Title

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ABSTRACT

A calculation is made of the interaction of a beam of particles in an accelerator with the radio-frequency cavity that provides the accelerating mechanism of the machine. A Hamiltonian for synchrotron motion is employed that makes possible the simultaneous solution of Maxwell's equations and the Vlasov equation, so that a self-consistent distribution of particles in synchrotron phase space is determined.

The effective voltage on the cavity due to the beam of charged particles is of the order of magnitude of the product of the total circulating current in the accelerator and the shunt impedance of the rf cavity. It has the net effect of producing a total voltage on the cavity which is both less than the applied voltage, and shifted in phase with respect to it. The increase in the stable phase angle required so the particles will remain in phase with the accelerating radio frequency is calculated. The decrease in total voltage and increase in stable phase angle, result in a decrease in stable phase space available for acceleration, and convenient expressions are given for these quantities in terms of parameters of the accelerator. It is shown that the consequences of the induced voltage may be alleviated by increasing the voltage applied to the cavity.

COHERENT ELECTROMAGNETIC EFFECTS IN HIGH-CURRENT PARTICLE ACCELERATORS: I. INTERACTION OF A PARTICLE BEAM WITH AN EXTERNALLY DRIVEN RADIO-FREQUENCY CAVITY*

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I. INTRODUCTION

As a beam of charged particles circulates in an accelerator, it may pass through one or more radio-frequency cavities. At least one of the cavities is usually driven externally and provides the accelerating electric field that bunches the beam azimuthally. We shall show in the following treatment of the interaction of a charged-particle beam with an externally driven rf cavity that the particles induce a periodic voltage on the cavity, which from consideration of Lenz's law always opposes the particle motion. The particles lose energy to the induced field and must therefore shift in phase relative to the applied voltage, in order to gain more energy per turn. In this manner, particles that are phase-stable adjust their net energy gain to remain in step with the applied radiofrequency.

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The phase shift of the distribution reduces the stable phase area just as if the modulation rate were correspondingly increased.

This effect has been treated by different authors.[1, 2] Although small in existing machines, it will become a problem in accelerators with circulating currents of several amperes. The action of a beam passing repeatedly through the rf cavity is not unlike the action of bunched electrons in a klystron. A periodic voltage is induced across the gap, and we shall see in the following treatment that this voltage can be large. The effect is enhanced because Fourier coefficients of the particle distribution are large for the resonance harmonic. If the cavity in question is the accelerating device, or if it is merely maintaining the beam (no modulation), beam and cavity are precisely in resonance.

Transverse particle motion is neglected. We simultaneously solve Maxwell's equations and the Vlasov equation, thus obtaining a self-consistent distribution of particles in synchrotron phase space. In Section II we develop expressions for the induced voltage and effective electric field as functions of the Fourier components of the particle distribution in azimuth. These expressions are then employed in the Hamiltonian for synchrotron motion (Section IIIA), and in Section IIIB we solve the Vlasov equation for the particle distribution. The Hamiltonian formalism is that of Symon and Sessler,[3] and the notation closely follows that of Nielsen and Sessler.[4] In Section IV we present a solution that is valid when there is no modulation of the rf, and when longitudinal space-charge effects are neglected.

The problem specified by the complete Hamiltonian is treated in Section V. Certain integrations that cannot be performed analytically are encountered, and results are given in graphical form. A simple expression is derived for the

ratio of applied to total voltage, which is a function of the operating parameters of the machine. The only parameter of the rf cavity entering into the results is the shunt or input impedance, which may be determined experimentally. Numerical examples are given in Section VC, and Section VD is devoted to the problem of maximizing phase flux.

The paper is summarized in Section VI in a way that should allow the use of the results without a careful study of the body of the paper.

An rf cavity will have an effect on a beam of particles even if it is not externally driven. The influence of such a cavity on the stability of a coasting particle beam will be considered in the third paper of this series.

II. INDUCED VOLTAGE AND EFFECTIVE ELECTRIC FIELD

The rf cavity is taken to be located with the center of the gap at $\theta = 0$. We shall use cylindrical coordinates (r, θ, z) exclusively in this paper. The gap has a width d which may be expressed in terms of the half angle of the gap $\theta_1 = d/2R$, where R is the radius of the accelerator. We assume that the electric field E is (to good approximation) constant in r and z, over the region of the gap in which particles move.

We take the beam current density circulating at a radius R as

$$j_\theta = e \omega_0 \delta (r - R) \delta (z - z_0) N (\theta - \omega_0 t) \tag{2.1}$$

where ω_0 is the average angular frequency of the bunch. The assumption that the beam has negligible cross-sectional area is merely convenient. Because of the assumed uniformity of E, a current with a small spread in r and z would lead to the same results as the current given by Eq. (2.1). Expanding $N (\theta - \omega_0 t)$ in a Fourier series gives

$$N (\theta - \omega_0 t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n (\theta - \omega_0 t) + b_n \sin n (\theta - \omega_0 t) . \tag{2.2}$$

The coefficients a_n and b_n are not time-dependent and may be evaluated at $t = 0$ giving:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} N (\theta) \cos n\theta \, d\theta \tag{2.3a}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} N (\theta) \sin n\theta \, d\theta \tag{2.3b}$$

The nth Fourier component of the current at $\theta = 0$ is obtained from Eqs. (2.1) and (2.2):

$$I_n (\theta = 0) = e\ \omega_0 (a_n \cos n\,\omega_0 t - b_n \sin n\,\omega_0 t) \tag{2.4}$$

If the rf cavity has a natural frequency $n\,\omega_0$, the Fourier component of the current that exhibits this frequency will be precisely in resonance with the cavity. This Fourier component will thus induce a voltage 180° out of phase with the current. Other Fourier components of the current will induce voltage components whose phase is not simply related to the phase of the current. We shall consider only the voltage induced by the resonance component of the current, and define the shunt impedance, Z, of the rf cavity by $Z = - V_n (\theta = 0)/I_n (\theta = 0)$, where V_n is the voltage induced by I_n.[5] Since $V_n (\theta = 0) = E_\theta(\theta = 0)\, d$, where $E_\theta (\theta = 0)$ is the azimuthal electric field across the gap, we have

$$E_\theta(\theta = 0) = - \frac{Z}{d} I_n (\theta = 0) \tag{2.5}$$

and consequently

$$E_\theta (\theta = 0) = - \frac{e\omega_0 Z}{d} (a_n \cos n\,\omega_0 t - b_n \sin n\,\omega_0 t). \tag{2.6}$$

It should be emphasized that the cavity mode excited by the current is, in this calculation, the same as that being driven externally.

We have in Eq. (2.6) an expression for the induced electric field in terms of the Fourier coefficients of the particle distribution. An effective electric field will now be found from Eq. (2.6) so that it may be inserted into the Vlasov equation to complete the calculation. Following the standard formalism, we decompose the electric field across the gap into standing waves around the azimuth of the machine. Thus we expand E_θ in a Fourier series and keep only the nth

harmonic. This Fourier decomposition brings in a factor of $2 \sin n\theta_1/\pi n$ which is approximately $2\theta_1/\pi = d/\pi R$. Thus, for the nth harmonic of the electric field, we have

$$E_n = - \frac{Ze\omega_0}{\pi R} \cos n\theta \left[a_n \cos n\omega_0 t - b_n \sin n\omega_0 t \right] \tag{2.7}$$

The effective electric field is found by decomposing E_n into traveling waves, and keeping only the wave traveling with the particles.

It will be convenient in what follows to introduce a new angle variable. Following the notation of Nielsen and Sessler;[4]

$$\phi = n(\theta - \omega_0 t) + \pi . \tag{2.8}$$

This substitution transforms the calculation into a coordinate system rotating with angular velocity $n\omega_0$. The arbitrary addition of π is conventional. In terms of this new variable our effective electric field becomes

$$\mathcal{E} = \frac{e\omega_0 Z}{2\pi R} \left[a_n \cos\phi + b_n \sin\phi \right] . \tag{2.9}$$

III. PARTICLE DISTRIBUTION IN SYNCHROTRON PHASE SPACE

For the remainder of this calculation, we follow closely the development of Ref. 4. So far, the externally-applied voltage, V, has not entered the calculation. The frequency of V need not be $\omega_0/2\pi$, but will generally be an integral multiple n of this value, so that the cavity is operating on the nth harmonic of the particle circulation frequency. It is possible to decompose this voltage into traveling waves as was done with the beam-induced electric field. In this way, the abrupt loss or gain of energy by a particle as it crosses the cavity gap is replaced by a continuous change as the particle travels around the machine. We define the phase of the voltage wave such that a particle at phase ϕ gains energy at the rate $eV \sin \phi$ per turn. Clearly, from this definition, the angle ϕ is the phase of the particle relative to the phase of V. If the frequency of the cavity f_c is constant, a particle at $\phi = 3\pi/2$ is riding the trough of the wave, while a particle at $\phi = \pi/2$ is riding the crest of the wave. For constant f_c we shall call the particle at $\phi = \pi$ the synchronous particle. Particles at other phase angle ϕ will be oscillating back and forth in the trough about the value $\phi = \pi$. Modulation of the cavity frequency displaces the synchronous particle to a position ϕ_s such that it gains energy at the rate $eV \sin \phi_s$ per turn. The phase angles of the nonsynchronous particles now oscillate about ϕ_s.

