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Abstract

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In this thesis we introduce and study two probabilistic models of competition and their applications. The first model is a particular contact process, and is intended to simulate propagation dynamics in real social networks. The second one stems from game theory and is of more theoretical value, as it is used to prove the existence of solutions to a certain non-linear partial differential equation.

The first model consists of two competing first passage percolation processes started from uniformly chosen subsets of a random regular graph on N vertices. The processes are allowed to spread with different rates, start from vertex subsets of different sizes or at different times. We obtain tight results regarding the sizes of the vertex sets occupied by each process, showing that in the generic situation one process will occupy $\Theta(1)N^{\alpha}$ vertices, for some $0 < \alpha < 1$. The value of α is calculated in terms of the relative rates of the processes, as well as the sizes of the initial vertex sets and the possible time advantage of one process.

The second model is a version of the stochastic "Tug-of-War" game, played on graphs and smooth domains, with the empty set of terminal states. We prove that, when the running payoff function is shifted by an appropriate constant, the values of the game after n steps converge in the continuous case and the case of finite graphs with loops. Using this we prove the existence of solutions to the infinity Laplace equation with vanishing Neumann boundary condition.

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Chapter 1

Introduction

Modern probability theory provides us with numerous models which simulate competitive aspects of interesting systems. These models are particularly important for studying highly complex systems whose structure is not well understood. For instance, many prominent examples arise as stochastic evolutionary models which have recently been a focus of substantial amount of research. In this thesis we will introduce and analyze two stochastic competition models.

The first one is a competing first passage percolation model, and is intended to model the spread of two competing products in a social network (or a spread of two viruses in a large population). We consider this model quite natural, as it is a finite graph version of the well studied Two Type Richardson Model introduced by Häggstrom and Pemantle [34] (which was also introduced to model similar dynamics). Apart from studying this process on a uniformly chosen d-regular graph, we also show striking difference in the behavior with respect to the same process on the large tori. This part of the thesis is based on a joint work with Yael Dekel, Elchanan Mossel and Yuval Peres [1].

The second process is a version of the stochastic "Tug-of-War" game, first introduced by Peres, Schramm, Sheffield and Wilson in [52]. While certainly an interesting model from the game-theoretic viewpoint, the main motivation for studying versions of this game is its close connection to the non-linear infinity Laplace equation. This observation, first made and exploited in [52], has lead to breakthrough results and completely new level of understanding of the solutions to this equation. In this text we introduce a study a new version of this game, and then use the obtained results to obtain novel results for the solutions of the infinity Laplace equation with pure Neumann boundary conditions. This part of the thesis is based on a joint work with Yuval Peres, Scott Sheffield and Stephanie Somersille [2].

1.1 Competing first-passage percolation

First passage percolation is one of the best studied discrete models in probability theory. It can be realized as a random graph metric when edges have independent identically distributed weights. Often the distribution is assumed to be exponential and then the ball of a radius t (from a fixed vertex) is a Markov set process $\mathcal{R}$, in which new vertices are occupied at a rate proportional to the number of their neighbors already in $\mathcal{R}(t)$. Apart from the classical shape problem on infinite transitive graphs (see [18]), recently there was substantial interest in estimating diameter, typical distance, flooding times and related quantities for the process on large finite (and possibly random) graphs [41, 38, 7, 8, 10, 9].

In a related two type Richardson model, introduced in [34], one considers two first passage percolation processes on $\mathbb{Z}^d$, a blue and a red one, with possibly different rates, spreading through the graph and capturing non-colored vertices. Each non-colored vertex becomes colored with color c at the rate proportional to the number of c colored neighbors (this can also be viewed as the Voronoi tessellation of two independent first passage percolation metrics). A significant amount of work on this model has been devoted to identifying the cases in which both colors grow indefinitely [33, 20, 21, 22, 28, 36, 37]. Here we initiate the study of this model on large finite graphs. We are interested in the sizes of each colored component, while allowing the processes to start at different times, from sets of different sizes and spread with different rates. We mostly focus on the case of large d-regular graphs (which are objects of independent interest [13, 42]), due to the fact that these graphs exhibit certain properties observed in real world networks and are amenable to theoretical study via the so-called configuration model.

From an applied point of view, this model can be viewed to simulate spreading of two products (or viruses) through a social network. In recent years, diffusion processes on social networks have been the focus of intense study in a variety of areas. Traditionally these processes have been of major interest in epidemiology where they model the spread of diseases and immunization [49, 47, 48, 25, 6, 26]. Much of the recent interest has resulted from applications in sociology, economics, and engineering [15, 5, 31, 30, 23, 53, 44, 43].

The interpretations of the diffusion process in terms of product marketing and in terms of virus spread lead to some natural questions we address in this paper. What is the advantage that the first product (the first virus) has in terms of the initial time it can spread with no competition? What is the effect of one of them starting with larger initial size (initial seed sets) than the other one or having a larger rate (higher quality of a product)? What is the effect of the structure of the social network on the outcome of the competition between the two products? To answer the last question we compare the results for the model on large random regular graphs to the same model on large d dimensional tori. The first family of graphs model some (but not all) features of current social networks (small diameter, expansion etc.) while the second family models traditional spatial graph processes that are traditionally studies in epidemiology, ecology and statistical physics.

Definition of the process and the results

Let $G = (V, E)$ be a graph with $|V|$ vertices and $|E|$ edges, and let $\mathcal{B}_0$ and $\mathcal{R}_0$ be disjoint sets of vertices (we think of $\mathcal{B}_0$ as a set of *blue* vertices and of $\mathcal{R}_0$ as a set of *red* vertices). Denote by $N(v)$ the set of neighbors of v. Competing first passage percolation (CFPP) considered in this thesis is a Markov process, whose state space is the family of subsets of V, which evolves by coloring an uncolored vertex blue (red) at the rate equal to β (ρ) times the number of neighbors of v which are already blue (red). That is, at any time t, each vertex $v \notin \mathcal{B}_t \cup \mathcal{R}_t$ becomes an element of $\mathcal{B}_t$ at the rate equal to $\beta|N(v) \cap \mathcal{B}_t|$, and an element of $\mathcal{R}_t$ at the rate equal to $\rho|N(v) \cap \mathcal{B}_t|$. Here β and ρ are parameters fixed throughout, called *rates* of $\mathcal{B}$ and $\mathcal{R}$ respectively. Sets $\mathcal{B}_t$ and $\mathcal{R}_t$ are increasing in t, that is once a vertex gets colored with a certain color it does not change its state again.

A more precise description of the process requires assigning, for every pair $(u, e) \in V \times E$ such that u is incident to e, two exponential random variables, $\tau^{\beta}_{u,e}$ with mean $1/\beta$ and $\tau^{\rho}_{u,e}$ with mean $1/\rho$. Assume that the clocks are all independent. Up to a time parametrization the above process can be realized as follows. Set $\tilde{\mathcal{B}}_0 = \mathcal{B}_0$ and $\tilde{\mathcal{R}}_0 = \mathcal{R}_0$ and $T(u) = 0$ for all $u \in \mathcal{B}_0 \cup \mathcal{R}_0$. At every time step $n \geq 0$ choose the vertex v which minimizes the value

$$\min\Big(\{T(u_1) + \tau^{\beta}_{u_1,e} : u_1 \in \tilde{\mathcal{B}}_n \cap N(v)\} \cup \{T(u_2) + \tau^{\rho}_{u_2,e} : u_2 \in \tilde{\mathcal{R}}_n \cap N(v)\}\Big).$$

Then set $T(v)$ to be this minimal value and $\tilde{\mathcal{B}}_{n+1} = \tilde{\mathcal{B}}_n \cup \{v\}$, $\tilde{\mathcal{R}}_{n+1} = \tilde{\mathcal{R}}_n$ if the minimum is achieved for some $u_1 \in \tilde{\mathcal{B}}_n \cap N(v)$ and $\tilde{\mathcal{B}}_{n+1} = \tilde{\mathcal{B}}_n$, $\tilde{\mathcal{R}}_{n+1} = \tilde{\mathcal{R}}_n \cup \{v\}$ otherwise. Finally define $\mathcal{B}_t = \bigcup_n \{v \in \tilde{\mathcal{B}}_n : T(v) \leq t\}$ and $\mathcal{R}_t = \bigcup_n \{v \in \tilde{\mathcal{R}}_n : T(v) \leq t\}$.

Note that $(\tilde{\mathcal{B}}_n, \tilde{\mathcal{R}}_n)$ above is a discretized version of this process, which records its state only at times when a change happens. This discretized process has very simple jump rules. At each integer n, choose an edge connecting a vertex u in $\tilde{\mathcal{B}}_n \cup \tilde{\mathcal{R}}_n$ to a vertex v in the complement $\big(\tilde{\mathcal{B}}_n \cup \tilde{\mathcal{R}}_n\big)^c$. The edges incident to a vertex in $\tilde{\mathcal{B}}_n$ are chosen with probability proportional to β and those incident to a vertex in $\tilde{\mathcal{R}}_n$ with probability proportional to ρ. If $u \in \tilde{\mathcal{B}}_n$ then set $\tilde{\mathcal{B}}_{n+1} = \tilde{\mathcal{B}}_n \cup \{v\}$ and $\tilde{\mathcal{R}}_{n+1} = \tilde{\mathcal{R}}_n$. If $u \in \tilde{\mathcal{R}}_n$ then set $\tilde{\mathcal{R}}_{n+1} = \tilde{\mathcal{R}}_n \cup \{v\}$ and $\tilde{\mathcal{B}}_{n+1} = \tilde{\mathcal{B}}_n$.

By $\mathcal{B}_{\text{fin}}$ and $\mathcal{R}_{\text{fin}}$ denote the final set of blue and red vertices when the whole graph is exhausted, and their sizes by B_{fin} and R_{fin} respectively. We are interested in the asymptotic behavior of B_{fin} and R_{fin} as the size of the graph tends to infinity, and how it depends on the choice of initial sets $\mathcal{B}_0$ and $\mathcal{R}_0$ and rates β and ρ. Observe that time parametrization is irrelevant for the sets $\mathcal{B}_{\text{fin}}$ and $\mathcal{R}_{\text{fin}}$. In particular, we will be mainly studying the process through its discretized version $(\tilde{\mathcal{B}}_n, \tilde{\mathcal{R}}_n)$, which will be denoted by $(\mathcal{B}_n, \mathcal{R}_n)$ (as opposed to $(\mathcal{B}_t, \mathcal{R}_t)$ for the continuous process).

Consider the (finite) set of all simple d-regular vertex-labeled graphs with the vertex set $\{1, \ldots, N\}$. The random d-regular graph on N vertices is a random graph chosen uniformly from this set (here we assume that dN is even, as otherwise such graphs do not exist). We will study the above process on the random d-regular graphs. Sets $\mathcal{B}_0$ and $\mathcal{R}_0$ will be chosen random as well. This all means that we will first choose a d-regular graph graph on N vertices from the uniform distribution, conditioned on its realization we will sample sets $\mathcal{B}_0$ and $\mathcal{R}_0$ using a certain rule, and conditioned on the realization of this coupling we will run the competing first passage percolation process (CFPP) described above. Note that we will always assume that $d \geq 3$. The reason for this assumption is that 2-regular graphs are just disjoint unions of cycles. As these graphs (except in the case of one cycle) are not connected, the process can not spread throughout the whole graph.

Here we state one of our results which is a special case of Theorem 2.1.4, but which nicely describes the type of results we obtain. It refers to the case when the sets $\mathcal{B}_0$ and $\mathcal{R}_0$ are chosen uniformly of large prescribed size (in Section 2.1 we will allow more general rules for choosing the initial sets to model certain aspects of competitive behavior in real-world networks).

As our theorems give the asymptotics of B_{fin} and R_{fin} as the graph sizes $N \to \infty$, values such as B_0, R_0, B_{fin} and R_{fin} will in general depend on N. However, to keep the formulas more readable, we will not always emphasize this dependence explicitly.

Theorem 1.1.1. *For $d \geq 3$ and a random d-regular graph on N vertices, assume that the sets $\mathcal{B}_0 = \mathcal{B}_0(N)$ and $\mathcal{R}_0 = \mathcal{R}_0(N)$ are chosen uniformly at random among all disjoint vertex subsets of sizes $B_0 = B_0(N)$ and $R_0 = R_0(N)$ respectively. Assume that there are constants c_1, C_1 such that $c_1 N^{\alpha_b} \leq B_0 \leq C_1 N^{\alpha_b}$ and $c_1 N^{\alpha_r} \leq R_0 \leq C_1 N^{\alpha_r}$. Then there exist constants c_2, C_2 such that with probability converging to 1 as $N \to \infty$*

i) $c_2 N^{\alpha_b + (1-\alpha_r)\beta/\rho} \leq B_{\mathit{fin}} \leq C_2 N^{\alpha_b + (1-\alpha_r)\beta/\rho}$, *in the case* $\beta(1-\alpha_b) \leq \rho(1-\alpha_r)$,

i) $c_2 N^{\alpha_r + (1-\alpha_b)\rho/\beta} \leq R_{\mathit{fin}} \leq C_2 N^{\alpha_r + (1-\alpha_b)\rho/\beta}$, *in the case* $\beta(1-\alpha_b) \geq \rho(1-\alpha_r)$.

This result shows that typically one process occupies only $o(N)$ vertices, and the other one everything else. From the applied perspective, the results of this type can be interpreted as one of the two products taking the lion share of the market. This result stands in striking contrast with the ones that we obtained in the case when the underlying graph is a large torus. Our results (see Theorem 2.1.6) show that even if we start one of the processes earlier than the other and we give it a much higher rate, the other process will still occupy a linear fraction of vertices with high probability.

1.2 Tug-of-War

For a (possibly infinite) graph $G = (V, E)$, the stochastic tug-of-war game, as introduced in [52], is a two-player zero-sum game defined as follows. At the beginning there is a token located at a vertex $x \in V$. At each step of the game players toss a fair coin and the winning player gets to move the token to an arbitrary neighbor of x. At the same time Player II pays Player I the value $f(x)$, where $f\colon V \to \mathbb{R}$ is a given function on the set of vertices, called the *running payoff*. The game stops when the token reaches any vertex in a given set $W \subset V$, called the *terminal set*. If $y \in W$ is the final position of the token, then Player II pays Player I a value of $g(y)$ for a given function $g\colon W \to \mathbb{R}$ called the *terminal payoff*. One can show that, when g is bounded and either $f = 0$, $\inf f > 0$ or $\sup f < 0$, this game has a value (Theorem 1.2 in [52]), which corresponds to the expected total amount that Player II pays to Player I when both players "play optimally".

Among other reasons, these games are interesting because of a connection between the game values and viscosity solutions of the infinity Laplace equation. Let $\Omega \subset \mathbb{R}^d$ be a domain (open, bounded and connected set) with C^1 boundary $\partial\Omega$, and let $f\colon \Omega \to \mathbb{R}$ and $g\colon \partial\Omega \to \mathbb{R}$ be continuous functions. Define the graph with the vertex set $\overline{\Omega}$ so that two points $x, y \in \overline{\Omega}$ are connected by an edge if and only if the intrinsic path distance between x and y in $\overline{\Omega}$ is less than ϵ. Playing the game on this graph corresponds to moving the token from a position $x \in \Omega$ to anywhere inside the ball with the center in x and radius ϵ, defined with respect to the intrinsic path metric in $\overline{\Omega}$. Consider this game with the running payoff $\epsilon^2 f$, the terminal set $\partial\Omega$ and the terminal payoff g. By Dynamic programming principle if the value of this game exists then it is a solution to the finite difference equation

$$u(x) - \frac{1}{2}\left(\min_{B(x,\epsilon)} u + \max_{B(x,\epsilon)} u\right) = \epsilon^2 f(x),$$

for all $x \in \Omega$, and $u(y) = g(y)$, for all $y \in \partial\Omega$. In [52] it was shown that, under certain assumptions on the payoff function f, the game values with step size ϵ converge as ϵ converges to zero appropriately. Moreover the limit u is shown to be a viscosity solution to the non-linear partial differential equation

$$\begin{cases} -\Delta_\infty u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

(one can intuitively think of $\Delta_\infty u$ as the second derivative of u in the direction of the gradient of u, see Definition 3.1.4 for the precise definition). Using finite difference approach and avoiding probabilistic arguments, Armstrong and Smart [3] obtained general existence results for this equation and the uniqueness for typical shifts of the function f. Several modifications of this game have also been studied, including using biased coins, which corresponds to adding a gradient term to the equation (see [51]) and taking the terminal set W to be a non-empty subset of $\partial\Omega$, which corresponds to Dirichlet boundary conditions on W and vanishing Neumann boundary conditions on $\partial\Omega \setminus W$ (see [4] and [16]).

