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Cycle Spaces of Infinite Dimensional Flag Domains

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Abstract

Let G be a complex simple direct limit group, specifically $SL(\infty;\mathbb{C})$, $SO(\infty;\mathbb{C})$ or $Sp(\infty;\mathbb{C})$. Let $\mathcal{F}$ be a (generalized) flag in $\mathbb{C}^\infty$. If G is $SO(\infty;\mathbb{C})$ or $Sp(\infty;\mathbb{C})$ we suppose further that $\mathcal{F}$ is isotropic. Let $\mathcal{Z}$ denote the corresponding flag manifold; thus $\mathcal{Z} = G/Q$ where Q is a parabolic subgroup of G. In a recent paper [7] we studied real forms G_0 of G and properties of their orbits on $\mathcal{Z}$. Here we concentrate on open G_0–orbits $D \subset \mathcal{Z}$. When G_0 is of hermitian type we work out the complete G_0–orbit structure of flag manifolds dual to the bounded symmetric domain for G_0. Then we develop the structure of the corresponding cycle spaces $\mathcal{M}_D$. Finally we study the real and quaternionic analogs of these theories. All this extends results from the finite dimensional cases on the structure of hermitian symmetric spaces and cycle spaces (in chronological order: [12], [17], [14], [15], [18], [16], [4], [5], [19], [6]).

1 Introduction.

The object of this paper is the study of certain infinite dimensional bounded symmetric domains and the related cycle spaces for open real group orbits on complex flag manifolds. The cycle space theory is well understood in the finite dimensional setting (in chronological order: [12], [17], [14], [15], [18], [16], [4], [5], [19], [6]). Here we initiate its extension to infinite dimensions. Specifically, we look at the action of real reductive direct limit groups, G_0 such as $SL(\infty;\mathbb{R})$, $SO(\infty,\infty)$, $Sp(\infty,q)$, or $Sp(\infty;\mathbb{R})$, on a class of direct limit complex flag manifolds $\mathcal{Z} = G/Q$, where G is the complexification of G_0. While the classical finite dimensional setting [12] is the guide, the results in infinite dimensions are much more delicate, and often different. See [7], as indicated below. In fact there are even stringent requirements for the existence of open G_0–orbits on $\mathcal{Z}$. In all cases where G_0 is the group of an hermitian symmetric space we work out a complete structure theory for the cycle spaces of open orbits in our class of flag manifolds. That structure is explicit in terms of the bounded symmetric domains of the G_0.

In Section 2 we review the basic facts about our class of infinite dimensional complex Lie groups, their construction, their flag manifolds, and their real forms. We note [7] that every G_0–orbit on $\mathcal{Z}$ is infinite dimensional, and we describe just when the number of G_0–orbits on $\mathcal{Z}$ is finite.

In Section 3 we concentrate on the cases where G_0 is a special linear group or is defined by a bilinear or hermitian form. We then recall foundational results from [7] and describe a notion of nondegeneracy for flags $\mathcal{F} \in \mathcal{Z}$ (even in the cases $G_0 = SL(\infty;\mathbb{R})$ and $G_0 = SL(\infty;\mathbb{H})$). We

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use nondegeneracy to determine which G_0–orbits are open, and in fact and whether there are any open G_0–orbits.

In Section 4 we develop a complete structure theory for the finitary infinite dimensional bounded symmetric domains. The results are similar to the classical finite dimensional results, but one has to be careful about the details. We obtain complete extensions of the orbit structure (in particular the boundary structure) from the finite dimensional cases ([8], [11], [12]).

Then in Section 5 we initiate the study of cycle spaces of the open G_0–orbits on $\mathcal{Z}$. We start with the important case of $G_0 = SU(\infty, q)$, $q \leqq \infty$, using an idea from the finite dimensional setting. We show how that idea leads to a precise description of the cycle space more generally. This is the start of a program to extend results of [3] to infinite dimensions. This study raises many important questions and initiates several promising lines of research. Compare [3].

One could carry out the considerations of Sections 4 and 5 in a more unified way, but there are many small differences of technical detail, so it would not be advantageous.

Finally in Section 6 we carry some of the results of Sections 4 and 5 over to certain real and quaternionic bounded symmetric domains. As noted in [10] this has some physical interest.

This study grew out of a joint project [7] with Ivan Penkov and Mikhail Ignatyev, where we studied real forms G_0 of $SL(\infty;\mathbb{C})$ and the basic properties of their orbits on flag varieties $\mathcal{Z}$. I thank Ivan Penkov for important discussions on early versions of this manuscript, and I thank the referee for comments and observations that led to improvements in this paper.

2 Basics.

In this section we review some basic facts about our class of infinite dimensional real and complex Lie groups, complex flag manifolds, and real group orbits.

2.1 Direct Limit Groups.

Let V be a countable dimensional complex vector space and E a fixed basis of V. We fix a linear order on E, specifically by $\mathbb{N} = \mathbb{Z}^+$, where $E = \{e_1, e_2, \dots\}$. When we come to flags and parabolics we will consider other orders on E, but we use the given order by $\mathbb{Z}^+$ to define our groups and our exhaustions of V.

Let V_* denote the span of the dual system $\{e_1^*, e_2^*, \dots\}$; we view V_* as the restricted dual of V. The group $GL(V, E)$ is the group of invertible linear transformations on V that keep fixed all but finitely many elements of E. It is easy to see that $GL(V, E)$ depends only on the pair (V, V_*) as long as V_* is constructed from E.

Express the basis E as an increasing union $E = \bigcup E_n$ of finite subsets. That exhausts V by finite dimensional subspaces $V_n = \mathrm{Span}\,\{E_n\}$, $V = \varinjlim V_n$, and thus expresses $GL(V, E)$ as $\varinjlim GL(V_n)$ and $SL(V, E)$ as $\varinjlim SL(V_n)$. When we write $GL(\infty;\mathbb{C})$ or $SL(\infty;\mathbb{C})$ we must have in mind such an associated exhaustion of V by finite dimensional subspaces.

For the orthogonal and symplectic groups, V is endowed with a nondegenerate symmetric or antisymmetric bilinear form b that is related to E as follows: We can choose the increasing union $E = \bigcup E_n$ so that the $V_n = \mathrm{Span}\,\{E_n\}$ are nondegenerate for b, and so that $b(e_m, V_n) = 0$ for $e_m \notin E_n$. Thus $O(V, E, b) = \varinjlim O(V_n, b|_{V_n})$ when b is symmetric, and $Sp(V, E, b) = \varinjlim Sp(V_n, b|_{V_n})$ when b is antisymmetric. Again, when we write $O(\infty;\mathbb{C})$, $SO(\infty;\mathbb{C})$ or $Sp(\infty;\mathbb{C})$ we must have in mind such an associated exhaustion of V by finite dimensional b–nonsingular subspaces.

2.2 Flags.

We now recall some basic definitions from [2]. A **chain** of subspaces in V is a set $\mathcal{C}$ of distinct subspaces such that if $F, F' \in \mathcal{C}$ then either $F \subset F'$ or $F' \subset F$. We write $\mathcal{C}'$ (resp. $\mathcal{C}''$) for the subchain of all $F \in \mathcal{C}$ with an immediate successor (resp. immediate predecessor). Also, we write $\mathcal{C}^\dagger$ for the set of all pairs (F', F'') where $F'' \in \mathcal{C}''$ is the immediate successor of $F' \in \mathcal{C}'$.

Let $\mathcal{F}$ be a chain, and let $\mathcal{F}'$ and $\mathcal{F}''$ be defined as just above. Then $\mathcal{F}$ is a **generalized flag** if $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$ and $V \setminus \{0\} = \bigcup_{(F',F'')\in\mathcal{F}^\dagger}(F'' \setminus F')$. Note that $0 \neq v \in V$ determines $(F', F'') = (F'_v, F''_v) \in \mathcal{F}^\dagger$ such that $v \in F'' \setminus F'$. If $\mathcal{F}$ is a generalized flag then each of $\mathcal{F}'$ and $\mathcal{F}''$ determines $\mathcal{F}$:

$$\text{if } (F', F'') \in \mathcal{F}^\dagger \text{ then } F' = \bigcup_{G''\in\mathcal{F}'', G''\subsetneqq F''} G'' \text{ and } F'' = \bigcap_{G'\in\mathcal{F}', G'\supsetneqq F'} G'.$$

A generalized flag $\mathcal{F}$ is **maximal** if it is not properly contained in another generalized flag. This is equivalent to the condition that $\dim F''_v/F'_v = 1$ for all $0 \neq v \in V$.

A generalized flag is a **flag** if, as a linearly ordered set, the proper subspaces of $\mathcal{F}$ are isomorphic to a linearly ordered subset of $\mathbb{Z}$, so that we don't have to deal with limit ordinals.

In the orthogonal and symplectic cases, we say that a generalized flag $\mathcal{F}$ in V is **isotropic** (relative to b) if $b(F, F) = 0$ for every $F \in \mathcal{F}$. This is equivalent to the notion in [7], where "isotropic" is defined to mean that $\tau : F \mapsto F^\perp$ (relative to b) is an order–reversing involution of $\mathcal{F}$, so that $(F', F'') \in \mathcal{F}^\dagger$ if and only if $((F'')^\perp, (F')^\perp) \in \mathcal{F}^\dagger$. In effect, if $\mathcal{F} = (F_\alpha)$ is isotropic in the sense of this paper then $\mathcal{F} \cup \mathcal{F}^\perp := \mathcal{F} \bigcup \{F^\perp \mid F \in \mathcal{F}\}$ is isotropic in the sense of [7], and if $\mathcal{J} = \{J_\beta\}$ is isotropic in the sense of [7] then $\{J_\alpha \mid J_\alpha \subset J_\alpha^\perp\}$ is isotropic in the current sense.

A partial order $\prec$ on a basis E of V is called **strict** if $\beta \prec \alpha$ implies $\beta \neq \alpha$, and $\beta \preceq \alpha$ means that either $\beta \prec \alpha$ or $\beta = \alpha$. We emphasize that this is only a partial order, not a linear order, and there may be elements of the index set that are not comparable under $\prec$. In particular $\prec$ need not be the same as any order with which E is presented. See Example 2.2.2 below.

Definition 2.2.1. A generalized flag $\mathcal{F}$ is **compatible** with E if there exists a strict partial order $\prec$ on E for which every pair (F', F'') is a pair $(\mathrm{Span}\{e_\beta \mid \beta \prec \alpha\}, \mathrm{Span}\{e_\beta \mid \beta \preceq \alpha\})$ or a pair $(0, \mathrm{Span}\{e_\beta \mid \beta \preceq \alpha\})$. If $\mathcal{F}$ is isotropic in the sense that each F_α is either isotropic or coisotropic, then in addition we require that E be isotropic.

A generalized flag $\mathcal{F}$ is **weakly compatible** with E if it is compatible with a basis L of V where $E \setminus (E \cap L)$ is finite.

A subspace $F \subset V$ is **(weakly) compatible** with E if the generalized flag $(0, F, V)$ is (weakly) compatible with E.

Generalized flags $\mathcal{F}$ and $\mathcal{G}$ are E–**commensurable** if they are both weakly compatible with E and there is a bijection $\varphi : \mathcal{F} \to \mathcal{G}$ and a finite dimensional $U \subset V$ such that each $F \subset \varphi(F) + U$, $\varphi(F) \subset F + U$, and $\dim(F \cap U) = \dim(\varphi(F) \cap U)$. E–commensurability is an equivalence relation. ♢

Example 2.2.2. This is the example that we'll need to discuss bounded symmetric domains. Let $\mathcal{F} = (0 \subset F \subset V)$. We divide the index set A of the basis E as $A = A_1 \cup A_2$ where $A_1 = \{\alpha \mid e_\alpha \in F\}$. Let $\prec$ be any partial order on A such that (i) $\alpha_1 \prec \alpha_2$ whenever $\alpha_1 \in A_1$ and $\alpha_2 \in A_2$ and (ii) A_i has a maximal element γ_i in the sense that $\alpha \prec \gamma_i$ whenever $\gamma_i \neq \alpha \in A_i$. Then $(0, F) = (0, \mathrm{Span}\{e_\beta \mid \beta \preceq \gamma_1\})$ (by convention on pairs with $F' = 0$) and $(F, V) = (\mathrm{Span}\{e_\beta \mid \beta \prec \gamma_2\}, \mathrm{Span}\{e_\beta \mid \beta \preceq \gamma_2\})$, so $\mathcal{F}$ is compatible with E.

This example extends to generalized flags of the form $(0 \subset F_1 \subset \cdots \subset F_\ell \subset V)$, with only the obvious changes. ♢

Fix a generalized flag $\mathcal{F}$ compatible with E. If E is b–isotropic suppose that $\mathcal{F}$ is isotropic. Then $\mathcal{Z} = \mathcal{Z}_{\mathcal{F},E}$ denotes the **flag manifold** G/Q where Q is the parabolic $\{g \in G \mid g(F) = F \text{ for all } F \in \mathcal{F}\}$. If E is isotropic we'll write $\mathcal{Z} = \mathcal{Z}_{\mathcal{F},b,E}$ for G/Q where $Q = Q_{\mathcal{F}}$ is the stabilizer of $\mathcal{F}$ in G. As noted in Section 2.3, $\mathcal{Z}_{\mathcal{F},E}$ is a holomorphic direct limit of finite dimensional complex flag manifolds, so $\mathcal{Z}_{\mathcal{F},E}$ has the structure of complex manifold.

Theorem 6.2 in [2] says

Lemma 2.2.3. *Let $\mathcal{F}$ be a generalized flag that is weakly compatible with E. If $G = SO(V, E, b)$ or $G = Sp(V, E, b)$ suppose further that $\mathcal{F}$ is isotropic. If $g \in G$, then $g(\mathcal{F})$ is E–commensurable to $\mathcal{F}$. If $\mathcal{F}$ and $\mathcal{L}$ are E–commensurable then there is an element $g \in G$ such that $\mathcal{L} = g(\mathcal{F})$.*

Proof. (Compare with Theorem 6.1 of [2].) If $g \in G$ then $V = U + W$ where g is the identity on W, $g(U) = U$, and $\dim U < \infty$. If $F_\alpha \in \mathcal{F}$ then $g(F_\alpha) \subset F_\alpha + U$. In particular $g(\mathcal{F})$ is weakly compatible with E. This proves the first statement.

Let $\mathcal{F}$ and $\mathcal{L}$ be E–commensurable, and let U be a finite dimensional subspace of V, such that each $F \subset \varphi(F) + U$, $\varphi(F) \subset F + U$ and $\dim(F \cap U) = \dim(\varphi(F) + U)$. They are weakly compatible with E so they are compatible with bases X and Y such that $E \setminus (E \cap X)$ and $E \setminus (E \cap Y)$ are finite. Now $E \setminus (E \cap (X \cup Y))$ is finite; let U denote its span and let W be the span of its complement in E. Let $g \in G$ be the identity on W, and define $g : U \to U$ by $g(x_\alpha) = y_\alpha$ for α an index of $E \setminus (E \cap (X \cup Y))$. □

2.3 Flag Manifolds.

Let $\mathcal{F}$ be a generalized flag weakly compatible with E. If $G = SO(V, E, b)$ or $G = Sp(V, R, b)$, suppose that $\mathcal{F}$ is isotropic. In view of Lemma 2.2.3,

Remark 2.3.1. The **flag manifold** $\mathcal{Z}_{\mathcal{F},E}$ consists of all generalized flags in V that are E–commensurable to $\mathcal{F}$. ♢

Lemma 2.2.3 says that $\mathcal{Z}_{\mathcal{F},E}$ is a homogeneous space for the complex group G. Realize $V = \varinjlim V_n$ according to an exhaustion $E = \bigcup E_n$ by finite subsets. Denote $\mathcal{F}_n = \mathcal{F} \cap V_n$. In other words, if $\mathcal{F} = \{F_\alpha\}_{\alpha \in A}\}$ then $\mathcal{F}_n$ is $\{F_\alpha \cap V_n\}_{\alpha \in A}\}$ with repetitions allowed. Now $\mathcal{F}_n$ is a flag in V_n so we have the flag manifold $\mathcal{Z}_{\mathcal{F}_n,E_n}$. Note that the $F_\alpha \cap V_n \hookrightarrow F_\alpha \cap V_m$, $m \geqq n$, define maps $\mathcal{Z}_{\mathcal{F}_n,E_n} \to \mathcal{Z}_{\mathcal{F}_m,E_m}$ and give us a direct system $\{\mathcal{Z}_{\mathcal{F}_n,E_n}\}$ for which $\mathcal{Z}_{\mathcal{F},E} = \varinjlim\{\mathcal{Z}_{\mathcal{F}_n,E_n}\}$. Since the finite dimensional flag manifold $\mathcal{Z}_{\mathcal{F}_n,E_n}$ has the natural structure of homogeneous projective variety under the action of G_n, and the $\mathcal{Z}_{\mathcal{F}_n,E_n} \to \mathcal{Z}_{\mathcal{F}_m,E_m}$ are equivariant rational maps and equivariant for $G_n \hookrightarrow G_m$, the infinite dimensional flag manifold $\mathcal{Z}_{\mathcal{F},E}$ is a G–homogeneous ind–variety. We emphasize the connection with the (finite dimensional) $\mathcal{Z}_{\mathcal{F}_n,E_n}$ by viewing $\mathcal{Z}_{\mathcal{F},E}$ as a complex ind–manifold referring to it simply as a complex flag manifold.

2.4 Real Forms of the Complex Groups.

Corresponding to the complex classical groups G mentioned above, we have their real forms as follows. Here note that a local isomorphism to one of the groups on the following list implies an isomorphism of Lie algebras, so the local isomorphism is compatible with the ind–structure specified as direct limit of finite dimensional Lie groups.

<u>If $G = SL(\infty; \mathbb{C})$</u>, then G_0 is locally isomorphic to one of

$SL(\infty; \mathbb{R}) = \lim_{n\to\infty} SL(n; \mathbb{R})$ the real special linear group,
$SL(\infty; \mathbb{H}) = \lim_{n\to\infty} SL(n; \mathbb{H})$ the quaternion special linear group,
$SU(p, \infty) = \lim_{n\to\infty} SU(p, n)$ the complex special unitary group of finite real rank p, and
$SU(\infty, \infty) = \lim_{p,q\to\infty} SU(p, q)$ the complex special unitary group of infinite real rank.

If $G = GL(\infty;\mathbb{C})$, then G_0 is locally isomorphic to one of

$GL(\infty;\mathbb{R}) = \lim_{n\to\infty} GL(n;\mathbb{R})$ the real general linear group,
$GL(\infty;\mathbb{H}) = \lim_{n\to\infty} GL(n;\mathbb{H}) = SL(\infty;\mathbb{H}) \times \mathbb{R}$ the quaternion general linear group,
$U(p,\infty) = \lim_{n\to\infty} U(p,n)$ the complex unitary group algebra of finite real rank p, and
$U(\infty,\infty) = \lim_{p,q\to\infty} U(p,q)$ the complex unitary group of infinite real rank.