A. Hamiltonian for Synchrotron Motion

Following Symon and Sessler,[3] we define an action variable w, that is canonically conjugate to ϕ as

$$w = \int_{E_0}^{E} \frac{dE}{f_p(E)} \quad , \tag{3.1}$$

where E is the energy of the particle, and f_p is the instantaneous particle frequency. The introduction of w allows us to write a single-particle Hamiltonian in terms of canonically conjugate variables. Each particle gains an amount of energy per turn given by

$$\delta E = e\,V \sin\phi + 2\pi\, e\, R\, \mathcal{E} \quad .$$

Since the energy gain is

$$\delta E = f_p\, \delta w = \frac{dw}{dt}$$

we have the first-order differential equation for w,

$$dw/dt = e\,V \sin\phi + 2\pi\, e\, R\,\mathcal{E} \tag{3.2}$$

The angle variable, ϕ obeys the equation

$$d\phi/dt = n\, d\theta/dt = 2\pi m\,(f_p - f_s)\,, \tag{3.3}$$

where f_s is the frequency of the synchronous particle. A change of variable defined by $W = w - w_s$ allows the use of a Hamiltonian for $W = \phi$ motion of the form:

$$\mathcal{H}(W,\phi) = \pi m \left(f \frac{df}{dE}\right)_s W^2 + e\,V\cos\phi - 2\pi eR \int \mathcal{E}\,d\phi + w_s\,\phi + 2\pi en\,U(\phi).$$

The last term gives the forces due to longitudinal space-charge effects. In the first term, fdf/dE is to be evaluated at the synchronous energy. This approximate

form of the Hamiltonian is derived in Ref. 4, and the derivation will not be repeated here. When Eq. (2.9) is used, the Hamiltonian becomes

$$\mathcal{H}(W, \phi) = \pi n \left(f \frac{df}{dE}\right)_s W^2 + e V \cos \phi + \dot{w}_s \phi + 2\pi e n U(\phi) - e^2 \omega_0 Z [a_n \sin \phi - b_n \cos \phi] . \quad (3.4)$$

B. Solution of the Vlasov Equation

Having obtained a Hamiltonian in which the forces are functions of the spatial distribution of particles, we are in a position to determine a stationary distribution function $\psi(W, \phi)$ that obeys the Vlasov equation

$$\frac{\partial \psi}{\partial W} \frac{dW}{dt} + \frac{\partial \psi}{\partial \phi} \frac{d\phi}{dt} = 0 . \quad (3.5)$$

A particular solution is a distribution $\psi(W, \phi)$, which is constant within a certain bounding curve $W_b(\phi)$ and zero outside. By Liouville's theorem, density in phase space is a constant of the motion, which is determined in this instance by the injector of the machine. Although a uniform density is an idealization, it is a reasonably valid assumption for most injection devices. The solution may be written

$$\psi(W, \phi) = \sigma \Theta [|W_b(\phi)| - |W|] , \quad (3.6)$$

in which σ is the number density in $W - \phi$ space and Θ is the step function which is unity for positive argument and zero otherwise. With this form for ψ, Eq. (3.5) yields

$$\delta(W_b - W) \left[\frac{\partial \mathcal{H}}{\partial \phi} + \frac{\partial \mathcal{H}}{\partial W} \frac{dW_b}{d\phi} \right] = 0 ,$$

which is satisfied for $W \neq W_b$. The term in brackets is zero for $W = W_b$ if we have

$$\mathcal{H} [W_b(\phi), \phi] = \text{constant.} \quad (3.7)$$

Equation (3.4) may then be used to determine $W_b(\phi)$, the bounding curve of the distribution of particles in synchrotron phase space.

IV. SOLUTION IN THE ABSENCE OF FREQUENCY MODULATION

Before solving the complete problem specified by the Hamiltonian, Eq. (3.4), we shall treat a somewhat simpler situation that arises in particle storage rings. Consider a stationary distribution of particles in the absence of cavity modulation. Particles are being held at a constant energy and not accelerated. The rf voltage merely provides stabilizing potential troughs and compensates for any energy losses. For simplicity, longitudinal space charge effects will be neglected here. Using Eqs. (3.7), and (3.4), without the space-charge and modulation terms, we have the following equation for the boundary of our distribution:

$$\pi n \left(f \frac{df}{dE}\right)_s W_b^2 + e V \cos \phi - e^2 \omega_0 Z \left[a_n \sin \phi - b_n \cos \phi\right] = C, \tag{4.1}$$

where C is a constant. For the present, we shall discuss the problem below transition energy, where df/dE is positive. A slight modification of the treatment is necessary for negative df/dE, which will be considered subsequently.

We define the new quantities

$$Y_b(\phi) = K W_b(\phi), \tag{4.2}$$

where

$$K^2 = \frac{2\pi n}{eV} \left(f \frac{df}{dE}\right)_s \tag{4.3}$$

and

$$\xi = \frac{e \omega_0 Z}{V} . \tag{4.4}$$

Since the impedance of the cavity may be determined experimentally, we can now solve for Y_b in terms of the operating parameters of the machine and the Fourier coefficients of the distribution. It will then be possible to calculate a_n and b_n as self-consistent functions of σ, K, and ξ by performing the integrals indicated in Eq. (2.3a-b).

The definitions introduced by Eqs. (4.2), (4.3) and (4.4) allow us to write Eq. (4.1) for the boundary curve as

$$1/2\ Y_b^2 + (1 + b_n \xi) \cos \phi - a_n \xi \sin \phi = C$$

or

$$Y_b = \sqrt{2}\ [\ C - (1 + b_n \xi) \cos \phi + a_n \xi \sin \phi\ ]^{1/2} . \tag{4.5}$$

Constant C is selected so as to include the maximum area within the bounding curve. It can easily be chosen in this simple case without resorting to topological methods. If the reaction of beam and cavity were neglected, we would have $Y_b = \sqrt{2}\ [\ C - \cos \phi]^{1/2}$. The separatrix, or closed curve which includes maximum area, is obviously obtained by setting C equal to 1. Our distribution in $Y - \phi$ space is then bounded by the curve $Y_b = 2 \sin \phi/2$. The distribution extends from $\phi = 0$ to $\phi = 2\pi$ and is centered at $\phi = \pi$.

When $\xi \neq 0$, we may define an angle η which represents the shift of the total voltage wave relative to the driving voltage.

Let

$$\tan \eta = \frac{a_n \xi}{1 + b_n \xi} , \tag{4.6}$$

and

$$u = \phi + \eta$$

so that u represents the phase of a particle relative to the total voltage. By arguments analogous to those above we may determine C so as to obtain the expression

$$Y_b(u) = \sqrt{2}\ [(1 + b_n \xi)^2 + a_n^2 \xi^2]^{1/4} (1 - \cos u)^{1/2} . \tag{4.7}$$

The distribution extends from $u = 0$ to $u = 2\pi$. It has been shifted by an angle η and the area has been changed by a factor $[(1 \pm b_n \xi)^2 + a_n^2 \xi^2]^{1/4}$. We introduce the quantity Ω by the definition

$$\Omega^2 \equiv (1 + b_n \xi)^2 + a_n^2 \xi^2 . \tag{4.8}$$

This phase shift cannot be compensated by altering the phase of V, because the angle η is the phase of the synchronous particle relative to the applied voltage.

It will be convenient to calculate the Fourier coefficients by integrating over angle u rather than over θ. The density σ_θ of particles in $(W - \theta)$ space is related to the density σ in $(W - \phi)$ space by $\sigma_\theta = n\sigma$. Taking this relationship into account, we find:

$$N(\theta) = 2\sigma_\theta W_b(\theta) = \frac{2n\sigma}{K} Y_b(n\theta). \tag{4.9}$$

The Fourier coefficient a_n may then be written:

$$a_n = - \frac{2n\sigma}{K\pi} \int_{-\pi}^{\pi} Y_b(\phi) \cos \phi \, d\phi .$$

Since $Y_b(u)$ is an even function, we may use the relation between u and ϕ to obtain

$$a_n = - \frac{4n\sigma}{K\pi} \cos \eta \int_0^{\pi} Y_b(u) \cos u \, du . \tag{4.10}$$

When Eq. (4.7) for $Y_b(u)$ is employed, the integral is easily evaluated, with the result:

$$a_n = (16n\sigma/3 K\pi) \Omega^{1/2} \cos \eta . \tag{4.11}$$

By a similar manipulation, we obtain

$$b_n = - (16 n\sigma/3K\pi) \Omega^{1/2} \sin \eta . \tag{4.12}$$

From Eqs. (4.11) and (4.12) we may eliminate a_n and b_n in Eqs. (4.6) and (4.8). We may then solve the resulting simultaneous equations for Ω and η to obtain:

$$\Omega = -1/2\, B^2 + [\,1 + 1/4\, B^4]^{1/2} \tag{4.13a}$$

and

$$\cos \eta = \Omega \tag{4.13b}$$

with

$$B = \frac{16 n \sigma \xi}{3 \pi K} = \frac{16 n \sigma e \omega_0 Z}{2 \pi K V} . \tag{4.14}$$

Examination of the Hamiltonian Eq. (3.4) reveals the physical meaning of the phase shift. Eliminating space-charge and modulation terms, and using the definition of ξ, we write:

$$\mathcal{H}(W, \phi) = \pi n \left(f \frac{df}{dE}\right)_s W^2 + e V (1 + b_n \xi) \cos \phi - e V a_n \xi \sin \phi . \tag{4.15}$$

From the equation of motion, $dW/dt = - \partial \mathcal{H}/\partial \phi$, we have

$$\frac{dW}{dt} = e V (1 + b_n \xi) \sin \phi + e V a_n \xi \cos \phi, \tag{4.16}$$

which is zero for the synchronous particle. Therefore, the synchronous phase-angle is given by

$$\tan \phi_s = - \frac{a_n \xi}{1 + b_n \xi} = - \tan \eta . \tag{4.17}$$

The synchronous particle gains energy from the applied rf at a rate

$$\delta E = e V \sin (\pi - \eta) \tag{4.18}$$

in order to compensate for losses to the induced voltage and maintain w at a constant value.

