$$

Now we combine (4.54) with Section 4.4.1. First, note that $\xi_{34}^3 = \Gamma_{\alpha\beta}^0 e_4^\alpha e_4^\beta = \varepsilon\mathfrak{R}_{-1,-1}^m$ by Lemma 4.34, so $|V^I(\xi_{34}^3)| \lesssim \varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1}$ whenever $|I| \leq m$. There is no need to apply (4.54).

Next, we consider $\chi_{ab} = \xi_{a4}^b$.

Proposition 4.38. *Under our induction hypotheses* (4.45), *for* $|I| = m$ *we have*

$$|V^I(\chi_{ab})| \lesssim t^{-1+C\varepsilon}, \qquad |V^I(\chi_{ab} - r^{-1}\delta_{ab})| \lesssim t^{-2+C\varepsilon}.$$

So $\chi_{ab} = \mathfrak{R}_{-1,0}^m$ *and* $\chi_{ab} - r^{-1}\delta_{ab} = \mathfrak{R}_{-2,0}^m$.

Proof. We first prove that $V^I(\chi_{ab}) = O(t^{-1+C\varepsilon})$ whenever $|I| = m$. Fix I such that $|I| = m$ and $n_{I,3} = n \leq m$. Recall from (4.45) that $\chi_{ab} = \mathfrak{R}_{-1,0}^{m-1}$ and $\chi_{ab} - r^{-1}\delta_{ab} = \mathfrak{R}_{-2,0}^{m-1}$. Suppose that we have proved $V^J(\chi_{ab}) = O(t^{-1+C\varepsilon})$ for all J such that $|J| = m$ and $n_{J,3} < n$. Note that

$$\chi_{ac}\chi_{cb} = \delta_{ab}r^{-2} + 2(\chi_{ab} - \delta_{ab}r^{-1})r^{-1} + (\chi_{ac} - \delta_{ac}r^{-1})(\chi_{cb} - \delta_{cb}r^{-1}).$$

By Lemma 4.34, we have $r^{-1} = \mathfrak{R}_{-1,0}^{m+1}$ and $t = \mathfrak{R}_{1,0}^{m+1}$. Also note that $V(tr^{-1}) = V((t-r)r^{-1}) = \mathfrak{R}_{-1,1}^m$. Thus,

$$\begin{aligned} &|\sum_c V^I(t\chi_{ac}\chi_{cb}) - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}t)| \\ \lesssim &\sum_{\substack{|J_1|+|J_2|=m \\ |J_1|>0}} |V^{J_1}(tr^{-1})V^{J_2}(\chi_{ab} - r^{-1}\delta_{ab})| + t|\chi_{**} - \delta_{**}r^{-1}||V^I(\chi_{**} - \delta_{**}r^{-1})| \\ &+ \sum_{\substack{|J_1|+|J_2|+|J_3|=m \\ |J_2|<m,\ |J_3|<m}} |V^{J_1}(t)V^{J_2}(\chi_{**} - \delta_{**}r^{-1})V^{J_3}(\chi_{**} - \delta_{**}r^{-1})| \\ \lesssim &\langle q\rangle t^{-3+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**} - \delta_{**}r^{-1})|. \end{aligned}$$

By the Raychaudhuri equation, we have

$$\begin{aligned}
&V^I(V_4(\chi_{ab})) = V^I(t\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{ab}) - \sum_c V^I(t\chi_{ac}\chi_{cb}) + V^I(t\langle R(e_4,e_a)e_4,e_b\rangle)\\
&= t\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta V^I(\chi_{ab}) + O(\sum_{\substack{|J_1|+|J_2|=m\\|J_2|<m}}|V^{J_1}(\varepsilon\mathfrak{R}^m_{0,-1})V^{J_2}(\chi_{ab})|)\\
&\quad - 2tr^{-1}V^I(\chi_{ab}-\delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}t) + O(\langle q\rangle t^{-3+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**}-\delta_{**}r^{-1})|)\\
&\quad + V^I(\varepsilon t\mathfrak{R}^m_{-2,-1})\\
&= -2tr^{-1}V^I(\chi_{ab}) + O((\varepsilon+t^{-1+C\varepsilon})|V^I(\chi_{**})|) + O(t^{-1+C\varepsilon}).
\end{aligned}$$

Besides, by (4.55) and our induction hypotheses, we have

$$\begin{aligned}
|Q_1 - nV_4(\ln(3R-r+t))V^I(\chi_{ab})| &\lesssim \varepsilon\sum_{\substack{|J|=m,\\n_{J,3}=n}}|V^J(\chi_{ab})| + \langle q\rangle t^{-1+C\varepsilon}\sum_{\substack{|J|=m,\\n_{J,3}<n}}|V^J(\chi_{ab})|\\
&\lesssim \varepsilon\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\chi_{ab})| + \langle q\rangle t^{-2+C\varepsilon}.
\end{aligned}$$

By (4.56) and our induction hypotheses, we have

$$|Q_2| \lesssim t^{-1+C\varepsilon}\langle q\rangle\sum_{|J|<m}|V^J(\chi_{ab})| \lesssim t^{-2+C\varepsilon}\langle q\rangle.$$

In conclusion, by (4.54) we have

$$\begin{aligned}
&|e_4(V^I(\chi_{ab})) + (-ne_4(\ln(3R-r+t)) + 2r^{-1})V^I(\chi_{ab})|\\
&\lesssim t^{-1}(|Q_1 - nV_4(\ln(3R-r+t))V^I(\chi_{ab})| + |Q_2| + |V^I(V_4(\chi_{ab})) + 2tr^{-1}V^I(\chi_{ab})|)\\
&\lesssim \varepsilon t^{-1}\sum_{c,c'}\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\chi_{cc'})| + t^{-2+C\varepsilon} + \langle q\rangle t^{-3+C\varepsilon} \lesssim \varepsilon t^{-1}\sum_{c,c'}\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\chi_{cc'})| + t^{-2+C\varepsilon}.
\end{aligned}$$

The last inequality holds as $\langle q\rangle \lesssim t$. By Lemma 4.32 with $n_0 = 2$, $n_1 = n$ and Lemma 4.30, we conclude that

$$\begin{aligned}
\sum_{a,b}\sum_{\substack{|I|=m\\n_{I,3}=n}}|V^I(\chi_{ab})| &\lesssim t^{-2+C\varepsilon}(x^0(0)^2\cdot x^0(0)^{-1+C\varepsilon} + \int_{x^0(0)}^t \tau^{2+C\varepsilon}\cdot\tau^{-2+C\varepsilon}\,d\tau)\\
&\lesssim t^{-2+C\varepsilon}\cdot t^{1+C\varepsilon} \lesssim t^{-1+C\varepsilon}.
\end{aligned}$$

By induction we obtain $\chi_{ab} = \mathfrak{R}^m_{-1,0}$.

Next we prove $V^I(\chi_{ab} - r^{-1}\delta_{ab}) = O(t^{-2+C\varepsilon})$ whenever $|I| = m$. Again fix I such that $|I| = m$ and $n_{I,3} = n \leq m$. Suppose we have proved that $V^J(\chi_{ab} - r^{-1}\delta_{ab}) = O(t^{-2+C\varepsilon})$ for

$|J| = m$ and $n_{J,3} < n$. Now we can apply Lemma 4.37. We have

$$\begin{aligned}
&V^I(V_4(\chi_{ab})) = V^I(t\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta \chi_{ab}) - \sum_c V^I(t\chi_{ac}\chi_{cb}) + V^I(t\langle R(e_4, e_a)e_4, e_b\rangle) \\
&= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t\chi_{ab} + t\varepsilon\mathfrak{R}^m_{-3,0}) + V^I(V_4(f_{ab}) + \frac{1}{4}e_4^\alpha e_4^\beta tr^{-1}\delta_{ab}e_3(g_{\alpha\beta}) + t\varepsilon\mathfrak{R}^m_{-3,0}) \\
&\quad - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}t) + O(t^{-3+C\varepsilon}\langle q\rangle + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|) \\
&= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t(\chi_{ab} - r^{-1}\delta_{ab})) + V^I(V_4(f_{ab})) + O(\varepsilon t^{-2+C\varepsilon}) \\
&\quad - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) - V^I(\delta_{ab}r^{-2}t) + O(t^{-3+C\varepsilon}\langle q\rangle + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|).
\end{aligned}$$

Also note that

$$V^I(V_4(r^{-1})) = V^I(te_4(r^{-1})) = V^I(-tr^{-2}e_4(r))$$

and that $e_4(r) - 1 = \varepsilon\mathfrak{R}^m_{-1,0}$ by Lemma 4.35. In conclusion,

$$\begin{aligned}
&V^I(V_4(\chi_{ab} - r^{-1}\delta_{ab} - f_{ab})) \\
&= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t(\chi_{ab} - r^{-1}\delta_{ab})) - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) + V^I(\delta_{ab}r^{-2}t(e_4(r) - 1)) \\
&\quad + O(t^{-3+C\varepsilon}\langle q\rangle + \varepsilon t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|) \\
&= V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t(\chi_{ab} - r^{-1}\delta_{ab})) - 2tr^{-1}V^I(\chi_{ab} - \delta_{ab}r^{-1}) \\
&\quad + O(t^{-3+C\varepsilon}\langle q\rangle + \varepsilon t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\chi_{**} - r^{-1}\delta_{**})|).
\end{aligned}$$

Besides, we note that

$$V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t(\chi_{ab} - r^{-1}\delta_{ab})) + \frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})tV^I(\chi_{ab} - r^{-1}\delta_{ab})$$

is a linear combination of terms of the form

$$V^{I_1}(e_4^\alpha e_4^\beta t(3R - r + t)^{-1}V_3(g_{\alpha\beta}))V^{I_2}(\chi_{ab} - r^{-1}\delta_{ab})$$

where $|I_1| + |I_2| = |I| = m$ and $|I_2| < m$. By the induction hypotheses and since

$$e_4^\alpha e_4^\beta t(3R - r + t)^{-1}V_3(g_{\alpha\beta}) = \mathfrak{R}^m_{1,-1} \cdot \varepsilon\mathfrak{R}^m_{-1,0} = \varepsilon\mathfrak{R}^m_{0,-1}$$

by Lemma 4.34, we conclude that

$$V^I(-\frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})t(\chi_{ab} - r^{-1}\delta_{ab})) + \frac{1}{4}e_4^\alpha e_4^\beta e_3(g_{\alpha\beta})tV^I(\chi_{ab} - r^{-1}\delta_{ab}) = O(\varepsilon t^{-2+C\varepsilon}\langle q\rangle^{-1}).$$

Thus, by setting $F_{ab} = \chi_{ab} - r^{-1}\delta_{ab} - f_{ab} = \mathfrak{R}^{m-1}_{-2,0}$ and noting that $f_{ab} = \varepsilon\mathfrak{R}^m_{-2,0}$, we have

$$\begin{aligned}
V^I(V_4(F_{ab})) &= -2tr^{-1}V^I(F_{ab} + f_{ab}) + O(\varepsilon|V^I(F_{ab} + f_{ab})|) \\
&\quad + O(\varepsilon t^{-2+C\varepsilon} + t^{-3+C\varepsilon}\langle q\rangle + t^{-1+C\varepsilon}|V^I(F_{**} + f_{**})|) \\
&= -2tr^{-1}V^I(F_{ab}) + O(\varepsilon|V^I(F_{ab})| + \varepsilon t^{-2+C\varepsilon} + t^{-3+C\varepsilon}\langle q\rangle + t^{-1+C\varepsilon}|V^I(F_{**})|).
\end{aligned}$$

In (4.54), (4.55) and (4.56), we can replace $\xi^{l_1}_{k_1k_2}$ with F_{ab}. Thus, we have $V_4(V^I(F_{ab})) = Q_1 + Q_2 + V^I(V_4(F_{ab}))$, where by the induction hypotheses we have

$$\begin{aligned} Q_1 &= nV_4(\ln(3R-r+t))V^I(F_{ab}) + O(\varepsilon \sum_{\substack{|J|=m\\ n_{J,3}=n}} |V^J(F_{ab})|) + O(\langle q\rangle t^{-1+C\varepsilon} \sum_{\substack{|J|=m\\ n_{J,3}<n}} |V^J(F_{ab})|) \\ &= nV_4(\ln(3R-r+t))V^I(F_{ab}) + O(\varepsilon \sum_{\substack{|J|=m\\ n_{J,3}=n}} |V^J(F_{ab})|) + O(\langle q\rangle t^{-3+C\varepsilon}), \end{aligned}$$

$$|Q_2| \lesssim \langle q\rangle t^{-1+C\varepsilon} \sum_{0<|J|<m} |V^J(F_{ab})| \lesssim \langle q\rangle t^{-3+C\varepsilon}.$$

Thus,

$$\begin{aligned} &|e_4(V^I(F_{ab})) - ne_4(\ln(3R-r+t))V^I(F_{ab}) + 2r^{-1}V^I(F_{ab})| \\ &\lesssim \varepsilon t^{-1} \sum_{\substack{|J|=m\\ n_{J,3}=n}} |V^J(F_{ab})| + t^{-2+C\varepsilon}|V^I(F_{**})| + t^{-4+C\varepsilon}\langle q\rangle + \varepsilon t^{-3+C\varepsilon}. \end{aligned}$$

By Lemma 4.32 with $n_0 = 2$, $n_1 = n$ and Lemma 4.30, we have

$$\begin{aligned} \sum_{a,b} \sum_{\substack{|I|=m\\ n_{I,3}=n}} |V^I(F_{ab})| &\lesssim t^{-2+C\varepsilon}(x^0(0)^{C\varepsilon} + \int_{x^0(0)}^t \langle q\rangle \tau^{-2+C\varepsilon} + \varepsilon\tau^{-1+C\varepsilon}\, d\tau) \\ &\lesssim t^{-2+C\varepsilon}(x^0(0)^{C\varepsilon} + \langle q\rangle (x^0(0))^{-1+C\varepsilon} + t^{C\varepsilon}) \lesssim t^{-2+C\varepsilon}. \end{aligned}$$

Here we recall that $t \geq x^0(0) \sim T_0 + \langle q\rangle$. We then finish the proof by induction. $\square$

Next, we consider ξ^a_{12}.

Proposition 4.39. *Under our induction hypotheses* (4.45), *for* $|I| = m$, *we have*

$$|V^I(\xi^a_{12})| \lesssim t^{-1+C\varepsilon}.$$

So $\xi^a_{12} = \mathfrak{R}^m_{-1,0}$.

Proof. Fix I such that $|I| = m$ and $n_{I,3} = n \leq m$. Recall from (4.45) that $\xi^a_{12} = \mathfrak{R}^{m-1}_{-1,0}$. Suppose that $V^J(\xi^a_{12}) = O(t^{-1+C\varepsilon})$ for $|J| = m$ and $n_{J,3} < n$. By the equation in Section 4.4.1 we have

$$V^I(V_4(\xi^a_{12})) = V^I(t\Gamma^0_{\alpha\beta}e^\alpha_4 e^\beta_2 \chi_{a1} - t\Gamma^0_{\alpha\beta}e^\alpha_4 e^\beta_1 \chi_{a2}) - V^I(t\chi_{ac}\xi^c_{12}) + V^I(t\langle R(e_4, e_a)e_2, e_1\rangle). \tag{4.57}$$

By Lemma 4.36, the last term is $O(\varepsilon\langle q\rangle^{-1}t^{-1+C\varepsilon})$. By Lemma 4.34 and Proposition 4.38, we note that

$$t\Gamma^0_{\alpha\beta}e^\alpha_4 e^\beta_2 \chi_{a1} - t\Gamma^0_{\alpha\beta}e^\alpha_4 e^\beta_1 \chi_{a2} = \mathfrak{R}^{m+1}_{1,0} \cdot \varepsilon\mathfrak{R}^{m+1}_{-1,-1} \cdot \mathfrak{R}^m_{0,0} \cdot \mathfrak{R}^m_{0,0} \cdot \mathfrak{R}^m_{-1,0} = \varepsilon\mathfrak{R}^m_{-1,-1}.$$

Thus, the first term in (4.57) is also $O(\varepsilon\langle q\rangle^{-1}t^{-1+C\varepsilon})$. Next, by the Leibniz's rule we have

$$\begin{aligned}|V^I(t\chi_{ac}\xi_{12}^c)-t\chi_{ac}V^I(\xi_{12}^c)| &\lesssim \sum_{\substack{|J_1|+|J_2|=m\\|J_1|>0}}|V^{J_1}(t\chi_{ac})V^{J_2}(\xi_{12}^c)|\\ &\lesssim \sum_{\substack{|J_1|+|J_2|=m\\|J_1|>0}}(|V^{J_1}(t(\chi_{ac}-\delta_{ac}r^{-1}))V^{J_2}(\xi_{12}^c)|+|V^{J_1}(tr^{-1})V^{J_2}(\xi_{12}^a)|).\end{aligned}$$

By Proposition 4.38 we have $t(\chi_{ac}-\delta_{ac}r^{-1})=\mathfrak{R}^m_{-1,0}$. Also recall that $V(tr^{-1})=V((t-r)r^{-1})=\mathfrak{R}^m_{-1,1}$. Thus,

$$\begin{aligned}|V^I(t\chi_{ac}\xi_{12}^c)-tr^{-1}V^I(\xi_{12}^a)| &\lesssim |V^I(t\chi_{ac}\xi_{12}^c)-t\chi_{ac}V^I(\xi_{12}^c)|+|t(\chi_{ac}-r^{-1}\delta_{ac})V^I(\xi_{12}^c)|\\ &\lesssim t^{-2+C\varepsilon}\langle q\rangle+t^{-1+C\varepsilon}|V^I(\xi_{12}^*)|.\end{aligned}$$

In conclusion, we have

$$V^I(V_4(\xi_{12}^a))=-tr^{-1}V^I(\xi_{12}^a)+O(t^{-1+C\varepsilon}|V^I(\xi_{12}^*)|+t^{-2+C\varepsilon}\langle q\rangle+\varepsilon\langle q\rangle^{-1}t^{-1+C\varepsilon}).$$

Moreover, by (4.55), we have

$$\begin{aligned}|Q_1-nV_4(\ln(3R-r+t))V^I(\xi_{12}^a)| &\lesssim \varepsilon\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi_{12}^a)|+\langle q\rangle t^{-1+C\varepsilon}\sum_{\substack{|J|=m\\n_{J,3}<n}}|V^J(\xi_{12}^a)|\\ &\lesssim \varepsilon\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi_{12}^a)|+\langle q\rangle t^{-2+C\varepsilon}.\end{aligned}$$

By (4.56), we have

$$|Q_2|\lesssim t^{-1+C\varepsilon}\langle q\rangle\sum_{0<|J|<m}|V^J(\xi_{12}^a)|\lesssim t^{-2+C\varepsilon}\langle q\rangle.$$

Thus,

$$\begin{aligned}&|e_4(V^I(\xi_{12}^a))+(-ne_4(\ln(3R-r+t))+r^{-1})V^I(\xi_{12}^a)|\\ &\lesssim \varepsilon t^{-1}\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi_{12}^a)|+t^{-2+C\varepsilon}|V^I(\xi_{12}^*)|+t^{-3+C\varepsilon}\langle q\rangle+\varepsilon\langle q\rangle^{-1}t^{-2+C\varepsilon}.\end{aligned}$$

We now apply Lemma 4.32 with $n_0=1$, $n_1=n$ and Lemma 4.30. Then,

$$\begin{aligned}\sum_a\sum_{\substack{|I|=m\\n_{I,3}=n}}|V^I(\xi_{12}^a)| &\lesssim t^{-1+C\varepsilon}(x^0(0)^{C\varepsilon}+\int_{x^0(0)}^t\tau^{-2+C\varepsilon}\langle q\rangle+\varepsilon\langle q\rangle^{-1}\tau^{-1+C\varepsilon}\ d\tau)\\ &\lesssim t^{-1+C\varepsilon}(x^0(0)^{C\varepsilon}+x^0(0)^{-1+C\varepsilon}\langle q\rangle+\langle q\rangle^{-1}t^{C\varepsilon})\lesssim t^{-1+C\varepsilon}.\end{aligned}$$

Again recall that $t\geq x^0(0)\sim\langle q\rangle+T_0$. We finish the proof by induction. □

Next we study ξ_{34}^a. The proof of the following proposition is very similar to that of the previous one.

Proposition 4.40. *Under our induction hypotheses* (4.45)*, for* $|I| = m$*, we have*

$$|V^I(\xi_{34}^a)| \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

So $\xi_{34}^a = \mathfrak{R}_{-1,-1}^m$.

Proof. Fix I such that $|I| = m$ and $n_{I,3} = n \leq m$. Recall from (4.45) that $\xi_{34}^a = \mathfrak{R}_{-1,-1}^{m-1}$. Suppose that $V^J(\xi_{34}^a) = O(t^{-1+C\varepsilon}\langle q\rangle^{-1})$ for $|J| = m$ and $n_{J,3} < n$. By the equation in Section 4.4.1 we have

$$V^I(V_4(\xi_{34}^a)) = -V^I(t\chi_{ba}\xi_{34}^b) + V^I(t\langle R(e_4, e_3)e_4, e_a\rangle) + 2V^I(V_4(\Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta)).$$

By Lemma 4.36, the second term is $O(\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1})$. In the third term, we note that

$$\begin{aligned}V_4(\Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta) &= V_4(\Gamma_{\alpha\beta}^0)e_4^\alpha e_a^\beta + \Gamma_{\alpha\beta}^0 V_4(e_4^\alpha)e_a^\beta + \Gamma_{\alpha\beta}^0 e_4^\alpha V_4(e_a^\beta) \\ &= \varepsilon\mathfrak{R}_{-1,-1}^m + \varepsilon\mathfrak{R}_{-1,-1}^m \cdot \varepsilon\mathfrak{R}_{-1,0} + \varepsilon\mathfrak{R}_{-1,-1}^m \cdot \varepsilon\mathfrak{R}_{-1,0} = \varepsilon\mathfrak{R}_{-1,-1}^m.\end{aligned}$$

We recall from Remark 4.34.1 that $e_4(e_*^*) = \varepsilon\mathfrak{R}_{-2,0}^m$. Thus, $V^I(V_4(\Gamma_{\alpha\beta}^0)) = O(\varepsilon\langle q\rangle^{-1}t^{-1+C\varepsilon})$. Following the computation in Proposition 4.39, we can prove that

$$\begin{aligned}&|V^I(t\chi_{ba}\xi_{34}^b) - tr^{-1}V^I(\xi_{34}^a)| \lesssim |V^I(t\chi_{ab}\xi_{34}^b) - t\chi_{ab}V^I(\xi_{34}^b)| + |t(\chi_{ab} - r^{-1}\delta_{ab})V^I(\xi_{34}^b)| \\ &\lesssim \sum_{\substack{|J_1|+|J_2|=m \\ |J_1|>0}} (|V^{J_1}(t(\chi_{ab} - \delta_{ab}r^{-1}))V^{J_2}(\xi_{34}^b)| + |V^{J_1}(tr^{-1})V^{J_2}(\xi_{34}^a)|) + t^{-1+C\varepsilon}|V^I(\xi_{34}^b)| \\ &\lesssim t^{-2+C\varepsilon} + t^{-1+C\varepsilon}|V^I(\xi_{34}^*)|.\end{aligned}$$

Moreover, by (4.55) we have

$$\begin{aligned}|Q_1 - nV_4(\ln(3R - r + t))V^I(\xi_{34}^a)| &\lesssim \varepsilon \sum_{\substack{|J|=m \\ n_{J,3}=n}} |V^J(\xi_{34}^a)| + \langle q\rangle t^{-1+C\varepsilon} \sum_{\substack{|J|=m \\ n_{J,3}<n}} |V^J(\xi_{34}^a)| \\ &\lesssim \varepsilon \sum_{\substack{|J|=m \\ n_{J,3}=n}} |V^J(\xi_{34}^a)| + t^{-2+C\varepsilon}.\end{aligned}$$

By (4.56), we have

$$|Q_2| \lesssim t^{-1+C\varepsilon}\langle q\rangle \sum_{0<|J|<m} |V^J(\xi_{34}^a)| \lesssim t^{-2+C\varepsilon}.$$

Thus,

$$\begin{aligned}&|e_4(V^I(\xi_{34}^a)) + (-ne_4(\ln(3R - r + t)) + r^{-1})V^I(\xi_{34}^a)| \\ &\lesssim \varepsilon t^{-1} \sum_{\substack{|J|=m \\ n_{J,3}=n}} |V^J(\xi_{34}^a)| + t^{-2+C\varepsilon}|V^I(\xi_{34}^*)| + t^{-3+C\varepsilon} + \varepsilon\langle q\rangle^{-1}t^{-2+C\varepsilon}.\end{aligned}$$

We now apply Lemma 4.32 with $n_0 = 1$, $n_1 = n$ and Lemma 4.30. Then,

$$\sum_a \sum_{\substack{|I|=m \\ n_{I,3}=n}} |V^I(\xi_{12}^a)| \lesssim t^{-1+C\varepsilon}(x^0(0)^{C\varepsilon}\langle q\rangle^{-1} + \int_{x^0(0)}^t \tau^{-2+C\varepsilon} + \varepsilon\langle q\rangle^{-1}\tau^{-1+C\varepsilon}\, d\tau)$$
$$\lesssim t^{-1+C\varepsilon}(x^0(0)^{C\varepsilon}\langle q\rangle^{-1} + x^0(0)^{-1+C\varepsilon} + \langle q\rangle^{-1}t^{C\varepsilon}) \lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

Again recall that $t \geq x^0(0) \sim \langle q\rangle + T_0$. We finish the proof by induction. □

Finally, we consider ξ_{a3}^l. The case when $l \in \{a, 3\}$ is easy.

Proposition 4.41. *Under our induction hypotheses* (4.45), *for* $|I| = m$, *we have*

$$\langle q\rangle|V^I(\xi_{a3}^3)| + |V^I(\xi_{a3}^a)| \lesssim t^{-1+C\varepsilon}.$$

So $\xi_{a3}^3 = \mathfrak{R}_{-1,-1}^m$ *and* $\xi_{a3}^a = \mathfrak{R}_{-1,0}^m$.

Proof. Recall from Section 4.4.1 that

$$\xi_{a3}^3 = -2\Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta + \frac{1}{2}\xi_{34}^a, \qquad \xi_{a3}^a = \chi_{aa} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_a^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_a^\nu.$$

Now we apply Lemma 4.34. Since $\Gamma = \varepsilon\mathfrak{R}_{-1,-1}^{m+1}$ and $(g, e_*^*) = \mathfrak{R}_{0,0}^m$, we have $\Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta = \varepsilon\mathfrak{R}_{-1,-1}^m$ and $g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_a^\nu = \varepsilon\mathfrak{R}_{-1,-1}^m$. Since $e_4(g^{0\alpha}) = t^{-1}V_4(g) = \varepsilon\mathfrak{R}_{-2,0}^m$ and $e_a(g^{0\alpha}) = r^{-1}V_a(g) = \varepsilon\mathfrak{R}_{-2,0}^m$, we have $e_a(g^{0\alpha})g_{\alpha\beta}e_a^\beta = \varepsilon\mathfrak{R}_{-2,0}^m$. We thus conclude that

$$(\xi_{a3}^3, \xi_{a3}^a) = (\frac{1}{2}\xi_{34}^a, \chi_{aa}) + \varepsilon\mathfrak{R}_{-1,-1}^m.$$

We finally apply Proposition 4.38, Proposition 4.39 and Proposition 4.40 to conclude that $\xi_{a3}^3 = \mathfrak{R}_{-1,-1}^m$ and $\xi_{a3}^a = \mathfrak{R}_{-1,0}^m$. □

The case $l = a'$ where $\{a, a'\} = \{1, 2\}$ is harder.

Proposition 4.42. *Under our induction hypotheses* (4.45), *for* $|I| = m$, *we have*

$$|V^I(\xi_{a3}^{a'})| \lesssim \langle q\rangle^{-1}t^{C\varepsilon}.$$

So $\xi_{a3}^{a'} = \mathfrak{R}_{0,-1}^m$.