If $G = SO(\infty;\mathbb{C})$, then G_0 is locally isomorphic to one of

$SO(p,\infty) = \lim_{n\to\infty} SO(p,n)$ the real orthogonal group of finite real rank p,
$SO(\infty,\infty) = \lim_{p,q\to\infty} SO(p,q)$ the real orthogonal group of infinite real rank, and
Caveat: when we write $SO(—)$ we mean the topological identity component of $O(—)$.
$SO^*(\infty) = \lim_{n\to\infty}(SO^*(2n) = \{g \in SL(n;\mathbb{H}) \mid g \text{ preserves } \kappa(x,y) := \sum \bar{x}^\ell i y^\ell = {}^t\bar{x} i y\})$.

If $G = Sp(\infty;\mathbb{C})$, then G_0 is locally isomorphic to one of

$Sp(\infty;\mathbb{R}) = \lim_{n\to\infty} Sp(n;\mathbb{R})$ the real symplectic group,
$Sp(p,\infty) = \lim_{n\to\infty} Sp(p,n)$ the quaternion unitary Lie algebra of finite real rank p, and
$Sp(\infty,\infty) = \lim_{p,q\to\infty} Sp(p,q)$ the quaternion unitary Lie algebra of infinite real rank.

As usual we use Roman letters for the Lie groups and the corresponding lower case fraktur for their Lie algebras. In order to be precise about the real groups we must be careful about two notions: nondegeneracy of subspaces, and the role of V and E in complex conjugation τ of $\mathfrak{g}$ over $\mathfrak{g}_0$ and G over G_0 .

3 Basis and Exhaustion.

We run through the real groups of Section 2.4, defining some particular bases, flags and signatures relevant to our results on cycle spaces.

3.1 $SU(\infty,q)$, $q \leqq \infty$.

In this case $V = \mathbb{C}^{\infty,q}$ with $q \leqq \infty$ and we start with an ordered basis

(3.1.1)
$$\begin{aligned} E =& \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots, e_q\} \text{ if } q < \infty, \\ E =& \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots\} \text{ if } q = \infty, \end{aligned}$$

where G_0 is defined by the hermitian form

(3.1.2)
$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

The corresponding exhaustion $V = \bigcup V_n$ realizes G_0 as $\lim_{k,\ell\to\infty} SU(k,\ell)$ or $\lim_{k\to\infty} SU(k,q)$.

$\mathcal{F}$ is a flag in V compatible with the ordered basis E of (3.1.1). *The partial order $\prec$ for this compatibility is not necessarily the order of* (3.1.1)*; it is a property of $\mathcal{F}$ relative to E rather than a property of the ordering* (3.1.1) *of E.* To each flag $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$ we assign the signature sequence $\{s_k = s_k(\mathcal{F}^{(1)}) := (\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}), \mathrm{nul}_k(\mathcal{F}^{(1)}))\}$ where pos_k is the dimension of the maximal positive definite subspace of $F_k^{(1)}$, neg_k is the dimension of the maximal negative definite subspace, and nul_k is the nullity. If $\mathrm{nul}_k = 0$ we write $(\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}))$ for $(\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}), 0)$. If the F_k all are finite dimensional, then pos_k, neg_k and nul_k all are finite. If $q < \infty$, i.e. if G_0 has finite real rank q, then every $\mathrm{nul}_k \leqq q$. However, when one or more of the F_k is infinite dimensional the signature sequence is not always useful.

3.2 $SO(\infty, q)\,, q \leqq \infty$.

Again $V = \mathbb{C}^{\infty,q}$. In terms of an ordered basis $E' = \{e'_i\}$ as in (3.1.1), G_0 is defined by a symmetric bilinear form b together with a hermitian form h, as follows:

$$\begin{aligned} &b(e'_i, e'_j) = +\delta_{i,j} = h(e'_i, e'_j) \text{ for } i < 0,\ b(e'_i, e'_j) = -\delta_{i,j} = h(e'_i, e'_j) \text{ for } i > 0, \\ &\text{the other } b(e'_k, e'_\ell) = 0 = h(e'_k, e'_\ell). \end{aligned} \tag{3.2.1}$$

To see that (3.2.1) defines $SO(\infty, q)$, we note that $SO(\infty, q)$ consists of all finitary real matrices (relative to the basis E') in the $SO(\infty; \mathbb{C})$ defined by b, and also consists of all real matrices in the $SU(\infty, q)$ defined by h. Write B and H for the matrices of b and h, so $SO(\infty; \mathbb{C})$ is given by $gB \cdot {}^t g = B$ and $SU(\infty, q)$ is given by $gH \cdot {}^t \bar{g} = H$. Since $B = H$, now $g \in SO(\infty; \mathbb{C}) \cap SU(\infty, q)$ implies $g = \bar{g}$ so $g \in SO(\infty, q)$, and obviously $g \in SO(\infty, q)$ implies $g \in SO(\infty; \mathbb{C}) \cap SU(\infty, q)$. For $k, \ell \leqq \infty$ we have verified

(3.2.2) $SO(k, \ell) = SO(k + \ell; \mathbb{C}) \cap SU(k, \ell)\,, SO(k + \ell; \mathbb{C})$ defined by $b\,, SU(k, \ell)$ defined by h.

A b–isotropic flag in V cannot be compatible with E' because a subspace of V spanned by some of the e'_i neither contains nor is contained in its b–orthocomplement. So we define

$$\begin{aligned} E =& \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots, e_q\} \text{ if } q < \infty, \\ E =& \{\dots, e_{-2}, e_{-1}, e_1, e_2, \dots\} \text{ if } q = \infty, \end{aligned} \tag{3.2.3}$$

where

$$\begin{aligned} &h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0,\ h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0 \text{ and} \\ &b(e_i, e_j) = \delta_{i,j} \text{ for } i < -q,\ b(e_i, e_j) = \delta_{0,i+j} \text{ for } i \geqq -q\,. \end{aligned} \tag{3.2.4}$$

The transformation $e'_i \mapsto e_i$ is not finitary when $q = \infty$, but nonetheless every $g \in G$ is finitary relative to the basis E. $\mathcal{F}$ is an E–commensurable isotropic flag in V. We use h for the signature sequence $\{s_k = s_k(\mathcal{F}^{(1)}) := (\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}), \mathrm{nul}_k(\mathcal{F}^{(1)}))\}$ for a flag $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$, where pos_k is the dimension of the maximal h–positive definite subspace of $F^{(1)}_k$, neg_k is the dimension of the maximal h–negative definite subspace, and nul_k is the h–nullity. As in Section 3.1 above, if $\mathrm{nul}_k = 0$ we write $(\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}))$ for $(\mathrm{pos}_k(\mathcal{F}^{(1)}), \mathrm{neg}_k(\mathcal{F}^{(1)}), 0)$, and if the F_k all are finite dimensional, then pos_k, neg_k and nul_k all are finite, and if $q < \infty$ then every $\mathrm{nul}_k \leqq q$.

Remark 3.2.5. Orientation can be a consideration for $SO(\infty, q)$. Following [1, Theorem 2.8], the stabilizer of a b–isotropic flag $\mathcal{F}$ determines all the subspaces in $\mathcal{F}$ except when some there is an isotropic subspace $L \in \mathcal{F}$ with $\dim L^\perp / L = 2$. In that case there are two maximal isotropic subspaces M_1 and M_2 of (V, b) that contain L, and there are three flags with the same stabilizer as $\mathcal{F}$, and of course $\mathcal{F}$ is one of them. We list them with *ad hoc* designations. (i) (undecided orientation) $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\}$ and neither of the M_i is contained in $\mathcal{F}$, (ii) (positive orientation) $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\} \cup \{M_1\}$, i.e. $M_1 \in \mathcal{F}$, and (iii) (negative orientation) $\{F^{(1)} \in \mathcal{F} \mid F^{(1)} \subset L \text{ or } L^\perp \subset F^{(1)}\} \cup \{M_2\}$, i.e. $M_2 \in \mathcal{F}$. Signature does not distinguish these three flags, for example (ii) and (iii) have the same signatures for all q, and sometimes all three have the same signature with $q = \infty$.

3.3 $Sp(\infty, q)\,, q \leqq \infty$.

Here $V = \mathbb{C}^{\infty,2q}$ and we use the basis (3.1.1) with q replaced by $2q$. Then G_0 is defined by both an antisymmetric bilinear form b and an hermitian form h.

$$\begin{aligned} &b(e_{2i-1}, e_{2i}) = -1,\ b(e_{2i}, e_{2i-1}) = +1, \text{ for } i > 0, \\ &b(e_{2i+1}, e_{2i}) = +1,\ b(e_{2i}, e_{2i+1}) = -1 \text{ for } i < 0, \text{ all other } b(e_a, e_b) = 0; \\ &h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0, \text{ all other } h(e_a, e_b) = 0. \end{aligned} \tag{3.3.1}$$

To see this we need the analog of (3.2.2), and for that we need to find the quaternion algebra that realizes $\mathbb{C}^{2p,2r}$ as $\mathbb{H}^{p,r}$.

Lemma 3.3.2. *$Sp(p,r) = Sp(p+r;\mathbb{C}) \cap SU(2p,2r)$ where $SO(p+r;\mathbb{C})$ is defined by b as in* (3.3.1) *and $SU(2p,2r)$ is defined by h as in* (3.3.1).

Proof. We work in matrices relative to the portion $E = \{e_{-2p}, \dots, e_{2r}\}$ of (3.1.1). Then b has matrix $B = \mathrm{diag}\{J, \dots, J\}$ where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and h has matrix $H = \left(\begin{smallmatrix} I_{2p} & 0 \\ 0 & -I_{2r} \end{smallmatrix}\right)$. So $Sp(p+r;\mathbb{C})$ is given by $gB{}^tg = B$ and $SU(2p,2r)$ is given by $gH{}^t\bar{g} = H$. Define $\mathbb{R}$–linear transformations of V by

$$\mathcal{I}(v) = \sqrt{-1}\,v \text{ and } \mathcal{J}(v) = \sqrt{-1}BH\bar{v} \text{ for } v \in V.$$

Compute

$$\mathcal{I}^2 = -I,\ \mathcal{J}^2 = -I \text{ and } \mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I} = 0$$

so $\mathcal{I}$ and $\mathcal{J}$ generate a quaternion algebra; call it $\mathbb{H}$. If $g \in Sp(p+r;\mathbb{C}) \cap SU(2p,2r)$, so ${}^tg = B^{-1}g^{-1}B$ and ${}^t\bar{g} = H^{-1}g^{-1}H$, then $B^{-1} = -B$, $H^{-1} = H$, and we compute

$$\begin{aligned}\mathcal{J}g\mathcal{J}^{-1}v &= (\sqrt{-1}BH)(\bar{g})(\overline{-\sqrt{-1}HB\bar{v}}) = -BH\bar{g}HBv \\ &= -BH \cdot H{}^tg^{-1}H \cdot HBv = B{}^tg^{-1}Bv = B \cdot BgB^{-1} \cdot Bv = gv\end{aligned}$$

for $v \in V$. Thus $\mathcal{J}$ commutes with every $g \in Sp(p+r;\mathbb{C}) \cap SU(2p,2r)$, in other words every $g \in Sp(p+r;\mathbb{C}) \cap SU(2p,2r)$ is $\mathbb{H}$–linear. That shows $Sp(p+r;\mathbb{C}) \cap SU(2p,2r) \subset Sp(p,r)$.

On the other hand, $\sigma : g \mapsto \mathcal{J}g\mathcal{J}^{-1}$ is an involutive automorphism on the underlying real structure of $Sp(p+r;\mathbb{C})$. The latter is simply connected, so its fixed point set is connected. But σ fixes every element of $Sp(p,r)$, which is maximal among the connected subgroups of $Sp(p+r;\mathbb{C})$. So now $Sp(p,r) \subset Sp(p+r;\mathbb{C}) \cap SU(2p,2r)$. That completes the proof. □

Now take the limit on p, or on p and r, to see how G_0 is defined by the two forms b and h of (3.3.1). Let $\mathcal{F}$ be a b–isotropic flag in V compatible with the basis E of (3.1.1). For that, note that $\mathrm{Span}\,\{e_i \mid i \text{ even}\}$ and $\mathrm{Span}\,\{e_i \mid i \text{ odd}\}$ are b–isotropic subspaces. As in the previous cases one can discuss signature for flags $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$.

3.4 $SO^*(\infty)$.

This case is similar to the case of $SO(\infty,\infty)$, except that we use a different bilinear form b. The basis is

(3.4.1) $$E = \{\dots e_{-3}, e_{-2}, e_{-1}, e_1, e_2, e_3, \dots\} = \bigcup E_n \text{ where } E_n = \{e_{-n}, \dots, e_n\}.$$

G_0 is defined by the symmetric bilinear form b and the hermitian form h:

(3.4.2) $$b(e_i, e_j) = \delta_{i+j,0},\ h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

Thus $V = \bigcup V_n$ where $V_n = \mathrm{Span}\,\{E_n\}$ and $G_0 = SO^*(\infty) = \varinjlim SO^*(2n)$ where $SO^*(2n)$ is the subgroup of $SL(2n;\mathbb{C})$ defined by the forms b and h of (3.4.2). To check this, note that that subgroup of $SL(2n;\mathbb{C})$ itself has maximal compact subgroup isomorphic to $U(n)$.

3.5 $Sp(\infty;\mathbb{R})$.

This case is similar to the case of $SO^*(\infty)$, except that the bilinear form b is antisymmetric. We use the same basis (3.4.1), with bilinear form b and hermitian form h:

$$\begin{aligned} &b(e_i,e_j)=\delta_{i+j,0} \text{ for } i<0 \text{ and } b(e_i,e_j)=-\delta_{i+j,0} \text{ for } i>0;\\ &h(e_i,e_j)=\delta_{i,j} \text{ for } i<0 \text{ and } h(e_i,e_j)=-\delta_{i,j} \text{ for } i>0. \end{aligned} \tag{3.5.1}$$

Here $G=Sp(\infty;\mathbb{C})$ is defined by b. One can view G_0 as the real (relative to $\mathrm{Span}_{\mathbb{R}}(E)$) elements of G, but for our purposes it is better to view it as $G\cap SU(\infty,\infty)$ where $U(\infty,\infty)$ is defined by the hermitian form h. For that, it suffices to check that $Sp(n;\mathbb{R})=Sp(n;\mathbb{C})\cap U(n,n)$, and to check that it suffices to note that $Sp(n;\mathbb{C})\cap SU(n,n)$ contains a $U(n)$ in the form $\left(\begin{smallmatrix} A & 0\\ 0 & {}^t\bar{A}^{-1}\end{smallmatrix}\right)$.

As in the previous cases one can discuss signature for flags $\mathcal{F}^{(1)}\in\mathcal{Z}_{\mathcal{F},E}$.

3.6 $SL(\infty;\mathbb{R})$ and $SL(\infty;\mathbb{H})$.

Fix a real form V_0 of V and an ordered basis $E=\{e_1,e_2,e_3,\dots\}$ of V_0. Then $SL(\infty;\mathbb{R})$ is defined by complex conjugation $\tau: v\mapsto\overline{v}$ of V over a real form V_0, while $SL(\infty;\mathbb{H})$ is defined by a conjugate linear map $\tau: v\mapsto\left(\begin{smallmatrix} 0 & I\\ -I & 0\end{smallmatrix}\right)\overline{v}$ on V. In terms of E,

Case $\mathbb{F}=\mathbb{R}$: each $\tau e_a=e_a$, so E is an $\mathbb{R}$–basis of a real form V_0 of V

Case $\mathbb{F}=\mathbb{H}$: $\tau e_{2a-1}=-e_{2a}$ and $\tau e_{2a}=e_{2a-1}$, so each $\{e_{2a-1},ie_{2a-1},e_{2a},ie_{2a}\}$ is an $\mathbb{R}$–basis of an $\mathbb{H}$–subspace of V

In the finite dimensional case the signature for a generalized flag $\mathcal{F}^{(1)}$ is $\{s_{i,j}=s_{i,j}(\mathcal{F}^{(1)})\}$ where $s_{i,j}(\mathcal{F}^{(1)})$ is the dimension of the maximal complex subspace of $F_i^{(1)}\cap\tau F_j^{(1)}$ ([4] and [5]). In the infinite dimensional cases we will have to be more precise [7, §5].

3.7 Nondegeneracy and Open Orbits.

Fix a basis E of V as in Sections 3.1 through 3.6, and a flag $\mathcal{F}$ in V that is compatible with E. Except in the cases of $SL(\infty;\mathbb{R})$ and $SL(\infty;\mathbb{H})$, we use signatures of generalized flags to distinguish real group orbits on $\mathcal{Z}_{\mathcal{F},E}$, as follows.

Definition 3.7.1. Let G_0 be defined by a nondegenerate bilinear form b or an hermitian form h or both. Then we say that a flag $\mathcal{F}^{(1)}\in\mathcal{Z}_{\mathcal{F},E}$ is **nondegenerate** if (i) for $SU(\infty,q)$, $SO(\infty,q)$ or $Sp(\infty,q)$ each $F_\alpha^{(1)}$ is h– (or b–) nondegenerate; and (ii) for $Sp(n;\mathbb{R})$ or $SO^*(\infty)$ each $F_\alpha^{(1)}$ is h–nondegenerate. ♢

Theorem 3.7.2. *Let G_0 be $SU(\infty,q)$, $SO(\infty,q)$, $Sp(\infty,q)$, $SO^*(\infty)$ or $Sp(\infty;\mathbb{R})$ and consider a flag $\mathcal{F}^{(1)}\in\mathcal{Z}_{\mathcal{F},E}$. Then $G_0(\mathcal{F}^{(1)})$ is open in $\mathcal{Z}_{\mathcal{F},e}$ if and only if $\mathcal{F}^{(1)}$ is nondegenerate.*

Proof. An orbit $G_0(\mathcal{F}^{(1)})$ in $\mathcal{Z}_{\mathcal{F},e}$ is open just when one stays inside the orbit under any sufficiently small perturbation of a finite number of the $F^{(1)}$ in $\mathcal{F}^{(1)}$. Using the direct limit topology on $\mathcal{Z}_{\mathcal{F},e}$ and the finite dimensional analog ([12], [3]), the assertion follows. This is the same argument as that of the first part of [7, Proposition 5.1]. □

Corollary 3.7.3. *There are open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$ if and only if $\mathcal{Z}_{\mathcal{F},E}$ contains a nondegenerate flag. In particular, if G_0 is $SU(\infty,q)$, $SO(\infty,q)$ or $Sp(\infty,q)$ with $q<\infty$ then there are open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$.*

The matter is subtler for the special and general linear groups, where we don't have b– or h–nondegeneracy for subspaces of V, and where if $\dim V=\infty$ then the $\dim(F_i^{(1)}\cap\tau F_j^{(1)})$ do not suffice. Instead we use [7, Definition 5.1] as follows.