From Eqs. (4.7) and (4.13) we see that the induced voltage has the effect of reducing the height (and therefore the total area) of the stable region of $(W - \phi)$ phase-space. When this area is reduced by the factor $\Omega^{1/2}$, the total number of particles that can be held in stable phase is also reduced by this factor. This number of particles can be found from

$$N_t = \int_0^{2\pi} N(\theta) d\theta \ , \tag{4.19}$$

which may be expressed in terms of an integral over u as

$$N_t = 2n\sigma/K \int_0^{2\pi} Y_b(u)\, du \approx (16\, n\sigma/K)\, \Omega^{1/2} \ . \tag{4.20}$$

The quantity $\Omega^{1/2}$ takes on the value $1/2$ when the phase is shifted by an angle η of 78°. For $B = 0$, Ω is unity, and Ω approaches zero as B approaches infinity. The phase shift η approaches $\pi/2$ in this limit, and for small values of B we have $\sin\eta \approx B$.

Perhaps more appropriate to storage schemes and beam stacking is the situation above transition energy. When df/dE is negative, we must redefine K by

$$K^2 = 2\pi m/eV \left| \left(f \frac{df}{dE}\right)_s \right| \ , \tag{4.21}$$

which modifies the Y_b equation so that

$$-1/2\, Y_b^2(\phi) + (1 + b_n \xi)\cos\phi - a_n \xi \sin\phi = C \ .$$

A transformation $\phi = \psi + \pi$ restores this expression to the original form

$$1/2\, Y_b^2(\psi) + (1 + b_n \xi)\cos\psi - a_n \xi \sin\psi = C \ . \tag{4.22}$$

The analysis proceeds just as above, with the distribution shifted in phase by an angle π. The stable phase area is reduced by the same factor $\Omega^{1/2}$ given by Eq. (4.13a-b).

V. SPACE-CHARGE AND FREQUENCY MODULATION

We now turn our attention to the complete Hamiltonian of Eq. (3.4). Although we could proceed analytically in Section IV, we must resort to numerical computations in the general case. A limitation on the validity of the treatment arises from the fact that, when longitudinal space-charge effects are included, the theory is valid only below the transition energy; this failure is discussed in detail in Ref. 4. When the space-charge term is neglected (as in certain special cases below), the theory is also valid for df/dE negative. Although space-charge effects constitute a problem separate from the cavity interaction, they are included in order to present a complete theory.

A. Equation for the Separatrix

Again following Ref. 4, we replace the term $2\pi enU(\phi)$ in Eq. (3.4) by the approximate expression $4\pi e^2 n^2 \sigma g |W(\phi)|/R$, where $g = 1 + 2\ell n (2G/\pi a)$. The subscript b will be omitted in the remainder of this paper, it being understood that W and Y always refer to values on the boundary. The cross-sectional radius of the beam, a, enters the calculation only through the factor g. The height of the accelerator vacuum tank is here indicated by G in order to conform to the notation of Ref. 4. With this alteration, the complete Hamiltonian of Eq. (3.4) becomes

$$\mathcal{H}[W(\phi), \phi] = \pi n \left(f \frac{df}{dE}\right)_s W^2 + \dot{w}_s \phi + eV(1 + b_n \xi)\cos\phi - eV a_n \xi \sin\phi + 4\pi e^2 n^2 \sigma g |W|/R . \qquad (5.1)$$

Evidently the phase shift η may be defined as in the previous section by Eq. (4.6). In terms of η and Ω as defined by Eq. (4.8), the Hamiltonian takes the form

$$\frac{\mathcal{H}[W(\phi), \phi]}{eV\Omega} = \frac{\pi n}{eV\Omega}\left(f \frac{df}{dE}\right)_s^2 W^2 + \frac{\dot{w}_s}{eV\Omega}(u - \xi) + \cos u + \frac{4\pi e^2 n^2 \sigma g}{eV\Omega R}|W| ,$$

with u again equal to $(\phi + \eta)$. (5.2)

We see that Ω represents the ratio of the total peak voltage on the cavity to the peak voltage, V, in the absence of the beam. It turns out that this calculation is most easily carried out in terms of $V_t = \Omega V$, which yields expressions for Ω and η as functions of V_t and the operating parameters of the machine. Using V_t as an independent variable is logical as well as convenient, since it is certainly the quantity of physical interest. However, this procedure necessitates one change in notation, namely we must redefine K as

$$K^2 = \frac{2\pi n}{eV_t} \left(f \frac{df}{dE}\right)_s \tag{5.3}$$

and introduce two new quantities:

$$A^2 = \frac{2\pi}{e} \frac{e^3 n^3}{R_t} \left[\frac{\left(\frac{df}{\pi}\right)}{V_t \left(f \frac{df}{dE}\right)_s}\right]^{1/2}, \tag{5.4a}$$

and

$$\Gamma = \frac{\dot{W}_s}{eV_t} . \tag{5.4b}$$

Space-charge effects are completely contained in A, while Γ contains the frequency modulation. These definitions allow us to write the equation for the boundary of the stable-phase area in the convenient form

$$\frac{1}{2} Y^2(u) + \sqrt{2} A \,|\, Y(u) \,| + \cos u + \Gamma u = C.$$

Solving for Y results in

$$Y = \sqrt{2} \left\{ -A + \left[A^2 + C - \cos u - \Gamma u \right]^{1/2} \right\} . \tag{5.5}$$

The evaluation of the constant C is not as simple as before. There are two values of u for which dY/du vanishes for any value of the constant. These are

$$u_s = \sin^{-1} \Gamma,$$

with

$$\pi/2 < u_s < \pi , \tag{5.6}$$

and $u_1 = \pi - u_s$. The first of these represents the phase of the synchronous particle, while u_1 gives one extreme of the stable-phase region. The value of the constant that gives the separatrix is then found by setting $Y(u_1)$ equal to zero. The value obtained in this manner is $C = \cos u_1 + \Gamma u_1$. For $\Delta = 0$, dY/du is undefined at u_1, but for non-zero Δ the separatrix has zero slope at this point. The other end of the stable phase region is located at $u_2 > u_s$ such that $\cos u_2 + \Gamma u_2 = \cos u_1 + \Gamma u_1$. At u_2, the separatrix has finite slope if Δ is different from zero. Parenthetically, the ends of the stable-phase region have peculiar shapes that are due to approximations in the space-charge theory. Space-charge effects at the ends of the bunch are not accurately treated.

We now see that the energy gain per turn is independent of Ω. From the equation of motion, Eq. (3.2), we have for the synchronous particle in the absence of induced voltage $\dot{w}_s = eV \sin \phi_{s0}$ in which ϕ_{s0} is the synchronous phase angle if the induced electric field is zero. We then define $\Gamma_0 = \sin \phi_{s0} = \dot{w}_s/eV$. When the induced electric field is included in the Hamiltonian, it can be shown that

$$\dot{\hat{w}}_s = e V_t \sin u_s , \tag{5.7}$$

which because of Eqs. (5.4b) and (5.6) and the relation $V_t = \Omega V$, is just equal to $eV \sin \phi_{s0}$. This result is not surprising, because otherwise the particles would soon be completely out of phase with the external voltage and no stationary distribution could exist. The maximum total number of particles in the accelerator is appreciably affected by Ω, and thus the total phase flux changes. This quantity is defined by

$$\Phi = N_t \dot{\hat{w}}_s , \tag{5.8}$$

and will be discussed in more detail later.

B. Phase Shift and Total Voltage

Although explicit analytic expressions for a_n and b_n cannot be found in this general case, we can obtain simple expressions for Ω and η in terms of two integrals. From Eq. (2.3a-b) we can again derive

$$a_n = -\frac{2n\sigma}{K\pi}\int_{-\pi}^{\pi} Y(\phi)\cos\phi\, d\phi \tag{5.9}$$

and

$$b_n = -\frac{2n\sigma}{K\pi}\int_{-\pi}^{\pi} Y(\phi)\sin\phi\, d\phi$$

The integrals are functions of Δ and Γ and cannot be performed analytically. If we introduce the quantities I_A (for later use), I_S, and I_C by these definitions:

$$I_A(\Gamma,\Delta) = \int_{u_1}^{u_2}\left\{-\Delta + \left[\Delta^2 + \cos u_1 + \Gamma u_1 - \cos u - \Gamma u\right]^{1/2}\right\} du;$$

$$I_S(\Gamma,\Delta) = \int_{u_1}^{u_2}\left\{-\Delta + \left[\Delta^2 + \cos u_1 + \Gamma u_1 - \cos u - \Gamma u\right]^{1/2}\right\} \sin u\, du;$$

$$I_C(\Gamma,\Delta) = \int_{u_1}^{u_2}\left\{-\Delta + \left[\Delta^2 + \cos u_1 + \Gamma u_1 - \cos u - \Gamma u\right]^{1/2}\right\} \cos u\, du; \tag{5.10}$$

we may express a_n and b_n in terms of I_C and I_S. These integrals have been evaluated numerically and are plotted in Figs. 1, 2, 3, and 4. Our definition of I_A differs from that of Ref. 4 by a factor $2^{-1/2}$.