Proof. Fix I such that $|I| = m$ and $n_{I,3} = n \leq m$. Recall from (4.45) that $\xi_{a3}^{a'} = \mathfrak{R}_{0,-1}^{m-1}$. Suppose that $V^J(\xi_{a3}^{a'}) = O(\langle q\rangle^{-1}t^{C\varepsilon})$ for $|J| = m$ and $n_{J,3} < n$. By the equation in Section 4.4.1 we have

$$V^I(V_4(\xi_{a3}^{a'})) = V^I((V_4 + t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu)(\chi_{aa'} + 2e_a(g^{0\alpha})g_{\alpha\beta}e_{a'}^\beta + 2g^{0\alpha}e_a^\beta\Gamma_{\beta\alpha}^\mu g_{\mu\nu}e_{a'}^\nu)) - V^I(t\Gamma_{\mu\nu}^0 e_4^\mu e_4^\nu\xi_{a3}^{a'})$$
$$- \sum_c V^I(t\xi_{34}^c\xi_{aa'}^c) - V^I(t\langle R(e_4, e_3)e_a, e_{a'}\rangle) - V^I(t\Gamma_{\alpha\beta}^0 e_4^\alpha e_a^\beta\xi_{34}^{a'} + t\Gamma_{\alpha\beta}^0 e_4^\alpha e_{a'}^\beta\xi_{34}^a).$$

By the Leibniz's rule and all the previous results, we conclude that the second line has an upper bound

$$t^{-1+C\varepsilon}\langle q\rangle^{-1}+\varepsilon\langle q\rangle^{-1}t^{-1+C\varepsilon}\lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.$$

In the first line, we note that

$$t\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu(2e_a(g^{0\alpha})g_{\alpha\beta}e^\beta_{a'}+2g^{0\alpha}e^\beta_a\Gamma^\mu_{\beta\alpha}g_{\mu\nu}e^\nu_{a'})=\varepsilon\mathfrak{R}^m_{0,-1}\cdot(\varepsilon\mathfrak{R}^m_{-2,0}+\varepsilon\mathfrak{R}^m_{-1,-1})=\varepsilon^2\mathfrak{R}^m_{-1,-2}.$$

Besides, since $\chi_{aa'}=\mathfrak{R}^m_{-2,0}$ and since $\sum_c\chi_{ac}\chi_{ca'}=\chi_{12}\mathrm{tr}\chi$, we have

$$\begin{aligned}&|V^I(V_4(\chi_{aa'})+t\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu\chi_{aa'})|\\ \lesssim&|V^I(2t\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta\chi_{aa'})|+|V^I(t\chi_{12}(\chi_{11}+\chi_{22}))|+|V^I(t\langle R(e_4,e_a)e_4,e_{a'}\rangle)|\\ \lesssim&|V^I(\varepsilon\mathfrak{R}^m_{-2,-1})|+|V^I(\mathfrak{R}^{m+1}_{1,0}\cdot\mathfrak{R}^m_{-2,0}\cdot\mathfrak{R}^m_{-1,0})|+|V^I(\varepsilon\mathfrak{R}^m_{-1,-1})|\lesssim t^{-2+C\varepsilon}+\varepsilon t^{-1+C\varepsilon}\langle q\rangle^{-1}\lesssim t^{-1+C\varepsilon}\langle q\rangle^{-1}.\end{aligned}$$

Moreover, recall that $V_4(e^*_*)=\varepsilon\mathfrak{R}^m_{-1,0}$. We also have $\partial g=\varepsilon\mathfrak{R}^{m+1}_{-1,-1}$ by Remark 4.34.1. Thus, we have

$$\begin{aligned}&V_4(2e_a(g^{0\alpha})g_{\alpha\beta}e^\beta_{a'}+2g^{0\alpha}e^\beta_a\Gamma^\mu_{\beta\alpha}g_{\mu\nu}e^\nu_{a'})=2V_4(e_a(g^{0\alpha}))g_{\alpha\beta}e^\beta_{a'}+\varepsilon\mathfrak{R}^m_{-1,-1}\\ =&2e^\sigma_aV_4(\partial_\sigma g^{0\alpha})g_{\alpha\beta}e^\beta_{a'}+2V_4(e^\sigma_a)(\partial_\sigma g^{0\alpha})g_{\alpha\beta}e^\beta_{a'}+\varepsilon\mathfrak{R}^m_{-1,-1}=\varepsilon\mathfrak{R}^m_{-1,-1}.\end{aligned}$$

In conclusion,

$$\begin{aligned}|V^I(V_4(\xi^{a'}_{a3})|&\lesssim|V^I(t\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu\xi^{a'}_{a3})|+t^{-1+C\varepsilon}\langle q\rangle^{-1}\\&\lesssim|t\Gamma^0_{\mu\nu}e_4^\mu e_4^\nu V^I(\xi^{a'}_{a3})|+\sum_{\substack{|J_1|+|J_2|=m\\|J_2|<m}}|V^{J_1}(\varepsilon\mathfrak{R}^m_{0,-1})V^{J_2}(\xi^{a'}_{a3})|+t^{-1+C\varepsilon}\langle q\rangle^{-1}\\&\lesssim\varepsilon|V^I(\xi^{a'}_{a3})|+\varepsilon\langle q\rangle^{-2}t^{C\varepsilon}+t^{-1+C\varepsilon}\langle q\rangle^{-1}.\end{aligned}$$

Next, by (4.55), we have

$$\begin{aligned}|Q_1-nV_4(\ln(3R-r+t))V^I(\xi^{a'}_{a3})|&\lesssim\varepsilon\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi^{a'}_{a3})|+\langle q\rangle t^{-1+C\varepsilon}\sum_{\substack{|J|=m\\n_{J,3}<n}}|V^J(\xi^{a'}_{a3})|\\&\lesssim\varepsilon\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi^{a'}_{a3})|+t^{-1+C\varepsilon}.\end{aligned}$$

By (4.56), we have

$$|Q_2|\lesssim t^{-1+C\varepsilon}\langle q\rangle\sum_{0<|J|<m}|V^J(\xi^{a'}_{a3})|\lesssim t^{-1+C\varepsilon}.$$

Thus,

$$|e_4(V^I(\xi^{a'}_{a3}))-ne_4(\ln(3R-r+t))V^I(\xi^{a'}_{a3})|\lesssim\varepsilon t^{-1}\sum_{\substack{|J|=m\\n_{J,3}=n}}|V^J(\xi^{a'}_{a3})|+\varepsilon\langle q\rangle^{-2}t^{-1+C\varepsilon}+t^{-2+C\varepsilon}.$$

By Lemma 4.32 with $n_0 = 0$, $n_1 = n$ and Lemma 4.30, we have

$$\sum_{a,a'} \sum_{\substack{|I|=m \\ n_{I,3}=n}} |V^I(\xi^{a'}_{a3})| \lesssim t^{C\varepsilon}(\langle q\rangle^{-1} x^0(0)^{C\varepsilon} + \int_{x^0(0)}^t \varepsilon\langle q\rangle^{-2}\tau^{-1+C\varepsilon} + \tau^{-2+C\varepsilon}\, d\tau)$$
$$\lesssim t^{C\varepsilon}(\langle q\rangle^{-1}t^{C\varepsilon} + \langle q\rangle^{-2}t^{C\varepsilon} + (x^0(0))^{-1+C\varepsilon}) \lesssim \langle q\rangle^{-1}t^{C\varepsilon}.$$

We finish the proof by induction. □

Combining Proposition 4.38-4.42, we finish the proof of Proposition 4.31 by induction.

4.4.4 Estimates for higher derivatives of q

Now we can prove the estimates for higher derivatives of q. We first note that (4.53) holds for each $m \geq 1$, as long as $\varepsilon \ll_m 1$. This is because (4.53) is a result of (4.45) which then results from Proposition 4.31.

Lemma 4.43. *In $\Omega \cap \{r - t < 2R\}$, we have $V^I q = O(\langle q\rangle t^{C\varepsilon})$ for each multiindex I.*

Proof. We induct on $|I|$. If $|I| = 0$, there is nothing to prove. If $|I| = 1$, the estimates are clear since $V_1(q) = V_2(q) = V_4(q) = 0$ and $V_3(q) = O((3R - r + t)|\partial q|) = O(\langle q\rangle t^{C\varepsilon})$.

In general, we fix an integer $m > 1$. By choosing $\varepsilon \ll_m 1$, we can assume that Proposition 4.31 holds for all $|I| \leq m$. Suppose we have proved the estiamtes for $|I| < m$, so $q = \mathfrak{R}^{m-1}_{0,1}$. Fix a multiindex I such that $|I| = m$. If $n_{I,4} > 0$, we can write $I = (J', 4, J)$. Here we can assume $|J| > 0$ since otherwise we have $V^I(q) = V_{J'}(V_4(q)) = 0$. By (4.53), we have

$$\begin{aligned} V^I(q) &= V^{J'}(V_4(V^J(q)) = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}[V_4, V_k]V^{J_2}(q) \\ &= \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(\eta^l_k V^{(l,J_2)}(q)) = \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(\mathfrak{R}^{m-1}_{-1,1} \cdot \mathfrak{R}^{m-1-(1+|J_2|)}_{0,1}) \\ &= \sum_{J=(J_1,k,J_2)} V^{(J',J_1)}(\mathfrak{R}^{m-1-(1+|J_2|)}_{-1,2}) = O(\langle q\rangle^2 t^{-1+C\varepsilon}) = O(\langle q\rangle t^{C\varepsilon}). \end{aligned}$$

Here we note that $|J_2| + 1 = |J| - |J_1| = m - 1 - |J'| - |J_1|$, so we are able to apply the definition of $\mathfrak{R}^*_{**}$ here.

Next suppose $n_{I,3} < m$ and $n_{I,4} = 0$. Thus we can write $I = (J', a, J)$ where $n_{J,3} = |J|$. Here we can assume $|J| > 0$ since $V_a(q) = 0$. Then

$$V^I(q) = V^{J'}V_a(V^J(q)) = \sum_{J=(J_1,3,J_2)} V^{(J',J_1)}[V_a, V_3]V^{J_2}(q).$$

Note that

$$\begin{aligned}[V_a,V_3]F &= V_a((3R-r+t)e_3(F)) - V_3(re_a(F)) \\ &= V_a(3R-r+t)e_3(F) - V_3(r)e_a(F) + (3R-r+t)r[e_a,e_3](F) \\ &= -(3R-r+t)^{-1}V_a(r)V_3(F) - r^{-1}V_3(r)V_a(F) \\ &\quad + (3R-r+t)\xi^b_{a3}V_b(F) + r\xi^3_{a3}V_3(F) + (3R-r+t)rt^{-1}\xi^4_{a3}V_4(F).\end{aligned}$$

By Lemma 4.34 and Remark 4.34.1, we have $V_a(r)=\mathfrak{R}^m_{0,0}$, $V_3(r)=(3R-r+t)e^i_3\omega_i=\mathfrak{R}^m_{0,1}$. By Proposition 4.31, we have

$$[V_a,V_3] = \sum_{k=1}^4 \mathfrak{R}^m_{0,0}\cdot V_k = \mathfrak{R}^m_{0,0}\cdot V.$$

Thus,

$$\begin{aligned}V^I(q) &= \sum_{J=(J_1,3,J_2)} V^{(J',J_1)}(\mathfrak{R}^m_{0,0}\cdot V(V^{J_2}(q))) \\ &= \sum_{J=(J_1,3,J_2)} V^{(J',J_1)}(\mathfrak{R}^m_{0,0}\cdot \mathfrak{R}^{m-1-(1+|J_2|)}_{0,1}) = O(t^{C\varepsilon}\langle q\rangle).\end{aligned}$$

Again, we have $m-1=1+|J_2|+|J_1|+|J'|$.

Finally, suppose $n_{I,3}=|I|$. We have

$$V_4(V^I(q)) = \sum_{\substack{I=(J_1,3,J_2)\\ n_{J_1,3}=|J_1|,\ n_{J_2,3}=|J_2|}} V^{J_1}[V_4,V_3]V^{J_2}(q) = \sum_{\substack{I=(J_1,3,J_2)\\ n_{J_1,3}=|J_1|,\ n_{J_2,3}=|J_2|}} V^{J_1}(\eta^l_3 V^{(l,J_2)}(q)).$$

By the Leibniz's rule, we can express $V^{J_1}(\eta^l_3V^{(l,J_2)}(q))$ as a linear combination of terms of the form $V^{K_1}(\eta^l_3)V^{K_2}(q)$, where $|K_1|+|K_2|=m$, K_2 contains l, and (K_1,K_2) is an rearrangement of (J_1,l,J_2). Now recall from (4.53) that $\eta^l_3=\mathfrak{R}^{m-1}_{-1,1}+\varepsilon\mathfrak{R}^m_{0,-1}$. Since $V^J(q)=O(\langle q\rangle t^{C\varepsilon})$ for $|J|=m$ and $n_{J,3}<|J|$, we have

$$\begin{aligned}&V^{J_1}(\eta^l_3V^{(l,J_2)}(q)) \\ &= \eta^3_3V^I(q) + O(\sum_{\substack{|K_1|+|K_2|=m,\ 0<|K_1|<m\\ n_{|K_1|,3}=|K_1|,\ n_{|K_2|,3}=|K_2|}} |V^{K_1}(\eta^3_3)V^{K_2}(q)|) \\ &\quad + O(\sum_{l\neq 3}\sum_{\substack{|K_1|+|K_2|=m,\ |K_2|>0\\ n_{K_1,3}=|K_1|,\ n_{K_2,3}=|K_2|-1}} |V^{K_1}(\eta^l_3)V^{K_2}(q)|) \\ &= (te_4(\ln(3R-r+t))+O(\varepsilon))V^I(q) + O(\sum_{0<|K_1|<m} |V^{K_1}(\varepsilon\mathfrak{R}^m_{0,-1}+\mathfrak{R}^{m-1}_{-1,1})|\cdot t^{C\varepsilon}\langle q\rangle) \\ &\quad + O(\sum_{|K_1|<m} |V^{K_1}(\varepsilon\mathfrak{R}^m_{0,-1}+\mathfrak{R}^{m-1}_{-1,1})\cdot\langle q\rangle t^{C\varepsilon}| \\ &= te_4(\ln(3R-r+t))V^I(q) + O(\varepsilon|V^I(q)|) + O(\varepsilon t^{C\varepsilon}+t^{-1+C\varepsilon}\langle q\rangle^2).\end{aligned}$$

Thus,

$$|e_4(V^I(q)) - me_4(\ln(3R-r+t))V^I(q)| \lesssim \varepsilon t^{-1}|V^I(q)| + \varepsilon t^{-1+C\varepsilon} + t^{-2+C\varepsilon}\langle q\rangle^2.$$

Recall from Remark 4.29.1 that $V^I(q) = O(t^{C\varepsilon}\langle q\rangle)$ on H. Then, by Lemma 4.32 with $n_0 = 0$ and $n_1 = |I|$, we have

$$\begin{aligned}|V^I(q)| &\lesssim t^{C\varepsilon}(\langle q\rangle x^0(0)^{C\varepsilon} + \int_{x^0(0)}^t \varepsilon\tau^{-1+C\varepsilon} + \tau^{-2+C\varepsilon}\langle q\rangle^2\ d\tau)\\ &\lesssim t^{C\varepsilon}(\langle q\rangle t^{C\varepsilon} + t^{C\varepsilon} + (x^0(0))^{-1+C\varepsilon}\langle q\rangle^2) \lesssim \langle q\rangle t^{C\varepsilon}.\end{aligned}$$

□

We have the following important corollary.

Corollary 4.43.1. *The function $q(t,x)$ is a smooth function (in the sense defined in Section 4.2.1) in Ω. Moreover, we have $Z^I q = O(\langle q\rangle t^{C\varepsilon})$ and $Z^I\Omega_{ij}q = O(t^{C\varepsilon})$ for each multiindex I and $1 \le i < j \le 3$.*

Proof. Fix an integer $m > 1$. We seek to prove that for $\varepsilon \ll_m 1$, q is a C^m function and $Z^I q = O(\langle q\rangle t^{C\varepsilon})$ for $|I| \le m$. By writing $Z = z^\nu(t,x)\partial_\nu$, we have

$$r^{-1}\langle Z, e_a\rangle = r^{-1}z^\alpha e_a^\beta g_{\alpha\beta} = \mathfrak{R}^m_{0,0}, \qquad t^{-1}\langle Z, e_3\rangle = t^{-1}z^\alpha e_3^\beta g_{\alpha\beta} = \mathfrak{R}^m_{0,0}.$$

Moreover,

$$\begin{aligned}\langle Z, e_4\rangle &= z^\alpha e_4^\beta g_{\alpha\beta} = z^\alpha e_4^\beta(g_{\alpha\beta} - m_{\alpha\beta}) + z^\alpha e_4^\beta m_{\alpha\beta}\\ &= \varepsilon\mathfrak{R}^m_{0,0} - z^0 + z^i(e_4^i - \omega_i) + z^i\omega_i = \mathfrak{R}^m_{0,1} + Z(r-t).\end{aligned}$$

We can easily check that $Z(r-t) = \mathfrak{R}^m_{0,1}$, so $(3R-r+t)^{-1}\langle Z, e_4\rangle = \mathfrak{R}^m_{0,0}$. Then, by (4.29), $Z = \mathfrak{R}^m_{0,0}\cdot V$, so $Z^I q$ is a linear combination of terms of the form

$$Z^{I_1}(\mathfrak{R}^m_{0,0})\cdots Z^{I_s}(\mathfrak{R}^m_{0,0})V^s(q), \qquad \sum|I_*| + s = |I|,\ s > 0.$$

Each of such terms is $O(t^{C\varepsilon}\langle q\rangle)$ if $|I| \le m$, so we have $Z^I q = O(t^{C\varepsilon}\langle q\rangle)$ for $|I| \le m$.

Moreover, for each $m > 1$, as long as $\varepsilon \ll_m 1$, we have $q = \mathfrak{R}^{m+1}_{0,1}$ by Lemma 4.43. Then we have

$$\begin{aligned}\Omega_{ij}q &= \frac{1}{2}\langle\Omega_{ij}, e_4\rangle e_3(q) = \frac{1}{2}(x_i g_{j\beta} - x_j g_{i\beta})e_4^\beta e_3(q)\\ &= \frac{1}{2}(x_i m_{jk} - x_j m_{ik})\omega_k e_3(q) + \frac{1}{2}(x_i(g_{jk} - m_{jk}) - x_j(g_{ik} - m_{ik}))\omega_k e_3(q)\\ &\quad + \frac{1}{2}(x_i g_{jk} - x_j g_{ik})(e_4^k - \omega_k)e_3(q)\\ &= 0 + \varepsilon\mathfrak{R}^m_{0,0} + \mathfrak{R}^m_{0,0} = \mathfrak{R}^m_{0,0}.\end{aligned}$$

Again, for each multiindex I with $|I| \le m$, we can write $Z^I\Omega_{ij}q$ as a linear combination of terms of the form

$$Z^{I_1}(\mathfrak{R}^m_{0,0})\cdots Z^{I_s}(\mathfrak{R}^m_{0,0})V^s\Omega_{ij}(q), \qquad \sum|I_*| + s = m,\ s > 0.$$

Each of such terms is $O(t^{C\varepsilon})$, so we have $Z^I\Omega_{ij}q = O(t^{C\varepsilon})$ for $|I| \le m$. □

4.4.5 More estimates

We end this section with some estimates derived from our original wave equation (1.1). We first introduce a new definition.

Definition 4.44. Let $F = F(t, x)$ be a function with domain $\Omega \cap \{r - t < 2R\}$. For any integer $m \geq 0$ and any real numbers s, p, we have defined $F = \mathfrak{R}^m_{s,p}$ in Section 4.4.3 prior to Lemma 4.34. We now define $F = \mathfrak{R}_{s,p}$, if $F = \mathfrak{R}^m_{s,p}$ for each $m \geq 0$.

Again, by the Leibniz's rule, we have $V^I(\mathfrak{R}_{s,p}) = \mathfrak{R}_{s,p}$ and $\mathfrak{R}_{s_1,p_1} \cdot \mathfrak{R}_{s_2,p_2} = \mathfrak{R}_{s_1+s_2,p_1+p_2}$. In addition, by Proposition 4.31, we have

$$(\xi^2_{13}, \xi^1_{23}) = \mathfrak{R}_{0,-1};\ \xi^a_{34} = \mathfrak{R}_{-1,-1};\ \xi^a_{k_1k_2} = \mathfrak{R}_{-1,0} \text{ for all other } k_1 < k_2 \text{ and } a = 1, 2;$$
$$\xi^3_{k_1k_2} = \mathfrak{R}_{-1,-1} \text{ for all } k_1 < k_2;\ \chi_{ab} - r^{-1}\delta_{ab} = \mathfrak{R}_{-2,0}.$$

There are many other estimates in Section 4.4.3 invovling $\mathfrak{R}^*_{*,*}$. They would still hold if all the superscripts are removed, because they all rely on Proposition 4.31. For example, by Lemma 4.34 we have

$$e^*_* = \mathfrak{R}_{0,0},\ (e^i_4 - \omega_i, e^i_3 - \omega_i) = \mathfrak{R}_{-1,0};\ \partial^s Z^I(g - m) = \varepsilon\mathfrak{R}_{-1,-s},\ \Gamma^*_{**} = \varepsilon\mathfrak{R}_{-1,-1};$$
$$\omega = \mathfrak{R}_{0,0},\ (t^s, r^s) = \mathfrak{R}_{s,0},\ (3R - r + t)^s = \mathfrak{R}_{0,s}.$$

We remark that this definition follows the spirits of the convention in Section 4.2.1. In the defintion of $\mathfrak{R}^m_{s,p}$, we require some estimates to hold for all $\varepsilon \ll_{s,p,m} 1$. The dependence on m here should be emphasized.

Our goal in this subsection is to prove that

$$e_4(e_3(u)) + r^{-1}e_3(u) = \varepsilon\mathfrak{R}_{-3,0}, \qquad e_4(e_3(u)) = \varepsilon\mathfrak{R}_{-2,0}; \tag{4.58}$$

$$e_4(e_3(q)) = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}. \tag{4.59}$$

We start our proof with the following lemma.

Lemma 4.45. *We have the following estimates.*

(a) $q_\alpha = \mathfrak{R}_{0,0}$, $q_r^{-1} = \mathfrak{R}_{0,0}$; $e_k(q_r) = \mathfrak{R}_{-1,-1}$, $e_k(q_r^{-1}) = \mathfrak{R}_{-1,-1}$ *for* $k \neq 3$.

(b) $q_i + \omega_i q_t = \mathfrak{R}_{-1,0}$, $u_i + \omega_i u_t = \varepsilon\mathfrak{R}_{-2,0}$.

(c) $e_k(q_i + \omega_i q_t) = \mathfrak{R}_{-2,0}$, $e_k(u_i + \omega_i u_t) = \varepsilon\mathfrak{R}_{-3,0}$, *for* $k \neq 3$.

(d) *In* (b) *and* (c) *we can replace* $q_i + \omega_i q_t$ *with* $q_t + q_r$ *or* $q_i - \omega_i q_r$, *and replace* $u_i + \omega_i u_t$ *with* $u_t + u_r$ *or* $u_i - \omega_i u_r$. *The results are the same.*

Proof. (a) By Lemma 4.43, we have $V_3(q) = \mathfrak{R}_{0,1}$ and $e_3(q) = V_3(q) = \mathfrak{R}_{0,0}$. Then,

$$q_\alpha = \frac{1}{2} g_{\alpha\beta} e_4^\beta e_3(q) = \mathfrak{R}_{0,0} \cdot \mathfrak{R}_{0,0} \cdot \mathfrak{R}_{0,0} = \mathfrak{R}_{0,0}.$$

Since $\omega_i = \mathfrak{R}_{0,0}$, we have $q_r = \mathfrak{R}_{0,0}$. Since $q_r \geq C^{-1}t^{-C\varepsilon}$ and since $V^I(q_r^{-1})$ is a linear combination of terms of the form

$$q_r^{-s-1} V^{I_1}(q_r) \cdots V^{I_s}(q_r), \quad \text{where } \sum |I_j| = |I|,\ |I_j| > 0, \tag{4.60}$$

we conclude that $V^I(q_r^{-1}) = O(t^{C\varepsilon})$ for each I and thus $q_r^{-1} = \mathfrak{R}_{0,0}$. Besides, we have

$$\begin{aligned} &e_k(e_3(q)) = [e_k, e_3]q = \xi_{k3}^3 e_3(q), \qquad k = 1, 2, 3, 4; \\ &2\omega_i g_{i\beta} e_4^\beta = \langle e_3 + e_4, e_4 \rangle + (2\omega_i - e_4^i - e_3^i) g_{i\beta} e_4^\beta = 2 + \mathfrak{R}_{-1,0}. \end{aligned}$$

Thus, for $k \neq 3$,

$$\begin{aligned} e_k(q_r) &= e_k(\frac{1}{2}\omega_i g_{i\beta} e_4^\beta e_3(q)) = e_k(\frac{1}{2}\omega_i g_{i\beta} e_4^\beta) e_3(q) + \frac{1}{2}\omega_i g_{i\beta} e_4^\beta e_k(e_3(q)) \\ &= e_k(\frac{1}{2} + \mathfrak{R}_{-1,0}) e_3(q) + (\frac{1}{2} + \mathfrak{R}_{-1,0}) \xi_{k3}^3 e_3(q) \\ &= \mathfrak{R}_{-1,0} \cdot V_k(\mathfrak{R}_{-1,0}) \cdot \mathfrak{R}_{0,0} + \mathfrak{R}_{-1,-1} = \mathfrak{R}_{-1,-1}. \end{aligned}$$

Now if we expand $V^I(e_k(q_r^{-1}))$, each term is still of the form (4.60) with $s > 0$ and $V^{I_s}(q_r)$ replaced by $V^{I_s}(e_k(q_r))$. We thus conclude that $e_k(q_r^{-1}) = \mathfrak{R}_{-1,-1}$ for $k \neq 3$.

(b) We have

$$q_i + \omega_i q_t = \frac{1}{2}(g_{i\beta} + \omega_i g_{0\beta}) e_4^\beta e_3(q)$$

and

$$\begin{aligned} u_i + \omega_i u_t &= \frac{1}{2}(g_{i\beta} + \omega_i g_{0\beta}) e_4^\beta e_3(u) + \frac{1}{2}(g_{i\beta} + \omega_i g_{0\beta}) e_3^\beta e_4(u) + \sum_a (g_{i\beta} + \omega_i g_{0\beta}) e_a^\beta e_a(u) \\ &= \frac{1}{2}(g_{i\beta} + \omega_i g_{0\beta}) e_4^\beta (3R - r + t)^{-1} V_3(u) + \varepsilon \mathfrak{R}_{-2,0}. \end{aligned}$$

Here we have

$$(g_{i\beta} + \omega_i g_{0\beta}) e_4^\beta = e_4^i - \omega_i + ((g_{i\beta} - m_{i\beta}) + \omega_i (g_{0\beta} - m_{0\beta})) e_4^\beta = \mathfrak{R}_{-1,0}.$$

We thus conclude that $q_i + \omega_i q_t = \mathfrak{R}_{-1,0}$ and $u_i + \omega_i u_t = \varepsilon \mathfrak{R}_{-2,0}$.

(c) Recall that $e_a(r) = \mathfrak{R}_{-1,0}$, $e_4(\omega_i) = r^{-1}(e_4^i - \omega_i + (1 - e_4^j \omega_j)\omega_i) = \mathfrak{R}_{-2,0}$ and $e_4(e_k^\alpha) = \varepsilon \mathfrak{R}_{-2,0}$ by Lemma 4.34 and Lemma 4.35. Besides, note that

$$\begin{aligned} e_a(\omega_i) &= r^{-1}(e_a^i - e_a(r)\omega_i) = r^{-1} e_a^i + \mathfrak{R}_{-2,0}, \\ e_4(\omega_i) &= (e_4^j - \omega_j)\partial_j \omega_i = r^{-1}(e_4^i - \omega_i - (e_4^j - \omega_j)\omega_j \omega_i) = \mathfrak{R}_{-2,0}. \end{aligned}$$

Thus we have

$$\begin{aligned}
&e_a((g_{i\beta}+\omega_i g_{0\beta})e_4^\beta) = e_a(g_{i\beta}+\omega_i g_{0\beta})e_4^\beta + (g_{i\beta}+\omega_i g_{0\beta})e_a(e_4^\beta)\\
&= (e_a(g_{i\beta}) + \omega_i e_a(g_{0\beta}) + e_a(\omega_i)g_{0\beta})e_4^\beta + (g_{i\beta}+\omega_i g_{0\beta})(\xi_{a4}^l e_l^\beta + e_4(e_a^\beta))\\
&= (\varepsilon\mathfrak{R}_{-2,0} + (r^{-1}e_a^i + \mathfrak{R}_{-2,0})g_{0\beta})e_4^\beta + (g_{i\beta}+\omega_i g_{0\beta})(\xi_{a4}^b e_b^\beta + \varepsilon\mathfrak{R}_{-2,0})\\
&= r^{-1}e_a^i g_{0\beta}e_4^\beta + r^{-1}(g_{i\beta}+\omega_i g_{0\beta})e_a^\beta + (g_{i\beta}+\omega_i g_{0\beta})(\chi_{ab}-\delta_{ab}r^{-1})e_b^\beta + \mathfrak{R}_{-2,0}\\
&= r^{-1}(-e_a^i + e_a^i(g_{0\beta}-m_{0\beta})e_4^\beta + e_a^i + ((g_{i\beta}-m_{i\beta}) + \omega_i(g_{0\beta}-m_{0\beta}))e_a^\beta) + \mathfrak{R}_{-2,0} = \mathfrak{R}_{-2,0},
\end{aligned}$$

and

$$\begin{aligned}
e_4((g_{i\beta}+\omega_i g_{0\beta})e_4^\beta) &= e_4(g_{i\beta}+\omega_i g_{0\beta})e_4^\beta + (g_{i\beta}+\omega_i g_{0\beta})e_4(e_4^\beta)\\
&= (e_4(g_{i\beta}) + \omega_i e_4(g_{0\beta}) + e_4(\omega_i)g_{0\beta})e_4^\beta + \varepsilon\mathfrak{R}_{-2,0}\\
&= \mathfrak{R}_{-2,0} + \varepsilon\mathfrak{R}_{-2,0} = \mathfrak{R}_{-2,0}.
\end{aligned}$$

Since $(g_{i\beta}+\omega_i g_{0\beta})e_4^\beta = \mathfrak{R}_{-1,0}$ and $e_k(e_3(q)) = \xi_{k3}^3 e_3(q) = \mathfrak{R}_{-1,-1}$, we conclude from the Leibniz's rule that for $k \neq 3$,

$$\begin{aligned}
e_k(q_i+\omega_i q_t) &= \frac{1}{2}e_k((g_{i\beta}+\omega_i g_{0\beta})e_4^\beta)e_3(q) + \frac{1}{2}(g_{i\beta}+\omega_i g_{0\beta})e_4^\beta e_k(e_3(q))\\
&= \mathfrak{R}_{-2,0}\cdot\mathfrak{R}_{0,0} + \mathfrak{R}_{-1,0}\cdot\mathfrak{R}_{-1,-1} = \mathfrak{R}_{-2,0}.
\end{aligned}$$

Besides,

$$u_i + \omega_i u_t = r^{-1}\sum_j \omega_j\Omega_{ji}u + r^{-1}\omega_i Su + r^{-1}\omega_i(t+r)^{-1}(tSu - \sum_j x_j\Omega_{0j}u) = \mathfrak{R}_{-1,0}\cdot Zu.$$

Note that $Zu = \varepsilon\mathfrak{R}_{-1,0}$ and $e_k = \mathfrak{R}_{-1,0}\cdot V$ for $k \neq 3$. We conclude that

$$\begin{aligned}
e_k(u_i+\omega_i u_t) &= e_k(\mathfrak{R}_{-1,0})\cdot Zu + \mathfrak{R}_{-1,0}\cdot e_k(Zu)\\
&= \mathfrak{R}_{-1,0}\cdot V_k(\mathfrak{R}_{-1,0})\cdot\varepsilon\mathfrak{R}_{-1,0} + \mathfrak{R}_{-1,0}\cdot\mathfrak{R}_{-1,0}\cdot V_k(\varepsilon\mathfrak{R}_{-1,0}) = \varepsilon\mathfrak{R}_{-3,0}.
\end{aligned}$$

(d) This part follows directly from

$$\partial_t + \partial_r = \sum \omega_i(\partial_i + \omega_i\partial_t), \qquad \partial_i - \omega_i\partial_r = \partial_i + \omega_i\partial_t - \sum \omega_i\omega_j(\partial_j + \omega_j\partial_t).$$

□

Proposition 4.46. *We have* $e_4(e_3(u)) + r^{-1}e_3(u) = \varepsilon\mathfrak{R}_{-3,0}$ *and* $e_4(e_3(ru)) = \varepsilon\mathfrak{R}_{-2,0}$.