Definition 3.7.4. Let G_0 be $SL(\infty;\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$. Then $G_0(\mathcal{F}^{(1)}) \subset \mathcal{Z}_{\mathcal{F},E}$ is **nondegenerate** if $F_i^{(1)} \cap \tau F_j^{(1)}$ fails to properly contain $F_i^{(2)} \cap \tau F_j^{(2)}$, whenever $\mathcal{F}^{(2)} \in \mathcal{Z}_{\mathcal{F},E}$ and $F_i^{(1)}, F_j^{(1)} \in \mathcal{F}^{(1)}$. ♢

The first consequence of this definition is

Theorem 3.7.5. ([7, Proposition 5.3]) *Let G_0 be $SL(\infty;\mathbb{R})$ or $SL(\infty;\mathbb{H})$, and consider a flag $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$. Then the orbit $G_0(\mathcal{F}^{(1)})$ is open in $\mathcal{Z}_{\mathcal{F},E}$ if and only if $\mathcal{F}^{(1)}$ is nondegenerate. In particular, if each $F_n^{(1)} \cap \tau F_n^{(1)} = 0$ then $G_0(\mathcal{F}^{(1)})$ is open in $\mathcal{Z}_{\mathcal{F},E}$.*

If n is odd, or if $n = 2m$ and $\dim F \neq m$ for all $F \in \mathcal{F} \cap \mathbb{C}^n$, then $G_{n,0} = SL(n;\mathbb{R})$ has only one open orbit on a flag manifold G_n/Q_n in $\mathbb{C}^n$; if $n = 2m$ even, and some $F \in \mathcal{F} \cap \mathbb{C}^n$ has dimension m, then there is an orientation question and $G_{n,0} = SL(n;\mathbb{R})$ has two open orbits on G_n/Q_n . See [4, Corollary 2.3]. Further $G_{n,0} = SL(n;\mathbb{H})$ has a unique open orbit on a flag manifold G_n/Q_n in $\mathbb{C}^{2n}$. See [5, Proposition 3.14]. This extends to infinite dimensions as follows.

Corollary 3.7.6. *Let G_0 be $SL(\infty;\mathbb{R})$ or $SL(\infty;\mathbb{H})$. Then there is an open G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$ if and only if $\mathcal{Z}_{\mathcal{F},E}$ contains a nondegenerate flag, and in that case there is exactly one open G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$.*

Proof. The first assertion is immediate from Theorem 3.7.5. For the second, let $\mathcal{O}_1 = G_0(\mathcal{F}^{(1)})$ and $\mathcal{O}_2 = G_0(\mathcal{F}^{(2)})$ be open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$. Then

$$\begin{aligned} \mathcal{Z}_{\mathcal{F},E} = \varinjlim \mathcal{Z}_{\mathcal{F}_n,E_n} &\text{ where, for } n \text{ in a cofinal subset } \mathbb{S} \subset \mathbb{Z}^+, \\ &E \text{ is an increasing union of finite subsets } E_n\,, \\ &V_n = \mathrm{Span}\,\{E_n\} \text{ and } \mathcal{F}_n = \mathcal{F} \cap V_n := (F_k \cap V_n), \\ &\mathcal{Z}_{\mathcal{F}_n,E_n} = G_n/Q_n \text{ flag manifold in } V_n \text{ with } Q_n \text{ parabolic in } G_n, \text{ and} \\ &\mathcal{O}_k \cap \mathcal{Z}_{\mathcal{F}_n,E_n} \text{ is an open } G_{n,0}\text{– orbit on } \mathcal{Z}_{\mathcal{F}_n,E_n} \text{ for } k = 1,\, 2. \end{aligned}$$

In the $SL(\infty;\mathbb{R})$ case we modify $\mathbb{S}$. If $n \in \mathbb{S}$ is even and $n+1 \notin \mathbb{S}$ we replace n by $n+1$. If $n \in \mathbb{S}$ is even and $n+1 \in \mathbb{S}$ we delete n. Thus we may assume that every element of $\mathbb{S}$ is odd. In the $SL(\infty;\mathbb{H})$ case we do not modify $\mathbb{S}$. Thus, in both cases, if $n \in \mathbb{S}$ then $G_{n,0}$ has a unique open orbit on $\mathcal{Z}_{\mathcal{F}_n,E_n}$, so $(\mathcal{O}_1 \cap \mathcal{Z}_{\mathcal{F}_n,E_n}) = (\mathcal{O}_2 \cap \mathcal{Z}_{\mathcal{F}_n,E_n})$. Thus $\mathcal{O}_1$ meets $\mathcal{O}_2$, so $\mathcal{O}_1 = \mathcal{O}_2$. □

Combining the argument of the proof of Corollary 3.7.6 with the uniqueness of closed orbits in the finite dimensional case [12], we have the related result

Proposition 3.7.7. (Compare [7, Proposition 5.6].) *Let G_0 be $SL(\infty;\mathbb{R})$ or $SL(\infty;\mathbb{H})$. Then there is closed G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$ if and only if each $\tau F_i = F_i$, and in that case there is exactly one closed G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$.*

4 Complex Bounded Symmetric Domains

The bounded symmetric domains are important cases of the orbits considered in Section 3. In finite dimensions they play a pivotal role in complex analysis, moduli theory, cycle space theory, automorphic function theory, and and both riemannian and complex differential geometry. In this section we extend parts of the finite dimensional bounded domain theory to our infinite dimensional setting, following the lines of the classical examples in [13].

In the classical theory one has the bounded symmetric domain $D_0 = G_0(z_0)$, its compact dual hermitian symmetric space $\mathcal{Z}$, the Borel embedding $D_0 \hookrightarrow \mathcal{Z}$, and the Harish-Chandra embedding $\xi^{-1}|_{D_0} : D_0 \hookrightarrow \mathfrak{m}^+$. In the Harish-Chandra embedding, $\mathfrak{m}^+ \subset \mathfrak{g}$ is a commutative subalgebra that represents the holomorphic tangent space and $\xi : \mathfrak{m}^+ \to \mathcal{Z}$ by $\xi(X) = \exp(X)z_0$. A maximal set of strongly orthogonal noncompact positive roots $\{\alpha_1, \dots, \alpha_r\}$, $r = \operatorname{rank} D_0$, defines a set $\{c_1, \dots, c_r\}$ of partial Cayley transforms, and the G_0–orbits on $\mathcal{Z}$ are exactly the $G_0(c_1 \dots c_k c_{k+1}^2 \dots c_{k+\ell}^2 z_0)$ where $k, \ell \geqq 0$ and $k+\ell \leqq r$. The open orbits are the $G_0(c_1^2 \dots c_\ell^2 z_0)$, i.e. the ones with $k = 0$, and $G_0(c_1 \dots c_r z_0)$ is the Bergman–Shilov boundary of D_0. See [12]. It is not so difficult to verify that this theory goes through *mutatis mutandis* for the infinite dimensional bounded symmetric domains as well, with the one restriction that $k + \ell < \infty$.

There are only four classes of (finitary) infinite dimensional complex bounded symmetric domains: the $SU(\infty, q)/S(U(\infty) \times U(q))$ with $q \leqq \infty$, the $Sp(\infty; \mathbb{R})/U(\infty)$, the $SO^*(\infty)/U(\infty)$, and the $SO(\infty, 2)/[SO(\infty) \times SO(2)]$. Their respective symmetric space ranks are q, ∞, ∞ and 2. In the all four cases it is easier to use some linear algebra, as in the examples worked out in [13], than to stick to the general theory. But of course we indicate the connection. The fourth case, however, where $\mathcal{Z}$ is a quadric in an infinite dimensional complex projective space, is not as straightforward as the others. Now we run through the cases.

4.1 The Complex Bounded Symmetric Domain for $SU(\infty, q)$.

We study the bounded symmetric domain D_0 associated to $G_0 = SU(\infty, q)$, $q \leqq \infty$. Start with

$$E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, \dots, e_q\} \text{ for } q < \infty$$
$$E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\} \text{ for } q = \infty$$

where the hermitian form h is given by

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0, \;\; h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

Let $F = \operatorname{Span}\{e_i \mid i > 0\}$. As in Example 2.2.2, $\mathcal{F} = (0, F, V)$ is compatible with E. Also, $G_0(\mathcal{F})$ is open in $\mathcal{Z}_{\mathcal{F},E}$. The bounded symmetric domain is $D_0 = G_0(\mathcal{F}) \in \mathcal{Z}_{\mathcal{F},E}$. Note that $\mathcal{Z}_{\mathcal{F},E}$ is a complex Grassmann manifold and the domain is

(4.1.1)
$$D_0 = \{\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in \mathcal{Z}_{\mathcal{F},E} \mid F^{(1)} \text{ is a maximal negative definite subspace of } V\}.$$

We go on to see why it is a bounded symmetric domain.

We will use the h–orthogonal decomposition $V = V_+ \oplus V_-$ where $V_+ = \operatorname{Span}\{e_i \mid i < 0\}$ and $V_- = \operatorname{Span}\{e_i \mid i > 0\}$ and the orthogonal projections $\pi_\pm : V \to V_\pm$. The kernel of π_- is h–positive definite so it has zero intersection with $F^{(1)}$ for any $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in D_0$. Thus $\pi_- : F^{(1)} \cong V_-$ is injective. Since $F^{(1)}$ is a maximal negative definite subspace $\pi_- : F^{(1)} \cong V_-$ is surjective as well. Now we have a well defined linear map

(4.1.2) $$Z_{F^{(1)}} : V_- \to V_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}.$$

Since $\mathcal{F}^{(1)}$ is weakly compatible with E, the matrix of $F^{(1)}$ relative to E has only finitely many nonzero entries. In other words $Z_{F^{(1)}}$ is finitary. Using $\pi_- : F^{(1)} \cong V_-$ and the basis $\{e_i \mid i > 0\}$ of $F = V_-$ we have a basis $\{e_i''\}$ of $F^{(1)}$ defined by $\pi_-(e_i'') = e_i$. Write $e_i'' = e_i + \sum_{j<0} z_{j,i} e_j$; then $(z_{j,i})$ is the matrix of $Z_{F^{(1)}}$. The fact that $F^{(1)}$ is h–negative definite, in other words $(h(e_i'', e_\ell'')) \ll 0$, translates to the matrix condition $I - (z_{j,i})^*(z_{j,i}) \gg 0$, equivalently the operator condition $I - Z_{F^{(1)}}^* Z_{F^{(1)}} \gg 0$.

Conversely if $Z : V_- \to V_+$ is finitary and satisfies $I - Z^*Z \gg 0$, then the column span of its matrix relative to E is a maximal negative definite subspace $F^{(1)}$, and $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in D_0$.

The block form matrices of elements of G_0 act by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} Z \\ I \end{smallmatrix}\right) \to \left(\begin{smallmatrix} AZ+B \\ CZ+D \end{smallmatrix}\right)$, which has the same column span as $\left(\begin{smallmatrix} (AZ+B)(CZ+D)^{-1} \\ I \end{smallmatrix}\right)$. So G_0 acts by linear fractional transformations, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : Z \to (AZ + B)(CZ + D)^{-1}$. Now we summarize.

Proposition 4.1.3. *D_0 is realized as the bounded domain consisting of all finitary $Z : V_- \to V_+$ such that $I - Z^*Z \gg 0$. In that realization the action of G_0 is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : Z \to (AZ+B)(CZ+D)^{-1}$.*

Orbits

There are $q+1$ open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$:

$$D_k = G_0((0, F_{(k)}, V)) \text{ where } F_{(k)} = \text{Span}\,\{e_{-k}, \dots, e_{-1}; e_{k+1}, \dots, e_q\} \text{ if } q < \infty,$$
$$F_{(k)} = \text{Span}\,\{e_{-k}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ if } q = \infty,$$

If $q = 1$ then D_1 and D_0 are the upper and lower "hemispheres" in an infinite version of the Riemann sphere; they are related by the square of a Cayley transform. If $q > 1$ then D_0 is the only convex D_k, but the others are reached by squares of partial Cayley transforms applied to $F = F_0$ as in [13], [8] and [11].

In this bounded symmetric domain setting, the G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$ of signature $(a, b, c) =$ (pos, neg, nul) have a and c finite and $\leqq q$ because each $\mathcal{F}^{(1)} = (0, F^{(1)}, V) \in \mathcal{Z}_{\mathcal{F},E}$ is weakly compatible with E. We denote those orbits by

$$\begin{aligned} D_{a,b,c} &= G_0((0, (F_+ + F_- + F_0), V)) \text{ where} \\ &F_0 = \text{Span}\,\{e_{-c} + e_c, \dots, e_{-1} + e_1\} \text{ (null)} \\ &F_+ = \text{Span}\,\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (positive)} \\ &F_- = \text{Span}\,\{e_{c+1}, \dots, e_{c+b}\} \text{ if } q < \infty, \text{Span}\,\{e_{c+1}, e_{c+2}, \dots\} \text{ if } q = \infty \text{ (negative).} \end{aligned} \tag{4.1.4}$$

The open orbits are the $D_a = D_{a,b,0}$, $a < \infty$ and $a + b = q$. In other words, they are the ones for $c = 0$. If $q < \infty$ there is a unique closed orbit, $D_{0,0,q}$, consisting of the $\mathcal{F}^{(1)} = \{F^{(1)}\} \in \mathcal{Z}_{\mathcal{F},E}$ for which $F^{(1)}$ is null. It is in the closure of every orbit. If $q = \infty$ there is no closed orbit.

One goes from the initial orbit $D_{0,q,0} = G_0(\mathcal{F})$ to any $D_{a,b,c}$ by applying a product of partial Cayley transforms to F. Specifically, $D_{a,b,c} = G_0(c_1 \dots c_c c_{c+1}^2 \dots c_{a+c}^2 \mathcal{F})$ where the partial Cayley transforms c_k (corresponding to $0 \to 1 \to \infty \to -1 \to 0$ in one variable) are given by

$$c_k(e_{-k}) = \tfrac{1}{\sqrt{2}}(e_{-k} - e_k),\ c_k(e_k) = \tfrac{1}{\sqrt{2}}(e_{-k} + e_k),\ c_k(e_j) = e_j \text{ for } j \neq \pm k. \tag{4.1.5}$$

Here $1 \leqq k \leqq q$ when $q < \infty$ and $1 \leqq k < \infty$ when $q = \infty$. In particular one reaches the boundary (of $D_0 = D_{0,q,0}$) orbits by a product without repetition of $\leqq q$ partial Cayley transforms, and if $q < \infty$ the closed orbit is $D_{0,0,q} = G_0(c_1 \dots c_q \mathcal{F})$. If $q < \infty$ the closed orbit is the Bergman-Shilov boundary of D_0.

4.2 The Complex Bounded Symmetric Domain for $Sp(\infty; \mathbb{R})$.

Now let $G_0 = Sp(\infty; \mathbb{R})$. It is defined relative to the basis $E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\}$ by the hermitian form h and the antisymmetric bilinear form b,

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0,\ h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0 \text{ and}$$
$$b(e_i, e_j) = \delta_{i+j,0} \text{ for } i < 0,\ b(e_i, e_j) = -\delta_{i+j,0} \text{ for } i > 0.$$

The domain D_0 consists of the maximal h–negative definite b–isotropic subspaces of V in $\mathcal{Z}_{\mathcal{F},E}$. In other words, let $F = \text{Span}\,\{e_i \mid i > 0\}$. Evidently $\mathcal{F} = (0, F, V)$ is compatible with E and $G_0(\mathcal{F})$ is open in $\mathcal{Z}_{\mathcal{F},E}$. The bounded symmetric domain is

$$D_0 := G_0(\mathcal{F}) \subset \mathcal{Z}_{\mathcal{F},E}.$$

Note that $\mathcal{Z}_{\mathcal{F},E}$ is contained in the complex Grassmann manifold of Section 4.1 for $q = \infty$.

In Lie group terms, $D_0 \cong Sp(\infty;\mathbb{R})/U(\infty)$ where $U(\infty)$ is the stabilizer of F. Let both $\mathcal{F}$ and $\mathcal{F}_{(0)}$ denote the flag $(0, F, V)$, so $D_0 = G_0(\mathcal{F}_{(0)})$. The open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$ are the $D_k = G_0(\mathcal{F}_{(k)})$ where

$$F_{(k)} = \text{Span}\,\{e_{-k}, e_{-k+1}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ and } \mathcal{F}_{(k)} = (0, F_{(k)}, V)$$

for integers $k \geqq 0$. Note that $D_k \cong Sp(\infty;\mathbb{R})/U(k, \infty - k)$ where the $\infty - k$ refers to the action on $\text{Span}\,\{e_{k+1}, e_{k+2}, \dots\}$. Also, $F_{(k)}^{\perp} = F_{(k)}$ relative to b, so $\mathcal{F}_{(k)}^{\perp} = \mathcal{F}_{(k)}$.

The corresponding D_∞ is the h–orthocomplement of D_0, orbit of $(0, F_{(\infty)}, V)$ where $F_{(\infty)} := \text{Span}\,\{e_i \mid i < 0\}$.

As in the SU setting, the partial Cayley transforms c_j are given by (4.1.5) and one passes from D_0 to D_k by $F_{(k)} = c_1^2 c_2^2 \dots c_k^2 F_{(0)}$. Compare [13]. Similarly, as in [11], the boundary of D_0 is the union of the orbits $G_0(c_1 c_2 \dots c_\ell F_{(0)})$, but here there is no closed G_0–orbit on $\mathcal{F}_{(k)}$, $k < \infty$, and thus no Bergman–Shilov boundary of D_0.

The calculations for D_0 to be a bounded symmetric domain are essentially the same as those in Section 4.1. The result is

Proposition 4.2.1. *D_0 is realized as the bounded domain consisting of all finitary $Z : V_- \to V_+$ such that the matrix of Z is symmetric and $I - Z^*Z \gg 0$. In that realization the action of G_0 is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : Z \to (AZ + B)(CZ + D)^{-1}$.*

Corollary 4.2.2. *The bounded symmetric domain for $Sp(\infty;\mathbb{R})$ is a totally geodesic submanifold of the bounded symmetric domain for $SU(\infty,\infty)$.*

As in Section 4.1 for $SU(\infty,\infty)$, the G_0–orbit of signature $(pos, neg, nul) = (a, b, c)$, a and c finite, is

$$\begin{aligned} D_{a,b,c} &= G_0((0, (F_+ + F_- + F_0), V)) \text{ where} \\ &F_0 = \text{Span}\,\{(e_{-c} + e_c), \dots, (e_{-1} + e_1)\} \ (h\text{–null}), \\ &F_+ = \text{Span}\,\{e_{-c-a}, \dots, e_{-c-1}\} \ (h\text{–positive definite}), \\ &F_- = \text{Span}\,\{e_{c+1}, e_{c+2}, \dots\} \ (h\text{–negative definite}), \end{aligned}$$

and every G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$ is one of those $D_{a,b,c}$. We always have $b = \dim F_- = \infty$. The open orbits are the case $c = 0$ mentioned above: $D_k = D_{k,b,0}$.

4.3 The Complex Bounded Symmetric Domain for $SO^*(\infty)$.

Next, we let $G_0 = SO^*(\infty)$. It is defined relative to the basis $E = \{\dots, e_{-3}, e_{-2}, e_{-1}; e_1, e_2, e_3, \dots\}$ by the hermitian form h and the symmetric bilinear form b,

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0,\ h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0,\ \text{and } b(e_i, e_j) = \delta_{i+j,0} \text{ for all } i, j.$$

The domain D_0 consists of the maximal h–negative definite b–isotropic subspaces of V that are weakly compatible with E. In other words, let $F = \text{Span}\,\{e_i \mid i > 0\}$. Evidently $\mathcal{F} = (0, F, V)$ is compatible with E and $G_0(\mathcal{F})$ is open in $\mathcal{Z}_{\mathcal{F},E}$. The bounded symmetric domain is

$$D_0 := G_0(\mathcal{F}) \subset \mathcal{Z}_{\mathcal{F},E}.$$

Again, $\mathcal{Z}_{\mathcal{F},E}$ is contained in the complex Grassmann manifold of Section 4.1 for $q = \infty$.