In terms of these quantities we have

$$a_n = - \frac{2\sqrt{2}\, n \sigma}{K \pi} (\cos \eta\, I_C + \sin \eta\, I_S) ,$$

$$b_n = - \frac{2\sqrt{2}\, n \sigma}{K \pi} (- \sin \eta\, I_C + \cos \eta\, I_S) . \qquad (5.11)$$

Proceeding as in the previous section we may solve to obtain

$$\Omega = [(DI_S + 1)^2 + D^2 I_C{}^2]^{-1/2} , \qquad (5.12a)$$

and

$$\tan \eta = - \frac{DI_C}{DI_S + 1} , \qquad (5.12b)$$

where

$$D = \frac{2\sqrt{2} e \omega_0 Z n \sigma}{\pi K V_t} = 2\sigma \omega_0 Z \left[\frac{e^3 n}{\pi^3 V_t (f \frac{df}{dE})_s} \right]^{1/2} \qquad (5.13)$$

When Γ and Δ are zero, we have $I_S = 0$ and $I_C = - (4/3)\sqrt{2}$. These results reduce to Eq. (4.13a-b) in this limit. If Γ is not zero, the angle η still represents the difference in phase between the total voltage and the applied voltage. There is no simple relationship such as Eq. (4.18) between η and the energy-gain per turn. Instead, the energy-gain per turn is not affected by the induded voltage. From Eq. (5.12a) we have the ratio V_t/V as a function of V_t. We might then choose a desired value for the total voltage and stable-phase angle from which Γ and Δ can be calculated. Then from Eq. (5.12a), together with the graphs of I_C and I_S, we may determine the necessary applied voltage. A criterion for selecting the stable phase angle will be given shortly.

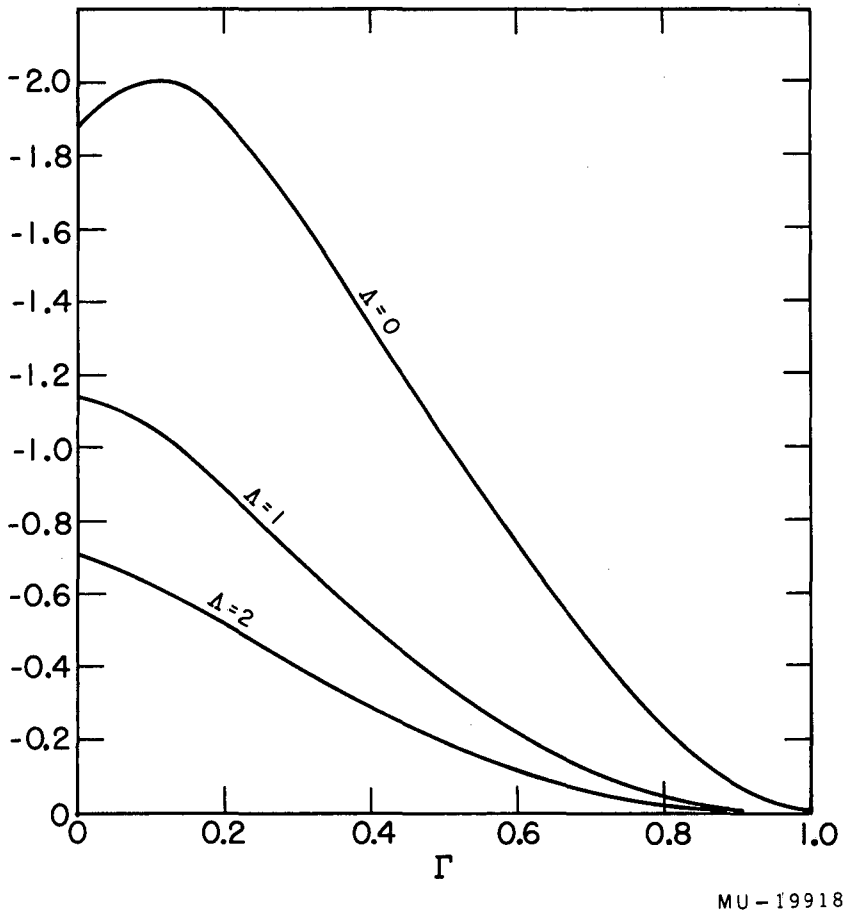

Fig. 1. The Fourier cosine transform $I_C(\Gamma, \Lambda)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Lambda = 0$, 1 and 2, where Λ characterizes the effect of longitudinal space charge.

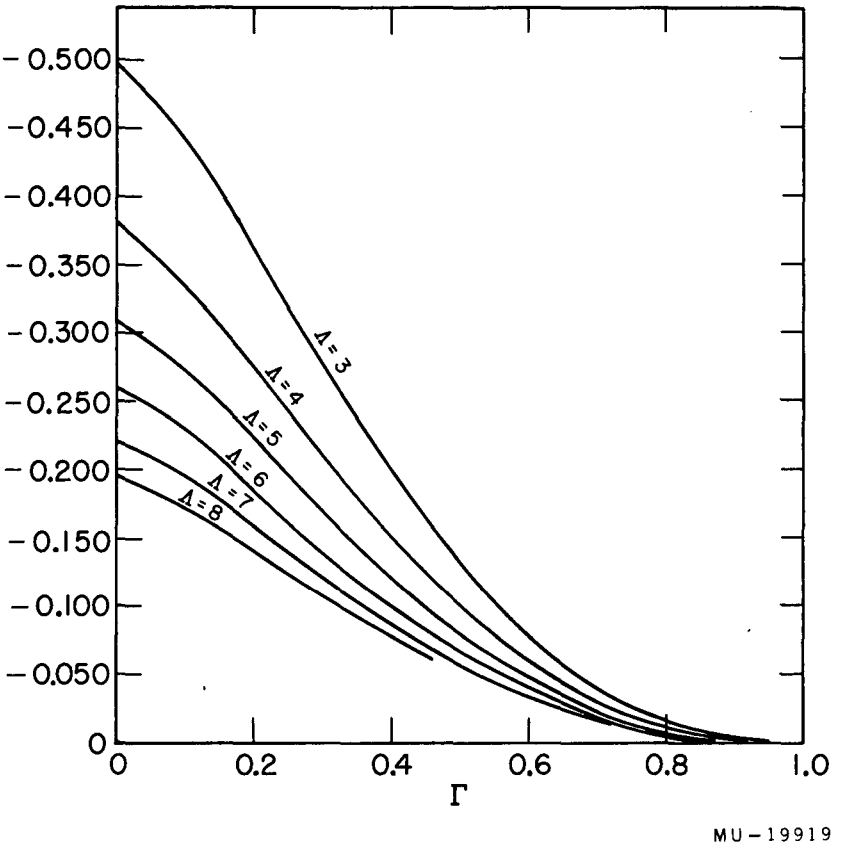

Fig. 2. The Fourier cosine transform $I_C(\Gamma, \Lambda)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Lambda = 3, 4, 5, 6, 7$, and 8, where Λ characterizes the effect of longitudinal space charge.

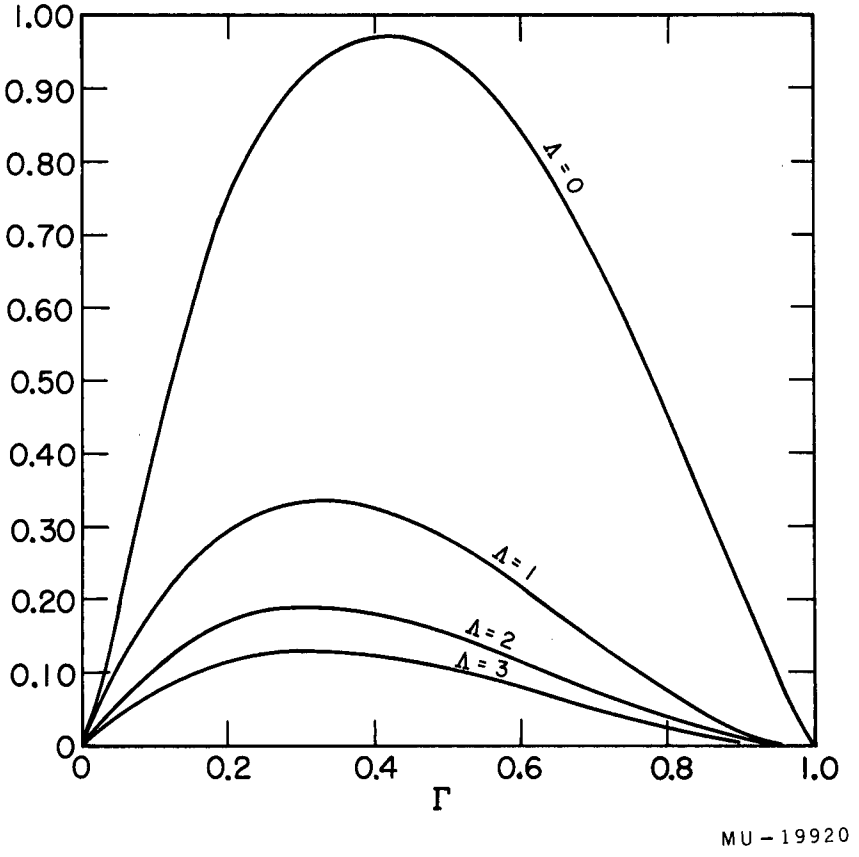

Fig. 3. The Fourier sine transform $I_S(\Gamma, \Lambda)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Lambda = 0, 1, 2$, and 3, where Λ characterizes the effect of longitudinal space charge.