A crucial property of these games is the fact that the terminal set is non-empty which ensures that the game can stop in finite time. However to use the above connection in order to study the infinity Laplace equation with pure vanishing Neumann boundary conditions, one would have to consider this game without the terminal set.

We circumvent this problem by considering a version of the game with *finite horizon*. This game is played on a graph (with empty terminal set) the same way as the original tug-of-war game described above, but only for a finite prescribed number of steps n. Each time Player II pays to Player I the value of f at the current position of the token. At the end of the game Player II pays to Player I value $u_0(x)$, where x is the final n-th position of the token. The value u_n of this game with horizon n then satisfies the following simple recursion (see Section 3.1 for the game theoretic background)

$$u_{n+1}(x) = \frac{1}{2}\Big(\min_{y\sim x} u_n(y) + \max_{y\sim x} u_n(y)\Big) + f(x). \tag{1.2.1}$$

The following result then gives the notion of the game value as the horizon n tends to ∞.

Theorem 1.2.1. *Let $G = (V, E)$ be a finite graph with a loop at each vertex, and let $f : V \to \mathbb{R}$ be a function on the set of vertices. Then there exists a constant c, such that for any function $u_0 : V \to \mathbb{R}$ the following holds. In the finite horizon tug-of-war game played on G with the terminal and running payoffs u_0 and $f - c$ respectively, the sequence of game values (u_n) converges.*

This theorem follows readily from Theorem 3.1.7. It is also true for adjacency graphs, which have uncountably many vertices of uncountable degree, which arise by connecting points of the closure of a bounded domain in $\mathbb{R}^d$ if their distance in the intrinsic metric is less than $\epsilon > 0$. Studying the limits (as $n \to \infty$) of the game values, and taking $\epsilon \downarrow 0$ we prove the following result.

Theorem 1.2.2. *Let Ω be a domain of finite diameter with C^1 boundary $\partial\Omega$ and $f_1 : \overline{\Omega} \to \mathbb{R}$ a continuous function. Then there exists a constant c such that the equation (3.1.5) with $f(x) = f_1(x) - c$ has a viscosity solution u which is Lipshitz continuous, with Lipshitz constant depending on Ω and the norm $\|f\|_\infty$. If Ω is convex, then c is unique.*

The above theorem is a corollary of Theorem 3.1.12.

Precise definitions are given in Section 3.1. In Section 3.2 we prove the convergence of game values as horizon tends to ∞. The results we obtain are used in Section 3.3 to prove the existence of solutions of infinity Laplace equations with pure vanishing Neumann boundary conditions. Section 3.4 contains a discussion about uniqueness.

Chapter 2

Competing first-passage percolation

2.1 Statements of results

In this chapter we study the CFPP model, as introduced in Section 1.1. Recall that we want from our model to handle situations which arise in cases when one of the processes starts earlier than the other. Therefore we start by discussing how we choose the initial random sets $\mathcal{B}_0$ and $\mathcal{R}_0$.

Definition 2.1.1. For a graph $G = (V, E)$ we say that the pair $(\mathcal{B}_0, \mathcal{R}_0)$ of subsets of V, is *uniform of size* (B_0, R_0) if it is chosen uniformly at random among all pairs of disjoint subsets of V of the sizes B_0 and R_0.

For the case when one process (say $\mathcal{B}$) starts earlier than the other, the idea is to let $\mathcal{B}$ evolve from a uniformly chosen subset of some size, until it reaches a certain prescribed size. Then we define $\mathcal{B}_0$ to be the occupied set and take $\mathcal{R}_0$ to be a uniform subset of $\mathcal{B}_0^c$. The first phase in which only $\mathcal{B}$ grows is simply the CFPP process in which $\mathcal{R}$ starts from the empty set of vertices. This leads to the following definition.

Definition 2.1.2. For a graph $G = (V, E)$ we say that the pair $(\mathcal{B}_0, \mathcal{R}_0)$ of subsets of V, is *uniform of size* (B_0, R_0) *with* $\mathcal{B}_0$ *center of size* k_0 if

i) $\mathcal{B}_0^0$ is a uniformly chosen subset of V of the size k_0,

ii) $\mathcal{B}_0 = \mathcal{B}_T^0$, where $(\mathcal{B}^0, \mathcal{R}^0)$ is the CFPP process ran from $(\mathcal{B}_0^0, \emptyset)$, and T is the first time k that $|\mathcal{B}_k^0| = B_0$,

iii) $\mathcal{R}_0$ is the uniformly chosen subset of $\mathcal{B}_0^c$ of size R_0.

For a sequence of graphs $G_N = (V_N, E_N)$ we say for $(\mathcal{B}_0(N), \mathcal{R}_0(N))$ a sequence of pairs of disjoint subsets of V_N, that $\mathcal{B}_0$ *has a small center* if

i) for every N, $(\mathcal{B}_0(N), \mathcal{R}_0(N))$ is uniform of some size $(B_0(N), R_0(N))$ with $\mathcal{B}_0(N)$ center of size $k_0(N)$,

ii) $\lim_{N\to\infty} k_0(N)/B_0(N) = 0$.

For our results we need to either choose the pair of initial sets uniformly of prescribed size, or always allow one of the processes to have a significant advantage. This is captured by the following definition.

Definition 2.1.3. For a sequence of graphs $G_N = (V_N, E_N)$ we say that $(\mathcal{B}_0(N), \mathcal{R}_0(N))$ a sequence of pairs of disjoint subsets of V_N is *admissible* if $\mathcal{B}_0$ has a small center, or $\mathcal{R}_0$ has a small center, or for every N the pair $(\mathcal{B}_0(N), \mathcal{R}_0(N))$ is uniform of some sizes $(B_0(N), R_0(N))$.

We now state our main results. In the statements of Theorems 2.1.4 and 2.1.5 we assume that G is a random d-regular graph on N vertices, and $(\mathcal{B}, \mathcal{R})$ a competing first passage percolation process on G with parameters (β, ρ).

The first theorem covers the case when both processes start from a large size.

Theorem 2.1.4. *Let $(L_N)_N$ be a sequence converging to ∞ and $B_0 = B_0(N)$ and $R_0 = R_0(N)$ be two sequences of positive integers such that*

$$B_0 \geq L_N, \quad R_0 \geq L_N, \quad \text{and} \quad \lim_{N\to\infty} \frac{B_0}{N} = \lim_{N\to\infty} \frac{R_0}{N} = 0.$$

Fix an integer $d \geq 3$ and rates $\beta > 0$ and $\rho > 0$. If $(\mathcal{B}, \mathcal{R})$ are started from admissible pairs of sizes (B_0, R_0) then there exists sequences $\overline{B} = \overline{B}(N)$ and $\overline{R} = \overline{R}(N)$, such that for every $\epsilon > 0$ the final sizes $B_{fin} = B_{fin}(N)$ and $R_{fin} = R_{fin}(N)$ satisfy

$$\lim_{N\to\infty} \mathbb{P}(|B_{fin} - \overline{B}| > \epsilon \overline{B}) = \lim_{N\to\infty} \mathbb{P}(|R_{fin} - \overline{R}| > \epsilon \overline{R}) = 0.$$

If $\beta = \rho$ then

$$\begin{aligned}
\overline{R} &= \frac{R_0}{B_0 + R_0} N, \quad \text{if } \mathcal{B}_0 \text{ and } \mathcal{R}_0 \text{ are chosen uniformly,} \\
\overline{R} &= \frac{dR_0}{(d-2)B_0 + dR_0} N, \quad \text{if } \mathcal{B}_0 \text{ has a small center,} \\
\overline{R} &= \frac{(d-2)R_0}{dB_0 + (d-2)R_0} N, \quad \text{if } \mathcal{R}_0 \text{ has a small center.}
\end{aligned}$$

For any $\beta \neq \rho$ there are positive constants c and C depending only on ρ/β and d such that

$$c \min\left(R_0 (N/B_0)^{\rho/\beta}, N\right) \leq \overline{R} \leq \min\left(C R_0 (N/B_0)^{\rho/\beta}, N\right).$$

The following theorem covers the case when both processes start from fixed sizes, that is both sequences $B_0 = B_0(N)$ and $R_0 = R_0(N)$ are constant. Here of course, we don't need to worry about the possibility of one process starting earlier - such a version wouldn't be admissible.

Theorem 2.1.5. *Assume that B_0, R_0 and $d \geq 3$ are fixed positive integers, and $\beta > 0$, $\rho > 0$ fixed rates. If $\beta = \rho$ then R_{fin}/N and B_{fin}/N converge in distribution, as $N \to \infty$ to $Beta(\frac{dR_0}{d-2}, \frac{dB_0}{d-2})$ and $Beta(\frac{dB_0}{d-2}, \frac{dR_0}{d-2})$ respectively. If $\rho < \beta$ then the sequence of random variables $R_{fin}/N^{\rho/\beta}$ is tight as $N \to \infty$.*

The following theorem about the behavior of the processes on the torus stands in contrast with the above two theorems. In its statement we assume that $G = \mathbb{T}(N, d)$ is a d-dimensional torus with N vertices, and $(\mathcal{B}, \mathcal{R})$ a CFP process on G with parameters (β, ρ).

Theorem 2.1.6. *Let $\mathbb{T}(N, d) = \left(\mathbb{Z}/n\mathbb{Z}\right)^d$ for n such that $N = n^d$, be the d-dimensional torus with N vertices, and fix the rates $\beta > 0$, $\rho > 0$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\liminf_{n\to\infty} \mathbb{P}(B_{fin} > \delta N, R_{fin} > \delta N) > 1 - \epsilon,$$

if one of the following two conditions hold

i) B_0 and R_0 are fixed positive integers, and $(\mathcal{B}_0, \mathcal{R}_0)$ are chosen uniformly of size (B_0, R_0),

ii) R_0 and k_0 are fixed positive integers, sequence B_0 converges to ∞ and it satisfies $\lim_N B_0/N = 0$ and $(\mathcal{B}_0, \mathcal{R}_0)$ are chosen uniformly of size (B_0, R_0), with $\mathcal{B}_0$ center of size k_0.

Remarks and follow up work

We note that the results of all the theorems above cannot hold if the sets $\mathcal{B}_0$ and $\mathcal{R}_0$ are arbitrary. Consider for example the case where $\mathcal{B}_0$ is the ball of radius r in the graph around a vertex v and $\mathcal{R}_0$ consists of all vertices at distance exactly $r + 1$ from v. While the set $\mathcal{R}_0$ is not much bigger than $\mathcal{B}_0$ - clearly the remaining vertices will all become red.

The fact that the results do not hold for arbitrary sets raise various game theoretic questions. For example, consider a game where player B has to choose the set $\mathcal{B}_0$ and player R has to choose the set $\mathcal{R}_0$. Suppose player B can choose up to N^{α_1} initial vertices and player R can choose up to N^{α_2} initial vertices. What are the Nash Equilibrea of this game? Are the payoffs in the Nash Equilibrea close to the payoffs obtained if the two players place the initial sets at random? Similar game theoretic questions may be asked if players alternate in placing the elements of $\mathcal{B}_0$ and $\mathcal{R}_0$.

As far as we know this game was first defined by Bharathi, Kempe and Salek in [11]. Their paper provides an approximation algorithm for the best response and shows that the social price of competition is at most 2 but does not analyze the utilities of each of the players in a Nash Equilibrea. A different direction of future study is extending the result in the current work to more realistic models of social networks and marketing. In particular it would be interesting to study the same question on preferential attachment random graphs and other more realistic models of social networks. We expect that for such graphs, game theoretic consideration can play an important role due to the different degrees and connectivity of different vertices.

Related work

As mentioned earlier, diffusion and growth processes have been studied intensely in the past few years in relation to many areas such as sociology, economics and engineering. Among the models studied are *stochastic cellular automata* (see, for example [55], [31], [30]), *the voter model* which was first introduced by Clifford and Sudbury in [17] and has been much studied since in, for example, [39], [24], *the contact process* (see, for example, [32]), *the stochastic Ising model* (see [29], [14]), and *the influence model* (see [5]).

Recently, a strong motivation for analyzing diffusion processes has emanated from the study of viral marketing strategies in data mining (see, for example, [23], [53], [44], [43]). In this model one takes into account the "network value" of potential customers, that is, it seeks to target a set of individuals whose influence on the social network through word-of-mouth effects is high. For a given diffusion process, we define the influence maximization problem. For each initial set of active nodes S, we define $\sigma(S)$ to be the expected size of the set of active nodes at the end of the process. In the influence maximization problem, we aim to find a set S of fixed size that maximizes $\sigma(S)$. In attempts to find a set of influential individuals, heuristic approaches such as picking individuals of high degree or picking individuals with short average distance to the rest of the network have been commonly used, typically with no theoretic guarantees (see [54]). In [44] it was shown that the influence maximization problem is NP-hard to approximate within a factor of $1-e^{-1}+\varepsilon$ for all $\varepsilon > 0$. On the other hand, in [43] it was shown that under the assumption that the function σ is *submodular*, for every $\varepsilon > 0$ it is possible to find a set S of fixed size that is a $(1-e^{-1}-\varepsilon)$-approximation of the maximum in random polynomial time. In [50] it was proven that the function σ is indeed submodular.

As mentioned earlier the paper [11] defines the competitive influence maximization problem on general graphs. We believe that an interesting research direction is to show that for random d-regular graphs, the payoffs of the two players at each Nash Equilibrea are essentially the same as the payoff obtained by playing according to random strategies.

2.2 Coupling with the configuration model

The configuration model (CM), introduced by Bollobás in [12], is a randomized algorithm used to construct a uniform random d-regular labeled graph on N vertices (we always assume that dN is even, as otherwise there is no such graph). In this model we view each vertex $i \in [N] = \{1, 2, \ldots, N\}$ of the graph as a set $H(i)$ of d *half-edges*. We pick a uniform perfect matching on the set $\cup_{i\in[N]} H(i)$ of all dN half-edges (recall that dN is even), and contract each d-tuple of half-edges $H(i)$ back to a single vertex. This yields a d-regular graph on N vertices with the vertex set $[N]$, and in which every coupled pair of a half-edge in $H(i)$ and $H(j)$ gives an edge connecting the vertices i and j. Note that this algorithm does not have to produce a simple graph. Each matching of a half-edge in $H(i)$ and $H(j)$ produces one edge between i and j, so the graph can have multiple edges. Also any matchings of two half-edges in

$H(i)$ will result in a loop at the vertex i. However, it is shown in [12] that with probability that tends to $e^{\frac{1-d^2}{4}}$ as $N \to \infty$, this process yields a simple d-regular graph. Moreover, conditioning on the event that the graph is simple, it is uniformly distributed among all simple d-regular labeled graphs on N vertices. The great power of configuration model (CM), for both simulations and theoretical considerations, comes partially from the fact that the uniform matching can be chosen by matching half-edges sequentially (for example, choosing an available half-edge in some way and matching it to a uniform available half-edge and then declaring both of them to be unavailable).

We will couple the configuration model and the competing process, and will prove the results for the coupled process. Recall that all the results stated in the previous section hold asymptotically almost surely, and that the probability of generating a simple graph using the configuration model is bounded away from zero. Thus our proofs will work asymptotically almost surely on a probability space that couples the graphs produced by CM and the competing first passage percolation process, and we don't need to worry about the possibility that the generated graph is not simple.

To make the coupling easier we will slightly modify the competing first passage percolation model we study (CFPP introduced in Section 1.1). Recall that $(\mathcal{B}_n, \mathcal{R}_n)$ in CFPP evolves by choosing an edge connecting a vertex u in $\mathcal{B}_n \cup \mathcal{R}_n$ with a vertex v in the complement $(\mathcal{B}_n \cup \mathcal{R}_n)^c$, with probabilities proportional to β and ρ depending on the color of u, and then coloring the vertex v in the corresponding color. In the modification of CFPP (call it MCFPP) we describe now, we also color the edges. We start like before with two disjoint subsets $\mathcal{B}_0$ and $\mathcal{R}_0$ of the vertex set colored blue and red respectively, and initially we set all the edges uncolored. At the n-th step we choose a pair (u, e) of a vertex u in $\mathcal{B}_n \cup \mathcal{R}_n$ and an incident uncolored edge e. We use the same probabilities as before; every pair for which $u \in \mathcal{B}_n$ is chosen with the probability proportional to β and every pair for which $u \in \mathcal{R}_n$ is chosen with the probability proportional to ρ. Then we color the edge e in the color of u. Furthermore, if the other end of e is uncolored, we also color it into the color of u. In this modification we can have steps that do not yield to coloring of new vertices, but it is easy to see that when $\mathcal{B}_n$ and $\mathcal{R}_n$ do grow, the transition probabilities are the same as in the original model. Thus, the distribution of $(\mathcal{B}_{\text{fin}}, \mathcal{R}_{\text{fin}})$ is unchanged.