Proof. Note that

$$\begin{aligned}
g^{\alpha\beta}(u)\partial_\alpha\partial_\beta u &= \sum_a e_a^\alpha e_a^\beta\partial_\alpha\partial_\beta u + \frac{1}{2}e_4^\alpha e_3^\beta\partial_\alpha\partial_\beta u + \frac{1}{2}e_3^\alpha e_4^\beta\partial_\alpha\partial_\beta u\\
&= \sum_a (e_a(e_a(u)) - e_a(e_a^\alpha)\partial_\alpha u) + e_4(e_3(u)) - e_4(e_3^\alpha)\partial_\alpha u.
\end{aligned}$$

Here we have

$$\begin{aligned}
&e_a(e_a^\alpha)\partial_\alpha u\\
&= -\xi_{aa'}^a e_{a'}(u) - \frac{1}{2}\chi_{aa}(e_3(u)+e_4(u)) - \langle e_a, e_a(g^{0\beta})\partial_\beta + g^{0\beta}e_a^\alpha\Gamma_{\alpha\beta}^\nu\partial_\nu\rangle e_4(u) - e_a^\mu e_a^\nu\Gamma_{\mu\nu}^\alpha u_\alpha\\
&= -\xi_{aa'}^a e_{a'}(u) - \frac{1}{2}\chi_{aa}(e_3(u)+e_4(u)) - (e_a^\alpha g_{\alpha\beta}e_a(g^{0\beta}) + e_a^\mu g_{\mu\nu}g^{0\beta}e_a^\alpha\Gamma_{\alpha\beta}^\nu)e_4(u) - e_a^\mu e_a^\nu\Gamma_{\mu\nu}^\alpha u_\alpha\\
&= -\frac{1}{2}\chi_{aa}e_3(u) - e_a^\mu e_a^\nu\Gamma_{\mu\nu}^\alpha u_\alpha + \varepsilon\mathfrak{R}_{-3,0}
\end{aligned}$$

and

$$e_4(e_3^\alpha)\partial_\alpha u = \varepsilon\mathfrak{R}_{-2,0}\cdot\varepsilon\mathfrak{R}_{-1,-1} = \varepsilon^2\mathfrak{R}_{-3,-1}.$$

In addition, for $k, l \neq 3$, we have

$$\begin{aligned}
e_k^\mu e_l^\nu\Gamma_{\mu\nu}^\alpha u_\alpha &= \frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\alpha} - \partial_\beta g_{\mu\nu})e_k^\mu e_l^\nu u_\alpha\\
&= \frac{1}{2}g^{\alpha\beta}e_k(g_{\nu\beta})e_l^\nu u_\alpha + \frac{1}{2}g^{\alpha\beta}e_l(g_{\mu\alpha})e_k^\mu u_\alpha - \frac{1}{2}g^{\alpha\beta}\partial_\beta g_{\mu\nu}e_k^\mu e_l^\nu u_\alpha\\
&= \varepsilon^2\mathfrak{R}_{-3,-1} - \frac{1}{2}\sum_c e_c(g_{\mu\nu})e_c(u)e_k^\mu e_l^\nu - \frac{1}{4}e_3(g_{\mu\nu})e_4(u)e_k^\mu e_l^\nu - \frac{1}{4}e_4(g_{\mu\nu})e_3(u)e_k^\mu e_l^\nu\\
&= \varepsilon^2\mathfrak{R}_{-3,-1}.
\end{aligned}$$

Since $\chi_{ab} - \delta_{ab}r^{-1} = \mathfrak{R}_{-2,0}$ and $e_3(u) = (3R-r+t)^{-1}V_3(u) = \varepsilon\mathfrak{R}_{-1,-1}$, their product is $\varepsilon\mathfrak{R}_{-3,-1}$. Thus we have

$$\begin{aligned}
0 &= \sum_a e_a(e_a(u)) + e_4(e_3(u)) + \frac{1}{2}\mathrm{tr}\chi e_3(u) + \varepsilon\mathfrak{R}_{-3,0}\\
&= \sum_a e_a(e_a(u)) + e_4(e_3(u)) + r^{-1}e_3(u) + \varepsilon\mathfrak{R}_{-3,0}.
\end{aligned}$$

Next, as in Lemma 4.20, we set

$$h_i := r(\partial_i(ru) - q_iq_r^{-1}\partial_r(ru)) = -r(u+ru_r)q_r^{-1}(q_i - \omega_iq_r) + r^2(u_i - \omega_iu_r).$$

Recall from Lemma 4.20 that

$$e_a(ru) = \sum_i e_a(\omega_i)h_i.$$

We claim that $h_i = \varepsilon\mathfrak{R}_{0,0}$ and $e_a(h_i) = \varepsilon\mathfrak{R}_{-1,0}$. In fact, note that $u + ru_r = \varepsilon\mathfrak{R}_{-1,0} + \mathfrak{R}_{1,0}\cdot\varepsilon\mathfrak{R}_{-1,-1} = \varepsilon\mathfrak{R}_{0,-1}$. We also recall that $e_a(r) = \mathfrak{R}_{-1,0}$, so $e_a(r^{-1}) = -r^{-2}e_a(r) = \mathfrak{R}_{-3,0}$. Thus by Lemma 4.45, we have $h_i = \varepsilon\mathfrak{R}_{0,0}$ and $e_a(h_i) = \varepsilon\mathfrak{R}_{-1,0}$. We thus have

$$\begin{aligned}
&e_a(e_a(u))\\
&= e_a(r^{-1}e_a(ru) + e_a(r^{-1})ru)\\
&= r^{-1}e_a(e_a(ru)) + 2e_a(r^{-1})e_a(ru) + e_a(e_a(r^{-1}))ru\\
&= r^{-1}e_a(e_a(ru)) + \mathfrak{R}_{-3,0}\cdot r^{-1}V_a(\varepsilon\mathfrak{R}_{0,0}) + V_a(\mathfrak{R}_{-3,0})\cdot\varepsilon\mathfrak{R}_{-1,0}\\
&= r^{-1}\sum_i e_a(e_a(\omega_i))h_i + r^{-1}\sum_i e_a(\omega_i)e_a(h_i) + \varepsilon\mathfrak{R}_{-4,0}\\
&= r^{-1}\sum_i e_a(r^{-1}(e_a^i - \omega_i\omega_j e_a^j))h_i + \mathfrak{R}_{-1,0}\cdot r^{-1}V_a(\mathfrak{R}_{0,0})\cdot\varepsilon\mathfrak{R}_{-1,0} + \varepsilon\mathfrak{R}_{-4,0}\\
&= r^{-2}\sum_i e_a(e_a^i - \omega_i\omega_j e_a^j)h_i + r^{-1}\sum_i e_a(r^{-1})(e_a^i - \omega_i\omega_j e_a^j)h_i + \varepsilon\mathfrak{R}_{-3,0}\\
&= r^{-2}e_a(\mathfrak{R}_{0,0})\cdot\varepsilon\mathfrak{R}_{0,0} + r^{-1}\mathfrak{R}_{-3,0}\cdot\varepsilon\mathfrak{R}_{0,0} + \varepsilon\mathfrak{R}_{-3,0} = \varepsilon\mathfrak{R}_{-3,0}.
\end{aligned}$$

Thus,

$$0 = e_4(e_3(u)) + r^{-1}e_3(u) + \varepsilon\mathfrak{R}_{-3,0}.$$

Finally, we have

$$\begin{aligned}
e_4(e_3(ru)) &= e_4(re_3(u)) + e_4(e_3(r)u) = re_4(e_3(u)) + e_4(r)e_3(u) + e_3(r)e_4(u) + e_4(e_3(r))u\\
&= -e_3(u) + e_4(r)e_3(u) + e_4(e_3^i\omega_i)u + \varepsilon r\mathfrak{R}_{-3,0} + \varepsilon\mathfrak{R}_{-2,0}\\
&= (e_4(r)-1)e_3(u) + t^{-1}V_4(1 + (e_3^i - \omega_i)\omega_i)u + \varepsilon\mathfrak{R}_{-2,0}\\
&= \mathfrak{R}_{-1,0}\cdot\varepsilon\mathfrak{R}_{-1,-1} + \mathfrak{R}_{-1,0}\cdot V_4(\mathfrak{R}_{-1,0})\cdot\varepsilon\mathfrak{R}_{-1,0} + \varepsilon\mathfrak{R}_{-2,0} = \varepsilon\mathfrak{R}_{-2,0}.
\end{aligned}$$

□

Next we prove an estimate for $e_3(q)$. We start with the following lemma.

Lemma 4.47. *Fix a function $f \in C^\infty(\mathbb{R})$. Then, for $\varepsilon \ll 1$, $f(u) - f(0) - f'(0)u = \varepsilon^2\mathfrak{R}_{-2,0}$ where u is a solution to* (1.1).

Proof. For $\varepsilon \ll 1$, we have $f(u) - f(0) - f'(0)u = O(|u|^2) = O(\varepsilon^2 t^{-2+C\varepsilon})$. Now, for each I with $|I| > 0$, we can write $V^I(f(u)) - f'(u)(V^I u)$ as a linear combination of terms of the form

$$f^{(s)}(u)V^{I_1}u\cdots V^{I_s}u, \qquad \sum |I_*| = |I|,\ s \geq 2,\ |I_*| > 0.$$

Since $u = \varepsilon\mathfrak{R}_{-1,0}$, we can prove that each of these terms are $O((\varepsilon t^{-1+C\varepsilon})^s) = O(\varepsilon^2 t^{-2+C\varepsilon})$. Finally, note that $f'(u)V^I u - f'(0)V^I u = O(|u|\cdot|V^I u|) = O(\varepsilon^2 t^{-1+C\varepsilon})$. This finishes the proof. □

Our main result is as follows.

Proposition 4.48. *In* $\Omega \cap \{r - t < 2R\}$, *we have*

$$e_4(e_3(q)) = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}.$$

Proof. We recall that

$$e_4(e_3(q)) = -\Gamma^0_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q) = -\frac{1}{2}g^{0\nu}(e_4^\beta e_4(g_{\nu\beta}) + e_4^\alpha e_4(g_{\nu\alpha}))e_3(q) + \frac{1}{2}g^{0\nu}\partial_\nu g_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q).$$

Here $e_3(q) = (3R - r + t)^{-1}V_3(q) = \mathfrak{R}_{0,0}$ and $e_4(g) = t^{-1}V_4(g) = \varepsilon\mathfrak{R}_{-2,0}$. Thus,

$$\begin{aligned} e_4(e_3(q)) &= \frac{1}{2}g^{0\nu}\partial_\nu g_{\alpha\beta}e_4^\alpha e_4^\beta e_3(q) + \varepsilon\mathfrak{R}_{-2,0} = \frac{1}{4}(e_3 - e_4)(g_{\alpha\beta})e_4^\alpha e_4^\beta e_3(q) + \varepsilon\mathfrak{R}_{-2,0} \\ &= \frac{1}{4}e_3(g_{\alpha\beta})e_4^\alpha e_4^\beta e_3(q) + \varepsilon\mathfrak{R}_{-2,0}. \end{aligned}$$

Recall that the coefficients $(g^{\alpha\beta}(v))$ in (1.1) are known smooth functions, and that for all $|v| \ll 1$ the matrix $(g^{\alpha\beta}(v))$ has a smooth inverse $(g_{\alpha\beta}(v))$. We differentiate $g^{\alpha\sigma}(v)g_{\sigma\beta}(v) = \delta_{\alpha\beta}$ with respect to v and then set $v = 0$. Thus,

$$\frac{d}{dv}g^{\alpha\sigma}|_{v=0} \cdot m_{\sigma\beta} + m^{\alpha\sigma} \cdot \frac{d}{dv}g_{\sigma\beta}|_{v=0} = 0.$$

By setting $g^0_{\alpha\beta} = \frac{d}{dv}g_{\alpha\beta}|_{v=0}$ and $g_0^{\alpha\beta} = \frac{d}{dv}g^{\alpha\beta}|_{v=0}$, we conclude that

$$g^0_{\alpha\beta} = -m_{\alpha\alpha}m_{\beta\beta}g_0^{\alpha\beta}.$$

Here we do not take sum over α, β. Thus we have

$$\begin{aligned} g^0_{\alpha\beta}e_4^\alpha e_4^\beta &= -g^{00}e_4^0 e_4^0 + 2g_0^{0i}e_4^0 e_4^i - g_0^{ij}e_4^i e_4^j \\ &= -G(\omega) + 2g_0^{0i}(e_4^i - \omega_i) - g_0^{ij}e_4^i(e_4^j - \omega_j) - g_0^{ij}(e_4^i - \omega_i)\omega_j = -G(\omega) + \mathfrak{R}_{-1,0}. \end{aligned}$$

By the previous lemma we have

$$e_4(e_3(q)) = \frac{1}{4}e_3(g^0_{\alpha\beta}u)e_4^\alpha e_4^\beta e_3(q) + \varepsilon\mathfrak{R}_{-2,0} = -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}.$$

□

4.5 The asymptotic equations and the scattering data

In Section 4.3, we have constructed a global optical function $q(t,x)$ in Ω such that $-q_t, q_r \geq C^{-1}t^{-C\varepsilon} > 0$. By setting

$$\Omega' := \{(s,q,\omega) : \ s > 0, \ q > (\exp(\delta/\varepsilon) - \exp((s+\delta)/\varepsilon))/2 + 2R, \ \omega \in \mathbb{S}^2\},$$

we have an invertible map from Ω to Ω', defined by

$$\Phi(t,r,\omega)=(s,q,\omega):=(\varepsilon\ln(t)-\delta,q(t,r\omega),\omega).$$

In fact, we have $t=\exp((s+\delta)/\varepsilon)$ and the map $r\mapsto q(t,r\omega)$ is strictly increasing for each fixed (t,ω). Thus, Φ is injective. Since $q=r-t$ when $r\geq t+2R$, we have $\lim_{r\to\infty}q(t,r\omega)=\infty$. Thus, Φ is surjective. This gives us a new coordinate (s,q,ω) on Ω.

In addition, Φ is smooth since q is a smooth function. Its inverse Φ^{-1} is also smooth, since we have $q_r>0$. So, any smooth function $F(t,x)$ induces a smooth function $F\circ\Phi^{-1}$. With an abuse of notation, we still write $F\circ\Phi^{-1}(s,q,\omega)$ as $F(s,q,\omega)$.

We define

$$(\mu,U)(t,x)=(q_t-q_r,\varepsilon^{-1}ru)(t,x),\quad (t,x)\in\Omega.$$

Since q and u are both smooth, $\mu(t,x)$ and $U(t,x)$ are smooth. As discussed above, we also obtain two smooth functions $\mu(s,q,\omega)$ and $U(s,q,\omega)$ in Ω'. Our goal in this section is to derive a system of asymptotic equations for (μ,U) in the coordinate set (s,q,ω). Our main result is the following proposition.

Proposition 4.49. *Let $(\mu,U)(s,q,\omega)$ be defined as above. Then, by writing $t=\exp(\varepsilon^{-1}(s+\delta))$ we have*

$$\begin{cases}\partial_s\mu=\dfrac14 G(\omega)\mu^2U_q+\varepsilon^{-1}\mathfrak{R}_{-1,0},\\[2mm] \partial_sU_q=-\dfrac14 G(\omega)\mu U_q^2+\varepsilon^{-1}\mathfrak{R}_{-1,0}.\end{cases}$$

In addition, the following three limits exist for all $(q,\omega)\in\mathbb{R}\times\mathbb{S}^2$:

$$\begin{cases}A(q,\omega):=-\dfrac12\lim\limits_{s\to\infty}(\mu U_q)(s,q,\omega),\\ A_1(q,\omega):=\lim\limits_{s\to\infty}\exp(\dfrac12 G(\omega)A(q,\omega)s)\mu(s,q,\omega),\\ A_2(q,\omega):=\lim\limits_{s\to\infty}\exp(-\dfrac12 G(\omega)A(q,\omega)s)U_q(s,q,\omega).\end{cases}$$

All of them are smooth functions of (q,ω) for $\varepsilon\ll1$. By setting

$$\begin{cases}\widetilde{\mu}(s,q,\omega):=A_1\exp(-\dfrac12 GAs),\\[2mm] \widetilde{U}_q(s,q,\omega):=A_2\exp(\dfrac12 GAs).\end{cases}$$

we obtain an exact solution to our reduced system

$$\begin{cases}\widetilde{\mu}_s=\dfrac14 G(\omega)\widetilde{\mu}^2\widetilde{U}_q,\\[2mm] \widetilde{U}_{sq}=-\dfrac14 G(\omega)\widetilde{\mu}\widetilde{U}_q^2.\end{cases}$$

We also have the following estimates:

$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q+2A)=O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m\partial_\omega^n A=O(\langle q\rangle^{-1+C\varepsilon});$$
$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\exp(\tfrac12 GAs)\mu-A_1)=O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m\partial_\omega^n A_1=O(\langle q\rangle^{C\varepsilon}),$$
$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\exp(-\tfrac12 GAs)U_q-A_2)=O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m\partial_\omega^n A_2=O(\langle q\rangle^{-1+C\varepsilon});$$
$$\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{\mu}-\mu,\widetilde{U}_q-U_q)=O(\varepsilon^{-p}t^{-1+C\varepsilon}), \qquad \partial_s^p\partial_\omega^n(\widetilde{U}-U)=O(\varepsilon^{-p}\langle q\rangle t^{-1+C\varepsilon}).$$

Remark 4.49.1. Here A is called the *scattering data*.

After some preliminary computations in the new coordinate set (s,q,ω) in Section 4.5.1, we derive the asymptotic equations for μ and U in Section 4.5.2 and Section 4.5.3, respectively. Next, in Section 4.5.4, we make use of the asymptotic equations to construct our scattering data. The main propositions in this subsection are Proposition 4.53 and Proposition 4.55. Finally, in Section 4.5.5, we define an exact solution $(\widetilde{\mu},\widetilde{U})(s,q,\omega)$ to our reduced system and we show that it provides a good approximation of $(\mu,U)(s,q,\omega)$.

4.5.1 Derivatives under the new coordinate

For convenience, from now on we make the following convention. For a function $F=F(s,q,\omega)$ where $\omega\in\mathbb{S}^2$, we extend it to all $\omega\neq 0$ by setting $F(s,q,\lambda\omega)=F(s,q,\omega)$ for each $\lambda>0$. Under such a setting, it is easy to compute the angular derivatives of F since we can now define ∂_{ω_i}. To avoid ambiguity, we will only use ∂_{ω_i} in the coordinate (s,q,ω) and will never use it in the coordinate (t,r,ω).

First we explain how to compute the derivatives of U in (s,q,ω). Note by the chain rule, for any function $F=F(s,q,\omega)=F(t,r,\omega)$ we have

$$\begin{cases} F_t=\varepsilon t^{-1}F_s+q_tF_q \\ F_r=q_rF_q \end{cases} \Longrightarrow \begin{cases} F_s=\varepsilon^{-1}t(F_t-q_tq_r^{-1}F_r) \\ F_q=q_r^{-1}F_r \end{cases}.$$

In addition, by the homogeneity, we have $F(s,q,\omega)=F(s,q,\lambda\omega)$ and $\partial_{\omega_i}F(s,q,\omega)=\lambda\partial_{\omega_i}F(s,q,\lambda\omega)$ for each $\lambda>0$. At (t,x), we set $\lambda=|x|$ which gives

$$F_i=q_iF_q+r^{-1}F_{\omega_i}\Longrightarrow F_{\omega_i}=r(F_i-q_iq_r^{-1}F_r).$$

Now we can explain the meaning of the function h_i defined in Lemma 4.20; it is the derivative of ru with respect to ω_i under the coordinate (s,q,ω).

To simplify our future computations, we note that ∂_q, ∂_s and ∂_{ω_i} commute with each other. In fact,

$$\begin{aligned}
[\partial_q,\partial_{\omega_i}]&=[q_r^{-1}\partial_r, r\partial_i-rq_iq_r^{-1}\partial_r]\\
&=q_r^{-1}\partial_i-q_r^{-1}\partial_r(rq_iq_r^{-1})\partial_r-r\partial_i(q_r^{-1}\omega_j)\partial_j+rq_iq_r^{-1}\partial_r(q_r^{-1})\partial_r\\
&=q_r^{-1}\partial_i-q_r^{-2}\partial_r(rq_i)\partial_r-r\partial_i(q_r^{-1})\partial_r-q_r^{-1}(\partial_i-\omega_i\partial_r)\\
&=-q_r^{-2}(q_i+r\partial_rq_i)\partial_r+rq_r^{-2}(\partial_r(q_i)+r^{-1}(q_i-\omega_iq_r))\partial_r+q_r^{-1}\omega_i\partial_r\\
&=0,
\end{aligned}$$

$$
\begin{aligned}
[\partial_s, \partial_q] &= [\varepsilon^{-1}t\partial_t - \varepsilon^{-1}tq_tq_r^{-1}\partial_r, q_r^{-1}\partial_r] \\
&= \varepsilon^{-1}t\partial_t(q_r^{-1})\partial_r - \varepsilon^{-1}tq_tq_r^{-1}\partial_r(q_r^{-1})\partial_r + \varepsilon^{-1}tq_r^{-1}\partial_r(q_tq_r^{-1})\partial_r \\
&= \varepsilon^{-1}t\partial_t(q_r^{-1})\partial_r + \varepsilon^{-1}tq_r^{-2}q_{tr}\partial_r = 0, \\
[\partial_s, \partial_{\omega_i}] &= [\varepsilon^{-1}t\partial_t - \varepsilon^{-1}tq_tq_r^{-1}\partial_r, r\partial_i - rq_iq_r^{-1}\partial_r] \\
&= -\varepsilon^{-1}tr\partial_t(q_iq_r^{-1})\partial_r - \varepsilon^{-1}tq_tq_r^{-1}(\partial_i - \partial_r(rq_iq_r^{-1})\partial_r) \\
&\quad + \varepsilon^{-1}tr\partial_i(q_tq_r^{-1}\omega_j)\partial_j - \varepsilon^{-1}trq_iq_r^{-1}\partial_r(q_tq_r^{-1})\partial_r \\
&= -\varepsilon^{-1}trq_{it}q_r^{-1}\partial_r - \varepsilon^{-1}tq_tq_r^{-1}\partial_i + \varepsilon^{-1}tq_tq_r^{-2}q_i\partial_r + \varepsilon^{-1}trq_tq_r^{-2}\partial_r(q_i)\partial_r \\
&\quad + \varepsilon^{-1}trq_{ti}q_r^{-1}\partial_r - \varepsilon^{-1}trq_tq_r^{-2}\partial_i(q_r)\partial_r + \varepsilon^{-1}tq_tq_r^{-1}(\partial_i - \omega_i\partial_r) \\
&= \varepsilon^{-1}tq_tq_r^{-2}q_i\partial_r - \varepsilon^{-1}tq_tq_r^{-2}(q_i - \omega_iq_r)\partial_r - \varepsilon^{-1}tq_tq_r^{-1}\omega_i\partial_r = 0.
\end{aligned}
$$

Moreover, we can express $(\partial_s, \partial_q, \partial_{\omega_i})$ in terms of the weighted null frame $\{V_k\}$.

Lemma 4.50. *We have*

$$
\begin{aligned}
\partial_s &= \sum_a \varepsilon^{-1}\mathfrak{R}_{-1,0}V_a + (\varepsilon^{-1} + \mathfrak{R}_{-1,0})V_4, \\
\partial_{\omega_i} &= \sum_{k\neq 3}\mathfrak{R}_{-1,0}V_k + \sum_a e_a^iV_a = \sum_{k\neq 3}\mathfrak{R}_{0,0}V_k, \\
\partial_q &= \sum_k \mathfrak{R}_{0,-1}V_k.
\end{aligned}
$$

Proof. We can express $\partial_s, \partial_{\omega_i}$ in terms of the null frame:

$$
\begin{aligned}
\partial_s &= \varepsilon^{-1}t(g_{0\beta}e_a^\beta e_a + \frac{1}{2}g_{0\beta}e_4^\beta e_3 + \frac{1}{2}g_{0\beta}e_3^\beta e_4) - \varepsilon^{-1}tq_tq_r^{-1}(\omega_ig_{i\beta}e_a^\beta e_a + \frac{1}{2}\omega_ig_{i\beta}e_4^\beta e_3 + \frac{1}{2}\omega_ig_{i\beta}e_3^\beta e_4) \\
&= \varepsilon^{-1}t((g_{0\beta} - q_tq_r^{-1}\omega_ig_{i\beta})e_a^\beta e_a + \frac{1}{2}(g_{0\beta} - q_tq_r^{-1}\omega_ig_{i\beta})e_3^\beta e_4), \\
\partial_{\omega_i} &= r(g_{i\beta}e_a^\beta e_a + \frac{1}{2}g_{i\beta}e_4^\beta e_3 + \frac{1}{2}g_{i\beta}e_3^\beta e_4) - rq_iq_r^{-1}(\omega_jg_{j\beta}e_a^\beta e_a + \frac{1}{2}\omega_jg_{j\beta}e_4^\beta e_3 + \frac{1}{2}\omega_jg_{j\beta}e_3^\beta e_4) \\
&= r((g_{i\beta} - q_iq_r^{-1}\omega_jg_{j\beta})e_a^\beta e_a + \frac{1}{2}(g_{i\beta} - q_iq_r^{-1}\omega_jg_{j\beta})e_3^\beta e_4).
\end{aligned}
$$

We note that there is no term with e_3 in ∂_s and ∂_{ω_i}, since

$$
\begin{aligned}
(g_{0\beta} - q_tq_r^{-1}\omega_ig_{i\beta})e_4^\beta &= q_r^{-1}(q_rg_{0\beta} - q_t\omega_ig_{i\beta})e_4^\beta = \frac{1}{2}q_r^{-1}e_3(q)(\omega_ig_{i\nu}e_4^\nu g_{0\beta}e_4^\beta - g_{0\nu}e_4^\nu\omega_ig_{i\beta}e_4^\beta) = 0, \\
(g_{i\beta} - q_iq_r^{-1}\omega_jg_{j\beta})e_4^\beta &= q_r^{-1}(q_rg_{i\beta} - q_i\omega_jg_{j\beta})e_4^\beta = \frac{1}{2}q_r^{-1}e_3(q)(\omega_jg_{j\nu}e_4^\nu g_{i\beta}e_4^\beta - g_{i\nu}e_4^\nu\omega_jg_{j\beta}e_4^\beta) = 0.
\end{aligned}
$$

In these computations we use the equality $q_\alpha = \frac{1}{2}g_{\alpha\beta}e_4^\beta e_3(q)$. In addition, we have

$$
\begin{aligned}
\varepsilon^{-1}t(g_{0\beta} - q_tq_r^{-1}\omega_ig_{i\beta})e_a^\beta &= \varepsilon^{-1}t((g_{0j} - m_{0j}) - q_tq_r^{-1}\omega_i(g_{ij} - m_{ij}))e_a^j - \varepsilon^{-1}tq_tq_r^{-1}e_a(r) \\
&= \mathfrak{R}_{0,0} + \varepsilon^{-1}\mathfrak{R}_{0,0} = \varepsilon^{-1}\mathfrak{R}_{0,0}, \\
r(g_{i\beta} - q_iq_r^{-1}\omega_jg_{j\beta})e_a^\beta &= r((g_{ij'} - m_{ij'}) - q_iq_r^{-1}\omega_j(g_{jj'} - m_{jj'}))e_a^{j'} + r(e_a^i - q_iq_r^{-1}e_a(r)) \\
&= \mathfrak{R}_{0,0} + re_a^i.
\end{aligned}
$$

Besides, since $e_3^i\omega_i = 2g^{0i}\omega_i + e_4^i\omega_i = 1 + \varepsilon\mathfrak{R}_{-1,0}$, we have

$$\begin{aligned}\varepsilon^{-1}t(g_{0\beta} - q_t q_r^{-1}\omega_i g_{i\beta})e_3^\beta &= \varepsilon^{-1}t((g_{0\beta} - m_{0\beta}) - q_t q_r^{-1}\omega_i(g_{i\beta} - m_{i\beta}))e_3^\beta + \varepsilon^{-1}t(1 - q_t q_r^{-1}e_3^i\omega_i)\\ &= \mathfrak{R}_{0,0} + \varepsilon^{-1}tq_r^{-1}(2q_r - (q_t + q_r) - q_t(e_3^i\omega_i - 1)) = \mathfrak{R}_{0,0} + 2\varepsilon^{-1}t,\\ r(g_{i\beta} - q_i q_r^{-1}\omega_j g_{j\beta})e_3^\beta &= r((g_{i\beta} - m_{i\beta}) - q_i q_r^{-1}\omega_j(g_{j\beta} - m_{j\beta}))e_3^\beta + r(e_3^i - q_i q_r^{-1}\omega_j e_3^j)\\ &= \varepsilon\mathfrak{R}_{0,0} + rq_r^{-1}((e_3^i - \omega_i)q_r - (q_i - \omega_i q_r) - q_i(e_3^j\omega_j - 1)) = \mathfrak{R}_{0,0}.\end{aligned}$$

Thus,

$$\begin{aligned}\partial_s &= \sum_a \varepsilon^{-1}\mathfrak{R}_{0,0}e_a + (\varepsilon^{-1}t + \mathfrak{R}_{0,0})e_4 = \sum_a \varepsilon^{-1}\mathfrak{R}_{-1,0}V_a + (\varepsilon^{-1} + \mathfrak{R}_{-1,0})V_4,\\ \partial_{\omega_i} &= \sum_{k\neq 3}\mathfrak{R}_{0,0}e_k + \sum_a re_a^i e_a = \sum_{k\neq 3}\mathfrak{R}_{-1,0}V_k + \sum_a e_a^i V_a = \sum_{k\neq 3}\mathfrak{R}_{0,0}V_k.\end{aligned}$$

It is also clear that

$$\partial_q = \sum_k \mathfrak{R}_{0,0}e_k = \sum_k \mathfrak{R}_{0,-1}V_k.$$

□

We end this subsection with the following estimates for U.