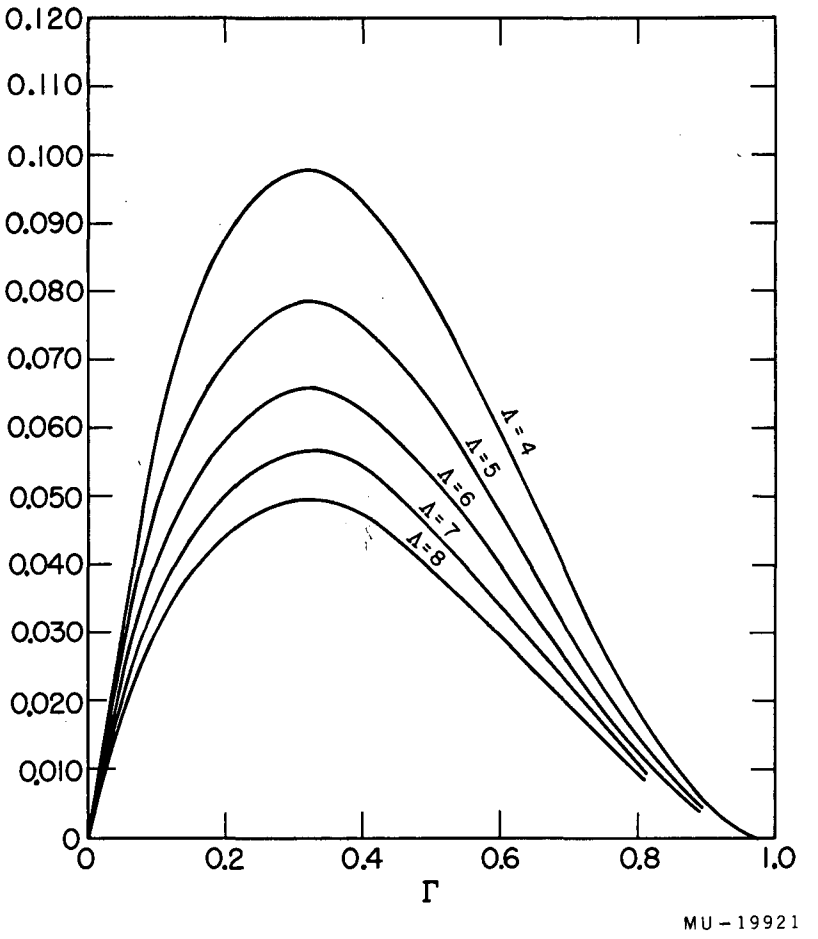

Fig. 4. The Fourier sine transform $I_S(\Gamma, \Lambda)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Lambda = 4, 5, 6, 7,$ and 8, where Λ characterizes the effect of longitudinal space charge.

Although Eq. (5.12a) is the important result of the calculation, the equation does not give a clear picture of what is actually happening. Examination of Fig. 5 may be helpful. The total voltage wave is shifted to the left by an amount η, and the amplitude is reduced by the factor Ω. In the absence of induced voltage, the synchronous particle would ride at a phase angle ϕ_{s0} and gain energy at the rate $eV \sin \phi_{s0}$ per turn. When we "turn on" the induced voltage, the synchronous particle must move to a phase u_s relative to the total voltage-wave. This angle is determined by the relation $V \sin \phi_{s0} = V_t \sin u_s$. The synchronous particle is now at a phase $\phi_s = u_s - \eta$ relative to the applied voltage. The over-all result is a reduction in the bucket area, which is caused by an increase in the energy per turn taken from the rf. This increase has the same effect as a corresponding increase in the modulation rate. As the strength of the beam-cavity interaction increases, perhaps through a larger shunt impedance, the angle ϕ_s approaches $\pi/2$, and the stable phase area approaches zero. The phase shift η also approaches zero in this limit.

In Fig. 6 we have plotted Ω and than η for $\dot{w}_s = 0.3$ eV. Notice that the abscissa in Fig. 6 is $/\Omega D$, which is proportional to $V^{-1/2}$ rather than $V_t^{-1/2}$ and thus is a function only of the operating parameters. The method involved in obtaining these curves is quite tedious, and they are presented only as an illustration. Equation (5.12) is more easily used for numerical computation. We see that the limiting value of V_t is $0.3\ V = \dot{w}_s/e$. Since the phase shift η goes to zero in the limit of infinite D, we have $u_s = \phi_s$ in this limit.

All the parameters in D, with the possible exception of σ, are well-known. The density in $W - \theta$ space at injection may be found from

$$n\sigma = \frac{N_i}{2\pi \Delta W} = \frac{N_i f_i}{2\pi \Delta E} \qquad (5.14)$$

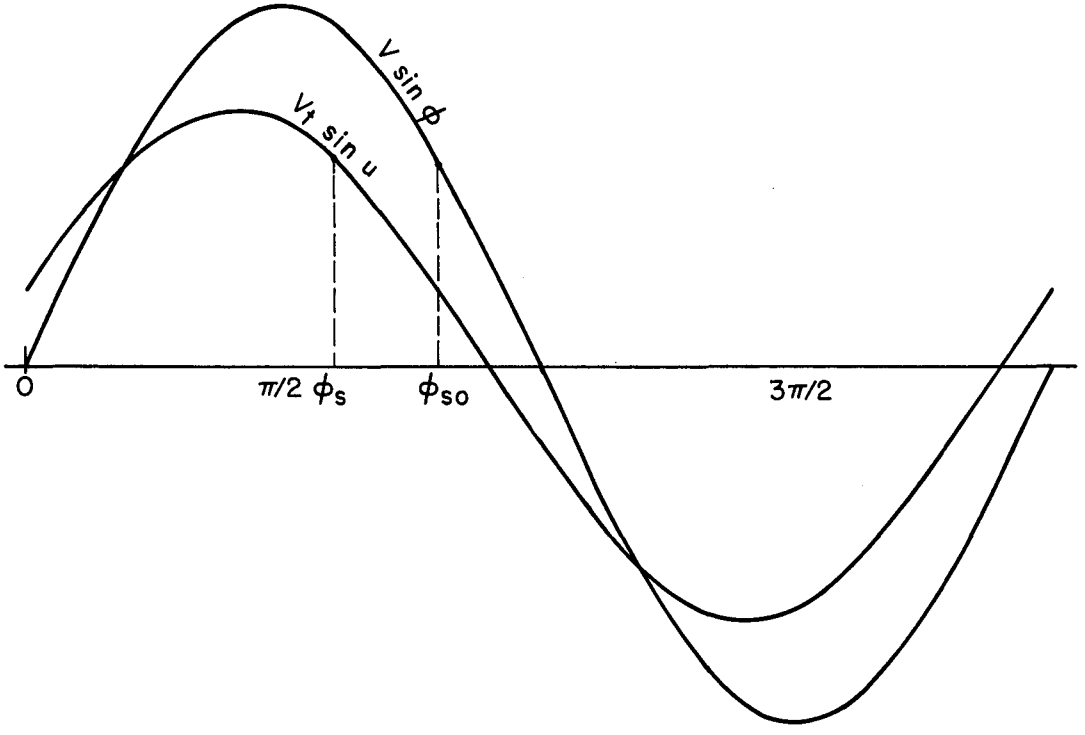

Fig. 5. Applied voltage wave and total voltage vs the phase angle ϕ for phase shift $\eta = \pi/10$