Therefore, it can be assumed for the competing first passage percolation model on a random regular graph (in Theorems 2.1.4 and 2.1.5) to first generate a random graph according to the configuration model (CM), and conditioned on its realization run the MCFPP process. This will be denoted by CM×MCFPP.

We now describe the coupling which we will refer to as CP. We first focus on the case when $(\mathcal{B}_0, \mathcal{R}_0)$ is chosen uniformly of size (B_0, R_0). Have in mind that the described coupling also works when one of $\mathcal{B}_0$ or $\mathcal{R}_0$ is empty, that is one of B_0 or R_0 is equal to zero (this is important as it will correspond to the evolution of the process when one set is given an advantage). The coupling CP goes as follows.

i) Start with the pair of disjoint sets $(\mathcal{B}_0, \mathcal{R}_0)$ which are uniformly chosen subsets of $[N]$ of size (B_0, R_0) in the spirit of Definition 2.1.1. Denote by $\mathcal{X}_0$ and $\mathcal{Y}_0$ the half-edges corresponding to vertices in $\mathcal{B}_0$ and $\mathcal{R}_0$ respectively, that is $\mathcal{X}_0 = \cup_{i\in\mathcal{B}_0} H(i)$ and $\mathcal{Y}_0 = \cup_{i\in\mathcal{R}_0} H(i)$. Color the half edges in $\mathcal{X}_0$ and $\mathcal{Y}_0$ in blue and red respectively. Define the set of uncolored half-edges initially as $\mathcal{Z}_0 = \left(\mathcal{X}_0 \cup \mathcal{Y}_0\right)^c$, and the set of inactive half-edges initially as $\mathcal{W}_0 = \emptyset$.

ii) At every time step $n \geq 0$ choose a half edge x in $\mathcal{X}_n \cup \mathcal{Y}_n$, the ones in $\mathcal{X}_n$ with probability proportional to β, and the ones in $\mathcal{Y}_n$ with probability proportional to ρ. Match the chosen half edge to a uniformly chosen half edge y in $\mathcal{X}_n \cup \mathcal{Y}_n \cup \mathcal{Z}_n \backslash \{x\}$. Let $x \in H(i)$ and $y \in H(j)$, for some $i, j \in [N]$. Make x and y inactive. If $y \in \mathcal{Z}_n$ then color all the half-edges in $H(j)$ with the color of x. To state it more precisely, assume $x \in \mathcal{X}_n$ and set

$$\mathcal{X}_{n+1} = \mathcal{X}_n \cup H(i)\backslash\{x, y\},\ \mathcal{Y}_{n+1} = \mathcal{Y}_n,\ \mathcal{Z}_{n+1} = \mathcal{Z}_n\backslash H(i),\quad \text{if } y \in \mathcal{Z}_n,$$

$$\mathcal{X}_{n+1} = \mathcal{X}_n\backslash\{x, y\},\ \mathcal{Y}_{n+1} = \mathcal{Y}_n,\ \mathcal{Z}_{n+1} = \mathcal{Z}_n,\quad \text{if } y \in \mathcal{X}_n,$$

$$\mathcal{X}_{n+1} = \mathcal{X}_n\backslash\{x\},\ \mathcal{Y}_{n+1} = \mathcal{Y}_n\backslash\{y\},\ \mathcal{Z}_{n+1} = \mathcal{Z}_n,\quad \text{if } y \in \mathcal{Y}_n,$$

and $\mathcal{W}_{n+1} = \mathcal{W}_n \cup \{x, y\}$ in every case. If $x \in \mathcal{Y}_n$ we proceed equivalently.

iii) Connect vertices i and j with an edge and color this edge in the color of i (which is the same as the color of x). Furthermore, if j is uncolored color it into the color of i. More precisely set

$$\mathcal{B}_{n+1} = \mathcal{B}_n, \mathcal{R}_{n+1} = \mathcal{R}_n,\quad \text{if } j \in \mathcal{B}_n \cup \mathcal{R}_n\ (\Leftrightarrow y \in \mathcal{X}_n \cup \mathcal{Y}_n),$$

$$\mathcal{B}_{n+1} = \mathcal{B}_n \cup \{j\}, \mathcal{R}_{n+1} = \mathcal{R}_n,\ \text{if } j \notin \mathcal{B}_n \cup \mathcal{R}_n\ (\Leftrightarrow y \notin \mathcal{X}_n \cup \mathcal{Y}_n),\ \text{and } i \in \mathcal{B}_n\ (\Leftrightarrow x \in \mathcal{X}_n),$$

$$\mathcal{B}_{n+1} = \mathcal{B}_n, \mathcal{R}_{n+1} = \mathcal{R}_n \cup \{j\},\ \text{if } j \notin \mathcal{B}_n \cup \mathcal{R}_n\ (\Leftrightarrow y \notin \mathcal{X}_n \cup \mathcal{Y}_n),\ \text{and } i \in \mathcal{R}_n\ (\Leftrightarrow x \in \mathcal{Y}_n).$$

iv) We stop the algorithm when $\mathcal{X}_0 = \mathcal{Y}_0 = \mathcal{Z}_0 = \emptyset$.

Note the algorithm can fail to reach the stopping state in iv) if, for some n, we have $\mathcal{X}_n = \mathcal{Y}_n = \emptyset$ and $\mathcal{Z}_n \neq \emptyset$. If this happens for some n then simply color some uncolored vertices into blue or red and proceed. This is of no concern, since after erasing the colors the above algorithm produces a uniform matching on the set of half-edges (see also the computations that justify the coupling below). If for some n we indeed have $\mathcal{X}_n = \mathcal{Y}_n = \emptyset$ and $\mathcal{Z}_n \neq \emptyset$, then the random graph produced by CM would be disconnected. However, for any $d \geq 3$ the probability of this event converges to 0, as $N \to \infty$, see [13, 42].

By X_n, Y_n and Z_n denote the sizes of $\mathcal{X}_n$, $\mathcal{Y}_n$ and $\mathcal{Z}_n$ respectively. Denoting $M = X_0 + Y_0 + Z_0$, and observing that at each time two half-edges become inactive we have $X_n + Y_n + Z_n = M - 2n$. The process (X_n, Y_n, Z_n) is a Markov chain with the following transition probabilities

$$\mathbb{P}(X_{n+1} = X_n + d - 2, Y_{n+1} = Y_n, Z_{n+1} = Z_n - 2) = \frac{\beta X_n}{\beta X_n + \rho Y_n} \frac{Z_n}{M - 2n - 1}$$
$$\mathbb{P}(X_{n+1} = X_n, Y_{n+1} = Y_n + d - 2, Z_{n+1} = Z_n - 2) = \frac{\rho Y_n}{\beta X_n + \rho Y_n} \frac{Z_n}{M - 2n - 1}$$
$$\mathbb{P}(X_{n+1} = X_n - 2, Y_{n+1} = Y_n, Z_{n+1} = Z_n) = \frac{\beta X_n}{\beta X_n + \rho Y_n} \frac{X_n - 1}{M - 2n - 1}$$
$$\mathbb{P}(X_{n+1} = X_n, Y_{n+1} = Y_n - 2, Z_{n+1} = Z_n) = \frac{\rho Y_n}{\beta X_n + \rho Y_n} \frac{Y_n - 1}{M - 2n - 1}$$
$$\mathbb{P}(X_{n+1} = X_n - 1, Y_{n+1} = Y_n - 1, Z_{n+1} = Z_n) = \frac{(\rho + \beta) X_n Y_n}{(\beta X_n + \rho Y_n)(M - 2n - 1)}.$$

In case that $(\mathcal{B}_0, \mathcal{R}_0)$ are chosen uniformly of size (B_0, R_0) the initial condition is $X_0 = dB_0$ and $Y_0 = dR_0$. One advantage of this coupling is that the process (B_n, R_n) can be studied through the process (X_n, Y_n, Z_n). This process in turn, is completely described by the above transition probabilities. Indeed, most of the technical work in this chapter is devoted to establishing maximal inequalities for the process (X_n, Y_n, Z_n).

Next we justify the coupling, that is explain why stage iii) of CP produces the graph with the set of blue and red vertices, which is equal in distribution to $(G, \mathcal{B}_{\text{fin}}, \mathcal{R}_{\text{fin}})$ produced by CM×MCFPP. To see this, denote by $\mathbf{A}_n$ the cluster formed by the colored edges and colored vertices at time n in CM×MCFPP. Also denote by $\mathbf{A}'_n$ the cluster formed by the colored edges and colored vertices at time n in CP (both $\mathbf{A}_n$ and $\mathbf{A}'_n$ contain the information about the color of each edge and vertex). It suffices to show that $\mathbf{A}_n$ and $\mathbf{A}'_n$ have the same distribution for every n. We show this inductively. For $n = 0$ the claim is obvious, as both $\mathbf{A}_0$ and $\mathbf{A}'_0$ consist of uniform disjoint subsets of $[N]$ of prescribed size colored blue and red respectively, and all the edges are uncolored. Condition on some realization $\mathbf{A}'_n = A$. Observe that the probability that $\mathbf{A}'_{n+1}$ is formed by connecting a vertex $i \in A$ to an uncolored vertex j (which results in coloring both j and edge (i, j)) is given by

$$\frac{\tau E_n(i)}{\beta X_n + \rho Y_n} \frac{d}{M - 2n - 1},$$

where $\tau = \beta$ if i is blue and $\tau = \rho$ if i red, and $E_n(i)$ is the number half-edges incident to i and not present in A (that is $|(\mathcal{X}_n \cup \mathcal{Y}_n) \cap H(i)|$). In other words $E_n(i) = d$ minus the degree of i in A.

To study the same conditional probability for CM×MCFPP observe that the event $\{\mathbf{A}_n = A\}$ happens if and only if the graph generated by CM supports the cluster A, and MCFPP on this graph generates A in the n-th step. The event that $\mathbf{A}_{n+1}$ is formed by joining a vertex $i \in A$ with an uncolored vertex j happens if and only if CM produces a graph in which there is at least one edge connecting i and j (as j is uncolored, such edges can not be a part of the cluster A), and in the next step MCFPP spreads along one of this edges. From the configuration model we know that, conditioned on the event that CM supports the cluster A, the probability that i and j are connected by k edges is just the probability CM has created k pairs of half-edges, which consists of one of d half-edges in $H(j)$ and one of the $E_n(i)$ half-edges in $H(i)$ not already matched into an edge of A. This probability is simply

$$\binom{E_n(i)}{k}\frac{d\cdots(d-k+1)(M-2n-d-1)\cdots(M-2n-E_n(i)+k)}{(M-2n-1)(M-2n-2)\cdots(M-2n-E_n(i))},$$

If there are k of such matchings, the probability that one of them is chosen (together with i) in the $n+1$-st step is equal to

$$\frac{\tau k}{\beta X_n+\rho Y_n}.$$

To see that these probabilities are the same one only has to check that

$$\sum_{k=1}^{E_n(i)}\binom{E_n(i)}{k}\frac{d\cdots(d-k+1)(M-2n-d-1)\cdots(M-2n-E_n(i)+k)}{(M-2n-1)(M-2n-2)\cdots(M-2n-E_n(i))}\frac{\tau k}{\beta X_n+\rho Y_n}$$
$$=\frac{\tau E_n(i)}{\beta X_n+\rho Y_n}\frac{d}{M-2n-1},$$

which follows by a simple algebra. Similar calculations show that the conditional probabilities agree for the events that j is already chosen red or blue.

The situation when one process starts earlier than the other is handled in almost exactly the same way. First observe that the above coupling makes sense even if $\mathcal{R}_0=\emptyset$. To generate random subsets $(\mathcal{B}_0,\mathcal{R}_0)$ as in Definition 2.1.2 simply run CP with $\mathcal{B}_0^0$ as a uniformly chosen subset of $[N]$ of size k_0, $\mathcal{R}_0^0=\emptyset$ until $\mathcal{B}_k^0$ grows to the prescribed size B_0. More precisely we define the stage 0 of CP as follows.

i) Take $\mathcal{B}_0^0$ as a uniformly chosen subset of $[N]$ of size k_0, $\mathcal{R}_0^0=\emptyset$, $\mathcal{X}_0^0=\cup_{i\in\mathcal{B}_0^0}H(i)$, $\mathcal{Y}_0^0=\emptyset$, $\mathcal{Z}_0^0=(\mathcal{X}_0^0)^c$, $\mathcal{W}_0^0=\emptyset$.

ii) Run CP with the above initial conditions.

iii) Stop CP at T when $|\mathcal{B}_T^0|=B_0$. Set $\mathcal{B}_0=\mathcal{B}_T^0$, $\mathcal{X}_0=\mathcal{X}_T^0$, $\mathcal{W}_0=\mathcal{W}_T^0$.

By the strong Markov property, CP for the initial conditions when $(\mathcal{B}_0,\mathcal{R}_0)$ is chosen as a uniform subset of size (B_0,R_0), with $\mathcal{B}_0$ center of size k_0 goes as follows

i) Run stage 0 of CP.

ii) Take $\mathcal{B}_0$, $\mathcal{X}_0$ and $\mathcal{W}_0$ as produced in stage 0, $\mathcal{R}_0$ as a uniform subset of $\mathcal{B}_0^c$ of size R_0, $\mathcal{Y}_0=\cup_{i\in\mathcal{R}_0}H(i)$ and $\mathcal{Z}_0=\left(\mathcal{X}_0\cup\mathcal{Y}_0\cup\mathcal{W}_0\right)^c$.

iii) Run CP with the initial conditions from ii).

As in the usual CP we denote the sizes of corresponding sets by the same letter in the normal font and set $M = X_0 + Y_0 + Z_0 = dN - W_0$.

The following strong result shows how one can estimate the final sizes simply from $X_0 = |\mathcal{X}_0|$ and $Y_0 = |\mathcal{Y}_0|$. From this theorem we will derive all our results about the competing process on the random regular graph. Note that the notation $x = (1 \pm \epsilon)y$ means that $(1-\epsilon)y \le x \le (1+\epsilon)y$, and that the quantity $\frac{R_{\text{fin}} - R_0}{N - R_0 - B_0}$ represents the fraction of added red vertices and the total number of added vertices.

Theorem 2.2.1. *Let (L_N) be a sequence converging to ∞. Assume that in a CP process, with the admissible initial conditions, sequences $X_0 = X_0(N)$, $Y_0 = Y_0(N)$ and $Z_0 = Z_0(N)$ satisfy $\min(X_0, Y_0) \ge L_N$ and $Z_0 \ge M/L_N$. Then for any $\varepsilon > 0$ asymptotically almost surely (as $N \to \infty$) we have*

$$\frac{R_{\mathit{fin}} - R_0}{N - R_0 - B_0} = (1 \pm \epsilon) \frac{Y_0}{X_0 + Y_0}, \ \text{for } \beta = \rho, \tag{2.2.1}$$

and

$$\frac{R_{\mathit{fin}} - R_0}{N - R_0 - B_0} = (1 \pm o(1)) \int_0^1 \phi^{-1}_{\beta,\rho} \left(M X_0^{\frac{\rho/\beta}{1-\rho/\beta}} Y_0^{\frac{\beta/\rho}{1-\beta/\rho}} \left(t^{1/d} - \frac{Z_0}{M} t^{(d-1)/d} \right) \right) dt, \tag{2.2.2}$$

for $\beta \ne \rho$, where $\phi_{\beta,\rho}\colon (0,1) \to \infty$ is a one-to-one function defined as

$$\phi_{\beta,\rho} = \left(\frac{\beta s}{\rho(1-s)} \right)^{\frac{1}{1-\beta/\rho}} + \left(\frac{\rho(1-s)}{\beta s} \right)^{\frac{1}{1-\rho/\beta}}.$$

The speed of convergence depends only on the values of β, ρ, d and the sequence (L_N).

To prove the main theorems from Theorem 2.2.1 one needs to relate X_0 and Y_0 to B_0 and R_0. In the setting of Theorem 2.1.4 when $(\mathcal{B}_0, \mathcal{R}_0)$ are chosen uniformly, this is trivial as $X_0 = dB_0$ and $Y_0 = dR_0$. If one of the initial sets, say $\mathcal{B}_0$, is assumed to have a small center then we still have $Y_0 = dR_0$. However, to estimate X_0 one needs to understand the evolution of the number of active edges, in the stage 0 of CP when only the blue set evolves. The small center assumption yields $X_0 = (1 + o(1))(d-2)B_0$ with high probability, see Lemma 2.2.5 below.

If both of the processes start from a small size (Theorem 2.1.5), one needs to be a bit more careful. The idea for this case is to control the processes X_n and Y_n by a comparison to a certain urn model, as long as $X_n = o(\sqrt{N})$ and $Y_n = o(\sqrt{N})$. Then, by the strong Markov property, we can apply Theorem 2.2.1.

First we introduce the notion of an urn model that will be used throughout the chapter.