Lemma 4.51. *We have*

$$(U, U_q, U_s, U_{\omega_i}) = (\mathfrak{R}_{0,0}, \mathfrak{R}_{0,-1}, \varepsilon^{-1}\mathfrak{R}_{0,0}, \mathfrak{R}_{0,0}).$$

In conclusion, we have $\mu U_q = \mathfrak{R}_{0,-1}$.

Proof. We have

$$\begin{aligned}U &= \varepsilon^{-1}ru,\\ U_q &= q_r^{-1}\partial_r(\varepsilon^{-1}ru) = \varepsilon^{-1}q_r^{-1}(u + ru_r),\\ U_s &= \varepsilon^{-2}tr(u_t + u_r - q_r^{-1}(q_t + q_r)u_r) - \varepsilon^{-2}tq_t q_r^{-1}u,\\ U_{\omega_i} &= -\varepsilon^{-1}r(q_i - \omega_i q_r)q_r^{-1}(u + ru_r) + \varepsilon^{-1}r^2(u_i - \omega_i u_r).\end{aligned}$$

It follows directly from Lemma 4.34, Lemma 4.45 and the proof of Proposition 4.46 that $(U, U_q, U_s, U_{\omega_i}) = (\mathfrak{R}_{0,0}, \mathfrak{R}_{0,-1}, \varepsilon^{-1}\mathfrak{R}_{0,0}, \mathfrak{R}_{0,0})$. Since $\mu = \mathfrak{R}_{0,0}$, we have $\mu U_q = \mathfrak{R}_{0,-1}$. □

4.5.2 The asymptotic equation for μ

We start with several estimates for $\mu = q_t - q_r$. By Proposition 4.48, we have

$$\begin{aligned}e_4(e_3(q)) &= -\frac{1}{4}e_3(u)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}\\ &= -\frac{1}{4}(\varepsilon r^{-1}e_3(U) - \varepsilon r^{-2}e_3(r)U)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}\\ &= -\frac{\varepsilon}{4r}e_3(U)G(\omega)e_3(q) + \varepsilon\mathfrak{R}_{-2,0}.\end{aligned}$$

Since $e_3^i - \omega_i = \mathfrak{R}_{-1,0}$, we have

$$e_3(q) = -\mu + \mathfrak{R}_{-1,0} \cdot \partial q = -\mu + \mathfrak{R}_{-1,0}.$$

Moreover,

$$\begin{aligned} e_4(e_3(q)+\mu) &= e_4((e_3^i - \omega_i)q_i) = e_4(e_3^i - \omega_i)q_i + (e_3^i - \omega_i)e_4(q_i) \\ &= -(e_4^j - \omega_j)r^{-1}(\delta_{ij} - \omega_i\omega_j)q_i + (e_3^i - \omega_i)e_4(\frac{1}{2}g_{i\beta}e_4^\beta e_3(q)) + \varepsilon\mathfrak{R}_{-2,0} \\ &= -r^{-1}(-q_t - q_r - q_r(e_4(r)-1)) + \frac{1}{2}g_{i\beta}(e_3^i - \omega_i)e_4^\beta e_4(e_3(q)) + \varepsilon\mathfrak{R}_{-2,0} = \varepsilon\mathfrak{R}_{-2,0}. \end{aligned}$$

To get the last equality, we use the following estimates: $e_4(r) - 1 = \varepsilon\mathfrak{R}_{-1,0}$ by Lemma 4.35, $e_4(e_3(q)) = \xi_{43}^3 e_3(q) = \varepsilon\mathfrak{R}_{-1,-1}$, and

$$q_t + q_r = \frac{1}{2}(g_{0\beta} + \omega_i g_{i\beta})e_4^\beta e_3(q) = \frac{1}{2}(-1 + e_4^i\omega_i)e_3(q) + (g_{**} - m_{**}) \cdot \mathfrak{R}_{0,0} = \varepsilon\mathfrak{R}_{-1,0}.$$

Besides, by the chain rule, we have

$$e_3(U) = e_3(q)U_q - \varepsilon t^{-1}U_s + \sum_i e_3(\omega_i)U_{\omega_i} = -\mu U_q + \mathfrak{R}_{-1,0}.$$

Here we apply Lemma 4.51 and we note that $e_3(\omega_i) = (e_3^j - \omega_j)r^{-1}(\delta_{ij} - \omega_i\omega_j) = \mathfrak{R}_{-2,0}$. Thus, we have

$$\begin{aligned} e_4(-\mu) + \varepsilon\mathfrak{R}_{-2,0} &= -\frac{\varepsilon}{4r}G(\omega)(-\mu U_q + \mathfrak{R}_{-1,0})(-\mu + \mathfrak{R}_{-1,0}) + \varepsilon\mathfrak{R}_{-2,0} \\ &= -\frac{\varepsilon}{4r}G(\omega)\mu^2 U_q + \varepsilon\mathfrak{R}_{-2,0}. \end{aligned}$$

Then,

$$e_4(\mu) = \frac{\varepsilon}{4r}G(\omega)\mu^2 U_q + \varepsilon\mathfrak{R}_{-2,0}. \tag{4.61}$$

By Lemma 4.50 we have

$$\begin{aligned} \mu_s &= \varepsilon^{-1}te_4(\mu) + \sum_{k\neq 3}\varepsilon^{-1}\mathfrak{R}_{-1,0}V_k(\mu) = \varepsilon^{-1}t(\frac{\varepsilon}{4r}G(\omega)\mu^2 U_q + \varepsilon\mathfrak{R}_{-2,0}) + \sum_{k\neq 3}\varepsilon^{-1}\mathfrak{R}_{-1,0}V_k(\mathfrak{R}_{0,0}) \\ &= \frac{t}{4r}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0} = \frac{1}{4}G(\omega)\mu^2 U_q + \frac{\varepsilon(t-r)}{4r}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0} \\ &= \frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon\mathfrak{R}_{-1,1} \cdot \mathfrak{R}_{0,0} \cdot \mathfrak{R}_{0,-1} + \varepsilon^{-1}\mathfrak{R}_{-1,0} = \frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{aligned}$$

We thus obtain the first asymptotic equation

$$\mu_s = \frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \tag{4.62}$$

4.5.3 The asymptotic equation for U

By Proposition 4.46, we have

$$e_4(e_3(U)) = \varepsilon^{-1}e_4(e_3(ru)) = \mathfrak{R}_{-2,0}.$$

Meanwhile, by Lemma 4.51 we have

$$\begin{aligned} e_4(e_3(U)) &= e_4(e_3(q)U_q + \varepsilon t^{-1}U_s + e_3(\omega_i)U_{\omega_i}) \\ &= -e_4(\mu U_q) + e_4((e_3^i - \omega_i)q_i U_q + \varepsilon t^{-1}U_s + (e_3^j - \omega_j)r^{-1}(\delta_{ij} - \omega_i\omega_j)U_{\omega_i}) \\ &= -e_4(\mu U_q) + \mathfrak{R}_{-1,0} \cdot V_4(\mathfrak{R}_{-1,-1} + \varepsilon t^{-1} \cdot \varepsilon^{-1}\mathfrak{R}_{0,0} + \mathfrak{R}_{-1,0} \cdot r^{-1} \cdot \mathfrak{R}_{0,0}) \\ &= -e_4(\mu U_q) + \mathfrak{R}_{-2,0}. \end{aligned}$$

Thus, $e_4(\mu U_q) = \mathfrak{R}_{-2,0}$.

Now, we compute $\partial_s(\mu U_q)$. By Lemma 4.50 we have

$$\begin{aligned} \partial_s(\mu U_q) &= \sum_a \varepsilon^{-1}\mathfrak{R}_{-1,0}V_a(\mu U_q) + (\varepsilon^{-1} + \mathfrak{R}_{-1,0})V_4(\mu U_q) \\ &= \sum_a \varepsilon^{-1}\mathfrak{R}_{-1,0}V_a(\mathfrak{R}_{0,-1}) + (\varepsilon^{-1} + \mathfrak{R}_{-1,0})\mathfrak{R}_{-1,0} = \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mu U_{sq} = \partial_s(\mu U_q) - \mu_s U_q &= \varepsilon^{-1}\mathfrak{R}_{-1,0} - (\frac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,-1} + \mathfrak{R}_{-1,0})U_q \\ &= -\frac{1}{4}G(\omega)\mu^2 U_q^2 + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{aligned}$$

Since $|\mu| > C^{-1}t^{-C\varepsilon}$, we have $\mu^{-1} = \mathfrak{R}_{0,0}$. Thus we obtain the second asymptotic equation

$$U_{sq} = -\frac{1}{4}G(\omega)\mu U_q^2 + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \tag{4.63}$$

In summary, by (4.62) and (4.63), we have proved the following proposition.

Proposition 4.52. *We have*

$$\begin{cases} \partial_s\mu = \dfrac{1}{4}G(\omega)\mu^2 U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}, \\ \partial_s U_q = -\dfrac{1}{4}G(\omega)\mu U_q^2 + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{cases} \tag{4.64}$$

In other words, $(\mu, U_q)(s, q, \omega)$ is an apporximate solution to the reduced system of ODE's

$$\begin{cases} \partial_s\widetilde{\mu} = \dfrac{1}{4}G(\omega)\widetilde{\mu}^2\widetilde{U}_q, \\ \partial_s\widetilde{U}_q = -\dfrac{1}{4}G(\omega)\widetilde{\mu}\widetilde{U}_q^2. \end{cases} \tag{4.65}$$

We remark that this proposition verifies the nonrigorous derivation in Section 3 of the author's previous paper [34].

4.5.4 The scattering data

From the previous subsections, we have proved that $(\mu, U_q)(s, q, \omega)$ is an approximate solution to the reduced system (4.65). In this subsection, we seek to construct an exact solution $(\widetilde{\mu}, \widetilde{U}_q)$ to (4.65) which is a good approximation of (μ, U_q).

We start with the following key proposition. In this proposition, we define the *scattering data* $A = A(q, \omega)$ for each $(q, \omega) \in \mathbb{R} \times \mathbb{S}^2$ and we show that it is a smooth function (in the sense defined in Section 4.2.1).

Proposition 4.53. *In Ω', we have*

$$(\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q) = O(\langle q\rangle^{-1} t^{C\varepsilon}), \qquad \partial_s^p (\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q) = O(\varepsilon^{-p} t^{-1+C\varepsilon}),\ p \geq 1.$$

Moreover, for each m, n, the limit

$$A_{m,n}(q, \omega) := -\frac{1}{2} \lim_{s\to\infty} (\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q)(s, q, \omega)$$

exists for all $(q, \omega) \in \mathbb{R}\times\mathbb{S}^2$, and the convergence is uniform in (q, ω). So $A(q, \omega) := A_{0,0}(q, \omega)$ is a smooth function of (q, ω) in $\mathbb{R} \times \mathbb{S}^2$ such that $(\langle q\rangle\partial_q)^m \partial_\omega^n A = A_{m,n}$. We call this function A the scattering data. *It is clear that $A \equiv 0$ for $q > R$.*

Finally, we have

$$(\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q + 2A) = O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m \partial_\omega^n A = O(\langle q\rangle^{-1+C\varepsilon}).$$

Proof. First we note that in the region $r - t > R$, we have $q = r - t$ and $u = 0$. In this case, every estimate in the statement of this proposition is equal to 0, so there is nothing to prove. Thus, we can assume that $q < 2R$ and $r - t < 2R$ in the rest of this proof.

We need to derive an estimate for $\partial_s \partial_q^m \partial_\omega^n (\mu U_q)$. Here we apply Lemma 4.50. Recall that $\mu U_q = \mathfrak{R}_{0,-1}$ and $V_4(\mu U_q) = \mathfrak{R}_{-1,0}$. By the Leibniz's rule, we have

$$(\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q) = (\sum_k \mathfrak{R}_{0,0} V_k)^{m+n} (\mathfrak{R}_{0,-1}) = O(\langle q\rangle^{-1+C\varepsilon} t^{C\varepsilon}) = O(\langle q\rangle^{-1} t^{C\varepsilon}).$$

In addition, for $p \geq 1$ we have

$$\begin{aligned}
\partial_s^p (\langle q\rangle\partial_q)^m \partial_\omega^n (\mu U_q) &= \partial_s^{p-1} (\langle q\rangle\partial_q)^m \partial_\omega^n \partial_s (\mu U_q) \\
&= \partial_s^{p-1} (\langle q\rangle\partial_q)^m \partial_\omega^n (\sum_{k\neq 3} \varepsilon^{-1} \mathfrak{R}_{-1,0} \cdot V_k(\mu U_q) + \varepsilon^{-1} V_4(\mu U_q)) \\
&= \partial_s^{p-1} (\langle q\rangle\partial_q)^m \partial_\omega^n (\sum_a \varepsilon^{-1} \mathfrak{R}_{-1,0} \cdot \mathfrak{R}_{0,-1} + \varepsilon^{-1} \mathfrak{R}_{-1,0}) \\
&= \varepsilon^{1-p} (\sum \mathfrak{R}_{0,0} V_k)^{p+m+n-1} (\varepsilon^{-1} \mathfrak{R}_{-1,0}) = O(\varepsilon^{-p} t^{-1+C\varepsilon}).
\end{aligned} \tag{4.66}$$

In both these estimates, we view t as a function of s.

For fixed $q < 2R$ and $\omega \in \mathbb{S}^2$, by the definition of Ω', we have $(s, q, \omega) \in \Omega'$ if and only if $s > 0$ and

$$\exp((s+\delta)/\varepsilon) > \exp(\delta/\varepsilon) - 2q + 4R. \tag{4.67}$$

We can write this condition as $s > s_{q,\delta,\varepsilon}$ where $s_{q,\delta,\varepsilon} \geq 0$ is a constant depending on its subscripts, such that $(s_{q,\delta,\varepsilon}, q, \omega) \in \partial\Omega'$ corresponds with a point on H. Thus, for each fixed (q, ω) and $s_2 > s_1 \geq s_{q,\delta,\varepsilon} = \exp(\delta/\varepsilon) - 2q + 4R$, by (4.66) with $p = 1$, we have

$$\begin{aligned} &|(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q)(s_2, q, \omega) - (\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q)(s_1, q, \omega)| \\ &\lesssim \int_{s_1}^{s_2} \varepsilon^{-1}\exp((-1+C\varepsilon)\varepsilon^{-1}(s+\delta))\, ds \lesssim \exp((-1+C\varepsilon)\varepsilon^{-1}(s_1+\delta)). \end{aligned}$$

In conclusion, $\{(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q)(s, q, \omega)\}_{s\geq s_{q,\delta,\varepsilon}}$ is uniformly Cauchy for each (q, ω). Thus, the limit

$$A_{m,n}(q, \omega) := -\frac{1}{2}\lim_{s\to\infty}(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q)(s, q, \omega)$$

exists, and the convergence is uniform in (q, ω). Besides, for each $s \geq s_{q,\delta,\varepsilon}$, we have

$$|(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q) + 2A_{m,n}| \lesssim t^{-1+C\varepsilon} = \exp((-1+C\varepsilon)\varepsilon^{-1}(s+\delta)). \tag{4.68}$$

By evaluating (4.68) at $(s_{q,\delta,\varepsilon}, q, \omega)$, we have

$$\begin{aligned} |A_{m,n}(q, \omega)| &\lesssim |(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q) + 2A_{m,n}| + |(\langle q\rangle\partial_q)^m\partial_\omega^n(\mu U_q)| \\ &\lesssim (\exp(\delta/\varepsilon) - 2q + 4R)^{-1+C\varepsilon} + \langle q\rangle^{-1}(\exp(\delta/\varepsilon) - 2q + 4R)^{C\varepsilon} \lesssim \langle q\rangle^{-1+C\varepsilon}. \end{aligned}$$

In the last inequality, we note that $(a+b)^{C\varepsilon} \leq 2^{C\varepsilon}\max\{a, b\}^{C\varepsilon} \leq 2(a^{C\varepsilon} + b^{C\varepsilon})$ for each pairs $a, b \geq 0$. Since the convergence is uniform in (q, ω), if we define $A := A_{0,0}$, then we have

$$(\langle q\rangle\partial_q)^m\partial_\omega^n A = A_{m,n} = O(\langle q\rangle^{-1+C\varepsilon}).$$

□

Note that each function of (s, q, ω) can be viewed as a function of (t, x). We then have the following lemma.

Lemma 4.54. *By viewing each function of (s, q, ω) as a function of $(t, x) \in \Omega \cap \{r - t < 2R\}$, we have $(A, \partial_\omega A) = \mathfrak{R}_{0,-1}$, $\mu U_q + 2A = \mathfrak{R}_{-1,0}$ and $\exp(\pm\frac{1}{2}G(\omega)As) - 1 = \mathfrak{R}_{0,-1}$.*

Proof. Note that $V^I A$ is a linear combination of terms of the form

$$\partial_q^m\partial_\omega^n A \cdot V^{I_1}q \cdots V^{I_m}q \cdot V^{J_1}\omega \cdots V^{J_n}\omega, \qquad \sum|I_*| + \sum|J_*| = |I|.$$

Each of these terms is $O(\langle q\rangle^{-1-m+C\varepsilon} \cdot \langle q\rangle^m t^{C\varepsilon}) = O(\langle q\rangle^{-1}t^{C\varepsilon})$, so $A = \mathfrak{R}_{0,-1}$. The proof of $\partial_\omega A = \mathfrak{R}_{0,-1}$ is essentially the same.

Moreover, $V^I(\mu U_q + 2A)$ is a linear combination of terms of the form

$$\partial_q^m \partial_\omega^n (\mu U_q + A) \cdot V^{I_1} q \cdots V^{I_m} q \cdot V^{J_1}\omega \cdots V^{J_n}\omega, \qquad \sum |I_*| + \sum |J_*| = |I|;$$

$$\partial_s^p \partial_q^m \partial_\omega^n (\mu U_q) \cdot V^{K_1} s \cdots V^{K_p} s \cdot V^{I_1} q \cdots V^{I_m} q \cdot \cdot V^{J_1}\omega \cdots V^{J_n}\omega, \quad \sum |I_*| + \sum |J_*| + \sum |K_*| = |I|,\ p > 0.$$

By applying (4.68) to the first row and (4.66) to the second row, we conclude that $V^I(\mu U_q + 2A) = O(t^{-1+C\varepsilon})$ and thus $\mu U_q + 2A = \mathfrak{R}_{-1,0}$.

Finally, by the chain rule, for each $|I| > 0$ we can write $V^I(\exp(\pm\frac{1}{2}G(\omega)As) - 1)$ as a linear combination of terms of the form

$$\exp(\pm\frac{1}{2}G(\omega)As) \cdot V^{I_1}(\pm\frac{1}{2}G(\omega)As) \cdots V^{I_m}(\pm\frac{1}{2}G(\omega)As), \qquad \sum |I_*| = |I|,\ |I_*| > 0.$$

The first term in this product is $O(t^{C\varepsilon})$, and each of the rest terms are $O(V^{I_*}(\mathfrak{R}_{0,-1})) = O(\langle q\rangle^{-1}t^{C\varepsilon})$, so we conclude that $V^I(\exp(\pm\frac{1}{2}G(\omega)As) - 1) = O(\langle q\rangle^{-1}t^{C\varepsilon})$ for $|I| > 0$. When $|I| = 0$, since $|e^\rho - 1| \lesssim |\rho| e^{|\rho|}$, we have

$$|\exp(\pm\frac{1}{2}G(\omega)As) - 1| \lesssim \langle q\rangle^{-1+C\varepsilon} s \exp(C\langle q\rangle^{-1+C\varepsilon} s) \lesssim \langle q\rangle^{-1} t^{C\varepsilon}.$$

Here we note that $s = \varepsilon \ln(t) - \delta = O(t^{C\varepsilon})$. In conclusion, $\exp(\pm\frac{1}{2}GAs) - 1 = \mathfrak{R}_{0,-1}$. □

By (4.64) and Lemma 4.54, we have

$$\begin{cases} \partial_s \mu = -\frac{1}{2}G(\omega)A(q,\omega)\mu + \varepsilon^{-1}\mathfrak{R}_{-1,0}, \\ \partial_s U_q = \frac{1}{2}G(\omega)A(q,\omega)U_q + \varepsilon^{-1}\mathfrak{R}_{-1,0}. \end{cases}$$

With the remainder terms omitted, we obtain two linear ODE's for μ and U_q. They motivate us to define

$$\begin{cases} \widetilde{V}_1 := \exp(\frac{1}{2}G(\omega)A(q,\omega)s)\mu, \\ \widetilde{V}_2 := \exp(-\frac{1}{2}G(\omega)A(q,\omega)s)U_q. \end{cases} \tag{4.69}$$

Now we can prove the following proposition.

Proposition 4.55. *We have*

$$(\langle q\rangle\partial_q)^m \partial_\omega^n \widetilde{V}_1 = O(t^{C\varepsilon}), \qquad \partial_s^p (\langle q\rangle\partial_q)^m \partial_\omega^n \widetilde{V}_1 = O(\varepsilon^{-p} t^{-1+C\varepsilon}),\ p \geq 1;$$

$$(\langle q\rangle\partial_q)^m \partial_\omega^n \widetilde{V}_2 = O(\langle q\rangle^{-1} t^{C\varepsilon}), \qquad \partial_s^p (\langle q\rangle\partial_q)^m \partial_\omega^n \widetilde{V}_2 = O(\varepsilon^{-p} t^{-1+C\varepsilon}),\ p \geq 1.$$

Moreover, for each m, n, the limit

$$A_{j,m,n}(q,\omega) := \lim_{s\to\infty} \widetilde{V}_j(s,q,\omega), \qquad j = 1,2$$

exists for all $(q,\omega) \in \mathbb{R}\times\mathbb{S}^2$, and the convergence is uniform in (q,ω). So, for $j = 1,2$, $A_j := A_{j,0,0}$ is smooth functions of (q,ω) in $\mathbb{R}\times\mathbb{S}^2$ such that $(\langle q\rangle\partial_q)^m\partial_\omega^n A_j = A_{j,m,n}$. It is clear that $A_1 \equiv -2$ and $A_2 \equiv 0$ for $q > R$. Besides, we have $A_1A_2 = -2A$ everywhere.

Finally, we have

$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{V}_1 - A_1) = O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m\partial_\omega^n A_1 = O(\langle q\rangle^{C\varepsilon}),$$

$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{V}_2 - A_2) = O(t^{-1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^m\partial_\omega^n A_2 = O(\langle q\rangle^{-1+C\varepsilon}).$$

Proof. By (4.61) and since $t/r = 1 + \mathfrak{R}_{-1,1}$, we have

$$V_4(\mu) = \frac{\varepsilon t}{4r}G(\omega)\mu^2U_q + \varepsilon\mathfrak{R}_{-1,0} = \frac{\varepsilon}{4}G(\omega)\mu^2U_q + \varepsilon\mathfrak{R}_{-1,0}.$$

Moreover, by viewing (s,q,ω) as functions of (t,x), we have

$$e_4(G(\omega)A(q,\omega)s) = \varepsilon G(\omega)At^{-1} + e_4(\omega_j)\partial_{\omega_j}(GA)s = \varepsilon G(\omega)At^{-1} + \mathfrak{R}_{-2,-1}.$$

Here we note that $\partial_{\omega_j}(GA) = \mathfrak{R}_{0,-1}$ by Lemma 4.54 and $e_4(\omega_i) = (e_4^j - \omega_j)\partial_j\omega_i = \mathfrak{R}_{-2,0}$. Then, by Lemma 4.54, we have $\widetilde{V}_1 = \mathfrak{R}_{0,0}\cdot\mathfrak{R}_{0,0} = \mathfrak{R}_{0,0}$ and

$$\begin{aligned}
V_4(\widetilde{V}_1) &= \frac{1}{2}V_4(GAs)\widetilde{V}_1 + \exp(\frac{1}{2}GAs)V_4(\mu)\\
&= \frac{1}{4}(2\varepsilon GA + \varepsilon G\mu U_q + \mathfrak{R}_{-1,-1})\widetilde{V}_1 + \varepsilon\mathfrak{R}_{-1,0}\cdot\exp(\frac{1}{2}GAs)\\
&= \frac{1}{4}(\varepsilon\mathfrak{R}_{-1,0} + \mathfrak{R}_{-1,-1})\cdot\mathfrak{R}_{0,0} + \varepsilon\mathfrak{R}_{-1,0}\cdot\mathfrak{R}_{0,0} = \varepsilon\mathfrak{R}_{-1,0} + \mathfrak{R}_{-1,-1} = \mathfrak{R}_{-1,0}.
\end{aligned}$$

Next, we have $\widetilde{V}_1\widetilde{V}_2 = \mu U_q$ and $\mu U_q = \mathfrak{R}_{0,-1}$, $V_4(\mu U_q) = \mathfrak{R}_{-1,0}$ from Proposition 4.53. Since $\mu = q_t - q_r \leq -2C^{-1}t^{-C\varepsilon}$ and $\exp(\frac{1}{2}GAs) \geq \exp(-Cs) = \exp(C\delta)t^{-C\varepsilon}$, we have $|\widetilde{V}_1| = -\widetilde{V}_1 \geq C^{-1}t^{-C\varepsilon}$. We can express $V^I(\widetilde{V}_2) = V^I((\mu U_q)/\widetilde{V}_1)$ as a linear combination of terms of the form

$$\widetilde{V}_1^{-m-1}\cdot V^{I_1}(\widetilde{V}_1)\cdots V^{I_m}(\widetilde{V}_1)\cdot V^{I_0}(\mu U_q), \qquad \sum|I_*| = |I|.$$

It is easy to conclude that $\widetilde{V}_2 = \mathfrak{R}_{0,-1}$ and $V_4(\widetilde{V}_2) = \mathfrak{R}_{-1,0}$.