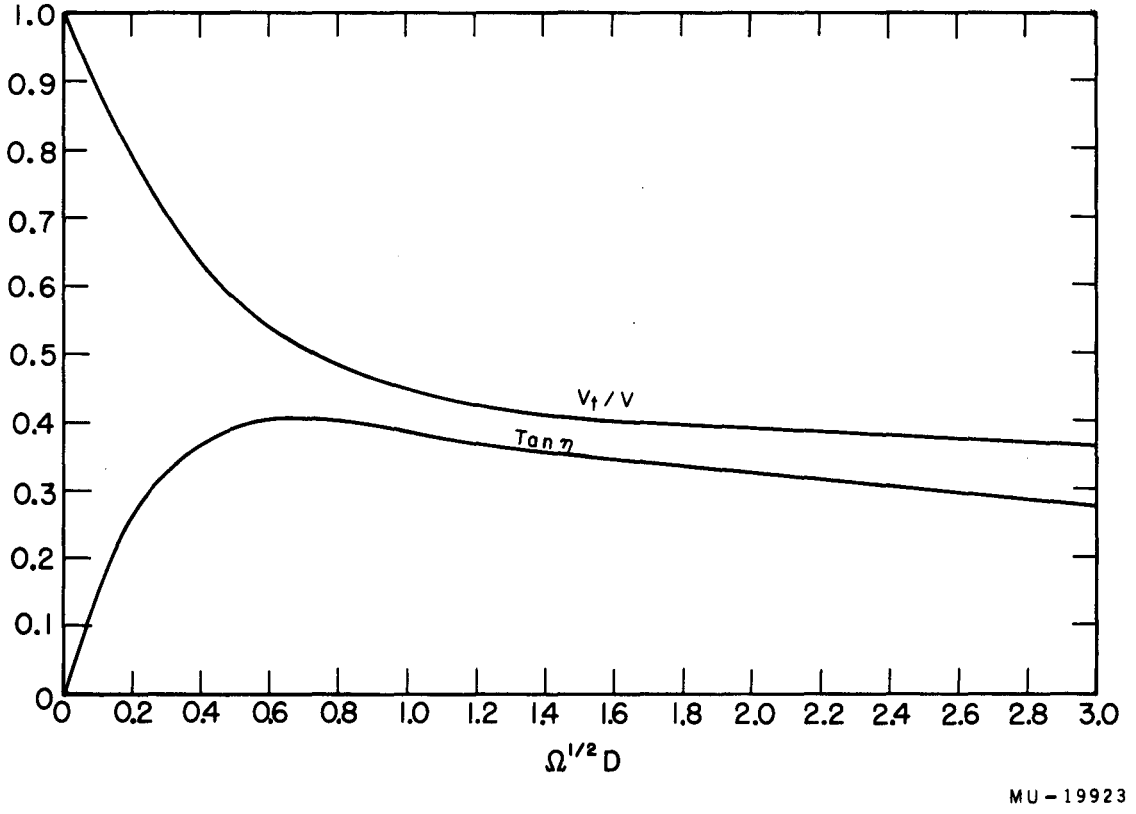

Fig. 6. The ratio of total peak voltage V_t to peak applied voltage V, and the tangent of the phase shift η as function of $\Omega^{1/2} D$ (Eqs. 5.12a and 5.13). The abscissa contains the operating parameters of the machine and is directly proportional to the shunt impedance of the cavity. The energy-gain per turn is 0.3 V.

where f_i is the frequency at injection, ΔE is the energy spread, and N_i is the number of particles injected per turn. The following numerical examples of two quite different accelerators illustrate the possible magnitudes of D.

C. Numerical Examples

As a first example we take the Bevatron. The configuration of the rf cavity in the Bevatron is that of a drift tube. The shunt impedance of the drift tube has not been measured but has a theoretical value of about 3000 ohms at the high-energy end of the accelerating cycle.[5] The peak applied voltage is 22 kv, with an energy of 15 kev per turn being imparted to the particles. Using the experimentally determined total number of particles, N_t, we may use the relation for the total number of particles in the accelerator

$$N_t = \frac{2\sqrt{2}}{K} n\sigma I_A(\Gamma, A), \tag{5.15}$$

to solve for $n\sigma$ rather than calculating it from Eq. (5.14). When the value of $\bar{\sigma}$ found in this manner is inserted into Eq. (5.13) for D, we find

$$D = \frac{N_t e \omega_0 Z}{\pi V_t I_A}. \tag{5.16}$$

In solving Eq. (5.15) for $n\sigma$, we have assumed that the particles are uniformly distributed over the entire stable-phase area. This is an idealization, because the stable phase area is probably not uniformly filled in this machine, and the approximate value of D obtained from Eq. (5.16) is slightly less than the accurate value.

We use $N_t = 2\times10^{11}$, which is typical for this machine, and $\omega_0/2\pi = 2.5\times10^6$ sec^{-1}. If we make the assumption (borne out by the result) that V_t is nearly equal to V, we find that the sine of the stable phase angle is 0.68.

From Fig. 7, for $\Delta = 0$ and $\Gamma = 0.68$, we find $I_A = 1$. Inserting these values into Eq. (5.16) for D, we obtain $D = 0.021$. From Figs. 1 and 3 we find $I_C = -0.51$ and $I_S = 0.7$. When these values are employed in Eq. (5.12a), the result is $V_t = 0.986\ V = 21.7$ kv. The effects of the induced voltage are evidently small for this current.

The transverse space-charge limit for the Bevatron has been estimated and found to correspond to 10^{13} particles, or a circulating current of 4 amp. Let us suppose that 10^{13} particles are circulating in this machine and that it is desired to maintain the total voltage at 22 kv with an energy gain of 15 kev per turn. Again using Eq. (5.16), we find $D = 1$. From Eq. (5.12a) we now obtain $V_t = 0.57\ V$. It will then be necessary to apply a peak voltage of 39 kv to the cavity in order to maintain a total voltage of 2 kv. This may not be an insurmountable difficulty, but may require a much larger rf power input. The additional power necessary to apply the higher voltage will depend upon the circuitry of the external power supply and its coupling to the rf cavity.

Our second example is the Cambridge electron accelerator.[6,7] The numerical results in this example will be approximate, because Liouville's theorem does not hold when the particles lose energy by radiation. The density of particles in phase space is therefore not a constant in time. We have further assumed that the phase-space density is uniform within a bounding curve $W_b(\phi)$, which is not true during the accelerating cycle of this machine. Even in this situation, the principles underlying our theoretical calculation remain valid. The theory should yield results that are accurate to within 50% if we correctly approximate σ. We use the technique of the previous example and determine the phase density from Eq. (5.15). Determining σ in this manner replaces the actual particle distribution with a region of phase-space of uniform density, and adjusts the density of particles in this region so as to give the correct total number of particles.

There are 16 rf cavities operating on the 360th harmonic of the particle circulation frequency. Each cavity has a shunt impedance estimated at 10 megohms and a peak voltage of 1 Mv. At an energy of 5 Bev the incoherent radiation amounts to an energy loss of 2 Mev per turn. If, as proposed, the particles gain 0.8 Mev per turn, the stable-phase angle must be such that $\Gamma = 2.8/16$, or 0.175. From Fig. 7 we find that $I_A = 4$, and from Figs. 1 and 4 we find $I_C = -1.94$ and $I_S = 0.68$. The total number of particles is expected to be 10^{11}, and the circulation frequency is 1.32 Mc. Eq. (5.16) yields $D = 0.105$, and with the use of Eq. (5.12a) we find that an applied voltage of 1.09 Mv is necessary to maintain a peak voltage of 1 Mv on each rf cavity.

Let us consider the effects of the induced voltage in this machine when the particle energy reaches 7 Bev. At this energy, the energy-loss per turn due to incoherent radiation reaches 8 Mev. This loss necessitates a stable-phase angle such that Γ is at least 0,5. Assuming that there are still 10^{11} particles in the accelerator, we may repeat the calculation to find that an applied voltage of 1.25 Mv is necessary to maintain a total voltage of 1 Mv.

In the total voltage in this machine is not maintained at a high level, particles will be lost from the stable-phase region. A reduction in N_t will lower the value of D, and thus tend to reduce the magnitude of the induced voltage. The ultimate result of the beam-cavity interaction should then be a loss of particles from the beam at the high-energy end of the accelerating cycle.

D. Maximum Phase Flux

Normally one attempts to operate an accelerator so as to maximize the total phase flux:

$$\bar{\Phi} = N_t \dot{w}_s \quad . \tag{5.8}$$

From Eq. (5.15) we obtain the expression:

$$\bar{\Phi} = \frac{2\sqrt{2}}{K} \; n \sigma e V_t \Gamma I_A(\Gamma, \Delta) \; . \tag{5.17}$$

Thus for fixed σ and a given V_t, we must maximize $\Gamma I_A(\Gamma, \Delta)$. This can be done by using the curves (Figs. 7 and 8) for I_A vs. Γ. For $\Delta = 0$ we get the well-known result that the maximum occurs at $\Gamma = 0.43$.[1] As Δ increase to unity, the optimum Γ falls to about 0.3 and remains fairly constant at this value as Δ increases to 10.

It is of interest that Δ is directly proportional to σ, and therefore when the longitudinal space charge is considered, there also exists an optimum σ that maximizes the phase flux. If we solve Eq. (5.4a) for σ and employ the result in Eq. (5.17) for $\bar{\Phi}$, we obtain

$$\bar{\Phi} = \frac{\sqrt{2R}}{K} \left[\frac{V_t^3 \left(f \frac{df}{dE}\right)_s}{e^3 n^3 \pi} \right]^{1/2} \Gamma \Delta \, I_A(\Gamma, \Delta) \; . \tag{5.18}$$

We may, for fixed Γ and V_t, find a value, Δ_m, which maximizes $\bar{\Phi}$. The problem is to maximize $\Delta I_A(\Gamma, \Delta)$, which may be accomplished by inspection of the curves of Figs. 9 and 10. For example, when $\Gamma = 0.309$, we find $\Delta_m = 1.3$, while for $\Gamma = 0.453$ a value of 1.6 is found. As Γ increases to a value of 0.7, Δ_m increases to about 1.8. The optimum σ is then found from

$$\sigma_m = \frac{R}{2g} \left[\frac{V_t \left(f \frac{df}{dE}\right)_s}{e^3 n^3 \pi} \right]^{1/2} \Delta_m \; . \tag{5.19}$$