Definition 2.2.2. We say that a process $(S_n, Z_n)_n$ is a Pólya urn process with a replacement matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ if conditioned on (S_n, Z_n) with probability $S_n/(S_n + Z_n)$ we have $(S_{n+1}, Z_{n+1}) = (S_n, Z_n) + (a_{11}, a_{12})$ and otherwise $(S_{n+1}, Z_{n+1}) = (S_n, Z_n) + (a_{21}, a_{22})$.

From the jump probabilities of (X_n, Y_n, Z_n) we obtain the following lemma, which is true even in the stage 0 of CP, that is for $Y_0 = 0$.

Lemma 2.2.3. *The process $(X_n + Y_n - 1, Z_n)$ is an urn model with the replacement matrix*

$$A = \begin{pmatrix} -2 & 0 \\ d-2 & -d \end{pmatrix} .$$

The following result will be of crucial importance. Having in mind Lemma 2.2.3 it follows directly from Theorem 2.6.1. Note that this theorem is simply a statement on the process (X_n, Y_n, Z_n) which can be studied through its jump probabilities given above. Also the role of N is replaced by $M = X_0 + Y_0 + Z_0$.

Corollary 2.2.4. *In the CP model started with X_0 blue and Y_0 red half-edges and $Z_0 = M - X_0 - Y_0$ uncolored half-edges, for any $\epsilon > 0$ we have that the events*

$$\left\{ Z_n = (1 \pm \epsilon) Z_0 (1 - 2n/M)^{d/2}, \text{ for all } 0 \le n \le (M - M Z_0^{-2/d} \log M)/2, \right\}$$

and

$$\left\{ X_n + Y_n = (1 \pm \epsilon)\Big((M - 2n) - Z_0 (1 - 2n/M)^{d/2}\Big), \text{ for all } 0 \le n < M/2 \right\}$$

have probabilities converging to 1, as $M \to \infty$.

We end this section by proving an estimate on the number of active half-edges in the stage 0 of CP, when the active processes starts from a small center.

Lemma 2.2.5. *Assume that $(\mathcal{B}_0, \mathcal{R}_0) = (\mathcal{B}_0(N), \mathcal{R}_0(N))$ is a sequence of uniform subsets of size (B_0, R_0) with $\mathcal{B}_0$ of small center (as in Definition 2.1.2). Then, for any $\epsilon > 0$ the probability that the stage 0 of CP that generates $\mathcal{B}_0$ ends with*

$$X_0 = (1 \pm \epsilon)(d-2) B_0,$$

converges to 1, as $N \to \infty$.

Proof. Recall that the stage 0 of CP consists of choosing $\mathcal{B}_0^0$, a uniform vertex subset of size $B_0^0 = k_0$, and $\mathcal{R}_0^0 = \emptyset$. Then we run CP until the stopping time T which is the first time that $|\mathcal{B}_T^0| = B_0$. Recall that X_n^0 denotes the number of active half-edges in stage 0 CP (therefore $X_0^0 = dk_0$) and $B_n^0 = |\mathcal{B}_n^0|$. If at the n-th step of stage 0 CP we spread to a new vertex then we have $B_{n+1}^0 = B_n^0 + 1$ and $X_{n+1}^0 = X_n^0 + d - 2$. Otherwise, $B_{n+1}^0 = B_n^0$ and $X_{n+1}^0 = X_n^0 - 2$. This leads to a simple deterministic relation

$$X_n^0 - X_0^0 = d(B_n^0 - B_0^0) - 2n. \tag{2.2.3}$$

In particular we have

$$X_0 = X_T^0 = d(B_T^0 - B_0^0) - 2T + X_0^0 = d(B_0 - B_0^0) - 2T + X_0^0. \tag{2.2.4}$$

Next we will show that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$X_n^0 - X_0^0 = (1 \pm \epsilon)(d-2)n, \text{ for all } 0 \le n \le \delta N, \tag{2.2.5}$$

has probability which converges to 1, as $N \to \infty$. This will suffice since then by (2.2.3) the event in (2.2.5) will imply

$$(1-\epsilon)(d-2)n \le d(B_n^0 - B_0^0) - 2n \le (1+\epsilon)(d-2)n, \text{ for all } 0 \le n \le \delta N,$$

which, for such n yields

$$(1-\epsilon)n \le B_n^0 - B_0^0 \le (1+\epsilon)n. \tag{2.2.6}$$

The lower bound above can reach the value of $(1-\epsilon)\delta N$ and since $\lim_N B_0/N = 0$, for N large enough we have $B_0 < (1-\epsilon)\delta N$. This means that the inequality in (2.2.6) is also true for $n = T$ and thus

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{B_0 - B_0^0}{1+\epsilon} \le T \le \frac{B_0 - B_0^0}{1-\epsilon}\right) = 1.$$

Then (2.2.4) gives that with the probability converging to 1

$$(B_0 - B_0^0)\Big(d - \frac{2}{1-\epsilon}\Big) + X_0^0 \le X_0 \le (B_0 - B_0^0)\Big(d - \frac{2}{1+\epsilon}\Big) + X_0^0.$$

Recalling that $X_0^0 = dB_0^0$ and that $B_0^0/B_0 \to 0$ the claim follows.

Thus we are left to prove the claim in (2.2.5). Observe that from Corollary 2.2.4 we have

$$X_n^0 = (1 \pm \epsilon/2)\Big((M^0 - 2n) - Z_0^0(1 - 2n/M^0)^{d/2}\Big), \text{ for all } 0 \le n < M/2,$$

with the probability converging to 1. Recalling that $Z_0^0 = M^0 - X_0^0$ the above event yields

$$\begin{aligned}(1-\epsilon/2)(M^0 - 2n)\left(1 - \left(1 - \frac{2n}{M^0}\right)^{d/2-1}\right) - X_0^0 &\le X_n^0 - X_0^0 \\ &\le (1+\epsilon/2)(M^0 - 2n)\left(1 - \left(1 - \frac{2n}{M^0}\right)^{d/2-1}\right).\end{aligned} \tag{2.2.7}$$

As $M/N \to d$, we can choose δ small enough so that for all $0 \le n \le \delta N$ we have

$$(1-\epsilon/2)\frac{(d-2)n}{M} \le 1 - \left(1 - \frac{2n}{M}\right)^{d/2-1} \le (1+\epsilon/2)\frac{(d-2)n}{M}.$$

Then the event in (2.2.7) yields

$$(1-\epsilon/2)^2\Big(1 - \frac{2n}{M}\Big)(d-2)n \le X_n^0 - X_0^0 \le (1+\epsilon/2)^2\Big(1 - \frac{2n}{M}\Big)(d-2)n.$$

Reducing δ if needed yields (2.2.5). □

2.3 Deducing the main theorems

The main goal of this section is to deduce Theorems 2.1.4 and 2.1.5 from Theorem 2.2.1. First, we rewrite Theorem 2.2.1 in the form in which assume that the red rate $\rho = 1$. We can do this without the loss of generality, as we simply need to scale the rates from (β, ρ) to $(\beta/\rho, 1)$. The assumption $\rho = 1$ will used throughout the whole chapter (except in the proofs of Theorems 2.1.4 and 2.1.5).

Theorem 2.3.1. *Let (L_N) be a sequence converging to ∞. Assume that in a CP process, with the admissible initial conditions, sequences $X_0 = X_0(N)$, $Y_0 = Y_0(N)$ and $Z_0 = Z_0(N)$ satisfy $\min(X_0, Y_0) \geq L_N$ and $Z_0 \geq M/L_N$. Then for any $\varepsilon > 0$ asymptotically almost surely (as $N \to \infty$) we have*

$$\frac{R_{fin} - R_0}{N - R_0 - B_0} = (1 \pm \epsilon)\frac{Y_0}{X_0 + Y_0}, \text{ for } \beta = 1, \tag{2.3.1}$$

and

$$\frac{R_{fin} - R_0}{N - R_0 - B_0} = (1 \pm \epsilon)\int_0^1 \phi_\beta^{-1}\left(\frac{MX_0^{1/(\beta-1)}}{Y_0^{\beta/(\beta-1)}}\Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)\right) dt, \text{ when } \beta \neq 1, \tag{2.3.2}$$

where $\phi_\beta \colon (0,1) \to \infty$ is a one-to-one function defined as

$$\phi_\beta(s) = (\beta s + (1-s))\Big(\frac{1-s}{(\beta s)^\beta}\Big)^{1/(\beta-1)}. \tag{2.3.3}$$

The speed of convergence depends only on the values of β, d and the sequence (L_N).

First, observe that we could redefine the function ϕ_β in (2.3.2) and the expression inside the ϕ_β appearing in the integral at our convenience. One reason we choose this form is because these expressions behave naturally if we switch the roles of processes X and Y. More precisely, by symmetry and scaling of the rates we expect from (2.3.2) to obtain

$$\frac{B_{\text{fin}} - B_0}{N - R_0 - B_0} = (1 \pm o(1))\int_0^1 \phi_{1/\beta}^{-1}\left(\frac{MY_0^{1/(\beta^{-1}-1)}}{X_0^{(\beta)^{-1}/(\beta^{-1}-1)}}\Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)\right) dt, \text{ when } \beta \neq 1. \tag{2.3.4}$$

Indeed, the above formula is equivalent to (2.3.2). To see this first observe that

$$\frac{MY_0^{1/(\beta^{-1}-1)}}{X_0^{\beta^{-1}/(\beta^{-1}-1)}} = \frac{MY_0^{\beta/(\beta-1)}}{X_0^{1/(\beta-1)}} = \frac{MX_0^{1/(1-\beta)}}{Y_0^{\beta/(1-\beta)}}.$$

Next (2.3.4) will follow from (2.3.2) if we show that $\phi_{1/\beta}^{-1}(s) = 1 - \phi_\beta^{-1}(s)$ (because $B_{\text{fin}} + R_{\text{fin}} = N$). This in turn follows from the fact $\phi_\beta(t) = \phi_{\beta^{-1}}(1-t)$, which is easy to check.

The following corollary clarifies the asymptotic behavior of $\frac{R_{\text{fin}} - R_0}{N - B_0 - R_0}$. Note that $s \wedge t = \min(s, t)$.

Corollary 2.3.2. *Let (L_N) be a sequence converging to ∞, and assume that* $\min(X_0, Y_0) \geq L_N$ *and* $Z_0 \geq M/L_N$. *Then there are constants c and C such that asymptotically almost surely (as $N \to \infty$)*

$$c\frac{Y_0}{M}\Big(\frac{M}{X_0}\Big)^{1/\beta} \wedge c \leq \frac{R_{fin} - R_0}{N - R_0 - B_0} \leq C\frac{Y_0}{M}\Big(\frac{M}{X_0}\Big)^{1/\beta} \wedge 1. \tag{2.3.5}$$

To prove Corollary 2.3.2. we need the following simple estimate.

Lemma 2.3.3. *Let $\beta \neq 1$ and the function $\phi_\beta \colon (0,1) \to \mathbb{R}^+$ as in* (2.3.3)*. Then ϕ_β is one-to-one and onto and there are constants $c_1 < c_2$ such that the inverse function $\phi_\beta^{-1} \colon \mathbb{R}^+ \to (0,1)$ satisfies*

$$c_1\Big(s^{1/\beta-1} \wedge 1\Big) \leq \phi_\beta^{-1}(s) \leq \Big(c_2 s^{1/\beta-1}\Big) \wedge 1.$$

Proof. For $\beta > 1$ the function ϕ_β is decreasing and $\phi_\beta(1) = 0$ and $\lim_{t\downarrow 0} \frac{\phi_\beta(t)}{t^{\beta/(1-\beta)}} = 1$. Therefore ϕ_β^{-1} is decreasing with $\phi_\beta^{-1}(0) = 1$ and $\lim_{s\to\infty} \frac{\phi_\beta^{-1}(s)}{s^{1/\beta-1}} = 1$. This proves the claim for $\beta > 1$. For $\beta < 1$ the function ϕ_β is increasing with $\phi(0) = 0$, $\lim_{t\downarrow 0} \frac{\phi_\beta(t)}{t^{\beta/(1-\beta)}} = 1$ and $\lim_{t\uparrow 1} \phi_\beta(t) = \infty$. This implies that ϕ_β^{-1} is also increasing and $\phi_\beta^{-1}(0) = 0$, $\lim_{s\downarrow 0} \frac{\phi_\beta(s)}{s^{1/\beta-1}} = 1$ and $\lim_{s\to\infty} \phi_\beta^{-1}(s) = 1$, which is enough to deduce the claim in the case $\beta < 1$. □

Proof of Corollary 2.3.2. For $\beta = 1$ the inequalities in (2.3.5) are easy to check from (2.3.1), so we focus on the case $\beta \neq 1$. By (2.3.2) and Lemma 2.3.3 we have

$$c_1 \int_0^1 \frac{M^{1/\beta-1}Y_0}{X_0^{1/\beta}}\Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)^{1/\beta-1} \wedge 1 \; dt \leq \frac{R_{\text{fin}} - R_0}{N - R_0 - B_0}$$
$$\leq \int_0^1 c_2 \frac{M^{1/\beta-1}Y_0}{X_0^{1/\beta}}\Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)^{1/\beta-1} \wedge 1 \; dt.$$

The trivial inequality $R_{\text{fin}} - R_0 \leq N - R_0 - B_0$ yields 1 for the upper bound in (2.3.5). For the second part of the upper bound in (2.3.5), it is enough to prove that the integral

$$\int_0^1 \Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)^{1/\beta-1} \; dt \tag{2.3.6}$$

is bounded from above by a constant depending on β and d only. This follows from the inequalities

$$t^{1/d} - t^{(d-1)/d} \leq t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d} \leq t^{1/d},$$

and the fact that both functions $t \mapsto t^{(1/\beta-1)/d}$ and $t \mapsto (t^{1/d} - t^{(d-1)/d})^{(1/\beta-1)}$ are integrable on $(0,1)$, for any $\beta > 0$. Actually the inequalities above imply that the minimum of the function $\Big(t^{1/d} - \frac{Z_0}{M}t^{(d-1)/d}\Big)^{1/\beta-1}$ is bounded from below on $[1/2, 3/4]$. Thus the integral in (2.3.6) is bounded from below by a positive constant depending only on d and β. By considering the cases when $M^{1/\beta-1}Y_0X_0^{-1/\beta}$ is smaller or greater than 1, the lower bound in (2.3.5) follows. □

Now we prove Theorem 2.1.4.

Proof of Theorem 2.1.4. First we consider the case of equal rates. If the pair of initial sets is chosen uniformly then the claim follows from (2.3.1) and the fact that $X_0 = dB_0$ and $Y_0 = dR_0$. If one of the initial sets has a small center, then use (2.3.1) together with Lemma 2.2.5. If the rates are different, in either case, the claim follows from Corollary 2.3.2.

□

To prove Theorem 2.1.5 we compare the process (X_n, Y_n) from the coupled process CP to an urn model. For the case of equal rates $\beta = \rho$, recall that in the Pólya urn process (S_n, Z_n) with the replacement matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ random variable $S_n/(\alpha n)$ converges in distribution to $Beta(S_0/\alpha, Z_0/\alpha)$. If the rates are different, we use the following result by Svante Janson (part of Theorem 1.4 in [40]). Here $\Gamma(m,1)$ denotes the Gamma distribution with parameter m, that is a probability distribution with the density $t^{m-1}e^{-t}\Gamma(m)^{-1}$, $t > 0$, where $\Gamma(m)$ is the Gamma function.

Theorem 2.3.4 (Janson)**.** *Consider the Pólya urn process (S_n, Z_n) with the replacement matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$, where $\alpha > 0$, $\delta > 0$, $S_0 > 0$ and $Z_0 > 0$. Let $U \sim \Gamma(S_0/\alpha, 1)$ and $V \sim \Gamma(Z_0/\delta, 1)$ be two independent random variables with Gamma distribution and parameters S_0/α and Z_0/δ respectively. If $\alpha < \delta$ then in distribution*

$$\frac{S_n}{n^{\alpha/\delta}} \to \alpha \frac{U}{V^{\alpha/\delta}}.$$

Remark 2.3.5. The process (X_n, Y_n) such that $(X_{n+1}, Y_{n+1}) = (X_n + a_1, Y_n)$, with the probability $\alpha_1 X_n/(\alpha_1 X_n + \alpha_2 Y_n)$ and $(X_{n+1}, Y_{n+1}) = (X_n, Y_n + a_2)$ otherwise, can be thought of as an urn process in which we draw balls with different weights. It is easy to observe that the process $(S_n, Z_n) = (\alpha_1 X_n, \alpha_2 Y_n)$ is indeed an urn process with $S_0 = \alpha_1 X_0$, $Z_0 = \alpha_2 Y_0$ and the replacement matrix $\begin{pmatrix} \alpha_1 a_1 & 0 \\ 0 & \alpha_2 a_2 \end{pmatrix}$.