In Lie group terms, $D_0 \cong SO^*(\infty)/U(\infty)$ where $U(\infty)$ is the stabilizer of F. Let both $\mathcal{F}$ and $\mathcal{F}_{(0)}$ denote the flag $(0, F, V)$, so $D_0 = G_0(\mathcal{F}_{(0)})$. The open G_0–orbits on $\mathcal{Z}_{\mathcal{F},E}$ are the $D_k = G_0(\mathcal{F}_{(k)})$ where

$$F_{(k)} = \text{Span}\,\{e_{-k}, e_{-k+1}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ and } \mathcal{F}_{(k)} = (0, F_{(k)}, V)$$

for integers $k \geqq 0$. Note that $D_k \cong SO^*(\infty)/U(k, \infty - k)$ where the $\infty - k$ refers to the action on Span $\{e_{k+1}, e_{k+2}, \dots\}$. Also, $F_{(k)}^\perp = F_{(k)}$ relative to b, so $\mathcal{F}_{(k)}^\perp = \mathcal{F}_{(k)}$.

The corresponding $D_\infty := G_0(\mathcal{F}_{(\infty)})$ where $F_{(\infty)}$ is the h–orthocomplement Span $\{e_i \mid i < 0\}$ of F_0.

Here the partial Cayley transforms are not given by (4.1.5), but rather by

$$\begin{aligned} &c_k(e_{-2k}) = \tfrac{1}{\sqrt{2}}(e_{-2k} - e_{2k}), && c_k(e_{-2k+1}) = \tfrac{1}{\sqrt{2}}(e_{-2k+1} + e_{2k-1}),\\ &c_k(e_{2k-1}) = \tfrac{1}{\sqrt{2}}(-e_{-2k+1} + e_{2k-1}), && c_k(e_{2k}) = \tfrac{1}{\sqrt{2}}(e_{-2k} + e_{2k}),\\ &c_k(e_j) = e_j \text{ for } j \notin \{-2k, -2k+1, 2k-1, 2k\}. \end{aligned} \tag{4.3.1}$$

As in the SU and Sp settings, one passes from D_0 to D_k using $F_{(k)} = c_1^2 c_2^2 \dots c_k^2 F_{(0)}$ where the c_j are partial Cayley transforms defined by (4.3.1), as in [13]. Similarly, as in [11], the boundary of D_0 is the union of the orbits $G_0(c_1 c_2 \dots c_\ell F_{(0)})$, but here there is no closed G_0–orbit on $\mathcal{F}_{(k)}$, $k < \infty$, and thus no Bergman–Shilov boundary of D_0.

The calculations for D_0 to be a bounded symmetric domain are essentially the same as those in Section 4.1. The result is

Proposition 4.3.2. *D_0 is realized as the bounded domain consisting of all finitary $Z : V_- \to V_+$ such that the matrix of Z is antisymmetric and $I - Z^*Z \gg 0$. In that realization the action of G_0 is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : Z \to (AZ + B)(CZ + D)^{-1}$.*

Corollary 4.3.3. *The bounded symmetric domain for $SO^*(\infty)$ is a totally geodesic submanifold of the bounded symmetric domain for $SU(\infty, \infty)$.*

As in Section 4.2 for $Sp(\infty; \mathbb{R})$, the G_0–orbit of signature $(pos, neg, nul) = (a, b, c)$, a and c finite, is

$$\begin{aligned} D_{a,b,c} = G_0((0, (F_+ + F_- + F_0), V)) &\text{ where}\\ F_0 = \text{Span}\,\{(e_{-c} + e_c), \dots, (e_{-1} + e_1)\} &\ (h\text{–null}),\\ F_+ = \text{Span}\,\{e_{-c-a}, \dots, e_{-c-1}\} &\ (h\text{–positive definite}),\\ F_- = \text{Span}\,\{e_{c+1}, e_{c+2}, \dots\} &\ (h\text{–negative definite}), \end{aligned}$$

and every G_0–orbit on $\mathcal{Z}_{\mathcal{F},E}$ is one of those $D_{a,b,c}$. We always have $b = \dim F_- = \infty$. The open orbits are the case $c = 0$ mentioned above: $D_k = D_{k,b,0}$.

4.4 The Complex Bounded Symmetric Domain for $SO(\infty, 2)$.

This one is more delicate because the bounded domain for $SO(\infty, 2)$ does not sit as an easily described totally geodesic submanifold of the bounded domain for any of the $SU(\infty, q)$. Specifically, it is a bounded domain in a nondegenerate complex quadric in an infinite dimensional complex projective space.

We use a basis

$$E = \{\dots, e_{-3}, e_{-2}, e_{-1}, e_1, e_2\} \tag{4.4.1}$$

of V. $G_0 = SO(\infty, 2) = SO(\infty; \mathbb{C}) \cap U(\infty, 2)$ is the connected real semisimple Lie group defined by the following hermitian form h and the symmetric bilinear form b:

$$h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0,\ h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0,\ b(e_i, e_j) = \delta_{i,j} \text{ for all } i, j. \tag{4.4.2}$$

This is a finitary change from (3.2.3) and (3.2.4). The only effect of the change is to facilitate our study of bounded domains and cycle spaces for $SO(\infty, 2)$.

For $n > 0$ we have $E_n = \{e_{-n}, e_{-n+1}, \dots, e_{-1}, e_1, e_2\}$, the $(n+2)$–dimensional subspace $V_n = \mathrm{Span}\,(E_n)$ of V, the $(n+1)$–dimensional complex projective space $\mathcal{P}^{n+1} = \mathcal{P}(V_n)$, and the nondegenerate complex quadric $\mathcal{Z}_n = \{[v] \in \mathcal{P}^{n+1} \mid b(v,v) = 0\}$. They define the infinite dimensional complex projective space $\mathcal{P}^\infty = \mathcal{P}(V) = \varinjlim \mathcal{P}^{n+1}$ and the nondegenerate complex quadric $\mathcal{Z} = \{[v] \in \mathcal{P}^\infty \mid b(v,v) = 0\} = \varinjlim \mathcal{Z}_n$ in $\mathcal{P}^\infty$. Note that everything here is finitary. The complex group $G = SO(\infty;\mathbb{C})$ is transitive on $\mathcal{Z}$ because $SO(n+2;\mathbb{C})$ is transitive on $\mathcal{Z}_n$.

Our bounded symmetric domain will be $D_0 = G_0(z_0) \subset \mathcal{Z}$ where $z_0 = [e_1 + \sqrt{-1}\, e_2]$. We now look at the Harish-Chandra embedding of D_0 in its holomorphic tangent space. The Lie algebra

$$\mathfrak{g} = \left\{ \left(\begin{smallmatrix} A & B \\ -{}^tB & D \end{smallmatrix} \right) \Big| \ {}^tA = -A,\ {}^tD = -D \right\} \text{ where } A \in \mathbb{C}^{\infty\times\infty},\ B \in \mathbb{C}^{\infty\times 2},\ D \in \mathbb{C}^{2\times 2}$$

and the isotropy subalgebra at z_0 is the parabolic

$$\mathfrak{p} = \left\{ \left(\begin{smallmatrix} A & B \\ -{}^tB & D \end{smallmatrix} \right) \in \mathfrak{g} \ \Big| \ B = (B'', \sqrt{-1}\, B'') \right\} \text{ where } B'' \in \mathbb{C}^{\infty\times 1}.$$

The holomorphic tangent space to $\mathcal{Z}$ at z_0 is

$$\mathfrak{m}^+ = \left\{ \left(\begin{smallmatrix} 0 & B \\ -{}^tB & 0 \end{smallmatrix} \right) \Big| \ B = (\sqrt{-1}\, B'', B'') \text{ with } B'' \in \mathbb{C}^{\infty\times 1} \right\}.$$

We view $\mathfrak{m}^+$ as $\mathbb{C}^\infty$ (column vectors) Z under the correspondence

$$Z \mapsto \widetilde{Z} := \begin{pmatrix} 0 & Z' \\ -{}^tZ' & 0 \end{pmatrix} \text{ where } Z' = (\sqrt{-1}\, Z, Z) \in \mathbb{C}^{\infty\times 2}.$$

Computing as in [13], the composition $\xi : \mathbb{C}^\infty \to \mathfrak{m}^+ \to \mathcal{Z}$ corresponding to the Harish-Chandra embedding is

$$\xi(Z) = (\exp(\widetilde{Z})(z_0) = \begin{bmatrix} 2\sqrt{-1}\, Z \\ 1 + {}^tZ\, Z \\ \sqrt{-1}\,(1 - {}^tZ\, Z) \end{bmatrix} \in \mathcal{Z} \subset \mathcal{P}(V).$$

Now $h(\xi(Z), \xi(Z)) = 2Z^* \cdot 2Z - |1 +^t Z\, Z|^2 - |1 -^t Z\, Z|^2 < 0$, so

$$h(\xi(Z), \xi(Z)) < 0 \Leftrightarrow 1 + |{}^tZ\, Z|^2 - 2Z^*\, Z > 0 \Leftrightarrow (1 - Z^*\, Z)^2 > (Z^*\, Z)^2 - |{}^tZ\, Z|^2.$$

Using $Z^*\, Z \geqq |{}^tZ\, Z| \geqq 0$ we take positive square roots to see

$$\{Z \in \mathbb{C}^\infty \mid h(\xi(Z), \xi(Z)) < 0\} = D_0' \cup D_1' \text{ (disjoint)}$$

where D_0' is the nonempty bounded domain star shaped from 0,

$$D_0' = \{Z \in \mathbb{C}^\infty \mid 1 - Z^*\, Z > \left((Z^*\, Z)^2 - |{}^tZ\, Z|^2\right)^{1/2}\},$$

and D_1' is the nonempty unbounded domain star shaped from ∞,

$$D_1' = \{Z \in \mathbb{C}^\infty \mid Z^*\, Z - 1 > \left((Z^*\, Z)^2 - |{}^tZ\, Z|^2\right)^{1/2}\}.$$

Using Witt's Theorem on the finite dimensional approximations, $\xi^{-1}(D_0)$ is the topological component of $\{Z \in \mathbb{C}^\infty \mid h(\xi(Z), \xi(Z)) < 0\}$ containing 0 so $D_0' = \xi^{-1}(D_0)$. We have proved

Proposition 4.4.3. *The bounded symmetric domain D_0 for $SO(\infty, 2)$ is given by*

$$\begin{aligned} \xi^{-1}(D_0) &= \{Z \in \mathbb{C}^\infty \mid 1 - Z^*\, Z > \left((Z^*\, Z)^2 - |{}^tZ\, Z|^2\right)^{1/2}\} \\ &= \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZ\, Z|^2 - 2Z^*\, Z > 0 \textit{ and } Z^*\, Z < 1\}. \end{aligned}$$

Shortly we will see this in terms of partial Cayley transforms, but for the moment we mention that $D_1' = \xi^{-1}(D_1)$ where $z_1 = [e_1 - \sqrt{-1}\, e_2]$ and $D_1 = G_0(z_1)$ is given by

$$\begin{aligned}\xi^{-1}(D_1) &= \{Z \in \mathbb{C}^\infty \mid Z^* Z - 1 > \left((Z^* Z)^2 - |{}^t Z Z|^2\right)^{1/2}\} \\ &= \{Z \in \mathbb{C}^\infty \mid 1 + |{}^t Z Z|^2 - 2 Z^* Z > 0 \text{ and } Z^* Z > 1\}.\end{aligned}$$

The action of G_0 on D_0 is somewhat complicated because of the quadratic term $q : \mathbb{C}^\infty \to \mathbb{C}$ given by $q(Z) ={}^t Z Z$. If $Z \in \xi^{-1}(D_0)$ the $Z^* Z < 1$ so $|q(Z)| < 1$, and the formula for $\xi(Z)$ says

$$\text{if } q = q(Z) \neq 1 \text{ then } \xi(Z) = (\exp(\widetilde{Z})(z_0) = \begin{bmatrix} \frac{2\sqrt{-1}\, Z}{1+q(Z)} \\ 1 - \sqrt{-1}\, q(Z) \end{bmatrix} \in \mathcal{Z} \subset \mathcal{P}(V).$$

Now, by straightforward computation,

Proposition 4.4.4. *The action $g(Z) = \xi^{-1} g \xi(Z)$ of G_0 on the open orbit D_0 is given by*

$$\textit{if } g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \textit{ and } \left(\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right) = \begin{pmatrix} \frac{2\sqrt{-1}\, Z}{1+q(Z)} \\ 1-\sqrt{-1}\, q(Z) \end{pmatrix} \textit{ then } g(Z) = \frac{1}{(1,\sqrt{-1})(CZ_1 + DZ_2)}(AZ_1 + BZ_2).$$

Here $\frac{1}{(1,\sqrt{-1})(CZ_1+DZ_2)}$ is 1×1 and is viewed as a complex number.

Express $V = V_+ \oplus V_-$ where $V_+ = \mathrm{Span}\,\{e_i \mid i < 0\}$ and $V_- = \mathrm{Span}\,\{e_1, e_2\}$. Then $h(V_+, V_-) = 0 = b(V_+, V_-)$. Let $[v] \in \mathcal{Z} \subset \mathcal{P}(V)$ with $G_0([v])$ open in $\mathcal{Z}$, in other words with $h(v,v) \neq 0$. If $\pi_-(v) = a(e_1 + \sqrt{-1}\, e_2) + b(e_1 - \sqrt{-1}\, e_2)$ then $0 = b(v,v) = 2ab$. Replacing v within $[v]$ now the only possibilities are (i) $\pi_-(v) = (e_1 + \sqrt{-1}\, e_2)$, (ii) $\pi_-(v) = (e_1 - \sqrt{-1}\, e_2)$ and (iii) $v \in V_+$. The domains $D_0 = G_0([e_1 + \sqrt{-1}\, e_2])$ and $D_1 = G_0([e_1 - \sqrt{-1}\, e_2])$, so the possibilities (i) and (ii) correspond to D_0 and D_1. They are equivalent under complex conjugation and each has signature $(0,1,0)$. See Remark 3.2.5. The bounded symmetric domains D_0 and D_1 are of tube type.

The G_0–stabilizer of V_+, which is $SO(\infty) \times SO(2)$, is transitive on the projective light cone in V_+; thus the possibility (iii) corresponds to the domain $D_2 = G_0([e_{-1} + \sqrt{-1}\, e_{-2}])$, signature $(1,0,0)$. This completes the verification of

Lemma 4.4.5. *There are three open orbits for the action of $SO(\infty,2)$ on $\mathcal{Z}$: the two h–negative definite orbits $D_0 = G_0([e_1 + \sqrt{-1}\, e_2])$ and $D_1 = G_0([e_1 - \sqrt{-1}\, e_2])$, and the h–positive definite orbit $D_2 = G_0([e_{-1} + \sqrt{-1}\, e_{-2}])$.*

Now we do this more carefully with the partial Cayley transforms. Each $c_i(e_j) = e_j$ for $j \notin \{-2,-1,1,2\}$. In the basis $\{e_{-2}, e_{-1}, e_1, e_2\}$ of $\mathrm{Span}\,\{e_{-2}, e_{-1}, e_1, e_2\}$, the c_i have matrices

$$c_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \text{ and } c_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \tag{4.4.6}$$

Thus there are six G_0–orbits on $\mathcal{Z}$. Their base points are

$$\begin{array}{lll} z_{0,0} = z_0 & = [e_1 + \sqrt{-1}\, e_2] & \text{(negative)} \\ z_{0,1} = c_1^2 z_0 & = [e_{-2} + \sqrt{-1}\, e_{-1}] & \text{(positive)} \\ z_{0,2} = c_1^2 c_2^2 z_0 & = [e_1 - \sqrt{-1}\, e_2] & \text{(negative)} \\ z_{1,1} = c_1 z_0 & = [e_{-2} + \sqrt{-1}\, e_{-1} + e_1 + \sqrt{-1}\, e_2] & \text{(isotropic)} \\ z_{1,2} = c_1 c_2^2 z_0 & = [e_{-2} - \sqrt{-1}\, e_{-1} - e_1 + \sqrt{-1}\, e_2] & \text{(isotropic)} \\ z_{2,2} = c_1 c_2 z_0 & = [e_{-2} + \sqrt{-1}\, e_2] & \text{(isotropic)} \end{array} \tag{4.4.7}$$

That gives 3 open orbits $D_0 = G_0(z_{0,0})$, $D_2 = G_0(z_{0,1})$ and $D_1 = G_0(z_{0,2})$; it gives two intermediate orbits $G_0(z_{1,1})$ and $G_0(z_{1,2})$; and it gives one closed orbit $G_0(z_{2,2})$.

4.5 Bounded Symmetric Domains for $SL(\infty;\mathbb{R})$ and $SL(\infty;\mathbb{H})$.

There are no complex bounded symmetric domains for these $SL(m;\mathbb{F})$, $m < \infty$, except the unit disk in $\mathbb{C}$, corresponding to $SL(2;\mathbb{R}) \cong SU(1,1) \cong SL(1;\mathbb{H})$. In particular there is no complex bounded symmetric domain for $SL(\infty;\mathbb{R})$, and there is none for $SL(\infty;\mathbb{H})$.

5 Cycles and Cycle Spaces

In the finite dimensional setting, where D is an open G_0–orbit (flag domain) in $\mathcal{Z} = G/Q$, a maximal compact subgroup $K_0 \subset G_0$ has just one orbit Y on D that is a complex submanifold [12]. The G–translates of Y that are contained in D form the *cycle space* $\mathcal{M}_D$. That cycle space is sometimes called the *universal domain* or *crown* of the flag domain. It has many uses in harmonic analysis and algebraic geometry; see [3]. It also has remarkable complex–geometric and function–analytic properties; for example it is a contractible Stein manifold, it has an explicit geometric description, and it is the key ingredient for the double fibration transform of which one special case is the Penrose Transform. Here we extend some of the basic results on cycle spaces to an infinite dimensional setting.

5.1 Basic Results.

We fix an open G_0–orbit D in the complex flag manifold $\mathcal{Z}_{E,\mathcal{F}} \cong G/Q$ with E as described in Section 3 and $\mathcal{F}$ compatible with E. (The results of Section 3.7 show when there are open G_0–orbits in $\mathcal{Z}_{E,\mathcal{F}}$.) Let K_0 be a maximal lim–compact subgroup of G_0 and let $K \subset G$ be its complexification. As in the finite dimensional case [3, Theorem 4.3.1],

Theorem 5.1.1. *There is a unique orbit $K_0(z) \subset D$ such that $K_0(z)$ is a complex submanifold of the flag manifold $\mathcal{Z}_{\mathcal{F},E}$. Further, $K \cap Q_z$ is a parabolic subgroup of K and $K_0(z) = K(z) \cong K/(K \cap Q_z)$, so $K_0(z)$ is a complex flag manifold.*

If $C \subset D$ is a lim–compact complex submanifold then the following are equivalent: (i) *C is a K_0–orbit,* (ii) *C is a K–orbit, and* (iii) *$C = K_0(z)$.*

Proof. The idea is to use the bases of Section 3 together with the results of [2, Section 6] in order to take a direct limit using the finite dimensional flag domain result of [3, Theorem 4.3.1].