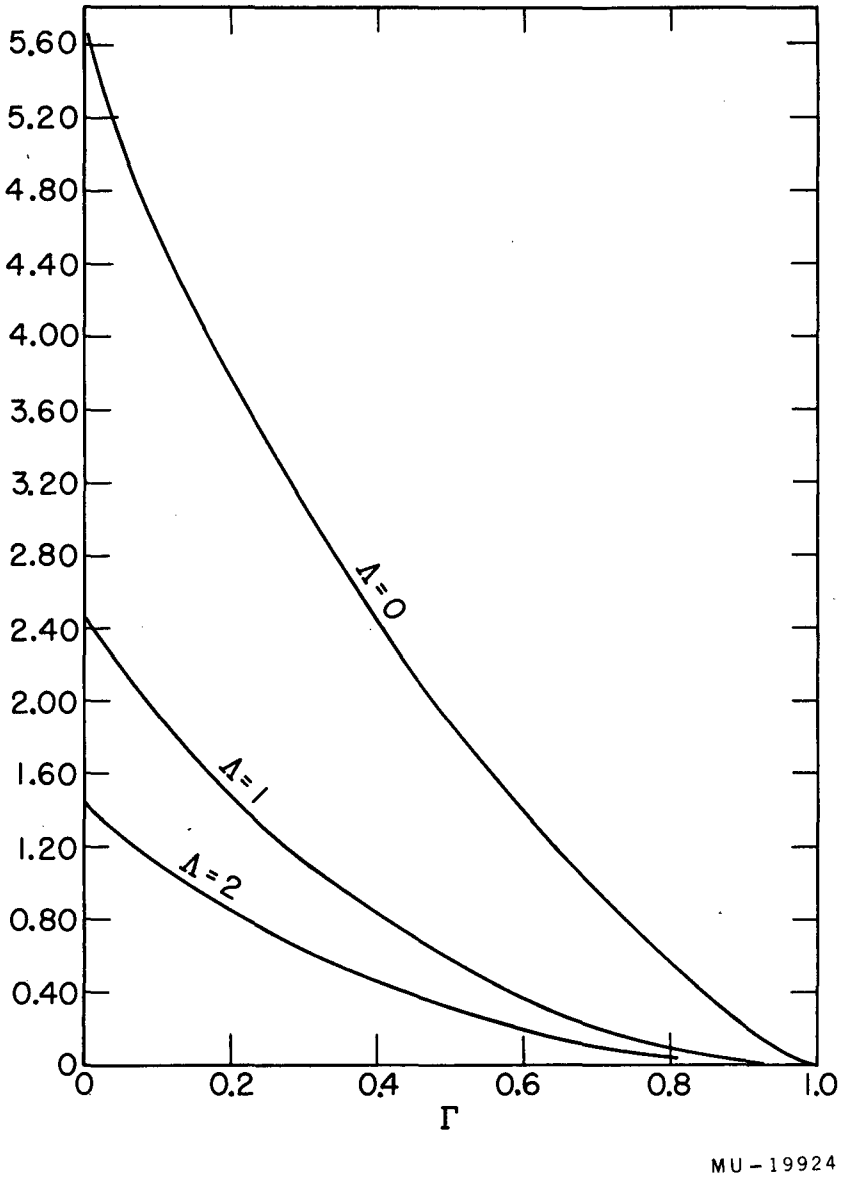

Fig. 7. The area $I_A(\Gamma, \Lambda)$ of the stable region of $Y - \phi$ space (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Lambda = 0, 1$, and 2, where Λ characterizes the effect of longitudinal space charge.

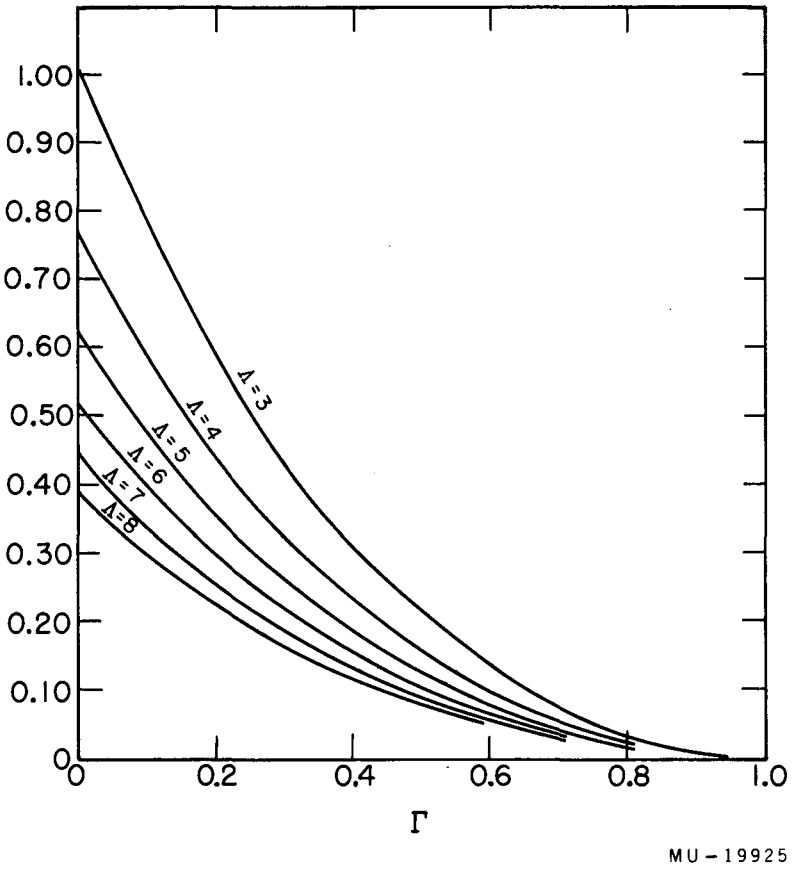

Fig. 8. The area $I_A(\Gamma, \Lambda)$ of the stable region of Y - ϕ space (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for Λ = 3, 4, 5, 6, 7, and 8, where Λ characterizes the effect of longitudinal space charge.

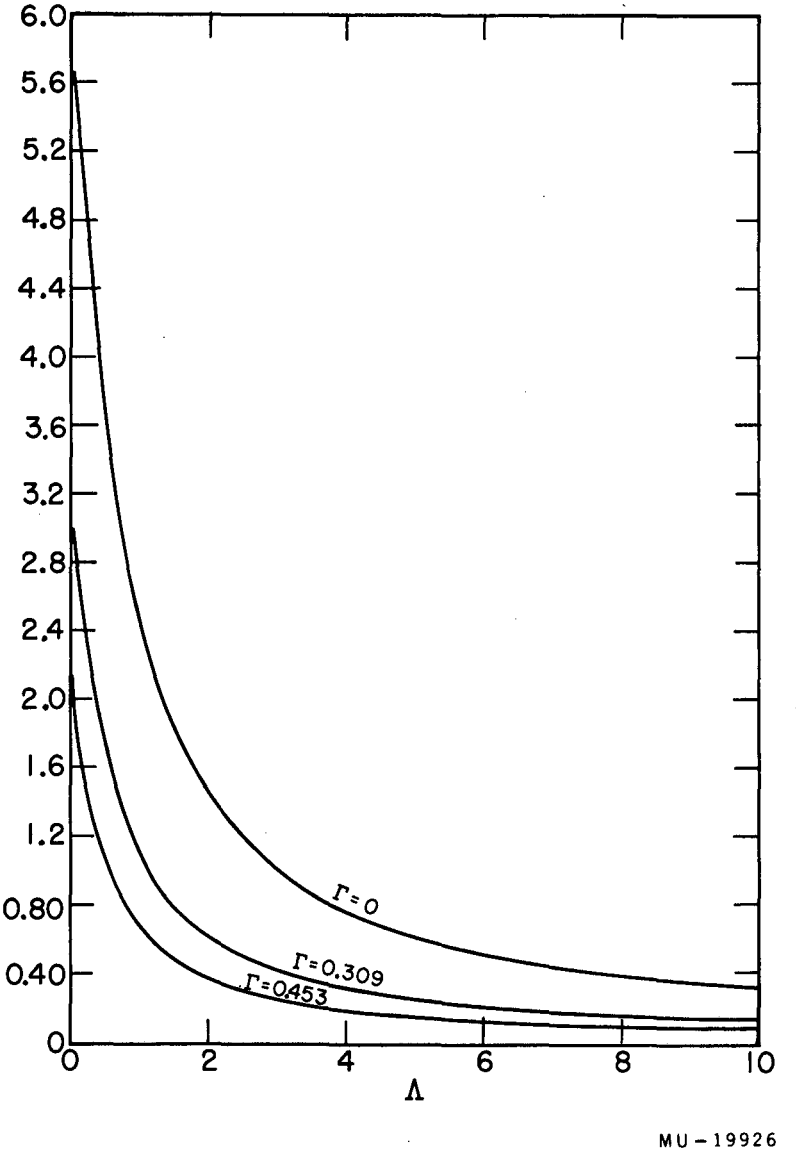

Fig. 9. The area $I_A(\Gamma, \Lambda)$ of the stable region of Y - ϕ space (Eq. 5.10) as a function of Λ which characterizes the effect of longitudinal space charge for $\Gamma = 0$, 0.309, and 0.453, where Γ characterizes the rate of frequency modulation. For $\Lambda > 5$, the dependence is approximately Λ^{-1}.