Proof of Theorem 2.1.5. Consider the process $(X_{n,1}, Y_{n,1})$ such that $(X_{0,1}, Y_{0,1}) = (dB_0, dR_0)$ and that $(X_{n+1,1}, Y_{n+1,1}) = (X_{n,1} + d - 2, Y_{n,1})$ happens with probability $\beta X_n/(\beta X_n + \rho Y_n)$, and $(X_{n+1,1}, Y_{n+1,1}) = (X_{n,1}, Y_{n,1} + d - 2)$ otherwise. First we show that we can couple the processes (X_n, Y_n) and $(X_{n,1}, Y_{n,1})$ so that with probability converging to 1, as $M \to \infty$, we have $(X_n, Y_n) = (X_{n,1}, Y_{n,1})$ for all $0 \le n \le M^{1/4}$. The construction of the coupling is simple: we set $(X_{n+1,1}, Y_{n+1,1}) = (X_{n,1} + d - 2, Y_{n,1})$ if in the step ii) of CP process the first chosen half-edge was blue (that is in $\mathcal{X}_n$) and we set $(X_{n+1,1}, Y_{n+1,1}) = (X_{n,1}, Y_{n,1} + d - 2)$ otherwise.

Then we have that $(X_n, Y_n) = (X_{n,1}, Y_{n,1})$ if for all $k \le n$, in the step ii) of the k-th round of CP we colored a new vertex, that is coupled a half-edge in $\mathcal{Z}_n$. The conditional probability that this does not happen is

$$\frac{\beta X_k(X_k-1) + Y_k(Y_k-1) + (1+\beta)X_kY_k}{(\beta X_k + Y_k)(X_k+Y_k+Z_k-1)} \le 2\frac{X_k+Y_k}{X_k+Y_k+Z_k} \le \frac{4(d-2)}{M^{3/4}}.$$

Here we used the fact that $X_n + Y_n$ can increase by at most $d-2$ in each step and that $2M^{1/4} < M/2$ for M large enough. Now the probability that in each of the first $M^{1/4}$ rounds of CP we color a new vertex is bounded from below by

$$1 - M^{1/4}\frac{4(d-2)}{M^{3/4}},$$

which converges to 1 as $M \to \infty$. Thus setting $n(M)$ to be the integer part of $M^{1/4}$, beta convergence of equal rate Pólya urns and Remark 2.3.5 yield that

$$\frac{X_{n(M)}}{X_{n(M)} + Y_{n(M)}}$$

converges to $Beta(\frac{dB_0}{d-2}, \frac{dR_0}{d-2})$ in distribution. Stopping the CP process at $n(M)$, and starting it again with the initial $X_{n(M)}$ and $X_{n(M)}$, apply the Markov property and Theorem 2.2.1 to get the claim. The proof for the different rates is analogous if one uses non-balanced urn result in Theorem 2.3.4 and Lemma 2.3.2.

□

2.4 Proof of the main estimate

This whole section is devoted to the proof of Theorem 2.3.1. The proof of Theorem 2.3.1 is based on a martingale method. In short we will identify two observables in our model that will "behave like martingales". More precisely, we will be able to effectively bound the conditional first and second moments of the step sizes in each process.

First we present a general lemma bounding the conditional expectation and variance of the differences in a general random process.

Lemma 2.4.1. *Let $(K_n)_{n\ge 0}$ be a positive process such that K_0 is a constant, and p_n and r_n positive real numbers defined for $n \ge 0$, such that*

$$|\mathbb{E}(K_{n+1} - K_n|\mathcal{F}_n)| \le p_nK_n, \text{ and } \mathbb{E}((K_{n+1}-K_n)^2|\mathcal{F}_n) \le r_nK_n.$$

Consider the process $I_0 = K_0$, $I_n = K_n - \sum_{k=0}^{n-1}\mathbb{E}(K_{k+1} - K_k|\mathcal{F}_k)$. Then process I_n is a martingale and for every positive integer n we have

$$|K_n - I_n| \le \sum_{k=0}^{n-1} p_kq_{k+1,n-1}I_k, \text{ and } \mathbb{E}((I_n - I_0)^2) \le K_0\sum_{k=0}^{n-1} r_kq_{0,k-1},$$

where $q_{\ell,k} = \prod_{i=\ell}^{k}(1+p_i)$ for $\ell \le k$ and $q_{k,k-1} = 1$, for all $k \ge 0$.

Proof. It is trivial to check that the process I_n is a martingale. Furthermore it can be shown by induction that for every $k \leq n$

$$q_{k,n} - 1 = \sum_{\ell=k}^{n} p_\ell q_{\ell+1,n} = \sum_{\ell=k}^{n} p_\ell q_{k,\ell-1}. \tag{2.4.1}$$

Using the first inequality in the statement we have that

$$|K_n - I_n| \leq \sum_{k=0}^{n-1} p_k K_k. \tag{2.4.2}$$

In particular we have

$$K_n - I_n \leq \sum_{k=0}^{n-1} p_k K_k = \sum_{k=0}^{n-1} p_k (K_k - I_k) + \sum_{k=0}^{n-1} p_k I_k. \tag{2.4.3}$$

Using (2.4.3) inductively we can show that $K_n - I_n \leq \sum_{k=0}^{n-1} a_{n,k} I_k$ whenever the sequence $(a_{n,k})_{0\leq k<n}$ satisfies $a_{n,n-1} = p_{n-1}$ and $a_{n,k} = \sum_{\ell=k+1}^{n-1} p_\ell a_{\ell,k} + p_k$. Using (2.4.1) it is easy to check that $a_{n,k} = p_k q_{k+1,n-1}$ satisfies these conditions. Thus we have

$$K_n \leq I_n + \sum_{k=0}^{n-1} p_k q_{k+1,n-1} I_k. \tag{2.4.4}$$

Plugging this back into (2.4.2) and using (2.4.1) we get

$$|K_n - I_n| \leq p_{n-1} I_{n-1} + \sum_{k=0}^{n-2} p_k \Big(1 + \sum_{\ell=k+1}^{n-1} p_\ell q_{k+1,\ell-1}\Big) I_k = \sum_{k=0}^{n-1} p_k q_{k+1,n-1} I_k,$$

which proves the first claim.

Note that (2.4.4) and (2.4.1) imply that

$$\mathbb{E}(K_n) \leq \Big(1 + \sum_{k=0}^{n-1} p_k q_{k+1,n-1}\Big) I_0 = q_{0,n-1} K_0. \tag{2.4.5}$$

Thus the condition in the statement implies that

$$\mathbb{E}((K_{n+1} - K_n)^2) \leq r_n \mathbb{E}(K_n) \leq r_n q_{0,n-1} K_0.$$

It is easy to check that $\mathbb{E}((I_{n+1} - I_n)^2 | \mathcal{F}_n) \leq \mathbb{E}((K_{n+1} - K_n)^2 | \mathcal{F}_n)$ which then yields

$$\mathbb{E}((I_n - I_0)^2) = \sum_{k=0}^{n-1} \mathbb{E}((I_{k+1} - I_k)^2) \leq K_0 \sum_{k=0}^{n-1} r_k q_{0,k-1}.$$

This concludes the proof. □

As we mentioned before, the key to the proof of Theorem 2.3.1 is to identify two processes for which we can estimate the conditional first and second moments or their step sizes. Then martingale methods (including the above lemma) will enable us to bound the maximal displacements of the processes throughout the whole relevant time regime. These processes are

$$K_n = \frac{X_n}{Y_n^{\beta}(1-2n/M)^{(1-\beta)/2}} \ . \tag{2.4.6}$$

and

$$L_n = \frac{X_n + Y_n}{(M-an) - Z_0(1-an/M)^{b/a}},$$

Note that once we managed to bound the values of processes we can "solve for X_n and Y_n" to estimate their values and obtain Theorem 2.3.1. The processes K_n and L_n tell us how $X_n + Y_n$ and $X_n Y_n^{-\beta}$ behave. It is actually quite natural to study these processes. The process $X_n + Y_n$ corresponds to the pure configuration model, that is to erasing the colors of half-edges, and can is equivalent to an urn model (see Lemma 2.2.3). The motivation for the considering the process $X_n Y_n^{-\beta}$ can be given as follows. After removing the interaction and self-interaction of the colored half-edges, the whole model reduces to an (unbalanced) urn model with a diagonal replacement matrix. Consider the Poissonized version of that model and define continuous time processes X_t and Y_t as the number of balls of each color. Then the process $X_t Y_t^{-\beta}$ is a continuous time martingale [40]. The factor $(1-2n/M)^{-(1-\beta)/2}$ thus accounts for the interaction and self-interaction of colors. From this discussion it follows for $\beta < 1$, that the value of the process $X_n Y_n^{-\beta}$ is smaller in our model than in the model with interactions and self-interactions removed. Thus, compared to the dynamics on disjoint d-regular trees, interactions and self-interactions on the random d-regular graph give the process with a faster rate X_n an additional "boost" relative to Y_n.

Estimates for the process L_n are given in Corollary 2.2.4 and proven in the final section of this chapter.. Process K_n will be estimated in this section. First we estimate its steps to be able to apply Lemma 2.4.1.

Lemma 2.4.2. *For the process K_n as defined in* (2.4.6)*, there exists a constant $C > 0$ depending on β and d, such that for all integers n, on the event that $Y_n \geq 2d$ we have both*

$$|\mathbb{E}(K_{n+1} - K_n | \mathcal{F}_n)| \leq \frac{CK_n}{Y_n(X_n + Y_n)}, \tag{2.4.7}$$

and

$$\mathbb{E}((K_{n+1} - K_n)^2 | \mathcal{F}_n) \leq \frac{CK_n}{Y_n^{1+\beta}(1-2n/M)^{(1-\beta)/2}}. \tag{2.4.8}$$

Proof. Throughout the proof we assume that $M - 2n \geq Y_n \geq 2d$. To prove (2.4.7) we calculate

$$\Big(1-\frac{2n+2}{M}\Big)^{(1-\beta)/2}(\beta X_n+Y_n)(M-2n-1)\mathbb{E}(K_{n+1}|\mathcal{F}_n) =$$
$$\frac{X_n+d-2}{Y_n^\beta}\beta X_n(M-2n-X_n-Y_n)+\frac{X_n}{(Y_n+d-2)^\beta}Y_n(M-2n-X_n-Y_n)$$
$$+\frac{X_n-2}{Y_n^\beta}\beta X_n(X_n-1)+\frac{X_n-1}{(Y_n-1)^\beta}(1+\beta)X_nY_n+\frac{X_n}{(Y_n-2)^\beta}Y_n(Y_n-1). \quad (2.4.9)$$

It can be easily verified that

$$\Big(1-\frac{2n+2}{M}\Big)^{(1-\beta)/2}(\beta X_n+Y_n)(M-2n-1)K_n$$
$$=\frac{X_n}{Y_n^\beta}(\beta X_n+Y_n)\Big(1-\frac{2}{M-2n}\Big)^{(1-\beta)/2}(M-2n-1)$$
$$=\frac{X_n}{Y_n^\beta}(\beta X_n+Y_n)(M-2n-2+\beta+O((M-2n)^{-1})), \quad (2.4.10)$$

where the absolute value of the term $O((M-2n)^{-1})$ is bounded by a constant multiple of $(M-2n)^{-1}$. To prove (2.4.7) it is enough to show that the absolute value of the difference of the terms in (2.4.9) and (2.4.10) is bounded by

$$\frac{CX_n(M-2n)}{Y_n^{\beta+1}} \quad (2.4.11)$$

for some constant C. First note that, since $X_n+Y_n\le M-2n$, the expression

$$X_nY_n^{-\beta}(\beta X_n+Y_n)(M-2n)^{-1}$$

is bounded by (2.4.11), for some $C>0$. Thus we can disregard the term $O((M-2n)^{-1})$ in (2.4.10).

By Taylor expansion we know that for any compact interval containing 1 there is a constant C_1 such that for all t in this interval

$$\Big|1-\beta t-\frac{1}{(1+t)^\beta}\Big|\le\frac{C_1t^2}{(1+t)^\beta}$$

(actually by a slightly more careful argument one can argue that C_1 does not depend on the interval). Now fix any $k\ge -2$ and choose $t=kY_n^{-1}$ and a constant C_1 to obtain

$$\Big|\frac{1}{Y_n^\beta}\Big(1-\frac{k\beta}{Y_n}\Big)-\frac{1}{(Y_n+k)^\beta}\Big|\le\frac{C_1k^2}{(Y_n+k)^\beta Y_n^2}.$$

For $k=d-2$ this in particular implies that

$$\Big|X_nY_n^{1-\beta}(M-2n-X_n-Y_n)\Big(1-\frac{\beta(d-2)}{Y_n}\Big)-\frac{X_nY_n(M-2n-X_n-Y_n)}{(Y_n+d-2)^\beta}\Big|$$

is at most

$$\frac{C_2 X_n(M-2n)}{Y_n^{\beta+1}},$$

for a constant $C_2 = (d-2)^2 C_1$. Therefore we can replace the term $\frac{X_n}{(Y_n+d-2)^\beta} Y_n(M-2n-X_n-Y_n)$ on the right hand side of (2.4.9) by $X_n Y_n^{-\beta}(M-2n-X_n-Y_n)(Y_n-\beta(d-2))$. Arguing similarly we see that we can replace the terms $\frac{X_n-1}{(Y_n-1)^\beta}(1+\beta)X_n Y_n$ and $\frac{X_n}{(Y_n-2)^\beta} Y_n(Y_n-1)$ on the right hand side of (2.4.9) by $\frac{X_n-1}{Y_n^\beta}(1+\beta)X_n(Y_n+\beta)$ and $\frac{X_n}{Y_n^\beta}(Y_n-1)(Y_n+2\beta)$ respectively. Therefore it is enough to prove

$$\begin{aligned}(M-2n-X_n-Y_n)\Big(\frac{X_n+d-2}{Y_n^\beta}\beta X_n + \frac{X_n}{Y_n^\beta}(Y_n-\beta(d-2)) - \frac{X_n}{Y_n^\beta}(\beta X_n+Y_n)\Big)\\ + \frac{X_n-2}{Y_n^\beta}\beta X_n(X_n-1) + \frac{X_n-1}{Y_n^\beta}(1+\beta)X_n(Y_n+\beta) + \frac{X_n}{Y_n^\beta}(Y_n-1)(Y_n+2\beta)\\ - \frac{X_n}{Y_n^\beta}(\beta X_n+Y_n)(X_n+Y_n-2+\beta) \le \frac{CX_n(M-2n)}{Y_n^{\beta+1}},\end{aligned}$$

for a large enough constant C. Expanding the expressions in the left hand side above we see that it is equal to $\beta(\beta+1)X_n Y_n^{-\beta}$. This proves the claim.

Now we prove (2.4.8). First note that it is enough to prove that

$$\mathbb{E}((K_{n+1}-K_n)^2|\mathcal{F}_n) \le \frac{CK_n^2}{X_n Y_n}. \tag{2.4.12}$$

Analyzing all the cases we see that the value of $|K_{n+1}-K_n|$ is

$$\left|\frac{X_n+d-2}{Y_n^\beta(1-\frac{2n+2}{M})^{(1-\beta)/2}} - \frac{X_n}{Y_n^\beta(1-2n/M)^{(1-\beta)/2}}\right| \le C_1 K_n\Big(\frac{1}{X_n}+\frac{1}{M-2n}\Big),$$

$$\left|\frac{X_n}{(Y_n+d-2)^\beta(1-\frac{2n+2}{M})^{(1-\beta)/2}} - \frac{X_n}{Y_n^\beta(1-2n/M)^{(1-\beta)/2}}\right| \le C_2 K_n\Big(\frac{1}{Y_n}+\frac{1}{M-2n}\Big),$$

$$\left|\frac{X_n-2}{Y_n^\beta(1-\frac{2n+2}{M})^{(1-\beta)/2}} - \frac{X_n}{Y_n^\beta(1-2n/M)^{(1-\beta)/2}}\right| \le C_3 K_n\Big(\frac{1}{X_n}+\frac{1}{M-2n}\Big),$$

$$\left|\frac{X_n-1}{(Y_n-1)^\beta(1-\frac{2n+2}{M})^{(1-\beta)/2}} - \frac{X_n}{Y_n^\beta(1-2n/M)^{(1-\beta)/2}}\right| \le C_4 K_n\Big(\frac{1}{X_n}+\frac{1}{Y_n}+\frac{1}{M-2n}\Big),$$

or

$$\left|\frac{X_n}{(Y_n-2)^\beta(1-\frac{2n+2}{M})^{(1-\beta)/2}} - \frac{X_n}{Y_n^\beta(1-2n/M)^{(1-\beta)/2}}\right| \le C_5 K_n\Big(\frac{1}{Y_n}+\frac{1}{M-2n}\Big),$$

with probabilities

$$\frac{\beta X_n(M-2n-X_n-Y_n)}{(\beta X_n+Y_n)(M-2n-1)},\ \frac{Y_n(M-2n-X_n-Y_n)}{(\beta X_n+Y_n)(M-2n-1)},\ \frac{\beta X_n(X_n-1)}{(\beta X_n+Y_n)(M-2n-1)},$$
$$\frac{(1+\beta)X_nY_n}{(\beta X_n+Y_n)(M-2n-1)},\ \frac{Y_n(Y_n-1)}{(\beta X_n+Y_n)(M-2n-1)},$$

respectively. Here C_1, C_2, C_3, C_4 and C_5 are constants depending only on β and d. Therefore for a large constant C_0 we have

$$\begin{aligned}\mathbb{E}((K_{n+1}-K_n)^2|\mathcal{F}_n) &\le C_0K_n^2\Big(\frac{1}{(M-2n)^2}+\frac{1}{X_n^2}\frac{X_n}{\beta X_n+Y_n}+\frac{1}{Y_n^2}\frac{Y_n}{\beta X_n+Y_n}\Big)\\ &\le C_0K_n^2\Big(\frac{1}{(M-2n)^2}+\frac{X_n+Y_n}{X_nY_n(\beta X_n+Y_n)}\Big),\end{aligned}$$

which, together with the fact $X_nY_n \le (M-2n)^2$, yields (2.4.12) .