We run through the cases of Section 3. In each case, the basis E of V is a disjoint union of finite sets E_ℓ where (i) if there is a hermitian form h then the Span $\{E_\ell\}$ are h–nondegenerate and mutually h–orthogonal, (ii) if there is a bilinear form b then the Span $\{E_\ell\}$ are b–nondegenerate and mutually b–orthogonal as well. Further, we may assume that ℓ runs over the positive integers,

Denote $\widetilde{E}_\ell = \bigcup_{k<\ell} E_k$ and $V_\ell = \text{Span}\,\{\widetilde{E}_\ell\}$. In view of (i) and (ii) just above, $G_0 = \varinjlim G_{\ell,0}$ where $G_{\ell,0} = \{g \in G_0 \mid gV_\ell = V_\ell\}|_{V_\ell}$ is a finite dimensional real simple Lie group, real form of the finite dimensional complex simple Lie group $G_\ell = \{g \in G \mid gV_\ell = V_\ell\}|_{V_\ell}$. Further, $Q = \varinjlim Q_\ell$ where Q_ℓ is the G_ℓ–stabilizer of $\mathcal{F}$.

We need a result of Dimitrov and Penkov [2, Proposition 6.1]. They assume that Q contains a splitting Cartan subgroup of G, but the argument is valid, as in our case, when each Q_ℓ contains a splitting Cartan subgroup H_ℓ of G_ℓ with $H_\ell \subset H_m$ for $\ell \leqq m$. Denote $\widetilde{E}_\ell = \bigcup_{k \leqq \ell} E_k$. Then $\mathcal{Z}_{\mathcal{F},E} = \varinjlim \mathcal{Z}_{\mathcal{F}\cap V_\ell, \widetilde{E}_\ell}$ where we either eliminate or ignore repetitions in the $\mathcal{F} \cap V_\ell$.

Since D is open in $\mathcal{Z}_{\mathcal{F},E}$, the flag $\mathcal{F}$ is nondegenerate in V, so by construction of the $\widetilde{E}_\ell$ each flag $\mathcal{F} \cap V_\ell$ is nondegenerate in V_ℓ. Thus $G_{\ell,0}(\mathcal{F} \cap V_\ell)$ is open in $\mathcal{Z}_{\mathcal{F}\cap V_\ell, \widetilde{E}_\ell}$ for each ℓ. It follows [12, Theorem 2.12] that, for each ℓ, Q_ℓ contains a fundamental Cartan subgroup $T_{\ell,0}$

of $G_{\ell,0}$. Any two fundamental Cartan subgroups of $G_{\ell,0}$ are conjugate, and if $k \leqq \ell$ then any fundamental Cartan subgroup of $G_{k,0}$ is contained in a fundamental Cartan subgroup of $G_{\ell,0}$. Thus we may assume $T_{k,0} \subset T_{\ell,0}$ for $k \leqq \ell$.

The fundamental Cartan $T_{\ell,0}$ determines a maximal compact subgroup $K_{\ell,0}$ of $G_{\ell,0}$ such that $T_{\ell,0} \cap K_{\ell,0}$ is a compact Cartan subgroup of $K_{\ell,0}$. Let K_ℓ denote the complexification of $K_{\ell,0}$. Now $K_{k,0} \subset K_{\ell,0}$ and $K_k \subset K_\ell$ for $k \leqq \ell$. Following [3, Theorem 4.3.1], $K_{\ell,0}(\mathcal{F} \cap V_\ell)$ is the unique $K_{\ell,0}$–orbit in $G_{\ell,0}(\mathcal{F} \cap V_\ell)$ that is a complex submanifold of $\mathcal{Z}_{\mathcal{F}\cap V_\ell, \widetilde{E}_\ell}$, and $K_{\ell,0}(\mathcal{F} \cap V_\ell) = K_\ell(\mathcal{F} \cap V_\ell)$.

Suppose for the moment that $K_0 = \varinjlim K_{\ell,0}$. Then $K_0(\mathcal{F})$ is the unique K_0–orbit in D that is a complex submanifold of $\mathcal{Z}_{\mathcal{F},E}$ and $K_0(\mathcal{F}) = K(\mathcal{F})$. Theorem 5.1.1 follows for this particular maximal lim–compact subgroup K_0 in G_0. But any two maximal lim–compact subgroups of G_0 are conjugate, so Theorem 5.1.1 follows for every choice of K_0. □

Let K_0 be the maximal lim–compact subgroup of G_0 constructed above in the proof of Theorem 5.1.1. We will use the notation

(5.1.2) $Y = K_0(\mathcal{F}),\ \ G\{Y\} = \{g \in G \mid gY \subset D\},\ \ G_Y = \{g \in G \mid gY = Y\},\ \ \mathcal{M}'_D = G\{Y\} \cdot Y.$

where Y is the complex K_0–orbit in the open G_0–orbit $D \subset \mathcal{Z}_{\mathcal{F},E}$. We refer to Y as the **base cycle** in D. Note that the elements of $G\{Y\}$ do not have to map D into itself; $G\{Y\}$ simply is the set of of all elements in the complex group that keep the base cycle Y inside D. Further, $\mathcal{M}'_D := G\{Y\} \cdot Y$ is the set of all such G–translates of Y.

Lemma 5.1.3. *$G\{Y\}$ is an open subset of G, G_Y is a closed complex subgroup of G, and $\mathcal{M}'_D = G\{Y\}/G_Y$ is an open subset of the complex manifold G/G_Y. In particular $\mathcal{M}'_D$ is an open complex submanifold of G/G_Y.*

Proof. For each ℓ, $G_Y \cap G_\ell$ is a closed complex subgroup of G_ℓ and $G\{Y\} \cap G_\ell$ is an open subset of G_ℓ. It follows that G_Y is a closed complex subgroup of G and $G\{Y\}$ is an open subset of G. Now $G\{Y\}/G_Y$ is open in the complex homogeneous space G/G_Y. □

The complex manifold structure of $\mathcal{M}'_D$ specifies its topology, and we define

Definition 5.1.4. *Let $\mathcal{M}_D$ denote the topological component of Y in $\mathcal{M}'_D$. Then $\mathcal{M}_D$ is the* **cycle space** *of the flag domain D.*

Note that $\mathcal{M}_D$ is not always the same as the Barlet cycle space [9] from the theory of complex analytic spaces. See [3, Part IV] for the comparison. Next, we discuss several cases where we can really pin down the structure of $\mathcal{M}_D$.

5.2 Cycle Spaces for $SU(\infty, q)$, $q \leqq \infty$.

In this section $G_0 = SU(\infty, q)$ and its maximal lim-compact subgroup is

$$\begin{aligned} K_0 &= S(U(\infty) \times U(q)) &&= \lim_{p\to\infty} S(U(p) \times U(q)) \text{ if } q < \infty, \\ K_0 &= S(U(\infty) \times U(\infty)) &&= \lim_{r,s\to\infty} S(U(r) \times U(s)) \text{ if } q = \infty. \end{aligned}$$

This corresponds to an h–orthogonal decomposition

$$(5.2.1)\quad \begin{aligned} \mathbb{C}^{\infty,q} &= V_+ \oplus V_- \text{ where } V_+ = \mathrm{Span}\,\{\dots, e_{-3}, e_{-2}, e_{-1}\},\ V_- = \mathrm{Span}\,\{e_1, \dots, e_q\} \text{ or} \\ \mathbb{C}^{\infty,\infty} &= V_+ \oplus V_- \text{ where } V_+ = \mathrm{Span}\,\{\dots, e_{-3}, e_{-2}, e_{-1}\},\ V_- = \mathrm{Span}\,\{e_1, e_2, e_3, \dots\}. \end{aligned}$$

Here we use the related orthogonal basis E given by (3.1.1) and the hermitian form h of (3.1.2) that defines G_0. Let $\mathcal{F} = (F_k)$ be a generalized flag in $V = \mathbb{C}^{\infty,q}$ that is weakly compatible with E. Let $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$ so that $D = G_0(\mathcal{F}^{(1)})$ is an open G_0–orbit. Then we may assume that $\mathcal{F}^{(1)}$ is compatible with our choice of E, so it fits the decomposition (5.2.1) in the sense that

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-). \tag{5.2.2}$$

Then $K_0(\mathcal{F}^{(1)})$ is the unique K_0–orbit in D that is a complex submanifold of the flag manifold $\mathcal{Z}_{\mathcal{F},E}$. Concretely, $K_0(\mathcal{F}^{(1)})$ is the product of "smaller" complex flag manifolds,

$$\begin{aligned} &Y = Y_1 \times Y_2 \text{ where} \\ &\qquad Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and} \\ &\qquad Y_2 = K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_- \end{aligned} \tag{5.2.3}$$

where $\mathcal{F}^{(1)} \cap V_+$ is the generalized flag of the $(F_k^{(1)} \cap V_+)$ and $\mathcal{F}^{(1)} \cap V_-$ is the generalized flag of the $(F_k^{(1)} \cap V_-)$, ignoring repetitions. The signature sequence $\{(a_k, b_k)\}$, where h has signature $(a_k, b_k, 0)$ on $F_k^{(1)}$, specifies the open orbit in $\mathcal{Z}_{\mathcal{F},E}$ and the factors of Y.

As in the finite dimensional case, this shows that the G–translates of Y contained in D correspond to the decompositions $V = W' \oplus W''$ where (i) W' is a maximal positive definite subspace such that $V_+ \cap W'$ has finite codimension in both in V_+ and in W', and (ii) W'' is a maximal negative definite subspace such that $V_- \cap W''$ has finite codimension in both in V_- and in W''. If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$ the correspondence depends only on W', and if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ it depends only on W''. Any two such decompositions $V = W' \oplus W''$ are G–equivalent.

Definition 5.2.4. The *positive bounded symmetric domain* $\mathcal{B}_E^+$ associated to (V, E) is the space of all maximal positive definite subspaces $W' \subset V$ such that $W' \cap V_+$ has finite codimension in both W' and V_+. The *negative bounded symmetric domain* $\mathcal{B}_E^-$ associated to (V, E) is the space of all maximal negative definite subspaces $W'' \subset V$ such that $W'' \cap V_-$ has finite codimension in both W'' and V_-.

As constructed, each element $W' \in \mathcal{B}_E^+$ is in the G–orbit of V_+. Relative to the basis E we look at $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ such that $gW' \in \mathcal{B}_E^+$, in other words such that the column span of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is positive definite. The column span is preserved under right multiplication by A, so the positive definite condition is $\left(\begin{smallmatrix} I \\ -CA^{-1} \end{smallmatrix}\right)^* \cdot \left(\begin{smallmatrix} I \\ CA^{-1} \end{smallmatrix}\right) \gg 0$. In other words $gW' \in \mathcal{B}_E^+$ simply means that gW' is the column span of an infinite matrix $\left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right)$ such that $I - Z_1^* Z_1 \gg 0$. Similarly $gW'' \in \mathcal{B}_E^-$ simply means that gW'' is the column span of an infinite matrix $\left(\begin{smallmatrix} Z_2 \\ I \end{smallmatrix}\right)$ such that $I - Z_2 Z_2^* \gg 0$. The distinction is that the G–stabilizer of $V_+ \in \mathcal{B}_E^+$ is the parabolic P consisting of all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, while the G–stabilizer of $V_- \in \mathcal{B}_E^-$ is the opposite parabolic ${}^tP = P^{opp}$ consisting of all $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$. Thus they have conjugate complex structures: $\mathcal{B}_E^+$ has holomorphic tangent space represented by the matrices $\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right) \in \mathfrak{g}$ while the holomorphic tangent space of $\mathcal{B}_E^-$ is represented by the $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{g}$.

Reformulating this,

Lemma 5.2.5. *Suppose that $G_0 = SU(\infty, q)$, $q \leqq \infty$. Then the positive bounded symmetric domain associated to the triple (V, G_0, E) is $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty \times q} \mid I - Z_1^* Z_1 \gg 0\}$ in G/P, and the negative bounded symmetric domain for (V, G_0, E) is the complex conjugate domain $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{q \times \infty} \mid I - Z_2^* Z_2 \gg 0\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is described in Section 4.1.

Now we are ready to prove the following theorem.

Theorem 5.2.6. *Let $G_0 = SU(\infty, q)$ with $q \leqq \infty$. Let D be an open G_0–orbit $G(\mathcal{F}^{(1)})$ in $\mathcal{Z}_{\mathcal{F},E}$. In the notation of* (5.2.1)*, the positive definite bounded symmetric domain $\mathcal{B}_E^+$ for (V, G_0, E) is* *the set of all positive definite G-translates of V_+ and the negative definite bounded symmetric domain $\mathcal{B}_E^-$ for (V, G_0, E) is the set of is the set of all negative definite G-translates of V_- . The $\mathcal{B}_E^{\pm}$ are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+$.

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^-$.

If some $F_k^{(1)} \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+ \times \mathcal{B}_E^-$.

Proof. Directly from Definition 5.2.4, gV_+ is h–positive definite if and only if $gV_+ \in \mathcal{B}_E^+$, gV_- is h–negative definite if and only if $gV_- \in \mathcal{B}_E^-$, and both properties hold for $gV_\pm$ if and only if $(gV_+, gV_-) \in \mathcal{B}_E^+ \times \mathcal{B}_E^-$.

First suppose that $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$, $g \in G$ and $k \in K_0$. Note $kV_+ = V_+$. If gV_+ is positive definite then $gk(\mathcal{F}^{(1)}) \in D$ because $gk(\mathcal{F}^{(1)})$ is nondegenerate and D is the only open G_0–orbit in $\mathcal{Z}_{\mathcal{F},E}$ consisting of positive definite subspaces. Thus $gY \subset D$, in other words $gY \in \mathcal{M}'_D$. Conversely if $gY \in \mathcal{M}'_D$, so $gY \subset D$, then gY consists of positive definite subspaces. If $0 \neq F^{(1)} \in \mathcal{F}^{(1)}$ then $\mathrm{Span}\, K_0(F^{(1)}) = V_+$, so $\mathrm{Span}\, gY = gV_+$ is positive definite. Now $gY \in \mathcal{M}'_D$ if and only if $gV_+ \in \mathcal{B}_E^+$.

Similarly, if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ and $g \in G$ then $gY \in \mathcal{M}'_D$ if and only if $gV_- \in \mathcal{B}_E^-$.

In the general case $\mathcal{F}^{(1)} \cap V_+ \neq \mathcal{F}^{(1)} \neq \mathcal{F}^{(1)} \cap V_-$ the arguments just above show that gV_+ is positive definite if and only if $gK_0(\mathcal{F}^{(1)} \cap V_+)$ consists of positive definite subspaces; and gV_- is negative definite if and only if $gK_0(\mathcal{F}^{(1)} \cap V_-)$ consists of negative definite subspaces. Thus $gY \in \mathcal{M}'_D$ if and only if gV_+ is positive definite and gV_- is negative definite, in other words if and only if $(gV_+, gV_-) \in \mathcal{B}_E^+ \times \mathcal{B}_E^-$.

In all three cases we note that $\mathcal{M}'_D$ is connected, so $\mathcal{M}'_D = \mathcal{M}_D$.

Finally, h–orthocomplementation is antiholomorphic and interchanges $\mathcal{B}_E^+$ with $\mathcal{B}_E^-$. □

5.3 Cycle Spaces for $Sp(\infty; \mathbb{R})$.

The case $G_0 = Sp(\infty; \mathbb{R}) = Sp(\infty; \mathbb{C}) \cap U(\infty, \infty)$ differs from the $SU(\infty, q)$ cases mainly in that we use b–isotropic flags where b is the antisymmetric bilinear form that defines $Sp(\infty; \mathbb{C})$. Specifically, we use the basis and forms described in Section 3, given by (3.4.1) and (3.5.1), where b defines $Sp(\infty; \mathbb{C})$ and h defines $U(\infty, \infty)$.

The maximal lim-compact subgroups K_0 of $G_0 = Sp(\infty; \mathbb{R})$ is the $U(\infty)$ constructed as follows. Relative to h,

(5.3.1) $\quad V = V_+ \oplus V_-$ where $V_+ = \mathrm{Span}\,\{\dots, e_{-3}, e_{-2}, e_{-1}\}$ and $V_- = \mathrm{Span}\,\{e_1, e_2, e_3, \dots\}$.

The maximal lim-compact subgroup of $U(\infty, \infty)$ is $U(V_+) \times U(V_-) = U(\infty) \times U(\infty)$, and K_0 is the subgroup $G_0 \cap \big(U(\infty) \times U(\infty)\big) \cong U(\infty)$. In the ordered basis $\{e_{-1}, e_{-2}, \dots; e_1, e_2, \dots\}$ it would be diagonally embedded in $U(\infty) \times U(\infty)$.

Let $\mathcal{F}$ be a b–isotropic generalized flag compatible with E. Let $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$ so that $D = G_0(\mathcal{F}^{(1)})$ is open in $\mathcal{Z}_{\mathcal{F},E}$. Again, we may assume that $\mathcal{F}^{(1)}$ is compatible with E, in other words, it fits the splitting (5.3.1) in the sense that

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

In particular $\mathcal{F}^{(1)}$ is h–nondegenerate, corresponding to the fact that $D = G_0(\mathcal{F})$ is open in $\mathcal{Z}_{\mathcal{F},E}$, and $K_0(\mathcal{F}^{(1)})$ is the unique K_0–orbit in D that is a complex submanifold of $\mathcal{Z}_{\mathcal{F},E}$.

Lemma 5.3.2. *Define* $\mathcal{F}^{(1)} \cap V_+ = (F_k^{(1)} \cap V_+)$ *and* $\mathcal{F}^{(1)} \cap V_- = (F_k^{(1)} \cap V_-)$*, and spaces* $W_+ = \bigcup_k (F_k^{(1)} \cap V_+)$ *and* $W_- = \bigcup_k (F_k^{(1)} \cap V_-)$*. Then the complex lim–compact group orbit* $Y = K_0(\mathcal{F}^{(1)})$ *is the subvariety of*

$$\widetilde{Y} = Y_1 \times Y_2 \text{ where } Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and}$$
$$Y_2 = K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_-$$

defined by $b(k_1W_+, k_2W_-) = 0$ *for* $k_1, k_2 \in K_0$ *. The signature sequence* $\{(a_k, b_k)\}$*, where* h *has signature* $(a_k, b_k, 0)$ *on* $F_k^{(1)}$*, specifies the open orbit in* $\mathcal{Z}_{\mathcal{F},E}$ *and the factors of* $\widetilde{Y}$*.*

Proof. The projections $r_1 : K_0 \to U(V_+)$ and $r_2 : K_0 \to U(V_-)$ are isomorphisms. Define $\mu : V \to V$ by $\mu(e_i) = e_{-i}$ and $\mu(e_{-i}) = -e_i$ for $i > 0$. Since $\mathcal{F}^{(1)}$ is b–isotropic and compatible with E, each $F_k^{(1)}$ is spanned by a subset $S_k \subset E$ that never contains a pair $\{e_i, e_{-i}\}$. Thus each $(\mathcal{F}_k^{(1)} \cap V_+) + \mu(\mathcal{F}_k^{(1)} \cap V_+)$ is b–nondegenerate and h–nondegenerate, and is orthogonal to $(\mathcal{F}_k^{(1)} \cap V_-) + \mu(\mathcal{F}_k^{(1)} \cap V_-)$ relative to both b and h. Now the action of $r_1(K_0)$ on $(\mathcal{F}_k^{(1)} \cap V_+) + \mu(\mathcal{F}_k^{(1)} \cap V_+)$ and the action of $r_2(K_0)$ on $(\mathcal{F}_k^{(1)} \cap V_-) + \mu(\mathcal{F}_k^{(1)} \cap V_-)$ only involve disjoint subsets of $S_k \cup -S_k$. Thus $Y \subset \widetilde{Y}$ and $b(k_1W_+, k_2W_-) = 0$ for $k_1, k_2 \in K_0$.