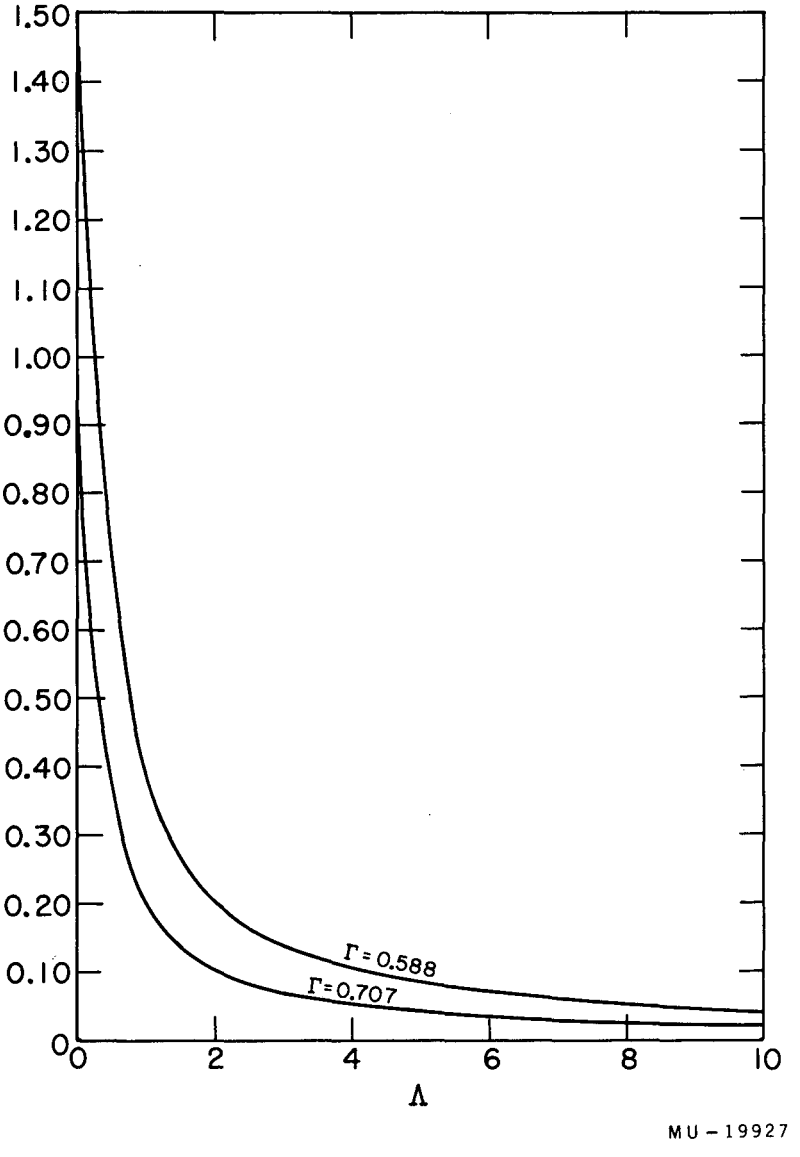

Fig. 10. The area $I_A(\Gamma, \Lambda)$ of the stable region of Y - ϕ space (Eq. 5.10) as a function of Λ which characterizes the effect of longitudinal space charge for Γ = 0.588 and 0.707, where Γ characterizes the rate of frequency modulation. For $\Lambda > 5$, the dependence is approximately Λ^{-1}.

Since the longitudinal space charge is a more restrictive effect at the beginning of the acceleration cycle than at other times,[4] we shall employ the nonrelativistic expression for $f\, df/dE$ in Eq. (5.19) to determine σ_m. If we neglect the change of radius with energy, it is clear that this nonrelativistic expression is

$$\left(f\frac{df}{dE}\right)_s = \frac{1}{(2\pi R)^2 m}$$

for particles of mass m circulating at a radius R. Equation (5.19) then becomes:

$$\sigma_m = \frac{A_m}{4g}\left[\frac{V_t}{(e n \pi)^3 m}\right]^{1/2} . \qquad (5.20)$$

The factor g is always of the order of unity. For a typical proton machine, with V_t of the order of 50 kv and $n = 10$, we find that $\sigma_m \sim 10^{19}$ Mev^{-1} sec^{-1}. This is a density slightly greater than the capability of most injectors currently in use.

VI. SUMMARY

We have considered a beam of particles which passes through an externally driven rf cavity in an accelerator. The particles induce a back voltage across the cavity gap, and an expression is developed for this voltage as a function of the Fourier coefficients of the particle distribution in azimuth. Clearly this voltage depends upon the total number of particles N_t and their azimuthal distribution as well as upon the shunt impedance of the rf cavity.

Describing the motion of particles by a Hamiltonian, we proceeded to find a self-consistent distribution of particles in synchrotron phase-space. The Hamiltonian is strictly valid only for a fixed magnetic guide field, because of the assumption that the particle frequency is not an explicit function of time. This assumption is inherent in the definition of the action variable w. As pointed out by Nielsen and Sessler,[4] if the variation of the magnetic field is slow compared with the synchrotron oscillation of the particles, the instantaneous solution to the Vlasov equation is the same as that for a fixed field. It may also be true that the azimuthal distribution of the particles varies slowly with time. If the characteristic time for this variation is very much longer than a period of the applied rf, it is a good approximation to assume that the distribution is constant.

The effects of induced voltage are to reduce the total voltage V_t across the cavity gap, and to shift its phase relative to the applied voltage V. The ratio of these is given by Eq. (5.12a):

$$\frac{V_t}{V} = \left[(DI_S + 1)^2 + D^2 I_C^2 \right]^{-1/2}$$

with D given by Eq. (5.13). In Eq. (5.13) the quantity σ is the number-density of particles in synchrotron phase space, and may be found from Eq. (5.14). A rough approximation for D is given by Eq. (5.16). The integrals I_C and I_S are plotted in Figs. 1 - 4.

Since the area of stable phase-space is proportional to the square root of the total voltage, this area is reduced by the presence of the back-voltage. It is further reduced by another consequence of the reduction of the total voltage; namely, that the stable-phase angle must shift toward $\pi/2$, so that the energy-gain per turn remains constant. The total number of particles that can be accelerated is given by Eq. (5.15). The quantity I_A is plotted in Figs. 7-10. It is a function of Γ, the sine of the stable-phase angle, and decreases as the stable phase-angle moves toward $\pi/2$.

We see from Fig. 6 that the ratio V_t/V increases as the quantity $\Omega^{1/2} D$ decreases. This quantity is proportional to $ZV^{-1/2}$. These difficulties may therefore be alleviated by increasing the applied voltage and (or) decreasing the shunt impedance Z. The shunt impedance is directly proportional to the Q of the cavity, which suggests that high-Q cavities may create some problems in high-current accelerators. We have not considered the effects of the induced voltage on the trapping efficiency of the rf system, but they may be important enough to preclude acceleration schemes in which the applied voltage rises slowly.

By the technique developed in this paper, one may easily calculate the effect on the beam of a cavity not driven externally. Such cavities might be present for use at some stage of the accelerating process. They have an effect on the beam even if their external power is turned off.

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2. J. Van Bladel, Midwestern Universities Research Association, Madison, Wis., Internal Report 210, November, 1956 (unpublished).
3. K. R. Symon and A. M. Sessler, in Proceedings of the CERN symposium on High Energy Accelerators, Geneva, 1956, (CERN, Geneva, 1956) p. 44.
4. C. E. Nielsen and A. M. Sessler, Rev. Sci. Instr. 30, 80 (1959).
5. The impedance as defined after Eq. (2.4) is a ratio of voltage in statvolts to current in esu per second, and has the dimensions of sec/cm. Since it is customary to measure impedance in ohms, we note that

$$1 \text{ sec/cm} = \frac{1 \text{ statvolt}}{1 \text{ esu/sec}} = \frac{300 \text{ volts}}{1/3\times10^{-9} \text{ coulomb/sec}} = 9\times10^{11} \text{ ohms.}$$

Thus to find Z in sec/cm, one divides the value of Z in ohms by 9×10^{11}.
6. M. S. Livingston, in Proceedings of the CERN Symposium on High Energy Accelerators, Geneva, 1956, (CERN, Geneva, 1956) p. 439.
7. M. S. Livingston, in Proceedings of the CERN Symposium on High Energy Accelerators, Geneva, 1959, p. 335.

LEGENDS

Fig. 1. The Fourier cosine transform $I_C(\Gamma,\Delta)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Delta = 0, 1$ and 2, where Δ characterizes the effect of longitudinal space charge.

Fig. 2. The Fourier cosine transform $I_C(\Gamma,\Delta)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Delta = 3, 4, 5, 6, 7$, and 8, where Δ characterizes the effect of longitudinal space charge.

Fig. 3. The Fourier sine transform $I_S(\Gamma,\Delta)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Delta = 0, 1, 2$, and 3, where Δ characterizes the effect of longitudinal space charge.

Fig. 4. The Fourier sine transform $I_S(\Gamma,\Delta)$ of the stable phase region (Eq. 5.10) as a function of Γ which characterizes the rate of frequency modulation for $\Delta = 4, 5, 6, 7$, and 8, where Δ characterizes the effect of longitudinal space charge.

Fig. 5. Applied voltage wave and total voltage *vs* the phase angle ϕ for phase shift $\eta = \pi/10$.

Fig. 6. The ratio of total peak voltage V_t to peak applied voltage V, and the tangent of the phase shift η as function of $\Omega^{1/2}D$ (Eqs. 5.12a and 5.13). The abscissa contains the operating parameters of the machine and is directly proportional to the shunt impedance of the cavity. The energy-gain per turn is 0.3 V.

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