Now we can follow the proof in Proposition 4.53 to prove every estimate involving A_2 in the statement. As for A_1, we note that

$$(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{V}_1) = (\sum_k \mathfrak{R}_{0,0}V_k)^{m+n}(\mathfrak{R}_{0,0}) = O(t^{C\varepsilon}).$$

In addition, for $p \geq 1$ we have

$$\begin{aligned}\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{V}_1) &= \partial_s^{p-1}(\langle q\rangle\partial_q)^m\partial_\omega^n\partial_s(\widetilde{V}_1)\\ &= \partial_s^{p-1}(\langle q\rangle\partial_q)^m\partial_\omega^n(\sum_{k\neq 3}\varepsilon^{-1}\mathfrak{R}_{-1,0}\cdot V_k(\widetilde{V}_1)+\varepsilon^{-1}V_4(\widetilde{V}_1))\\ &= \partial_s^{p-1}(\langle q\rangle\partial_q)^m\partial_\omega^n(\sum_a\varepsilon^{-1}\mathfrak{R}_{-1,0}\cdot\mathfrak{R}_{0,0}+\varepsilon^{-1}\mathfrak{R}_{-1,0})\\ &= \varepsilon^{1-p}(\sum\mathfrak{R}_{0,0}V_k)^{p+m+n-1}(\varepsilon^{-1}\mathfrak{R}_{-1,0}) = O(\varepsilon^{-p}t^{-1+C\varepsilon}).\end{aligned}$$

It is then clear that the estimates for $\widetilde{V}_1 - A_1$ are the same as those for $\mu U_q + 2A$. Finally, at $(s,q,\omega) = (s_{q,\delta,\varepsilon},q,\omega)$ we have

$$\begin{aligned}|(\langle q\rangle\partial_q)\partial_\omega^n A_1(q,\omega)| &\lesssim |(\langle q\rangle\partial_q)\partial_\omega^n(\widetilde{V}_1 - A_1)(s,q,\omega)| + |(\langle q\rangle\partial_q)\partial_\omega^n(\widetilde{V}_1)(s,q,\omega)|\\ &\lesssim (\exp(\delta/\varepsilon)-2q+4R)^{-1+C\varepsilon} + (\exp(\delta/\varepsilon)-2q+4R)^{C\varepsilon} \lesssim \langle q\rangle^{C\varepsilon}.\end{aligned}$$

In the last inequality, we note that $(a+b)^{C\varepsilon} \leq 2^{C\varepsilon}\max\{a,b\}^{C\varepsilon} \leq 2(a^{C\varepsilon}+b^{C\varepsilon})$ for each pairs $a,b \geq 0$. □

Remark 4.55.1. Following the proof of Lemma 4.54, we can show that $(A_1, \partial_\omega A_1) = \mathfrak{R}_{0,0}$, $\widetilde{V}_1 - A_1 = \mathfrak{R}_{-1,0}$, $(A_2, \partial_\omega A_2) = \mathfrak{R}_{0,-1}$ and $\widetilde{V}_2 - A_2 = \mathfrak{R}_{-1,0}$.

Moreover, we note that $A_1 \approx -2$ in the following sense.

Lemma 4.56. *Fix $0 < \kappa < 1$. For $\varepsilon \ll 1$ and for all $(q,\omega) \in \mathbb{R}\times\mathbb{S}^2$, we have $|A_1(q,\omega)+2| \leq \kappa\langle q\rangle^{-1+C\varepsilon}$. The constant in the power may depend on κ. As a result, we have $A_1(q,\omega) < -1 < 0$.*

Proof. Since $A_1 \equiv -2$ for $q > R$, we can assume $q < 2R$ in the proof. Recall from the proof of Proposition 4.55 that

$$e_4(\widetilde{V}_1) = \varepsilon\mathfrak{R}_{-2,0} + \mathfrak{R}_{-2,-1} = O(\varepsilon t^{-2+C\varepsilon} + t^{-2+C\varepsilon}\langle q\rangle^{-1}).$$

Next we consider $\widetilde{V}_1|_H$. On H we have $\mu = -2+O(|u|) = -2+O(\varepsilon t^{-1+C\varepsilon})$. As computed in Lemma 4.54, on H we have

$$\begin{aligned}|(\exp(\frac{1}{2}GAs)-1)\mu| &\lesssim \langle q\rangle^{-1+C\varepsilon}s\exp(C\langle q\rangle^{-1+C\varepsilon}s)\cdot(2+O(\varepsilon t^{-1+C\varepsilon}))\\ &\lesssim \langle q\rangle^{-1+C\varepsilon}s\exp(C\langle q\rangle^{-1+C\varepsilon}s).\end{aligned}$$

Thus, $\widetilde{V}_1|_H = -2+O(\varepsilon t^{-1+C\varepsilon} + \langle q\rangle^{-1+C\varepsilon}s\exp(C\langle q\rangle^{-1+C\varepsilon}s))$.

We integrate $e_4(\widetilde{V}_1)$ along the geodesic in $\mathcal{A}$ passing through $(t,x) \in \Omega \cap \{r-t<2R\}$. Then,

$$\begin{aligned}|\widetilde{V}_1(t,x)+2| &\lesssim \varepsilon(x^0(0))^{-1+C\varepsilon} + \langle q\rangle^{-1+C\varepsilon}(\varepsilon\ln x^0(0)-\delta)\exp(C\langle q\rangle^{-1+C\varepsilon}(\varepsilon\ln x^0(0)-\delta)) \\ &\quad + (\varepsilon+\langle q\rangle^{-1})\int_{x^0(0)}^t \tau^{-2+C\varepsilon}\, d\tau \\ &\lesssim \varepsilon(x^0(0))^{-1+C\varepsilon} + \langle q\rangle^{-1+C\varepsilon}(\varepsilon\ln x^0(0)-\delta)\exp(C\langle q\rangle^{-1+C\varepsilon}(\varepsilon\ln x^0(0)-\delta)) \\ &\quad + (\varepsilon+\langle q\rangle^{-1})(x^0(0))^{-1+C\varepsilon}.\end{aligned}$$

If $\varepsilon\ln x^0(0)-\delta \le c$ for some small constant $c>0$, we have

$$\begin{aligned}|\widetilde{V}_1(t,x)+2| &\le C\varepsilon\langle q\rangle^{-1+C\varepsilon} + Cc\langle q\rangle^{-1+C\varepsilon}\exp(Cc\langle q\rangle^{-1+C\varepsilon}) + C(\varepsilon+\langle q\rangle^{-1})(\langle q\rangle+\exp(\delta/\varepsilon))^{-1+C\varepsilon} \\ &\le C\varepsilon\langle q\rangle^{-1+C\varepsilon} + Cc\langle q\rangle^{-1+C\varepsilon}.\end{aligned}$$

By choosing $c,\varepsilon \ll_\kappa 1$, we can make $Cc+C\varepsilon<\kappa$. Thus, $|\widetilde{V}_1(t,x)+2| \le \kappa\langle q\rangle^{-1+C\varepsilon}$. If $\varepsilon\ln(x^0(0))-\delta>c$, we have $x^0(0)>\exp((c+\delta)/\varepsilon)$ and thus $q=(\exp(\delta/\varepsilon)-x^0(0))/2+2R< -C^{-1}\exp((c+\delta)/\varepsilon)$ for $\varepsilon \ll 1$. Then we have $\langle q\rangle^{C'\varepsilon} \ge C^{-C'\varepsilon}\exp(C'(c+\delta))$ and thus

$$\begin{aligned}|\widetilde{V}_1(t,x)+2| &\lesssim (\varepsilon+\langle q\rangle^{-1})(x^0(0))^{-1+C\varepsilon} + \langle q\rangle^{-1+C\varepsilon}(x^0(0))^{C\varepsilon} \\ &\lesssim (\varepsilon+\langle q\rangle^{-1})\langle q\rangle^{-1}(\exp(\delta/\varepsilon)+\langle q\rangle)^{C\varepsilon} + \langle q\rangle^{-1+C\varepsilon}(\exp(\delta/\varepsilon)+\langle q\rangle)^{C\varepsilon} \\ &\lesssim \langle q\rangle^{-1+C\varepsilon} \lesssim \langle q\rangle^{-1+(C+C')\varepsilon}C^{C'\varepsilon}\exp(-C'c).\end{aligned}$$

The second last inequality holds since $a^{C\varepsilon}+b^{C\varepsilon} \le (2\max\{a,b\})^{C\varepsilon} \le 2^{C\varepsilon}(a^{C\varepsilon}+b^{C\varepsilon})$ for $a,b>0$. By choosing $C' \gg_\kappa 1$ and $\varepsilon \ll_\kappa 1$, again we have $|\widetilde{V}_1(t,x)+2| \le \kappa\langle q\rangle^{-1+C\varepsilon}$.

We finish the proof by sending $s\to\infty$. ☐

4.5.5 An exact solution to the reduced system

For each $(s,q,\omega) \in \mathbb{R}\times\mathbb{R}\times\mathbb{S}^2$, we define

$$\begin{cases} \widetilde{\mu}(s,q,\omega) = A_1(q,\omega)\exp(-\dfrac{1}{2}G(\omega)A(q,\omega)s), \\ \widetilde{U}_q(s,q,\omega) = A_2(q,\omega)\exp(\dfrac{1}{2}G(\omega)A(q,\omega)s). \end{cases} \tag{4.70}$$

Since $\widetilde{\mu}\widetilde{U}_q = A_1A_2 = -2A$, it is easy to show that $(\widetilde{\mu},\widetilde{U}_q)$ is indeed a solution to the reduced system (4.65). To solve for $\widetilde{U}$ uniquely, we assume that $\lim_{q\to\infty}\widetilde{U}(s,q,\omega)=0$ (since $\lim_{q\to\infty}U(s,q,\omega)=0$). This also implies that $\widetilde{U}\equiv 0$ for $q\ge 2R$. At $(s,q,\omega)\in\Omega'\cap\{q<2R\}$ we have

$$\widetilde{\mu} = \mathfrak{R}_{0,0}\cdot(1+\mathfrak{R}_{0,-1}) = \mathfrak{R}_{0,0}, \qquad \widetilde{U}_q = \mathfrak{R}_{0,-1}(1+\mathfrak{R}_{0,0}) = \mathfrak{R}_{0,-1},$$

$$\widetilde{\mu}-\mu=\exp(-\frac{1}{2}G(\omega)A(q,\omega)s)(A_1-\widetilde{V}_1)=\mathfrak{R}_{-1,0},$$
$$\widetilde{U}_q-U_q=\exp(\frac{1}{2}G(\omega)A(q,\omega)s)(A_2-\widetilde{V}_2)=\mathfrak{R}_{-1,0}.$$

Thus, for each p,m,n, we have

$$\begin{aligned}
&\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n\widetilde{\mu}=\varepsilon^{-p}(\sum_k\mathfrak{R}_{0,0}V_k)^{p+m+n}(\mathfrak{R}_{0,0})=O(\varepsilon^{-p}t^{C\varepsilon}),\\
&\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n\widetilde{U}_q=\varepsilon^{-p}(\sum_k\mathfrak{R}_{0,0}V_k)^{p+m+n}(\mathfrak{R}_{0,-1})=O(\varepsilon^{-p}\langle q\rangle^{-1}t^{C\varepsilon}),\\
&\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n(\widetilde{\mu}-\mu,\widetilde{U}_q-U_q)=\varepsilon^{-p}(\sum_k\mathfrak{R}_{0,0}V_k)^{p+m+n}(\mathfrak{R}_{-1,0})=O(\varepsilon^{-p}t^{-1+C\varepsilon}).
\end{aligned}\tag{4.71}$$

Moreover, since $U=\varepsilon^{-1}ru=\mathfrak{R}_{0,0}$, we can also show that $\partial_s^p(\langle q\rangle\partial_q)^m\partial_\omega^n U=O(\varepsilon^{-p}t^{C\varepsilon})$. Now, by integrating $\partial_s^p\partial_\omega^n(\widetilde{U}_q-U_q)$ with respect to q, we have

$$\partial_s^p\partial_\omega^n(\widetilde{U}-U)=O(\varepsilon^{-p}\langle q\rangle t^{-1+C\varepsilon}),\qquad \partial_s^p\partial_\omega^n\widetilde{U}=O(\varepsilon^{-p}\langle q\rangle t^{-1+C\varepsilon}+\varepsilon^{-p}t^{C\varepsilon})=O(\varepsilon^{-p}t^{C\varepsilon}).\tag{4.72}$$

Here we note that $\langle q\rangle\lesssim t$ in $\Omega'\cap\{q<2R\}$. The estimates (4.71) and (4.72) will be used in Section 4.7.

4.6 Gauge independence

At the beginning of Section 4.3, we define a region Ω by (4.7) and then construct an optical function in Ω. If we replace (4.7) with

$$\Omega_{\kappa,\delta}:=\{(t,x):\ t>\exp(\delta/\varepsilon),\ |x|-\exp(\delta/\varepsilon)-2R>\kappa(t-\exp(\delta/\varepsilon))\}$$

for some fixed constants $\delta>0$ and $0<\kappa<1$, we are still able to construct an optical function in $\Omega_{\kappa,\delta}$ by following the proofs in Section 4.3 and Section 4.4. We are also able to construct a scattering data by following the proofs in Section 4.5. We do not expect that the scattering data to be independent of (κ,δ), but we have the next proposition.

Proposition 4.57. *Suppose $q(t,x)$ and $\bar{q}(t,x)$ are two solutions to the same eikonal equation*

$$g^{\alpha\beta}(u)q_\alpha q_\beta=0$$

in different regions $\Omega_{\kappa,\delta}$ and $\Omega_{\bar{\kappa},\bar{\delta}}$, respectively, as constructed in Section 4.3 and Section 4.4. Let $A(q,\omega)$ and $\bar{A}(\bar{q},\omega)$ be the corresponding scattering data constructed in Section 4.5.4. Under the change of coordinates $(s,q,\omega)=(\varepsilon\ln(t)-\delta,q(t,x),\omega)$, we can view $\bar{q}(t,x)$ as a function of (s,q,ω) in $\Omega_{\kappa,\delta}\cap\Omega_{\bar{\kappa}\cap\bar{\delta}}$. Then, the limit $\bar{q}_\infty(q,\omega):=\lim_{s\to\infty}\bar{q}(s,q,\omega)$ exists for which we have

$$A(q,\omega)=\bar{A}(\bar{q}_\infty(q,\omega),\omega).$$

Proof. We first recall several notations and estimates in Section 4.3. For example, we have $\mu = q_t - q_r = O(t^{C\varepsilon})$, $\nu = q_t + q_r = O(t^{-1+C\varepsilon})$, and we have similar definitions and estimates for $\bar{\mu}$ and $\bar{\nu}$. By viewing $\bar{q}(t,x)$ as a function of $(s,q,\omega) = (\varepsilon \ln(t) - \delta, q(t,x), \omega)$, we have

$$\partial_s \bar{q} = \varepsilon^{-1} t(\bar{q}_t - q_t q_r^{-1} \bar{q}_r) = t\varepsilon^{-1} \bar{q}_r (\bar{\nu} \bar{q}_r^{-1} - \nu q_r^{-1}).$$

By the eikonal equation, we have

$$0 = -(q_r - q_r)(q_r + q_r) + O(t^{-2+C\varepsilon}) + (g^{\alpha\beta}(u) - m^{\alpha\beta}) q_\alpha q_\beta = -\nu\mu + \frac{1}{4} uG(\omega)\mu^2 + O(t^{-2+C\varepsilon}).$$

Since $\mu \leq -C^{-1} t^{-C\varepsilon}$, we have

$$\nu = \frac{1}{4} uG(\omega)\mu + O(t^{-2+C\varepsilon})$$

and thus

$$\frac{\nu}{q_r} = \frac{1}{4} uG(\omega) \frac{\mu}{q_r} + O(t^{-2+C\varepsilon}) = \frac{1}{4} uG(\omega)(\frac{\nu}{q_r} - 2) + O(t^{-2+C\varepsilon}) = -\frac{1}{2} uG(\omega) + O(t^{-2+C\varepsilon}).$$

We conclude that

$$\begin{aligned} \partial_s \bar{q} &= t\varepsilon^{-1} \bar{q}_r^{-1} (-\frac{1}{2} uG(\omega) + O(t^{-2+C\varepsilon}) - (-\frac{1}{2} uG(\omega) + O(t^{-2+C\varepsilon}))) \\ &= O(\varepsilon^{-1} t^{-1+C\varepsilon}) = O(\varepsilon^{-1} \exp((-\varepsilon^{-1} + C)(s+\delta))). \end{aligned}$$

As computed in Section 4.5.4, we can show that $\bar{q}_\infty(q,\omega) := \lim_{s\to\infty} \bar{q}(s,q,\omega)$ exists for all (q,ω). Moreover, we can show that

$$|\bar{q}(s,q,\omega) - \bar{q}_\infty(q,\omega)| \lesssim t^{-1+C\varepsilon}.$$

Since $\lim_{s\to\infty}(\mu U_q)(s,q,\omega) = -2A(q,\omega)$ and $\lim_{\bar{s}\to\infty}(\bar{\mu}\bar{U}_q)(\bar{s},\bar{q},\omega) = -2\bar{A}(\bar{q},\omega)$ (recall that $\bar{s} + \bar{\delta} = s + \delta$), we have

$$\partial_r(\varepsilon^{-1} ru) = q_r U_q = -\frac{1}{2}\mu U_q + O(t^{-1+C\varepsilon}); \qquad \partial_r(\varepsilon^{-1} ru) = \bar{q}_r \bar{U}_{\bar{q}} = -\frac{1}{2}\bar{\mu}\bar{U}_{\bar{q}} + O(t^{-1+C\varepsilon}).$$

Then,

$$(\mu U_q)(s,q,\omega) = (\bar{\mu}\bar{U}_q)(s + \delta - \bar{\delta}, \bar{q}(s,q,\omega), \omega) + O(t^{-1+C\varepsilon}).$$

By sending s (and thus t) to infinity, we conclude that $A(q,\omega) = \bar{A}(\bar{q}_\infty(q,\omega),\omega)$. □

4.7 Approximation

Recall that we have constructed an exact solution to our reduced system in (4.70). In this section, we seek to prove that this exact solution gives a good approximation of the exact solution to (1.1).

To state the result, we first recall the solution $(\widetilde{\mu},\widetilde{U})(s,q,\omega)$ to the reduced system defined in Proposition 4.49, or in (4.70). We now solve

$$\widetilde{q}_t-\widetilde{q}_r=\widetilde{\mu}(\varepsilon\ln(t)-\delta,\widetilde{q}(t,x),\omega)\quad\text{in }\Omega\cap\{r-t<2R\};\qquad \widetilde{q}=r-t\quad\text{when }r-t\geq 2R$$

and set

$$\widetilde{u}(t,x)=\varepsilon r^{-1}\widetilde{U}(\varepsilon\ln(t)-\delta,\widetilde{q}(t,x),\omega)\quad\text{in }\Omega\cap\{r-t<2R\}.$$

We remark that the construction here is very similar to that in Section 4 of the author's prevoius paper [34]. We then have the following approximation result.

Proposition 4.58. *The function $\widetilde{u}=\widetilde{u}(t,x)$ is an approximate solution to* (1.1) *in the following sense:*

$$|Z^I(g^{\alpha\beta}(\widetilde{u})\partial_\alpha\partial_\beta\widetilde{u})(t,x)|\lesssim\varepsilon t^{-3+C\varepsilon},\qquad\forall(t,x)\in\Omega,\ \forall I.$$

Moreover, if we fix a constant $0<\gamma<1$ and a large integer N, then for $\varepsilon\ll_{\gamma,N}1$, at each $(t,x)\in\Omega$ such that $|r-t|\lesssim t^\gamma$, we have $|Z^I(u-\widetilde{u})|\lesssim_\gamma\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ for each $|I|\leq N$.

The estimates for $u-\widetilde{u}$ in this proposition is better than the estimates for u itself.

After making several definitions in Section 4.7.1, we introduce a simplification in Section 4.7.2. Instead of $(\widetilde{\mu},\widetilde{U}_q)$, the simplification in Section 4.7.2 allows us work with $(\hat{\mu},\widetilde{U}_q)$ which is an exact solution to the reduced system (4.70) with initial data $(-2,\hat{A})$. We thus get a new function $\hat{q}$ which is a solution to $\hat{q}_t-\hat{q}_r=\hat{\mu}$. In Section 4.7.3, we follow Section 4 of [34] to prove several estimates for $\hat{q}$ and $\hat{U}$. The most important result here is Proposition 4.68 which states that $\widetilde{u}=\hat{u}$ is indeed an approximate solution to (1.1). In Section 4.7.4, we show that $\hat{q}$ approximates the optical function q in a certain sense. Finally, in Section 4.7.5, we make use of the estimates in Section 4.7.4 to prove Proposition 4.58.

4.7.1 Definitions

We first define a function $\widetilde{q}(t,x)$ in Ω by solving the following equation

$$\widetilde{q}_t-\widetilde{q}_r=\widetilde{\mu}(\varepsilon\ln(t)-\delta,\widetilde{q}(t,x),\omega)\quad\text{in }\Omega\cap\{r-t<2R\};\qquad \widetilde{q}=r-t\quad\text{when }r-t\geq 2R.\tag{4.73}$$

Recall that $\widetilde{\mu}$ is defined by

$$\widetilde{\mu}(s,q,\omega):=A_1(q,\omega)\exp(-\frac{1}{2}G(\omega)A(q,\omega)s),\qquad\forall(s,q,\omega)\in\mathbb{R}\times\mathbb{R}\times\mathbb{S}^2.$$

In this section, when we write q, we usually mean a variable instead of the optical function $q(t,x)$.

As in [34], we can use the method of characteristics to solve (4.73). We fix $(t,x)\in\Omega\cap\{r-t<2R\}$ and set $z(\tau):=\widetilde{q}(\tau,r+t-\tau,\omega)$. Then, the function $z(\tau)$ is a solution to the autonomous system of ODE's

$$\dot{z}(\tau)=\widetilde{\mu}(\varepsilon s(\tau)-\delta,z(\tau),\omega),\qquad\dot{s}(\tau)=\varepsilon\tau^{-1}.$$

The initial data is given by $(z,s)((r+t)/2-R)=(2R,\varepsilon\ln((r+t)/2-R)-\delta)$. By Proposition 4.53, Proposition 4.55 and Lemma 4.56, we have $|A_1+2|=O(\langle q\rangle^{-1+C\varepsilon})$, $(A_2,A)(q,\omega)=O(\langle q\rangle^{-1+C\varepsilon})$ and $A_1<-1$ for all (q,ω). Thus,

$$\begin{aligned}0 &\geq \mu(\varepsilon s(\tau)-\delta,z(\tau),\omega)=A_1(z(\tau),\omega)\exp(-\frac{1}{2}G(\omega)A(z(\tau),\omega)(\varepsilon s(\tau)-\delta))\\ &\geq -C\tau^{C\varepsilon\langle z(\tau)\rangle^{-1+C\varepsilon}}\geq -C\tau^{C\varepsilon}.\end{aligned}$$

Then, $-C\tau^{C\varepsilon}\leq \dot z(\tau)\leq 0$, so $|z(\tau)|$ cannot blow up in finite time. By the Picard's theorem, the system of ODE's above has a solution for all $(r+t)/2-R\leq\tau<\frac{1}{3}(2(r+t)-4R-\exp(\delta/\varepsilon))$. The upper bound here guarantees that $(\tau,r+t-\tau,\omega)\in\Omega$. Thus, (4.73) has a solution $\widetilde{q}(t,x)$ in Ω.

Next, we define $\widetilde{U}(s,q,\omega)$ by

$$\widetilde{U}(s,q,\omega)=-\int_q^\infty A_2(p,\omega)\exp(\frac{1}{2}G(\omega)A(p,\omega)s)\ dp. \tag{4.74}$$

Note that $A_2(q,\omega)=0$ whenever $q>R$, so when $q<R$, we can replace ∞ with R in (4.74). In Ω we set

$$\widetilde{u}(t,x)=\varepsilon r^{-1}\widetilde{U}(\varepsilon\ln(t)-\delta,\widetilde{q}(t,x),\omega).$$

We seek to prove that $\widetilde{u}(t,x)$ provides a good approximation of $u(t,x)$.

4.7.2 Simplification

We aim to introduce some simplification in this subsection. Define a new function $F(q,\omega)$ on $\mathbb{R}\times\mathbb{S}^2$ by

$$F(q,\omega):=2R-\int_{2R}^q\frac{2}{A_1(p,\omega)}\ dp.$$

Then, we have

a) F is defined everywhere, and $2(q-R)\leq F(q,\omega)\leq 2(q+R)/3$ for all $q<2R$. This is because $A_1\in[-3,-1]$ by Lemma 4.56.

b) F is a smooth function of (q,ω), in the sense that for each large integer N and $\varepsilon\ll_N 1$, F is in C^N. This is because $A_1\in[-3,-1]$ and by Proposition 4.55.

c) $F(q,\omega)=q$ for $q>R$, and $\langle F(q,\omega)\rangle\sim\langle q\rangle$. This is because $A_1\equiv -2$ for $q>R$.

d) For each fixed ω, the map $q\mapsto F(q,\omega)$ has an inverse denoted by $\hat{F}(q,\omega)$ which is also smooth (in the same sense as in a) above) in $\mathbb{R}\times\mathbb{S}^2$. This is because $F_q=-2/A_1\in[2/3,2]$.

e) $\partial_q^a\partial_\omega^c F=O(\langle q\rangle^{1-a+C\varepsilon})$. Recall that $A_1<-1$ and $\partial_q^a\partial_\omega^c A_1=O(\langle q\rangle^{-a+C\varepsilon})$. If $a=0$, then $|\partial_\omega^c F|\lesssim\int_{[q,2R]}\langle p\rangle^{C\varepsilon}\ dp\lesssim\langle q\rangle^{1+C\varepsilon}$. If $a\geq 1$, then $|\partial_q^a\partial_\omega^c F|=|\partial_q^{a-1}\partial_\omega^c(2/A_1)|\lesssim\langle q\rangle^{1-a+C\varepsilon}$.

For each (s, q, ω), we set

$$\hat{A}(q, \omega) := A(\hat{F}(q, \omega), \omega)$$

and

$$\begin{cases} \hat{\mu}(s, q, \omega) := -2\exp(-\frac{1}{2}G(\omega)\hat{A}(q, \omega)s), \\ \hat{U}(s, q, \omega) := -\int_q^\infty \hat{A}(p, \omega)\exp(\frac{1}{2}G(\omega)\hat{A}(p, \omega)s)\ dp. \end{cases} \tag{4.75}$$

It is clear that $(\hat{\mu}, \hat{U})$ is a solution to the reduced system (4.65).

For each $(t, x) \in \Omega$, we set

$$\hat{q}(t, x) := F(\widetilde{q}(t, x), \omega), \qquad \hat{u}(t, x) := \varepsilon r^{-1}\hat{U}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega).$$

We then have the next key lemma.

Lemma 4.59. *In Ω, we have*

$$\hat{q}_t - \hat{q}_r = \hat{\mu}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega)$$

and $\hat{q} = r - t$ whenever $r - t > R$. Moreover, we have $\hat{u}(t, x) = \widetilde{u}(t, x)$ everywhere.

Proof. At $(t, x) \in \Omega$, we first have

$$\widetilde{q}(t, x) = \hat{F}(F(\widetilde{q}(t, x), \omega), \omega) = \hat{F}(\hat{q}(t, x), \omega).$$

Thus,

$$\begin{aligned} \hat{q}_t - \hat{q}_r &= (\partial_t - \partial_r)F(\widetilde{q}(t, x), \omega) = F_q(\widetilde{q}(t, x), \omega) \cdot \widetilde{\mu}(\varepsilon \ln t - \delta, \widetilde{q}(t, x), \omega) \\ &= (-2/A_1 \cdot A_1 \exp(-\frac{1}{2}GAs))(\varepsilon \ln t - \delta, \widetilde{q}(t, x), \omega) \\ &= -2\exp(-\frac{1}{2}G(\omega)A(\widetilde{q}(t, x), \omega)(\varepsilon \ln t - \delta)) \\ &= -2\exp(-\frac{1}{2}G(\omega)A(\hat{F}(\hat{q}(t, x), \omega), \omega)(\varepsilon \ln t - \delta)) \\ &= -2\exp(-\frac{1}{2}G(\omega)\hat{A}(\hat{q}(t, x), \omega)(\varepsilon \ln t - \delta)) = \hat{\mu}(\varepsilon \ln t - \delta, \hat{q}(t, x), \omega). \end{aligned}$$

Since $F(q, \omega) = q$ for all $q > R$, we have $\hat{q}(t, x) = \widetilde{q}(t, x) = r - t$ whenever $r - t > R$.

Moreover, if $\rho = \hat{F}(p, \omega)$, then we have $p = F(\rho, \omega)$ and thus

$$A(\rho, \omega) = A(\hat{F}(p, \omega), \omega) = \hat{A}(p, \omega).$$

Then by the change of variables ($\rho = \hat{F}(p,\omega)$ and thus $p = F(\rho,\omega)$), we have

$$\begin{aligned}\hat{U}(s,\hat{q},\omega) &= -\int_{\hat{q}}^{\infty} \hat{A}(p,\omega)\exp(\frac{1}{2}G(\omega)\hat{A}(p,\omega)s)\ dp \\ &= -\int_{\widetilde{q}}^{\infty} A(\rho,\omega)\exp(\frac{1}{2}G(\omega)A(\rho,\omega)s)F_\rho(\rho,\omega)\ d\rho \\ &= -\int_{\widetilde{q}}^{\infty} A_2(\rho,\omega)\exp(\frac{1}{2}G(\omega)A(\rho,\omega)s)\ d\rho = \widetilde{U}(s,\widetilde{q},\omega).\end{aligned}$$

Here we note that $AF_q = -2A/A_1 = A_2$. That is, for each (s,q,ω) (not viewed as functions of (t,x)),

$$\hat{U}(s,q,\omega) = \widetilde{U}(s,\hat{F}(q,\omega),\omega). \tag{4.76}$$

We thus have $\widetilde{u}(t,x) = \hat{u}(t,x)$. □

Because of Lemma 4.59, we can work with $(\hat{u},\hat{q})$ instead of $(\widetilde{u},\widetilde{q})$.