□

Unfortunately Lemma 2.4.1 will not allow us to directly estimate the process K_n. A reason for this is that using bounds in Lemma 2.4.2 depend on the values of X_n and Y_n themselves. Lemma 2.4.4 is a first attempt along these lines.

We introduce a function which will appear quite often in the analysis in this section. Define

$$f(t)=\sqrt{t}-\frac{Z_0}{M}t^{(d-1)/2}. \tag{2.4.13}$$

Clearly f is a positive concave function on $(0,1)$. The following estimate will be used in the proof of Lemma 2.4.4.

Lemma 2.4.3. *Let n be such that $M-2n\ge 1$, and $\gamma>1$. Then there is a constant $C_0=C_0(d,\gamma)$ such that*

$$\sum_{k=0}^{n-1}\frac{1}{(M-2k)f(1-2k/M)^\gamma}\le C_0\left(\Big(\frac{M}{M-2n}\Big)^{\gamma/2}+\Big(\frac{M}{X_0+Y_0}\Big)^{\gamma-1}\right).$$

Proof. It is easy to check that the summand corresponding to $k=0$ is equal to $M^{\gamma-1}(X_0+Y_0)^{-\gamma}$, and therefore we get neglect this term in the sum. Define

$$g(t)=\frac{1}{tf(t)^\gamma}=\frac{1}{t^{1+\gamma/2}\Big(1-\frac{Z_0}{M}t^{d/2-1}\Big)^\gamma}.$$

One can calculate

$$g'(x)=x^{-2-\gamma/2}\Big(1-\frac{Z_0}{M}x^{d/2-1}\Big)^{-\gamma-1}\left(x^{d/2-1}\Big(\frac{Z_0}{M}+\frac{Z_0\gamma d}{2M}-\frac{Z_0\gamma}{2M}\Big)-1-\frac{\gamma}{2}\right),$$

so $g(x)$ is either decreasing on $(0,1)$ or decreasing on an interval $(0,x_0)$ and increasing on $(x_0,1)$, for some $0<x_0<1$. Therefore, we can bound the sum by the integral

$$\sum_{k=1}^{n-1}\frac{1}{(M-2k)f(1-2k/M)^{\gamma}}=\frac{1}{2}\sum_{k=1}^{n-1}\frac{2}{M}g\Big(1-\frac{2k}{M}\Big)\le\frac{1}{2}\int_{1-2n/M}^{1}g(t)\,dt. \tag{2.4.14}$$

We split the integral into two parts. First we consider

$$\begin{aligned}\int_{(1-2n/M)\wedge 1/2}^{1/2}g(t)\,dt&=\int_{(1-2n/M)\wedge 1/2}^{1/2}\frac{1}{t^{1+\gamma/2}\Big(1-\frac{Z_0}{M}t^{d/2-1}\Big)^{\gamma}}\,dt\\&\le\frac{1}{\Big(1-2^{1-d/2}\Big)}\int_{(1-2n/M)\wedge 1/2}^{1/2}t^{-1-\gamma/2}\,dt\\&\le\frac{2}{\gamma\Big(1-2^{1-d/2}\Big)}\Big(\frac{M}{M-2n}\Big)^{\gamma/2},\end{aligned}$$

To analyze the other part we use a simple inequality $t^{d/2-1}\le(t+1)/2$, which holds for all $0\le t\le 1$ and $d\ge 3$. We get

$$\begin{aligned}\int_{1/2}^{1}g(t)\,dt&=\int_{1/2}^{1}\frac{1}{t^{1+\gamma/2}\Big(1-\frac{Z_0}{M}t^{d/2-1}\Big)^{\gamma}}\,dt\\&\le 2^{1+\gamma/2}\int_{1/2}^{1}\frac{1}{\Big(1-\frac{Z_0}{2M}(1+t)\Big)^{\gamma}}\,dt\\&\le\frac{2^{2+\gamma/2}M}{Z_0}\int_{1-Z_0/M}^{1}s^{-\gamma}\,ds\\&\le\frac{2^{2+\gamma/2}}{\gamma-1}\frac{M}{Z_0}\left(\Big(\frac{M}{M-Z_0}\Big)^{\gamma-1}-1\right).\end{aligned}$$

Now we use the inequality $((1-t)^{-\alpha}-1)t^{-1}\le(1\wedge\alpha)(1-t)^{-\alpha}$, which holds for all $0<t<1$ and all $\alpha>0$ (this inequality follows easily from the fact that $(1-t)^{\alpha}\ge 1-t$ for $\alpha\le 1$ and $(1-t)^{\alpha}\ge 1-\alpha t$ for $\alpha>1$). We apply this inequality for $t=Z_0/M$ and $\alpha=\gamma-1$. The above expression is then bounded by

$$\begin{aligned}\int_{1/2}^{1}g(t)\,dt&\le\frac{2^{2+\gamma/2}(1\wedge(\gamma-1))}{\gamma-1}\Big(\frac{M}{M-Z_0}\Big)^{\gamma-1}\\&=\frac{2^{2+\gamma/2}(1\wedge(\gamma-1))}{\gamma-1}\Big(\frac{M}{X_0+Y_0}\Big)^{\gamma-1}\end{aligned}$$

Adding both parts to (2.4.14) yields the claim. □

Lemma 2.4.4. *Let $0 < \varepsilon \leq 1/2$. For a positive real number c assume that the condition*

$$\frac{K_k}{M^{(1-\beta)/2}} \leq c(M-2k)^{(1-\beta)/2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta} \tag{2.4.15}$$

is satisfied for $k = 0$ and define the stopping time τ as the smallest positive integer k for which (2.4.15) *is not satisfied. Then there exists a sequence (δ_M) converging to 0 and depending only on M, and a constant C depending only on β, d and c such that for any positive integer n*

$$\mathbb{P}(|K_{k\wedge\tau} - K_0| \geq \varepsilon K_0, \text{ for some } 0 \leq k \leq n) \leq \frac{C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big) + \delta_M, \tag{2.4.16}$$

whenever

$$\varepsilon \geq C\Big(\frac{1}{M-2n} + \frac{1}{X_0+Y_0}\Big). \tag{2.4.17}$$

Proof. We begin by showing that for any $C_0 > 0$ we can choose C so that (2.4.17) implies

$$\Big(1 - \frac{2k}{M}\Big)\Big(1 - \frac{Z_0}{M}\Big(1 - \frac{2k}{M}\Big)^{d/2-1}\Big) \geq \frac{C_0}{M\varepsilon}, \tag{2.4.18}$$

for all $0 \leq k \leq n$. Since the function $\phi(t) = t - t^{d/2}Z_0/M$ is concave on $[0,1]$ the minimum of the left hand side in (2.4.18) is either

$$\phi(1-2n/M) \geq (1-2n/M) - (1-2n/M)^{d/2}, \quad \text{or } \phi(1) = (M-Z_0)/M = (X_0+Y_0)/M.$$

Clearly both of these values are bounded from below by the right hand side of (2.4.18) when the constant C is chosen to be large enough.

Now define σ as the first time k that

$$X_k + Y_k \leq \frac{M-2k}{2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big) \text{ or } Y_k \leq 2d,$$

and define the process $K'_k = K_{k\wedge\tau\wedge\sigma}$. Since σ and τ are stopping times with respect to the filtration $\mathcal{F}_k$, the process K'_k is adapted to this filtration.

Next we show that there is a positive constant c_1 such that for all $k < \sigma \wedge \tau$ we have

$$Y_k \geq c_1(M-2k)\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big). \tag{2.4.19}$$

Assume, for the sake of contradiction, that for some $k < \sigma \wedge \tau$ we have

$$Y_k < c_1(M-2k)\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big).$$

Then since $k < \tau$ we have

$$\begin{aligned} X_k &\leq Y_k^{\beta}(M-2k)^{(1-\beta)/2}c(M-2k)^{(1-\beta)/2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta} \\ &< cc_1^{\beta}(M-2k)\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big). \end{aligned}$$

Since $k < \sigma$ we have $X_k + Y_k > \frac{M-2k}{2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)$ which implies $cc_1^\beta + c_1 \geq 1/2$. When c_1 is small enough we obtain a contradiction and prove (2.4.19). Lemma 2.4.2 now implies that for all $0 \leq k \leq n$

$$|\mathbb{E}(K'_{k+1} - K'_k|\mathcal{F}_k)| \leq \frac{C_1 K'_k}{(M-2k)^2\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^2} = \frac{C_1 K'_k}{M(M-2k)f(1-2k/M)^2}, \tag{2.4.20}$$

and

$$\begin{aligned}\mathbb{E}((K'_{k+1} - K'_k)^2|\mathcal{F}_k) &\leq \frac{C_1 M^{(1-\beta)/2} K'_k}{(M-2k)^{(3+\beta)/2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1+\beta}} \\ &= \frac{C_1 K'_k M^{-\beta}}{(M-2k)f(1-2k/M)^{1+\beta}},\end{aligned} \tag{2.4.21}$$

for some constant $C_1 = C_1(\beta, d, c)$. Define

$$p_k = \frac{C_1}{M(M-2k)f(1-2k/M)}.$$

By Lemma 2.4.3 for $\gamma = 2$

$$\sum_{k=0}^{n-1} p_k = \sum_{k=0}^{n-1} \frac{C_1}{M(M-2k)f(1-2k/M)^2} \leq \frac{C_0 C_1}{M}\left(\frac{M}{M-2n} + \frac{M}{X_0 + Y_0}\right) \tag{2.4.22}$$

Combining this with (2.4.17) yields $\sum_{k=0}^{n-1} p_k \leq \varepsilon/3$, for a large enough constant C. Defining $q_{k,l} = \prod_{i=k}^{\ell}(1+p_i)$ as in Lemma 2.4.1 we have for all $1 \leq k \leq \ell \leq n-1$

$$q_{k,\ell} \leq e^{\sum_{k=0}^{n-1} p_k} \leq e^{\varepsilon/3} \leq \tfrac{3}{2}. \tag{2.4.23}$$

Define the martingale $I_0 = K'_0$, $I_k = K'_k - \sum_{\ell=0}^{k-1} \mathbb{E}(K'_{\ell+1} - K'_\ell|\mathcal{F}_\ell)$ as in Lemma 2.4.1, which together with (2.4.23) implies

$$|K'_k - I_k| \leq \frac{3}{2}\sum_{\ell=0}^{k-1} p_\ell I_\ell. \tag{2.4.24}$$

Next estimate the second moment of jumps. Define

$$r_k = \frac{C_1 M^{(1-\beta)/2}}{(M-2k)f(1-2k/M)^{1+\beta}},$$

so that by Lemma 2.4.3 for $\gamma = 1 + \beta$ we have

$$\sum_{k=0}^{n-1} r_k \leq C_0 C_1 M^{(1-\beta)/2}\left(\left(\frac{M}{M-2n}\right)^{(1+\beta)/2} + \left(\frac{M}{X_0+Y_0}\right)^\beta\right).$$

Lemma 2.4.1 now yields

$$\mathbb{E}((I_n - I_0)^2) \le \frac{3C_0C_1I_0}{2M^\beta}\left(\Big(\frac{M}{M-2n}\Big)^{(1+\beta)/2} + \Big(\frac{M}{X_0+Y_0}\Big)^{\beta}\right)$$

Combined with Doob's maximal inequality, this implies

$$\begin{aligned}\mathbb{P}(|I_k - I_0| \ge \frac{\varepsilon}{3}I_0, \text{ for some } 0 \le k \le n) &\le \frac{27C_0C_1Y_0^\beta}{2M^\beta X_0\epsilon^2}\left(\Big(\frac{M}{M-2n}\Big)^{(1+\beta)/2} + \Big(\frac{M}{X_0+Y_0}\Big)^{\beta}\right)\\ &\le \frac{C_2}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big),\end{aligned}$$

for some constant $C_2 = C_2(\beta, d, c)$. If $|I_k - I_0| \le \frac{\varepsilon}{3}I_0$, for all $0 \le k \le n$, then (2.4.24) and the inequality $\sum_{k=0}^{n-1} p_k \le \varepsilon/3$ imply

$$|K'_k - K'_0| \le |K'_k - I_k| + |I_k - I_0| \le \frac{3}{2}\Big(1 + \frac{\varepsilon}{3}\Big)I_0\frac{\varepsilon}{3} + \frac{\varepsilon}{3}I_0 \le \varepsilon K_0.$$

Thus we have

$$\mathbb{P}(|K'_k - K'_0| \ge \varepsilon K_0, \text{ for some } 0 \le k \le n) \le \frac{C_2}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big).$$

Define $1 - \delta_M$ to be the probability that

$$X_k + Y_k \ge \frac{1}{2}\Big(M - 2k - Z_0(1 - 2k/M)^{d/2}\Big) \tag{2.4.25}$$

holds for all $0 \le k \le n$. By Theorem 2.6.1 we have that $\lim_{M\to\infty}\delta_M = 0$. Since $K'_k = K_{k\wedge\tau}$ for $k \le \sigma \wedge n$ it is enough to show that $\mathbb{P}(\sigma < \tau \wedge n) \le \delta_M$. To this end simply observe that on the event in (2.4.25), inequality (2.4.19), $\sigma < \tau \wedge n$ and the fact that $Y_\sigma \le 2d$ imply

$$c_1(M - 2k)\Big(1 - \frac{Z_0}{M}(1 - 2k/M)^{d/2-1}\Big) \le Y_k \le 2d + 2,$$

for $k = \sigma - 1$. However, by (2.4.18) and the fact that $\varepsilon \le 1/2$, this is impossible for C large enough in (2.4.17) (recall that the value of c_1 depended only on c and β). □

Remark 2.4.5. A more careful analysis of the process K_n would allow one to replace Doob's maximal inequality with Freedman's inequality (see [27]), and obtain exponential bound in the statement of Lemma 2.4.4. However the bound above suffices to our purposes and, to avoid even more tedious analysis, we use Doob's maximal inequality.

It is perhaps inconvenient to apply Lemma 2.4.4 as the assumption (2.4.15) already involves an estimate on K_k one would have to check. Fortunately, the following easy corollary handles this problem.

Corollary 2.4.6. *Let $0 < \varepsilon \leq 1/2$. Assume that for a positive real number C_0 and an integer n the inequality*

$$\frac{K_0}{M^{(1-\beta)/2}} \leq C_0 f\Big(1 - \frac{2k}{M}\Big)^{1-\beta}$$

is satisfied for $0 \leq k \leq n$. Then there exists a sequence (δ_M) converging to 0 and depending only on M, and a constant C, depending on β, d and C_0, such that

$$\mathbb{P}(|K_k - K_0| \geq \varepsilon K_0,\ \textit{for some } 0 \leq k \leq n) \leq \frac{C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big) + \delta_M, \quad (2.4.26)$$

whenever

$$\varepsilon \geq C\Big(\frac{1}{M-2n} + \frac{1}{X_0+Y_0}\Big).$$

Proof. The assumption in the statement simply reads

$$\frac{K_0}{M^{(1-\beta)/2}} \leq C_0(M-2k)^{(1-\beta)/2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta}.$$

Define stopping time τ as the smallest integer k such that

$$\frac{K_k}{M^{(1-\beta)/2}} > 2C_0(M-2k)^{(1-\beta)/2}\Big(1 - \frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta}.$$

Applying Lemma 2.4.4 we conclude that the event

$$|K_{k\wedge\tau} - K_0| \leq \varepsilon K_0,\ \text{for all } 0 \leq k \leq n, \quad (2.4.27)$$

has probability of at least

$$1 - \frac{C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big) - \delta_M,$$

for an appropriately chosen constant C. Since $\tau < n$ implies $K_\tau > 2K_0$, on the event in (2.4.27) we have that $\tau \geq n$. Thus in the event in (2.4.27) we can replace $K_{k\wedge\tau}$ by K_k which completes the proof. □

As we said the assumption in the previous corollary is quite an improvement over (2.4.15). However, the problem is that it might fail to hold throughout the time regime. To overcome this problem, we switch the roles of processes X and Y and the value of the parameter β to $1/\beta$ when this condition fails to hold. This gives Lemma 2.4.7.