Conversely, if $(k_1(\mathcal{F}_k^{(1)} \cap V_+), k_2(\mathcal{F}_k^{(1)} \cap V_+)) \in Y$, so it has form $(k(\mathcal{F}_k^{(1)} \cap V_+), k(\mathcal{F}_k^{(1)} \cap V_+))$, then $k_1W_+ = kW_+ \perp_b kW_- = k_2W_-$. Given $kW_+ \perp_b kW_-$, K_0 moves $(\mathcal{F}_k^{(1)} \cap V_+)$ freely within $V_+ \cap (W_-)^\perp$ and moves $(\mathcal{F}_k^{(1)} \cap V_-)$ freely within $V_+ \cap (W_+)^\perp$. That proves the first assertion. The signature sequence assertion is contained in Theorem 3.7.2. □

These considerations show that the G–translates of Y contained in D correspond to decompositions $V = W' \oplus W''$ where (i) W' and W'' are maximal b–isotropic subspaces of V, (ii) W' is a maximal h–positive definite subspace such that $W' \cap V_+$ has finite codimension in both W' and V_+ , and (iii) W'' is a maximal h–negative definite subspace such that $W'' \cap V_-$ has finite codimension in both W'' and V_- . If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$ the correspondence depends only on W', and if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ it depends only on W''. Any two such decompositions $V = W' \oplus W''$ are G–equivalent.

Definition 5.3.3. The *positive bounded symmetric domain* $\mathcal{B}_E^+$ associated to (V, b, E) is the space of all maximal b–isotropic h–positive definite subspaces $W' \subset V$ such that $W' \cap V_+$ has finite codimension in both W' and V_+ . The *negative bounded symmetric domain* $\mathcal{B}_E^-$ associated to (V, b, E) is the space of all maximal b–isotropic h–negative definite subspaces $W'' \subset V$ such that $W'' \cap V_-$ has finite codimension in both W'' and V_- .

As constructed, each element $W' \in \mathcal{B}_E^+$ is in the G–orbit of V_+ . Relative to the basis E we look at $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ such that $gW' \in \mathcal{B}_E^+$, in other words such that the column span of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is h–positive definite. That span is b–isotropic by definition of G, and the column span is preserved under right multiplication by A. Let $Z_1 = CA^{-1}$. Then the h–positive definite condition is $\left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right)^* \cdot \left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right) \gg 0$, in other words $I - Z_1^*Z_1 \gg 0$. Let $Z_1 = (z_{i,j})$ where $i, j > 0$. The column span of $\left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right)$ has basis consisting of the $z_j := e_{-j} + \sum_{i>0} z_{i,j}e_i$, Compute $b(z_j, z_\ell) = z_{j,\ell} - z_{\ell,j}$. So the b–isotropic condition is $Z_1 =^t Z_1$. In other words $gW' \in \mathcal{B}_E^+$ simply means that gW' is the column span of an infinite matrix $\left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right)$ such that $I - Z_1^*Z_1 \gg 0$ and Z_1 is symmetric.

Similarly $gW'' \in \mathcal{B}_E^-$ simply means that gW'' is the column span of an infinite matrix $\left(\begin{smallmatrix} Z_2 \\ I \end{smallmatrix}\right)$ such that $I - Z_2Z_2^* \gg 0$ and Z_2 is symmetric. The distinction is that the G–stabilizer of $V_+ \in \mathcal{B}_E^+$ is the parabolic P consisting of all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ in $\mathfrak{g}$ while the G–stabilizer of $V_- \in \mathcal{B}_E^-$ is the opposite parabolic ${}^tP = P^{opp}$ consisting of all $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ in $\mathfrak{g}_\mathbb{C}$. Thus they have conjugate complex structures: $\mathcal{B}_E^+$ has holomorphic tangent space represented by the matrices $\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right)$ with

C symmetric while the holomorphic tangent space of $\mathcal{B}_E^-$ is represented by the $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ with B symmetric.

Reformulating this,

Lemma 5.3.4. *Let $G_0 = Sp(\infty;\mathbb{R})$. Then the positive bounded symmetric domain associated to the triple (V, G_0, E) is $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty\times\infty} \mid I - Z_1^* Z_1 \gg 0 \text{ and } Z_1 = {}^tZ_1\}$ in G/P, and the negative bounded symmetric domain for (V, G_0, E) is the complex conjugate domain $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{\infty\times\infty} \mid I - Z_2^* Z_2 \gg 0 \text{ and } Z_1 = {}^tZ_1\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is described in Section 4.2.

In any K_0–invariant Riemannian metric on $\widetilde{Y}$, Y_1 and Y_2 are the factors in the de Rham decomposition. The spaces $k(F_\ell^{(1)} \cap V_+)$ of the elements of Y_1 generate V_+ (or are zero), so either Y determines Y_1 determines V_+ , or the $F_\ell^{(1)} \cap V_+ = 0$. Similarly either Y determines V_- , or the $F_\ell^{(1)} \cap V_- = 0$. Now apply g^{-1} whenever $g \in G\{Y\}$ to see that gY determines gV_+ or gV_- or both, as appropriate. Exactly as in the proof of Theorem 5.2.6 we arrive at the following structure theorem.

Theorem 5.3.5. *Let $G_0 = Sp(\infty;\mathbb{R})$ and let D be an open G_0–orbit $G(\mathcal{F}^{(1)})$ in $\mathcal{Z}_{\mathcal{F},E}$. In the notation of (5.3.1), the positive definite bounded symmetric domain $\mathcal{B}_E^+$ for (V, G_0, E) is the set of all positive definite G-translates of V_+ and the negative definite bounded symmetric domain $\mathcal{B}_E^-$ for (V, G_0, E) is the set of is the set of all negative definite G-translates of V_- . The $\mathcal{B}_E^\pm$ are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+$.

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^-$.

If some $F_k^{(1)} \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+ \times \mathcal{B}_E^-$.

5.4 Cycle Spaces for $SO^*(\infty)$.

The case $SO^*(\infty) = SO(\infty;\mathbb{C}) \cap U(\infty,\infty)$ is very similar to the case of $Sp(\infty;\mathbb{R})$. The main difference is that the bilinear form b is symmetric rather than antisymmetric. Concretely, we have

$$E = \{\dots, e_{-k}, e_{-k+1}, \dots, e_{-1}; e_1, \dots, e_{k-1}, e_k, \dots\}, \text{ ordered basis of } V;$$
$$b(e_i, e_j) = \delta_{i+j,0}\, , h(e_i, e_j) = \delta_{i,j} \text{ for } i < 0 \text{ and } h(e_i, e_j) = -\delta_{i,j} \text{ for } i > 0.$$

Again we use the h–orthogonal splitting

$$(5.4.1) \qquad V = V_+ \oplus V_- \text{ where } V_+ = \text{Span}\,\{\dots, e_{-3}, e_{-2}, e_{-1}\} \text{ and } V_- = \text{Span}\,\{e_1, e_2, e_3, \dots\}.$$

The maximal lim-compact subgroup of $U(\infty,\infty)$ is $U(V_+) \times U(V_-) = U(\infty) \times U(\infty)$. Exactly as in the $Sp(\infty;\mathbb{R})$ case, K_0 is the subgroup $G_0 \cap \big(U(\infty) \times U(\infty)\big) \cong U(\infty)$. In the ordered basis $\{e_{-1}, e_{-2}, \dots; e_1, e_2, \dots\}$ it would be diagonally embedded in $U(\infty) \times U(\infty)$.

Let $\mathcal{F}$ be a b–isotropic generalized flag compatible with E. Let $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E}$ such that $D = G_0(\mathcal{F}^{(1)})$ is open in $\mathcal{Z}_{\mathcal{F},E}$. We may assume that $\mathcal{F}$ is compatible with E, so

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap V_+) \oplus (F_k^{(1)} \cap V_-).$$

In particular $\mathcal{F}^{(1)}$ is h–nondegenerate and $K_0(\mathcal{F}^{(1)})$ is the unique K_0–orbit in D that is a complex submanifold of $\mathcal{Z}_{\mathcal{F},E}$. With no nontrivial change, the proof of Lemma 5.3.2 also proves

Lemma 5.4.2. *Define $\mathcal{F}^{(1)} \cap V_+ = (F_k^{(1)} \cap V_+)$ and $\mathcal{F}^{(1)} \cap V_- = (F_k^{(1)} \cap V_-)$, and spaces $W_+ = \bigcup_k (F_k^{(1)} \cap V_+)$ and $W_- = \bigcup_k (F_k^{(1)} \cap V_-)$. Then the complex lim–compact group orbit $Y = K_0(\mathcal{F}^{(1)})$ is the subvariety of*

$$\widetilde{Y} = Y_1 \times Y_2 \text{ where } Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = U(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and}$$
$$Y_2 = K_0(\mathcal{F}^{(1)} \cap V_-) = U(q)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_-$$

defined by $b(k_1 W_+, k_2 W_-) = 0$ for $k_1, k_2 \in K_0$. The signature sequence $\{(a_k, b_k)\}$, and (where relevant – see Remark 3.2.5) *the orientation, specifies the open orbit in $\mathcal{Z}_{\mathcal{F},E}$ and the factors of $\widetilde{Y}$.*

Now the G–translates of Y contained in D correspond to decompositions $V = W' \oplus W''$ where (i) W' and W'' are maximal b–isotropic subspaces of V, (ii) W' is a maximal h–positive definite subspace such that $W' \cap V_+$ has finite codimension in both W' and V_+, and (iii) W'' is a maximal h–negative definite subspace such that $W'' \cap V_-$ has finite codimension in both W'' and V_-. If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$ the correspondence depends only on W', and if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$ it depends only on W''. Any two such decompositions $V = W' \oplus W''$ are G–equivalent.

Definition 5.4.3. The *positive bounded symmetric domain* $\mathcal{B}_E^+$ associated to (V, b, E) consists of all maximal b–isotropic h–positive definite subspaces $W' \subset V$ such that $W' \cap V_+$ has finite codimension in both W' and V_+. The *negative bounded symmetric domain* $\mathcal{B}_E^-$ associated to (V, b, E) consists of all maximal b–isotropic h–negative definite subspaces $W'' \subset V$ such that $W'' \cap V_-$ has finite codimension in both W'' and V_-.

Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ relative to E, such that $gW' \in \mathcal{B}_E^+$. Computing as for $Sp(\infty; \mathbb{R})$, with $Z_1 = CA^{-1}$, we see that $gW' \in \mathcal{B}_E^+$ if and only if gW' is the column span of an infinite matrix $\left(\begin{smallmatrix} I \\ Z_1 \end{smallmatrix}\right)$ such that $I - Z_1^* Z_1 \gg 0$ and Z_1 is antisymmetric. Similarly $gW'' \in \mathcal{B}_E^-$ if and only if gW'' is the column span of an infinite matrix $\left(\begin{smallmatrix} Z_2 \\ I \end{smallmatrix}\right)$ such that $I - Z_2 Z_2^* \gg 0$ and Z_2 is antisymmetric. The G–stabilizer of $V_+ \in \mathcal{B}_E^+$ is the parabolic P consisting of all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$ in $\mathfrak{g}_\mathbb{C}$, and the G–stabilizer of $V_- \in \mathcal{B}_E^-$ is the opposite parabolic ${}^tP = P^{opp}$ consisting of all $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$ in $\mathfrak{g}_\mathbb{C}$. Thus they have conjugate complex structures: $\mathcal{B}_E^+$ has holomorphic tangent space represented by the matrices $\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right)$ with C antisymmetric while the holomorphic tangent space of $\mathcal{B}_E^-$ is represented by the matrices $\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right)$ with B antisymmetric.

Reformulating this,

Lemma 5.4.4. *Let $G_0 = SO^*(\infty)$. Then the positive bounded symmetric domain associated to the triple (V, G_0, E) is $\mathcal{B}_E^+ \cong \{Z_1 \in \mathbb{C}^{\infty \times \infty} \mid I - Z_1^* Z_1 \gg 0 \text{ and } Z_1 + {}^tZ_1 = 0\}$ in G/P, and the negative bounded symmetric domain for (V, G_0, E) is the complex conjugate domain $\mathcal{B}_E^- \cong \{Z_2 \in \mathbb{C}^{\infty q \times \infty} \mid I - Z_2^* Z_2 \gg 0 \text{ and } Z_1 + {}^tZ_1 = 0\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is described in Section 4.3.

Arguing just as for Theorems 5.2.6 and 5.3.5, we arrive at the following structure theorem.

Theorem 5.4.5. *Let $G_0 = SO^*(\infty)$ and let D be an open G_0–orbit $G(\mathcal{F}^{(1)})$ in $\mathcal{Z}_{\mathcal{F},E}$. In the notation of* (5.4.1)*, the positive definite bounded symmetric domain $\mathcal{B}_E^+$ for (V, G_0, E) is the set of all positive definite G-translates of V_+ and the negative definite bounded symmetric domain $\mathcal{B}_E^-$ for (V, G_0, E) is the set of is the set of all negative definite G-translates of V_-. The $\mathcal{B}_E^{\pm}$ are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+$.

If every $F_k^{(1)} \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^-$.

If some $F_k^{(1)} \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}_E^+ \times \mathcal{B}_E^-$.

5.5 Cycle Spaces for $SO(\infty, 2)$.

Now we come to the rather delicate case $G_0 = SO(\infty, 2)$, where the lim–compact dual of the complex bounded symmetric domain is a nondegenerate quadric in a complex projective space. We specify G_0 by the basis (4.4.1) and the forms (4.4.2). Let

$$V_{even} = \text{Span}\,(\{e_{2k} + \sqrt{-1}e_{2k+1} \mid k < 0\} \cup \{e_1 - \sqrt{-1}e_2\}),$$
$$V_{odd} = \text{Span}\,(\{e_{2k} - \sqrt{-1}e_{2k+1} \mid k < 0\} \cup \{e_1 + \sqrt{-1}e_2\}).$$

They are maximal b–isotropic subspaces of V, paired by $b(e_j + \sqrt{-1}e_{j+1}, e_j - \sqrt{-1}e_{j+1}) = 2$. This basis E leads to the same splitting of V as the one based on (3.1.1):

(5.5.1) $$V = V_+ \oplus V_- \text{ where } V_+ = \text{Span}\,\{\dots, e_{-3}, e_{-2}, e_{-1}\} \text{ and } V_- = \text{Span}\,\{e_1, e_2\}.$$

We denote

$$\mathcal{P}^\infty \text{ is the projective space } \mathcal{P}(V) \text{ and } \mathcal{Z} \text{ is the quadric } b(v,v) = 0 \text{ in } \mathcal{P}^\infty\,.$$

The maximal lim-compact subgroup of G_0 is $K_0 = SO(V_+) \times SO(V_-) = SO(\infty) \times SO(2)$. The complex K_0–orbits within the open G_0–orbits on $\mathcal{Z}$ (from Lemma 4.4.5 and (4.4.7) are

(5.5.2)
$$\begin{aligned}
&\text{in } D_0 = G_0([e_1 + \sqrt{-1}e_2]) : && K_0([e_1 + \sqrt{-1}e_2]) = \text{ (single point } [e_1 + \sqrt{-1}e_2]),\\
&\text{in } D_1 = G_0([e_1 - \sqrt{-1}e_2]) : && K_0([e_1 - \sqrt{-1}e_2]) = \text{ (single point } [e_1 - \sqrt{-1}e_2]),\\
&\text{in } D_2 = G_0([e_{-2} + \sqrt{-1}e_{-1}]) : && K_0([e_{-2} + \sqrt{-1}e_{-1}]) = \mathcal{Z} \cap \mathcal{P}(V_+) \text{ quadric in } \mathcal{P}(V_+).
\end{aligned}$$

Definition 5.5.3. The *positive bounded symmetric domain* $\mathcal{B}^+_{E'}$ associated to (V, b, E') consists of all maximal b–isotropic h–positive definite subspaces $W' \subset V$ such that $W' \cap V_+$ has finite codimension in both W' and V_+. Those subspaces have codimension 2 in V. The *negative bounded symmetric domain* $\mathcal{B}^-_{E'}$ associated to (V, b, E') consists of all maximal b–isotropic h–negative definite subspaces $W'' \subset V$. (Since $\dim W'' = 2 = \dim V_-$ the finite codimension condition is automatic.)

Now more generally let $\mathcal{F} = (F_k)$ be an isotropic generalized flag in V that is weakly compatible with E'. Let $\mathcal{F}^{(1)} \in \mathcal{Z}_{\mathcal{F},E'}$ for which $D = G_0(\mathcal{F}^{(1)})$ is an open G_0–orbit. We may assume that $\mathcal{F}^{(1)}$ is compatible with our choice of E', so it fits the decomposition (5.4.1) as before:

(5.5.4) $$\mathcal{F}^{(1)} = (F^{(1)}_k) \text{ where each } F^{(1)}_k = (F^{(1)}_k \cap V_+) \oplus (F^{(1)}_k \cap V_-).$$

Then $K_0(\mathcal{F}^{(1)})$ is the unique K_0–orbit in D that is a complex submanifold of the flag manifold $\mathcal{Z}_{\mathcal{F},E'}$. Somewhat trivially, $K_0(\mathcal{F}^{(1)})$ is the product of "smaller" complex flag manifolds,

(5.5.5)
$$\begin{aligned}
&Y = Y_1 \times Y_2 \text{ where}\\
&\qquad Y_1 = K_0(\mathcal{F}^{(1)} \cap V_+) = SO(\infty)(\mathcal{F}^{(1)} \cap V_+) \text{ in } V_+ \text{ and}\\
&\qquad Y_2 = K_0(\mathcal{F}^{(1)} \cap V_-) = SO(2)(\mathcal{F}^{(1)} \cap V_-) \text{ in } V_-
\end{aligned}$$

where $\mathcal{F}^{(1)} \cap V_+ = ((F^{(1)}_k \cap V_+))$ and $\mathcal{F}^{(1)} \cap V_- = ((F^{(1)}_1 \cap V_-))$. The signature sequence $\{(a_k, b_k)\}$, where h has signature $(a_k, b_k, 0)$ on $F^{(1)}_k$, specifies the open orbit in $\mathcal{Z}_{\mathcal{F},E'}$ and the factors of Y.