We end this subsection with several useful estimates for $(\hat{A},\hat{\mu},\hat{U})$.

Proposition 4.60. *For each (q,ω), we have*

$$(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{F}(q,\omega) = O(\langle q\rangle^{1+C\varepsilon}), \qquad (\langle q\rangle\partial_q)^a\partial_\omega^c\hat{A}(q,\omega) = O(\langle q\rangle^{-1+C\varepsilon}).$$

Besides, for each $(s,q,\omega)\in\Omega'\cap\{q<2R\}$, we have

$$\partial_s^b(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{U} = O(\varepsilon^{-b}t^{C\varepsilon}), \qquad \partial_s^b(\langle q\rangle\partial_q)^{a+1}\partial_\omega^c\hat{U} = O(t^{C\varepsilon});$$

$$\hat{\mu} = O(t^{C\varepsilon}), \qquad \partial_s^b(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{\mu} = O(\langle q\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|), \quad a+b+|c|>0.$$

Proof. First, it is clear that $\langle\hat{F}(q,\omega)\rangle \sim \langle q\rangle$ and that $\hat{F}_q(q,\omega) = 1/(F_q(\hat{F}(q,\omega),\omega)) = -A_1(\hat{F}(q,\omega),\omega)/2 \sim \langle q\rangle^{C\varepsilon}$. In general we induct on $m+|n|$. By differentiating $q = F(\hat{F}(q,\omega),\omega)$, for $(a,c)\notin\{(0,0),(1,0)\}$, we have

$$0 = F_q(\hat{F}(q,\omega),\omega)\cdot\partial_q^a\partial_\omega^c\hat{F}(q,\omega) + \sum C[(\partial_q^m\partial_\omega^{c'}F)(\hat{F}(q,\omega),\omega)\cdot\prod_{j=1}^m(\partial_q^{a_j}\partial_\omega^{c_j}\hat{F})(q,\omega)].$$

Here the sum on the right hand side is taken over all (m,c',a_*,c_*) such that $\sum a_j = a$, $c'+\sum c_j = c$, $a_j+|c_j| < a+|c|$. We can now apply the induction hypotheses to conclude that

$$\begin{aligned}0 &= F_q(\hat{F}(q,\omega),\omega)\cdot\partial_q^a\partial_\omega^c\hat{F}(q,\omega) + \sum O(\langle\hat{F}(q,\omega)\rangle^{1-m+C\varepsilon}\cdot\langle\hat{q}\rangle^{m-\sum a_j+C\varepsilon}) \\ &= F_q(\hat{F}(q,\omega),\omega)\cdot\partial_q^a\partial_\omega^c\hat{F}(q,\omega) + O(\langle q\rangle^{1-a+C\varepsilon}).\end{aligned}$$

And since $F_q\sim 1$, we conclude that $\partial_q^a\partial_\omega^c\hat{F}(q,\omega) = O(\langle q\rangle^{1-a+C\varepsilon})$.

Next, recall that

$$\hat{A}(q,\omega) = A(\hat{F}(q,\omega),\omega), \qquad \hat{U}(s,q,\omega) = \widetilde{U}(s,\hat{F}(q,\omega),\omega).$$

Then, $\partial_s^b\partial_q^a\partial_\omega^c\hat{U}(s,q,\omega)$ is a linear combination of terms of the form

$$\partial_s^b\partial_q^m\partial_\omega^{c'}\widetilde{U}(s,\hat{F}(q,\omega),\omega)\cdot\prod_{j=1}^m \partial_q^{a_j}\partial_\omega^{c_j}\hat{F}(q,\omega), \qquad \sum a_j = a,\ c' + \sum c_j = c.$$

By (4.71) and (4.72), we conclude that each of these terms are controlled by

$$\varepsilon^{-b}\langle\hat{F}(q,\omega)\rangle^{-m}t^{C\varepsilon}\cdot\langle q\rangle^{m-\sum a_j + C\varepsilon} \lesssim \varepsilon^{-b}\langle q\rangle^{-a}t^{C\varepsilon}.$$

Thus, $\partial_s^b(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{U}(s,q,\omega) = O(\varepsilon^{-b}t^{C\varepsilon})$. Following the same proof, we can show that $(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{A}(q,\omega) = O(\langle q\rangle^{-1+C\varepsilon})$.

Finally, by (4.75), we can write $\partial_s^b\partial_q^a\partial_\omega^c\hat{U}_q(s,q,\omega)$ as a linear combination of terms of the form

$$\partial_q^{a'}\partial_\omega^{c'}\hat{A}(q,\omega)\cdot\exp(\frac{1}{2}G\hat{A}s)\prod_{j=1}^m\partial_s^{b_j}\partial_q^{a_j}\partial_\omega^{c_j}(\frac{1}{2}G\hat{A}s)$$

where $a' + \sum a_j = a,\ \sum b_j = b,\ c' + \sum c_j = c$. Each of these terms are controlled by

$$\langle q\rangle^{-1-a'+C\varepsilon}\cdot t^{C\varepsilon}\cdot\langle q\rangle^{-m-\sum a_j}t^{C\varepsilon} \lesssim \langle q\rangle^{-1-a}t^{C\varepsilon}.$$

In conclusion, we have $\partial_s^b(\langle q\rangle\partial_q)^{a+1}\partial_\omega^c\hat{U}(s,q,\omega) = O(t^{C\varepsilon})$. Here we do not have the factor ε^{-b} which is better. Moreover, we have $\hat{\mu} = O(t^{C\varepsilon})$ and

$$(\hat{\mu}_s, \langle q\rangle\hat{\mu}_q, \hat{\mu}_\omega) = -\frac{1}{2}(GA, \langle q\rangle GA_q s, \partial_\omega(GA)s)\hat{\mu}.$$

Following the same proof, we can show that $\partial_s^b(\langle q\rangle\partial_q)^a\partial_\omega^c\hat{\mu}(s,q,\omega) = O(\langle q\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|)$ if $a + b + |c| > 0$. □

4.7.3 Estimates for $\hat{q}$ and $\hat{U}$

We now follow Section 4 in [34] to prove several useful estimates. In this subsection, all functions of $(s,q,\omega) \in [0,\infty)\times\mathbb{R}\times\mathbb{S}^2$ are viewed as functions of $(t,x)\in\Omega$ by setting $(s,q,\omega) = (\varepsilon\ln t - \delta, \hat{q}(t,x),\omega)$. This setting is different from that in the previous sections of this chapter, where we take $q = q(t,x)$.

Lemma 4.61. *In $\Omega\cap\{r-t<2R\}$, we have $\langle\hat{q}\rangle/\langle r-t\rangle = t^{O(\varepsilon)}$ and $\hat{q}(t,x) - r + t = O(\min\{\varepsilon^{-1},\langle\hat{q}\rangle\}t^{C\varepsilon})$.*

Proof. Fix $(t,x)\in\Omega\cap\{r-t<2R\}$. Then, we have

$$\begin{aligned}|\hat{q}(t,x)-2R| &= \int_{(r+t)/2-R}^{t}(-\hat{\mu}(\varepsilon\ln\tau-\delta,\hat{q}(\tau,r+t-\tau,\omega),\omega))\ d\tau\\ &\lesssim \int_{(r+t)/2-R}^{t}\exp(C\langle\hat{q}\rangle^{-1+C\varepsilon}s)(\tau,r+t-\tau,\omega)\ d\tau\\ &\lesssim ((r-t)/2+R)t^{C\varepsilon}\lesssim\langle r-t\rangle t^{C\varepsilon};\\ |\hat{q}(t,x)-2R| &= \int_{(r+t)/2-R}^{t}(-\hat{\mu}(\varepsilon\ln\tau-\delta,\hat{q}(\tau,r+t-\tau,\omega),\omega))\ d\tau\\ &\gtrsim \int_{(r+t)/2-R}^{t}\exp(-C\langle\hat{q}\rangle^{-1+C\varepsilon}s)(\tau,r+t-\tau,\omega)\ d\tau\\ &\gtrsim ((r-t)/2+R)t^{-C\varepsilon}\gtrsim\langle r-t\rangle t^{-C\varepsilon}.\end{aligned}$$

Thus, we have $t^{-C\varepsilon}\langle\hat{q}\rangle\lesssim\langle r-t\rangle\lesssim t^{C\varepsilon}\langle\hat{q}\rangle$. It follows that

$$|\hat{q}(t,x)-(r-t)|\le|\hat{q}-2R|+|r-t-2R|\lesssim t^{C\varepsilon}\langle\hat{q}\rangle+\langle r-t\rangle\lesssim\langle\hat{q}\rangle t^{C\varepsilon}.$$

To improve the estimate above, we note that

$$\begin{aligned}\hat{q}(t,x)&=2R+\int_{(r+t)/2-R}^{t}\hat{\mu}(\varepsilon\ln\tau-\delta,\hat{q}(\tau,r+t-\tau,\omega),\omega)\ d\tau\\ &=r-t+\int_{(r+t)/2-R}^{t}(\hat{\mu}(\varepsilon\ln\tau-\delta,\hat{q}(\tau,r+t-\tau,\omega),\omega)+2)\ d\tau.\end{aligned}$$

For each $(s,q,\omega)\in[0,\infty)\times\mathbb{R}\times\mathbb{S}^2$, by Proposition 4.54 and Lemma 4.56 we have

$$|\hat{\mu}(s,q,\omega)+2|\lesssim|1-\exp(-\frac{1}{2}GAs)|\lesssim\langle q\rangle^{-1+C\varepsilon}|s|\exp(C\langle q\rangle^{-1+C\varepsilon}s).$$

By setting $(s,q,\omega)=(\varepsilon\ln\tau-\delta,\hat{q}(\tau,r+t-\tau,\omega),\omega)$, we have

$$|\hat{\mu}+2|(\tau)\lesssim\langle r+t-2\tau\rangle^{-1+C\varepsilon}\tau^{C\varepsilon}\lesssim(3R-r-t+2\tau)^{-1+C\varepsilon}t^{C\varepsilon}$$

and then

$$|\hat{q}-r+t|\lesssim t^{C\varepsilon}\int_{(r+t)/2-R}^{t}(3R-r-t+2\tau)^{-1+C\varepsilon}\ d\tau\lesssim\varepsilon^{-1}t^{C\varepsilon}(3R-r+t)^{C\varepsilon}.$$

And since $0\le 3R-r+t\lesssim 1+t\lesssim t$, we have $|\hat{q}-r+t|\lesssim\varepsilon^{-1}t^{C\varepsilon}$. □

Lemma 4.62. *In Ω we have*

$$\hat{\nu}:=\hat{q}_t+\hat{q}_r=O(t^{-1+C\varepsilon}),\qquad\hat{\lambda}_i:=\hat{q}_i-\omega_i\hat{q}_r=O((1+\ln\langle r-t\rangle)t^{-1+C\varepsilon}).$$

It follows that $\hat{q}_r=(\hat{\nu}-\hat{\mu})/2>C^{-1}t^{-C\varepsilon}$ and $\hat{q}_t=(\hat{\nu}+\hat{\mu})/2<-C^{-1}t^{-C\varepsilon}$. Thus, for each fixed (t,ω) the function $r\mapsto\hat{q}(t,r\omega)$ is continuous and strictly increasing.

Proof. There is nothing to prove when $r-t>R$. Fix $(t,x)\in\Omega\cap\{r-t<2R\}$. Then,

$$\begin{aligned}(\partial_t-\partial_r)\hat{\nu}&=(\partial_t+\partial_r)\hat{\mu}=\hat{\mu}_q\hat{\nu}+\varepsilon t^{-1}\hat{\mu}_s=\hat{\mu}_q\hat{\nu}-\frac{\varepsilon}{2t}G(\omega)A(\hat{q},\omega)\hat{\mu}\\&=-\frac{1}{2}G\hat{A}_qs\hat{\mu}\hat{\nu}-\frac{\varepsilon}{2t}G\hat{A}\hat{\mu}.\end{aligned}$$

By setting $z(\tau):=\hat{q}(\tau,r+t-\tau,\omega)$, we have $\dot{z}=\hat{\mu}<0$ and thus

$$\int_{(r+t)/2-R}^{t}|G\hat{A}_qs\hat{\mu}|(\tau,r+t-\tau,\omega)\ d\tau\lesssim\int_{(r+t)/2-R}^{t}(\varepsilon\ln\tau+1)\langle\hat{q}\rangle^{-2+C\varepsilon}(-\hat{\mu})\ d\tau$$

$$\lesssim(\varepsilon\ln t+1)\int_{(r+t)/2-R}^{t}\langle z\rangle^{-2+C\varepsilon}(-\dot{z})\ d\tau\lesssim\varepsilon\ln t+1,$$

$$\int_{(r+t)/2-R}^{t}|\varepsilon\tau^{-1}G\hat{A}\hat{\mu}|(\tau,r+t-\tau,\omega)\ d\tau\lesssim\varepsilon((r+t)/2-R)^{-1}\int_{(r+t)/2-R}^{t}\langle\hat{q}\rangle^{-1+C\varepsilon}(-\hat{\mu})\ d\tau$$

$$\lesssim\varepsilon t^{-1}\int_{(r+t)/2-R}^{t}\langle z\rangle^{-1+C\varepsilon}(-\dot{z})\ d\tau\lesssim t^{-1}\langle\hat{q}\rangle^{C\varepsilon}\lesssim t^{-1+C\varepsilon}.$$

Here we note that $\langle\hat{q}\rangle\lesssim\langle r-t\rangle t^{C\varepsilon}\lesssim t^{1+C\varepsilon}$. Since $\hat{\nu}=0$ at $\tau=(r+t)/2-R$, by the Gronwall's inequality we conclude that $\hat{\nu}=O(t^{-1+C\varepsilon})$.

Next, we have

$$\begin{aligned}(\partial_t-\partial_r)\hat{\lambda}_i&=(\partial_i-\omega_i\partial_r)\hat{\mu}+r^{-1}\hat{\lambda}_i=(\hat{\mu}_q+r^{-1})\hat{\lambda}_i+\sum_l(\partial_{\omega_l}\hat{\mu})(\partial_i\omega_l)\\&=(\hat{\mu}_q+r^{-1})\hat{\lambda}_i-\frac{1}{2}\sum_l(\partial_{\omega_l}(G\hat{A}))(\varepsilon\ln t-\delta)\hat{\mu}r^{-1}(\delta_{il}-\omega_i\omega_l)\\&=(\hat{\mu}_q+r^{-1})\hat{\lambda}_i+O(\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-1+C\varepsilon}|\hat{\mu}|).\end{aligned}$$

We have proved that $\int_{(r+t)/2-R}^{t}|\mu_q|\ d\tau\lesssim\varepsilon\ln t+1$. Integrate along the characteristic $(\tau,r+t-\tau,\omega)$ and we have

$$\int_{(r+t)/2-R}^{t}(r+t-\tau)^{-1}\ d\tau=\ln\frac{(r+t)/2+R}{r}=O(1),$$

$$\begin{aligned}\int_{(r+t)/2-R}^{t}\langle\hat{q}\rangle^{-1+C\varepsilon}(-\hat{\mu})\tau^{-1+C\varepsilon}\ d\tau&\lesssim\int_{(r+t)/2-R}^{t}\langle\hat{q}\rangle^{-1}(-\hat{\mu})\tau^{-1+C\varepsilon}\ d\tau\\&\lesssim t^{-1+C\varepsilon}\int_{(r+t)/2-R}^{t}\langle z\rangle^{-1}(-\dot{z})\ d\tau\\&\lesssim(1+\ln\langle\hat{q}\rangle)t^{-1+C\varepsilon}\lesssim(1+\ln\langle r-t\rangle)t^{-1+C\varepsilon}.\end{aligned}$$

Here note that $\langle\hat{q}\rangle\lesssim t^{1+C\varepsilon}$ and $\ln\langle\hat{q}\rangle\lesssim\ln\langle r-t\rangle+C\varepsilon\ln t$ in $\Omega\cap\{r-t<2R\}$. Since $\hat{\lambda}_i=0$ at $\tau=(r+t)/2-R$, by Gronwall's inequality we conclude that $\hat{\lambda}_i=O((1+\ln\langle r-t\rangle)t^{-1+C\varepsilon})$. □

Lemma 4.63. *In Ω, we have*

$$\hat{\nu} = \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U} + O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle), \qquad \hat{\nu}_q = \frac{\varepsilon G(\omega)}{4t}(\hat{\mu}\hat{U}_q + \hat{\mu}_q\hat{U}) + O(\varepsilon(1+\ln\langle r-t\rangle)t^{-2+C\varepsilon}).$$

Proof. We have

$$\begin{aligned}
&(\partial_t - \partial_r)(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U}) \\
&= \hat{\mu}_q\hat{\nu} - \frac{\varepsilon}{2t}G\hat{A}\hat{\mu} + \frac{\varepsilon G\hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon G}{4t}(\hat{\mu}_q\hat{U} + \hat{\mu}\hat{U}_q)\hat{\mu} - \frac{\varepsilon G}{4t}(\hat{\mu}_s\hat{U} + \hat{\mu}\hat{U}_s)\varepsilon t^{-1} \\
&= \hat{\mu}_q\hat{\nu} - \frac{\varepsilon}{2t}G\hat{A}\hat{\mu} + \frac{\varepsilon G\hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon G}{4t}(\hat{\mu}_q\hat{U} - 2\hat{A})\hat{\mu} - \frac{\varepsilon G}{4t}(-\frac{1}{2}G\hat{A}\hat{\mu}\hat{U} + \hat{\mu}\hat{U}_s)\varepsilon t^{-1} \\
&= \hat{\mu}_q(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + \frac{\varepsilon G\hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu}.
\end{aligned}$$

Since $\hat{U} = O(t^{C\varepsilon})$ and $\hat{U}_s = O(\varepsilon^{-1}t^{C\varepsilon})$ by Proposition 4.60, we have

$$|\frac{\varepsilon G\hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu}| \lesssim \varepsilon t^{-2+C\varepsilon}.$$

Besides, we have

$$\int_{(r+t)/2-R}^{t} \varepsilon\tau^{-2+C\varepsilon} \lesssim ((r+t)/2-R)^{-2+C\varepsilon}\cdot\varepsilon((t-r)/2-R) \lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle.$$

And since $\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U} = 0$ at $\tau = (r+t)/2-R$, by Gronwall's inequality we conclude that

$$\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U} = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle).$$

Next, we have

$$\begin{aligned}
(\partial_t - \partial_r)\partial_r(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U}) &= \partial_r(\partial_t - \partial_r)(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U}) \\
&= \partial_r(\hat{\mu}_q(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + \frac{\varepsilon G\hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu}) \\
&= \hat{\mu}_q\partial_r(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + \hat{q}_r\hat{\mu}_{qq}(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + \frac{\varepsilon G\hat{q}_r\partial_q(\hat{\mu}\hat{U})}{4t^2} \\
&\quad - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu}_q\hat{q}_r - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\partial_q(\hat{A}\hat{U}) + \hat{U}_{sq})\hat{\mu}\hat{q}_r.
\end{aligned}$$

By Proposition 4.60, we have

$$|\hat{\mu}_{qq}(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U})| \lesssim |\partial_q(G\hat{A}_q s\hat{\mu})|\cdot\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle \lesssim \varepsilon t^{-2+C\varepsilon}\langle\hat{q}\rangle^{-2+C\varepsilon},$$

$$|\partial_q(\hat{\mu}\hat{U})| \lesssim |\hat{\mu}_q\hat{U}| + |2\hat{A}| \lesssim t^{C\varepsilon}\langle\hat{q}\rangle^{-2+C\varepsilon} + \langle\hat{q}\rangle^{-1+C\varepsilon} \lesssim t^{C\varepsilon}\langle\hat{q}\rangle^{-1+C\varepsilon},$$

$$|(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu}_q| \lesssim (\langle q\rangle^{-1+C\varepsilon}t^{C\varepsilon} + \varepsilon^{-1}t^{C\varepsilon}) \cdot \langle q\rangle^{-2+C\varepsilon}t^{C\varepsilon} \lesssim \varepsilon^{-1}\langle\hat{q}\rangle^{-2+C\varepsilon}t^{C\varepsilon},$$

$$|(-\frac{1}{2}G\partial_q(\hat{A}\hat{U}) + \hat{U}_{sq})\hat{\mu}| \lesssim |(-\frac{1}{2}G\partial_q(\hat{A}\hat{U}) + \frac{1}{2}G\hat{A}\hat{U}_q)\hat{\mu}| \lesssim |(-\frac{1}{2}G\hat{A}_q\hat{U})\hat{\mu}| \lesssim \langle\hat{q}\rangle^{-2+C\varepsilon}t^{C\varepsilon}.$$

In conclusion,

$$\begin{aligned}
&(\partial_t - \partial_r)\partial_r(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U})\\
&= \hat{\mu}_q\partial_r(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + O(|\hat{q}_r|\varepsilon\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-2+C\varepsilon})\\
&= \hat{\mu}_q\partial_r(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + O((-\hat{\mu})\varepsilon\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-2+C\varepsilon} + |\hat{\nu}|\varepsilon\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-2+C\varepsilon})\\
&= \hat{\mu}_q\partial_r(\hat{\nu} - \frac{\varepsilon G}{4t}\hat{\mu}\hat{U}) + O((-\hat{\mu})\varepsilon\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-2+C\varepsilon} + \varepsilon\langle\hat{q}\rangle^{-1+C\varepsilon}t^{-3+C\varepsilon}).
\end{aligned}$$

Take integral of the remainder terms along a charactersitic $(\tau, r+t-\tau, \omega)$ for $(r+t)/2-R \leq \tau \leq t$. We have

$$\int_{(r+t)/2-R}^{t} \tau^{-2+C\varepsilon}\varepsilon\langle z\rangle^{-1+C\varepsilon}(-\dot{z}) + \varepsilon\tau^{-3+C\varepsilon}\ d\tau \lesssim \varepsilon(1+\ln\langle r-t\rangle)t^{-2+C\varepsilon}.$$

The proof of this estimate can be found in the proof of Lemma 4.62. Since $\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U} = 0$ whenever $r - t > R$, we have $\partial_r(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U}) = 0$ at $\tau = (r+t)/2 - R$. By Gronwall's inequality, we conclude that $\partial_r(\hat{\nu} - \frac{\varepsilon G(\omega)}{4t}\hat{\mu}\hat{U}) = O(\varepsilon(1+\ln\langle r-t\rangle)t^{-2+C\varepsilon})$. To end the proof, we recall that $\partial_r = \hat{q}_r\partial_q$ where $\hat{q}_r > C^{-1}t^{-C\varepsilon}$ in $\Omega \cap \{r - t < 2R\}$. ☐

Before we state the next lemma, we recall the definition in Section 1.6.4. We set $\mathcal{D} = \Omega \cap \{r-t < 2R\}$ and define $\varepsilon^n S^{s,p} = \varepsilon^n S^{s,p}_{\mathcal{D}}$ as in Definition 1.8.

Following the proof of Corollary 4.43.1, we can show that $\mathfrak{R}_{s,p} \in S^{s,p}$. Here we prefer the notation $S^{*,*}$ since it does not rely on the optical function $q(t,x)$ and the corresponding null frames.

Lemma 4.64. *We have $\hat{q} \in S^{0,1}$. We also have $\Omega_{kk'}\hat{q} \in S^{0,\gamma}$ for each $1 \leq k < k' \leq 3$ and $0 < \gamma < 1$. In other words, in $\Omega \cap \{r - t < 2R\}$, for each I we have*

$$|Z^I\hat{q}| \lesssim_I \langle r-t\rangle t^{C_I\varepsilon}; \tag{4.77}$$

$$|Z^I\Omega_{kk'}\hat{q}| \lesssim_I t^{C_I\varepsilon}\langle r-t\rangle^{\gamma}. \tag{4.78}$$

As a result, we have $\partial_q^m\partial_\omega^n\hat{A} \in S^{0,-1-m}$, $\hat{\mu} \in S^{0,0}$, $\partial_s^p\partial_q^m\partial_\omega^n\hat{\mu} \in S^{0,-1-m}$ for $m+n+p > 0$, $\partial_s^p\partial_\omega^n\hat{U} \in \varepsilon^{-p}S^{0,0}$ and $\partial_s^p\partial_q^m\partial_\omega^n\hat{U}_q \in S^{0,-1-m}$. All functions here are of $(s,q,\omega) = (\varepsilon\ln t - \delta, \hat{q}(t,x), \omega)$.

Proof. We prove (4.77) by induction on $|I|$. The case $|I| = 0$ has been proved in Lemma 4.61. In general, suppose (4.77) holds for all $|I| \leq k$, and fix a multiindex I with $|I| = k+1$. By the chain rule and Leibniz's rule, we express $Z^I\hat{\mu}$ as a linear combination of terms of the form

$$(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu}) \cdot Z^{I_1}\hat{q}\cdots Z^{I_a}\hat{q} \cdot Z^{J_1}(\varepsilon\ln t - \delta)\cdots Z^{J_b}(\varepsilon\ln t - \delta) \cdot \prod_l Z^{K_{l,1}}\omega_l \cdots Z^{K_{l,c_l}}\omega_l \tag{4.79}$$

where $a+b+|c| > 0$, $|I_*|, |J_*|, |K_{*,*}|$ are nonzero, and the sum of all these multiindices is $k+1$. The only term with some $|I_*| > k$ is $\hat{\mu}_q Z^I\hat{q}$. All the other terms have an upper bound

$$\langle\hat{q}\rangle^{-1-a+C\varepsilon}t^{C\varepsilon}|\hat{\mu}| \cdot (\langle r-t\rangle t^{C\varepsilon})^a \cdot \varepsilon^b \cdot 1 \lesssim \langle\hat{q}\rangle^{-1}t^{C\varepsilon}|\hat{\mu}|.$$

Here we apply Proposition 4.60 and the induction hypotheses to control $Z^{I_*}\hat{q}$. In summary, we have $Z^I\hat{\mu} = \hat{\mu}_q Z^I\hat{q} + O(\langle\hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|)$. Following the same proof, we also have

$$\sum_{0<|J|\leq k} |Z^J\hat{\mu}| = O(\langle\hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|).$$

In addition, by the induction hypotheses and Lemma 1.4, we have

$$\begin{aligned}\sum_{|J|<|I|} |(\partial_i + \omega_i\partial_t)Z^J\hat{q}| &\lesssim \sum_{|J|<k+1} (1+t+r)^{-1}|ZZ^J\hat{q}| \\ &\lesssim (1+t+r)^{-1}\sum_{|J|=k+1}|Z^J\hat{q}| + t^{-1+C\varepsilon}\langle r-t\rangle.\end{aligned}$$

In summary, by (1.18) in Lemma 1.3 we have

$$|(\partial_t - \partial_r)Z^I\hat{q}| \lesssim |\hat{\mu}_q Z^I\hat{q}| + (1+t+r)^{-1}\sum_{|J|=k+1}|Z^J\hat{q}| + t^{C\varepsilon}(-\hat{\mu}) + t^{-1+C\varepsilon}\langle r-t\rangle.$$

Here we note that

$$\sum_{|J|\leq k} |Z^J\hat{\mu}| \lesssim |\hat{\mu}| + \langle\hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}| \lesssim t^{C\varepsilon}(-\hat{\mu}).$$

Now, we fix $(t,x) \in \Omega \cap \{r-t < 2R\}$, integrate $(\partial_t - \partial_r)Z^I\hat{q}$ along the characteristic $(\tau, r+t-\tau, \omega)$ for $(t+r)/2 - R \leq \tau \leq t$, and sum over all $|I| = k+1$. We then have

$$\begin{aligned}&\sum_{|I|=k+1} |Z^I\hat{q}(t,x) - Z^I\hat{q}|_{\tau=(r+t)/2-R}| \\ &\lesssim \int_{(r+t)/2-R}^t (|\hat{\mu}_q| + (1+t+r)^{-1})\sum_{|I|=k+1}|Z^I\hat{q}|(\tau) + \tau^{C\varepsilon}(-\hat{\mu}) + \tau^{C\varepsilon}\ d\tau \\ &\lesssim \int_{(r+t)/2-R}^t (|\hat{\mu}_q| + (1+t+r)^{-1})\sum_{|I|=k+1}|Z^I\hat{q}|(\tau)\ d\tau + t^{C\varepsilon}\langle\hat{q}\rangle + \langle r-t\rangle t^{C\varepsilon}.\end{aligned}$$

Moreover, we have $\hat{q} = r - t$ for $r - t > R$ and $\hat{q} = 2R$ at $\tau = (r+t)/2 - R$, so

$$|Z^I \hat{q}|_{\tau=(r+t)/2-R}| = |Z^I(r-t)|_{\tau=(r+t)/2-R}| \lesssim t^{C\varepsilon}.$$

By Gronwall's inequality, we conclude that $\sum_{|I|=k+1} |Z^I \hat{q}(t,x)| \lesssim \langle r-t\rangle t^{C\varepsilon}$.