Lemma 2.4.7. *Let β be any positive real number. Let (L_M) be a sequence of positive numbers converging to ∞ and such that $\lim_M L_M M^{-\gamma} = 0$, for any $\gamma > 0$. Assume that $L_M^{\frac{(1+\beta)^4}{\beta^2}} \leq X_0 \leq Y_0$. Define n_0 as the largest integer such that*

$$M - 2n_0 \geq L_M\Big(\frac{M^{(1-\beta)/2}}{K_0} \vee \frac{K_0}{M^{(1-\beta)/2}}\Big)^{2/(1+\beta)}.$$

Then for any $\varepsilon > 0$ there is a sequence of numbers η_M converging to zero such that

$$\mathbb{P}(|K_n - K_0| \le \varepsilon K_0, \quad \text{for all } 0 \le n \le n_0) \ge 1 - \eta_M.$$

Remark 2.4.8. The choice of the exponent $\frac{(1+\beta)^4}{\beta^2}$ in the lower bound for X_0 is just a technical condition. Actually given X_0 one can always decrease the value of L_M so that the inequality in the statement holds. Nevertheless the assumption that $Y_0 \ge L_M^{(1+\beta)^4/\beta^2}$ and $X_0 \ge L_M^{(1+\beta)^4/\beta^2}$ implies that

$$L_M^{(1+\beta)^4/\beta^2} M^{-\beta} \le K_0 \le M L_M^{-(1+\beta)^4/\beta}$$

and

$$M - 2n_0 \le L_M^{1-2(1+\beta)^3/\beta} M \vee L_M^{1-2(1+\beta)^3/\beta^2} M. \tag{2.4.28}$$

In particular we know that $1 - 2n_0/M$ is converging to 0, a fact which will be useful in our proofs.

First we prove a few technical details. We start by recalling Theorem 2.6.1 we know that for any $\varepsilon > 0$ there is a sequence (δ_M) converging to 0 such that with probability of at least $1 - \delta_M$ we have that for every $0 \le n \le \frac{M}{2} - 1$

$$1 - \varepsilon \le \frac{X_n + Y_n}{M - 2n - Z_0(1 - 2n/M)^{d/2}} = \frac{X_n + Y_n}{f(1 - 2n/M)\sqrt{M(M - 2n)}} \le 1 + \varepsilon \ . \tag{2.4.29}$$

The following two simple claims will be helpful when switching the roles of the processes X and Y.

Lemma 2.4.9. *There exists a sequence δ_M depending only on ϵ, converging to zero such that for any non-negative integers n and k such that $M - 2n - 2k > 1$ we have*

$$(1 - \varepsilon) \le \frac{1 - \frac{Z_n}{M-2n}\left(1 - \frac{2k}{(M-2n)}\right)^{d/2-1}}{1 - \frac{Z_0}{M}\left(1 - \frac{2(n+k)}{M}\right)^{d/2-1}} \le (1 + \varepsilon) \tag{2.4.30}$$

Proof. Using the fact that $X_n + Y_n = M - 2n - Z_n$ inequalities (2.4.29) can be rewritten as

$$(1+\varepsilon)\frac{Z_0}{M}\left(1 - \frac{2n}{M}\right)^{d/2-1} - \varepsilon \le \frac{Z_n}{M - 2n} \le \varepsilon + (1-\varepsilon)\frac{Z_0}{M}\left(1 - \frac{2n}{M}\right)^{d/2-1}.$$

It is not hard to check that this in turn implies (2.4.30). □

Lemma 2.4.10. *Suppose the assumptions of Lemma 2.4.7 hold. Assume that $\beta \neq 1$ and let $c > 1$. Then there is a positive constant $c' = c'(\beta, c, d)$ such that for any $0 \le k \le n_0$*

$$\frac{1}{c} f(1 - 2k/M) \le \frac{K_k^{1/(1-\beta)}}{M} \le c f(1 - 2k/M) \Rightarrow X_k \wedge Y_k \ge c' L_M. \tag{2.4.31}$$

Proof. We first show that if the assumptions hold then for a constant $c'' = c''(\beta, c, d)$

$$\frac{1}{c} f(1-2k/M) \leq \frac{K_k^{1/(1-\beta)}}{M} \leq c f(1-2k/M) \Rightarrow X_k \wedge Y_k \geq c''(X_k + Y_k).$$

If $X_k < c''(X_k+Y_k)$ then $Y_k > (1-c'')(X_k+Y_k)$ and

$$\frac{K_k^{1/(1-\beta)}}{M} = \left(\frac{X_k}{Y_k^{\beta}}\right)^{1/(1-\beta)} \frac{1}{\sqrt{M(M-2k)}} \lesseqgtr \frac{c''^{1/(1-\beta)}}{(1-c'')^{\beta/(1-\beta)}} \frac{X_k+Y_k}{\sqrt{M(M-2k)}},$$

where the inequality in $\lesseqgtr$ is $<$ for $\beta < 1$ and $>$ for $\beta > 1$. Using (2.4.29) to bound the term $(X_k+Y_k)(M(M-2k))^{-1/2}$ we obtain a contradiction with the left hand side of (2.4.31) for c'' such that $c''(1-c'')^{-\beta} \leq (c/(1-\varepsilon))^{-|1-\beta|}$, which yields $X_k \geq c''(X_k+Y_k)$. In the same way one can show that $Y_k \geq c''(X_k+Y_k)$ for an appropriately chosen c''.

To finish the proof we show that or every $0 \leq n \leq n_0$

$$X_n + Y_n \geq L_M/3 \ .$$

To check this, by (2.4.29) it is enough to check that $\phi(t) \geq \frac{2L_M}{3M}$ for $1 - 2n_0/M \leq t \leq 1$, where $\phi(t) = t - t^{d/2} Z_0/M$. By the concavity of ϕ and the fact that $M - 2n_0 \geq L_M$ it is enough to check the lower bound for $t = L_M/M$ and $t = 1$ for which the claim is obvious. □

Lemma 2.4.11. *Suppose the assumptions in Lemma 2.4.7 hold. Then,* $f(1) \geq K_0^{1/(1-\beta)} M^{-1}$ *for* $\beta < 1$*, and* $f(1) \leq 2K_0^{1/(1-\beta)} M^{-1}$ *for* $\beta > 1$*.*

Proof. To prove the statement for $\beta < 1$ simply observe that it is equivalent to $X_0 Y_0^{-\beta} \geq (X_0+Y_0)^{1-\beta}$ and to

$$\frac{X_0}{X_0+Y_0} \leq \Big(\frac{Y_0}{X_0+Y_0}\Big)^{\beta},$$

which, because of $X_0 \leq Y_0$ surely holds for $\beta < 1$. The statement for $\beta > 1$ is similarly equivalent to

$$\frac{2X_0}{X_0+Y_0} \leq \Big(\frac{2Y_0}{X_0+Y_0}\Big)^{\beta},$$

which again holds, since the left hand side is smaller than 1 and the right hand side is larger than 1. □

Proof of Lemma 2.4.7. Throughout the proof we assume that M is sufficiently large for the estimates to hold. We can assume $\varepsilon < 1/2$ and $\eta_M \geq C' L_M^{-1} \varepsilon^{-1}$, for any constant C' (at different stages in the proof we choose convenient values for C'). Since we can also assume that $\eta_M < 1$ (otherwise there is nothing to prove), we can assume that

$$\varepsilon \geq \frac{C'}{L_M}. \tag{2.4.32}$$

First we present the bound for the simplest case when $\beta = 1$. Because $X_0 \le Y_0$ we have $K_0 \le 1$ and in this case n_0 is the largest integer with the property that $M - 2n_0 \ge L_M/K_0$. By Corollary 2.4.6 applied with $C_0 = 1$ we have that with probability at least

$$1 - \frac{C}{\varepsilon^2}\Big(\frac{1}{(M-2n_0)K_0} + \frac{1}{X_0}\Big) - \delta_M \ge 1 - \frac{2C}{\varepsilon^2 L_M} - \delta_M,$$

we have

$$|K_k - K_0| \le \varepsilon K_0, \text{ for all } 0 \le k \le n_0,$$

which proves the claim when $\beta = 1$.

Next we analyze the case of $\beta \ne 1$. Fix numbers $0 < c_2 < 1$ and $c_1 > 2$. Note that f is a concave nonnegative function on $[0,1]$ and $f(0) = 0$. By Lemma 2.4.11 if $\beta < 1$ there is a unique point $0 < t_2 < 1$ such that $f(t_2) = c_2 K_0^{1/(1-\beta)}/M$. If $\beta > 1$ then in the case when $\max_{[0,1]} f \ge c_1 K_0^{1/(1-\beta)}/M$ denote by t_2 the smallest element in $f^{-1}(c_2 K_0^{1/(1-\beta)}/M)$ and by t_1 the largest element in $f^{-1}(c_1 K_0^{1/(1-\beta)}/M)$. Define $\overline{n}_1$ as the largest integer such that $M - 2\overline{n}_1 \ge M t_1$, and $\overline{n}_2$ the largest integer such that $M - 2\overline{n}_2 \ge t_2 M$. Furthermore, define $n_1 = \overline{n}_1 \wedge n_0$ and $n_2 = \overline{n}_2 \wedge n_0$.

We also need the following inequality which is a technical detail left as an exercise for the reader.

$$c_2 K_0^{1/(1-\beta)}/M \le f(1 - 2n_i/M) \le c_1 K_0^{1/(1-\beta)}/M, \tag{2.4.33}$$

whenever $n_i < n_0$, for $i = 1, 2$.

We separate the analysis into three cases:

a) $\beta > 1$ and $\max_{[0,1]} f \le c_1 K_0^{1/(1-\beta)}/M$,

b) $\beta > 1$ and $\max_{[0,1]} f > c_1 K_0^{1/(1-\beta)}/M$,

c) $\beta < 1$.

To summarize, in case a) we have

$$f(1 - 2k/M) \le c_1 K_0^{1/(1-\beta)}/M, \quad \text{for } 0 \le k \le M/2 - 1, \tag{2.4.34}$$

in case b)

$$f(1-2k/M) \begin{cases} \le c_1 K_0^{1/(1-\beta)}/M, & \text{for } 0 \le k \le n_1, \\ \ge c_2 K_0^{1/(1-\beta)}/M, & \text{for } n_1 \le k \le n_2, \text{ if } n_1 < n_0, \\ \le c_1 K_0^{1/(1-\beta)}/M, & \text{for } n_2 \le k \le M/2 - 1, \text{ if } n_2 < n_0, \end{cases} \tag{2.4.35}$$

and in the case c)

$$f(1-2k/M) \begin{cases} \ge c_2 K_0^{1/(1-\beta)}/M, & \text{for } 0 \le k \le n_2, \\ \le c_1 K_0^{1/(1-\beta)}/M, & \text{for } n_2 \le k \le M/2 - 1, \text{ if } n_2 < n_0. \end{cases} \tag{2.4.36}$$

For the case a) note that (2.4.34) can be rewritten as

$$\frac{K_0}{M^{(1-\beta)/2}} \le c_1^{\beta-1}(M-2k)^{(1-\beta)/2}\Big(1-\frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta}. \tag{2.4.37}$$

Because $M-2n_0 \ge L_M$ and (2.4.32) we can apply Corollary 2.4.6 to get that the event that $|K_k-K_0| \le \varepsilon K_0$ for $0 \le k \le n_0$ has probability at least

$$1-\frac{C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n_0)^{(1+\beta)/2}K_0}+\frac{1}{X_0}\Big)-\delta_M \ge 1-\frac{C}{\varepsilon^2}\Big(\frac{1}{L_M^{(1+\beta)/2}}+\frac{1}{L_M}\Big)-\delta_M.$$

The inequality above follows from the definition of n_0. This suffices for the case a).

Next we assume that we are under the assumptions of case b). From the first inequality in (2.4.35) we obtain that (2.4.37) holds for $0 \le k \le n_1$. Because $M-2n_1 \ge M-2n_0$ we can apply Corollary 2.4.6 like in the case a) and conclude that the event that $|K_k-K_0| \le \frac{\varepsilon}{3}K_0$ holds for all $0 \le k \le n_1$, has probability of at least

$$1-\frac{9C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n_1)^{(1+\beta)/2}K_0}+\frac{1}{X_0}\Big)-\delta_M \ge 1-\frac{9C}{\varepsilon^2}\Big(\frac{1}{L_M^{(1+\beta)/2}}+\frac{1}{L_M}\Big)-\delta_M.$$

Now if $n_1=n_0$ we are done with the analysis in the case b).

Otherwise assume that $|K_k-K_0| \le \frac{\varepsilon}{3}K_0$ holds for all $0 \le k \le n_1$ indeed, and note that (2.4.33) implies that

$$\frac{(1+\varepsilon)^{1/(1-\beta)}}{c_1}f(1-2n_1/M) \le \frac{K_{n_1}^{1/(1-\beta)}}{M} \le \frac{(1-\varepsilon)^{1/(1-\beta)}}{c_2}f(1-2n_1/M), \tag{2.4.38}$$

which then by (2.4.31) implies that both X_{n_1} and Y_{n_1} are at least $c'L_M$ for some constant c'.

Define $M'=M-2n_1$, $X'_k=X_{n_1+k}$, $Y'_k=Y_{n_1+k}$, $Z'_k=Z_{n_1+k}$ and

$$K'_k=\frac{Y'_k}{X_k'^{1/\beta}(1-2k/M')^{(1-1/\beta)/2}}.$$

It is easy to check that in fact

$$K'_k=K_{n_1+k}^{-1/\beta}\Big(\frac{M'}{M}\Big)^{(1-1/\beta)/2}. \tag{2.4.39}$$

Similarly to (2.4.37), the second inequality in (2.4.35) implies that for $n_1 \le k \le n_2$

$$\frac{K_0}{M^{(1-\beta)/2}} \ge c_2^{\beta-1}(M-2k)^{(1-\beta)/2}\Big(1-\frac{Z_0}{M}(1-2k/M)^{d/2-1}\Big)^{1-\beta}.$$

Combined with the inequality $K_{n_1} \ge (1-\varepsilon)K_0$,

$$K_{n_1} \ge (1-\varepsilon)c_2^{\beta-1}M^{(1-\beta)/2}(M-2(n_1+k))^{(1-\beta)/2}\Big(1-\frac{Z_0}{M}(1-2(n_1+k)/M)^{d/2-1}\Big)^{1-\beta},$$

for $0 < k \le n_2 - n_1$. Raising the above inequality to the power of $-1/\beta$ and using (2.4.30) and (2.4.39) we obtain

$$\frac{K_0'}{M'^{(1-1/\beta)/2}} \le \frac{(1+\varepsilon)^{1-1/\beta}}{(1-\varepsilon)^{1/\beta} c_2^{1-1/\beta}} (M' - 2k)^{(1-1/\beta)/2} \Big(1 - \frac{Z_0'}{M'}(1 - 2k/M')^{d/2-1}\Big)^{1-1/\beta}, \quad (2.4.40)$$

for all $0 \le k \le n_2 - n_1$. By Corollary 2.4.6 we have that the event $|K_k' - K_0'| \le \frac{\varepsilon}{2^{\beta+3}} K_0'$, for all $0 \le k \le n_2 - n_1$, is of probability at least

$$\begin{aligned} &1 - \frac{4^{\beta+3}C}{\varepsilon^2}\Big(\frac{M'^{(1-1/\beta)/2}}{(M' - 2(n_2 - n_1))^{(1+1/\beta)/2} K_0'} + \frac{1}{Y_0'}\Big) - \delta_{M'} \\ &\ge 1 - \frac{4^{\beta+3}C}{\varepsilon^2}\Big(\frac{(1+\varepsilon)^{1/\beta} K_0^{1/\beta}}{(M - 2n_2)^{\frac{1+\beta}{2\beta}} M^{\frac{1-\beta}{2\beta}}} + \frac{1}{Y_{n_1}}\Big) - \delta_{L_M} \\ &\ge 1 - \frac{4^{\beta+3}C}{\varepsilon^2}\Big((1+\varepsilon)^{1/\beta} L_M^{-\frac{1+\beta}{2\beta}} + (c' L_M)^{-1}\Big) - \delta_{L_M}, \quad (2.4.41) \end{aligned}$$

where we used the fact that $n_2 \ge n_0$, the definition of n_0 and the lower bound $Y_{n_1} \ge c' L_M$. Then this event can be rewritten as

$$\Big|\Big(\frac{K_{n_1+k}}{K_{n_1}}\Big)^{-1/\beta} - 1\Big| \le \frac{\varepsilon}{2^{\beta+3}}, \text{ for all } 0 \le k \le n_2 - n_1,$$

which, using the fact that $\varepsilon \le 1/2$ easily implies that

$$|K_{n_1+k} - K_{n_1}| \le \frac{\varepsilon}{4} K_{n_1}, \text{ for all } 0 \le k \le n_2 - n_1,$$

and

$$|K_k - K_0| \le |K_k - K_{n_1}| + |K_{n_1} - K_0| \le \frac{\varepsilon}{4}\Big(1 + \frac{\varepsilon}{3}\Big) K_0 + \frac{\varepsilon}{3} K_0 \le \frac{2\varepsilon}{3} K_0, \text{ for all } n_1 \le k \le n_2. \quad (2.4.42)$$

If $n_2 = n_0$ we are done.