If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_+$, in other words $D = D_2$ and the cycles are of the form $K_0(gV_+)$ with $g \in G$, then M_D consists of the maximal b–isotropic h–positive definite subspaces of V. If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$, in other words $D = D_0$ or $D = D_1$ and the cycles are single points, then then M_D consists of the maximal b–isotropic h–negative definite subspaces of V. If $\mathcal{F}^{(1)} \cap V_+ \neq \mathcal{F}^{(1)} \neq \mathcal{F}^{(1)} \cap V_-$ then M_D is the product. Thus

Lemma 5.5.6. *Let $G_0 = SO(\infty, 2)$. Then the positive bounded symmetric domain associated to the triple (V, G_0, E') is $\mathcal{B}^+_{E'} \cong \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZZ|^2 - 2Z^*Z > 0$ and $Z^*Z < 1\}$ in G/P, and the negative bounded symmetric domain for (V, G_0, E') is the complex conjugate domain $\mathcal{B}^-_{E'} \cong \{Z \in \mathbb{C}^\infty \mid 1 + |{}^tZZ|^2 - 2Z^*Z > 0$ and $Z^*Z > 1\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is described in Section 4.4.

The argument for Theorem 5.2.6 remains valid here, with one small modification. Recall Lemma 4.4.5 and (4.4.7). There is just one open orbit $D_2 = G_0([e_{-1} + \sqrt{-1}\, e_{-2}])$ consisting of h–positive definite subspaces, but there are two orbits, $D_0 = G_0([e_1 + \sqrt{-1}\, e_2])$ and $D_1 = G_0([e_1 - \sqrt{-1}\, e_2])$, consisting of negative definite subspaces. These last two are related by complex conjugation of V over the real span of E. Suppose that D is either D_0 or D_2, that $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap V_-$, and that gV_- is negative definite. Then $gY \subset (D_0 \cup D_1)$. As gY is connected, either $gY \subset D_0$ or $gY \subset D_1$. Thus $gY \in \mathcal{M}'_D$, and $gY \in \mathcal{M}_D$ just when $gY \subset D$. With this adjustment the proof of Theorem 5.2.6 holds here, and the result is

Theorem 5.5.7. *Let $G_0 = SO(\infty, 2)$ and let D be an open G_0–orbit $G(\mathcal{F}^{(1)})$ in $\mathcal{Z}_{\mathcal{F},E'}$. In the notation of* (5.4.1)*, the positive definite bounded symmetric domain $\mathcal{B}^+_{E'}$ for (V, G_0, E') is the set of all positive definite G-translates of V_+ and the negative definite bounded symmetric domain $\mathcal{B}^-_E$ for (V, G_0, E') is the set of is the set of all negative definite G-translates of V_-. The $\mathcal{B}^\pm_{E'}$ are antiholomorphically diffeomorphic, in other words each is the complex conjugate of the other. There are three cases for the structure of the cycle space, as follows.*

If every $F^{(1)}_k \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}^+_{E'}$.

If every $F^{(1)}_k \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}^-_{E'}$.

If some $F^{(1)}_k \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is holomorphically diffeomorphic to $\mathcal{B}^+_{E'} \times \mathcal{B}^-_{E'}$.

6 Real and Quaternionic Domains and Cycle Spaces

In Section 4 we worked out the structure of finitary complex bounded symmetric domains, and in Section 5 we applied those results to obtain the structure of cycle spaces on corresponding flag domains. In this section we develop a variation on those results for particular real and quaternionic flag manifolds and cycle spaces based on the groups $SO(\infty, q)$ and $Sp(\infty, q)$, $q \leqq \infty$. Those groups provide real and quaternionic analogs of the complex domains of $SU(\infty, q)$. The methods and results are similar to those of Section 4.1, Section 5.2, and the last part of [10].

6.1 The Real Bounded Symmetric Domain for $SO(\infty, q)$.

In Section 4.1 we looked at the bounded domain of maximal negative definite subspaces of (V, h) contained in $\mathcal{Z}_{\mathcal{F},E}$, where V has basis E given by (3.1.1) and where the hermitian form h is given by (3.1.2). We studied it as an $SU(\infty, q)$–orbit on the complex Grassmann manifold of q–dimensional subspaces of V weakly compatible with E. Here we look at the real analog, the (real – not complex) bounded symmetric domain of maximal negative definite subspaces of (V_0, b) where V_0 is the real span of E and the symmetric bilinear form b is the restriction of h to V_0. Then we use it to describe real cycle spaces for open orbits on the corresponding real flag manifolds.

We consider the real group $G_0 = SO(\infty, q)$, $q \leqq \infty$ and the flag $\mathcal{F} = (0, F, V_0)$ where $F = \mathrm{Span}_{\mathbb{R}}\{e_i \mid i > 0\}$. View G_0 as a closed subgroup of $G := SL(\infty + q; \mathbb{R})$. That gives us the *real* flag manifold

(6.1.1) $\quad \mathfrak{X}_{\mathcal{F},E} = \{$ subspaces $F^{(1)} \subset V_0 \mid (0, F^{(1)}, V)$ is E–commensurable to $\mathcal{F}\} = G(\mathcal{F})$

where the second equality follows as in the argument of Lemma 2.2.3. Note that $\mathfrak{X}_{\mathcal{F},E}$ is a real Grassmann manifold. The domain of interest to us in this context is

$$D_0 = \{\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in \mathfrak{X}_{\mathcal{F},E} \mid F^{(1)} \text{ maximal negative definite subspace of } V_0\}. \tag{6.1.2}$$

If $\tau : V \to V$ denotes complex conjugation of V over V_0 then the domain D_0 of (6.1.2) can be identified with the fixed point set of τ on the complex Grassmannian of Section 4.1.

We use the b–orthogonal decomposition $V_0 = (V_0)_+ \oplus (V_0)_-$ where $(V_0)_+ = \mathrm{Span}_{\mathbb{R}}\{e_i \mid i < 0\}$ and $(V_0)_- = \mathrm{Span}_{\mathbb{R}}\{e_i \mid i > 0\}$. Consider the corresponding b–orthogonal projections $\pi_\pm$. The kernel of π_- is b–positive definite so it has zero intersection with $F^{(1)}$ for any $\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in D_0$. Thus $\pi_- : F^{(1)} \cong (V_0)_-$ is injective, and it is surjective as well because $F^{(1)}$ is a maximal negative definite subspace. Now we have a well defined linear map

$$X_{F^{(1)}} : (V_0)_- \to (V_0)_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}. \tag{6.1.3}$$

As $\mathcal{F}^{(1)}$ is weakly compatible with E, the matrix of $X_{F^{(1)}}$ relative to E has only finitely many nonzero entries, i.e. $X_{F^{(1)}}$ is finitary. Further, $\pi_- : F^{(1)} \cong (V_0)_-$ defines an $\mathbb{R}$–basis $\{e_i''\}$ of $F^{(1)}$ by $\pi_-(e_i'') = e_i$. Write $e_i'' = e_i + \sum_{j<0} x_{j,i} e_j$; then $(x_{j,i})$ is the matrix of $X_{F^{(1)}}$. The fact that $F^{(1)}$ is b–negative definite, translates to the matrix condition $I - {}^t(x_{j,i})\,(x_{j,i}) \gg 0$, equivalently the operator condition $I - {}^tX_{F^{(1)}}\, X_{F^{(1)}} \gg 0$. Conversely if $X : (V_0)_- \to (V_0)_+$ is finitary and satisfies $I - {}^tX\,X \gg 0$, then the real column span of its matrix relative to E is a maximal negative definite subspace $F^{(1)}$, and $\mathcal{F}^{(1)} = (0, F^{(1)}, V_0) \in D_0$.

The same computation as in Section 4.1 shows that the block form matrices of elements of G_0 act by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} X \\ I \end{smallmatrix}\right) \to \left(\begin{smallmatrix} AX+B \\ CX+D \end{smallmatrix}\right)$, which has the same real column span as $\left(\begin{smallmatrix} (AX+B)(CX+D)^{-1} \\ I \end{smallmatrix}\right)$. So G_0 acts by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : X \to (AX+B)(CX+D)^{-1}$. In summary,

Proposition 6.1.4. *D_0 is realized as the bounded domain of all finitary $X : (V_0)_- \to (V_0)_+$ such that $I - {}^tX\,X \gg 0$, and there the action of G_0 is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : X \to (AX+B)(CX+D)^{-1}$.*

Again, there are $q+1$ open G_0–orbits on $\mathfrak{X}_{\mathcal{F},E}$ corresponding to nondegenerate signatures:

$$\begin{aligned} D_k = G_0((0, F_{(k)}, V_0)) \text{ where } F_{(k)} &= \mathrm{Span}_{\mathbb{R}}\{e_{-k}, \dots, e_{-1}; e_{k+1}, \dots, e_q\} \text{ if } q < \infty, \\ F_{(k)} &= \mathrm{Span}_{\mathbb{R}}\{e_{-k}, \dots, e_{-1}; e_{k+1}, e_{k+2}, \dots\} \text{ if } q = \infty, \end{aligned}$$

More generally the G_0–orbits on $\mathfrak{X}_{\mathcal{F},E}$ of signature $(a,b,c) = (\mathrm{pos}, \mathrm{neg}, \mathrm{nul})$ have a and c finite and $\leqq q$. We denote them by

$$\begin{aligned} &D_{a,b,c} = G_0((0, (F_+ + F_- + F_0), V)) \text{ where} \\ &\quad F_0 = \mathrm{Span}_{\mathbb{R}}\{e_{-c} + e_c, \dots, e_{-1} + e_1\} \text{ (null)} \\ &\quad F_+ = \mathrm{Span}_{\mathbb{R}}\{e_{-c-a}, \dots, e_{-c-1}\} \text{ (positive)} \\ &\quad F_- = \mathrm{Span}_{\mathbb{R}}\{e_{c+1}, \dots, e_{c+b}\},\ q < \infty;\ \mathrm{Span}_{\mathbb{R}}\{e_{c+1}, e_{c+2}, \dots\},\ q = \infty \text{ (negative).} \end{aligned} \tag{6.1.5}$$

As in the complex case, the open orbits are the $D_a = D_{a,b,0}$, $a < \infty$ and $a+b = q$, i.e. the ones for $c = 0$. If $q < \infty$ there is a unique closed orbit, $D_{0,0,q} = \{(0, F^{(1)}, V_0) \in \mathfrak{X}_{\mathcal{F},E} \mid b(F^{(1)}, F^{(1)}) = 0\}$; it is in the closure of every orbit. If $q = \infty$ there is no closed orbit.

The Cayley transforms are given by (4.1.5): $c_k(e_j) = e_j$ if $j \neq \pm k$ and, in the basis $\{e_{-k}, e_k\}$ of $\mathrm{Span}_{\mathbb{R}}\{e_{-k}, e_k\}$, c_k has matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. This sends real subspaces of V to real subspaces; that is why, in Section 4.1, we based (4.1.5) on the one variable Cayley transform that sends $0 \to 1 \to \infty \to -1 \to 0$ and maps the unit disk to the right half plane. As a riemannian symmetric space, the real Grassmannian $\mathfrak{X}_{\mathcal{F},E}$ has rank q. Just as in the complex case the G_0–orbits on $\mathfrak{X}_{\mathcal{F},E}$ are the $G_0(c_1 \dots c_s c_{s+1}^2 \dots c_{s+t}^2 \mathcal{F})$, and the open ones are those for which

$s = 0$. If $q < \infty$ then $G_0(c_1 \dots c_q\mathcal{F})$ is the closed orbit, and if $q = \infty$ then there is no closed G_0–orbit on $\mathfrak{X}_{\mathcal{F},E}$.

The maximal lim-compact subgroup of G_0 is

$$K_0 = SO(\infty) \times SO(q) \ = \big(\lim_{p\to\infty} SO(p)\big) \times SO(q) \text{ if } q < \infty,$$
$$K_0 = SO(\infty) \times SO(\infty) = \lim_{p,q\to\infty}\big(SO(p) \times SO(q)\big) \text{ if } q = \infty.$$

This corresponds to the b–orthogonal decomposition $\mathbb{R}^{\infty,q} = (V_0)_+ \oplus (V_0)_-$. Let $\mathcal{F} = (F_k)$ be a generalized flag in $V = \mathbb{R}^{\infty,q}$ that is weakly compatible with E. Let $\mathcal{F}^{(1)} \in \mathfrak{X}_{\mathcal{F},E}$ so that $D = G_0(\mathcal{F}^{(1)})$ is an open G_0–orbit. Then we may assume that $\mathcal{F}^{(1)}$ is compatible with our choice of E, so it fits the decomposition $\mathbb{R}^{\infty,q} = (V_0)_+ \oplus (V_0)_-$ in the sense that

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap (V_0)_+) \oplus (F_k^{(1)} \cap (V_0)_-). \tag{6.1.6}$$

Then $K_0(\mathcal{F}^{(1)})$ is the real analog – in fact a real form – of the base cycle in the complexification of D. Concretely, $K_0(\mathcal{F}^{(1)})$ is the product of "smaller" real flag manifolds,

$$\begin{aligned} &Y = Y_1 \times Y_2 \text{ where} \\ &\qquad Y_1 = K_0(\mathcal{F}^{(1)} \cap (V_0)_+) = SO(\infty)(\mathcal{F}^{(1)} \cap (V_0)_+) \text{ in } (V_0)_+ \text{ and} \\ &\qquad Y_2 = K_0(\mathcal{F}^{(1)} \cap (V_0)_-) = SO(q)(\mathcal{F}^{(1)} \cap (V_0)_-) \text{ in } (V_0)_- . \end{aligned} \tag{6.1.7}$$

where

$$\mathcal{F}^{(1)} \cap (V_0)_+ = ((F_1^{(1)} \cap (V_0)_+) \subset \dots \subset (F_n^{(1)} \cap (V_0)_+)),$$
$$\mathcal{F}^{(1)} \cap (V_0)_- = ((F_1^{(1)} \cap (V_0)_-) \subset \dots \subset (F_n^{(1)} \cap (V_0)_-)).$$

The signature sequence $\{(a_k, b_k)\}$, where h has signature $(a_k, b_k, 0)$ on $F_k^{(1)}$, specifies the open orbit in $\mathfrak{X}_{\mathcal{F},E}$ and the factors of Y.

This shows that the G–translates of Y contained in D correspond to the decompositions $V_0 = W_0' \oplus W_0''$ where (i) W_0' is a maximal positive definite subspace such that $(V_0)_+ \cap W_0'$ has finite codimension in both W_0' and $(V_0)_+$, and (ii) W_0'' is a maximal negative definite subspace such that $(V_0)_- \cap W_0''$ has finite codimension in both W_0'' and $(V_0)+$. If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_0)_+$ the correspondence depends only on W_0', and if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_0)_-$ it depends only on W_0''. Any two such decompositions $V_0 = W_0' \oplus W_0''$ are G–equivalent.

Definition 6.1.8. The *positive real bounded symmetric domain* $\mathcal{B}_E^+$ associated to (V_0, b, E) is the space of all maximal positive definite subspaces $W_0' \subset V_0$ such that $W_0' \cap (V_0)_+$ has finite codimension in both W_0' and $(V_0)_+$. The *negative bounded symmetric domain* $\mathcal{B}_E^-$ associated to (V_0, b, E) is the space of all maximal negative definite subspaces $W_0'' \subset V$ such that $W_0'' \cap (V_0)_-$ has finite codimension in both W_0'' and $(V_0)_-$.

As constructed, each element $W_0' \in \mathcal{B}_E^+$ is in the G–orbit of $(V_0)_+$. Relative to the basis E we look at $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ such that $gW_0' \in \mathcal{B}_E^+$, in other words such that the column span of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is positive definite. The column span is preserved under right multiplication by A, so the positive definite condition is ${}^t\!\left(\begin{smallmatrix} I \\ -CA^{-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} I \\ CA^{-1} \end{smallmatrix}\right) \gg 0$. In other words $gW_0' \in \mathcal{B}_E^+$ simply means that gW_0' is the column span of an infinite real matrix $\left(\begin{smallmatrix} I \\ X_1 \end{smallmatrix}\right)$ such that $I - {}^tX_1X_1 \gg 0$. Similarly $gW_0'' \in \mathcal{B}_E^-$ simply means that gW_0'' is the column span of an infinite real matrix $\left(\begin{smallmatrix} X_2 \\ I \end{smallmatrix}\right)$ such that $I - {}^tX_2X_2 \gg 0$. The G–stabilizer of $0 \in \mathcal{B}_E^+$ is the parabolic P consisting of all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, while the G–stabilizer of $0 \in \mathcal{B}_E^-$ is the opposite parabolic ${}^tP = P^{opp}$ consisting of all $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$. Reformulating this,

Lemma 6.1.9. *Suppose that $G_0 = SO(\infty, q)$, $q \leqq \infty$. Then the real positive bounded symmetric domain associated to (V, G_0, E) is $\mathcal{B}_E^+ \cong \{X_1 \in \mathbb{R}^{\infty \times q} \mid I - {}^tX_1X_1 \gg 0\}$ in G/P, and the negative real bounded symmetric domain for (V, G_0, E) is $\mathcal{B}_E^- \cong \{X_2 \in \mathbb{R}^{q \times \infty} \mid I - {}^tX_2X_2 \gg 0\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is linear fractional, as described in Section 4.1 for the complex case. The proof of Theorem 5.2.6 is valid here, giving us the following structure theorem.

Theorem 6.1.10. *Let $G_0 = SO(\infty, q)$ with $2 < q \leqq \infty$. Let D be an open G_0–orbit $G((0, F^{(1)}, V_0))$ in the real flag manifold $\mathfrak{X}_{\mathcal{F},E}$. Then the positive definite bounded symmetric domain $\mathcal{B}_E^+$ for (V, G_0, E) is the set of all positive definite G-translates of $(V_0)_+$ and the negative definite bounded symmetric domain $\mathcal{B}_E^-$ for (V, G_0, E) is the set of is the set of all negative definite G-translates of $(V_0)_-$. The $\mathcal{B}_E^\pm$ are diffeomorphic. There are three cases for the structure of the cycle space, as follows.*

If every space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_E^+$.
If every space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_E^-$.
If some space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_E^+ \times \mathcal{B}_E^-$.

6.2 The Quaternionic Bounded Symmetric Domain for $Sp(\infty, q)$.

We now look at the quaternionic analog of Section 6.1. For that, we consider a quaternionic vector space $V_{\mathbb{H}} = \mathbb{H}^{\infty,q}$, one of whose underlying complex structures is that of $V = \mathbb{C}^{\infty,2q}$. We look at the bounded symmetric domain of maximal negative definite quaternionic subspaces of $(V_{\mathbb{H}}, h)$. As suggested by Section 3.3, the complex basis E of V is replaced by an $\mathbb{H}$–basis

$$\begin{aligned} L =&\{\dots, v_{-2}, v_{-1}; v_1, v_2, \dots, v_q\} \text{ for } q < \infty, \\ L =&\{\dots, v_{-2}, v_{-1}; v_1, v_2, v_3, \dots\} \text{ for } q = \infty. \end{aligned} \tag{6.2.1}$$

The relation with E is $v_i = e_{2i}$ for $i < 0$ and $v_j = e_{2j-1}$ for $j > 0$. The $\mathbb{H}$–hermitian form h is defined by $h(v_i, v_j) = \delta_{i,j}$ for $i < 0$ and $h(v_i, v_j) = -\delta_{i,j}$ for $i > 0$.