Fix $\gamma > 0$. Now we prove (4.78) by induction on $|I|$. First, in Lemma 4.62 we have proved $\hat{\lambda}_i = O((1+\ln\langle r-t\rangle)t^{-1+C\varepsilon}) = O_\gamma(\langle r-t\rangle^\gamma t^{-1+C\varepsilon})$. So we have $\Omega_{kk'}\hat{q} = x_k\lambda_{k'} - x_{k'}\lambda_k = O(\langle r-t\rangle^\gamma r t^{-1+C\varepsilon}) = O(\langle r-t\rangle^\gamma t^{C\varepsilon})$, so the case $|I| = 0$ is proved. In general, we fix I with $|I| > 0$. As computed above, we have

$$Z^I\Omega_{kk'}\hat{\mu} = \hat{\mu}_q Z^I\Omega_{kk'}\hat{q} + O(\langle \hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|), \qquad \sum_{|J|\le|I|}|Z^J\hat{\mu}| = O(\langle \hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}|\hat{\mu}|);$$

$$\begin{aligned}\sum_{|J|<|I|}|(\partial_i + \omega_i\partial_t)Z^J\Omega_{kk'}\hat{q}| &\lesssim (1+t+r)^{-1}\sum_{|J|\le|I|}|Z^J\Omega_{kk'}\hat{q}| \\ &\lesssim (1+t+r)^{-1}\sum_{|J|=|I|}|Z^J\Omega_{kk'}\hat{q}| + t^{-1+C\varepsilon}\langle r-t\rangle^\gamma.\end{aligned}$$

Thus, by (1.19), we have

$$\begin{aligned}|(\partial_t - \partial_r)Z^I\Omega_{kk'}\hat{q}| \lesssim{}& |\hat{\mu}_q Z^I\Omega_{kk'}\hat{q}| + (1+t+r)^{-1}\sum_{|J|=|I|}|Z^J\Omega_{kk'}\hat{q}| \\ &+ \langle \hat{q}\rangle^{-1+C\varepsilon}t^{C\varepsilon}(-\hat{\mu}) + t^{-1+C\varepsilon}\langle r-t\rangle^\gamma.\end{aligned}$$

Fix $(t,x) \in \Omega \cap \{r - t < 2R\}$ and take integrals along a geodesic $(\tau, r+t-\tau, \omega)$. We note that

$$\begin{aligned}&\int_{(r+t)/2-R}^t \langle \hat{q}(\tau)\rangle^{-1+C\varepsilon}\tau^{C\varepsilon}(-\hat{\mu}(\tau)) + \tau^{-1+C\varepsilon}\langle r+t-2\tau\rangle^\gamma \, d\tau \\ &\lesssim t^{C\varepsilon}\int_{(r+t)/2-R}^t \langle z(\tau)\rangle^{-1}(-\dot{z}(\tau))\, d\tau + t^{-1+C\varepsilon}\langle r-t\rangle^{1+\gamma} \\ &\lesssim (1+\ln\langle r-t\rangle)t^{C\varepsilon} + t^{C\varepsilon}\langle r-t\rangle^\gamma \lesssim t^{C\varepsilon}\langle r-t\rangle^\gamma.\end{aligned}$$

In addition, recall that $Z^I q|_{\tau=(r+t)/2-R} = O(t^{C\varepsilon})$. We finish the proof by applying Gronwall.

Finally, if $Q = Q(s,q,\omega)$ is a given function of (s,q,ω) and if we take $(s,q,\omega) = (\varepsilon\ln t - \delta, \hat{q}(t,x), \omega)$, then $Z^I Q$ is a linear combination of terms of the form (4.79) with $\hat{\mu}$ replaced by Q. Thus,

$$|Z^I Q| \lesssim \sum_{a+b+|c|\le|I|} \varepsilon^b \langle r-t\rangle^a t^{C\varepsilon}|\partial_s^b\partial_q^a\partial_\omega^c Q|.$$

We combine this inequality with Proposition 4.60. As a result, we have $\partial_q^m\partial_\omega^n\hat{A} \in S^{0,-1-m}$, $\hat{\mu} \in S^{0,0}$, $\partial_s^p\partial_q^m\partial_\omega^n\hat{\mu} \in S^{0,-1-m}$ for $m+n+p > 0$, $\partial_s^p\partial_\omega^n\hat{U} \in \varepsilon^{-p}S^{0,0}$ and $\partial_s^p\partial_q^m\partial_\omega^n\hat{U}_q \in S^{0,-1-m}$. $\square$

Lemma 4.65. *Fix $\gamma \in (0,1)$. We have $\hat{\nu} \in \varepsilon S^{-1,0}$, $\hat{\nu}_q \in \varepsilon S^{-1,-1}$, $\hat{\lambda}_i \in S^{-1,\gamma}$ and*

$$\hat{\nu} - \frac{\varepsilon}{4t} G(\omega)\hat{\mu}\hat{U} \in \varepsilon S^{-2,1}, \qquad \hat{\nu}_q - \frac{\varepsilon}{4t} G(\omega)(\hat{\mu}_q \hat{U} - 2\hat{A}) \in \varepsilon S^{-2,0}.$$

All functions here are of $(s, q, \omega) = (\varepsilon \ln t - \delta, \hat{q}(t,x), \omega)$.

Proof. First, we have

$$\hat{\lambda}_i = \sum_j r^{-1}\omega_j \Omega_{ji}\hat{q} \in S^{-1,0} \cdot S^{0,\gamma} \subset S^{-1,\gamma}.$$

Next, we set $Q := \hat{\nu} - \varepsilon G(\omega)\hat{\mu}\hat{U}/(4t)$. We have proved $Q = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle)$ in Lemma 4.63. In general, we fix I with $|I| > 0$ and suppose $Z^J Q = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle)$ whenever $|J| < |I|$. As computed in Lemma 4.63, we have

$$Q_t - Q_r = \hat{\mu}_q Q + \frac{\varepsilon G \hat{\mu}\hat{U}}{4t^2} - \frac{\varepsilon^2 G}{4t^2}(-\frac{1}{2}G\hat{A}\hat{U} + \hat{U}_s)\hat{\mu} = \hat{\mu}_q Q + \varepsilon S^{-2,0}.$$

By (1.18) in Lemma 1.3, we have

$$\begin{aligned} |(\partial_t - \partial_r)Z^I Q| &\lesssim |Z^I(\hat{\mu}_q Q + \varepsilon S^{-2,0})| + \sum_{|J|<|I|} [|Z^J(\hat{\mu}_q Q + \varepsilon S^{-2,0})| + (1+t+r)^{-1}|ZZ^J Q|] \\ &\lesssim |\hat{\mu}_q Z^I Q| + (1+t+r)^{-1}\sum_{|J|=|I|}|Z^J Q| + \sum_{\substack{|K_1|+|K_2|\le|I| \\ |K_2|<|I|}} (|Z^{K_1}\hat{\mu}_q| + t^{-1})|Z^{K_2}Q| + \varepsilon t^{-2+C\varepsilon} \\ &\lesssim |\hat{\mu}_q Z^I Q| + (1+t+r)^{-1}\sum_{|J|=|I|}|Z^J Q| + \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-1} + \varepsilon t^{-2+C\varepsilon}. \end{aligned}$$

The last estimate follows from $\hat{\mu}_q \in S^{0,-2}$ and the induction hypotheses. Since $Q \equiv 0$ near $\tau = (r+t)/2 - R$, and since

$$\int_{(r+t)/2-R}^t \varepsilon\tau^{-2+C\varepsilon}\, d\tau \lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle,$$

we conclude by Gronwall that $Z^I Q = O(\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle)$. So $Q \in \varepsilon S^{-2,1}$.

Since $\hat{\mu}, \hat{U} \in S^{0,0}$ and since $\langle r-t\rangle \lesssim t$ in $\Omega \cap \{r - t < 2R\}$, we have $\hat{\nu} = Q + \varepsilon G(\omega)\hat{\mu}\hat{U}/(4t) \in \varepsilon S^{-2,1} + \varepsilon S^{-1,0} \subset \varepsilon S^{-1,0}$. Moreover, for each I we have

$$\begin{aligned} |Z^I Q_q| &\lesssim |Z^I(\hat{q}_r^{-1}\omega\cdot\partial Q)| \lesssim \sum_{|J|\le|I|} t^{C\varepsilon}|Z^J\partial Q| \\ &\lesssim \sum_{|J|\le|I|} t^{C\varepsilon}|\partial Z^J Q| \lesssim \langle r-t\rangle^{-1} t^{C\varepsilon}\sum_{|J|\le|I|+1}|Z^J Q| \lesssim \varepsilon t^{-2+C\varepsilon}. \end{aligned}$$

Here we use the estimate $\hat{q}_r^{-1} \in S^{0,0}$ which follows from $\hat{q}_r \in S^{0,0}$ and $\hat{q}_r > C^{-1}t^{-C\varepsilon}$. Thus,

$$Q_q = \hat{\nu}_q - \frac{\varepsilon}{4t}G(\omega)(\hat{\mu}_q\hat{U} - 2\hat{A}) \in \varepsilon S^{-2,0}.$$

Since $\hat{\mu}_q\hat{U} \in S^{0,-2}$ and $\hat{A} \in S^{0,-1}$, we conclude that $\hat{\nu}_q \in \varepsilon S^{-1,-1} + \varepsilon S^{-2,0} = \varepsilon S^{-1,-1}$. $\square$

Now we prove that $\hat{q}$ is an approximate optical function.

Proposition 4.66. *We have*

$$g^{\alpha\beta}(\hat{u})\hat{q}_\alpha\hat{q}_\beta \in S^{-2,1}.$$

Proof. Fix $\gamma \in (0,1/2)$ and suppose we have obtained $\hat{\lambda}_i \in S^{-1,\gamma}$ from the pervious lemma. We note that $\hat{q}_t = \frac{1}{2}(\hat{\mu}+\hat{\nu}) \in S^{0,0}$ and $\hat{q}_i = \frac{1}{2}(-\hat{\mu}+\hat{\nu})\omega_i + \hat{\lambda}_i \in S^{0,0}$. Thus,

$$\begin{aligned}
g_0^{\alpha\beta}\hat{q}_\alpha\hat{q}_\beta &= \frac{1}{4}g_0^{00}(\hat{\mu}+\hat{\nu})^2 + \frac{1}{2}g^{0i}(\hat{\mu}+\hat{\nu})((-\hat{\mu}+\hat{\nu})\omega_i + 2\hat{\lambda}_i) \\
&\quad + \frac{1}{4}g_0^{ij}((-\hat{\mu}+\hat{\nu})\omega_i + 2\hat{\lambda}_i)((-\hat{\mu}+\hat{\nu})\omega_j + 2\hat{\lambda}_j) \\
&= \frac{1}{4}G(\omega)\hat{\mu}^2 + \frac{1}{2}g_0^{00}\hat{\mu}\hat{\nu} + \frac{1}{4}g_0^{00}\hat{\nu}^2 + \frac{1}{2}g_0^{0i}(2\hat{\mu}\hat{\lambda}_i + \hat{\nu}^2\omega_i + 2\hat{\nu}\hat{\lambda}_i) \\
&\quad + \frac{1}{4}g_0^{ij}(-\hat{\mu}(2\hat{\nu}\omega_j\omega_i + 2\hat{\lambda}_j\omega_i + 2\hat{\lambda}_i\omega_j) + (\hat{\nu}\omega_i + 2\hat{\lambda}_i)(\hat{\nu}\omega_j + 2\hat{\lambda}_j)).
\end{aligned}$$

Since $\hat{\nu} \in \varepsilon S^{-1,0}$ and $\hat{\lambda}_i \in S^{-1,\gamma}$, we have $\hat{\nu}^2, \hat{\nu}\hat{\lambda}_i, \hat{\lambda}_i\hat{\lambda}_j \in S^{-2,2\gamma}$ and thus

$$\begin{aligned}
g_0^{\alpha\beta}\hat{q}_\alpha\hat{q}_\beta &= \frac{1}{4}G(\omega)\hat{\mu}^2 + \frac{1}{2}(g_0^{00} - g_0^{ij}\omega_i\omega_j)\hat{\mu}\hat{\nu} + g_0^{0i}\hat{\mu}\hat{\lambda}_i - \frac{1}{2}g_0^{ij}\hat{\mu}(\hat{\lambda}_j\omega_i + \hat{\lambda}_i\omega_j) \mod S^{-2,2\gamma} \\
&= \frac{1}{4}G(\omega)\hat{\mu}^2 \mod S^{-1,\gamma}.
\end{aligned}$$

If we replace $(g_0^{\alpha\beta})$ with $(m^{\alpha\beta})$ in the computations, we have

$$-\hat{q}_t^2 + \sum_j \hat{q}_j^2 = -\hat{\mu}\hat{\nu} - \frac{1}{2}m^{ij}\hat{\mu}(\hat{\lambda}_j\omega_i + \hat{\lambda}_i\omega_j) \mod S^{-2,2\gamma} = -\hat{\mu}\hat{\nu} \mod S^{-2,2\gamma}.$$

Here we note that $m^{ij}\hat{\lambda}_j\omega_i = m^{ij}\hat{\lambda}_i\omega_j = \sum_j \omega_j(\hat{q}_j - \omega_j\hat{q}_r) = 0$.

Moreover, note that $\hat{u} = \varepsilon r^{-1}\hat{U} \in \varepsilon S^{-1,0}$. Following the proof of Lemma 4.47 with V replaced by Z, we can prove that $f(\hat{u}) - f(0) - f'(0)\hat{u} \in \varepsilon^2 S^{-2,0}$ for each smooth function f. Thus,

$$\begin{aligned}
g^{\alpha\beta}(\hat{u})\hat{q}_\alpha\hat{q}_\beta &= -\hat{q}_t^2 + \sum_j \hat{q}_j^2 + g_0^{\alpha\beta}\hat{u}\hat{q}_\alpha\hat{q}_\beta + (g^{\alpha\beta}(\hat{u}) - g_0^{\alpha\beta}\hat{u} - m^{\alpha\beta})\hat{q}_\alpha\hat{q}_\beta \\
&= -\hat{\mu}(\hat{\nu} - \frac{\varepsilon}{4r}G(\omega)\hat{\mu}\hat{U}) \mod S^{-2,2\gamma} \\
&= -\hat{\mu}(\hat{\nu} - \frac{\varepsilon}{4t}G(\omega)\hat{\mu}\hat{U}) + \frac{\varepsilon(t-r)}{4rt}G(\omega)\hat{\mu}^2\hat{U} \mod S^{-2,2\gamma} \\
&= \varepsilon S^{-2,1} \mod S^{-2,2\gamma}.
\end{aligned}$$

Since $\gamma \in (0,1/2)$, we have $\varepsilon S^{-2,1} \subset S^{-2,1}$ and $S^{-2,2\gamma} \subset S^{-2,1}$. □

In order to prove that $\hat{u}$ is an approximate solution to (1.1), we need the following lemma.

Lemma 4.67. *For each $\gamma \in (0, 1/2)$, we have*

$$g^{\alpha\beta}(\hat{u})\partial_\alpha\partial_\beta\hat{q} = -r^{-1}\hat{\mu} + \frac{\varepsilon}{2t}G\hat{A}\hat{\mu} \mod S^{-2,\gamma}.$$

Proof. Fix $\gamma \in (0, 1/2)$ and suppose we have obtained $\hat{\lambda}_i \in S^{-1,\gamma}$. First we note that

$$\varepsilon t^{-1}\hat{\nu}_s = \hat{\nu}_t - \hat{\nu}_q\hat{q}_t = \hat{\nu}_t + \hat{\nu}_r - \hat{\nu}\hat{\nu}_q,$$
$$\sum_j(\partial_i\omega_j)\hat{\nu}_{\omega_j} = \hat{\nu}_i - \hat{\nu}_q\hat{q}_i = \hat{\nu}_i - \omega_i\hat{\nu}_r - \hat{\lambda}_i\hat{\nu}_q.$$

Note that

$$\partial_t + \partial_r = \frac{\sum_j \omega_j\Omega_{0j} + S}{r+t}, \qquad \partial_i - \omega_i\partial_r = r^{-1}\sum_j \omega_j\Omega_{ji},$$

and that $\hat{\nu} \in \varepsilon S^{-1,0}$. Thus, we conclude that $\hat{\nu}_t + \hat{\nu}_r, \hat{\nu}_i - \omega_i\hat{\nu}_r \in \varepsilon S^{-2,0}$. Besides, we have $\hat{\nu}\hat{\nu}_q \in \varepsilon^2 S^{-2,-1}$ and $\hat{\lambda}_i\hat{\nu}_q \in \varepsilon S^{-2,-1+\gamma}$. We conclude that $\varepsilon t^{-1}\hat{\nu}_s, \sum_j(\partial_i\omega_j)\hat{\nu}_{\omega_j} \in \varepsilon S^{-2,0}$.

Now, we have

$$\begin{aligned}\hat{q}_{tt} &= \partial_t(\frac{1}{2}(\hat{\mu}+\hat{\nu})) = \frac{1}{2}((\hat{\mu}_q + \hat{\nu}_q)\cdot\frac{1}{2}(\hat{\mu}+\hat{\nu}) + \varepsilon t^{-1}\hat{\mu}_s + \varepsilon t^{-1}\hat{\nu}_s)\\ &= \frac{1}{4}\hat{\mu}_q\hat{\mu} + \frac{1}{4}\hat{\mu}_q\hat{\nu} + \frac{1}{4}\hat{\nu}_q\hat{\mu} + \frac{\varepsilon}{2t}\hat{\mu}_s \mod \varepsilon S^{-2,0} = \frac{1}{4}\hat{\mu}_q\hat{\mu} \mod \varepsilon S^{-1,-1},\end{aligned}$$

$$\begin{aligned}\hat{q}_{ti} &= \partial_i(\frac{1}{2}(\hat{\mu}+\hat{\nu})) = \frac{1}{2}((\hat{\mu}_q+\hat{\nu}_q)\cdot(\frac{1}{2}(\hat{\nu}-\hat{\mu})\omega_i + \hat{\lambda}_i) + \sum_j(\partial_i\omega_j)\hat{\mu}_{\omega_j} + \sum_j(\partial_i\omega_j)\hat{\nu}_{\omega_j})\\ &= -\frac{1}{4}\hat{\mu}\hat{\mu}_q\omega_i \mod S^{-1,-1},\end{aligned}$$

$$\begin{aligned}\hat{q}_{ij} &= \partial_i(\frac{1}{2}(\hat{\nu}-\hat{\mu})\omega_j + \hat{\lambda}_j)\\ &= \frac{1}{2}(\hat{\nu}_q - \hat{\mu}_q)(\frac{1}{2}(\hat{\nu}-\hat{\mu})\omega_i + \hat{\lambda}_i)\omega_j + \frac{1}{2}\sum_k(\hat{\nu}_{\omega_k} - \hat{\mu}_{\omega_k})(\partial_i\omega_k)\omega_j + \frac{1}{2}(\hat{\nu}-\hat{\mu})\partial_i\omega_j + \partial_i\hat{\lambda}_j\\ &= \frac{1}{4}(\hat{\mu}\hat{\mu}_q - \hat{\mu}_q\hat{\nu} - \hat{\nu}_q\hat{\mu})\omega_i\omega_j - \frac{1}{2}\hat{\mu}_q\hat{\lambda}_i\omega_j - \frac{1}{2}\sum_k\hat{\mu}_{\omega_k}(\partial_i\omega_k)\omega_j - \frac{1}{2}\hat{\mu}\partial_i\omega_j + \partial_i\hat{\lambda}_j \mod \varepsilon S^{-2,0}\\ &= \frac{1}{4}\hat{\mu}\hat{\mu}_q\omega_i\omega_j \mod S^{-1,0}.\end{aligned}$$

In the last estimate, we note that $\partial_i\hat{\lambda}_j \in S^{-1,0}$ since for each I,

$$\begin{aligned}|Z^I\partial_i\hat{\lambda}_j| &\lesssim \sum_{|J|\le|I|}|\partial Z^J\hat{\lambda}_j| \lesssim \langle r-t\rangle^{-1}\sum_{|J|\le|I|+1}|Z^J\hat{\lambda}_j|\\ &\lesssim \langle r-t\rangle^{-1}\cdot t^{-1+C\varepsilon}\langle r-t\rangle^{\gamma} \lesssim t^{-1+C\varepsilon}\langle r-t\rangle^{1-\gamma}.\end{aligned}$$

Thus, we have $\partial^2\hat{q} \in S^{0,-2} + S^{-1,-1} = S^{0,-2}$ and

$$g_0^{\alpha\beta}\hat{q}_{\alpha\beta} = \frac{1}{4}G(\omega)\hat{\mu}_q\hat{\mu} \mod S^{-1,0}.$$

In addition,

$$\begin{aligned}\Box\hat{q} &= -(\frac{1}{4}\hat{\mu}_q\hat{\mu} + \frac{1}{4}\hat{\mu}_q\hat{\nu} + \frac{1}{4}\hat{\nu}_q\hat{\mu} + \frac{\varepsilon}{2t}\hat{\mu}_s) + [\frac{1}{4}(\hat{\mu}\hat{\mu}_q - \hat{\mu}_q\hat{\nu} - \hat{\nu}_q\hat{\mu}) - r^{-1}\hat{\mu} + \sum_i \partial_i\hat{\lambda}_i] \mod \varepsilon S^{-2,0}\\ &= -(\frac{1}{2}\hat{\mu}_q\hat{\nu} + \frac{1}{2}\hat{\nu}_q\hat{\mu} + \frac{\varepsilon}{2t}\hat{\mu}_s) - r^{-1}\hat{\mu} + \sum_i \partial_i\hat{\lambda}_i \mod \varepsilon S^{-2,0}.\end{aligned}$$

Since $\sum_i \omega_i\hat{\lambda}_i = 0$, we have $0 = \partial_r(\sum_i \omega_i\hat{\lambda}_i) = \sum_i \omega_i\partial_r\hat{\lambda}_i$. And since $\hat{\lambda}_i \in S^{-1,\gamma}$, we have

$$\sum_i \partial_i\hat{\lambda}_i = \sum_i(\partial_i - \omega_i\partial_r)\hat{\lambda}_i = \sum_{i,j} r^{-1}\omega_i\Omega_{ji}\hat{\lambda}_i \in S^{-2,\gamma}$$

Finally, we have

$$\begin{aligned}g^{\alpha\beta}(\hat{u})\partial_\alpha\partial_\beta\hat{q} &= \Box\hat{q} + g_0^{\alpha\beta}\hat{u}\partial_\alpha\partial_\beta\hat{q} + (g^{\alpha\beta}(\hat{u}) - g_0^{\alpha\beta}\hat{u} - m^{\alpha\beta})\partial_\alpha\partial_\beta\hat{q}\\ &= -(\frac{1}{2}\hat{\mu}_q\hat{\nu} + \frac{1}{2}\hat{\nu}_q\hat{\mu} + \frac{\varepsilon}{2t}\hat{\mu}_s) - r^{-1}\hat{\mu} + \frac{\varepsilon}{4r}G(\omega)\hat{\mu}\hat{\mu}_q\hat{U} \mod S^{-2,\gamma}\\ &= -\frac{1}{2}\hat{\mu}_q \cdot \frac{\varepsilon}{4t}G\hat{\mu}\hat{U} - \frac{1}{2}\hat{\mu} \cdot \frac{\varepsilon}{4t}G(\hat{\mu}_q\hat{U} - 2\hat{A}) + \frac{\varepsilon}{4t}G\hat{A}\hat{\mu} - r^{-1}\hat{\mu}\\ &\quad + \frac{\varepsilon}{4t}G\hat{\mu}\hat{\mu}_q\hat{U} + \frac{\varepsilon(t-r)}{4tr}G\hat{\mu}\hat{\mu}_q\hat{U} \mod S^{-2,\gamma}\\ &= -r^{-1}\hat{\mu} + \frac{\varepsilon}{2t}G\hat{A}\hat{\mu} \mod S^{-2,\gamma}.\end{aligned}$$

□

Now we claim that $\hat{u} = \varepsilon r^{-1}\hat{U}(\varepsilon\ln t - \delta, \hat{q}(t,x), \omega)$ is an approximate solution to (1.1).

Proposition 4.68. *We have*

$$g^{\alpha\beta}(\hat{u})\partial_\alpha\partial_\beta\hat{u} \in \varepsilon S^{-3,0}.$$

Proof. We have

$$\hat{u}_t = \varepsilon r^{-1}(\varepsilon t^{-1}\hat{U}_s + \hat{q}_t\hat{U}_q), \qquad \hat{u}_i = -\varepsilon r^{-2}\omega_i\hat{U} + \varepsilon r^{-1}(\hat{U}_q\hat{q}_i + \sum_k \hat{U}_{\omega_k}\partial_i\omega_k).$$

By Lemma 4.64, we have $\partial_s^b\partial_\omega^c\hat{U} \in \varepsilon^{-b}S^{0,0}$. Thus we have

$$\begin{aligned}\hat{u}_{tt} &= \varepsilon r^{-1}(-\varepsilon t^{-2}\hat{U}_s + \varepsilon^2 t^{-2}\hat{U}_{ss} + 2\varepsilon t^{-1}\hat{q}_t\hat{U}_{sq} + \hat{q}_{tt}\hat{U}_q + \hat{q}_t^2\hat{U}_{qq})\\ &= \varepsilon r^{-1}(2\varepsilon t^{-1}\hat{q}_t\hat{U}_{sq} + \hat{q}_{tt}\hat{U}_q + \hat{q}_t^2\hat{U}_{qq}) \mod \varepsilon S^{-3,0}\\ &= \varepsilon r^{-1}(\hat{q}_{tt}\hat{U}_q + \hat{q}_t^2\hat{U}_{qq}) \mod \varepsilon S^{-2,-1},\end{aligned}$$

$$\begin{aligned}
\hat{u}_{ti} &= -\varepsilon r^{-2}\omega_i(\varepsilon t^{-1}\hat{U}_s + \hat{q}_t\hat{U}_q) \\
&\quad + \varepsilon r^{-1}(\varepsilon t^{-1}\hat{U}_{sq}\hat{q}_i + \varepsilon t^{-1}\sum_k \hat{U}_{s\omega_k}\partial_i\omega_k + \hat{q}_{ti}\hat{U}_q + \hat{q}_t\hat{U}_{qq}\hat{q}_i + \hat{q}_t\sum_k \hat{U}_{q\omega_k}\partial_i\omega_k) \\
&= \varepsilon r^{-1}(\hat{q}_{ti}\hat{U}_q + \hat{q}_t\hat{U}_{qq}\hat{q}_i) \mod \varepsilon S^{-2,-1}, \\
\hat{u}_{ij} &= -\varepsilon\partial_i(r^{-2}\omega_j)\hat{U} - \varepsilon r^{-2}\omega_j(\hat{U}_q\hat{q}_i + \sum_k \hat{U}_{\omega_k}\partial_i\omega_k) - \varepsilon r^{-2}\omega_i(\hat{U}_q\hat{q}_j + \sum_k \hat{U}_{\omega_k}\partial_j\omega_k) \\
&\quad + \varepsilon r^{-1}[\hat{U}_{qq}\hat{q}_i\hat{q}_j + \sum_k \hat{U}_{q\omega_k}(\partial_i\omega_k)\hat{q}_j + \hat{U}_q\hat{q}_{ij} \\
&\qquad + \sum_k(\hat{U}_{\omega_k q}\hat{q}_i\partial_j\omega_k + \hat{U}_{\omega_k}\partial_i\partial_j\omega_k) + \sum_{k,k'}\hat{U}_{\omega_k\omega_{k'}}(\partial_i\omega_k)(\partial_j\omega_{k'})] \\
&= -\varepsilon r^{-2}\omega_j\hat{U}_q\hat{q}_i - \varepsilon r^{-2}\omega_i\hat{U}_q\hat{q}_j \\
&\quad + \varepsilon r^{-1}[\hat{U}_{qq}\hat{q}_i\hat{q}_j + \sum_k \hat{U}_{q\omega_k}((\partial_i\omega_k)\hat{q}_j + (\partial_j\omega_k)\hat{q}_i) + \hat{U}_q\hat{q}_{ij}] \mod \varepsilon S^{-3,0} \\
&= \varepsilon r^{-1}(\hat{U}_{qq}\hat{q}_i\hat{q}_j + \hat{U}_q\hat{q}_{ij}) \mod \varepsilon S^{-2,-1}.
\end{aligned}$$

Since $g^{\alpha\beta}(\hat{u}) - m^{\alpha\beta} = g_0^{\alpha\beta}\hat{u} \mod \varepsilon^2 S^{-2,0} \in \varepsilon S^{-1,0}$, we have

$$\begin{aligned}
&g^{\alpha\beta}(\hat{u})\partial_\alpha\partial_\beta\hat{u} = \Box\hat{u} + (g^{\alpha\beta}(\hat{u}) - m^{\alpha\beta})\partial_\alpha\partial_\beta\hat{u} \\
&= -\varepsilon r^{-1}(2\varepsilon t^{-1}\hat{q}_t\hat{U}_{sq} + \hat{q}_{tt}\hat{U}_q + \hat{q}_t^2\hat{U}_{qq}) - 2\varepsilon r^{-2}\hat{U}_q\hat{q}_r \\
&\quad + \varepsilon r^{-1}\sum_i[\hat{U}_{qq}\hat{q}_i^2 + \sum_k 2\hat{U}_{q\omega_k}(\partial_i\omega_k)\hat{q}_i + \hat{U}_q\hat{q}_{ii}] \\
&\quad + (g^{\alpha\beta}(\hat{u}) - m^{\alpha\beta})\cdot\varepsilon r^{-1}(\hat{q}_{\alpha\beta}\hat{U}_q + \hat{q}_\alpha\hat{q}_\beta\hat{U}_{qq}) \mod \varepsilon S^{-3,0} \\
&= -\varepsilon^2(tr)^{-1}\hat{q}_tG\hat{A}\hat{U}_q - 2\varepsilon r^{-2}\hat{U}_q\hat{q}_r + \varepsilon r^{-1}\sum_i\sum_k 2\hat{U}_{q\omega_k}(\partial_i\omega_k)(\hat{\lambda}_i + \omega_i\hat{q}_r) \\
&\quad + \varepsilon r^{-1}(g^{\alpha\beta}(\hat{u})\hat{q}_{\alpha\beta}\hat{U}_q + g^{\alpha\beta}(\hat{u})\hat{q}_\alpha\hat{q}_\beta\hat{U}_{qq}) \mod \varepsilon S^{-3,0} \\
&= -\varepsilon^2(rt)^{-1}\hat{q}_tG\hat{A}\hat{U}_q - 2\varepsilon r^{-2}\hat{U}_q\hat{q}_r - \varepsilon r^{-2}\hat{\mu}\hat{U}_q + \varepsilon^2(2tr)^{-1}G\hat{A}\hat{\mu}\hat{U}_q \mod \varepsilon S^{-3,0} \\
&= -\frac{1}{2}\varepsilon^2 r^{-2}\hat{\nu}G\hat{A}\hat{U}_q - \varepsilon r^{-2}\hat{\nu}\hat{U}_q \mod \varepsilon S^{-3,0} \in \varepsilon S^{-3,0}.
\end{aligned}$$

In the third equality, we note that

$$\begin{aligned}
\varepsilon r^{-1}[g^{\alpha\beta}(\hat{u})\hat{q}_{\alpha\beta} + r^{-1}\hat{\mu} - \frac{\varepsilon}{2t}G\hat{A}\hat{\mu}]\hat{U}_q &\in \varepsilon S^{-1,0}\cdot S^{-2,\gamma}\cdot S^{0,-1} \subset \varepsilon S^{-3,0}, \\
\varepsilon r^{-1}g^{\alpha\beta}(\hat{u})\hat{q}_\alpha\hat{q}_\beta\hat{U}_{qq} &\in \varepsilon S^{-1,0}\cdot S^{-2,1}\cdot S^{0,-2} \subset \varepsilon S^{-3,0}
\end{aligned}$$

and that

$$\begin{aligned}
\varepsilon r^{-1}\sum_i\sum_k 2\hat{U}_{q\omega_k}(\partial_i\omega_k)(\hat{\lambda}_i + \omega_i\hat{q}_r) &= \varepsilon r^{-1}\sum_i\sum_k 2\hat{U}_{q\omega_k}(\partial_i\omega_k)\hat{\lambda}_i + \varepsilon r^{-1}\sum_k 2\hat{U}_{q\omega_k}(\partial_r\omega_k)\hat{q}_r \\
&\in \varepsilon S^{-1,0}\cdot S^{0,-1}\cdot S^{-1,0}\cdot S^{-1,\gamma} + 0 \subset \varepsilon S^{-3,0}.
\end{aligned}$$

□

4.7.4 Approximation of the optical function

We set $p(t,x) := F(q(t,x),\omega) - \hat{q}(t,x)$ in Ω, where $q(t,x)$ is the optical function constructed in Section 4.3.

Proposition 4.69. *Fix a constant $\gamma \in (0,1)$. Then, for $\varepsilon \ll_\gamma 1$, at each $(t,x) \in \Omega$ such that $|r-t| \lesssim t^\gamma$, we have $|p(t,x)| \lesssim_\gamma t^{-1+C\varepsilon}\langle r-t\rangle$.*

Proof. It is clear that $p \equiv 0$ in the region $\{r-t > R\}$. In $\Omega \cap \{r-t < 2R\}$, by setting $s = \varepsilon \ln t - \delta$ we have

$$\begin{aligned} p_t - p_r &= F_q\mu(s, q(t,x),\omega) - \hat{\mu}(s,\hat{q}(t,x),\omega) \\ &= [F_q\mu(s,q(t,x),\omega) - \hat{\mu}(s, F(q(t,x),\omega),\omega)] + [\hat{\mu}(s,F(q(t,x),\omega),\omega) - \hat{\mu}(s,\hat{q}(t,x),\omega)] \\ &=: \mathcal{R}_1 + \mathcal{R}_2. \end{aligned} \tag{4.80}$$

Since $\hat{A}(F(q,\omega),\omega) = A(q,\omega)$, we have

$$\begin{aligned} \mathcal{R}_1 &= -\frac{2}{A_1(q(t,x),\omega)}\widetilde{V}_1(s,q(t,x),\omega)\exp(-\frac{1}{2}G(\omega)A(q(t,x),\omega)s) \\ &\quad + 2\exp(-\frac{1}{2}G(\omega)\hat{A}(F(q(t,x),\omega),\omega)s) \\ &= (-\frac{2}{A_1(q(t,x),\omega)}\widetilde{V}_1(s,q(t,x),\omega) + 2)\exp(-\frac{1}{2}G(\omega)A(q(t,x),\omega)s) \\ &= -\frac{2}{A_1(q(t,x),\omega)}(\widetilde{V}_1(s,q(t,x),\omega) - A_1(q(t,x),\omega))\exp(-\frac{1}{2}G(\omega)A(q(t,x),\omega)s). \end{aligned} \tag{4.81}$$

By Proposition 4.55, we have

$$|\mathcal{R}_1| \lesssim |\widetilde{V}_1(s,q(t,x),\omega) - A_1(q(t,x),\omega)|\exp(C\langle q\rangle^{-1+C\varepsilon}s) \lesssim t^{-1+C\varepsilon}.$$

Moreover,

$$\begin{aligned} |\mathcal{R}_2| &= |\int_{\hat{q}}^{F(q,\omega)} \hat{\mu}_\rho(s,\rho,\omega)\ d\rho| \lesssim \left|\int_{\hat{q}}^{F(q,\omega)} \langle\rho\rangle^{-2+C\varepsilon}s|\hat{\mu}(s,\rho,\omega)|\ d\rho\right| \\ &\lesssim (\varepsilon\ln t - \delta)|p| \cdot \max_{\kappa\in[0,1]}\big[\langle\hat{q}+\kappa p\rangle^{-2+C\varepsilon}\exp(-\frac{1}{2}G(\omega)\hat{A}(\hat{q}+\kappa p,\omega)s)\big]. \end{aligned}$$

We now use a continuity argument to end the proof. Fix $(t,x) \in \Omega\cap\{r-t<2R,\ |r-t| \lesssim t^\gamma\}$. Suppose that for some $t_0 \in [(r+t)/2-R, t)$, we have

$$|p(\tau, r+t-\tau,\omega)| \le \frac{\delta}{10\varepsilon\ln\tau}, \qquad \forall \tau \in [(r+t)/2-R, t_0]. \tag{4.82}$$

Note that (4.82) holds for $t_0 = (r+t)/2-R$, since $p((r+t)/2-R,(r+t)/2+R,\omega) = 0$. At $(\tau, r+t-\tau,\omega)$ for $(r+t)/2-R \le \tau \le t_0$ and for each $\kappa\in[0,1]$, we have

$$\langle\hat{q}+\kappa p\rangle \sim 1 + |\hat{q}+\kappa p| \ge 1 + |\hat{q}| - |\kappa p| \ge 1+|\hat{q}| - \frac{1}{10} \gtrsim \langle\hat{q}\rangle.$$

In the second last inequality we note that $\tau > \exp(\delta/\varepsilon)$, so $\varepsilon \ln \tau > \delta$ and thus $|p| \leq 1/10$. Moreover,

$$\exp(-\frac{1}{2}G(\omega)(\hat{A}(\hat{q}+\kappa p,\omega)-\hat{A}(\hat{q},\omega))s) \lesssim \exp(C\kappa|p|s) \lesssim \exp(\delta/10) \lesssim 1.$$

In conclusion, at $(\tau, r+t-\tau,\omega)$ for $(r+t)/2-R \leq \tau \leq t_0$, we have

$$\begin{aligned}|\mathcal{R}_2| &\lesssim (\varepsilon\ln\tau-\delta)[|p|\langle\hat{q}\rangle^{-2+C\varepsilon}\exp(-\frac{1}{2}G(\omega)\hat{A}(\hat{q},\omega)s)](\tau,r+t-\tau,\omega)\\ &\lesssim (\varepsilon\ln\tau-\delta)[|p|\langle\hat{q}\rangle^{-2+C\varepsilon}(-\hat{\mu})](\tau,r+t-\tau,\omega).\end{aligned}$$

If we fix any $t_1 \in [(r+t)/2-R, t_0]$, then

$$\begin{aligned}\int_{(r+t)/2-R}^{t_1}(\varepsilon\ln\tau-\delta)\langle\hat{q}\rangle^{-2+C\varepsilon}(-\hat{\mu})(\tau,r+t-\tau,\omega)\ d\tau &\lesssim \varepsilon\ln t_1\int_{(r+t)/2-R}^{t_1}\langle z\rangle^{-2+C\varepsilon}(-\dot{z})\ d\tau\\ &\lesssim \varepsilon \ln t_1\end{aligned}$$

and

$$\begin{aligned}\int_{(r+t)/2-R}^{t_1}|\mathcal{R}_1|(\tau,r+t-\tau,\omega)\ d\tau &\lesssim \int_{(r+t)/2-R}^{t_1}\tau^{-1+C\varepsilon}\ d\tau\\ &\lesssim ((r+t)/2-R)^{-1+C\varepsilon}(t_1-(r+t)/2+R)\\ &\lesssim t_1^{-1+C\varepsilon}\langle r-t\rangle.\end{aligned}$$

Here we recall that $[(r+t)/2-R] \sim t \sim t_1$. And since $p=0$ at $\tau=(r+t)/2-R$, by applying the Gronwall's inequality to $p_t - p_r = \mathcal{R}_1+\mathcal{R}_2$, we conclude that

$$\begin{aligned}|p(t_1,r+t-t_1,\omega)| \lesssim t_1^{-1+C\varepsilon}\langle r-t\rangle\cdot\exp(C\varepsilon\ln(Ct_1)) &\lesssim t_1^{-1+C\varepsilon}\langle r-t\rangle,\\ &\forall t_1\in[(r+t)/2-R,t_0].\end{aligned} \tag{4.83}$$

For $\varepsilon \ll_\gamma 1$ (where ε does not depend on (t,x)) and $t_1 \in [(r+t)/2-R,t_0]$, we have $|r-t| \lesssim t^\gamma \sim t_1^\gamma$ and thus

$$t_1^{-1+C\varepsilon}\langle r-t\rangle \lesssim t_1^{-1+\gamma+C\varepsilon} \leq t_1^{(\gamma-1)/2} \leq \delta/(20\varepsilon\ln t_1).$$

And since $\tau \mapsto \varepsilon(\ln\tau)p(\tau,r+t-\tau,\omega)$ is a continuous function, (4.82) holds with t_0 replaced by some $t_0' > t_0$. By the continuity argument we conclude that $|p(t,x)| \lesssim t^{-1+C\varepsilon}\langle r-t\rangle$. The constants here do not depend on (t,x). □

Next we consider Z^Ip. We need the following lemma.

Lemma 4.70. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be defined as in* (4.80)*. Then, we have $\mathcal{R}_1 \in S^{-1,0}$ and for $|I|>0$ we have*

$$|Z^I\mathcal{R}_2| \lesssim \langle r-t\rangle^{-2}t^{C\varepsilon}\sum_{|J|<|I|}|Z^Jp| + |\hat{\mu}_qZ^Ip|.$$

Proof. By (4.81), Remark 4.55.1 and Lemma 4.54, and since $A_1 < -1$ everywhere, we have $\mathcal{R}_1 = \mathfrak{R}_{0,0} \cdot \mathfrak{R}_{-1,0} \cdot \mathfrak{R}_{0,0} = \mathfrak{R}_{-1,0} \in S^{-1,0}$.

To estimate $\mathcal{R}_2$, we fix an arbitrary multiindex I with $|I| > 0$. By the chain rule and Leibniz's rule, we can express $Z^I\hat{\mu}(s, F(q(t,x),\omega),\omega) - Z^I\hat{\mu}(s,\hat{q}(t,x),\omega)$ as a linear combination of terms of the form

$$\begin{aligned}[(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,F(q,\omega),\omega)\cdot\prod_{i=1}^a Z^{I_i}(F(q,\omega)) - (\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,\hat{q},\omega)\cdot\prod_{i=1}^a Z^{I_i}\hat{q}]\\ \cdot\prod_{j=1}^b Z^{J_j}(\varepsilon\ln t-\delta)\cdot\prod_{l=1}^c Z^{K_{l,1}}\omega_l\cdots Z^{K_{l,c_l}}\omega_l\end{aligned} \tag{4.84}$$

where $|I_*|, |J_*|, |K_{*,*}|$ are nonzero, and the sum of all these multiindices is $|I|$. The only term with $|I_j| = |I|$ for some j is $\hat{\mu}_q Z^I p$, so from now on we assume $|I_j| < |I|$ for each j in (4.84). Here the second row in (4.84) is $O(\varepsilon^b)$. The first row is equal to the sum of

$$[(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,F(q,\omega),\omega) - (\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,\hat{q},\omega)]\cdot\prod_{i=1}^a Z^{I_i}(F(q,\omega)) \tag{4.85}$$

and for each $j = 1, 2, \dots, a$

$$(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,\hat{q},\omega)\cdot\prod_{i=1}^{j-1} Z^{I_i}(F(q,\omega))\cdot Z^{I_j}p\cdot\prod_{i=j+1}^a Z^{I_i}\hat{q}. \tag{4.86}$$

Since $|I| > 0$, we must have $a > 0$ if (4.86) does appear.

To control (4.85) and (4.86), we first recall from Lemma 4.64 and Proposition 4.60 that

$$Z^{I_*}(\hat{q}(t,x), F(q(t,x),\omega)) = O(\langle r-t\rangle t^{C\varepsilon});$$

$$(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,\hat{q},\omega) = O(\langle\hat{q}\rangle^{-a-1+C\varepsilon}t^{C\varepsilon}) = O(\langle r-t\rangle^{-a-1}t^{C\varepsilon}), \qquad \text{when } a+b+|c|>0.$$

It follows immediately that (4.86) is $O(\sum_{|J|<|I|} t^{C\varepsilon}\langle r-t\rangle^{-2}|Z^J p|)$. In addition, we have $\langle F(q,\omega)\rangle/\langle r-t\rangle \sim \langle q\rangle/\langle r-t\rangle = t^{O(\varepsilon)}$ and $\langle\hat{q}\rangle/\langle r-t\rangle = t^{O(\varepsilon)}$. Thus, for each $\tau\in[0,1]$,

$$\langle\tau\hat{q} + (1-\tau)F(q,\omega)\rangle \sim \tau\langle\hat{q}\rangle + (1-\tau)\langle F(q,\omega)\rangle \gtrsim \langle r-t\rangle t^{-C\varepsilon}. \tag{4.87}$$

Then, we have

$$\begin{aligned}&|(\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,F(q,\omega),\omega) - (\partial_s^b\partial_q^a\partial_\omega^c\hat{\mu})(s,\hat{q},\omega)| = |\int_{\hat{q}}^{F(q,\omega)}(\partial_s^b\partial_q^{a+1}\partial_\omega^c\hat{\mu})(s,\rho,\omega)\,d\rho|\\ &\lesssim |\int_{\hat{q}}^{F(q,\omega)}\langle\rho\rangle^{-2-a+C\varepsilon}\exp(Cs)\,d\rho| \lesssim |p(t,x)|t^{C\varepsilon}\langle r-t\rangle^{-a-2}.\end{aligned}$$

Thus, (4.85) is $O(|p|t^{C\varepsilon}\langle r-t\rangle^{-2})$.

In conclusion, for $|I| > 0$ we have

$$|Z^I \mathcal{R}_2| \lesssim \langle r-t\rangle^{-2} t^{C\varepsilon} \sum_{|J|<|I|} |Z^J p| + |\hat{\mu}_q Z^I p|.$$

□

Proposition 4.71. *Fix a constant $\gamma \in (0, 1/2)$ and a large integer N. Then, for $\varepsilon \ll_{\gamma,N} 1$, at each $(t, x) \in \Omega$ such that $|r-t| \lesssim t^\gamma$, we have $|Z^I p(t, x)| \lesssim_\gamma t^{-1+C\varepsilon}\langle r-t\rangle$ for each $|I| \leq N$.*

Proof. We prove by induction on $|I|$. The case $|I| = 0$ has been proved in Proposition 4.69. Fix a multiindex I with $|I| > 0$, and suppose that we have proved the proposition for all $|J| < |I|$. By Lemma 1.3, we have

$$(\partial_t - \partial_r)Z^I p = Z^I(p_t - p_r) + \sum_{|J|<|I|} [f_0 Z^J(p_t - p_r) + \sum_i f_0(\partial_i + \omega_i\partial_t)Z^J p].$$

By Lemma 4.70 and our induction hypotheses, in $\Omega \cap \{r - t < 2R,\ |r - t| \lesssim t^\gamma\}$ we have

$$\begin{aligned}
|(\partial_t - \partial_r)Z^I p| &\lesssim |Z^I(\mathcal{R}_1 + \mathcal{R}_2)| + \sum_{|J|<|I|} |Z^J(\mathcal{R}_1 + \mathcal{R}_2)| + t^{-1}|ZZ^J p|] \\
&\lesssim t^{-1+C\varepsilon} + \langle r-t\rangle^{-2} t^{C\varepsilon} \sum_{|J|<|I|} |Z^J p| + |\hat{\mu}_q Z^I p| + \sum_{|J|\leq|I|} t^{-1}|Z^J p| \\
&\lesssim t^{-1+C\varepsilon} + \langle r-t\rangle^{-2} \cdot t^{-1+C\varepsilon}\langle r-t\rangle + |\hat{\mu}_q Z^I p| + \sum_{|J|=|I|} t^{-1}|Z^J p| + t^{-2+C\varepsilon}\langle r-t\rangle \\
&\lesssim t^{-1+C\varepsilon} + |\hat{\mu}_q Z^I p| + \sum_{|J|=|I|} t^{-1}|Z^J p|.
\end{aligned}$$

The integral of $|\hat{\mu}_q|$ and t^{-1} along a characteristic $(\tau, r + t - \tau, \omega)$, $\tau \in [(r + t)/2 - R, t]$, is $O(\varepsilon \ln t + 1)$. Moreover,

$$\int_{(r+t)/2-R}^t \tau^{-1+C\varepsilon}\, d\tau \lesssim ((r + t)/2 - R)^{-1+C\varepsilon}((t - r)/2 + R) \lesssim t^{-1+C\varepsilon}\langle r-t\rangle.$$

Since $Z^I p \equiv 0$ in the region $\Omega \cap \{r - t > R\}$, by Gronwall's inequality we conclude that $|Z^I p| \lesssim t^{-1+C\varepsilon}\langle r-t\rangle$. □

4.7.5 Approximation of the solution to (1.1)

We can now discuss the difference $u - \hat{u}$ where u is a solution to (1.1) and $\hat{u}$ is defined in Section 4.7.2. Again, we fix a point in region $\Omega \cap \{|r - t| \lesssim t^\gamma\}$ for some $0 < \gamma < 1$. Note that

$$\begin{aligned} u-\hat{u} &= \varepsilon r^{-1}U(s,q(t,x),\omega)-\varepsilon r^{-1}\hat{U}(s,\hat{q}(t,x),\omega)\\ &= \varepsilon r^{-1}U(s,q(t,x),\omega)-\varepsilon r^{-1}\hat{U}(s,F(q(t,x),\omega),\omega)\\ &\quad +\varepsilon r^{-1}\hat{U}(s,F(q(t,x),\omega),\omega)-\varepsilon r^{-1}\hat{U}(s,\hat{q}(t,x),\omega)\\ &=: \mathcal{R}_3+\mathcal{R}_4. \end{aligned}$$

Now we estimate $\mathcal{R}_3$ and $\mathcal{R}_4$ separately.

Lemma 4.72. *Fix a constant* $0<\gamma<1$ *and a large integer* N*. Then, for* $\varepsilon \ll_{\gamma,N} 1$*, at each* $(t,x)\in\Omega$ *such that* $|r-t|\lesssim t^{\gamma}$*, we have* $|Z^I\mathcal{R}_3|\lesssim_\gamma \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ *for each* $|I|\le N$*.*

Proof. As computed in Lemma 4.59, by change of variables we can prove that

$$\hat{U}(s,F(q(t,x),\omega),\omega)=\widetilde{U}(s,q(t,x),\omega).$$

Thus,

$$\mathcal{R}_3=\varepsilon r^{-1}(U(s,q(t,x),\omega)-\widetilde{U}(s,q(t,x),\omega)).$$

By (4.72), we have $|U-\widetilde{U}|\lesssim \langle q\rangle t^{-1+C\varepsilon}$ at $(s,q,\omega)=(\varepsilon\ln t-\delta,q(t,x),\omega)$, so

$$|\mathcal{R}_3|\lesssim \varepsilon t^{-2+C\varepsilon}\langle q\rangle \lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle.$$

Next we fix a multiindex I with $|I|>0$. Then, $Z^I\mathcal{R}_3$ can be expressed as a linear combination of terms of the form

$$Z^{I'}(\varepsilon r^{-1})\cdot(\partial_s^b\partial_q^a\partial_\omega^c(U-\widetilde{U}))(s,q,\omega)\cdot\prod_{i=1}^a Z^{I_i}q\cdot\prod_{i=1}^b Z^{J_i}s\cdot\prod_{i=1}^c Z^{K_i}\omega. \tag{4.88}$$

The sum of all the $|I'|,|I_*|,|J_*|,|K_*|$ is $|I|$. If $a\ge 1$, by (4.71), we have

$$|\partial_s^b\partial_q^{a-1}\partial_\omega^c(U_q-\widetilde{U}_q)|\lesssim \varepsilon^{-b}\langle q\rangle^{1-a}t^{-1+C\varepsilon}.$$

Thus, the terms (4.88) with $a>0$ have an upper bound

$$\varepsilon t^{-1}\cdot\varepsilon^{-b}\langle q\rangle^{1-a}t^{-1+C\varepsilon}\cdot(\langle q\rangle t^{C\varepsilon})^a\cdot\varepsilon^b\lesssim \varepsilon\langle q\rangle t^{-2+C\varepsilon}\lesssim\varepsilon\langle r-t\rangle t^{-2+C\varepsilon}.$$

Moreover, by (4.72), we have

$$|\partial_s^b\partial_\omega^c(U-\widetilde{U})|\lesssim\varepsilon^{-b}\langle q\rangle t^{-1+C\varepsilon}.$$

Thus, the terms (4.88) with $a=0$ have an upper bound

$$\varepsilon t^{-1}\cdot\varepsilon^{-b}\langle q\rangle t^{-1+C\varepsilon}\cdot\varepsilon^b\lesssim\varepsilon\langle q\rangle t^{-2+C\varepsilon}\lesssim\varepsilon\langle r-t\rangle t^{-2+C\varepsilon}.$$

In conclusion, $|Z^I\mathcal{R}_3|\lesssim\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ for $|I|>0$. □

Lemma 4.73. *Fix a constant* $0<\gamma<1$ *and a large integer* N. *Then, for* $\varepsilon\ll_{\gamma,N}1$, *at each* $(t,x)\in\Omega$ *such that* $|r-t|\lesssim t^{\gamma}$, *we have* $|Z^I\mathcal{R}_4|\lesssim_\gamma \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ *for each* $|I|\leq N$.

Proof. First we consider the case $|I|=0$. We have

$$\begin{aligned}|\mathcal{R}_4|&\lesssim \varepsilon r^{-1}|\hat{U}(s,F(q(t,x),\omega),\omega)-\varepsilon r^{-1}\hat{U}(s,\hat{q}(t,x),\omega)|\\&\lesssim \varepsilon t^{-1}|\int_{\hat{q}}^{F(q,\omega)}|\hat{U}_\rho(s,\rho,\omega)|\ d\rho|\lesssim \varepsilon t^{-1}|\int_{\hat{q}}^{F(q,\omega)}(|\partial_\rho A_2|+|A_2||\partial_\rho A|)t^{C\varepsilon}\ d\rho|\\&\lesssim \varepsilon\langle r-t\rangle^{-2}t^{-1+C\varepsilon}|p(t,x)|\lesssim \varepsilon t^{-2+C\varepsilon}\langle r-t\rangle^{-1}.\end{aligned}$$

In the second last inequality, we apply (4.87) to see that the integrand is $O(\langle r-t\rangle^{-2}t^{C\varepsilon})$. In the last inequality we apply Proposition 4.69.

In general, fix a multiindex I with $|I|>0$. Then, we can express $Z^I\mathcal{R}_4$ as a linear combination of terms of the form

$$\begin{aligned}&[(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,F(q,\omega),\omega)\cdot\prod_{i=1}^a Z^{I_i}(F(q,\omega))-(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,\hat{q},\omega)\cdot\prod_{i=1}^a Z^{I_i}\hat{q}]\\&\qquad\qquad\cdot Z^{I'}(\varepsilon r^{-1})\cdot\prod_{j=1}^b Z^{J_j}(\varepsilon\ln t-\delta)\cdot\prod_{l=1}^c Z^{K_l}\omega\end{aligned}\tag{4.89}$$

where the sum of all these multiindices is $|I|$. The estimates for such terms are similar to those for (4.84). The second row is $O(\varepsilon^{b+1}t^{-1+C\varepsilon})$ while the first row is equal to the sum of

$$[(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,F(q,\omega),\omega)-(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,\hat{q},\omega)]\cdot\prod_{i=1}^a Z^{I_i}(F(q,\omega))\tag{4.90}$$

and for each $j=1,2,\dots,a$

$$(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,\hat{q},\omega)\cdot\prod_{i=1}^{j-1}Z^{I_i}(F(q,\omega))\cdot Z^{I_j}p\cdot\prod_{i=j+1}^{a}Z^{I_i}\hat{q}.\tag{4.91}$$

Since $|I|>0$, we must have $a+b+|c|>0$ if (4.91) appears.

Note that

$$Z^{I_*}(\hat{q},F(q,\omega))=O(\langle r-t\rangle t^{C\varepsilon}),\qquad Z^{I_*}p=O(t^{-1+\gamma+C\varepsilon});$$

$$(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,\hat{q},\omega)=O(\varepsilon^{-b}\langle\hat{q}\rangle^{1-a+C\varepsilon}t^{C\varepsilon})=O(\varepsilon^{-b}\langle r-t\rangle^{1-a}t^{C\varepsilon}),\qquad \text{when } a+b+|c|>0.$$

So (4.91) has an upper bound

$$\varepsilon^{-b}\langle r-t\rangle^{1-a}t^{C\varepsilon}\cdot(\langle r-t\rangle t^{C\varepsilon})^{a-1}\cdot t^{-1+C\varepsilon}\langle r-t\rangle\lesssim\varepsilon^{-b}t^{-1+C\varepsilon}\langle r-t\rangle.$$

Besides, by applying Proposition 4.60 and (4.87), we have

$$|(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,F(q,\omega),\omega)-(\partial_s^b\partial_q^a\partial_\omega^c\hat{U})(s,\hat{q},\omega)|\lesssim|\int_{\hat{q}}^{F(q,\omega)}|\partial_s^b\partial_q^{a+1}\partial_\omega^c\hat{U}|(s,\rho,\omega)\ d\rho|$$

$$\lesssim|\int_{\hat{q}}^{F(q,\omega)}\langle\rho\rangle^{-a-1+C\varepsilon}t^{C\varepsilon}\ d\rho|\lesssim|p(t,x)|\cdot\langle r-t\rangle^{-a-1+C\varepsilon}t^{C\varepsilon}\lesssim t^{-1+C\varepsilon}\cdot\langle r-t\rangle^{-a}.$$

In conclusion, (4.90) has an upper bound

$$t^{-1+C\varepsilon}\langle r-t\rangle^{-a}\cdot(\langle r-t\rangle t^{C\varepsilon})^a\lesssim t^{-1+C\varepsilon}.$$

Combine all the estimates above and we conclude that $|Z^I\mathcal{R}_4|\lesssim\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$. □

We thus conclude the following approximation result.

Proposition 4.74. *Fix a constant $0<\gamma<1$ and a large integer N. Then, for $\varepsilon\ll_{\gamma,N}1$, at each $(t,x)\in\Omega$ such that $|r-t|\lesssim t^\gamma$, we have $|Z^I(u-\widetilde{u})|\lesssim_\gamma\varepsilon t^{-2+C\varepsilon}\langle r-t\rangle$ for each $|I|\le N$.*

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