The real group is $G_0 = Sp(\infty, q)$, $q \leqq \infty$. We view G_0 as a closed subgroup of the quaternionic linear group $G := SL(\infty + q; \mathbb{H})$. The flag is $\mathcal{F} = \{F\}$ where $F = \mathrm{Span}_{\mathbb{H}}\{e_i \mid i > 0\}$. That gives us the *quaternionic* flag manifold

$$\mathfrak{X}_{\mathcal{F},L} = \{\text{subspaces } F^{(1)} \subset V_{\mathbb{H}} \mid (0, F^{(1)}, V_{\mathbb{H}}) \text{ is } L\text{–commensurable to } \mathcal{F}\} = G(\mathcal{F}) \tag{6.2.2}$$

where the second equality follows as in the argument of Lemma 2.2.3. Note that $\mathfrak{X}_{\mathcal{F},L}$ is a quaternionic Grassmann manifold. The domain of interest to us in this context is

$$D_0 = \{(0, F^{(1)}, V_{\mathbb{H}}) \in \mathfrak{X}_{\mathcal{F},L} \mid F^{(1)} \text{ is a maximal } h\text{–negative definite subspace of } V_{\mathbb{H}}\}. \tag{6.2.3}$$

Now consider the h–orthogonal decomposition $V_{\mathbb{H}} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$ where $(V_{\mathbb{H}})_+$ denotes $\mathrm{Span}_{\mathbb{H}}\{e_i \mid i < 0\}$ and $(V_{\mathbb{H}})_-$ denotes $\mathrm{Span}_{\mathbb{H}}\{e_i \mid i > 0\}$. Consider the corresponding orthogonal projections $\pi_+ : V_{\mathbb{H}} \to (V_{\mathbb{H}})_+$ and $\pi_- : V_{\mathbb{H}} \to (V_{\mathbb{H}})_-$, The kernel of π_- is h–positive definite so it has zero intersection with $F^{(1)}$ for any $\mathcal{F}^{(1)} = (0, F^{(1)}, V_{\mathbb{H}}) \in D_0$. Thus $\pi_- : F^{(1)} \cong (V_{\mathbb{H}})_-$ is injective. Since $F^{(1)}$ is a maximal h–negative definite subspace $\pi_- : F^{(1)} \cong (V_{\mathbb{H}})_-$ is surjective as well. Now we have a well defined $\mathbb{H}$–linear map

$$X_{F^{(1)}} : (V_{\mathbb{H}})_- \to (V_{\mathbb{H}})_+ \text{ defined by } \pi_-(x) \mapsto \pi_+(x) \text{ for } x \in F^{(1)}. \tag{6.2.4}$$

As $\mathcal{F}^{(1)}$ is weakly compatible with L, the matrix of $X_{F^{(1)}}$ relative to L has only finitely many nonzero entries, i.e. $X_{F^{(1)}}$ is finitary. Using $\pi_- : F^{(1)} \cong V_{\mathbb{H},-}$ defines an $\mathbb{H}$–basis $\{v_i''\}$ of $F^{(1)}$

by $\pi_-(v_i'') = v_i$. Write $v_i'' = v_i + \sum_{j<0} x_{j,i} v_j$; then $(x_{j,i})$ is the matrix of $X_{F^{(1)}}$. The fact that $F^{(1)}$ is h–negative definite, translates to the matrix condition $I - (x_{j,i})^* (x_{j,i}) \gg 0$, equivalently the operator condition $I - X^*_{F^{(1)}} X_{F^{(1)}} \gg 0$. Conversely if $X : (V_{\mathbb{H}})_- \to (V_{\mathbb{H}})_+$ is finitary and satisfies $I - X^* X \gg 0$, then the quaternionic column span of its matrix relative to L is a maximal negative definite subspace $F^{(1)}$, and $\mathcal{F}^{(1)} = (0, F^{(1)}, V_{\mathbb{H}}) \in D_0$.

The same computation as in Section 4.1 shows that the block form matrices of elements of G_0 act by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : \left(\begin{smallmatrix} X \\ I \end{smallmatrix}\right) \to \left(\begin{smallmatrix} AX+B \\ CX+D \end{smallmatrix}\right)$, which has the same quaternionic column span as $\left(\begin{smallmatrix} (AX+B)(CX+D)^{-1} \\ I \end{smallmatrix}\right)$. So G_0 acts by the linear fractional $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : X \to (AX+B)(CX+D)^{-1}$. In summary,

Proposition 6.2.5. *D_0 is realized as the bounded domain of all finitary $X : (V_{\mathbb{H}})_- \to (V_{\mathbb{H}})_+$ such that $I - X^* X \gg 0$, and there the action of G_0 is $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) : X \to (AX+B)(CX+D)^{-1}$.*

Again, there are $q+1$ open G_0–orbits on $\mathfrak{X}_{\mathcal{F},L}$ corresponding to nondegenerate signatures:

$$\begin{aligned} D_k = G_0((0, F_{(k)}, V_{\mathbb{H}})) \text{ where } F_{(k)} &= \mathrm{Span}_{\mathbb{H}}\{v_{-k}, \dots, v_{-1}; v_{k+1}, \dots, v_q\} \text{ if } q < \infty, \\ F_{(k)} &= \mathrm{Span}_{\mathbb{H}}\{v_{-k}, \dots, v_{-1}; v_{k+1}, v_{k+2}, \dots\} \text{ if } q = \infty, \end{aligned}$$

More generally the G_0–orbits on $\mathfrak{X}_{\mathcal{F},L}$ of signature $(a,b,c) = (\mathrm{pos}, \mathrm{neg}, \mathrm{nul})$ have a and c finite and $\leqq q$. We denote them by

$$\begin{aligned} D_{a,b,c} &= G_0((0, (F_+ + F_- + F_0), V_{\mathbb{H}})) \text{ where} \\ F_0 &= \mathrm{Span}_{\mathbb{H}}\{v_{-c} + v_c, \dots, v_{-1} + v_1\} \text{ (null)} \\ F_+ &= \mathrm{Span}_{\mathbb{H}}\{v_{-c-a}, \dots, v_{-c-1}\} \text{ (positive)} \\ F_- &= \mathrm{Span}_{\mathbb{H}}\{v_{c+1}, \dots, v_{c+b}\} \text{ if } q < \infty, \\ &\quad \mathrm{Span}_{\mathbb{H}}\{v_{c+1}, v_{c+2}, \dots\} \text{ if } q = \infty \text{ (negative).} \end{aligned} \tag{6.2.6}$$

The open orbits are the $D_a = D_{a,b,0}$, $a < \infty$ and $a + b = q$, i.e. the ones for $c = 0$. If $q < \infty$ there is a unique closed orbit, $D_{0,0,q} = \{(0, F^{(1)}, V_{\mathbb{H}}) \in \mathfrak{X}_{\mathcal{F},L} \mid h(F^{(1)}, F^{(1)}) = 0\}$; it is in the closure of every orbit. If $q = \infty$ there is no closed orbit.

The Cayley transforms are given by (4.1.5): $c_k(v_j) = v_j$ if $j \neq \pm k$ and, in the basis $\{v_{-k}, v_k\}$ of $\mathrm{Span}_{\mathbb{H}}\{v_{-k}, v_k\}$, c_k has matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. This sends quaternionic subspaces of V to quaternionic subspaces. As a riemannian symmetric space, the quaternion Grassmannian $\mathfrak{X}_{\mathcal{F},L}$ has rank q. Just as in the complex case the G_0–orbits on $\mathfrak{X}_{\mathcal{F},L}$ are the $G_0(c_1 \dots c_s c_{s+1}^2 \dots c_{s+t}^2 \mathcal{F})$, and the open ones are those for which $s = 0$. If $q < \infty$ then $G_0(c_1 \dots c_q \mathcal{F})$ is the closed orbit, and if $q = \infty$ then there is no closed G_0–orbit on $\mathfrak{X}_{\mathcal{F},L}$.

The maximal lim-compact subgroup of G_0 is

$$\begin{aligned} K_0 &= Sp(\infty) \times Sp(q) \ = \left(\lim_{p\to\infty} Sp(p)\right) \times Sp(q) \text{ if } q < \infty, \\ K_0 &= Sp(\infty) \times Sp(\infty) = \lim_{p,q\to\infty}\left(Sp(p) \times Sp(q)\right) \text{ if } q = \infty. \end{aligned}$$

This corresponds to the h–orthogonal decomposition $\mathbb{H}^{\infty,q} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$. Let $\mathcal{F} = (F_k)$ be a generalized flag in $V = \mathbb{H}^{\infty,q}$ that is weakly compatible with L. Let $\mathcal{F}^{(1)} \in \mathfrak{X}_{\mathcal{F},L}$ so that $D = G_0(\mathcal{F}^{(1)})$ is an open G_0–orbit. Then we may assume that $\mathcal{F}^{(1)}$ is compatible with our choice of L, so it fits the decomposition $\mathbb{H}^{\infty,q} = (V_{\mathbb{H}})_+ \oplus (V_{\mathbb{H}})_-$ in the sense that

$$\mathcal{F}^{(1)} = (F_k^{(1)}) \text{ where each } F_k^{(1)} = (F_k^{(1)} \cap (V_{\mathbb{H}})_+) \oplus (F_k^{(1)} \cap (V_{\mathbb{H}})_-). \tag{6.2.7}$$

Then $K_0(\mathcal{F}^{(1)})$ is the quaternionic analog – in fact a quaternion form – of the base cycle when the latter is viewed as a quaternionic manifold. Concretely, $K_0(\mathcal{F}^{(1)})$ is the product of "smaller" quaternionic flag manifolds,

$$(6.2.8)\qquad \begin{aligned} &Y = Y_1 \times Y_2 \text{ where}\\ &Y_1 = K_0(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+) = Sp(\infty)(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+) \text{ in } (V_{\mathbb{H}})_+ \text{ and}\\ &Y_2 = K_0(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-) = Sp(q)(\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-) \text{ in } (V_{\mathbb{H}})_- \end{aligned}$$

where

$$\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+ = ((F_1^{(1)} \cap (V_{\mathbb{H}})_+) \subset \cdots \subset (F_n^{(1)} \cap (V_{\mathbb{H}})_+)),$$
$$\mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_- = ((F_1^{(1)} \cap (V_{\mathbb{H}})_-) \subset \cdots \subset (F_n^{(1)} \cap (V_{\mathbb{H}})_-)).$$

The signature sequence $\{(a_k, b_k)\}$, where h has signature $(a_k, b_k, 0)$ on $F_k^{(1)}$, specifies the open orbit in $\mathfrak{X}_{\mathcal{F},L}$ and the factors of Y.

This shows that the G–translates of Y contained in D correspond to the decompositions $V_H = W'_{\mathbb{H}} \oplus W''_{\mathbb{H}}$ where (i) $W'_{\mathbb{H}}$ is a maximal positive definite $\mathbb{H}$–subspace such that $(V_{\mathbb{H}})_+ \cap W'_{\mathbb{H}}$ has finite codimension in both $(V_{\mathbb{H}})_+$ and $\cap W'_{\mathbb{H}}$, and (ii) $W''_{\mathbb{H}}$ is a maximal negative definite subspace such that $(V_{\mathbb{H}})_- \cap W''_{\mathbb{H}}$ has finite codimension in both $(V_{\mathbb{H}})_-$ and $W''_{\mathbb{H}}$. If $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_+$ the correspondence depends only on $W'_{\mathbb{H}}$, and if $\mathcal{F}^{(1)} = \mathcal{F}^{(1)} \cap (V_{\mathbb{H}})_-$ it depends only on $W''_{\mathbb{H}}$. Any two such decompositions $V_{\mathbb{H}} = W'_{\mathbb{H}} \oplus W''_{\mathbb{H}}$ are G–equivalent.

Definition 6.2.9. The *positive quaternionic bounded symmetric domain* $\mathcal{B}_L^+$ associated to $(V_{\mathbb{H}}, b, L)$ is the space of all maximal positive definite subspaces $W'_{\mathbb{H}} \subset V_{\mathbb{H}}$ such that $W'_{\mathbb{H}} \cap (V_{\mathbb{H}})_+$ has finite codimension in both $W'_{\mathbb{H}}$ and $(V_{\mathbb{H}})_+$. The *negative quaternionic bounded symmetric domain* $\mathcal{B}_L^-$ associated to $(V_{\mathbb{H}}, b, L)$ is the space of all maximal negative definite subspaces $W''_{\mathbb{H}} \subset V_{\mathbb{H}}$ such that $W''_{\mathbb{H}} \cap (V_{\mathbb{H}})_-$ has finite codimension in both $W''_{\mathbb{H}}$ and $(V_{\mathbb{H}})_-$.

As constructed, each element $W'_{\mathbb{H}} \in \mathcal{B}_L^+$ is in the G–orbit of $(V_{\mathbb{H}})_+$. Relative to the basis L we look at $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$ such that $gW'_{\mathbb{H}} \in \mathcal{B}_L^+$, in other words such that the column span of $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ is positive definite. The column span is preserved under right multiplication by A, so the positive definite condition is ${}^t\!\left(\begin{smallmatrix} I \\ -CA^{-1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} I \\ CA^{-1} \end{smallmatrix}\right) \gg 0$. In other words $gW'_{\mathbb{H}} \in \mathcal{B}_L^+$ simply means that $gW'_{\mathbb{H}}$ is the column span of an infinite matrix $\left(\begin{smallmatrix} I \\ X_1 \end{smallmatrix}\right)$ such that $I - {}^tX_1X_1 \gg 0$. Similarly $gW''_{\mathbb{H}} \in \mathcal{B}_L^-$ simply means that $gW''_{\mathbb{H}}$ is the column span of an infinite matrix $\left(\begin{smallmatrix} X_2 \\ I \end{smallmatrix}\right)$ such that $I - {}^tX_2X_2 \gg 0$. The G–stabilizer of $0 \in \mathcal{B}_L^+$ is the parabolic P consisting of all $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)$, while the G–stabilizer of $0 \in \mathcal{B}_L^-$ is the opposite parabolic ${}^tP = P^{opp}$ consisting of all $\left(\begin{smallmatrix} A & 0 \\ C & D \end{smallmatrix}\right)$. Reformulating this,

Lemma 6.2.10. *Suppose that $G_0 = Sp(\infty, q)$, $q \leqq \infty$. Then the quaternionic positive bounded symmetric domain associated to the triple (V, G_0, L) is $\mathcal{B}_L^+ \cong \{X_1 \in \mathbb{H}^{\infty \times q} \mid I - {}^tX_1X_1 \gg 0\}$ in G/P, and the corresponding quaternionic negative bounded symmetric domain for (V, G_0, L) is $\mathcal{B}_L^- \cong \{X_2 \in \mathbb{H}^{q \times \infty} \mid I - {}^tX_2X_2 \gg 0\}$ in G/P^{opp}.*

The action of G_0 on these bounded symmetric domains is linear fractional, as described in Section 4.1 for the complex case. The proof of Theorem 5.2.6 is valid here, giving us the following structure theorem.

Theorem 6.2.11. *Let $G_0 = Sp(\infty, q)$ with $q \leqq \infty$. Let D be an open G_0–orbit $G(\mathcal{F}^{(1)})$ in the quaternionic flag manifold $\mathfrak{X}_{\mathcal{F},L}$. Then the positive definite bounded symmetric domain all positive definite G-translates of $(V_{\mathbb{H}})_+$ and the negative definite bounded symmetric domain $\mathcal{B}_L^-$ for (V, G_0, L) is the set of is the set of all negative definite G-translates of $(V_{\mathbb{H}})_-$. The $\mathcal{B}_L^{\pm}$ are diffeomorphic. There are three cases for the structure of the cycle space, as follows.*

If every space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is positive definite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_L^+$.

If every space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is negative definite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_L^-$.

If some space $F_k^{(1)} \in \mathcal{F}^{(1)}$ is indefinite then $\mathcal{M}_D$ is diffeomorphic to $\mathcal{B}_L^+ \times \mathcal{B}_L^-$.

References

[1] E. Dan-Cohen, I. Penkov & J. A. Wolf, Parabolic subgroups of infinite dimensional real Lie groups. Contemporary Math. **499** (2009), 47–59. arXiv:0901.0295 (math.RT, math.RA).

[2] I. Dimitrov & I. Penkov, Ind–varieties of generalized flags as homogeneous spaces for classical ind-groups, IMRN **2004**, 2935–2953.

[3] G. Fels, A. T. Huckleberry & J. A. Wolf, Cycle spaces of flag domains: A complex geometric viewpoint Progress in Mathematics, vol. 245, Birkhäuser/Springer Boston, 2005.

[4] A. T. Huckleberry, A. Simon & D. Barlet, On cycle spaces of flag domains of SL_nR Journal für die reine und angewandte Mathematik **2001** (2001), 171–208.

[5] A. T. Huckleberry & J. A. Wolf, Cycle spaces of real forms of $SL_n(C)$. In "Complex Geometry: A Collection of Papers Dedicated to Hans Grauert," Springer-Verlag, 2002, 111–133.

[6] A. T. Huckleberry & J. A. Wolf, Cycle space constructions for exhaustions of flag domains, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), **9** (2010), 573–580.

[7] M. Ignatyev, I. Penkov & J. A. Wolf, Real group orbits on flag ind–varieties of $SL_\infty(\mathbb{C})$, to appear.

[8] A. Korányi & J. A. Wolf, Realization of hermitian symmetric spaces as generalized half–planes. Annals of Mathematics, vol. 81 (1965), 265–288.

[9] J. I. Magnússon, Lectures on Cycle Spaces, Available from Jón Ingólfur Magnússon, University of Iceland; download from ResearchGate.

[10] S. Sternberg & J. A. Wolf, Charge conjugation and Segal's cosmology. Il Nuovo Cimento, vol. 28A (1975), pp. 253-271.

[11] J. A. Wolf & A. Korányi, Generalized Cayley transformations of bounded symmetric domains American Journal of Mathematics, vol. 87 (1965), 899–939.

[12] J. A. Wolf, The action of a real semisimple group on a complex flag manifold, I: Orbit structure and holomorphic arc components. Bulletin of the American Mathematical Society, vol. 75 (1969), pp. 1121–1237.

[13] J. A. Wolf, Fine structure of hermitian symmetric spaces. Symmetric Spaces: Short Courses Presented at Washington University, ed. Boothby & Weiss. Marcel Dekker Inc., 1972, 271–357.

[14] J. A. Wolf, The Stein condition for cycle spaces of open orbits on complex flag manifolds, Annals of Math. **136** (1992), 541–555.

[15] J. A. Wolf, Compact subvarieties in flag domains. Lie Theory and Geometry: in Honor of B. Kostant, Birkhäuser, Progress in Math., vol. 13, 1994, 577–596.

[16] J. A. Wolf, Hermitian symmetric spaces, cycle spaces, and the Barlet–Koziarz method for holomorphic convexity. Math. Research Letters **7** (2000), 1–13.

[17] J. A. Wolf & R. O. Wells, Jr., Poincaré series and automorphic cohomology on flag domains, Annals of Math. **105** (1977), 397–448.

[18] J. A. Wolf & R. Zierau, Linear cycle spaces in flag domains, Math. Annalen **316** (2000), 529–545.

[19] J. A. Wolf & R. Zierau, A note on the linear cycle space for groups of hermitian type, J. Lie Theory **13** (2003), 189–191.

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