Otherwise, assume that the event in (2.4.42) holds and observe that (2.4.38) holds when n_1 is replaced by n_2. Thus again we have that $X_{n_2} \ge c' L_M$.

Define $M'' = M - 2n_2$, $X_k'' = X_{n_2+k}$, $Y_k'' = Y_{n_2+k}$, $Z_k'' = Z_{n_2+k}$ and

$$K_k'' = \frac{X_k''}{Y_k''^{\beta}(1 - 2k/M'')^{(1-\beta)/2}} = K_{n_2+k}\Big(\frac{M''}{M}\Big)^{(1-\beta)/2}.$$

Following the argument that lead to (2.4.40), and using the third inequality in (2.4.35) we can deduce that

$$\frac{K_0''}{M''^{(1-\beta)/2}} \le (1+\varepsilon)(1-\varepsilon)^{1-\beta} c_1^{\beta-1} (M'' - 2k)^{(1-\beta)/2}\Big(1 - \frac{Z_0''}{M''}(1 - 2k/M'')^{d/2-1}\Big)^{1-\beta}.$$

By Corollary 2.4.6 we have that with probability at least

$$\begin{aligned} 1 &- \frac{16C}{\varepsilon^2}\Big(\frac{M''^{(1-\beta)/2}}{(M''-2(n_0-n_2))^{(1+\beta)/2}K_0''} + \frac{1}{X_0''}\Big) - \delta_{M''} \\ &\geq 1 - \frac{16C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(1-\varepsilon_2)(M-2n_0)^{\frac{1+\beta}{2}}K_0} + \frac{1}{X_{n_1}}\Big) - \delta_{L_M} \\ &\geq 1 - \frac{16C}{\varepsilon^2}\Big((1-\varepsilon_2)^{-1}L_M^{-(1+\beta)/2} + (c'L_M^{-1})\Big) - \delta_{L_M}. \end{aligned}$$

the event

$$|K_k'' - K_0''| \geq \frac{\varepsilon}{4}K_0'', \text{ for some } 0 \leq k \leq n_0 - n_2$$

occurs. After a glance at the definition of K_k'' we proceed as in the previous step and finish the analysis in the case b).

The case c) is handled in the same way. The first inequality in (2.4.36), inequality (2.4.32), the fact that $M - 2n_0 \geq L_M$ and Corollary 2.4.6 imply that the event $|K_k - K_0| \leq \varepsilon K_0/3$, for $0 \leq k \leq n_2$, has probability at least

$$1 - \frac{9C}{\varepsilon^2}\Big(\frac{M^{(1-\beta)/2}}{(M-2n_2)^{(1+\beta)/2}K_0} + \frac{1}{X_0}\Big) - \delta_M \geq 1 - \frac{9C}{\varepsilon^2}\Big(\frac{1}{L_M^{(1+\beta)/2}} + \frac{1}{L_M}\Big) - \delta_M.$$

This finishes the proof if $n_2 = n_0$. Otherwise, observe that (2.4.38) holds and thus we have $Y_{n_2} \geq c'L_M$. Then define $X_k' = X_{n_2+k}$, $Y_k' = Y_{n_2+k}$, $Z_k' = Z_{n_2+k}$, $M' = M - 2n_2$ and

$$K_k' = \frac{Y_k'}{X_k'^{1/\beta}(1-2k/M')^{(1-1/\beta)/2}} = K_k^{-1/\beta}\Big(\frac{M}{M'}\Big)^{\frac{1-\beta}{2\beta}}.$$

The second inequality in (2.4.36) and (2.4.29) now imply

$$\frac{K_0'}{M'^{(1-1/\beta)/2}} \leq \frac{c_1^{(1-\beta)/\beta}(1+\varepsilon)^{1-1/\beta}}{(1-\varepsilon)^{1/\beta}}(M'-2k)^{(1-1/\beta)/2}\Big(1 - \frac{Z_0'}{M'}(1-2k/M')^{d/2-1}\Big)^{1-1/\beta}.$$

Now we can apply Corollary 2.4.6 and conclude that with probability at least

$$\begin{aligned} 1 &- \frac{4^{\beta+1}C}{\varepsilon^2}\Big(\frac{M'^{(1-1/\beta)/2}}{(M-2n_0)^{(1+1/\beta)/2}K_0'} + \frac{1}{Y_0'}\Big) - \delta_{M'} \\ &\geq 1 - \frac{4^{\beta+1}C}{\varepsilon^2}\Big(\frac{(1+\varepsilon)^{1/\beta}K_0^{1/\beta}}{(M-2n_0)^{\frac{1+\beta}{2\beta}}M^{\frac{1-\beta}{2\beta}}} + \frac{1}{c'L_M}\Big) - \delta L_M \\ &\geq 1 - \frac{4^{\beta+1}C}{\varepsilon^2}\Big(\frac{(1+\varepsilon)^{1/\beta}}{L_M^{\frac{1+\beta}{2\beta}}} + \frac{1}{c'L_M}\Big) - \delta L_M, \end{aligned}$$

we have that $|K_k' - K_0'| \leq \varepsilon 2^{-\beta-1}K_0'$ for all $0 \leq k \leq n_0 - n_2$. Using the analysis similar to the case b) we see that this event implies $|K_{n_2+k} - K_0| \leq \varepsilon K_0$, for $0 \leq k \leq n_0 - n_2$. This finishes the proof. □

The next lemma controls the size of processes for large times.

Remark 2.4.12. Starting in the proof of Lemma 2.4.13 and further we will use the following simple fact. Let L_n be any process adapted to a filtration $\mathcal{F}_n$. Assume that for any ϵ we have a sequence of events $(\Omega_{M,\epsilon})$ such that $\mathbb{P}(\Omega_{M,\epsilon}) \to 1$ as $M \to \infty$, and such that on the event $\Omega_{M,\epsilon}$ we have $p_M(n) - \epsilon \leq \mathbb{P}(L_{n+1} - L_n \in A|\mathcal{F}_n) \leq p_M(n) + \epsilon$. Then the number of indices $1 \leq n \leq N(M) \to \infty$ such that $L_{n+1} - L_n \in A$ is on $\Omega_{M,\epsilon}$ stochastically bounded from above (below) by a sum of $N(M)$ independent Bernoulli random variables with parameters $p_M(n)+\epsilon$ ($p_M(n)-\epsilon$). In particular, if $N(M) \to \infty$ then by Hoeffding's inequality (see [35]), the number of such indices is equal to $1 + o(1)$ times $\lim_{M\to\infty} \sum_{n=1}^{N(M)} p(n)$ asymptotically almost surely.

Lemma 2.4.13. *Suppose the conditions of Lemma 2.4.7 hold. Then there is a sequence* (η_M) *converging to zero such that with the probability at least* $1 - \eta_M$ *we have*

(i) $X_{n+1} \leq X_n$ *for all* $n \geq n_0$*, in the case* $K_0 \leq M^{(1-\beta)/2}$*,*

(ii) $Y_{n+1} \leq Y_n$ *for all* $n \geq n_0$*, in the case* $K_0 \geq M^{(1-\beta)/2}$*.*

Proof. When $K_0 \leq M^{(1-\beta)/2}$, the inequality $Y_{n_0} \leq M - 2n_0$ implies

$$K_{n_0} \geq X_{n_0} M^{(1-\beta)/2}(M - 2n_0)^{-(1+\beta)/2},$$

which, by the definition of n_0 and the fact that $K_{n_0} \leq (1+\varepsilon)K_0$, yields $X_{n_0} \leq (1+\varepsilon)L_M^{(1+\beta)/2}$. When $K_0 \geq M^{(1-\beta)/2}$ then the inequality $X_{n_0} \leq M - 2n_0$ implies

$$K_{n_0} \leq M^{(1-\beta)/2}(M - 2n_0)^{(1+\beta)/2} Y_{n_0}^{-\beta},$$

which, by the definition of n_0 and the fact that $K_{n_0} \geq (1-\varepsilon)K_0$, yields $Y_{n_0} \leq (1-\varepsilon)^{-1/\beta} L_M^{\frac{1+\beta}{2\beta}}$. If $K_0 \leq M^{(1-\beta)/2}$ denote by U the process X and $\tau = \frac{1+\beta}{2}$ and if $K_0 \geq M^{(1-\beta)/2}$ denote by U the process Y and $\tau = \frac{1+\beta}{2\beta}$. (if $K_0 = M^{(1-\beta)/2}$ do either). Furthermore denote by n' and n'' the largest integers such that

$$M - 2n' \geq M\Big(\frac{L_M^\tau}{Z_0}\Big)^{2/d}, \text{ and } M - 2n'' \geq M\Big(\frac{1}{L_M Z_0}\Big)^{2/d}.$$

First denote the event $\mathbf{U}_3 = \{U_{k+1} \leq U_k : n'' \leq k\}$. As the value of U can grow only if the value of Z decreases, the event $\mathbf{U}_3$ contains the event that $\{Z_k = 0 : k \geq n''\}$. By Theorem 2.6.1 i) the probability of this event converges to 1. This handles the time regime $k \geq n''$. In particular, the claim is proved if $n_0 \geq n''$, so we assume $n_0 < n''$.

To finish the proof denote the events

$$\mathbf{U}_1 = \{U_{k+1} \leq U_k : n_0 \leq k \leq n'\}, \mathbf{U}_2 = \{U_{k+1} \leq U_k : n' \leq k \leq n''\},$$

in the case $n_0 \leq n'$, and just

$$\mathbf{U}_2 = \{U_{k+1} \leq U_k : n_0 \leq k \leq n''\},$$

if $n_0 > n'$.

Since with high probability Z_k is bounded by a constant multiple of $Z_0(1-2k/M)^{d/2}$, for $k \le n'$, by Remark 2.4.12, the probability of the event $\mathbf{U}_1$ can be estimated as

$$\mathbb{P}(\mathbf{U}_1) \ge \prod_{k=n_0}^{n'} \left(1 - \frac{c_2 L_M^\tau Z_0(1-2k/M)^{d/2}}{(M-2k-Z_0(1-2k/M)^{d/2})(M-2k)}\right)$$
$$\ge 1 - \frac{c_2}{1-2^{-d/2+1}} \sum_{k=n_0}^{n'} \frac{L_M^\tau Z_0(1-2k/M)^{d/2}}{(M-2k)^2},$$

where we used that fact that $Z_0 \le M$ and $M - 2k \le M/2$, for $k \ge n_0$, which follows from (2.4.28). It suffices to prove that the above sum converges to 0. To estimate it calculate

$$\frac{L_M^\tau Z_0}{M^2} \sum_{k=n_0}^{n'} \left(1 - \frac{2k}{M}\right)^{d/2-2} \le \frac{L_M^\tau Z_0}{M} \int_{1-2(n'+1)/M}^{1-2(n_0-1)/M} t^{d/2-2} dt$$
$$\le \frac{2}{d-2} L_M^\tau \left(\frac{M-2n_0+2}{M}\right)^{d/2-1}, \tag{2.4.43}$$

where the first inequality follows by monotonicity of the function $t \mapsto t^{d/2-2}$. From (2.4.28) it is easy to see that the last term converges to 0 (the only thing to check is that $(d/2 - 1)(1-2(1+\beta)^3/\beta) + \tau < 0$ and $(d/2-1)(1-2(1+\beta)^3/\beta^2) + \tau < 0$).

Next we bound the probability of $\mathbf{U}_2$. Using the inequality

$$M - 2n'' \ge \frac{M}{Z_0^{2/d} L_M^{2/d}} \ge \frac{M^{1-2/d}}{L_M^{2/d}}, \tag{2.4.44}$$

we obtain $M - 2n'' \ge 2L_M$, for M large enough and, since with probability converging to 1 we have $Z_n \le 2L_M^\tau$, for $n \ge n'$

$$\mathbb{P}(\mathbf{U}_2) \ge \prod_{k=n'}^{n''} \left(1 - \frac{c_3 L_M^{2\tau}}{(M-2k)(M-2k-L_M)}\right) \ge 1 - 2c_3 L_M^{2\tau} \sum_{k=n'}^{n''} \frac{1}{(M-2k)^2}$$
$$\ge 1 - 4c_3 \frac{L_M^{2\tau}}{M-2n''-2} \ge 1 - 4c_3 \frac{L_M^{2\tau+2/d}}{M^{1-2/d}}.$$

The right hand side clearly converges to 1 which finishes the analysis of the event $\mathbf{U}_2$.

□

Remark 2.4.14. Note that the replacing the roles of processes X and Y and setting X to have rate 1 and Y to have the rate $1/\beta$ in Lemma 2.4.7 causes the process K_n to become $K_n^{-1/\beta}$ and the value of n_0 and the exponents on L_M to remain unchanged. Therefore if both X_0 and Y_0 are bounded from below by an appropriate power of L_M we do not need to assume that $X_0 \le Y_0$ for Lemma 2.4.7 to hold.

We are now ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. To have notation more compatible with the previous results in this section (which will be heavily used) we change the notation from L_N in the statement to L_M. Throughout the proof we assume that the sequence (L_M) grows slowly enough, so that it satisfies the assumptions in Lemma 2.4.7, and so that, similarly to Remark 2.4.8 we can conclude that $L_M^{\kappa} M^{-\beta} \leq K_0 \leq M L_M^{-\kappa}$ and $1 - 2n_0/M \leq L_M^{-\kappa}$ for $\kappa > 0$ appropriate for our calculations.

Define n_1 as the largest integer such that $M - 2n_1 \geq M(Z_0 L_M)^{-2/d}$. By the assumptions it follows that $\varepsilon L_M^{2/d} M Z_0^{-2/d}$ converges to ∞ and thus we can apply Theorem 2.6.1 to conclude that for any $\epsilon > 0$ asymptotically almost surely we have

$$\frac{Z_n}{Z_0} = (1 \pm \epsilon)\Big(1 - \frac{2n}{M}\Big)^{d/2}, \text{ for } 0 \leq n \leq n_1.$$

Also define $m = n_0 \wedge n_1$.

We will prove all the results with R_{fin} replaced by R_m. Then we show that $(R_{\text{fin}} - R_m)/Z_0$ is small and also small compared to $M^{1/\beta - 1} K^{-1/\beta}$, which by $Z_0/d = N - R_0 - B_0$ allows us to replace R_m with R_{fin}.

By Lemma 2.4.7 and Theorem 2.6.1 for any $\varepsilon > 0$ asymptotically almost surely we have

$$\frac{1 - Y_n/(X_n + Y_n)}{\big(Y_n/(X_n + Y_n)\big)^{\beta}} = (1 \pm \varepsilon)\frac{K_0}{M^{1-\beta}(1 - 2n/M)^{(1-\beta)/2}\Big(1 - \frac{Z_0}{M}(1 - 2n/M)^{d/2-1}\Big)^{1-\beta}}$$

for $0 \leq n \leq m$. For $\beta = 1$ this yields

$$\frac{Y_n}{X_n + Y_n} = (1 \pm \epsilon)\frac{1}{1 + K_0} = (1 \pm \epsilon)\frac{Y_0}{X_0 + Y_0}, \text{ for } 0 \leq n \leq m.$$

For $\beta \neq 1$, define the function $\varphi_\beta \colon (0,1) \to \mathbb{R}$ by $\varphi_\beta(t) = (1-t)^{1/\beta - 1} t^{-\beta/(\beta-1)}$ and observe that the derivative of φ_β is bounded from below. Then the derivative of the inverse function φ_β^{-1} is bounded from above, which gives that asymptotically almost surely

$$\frac{Y_n}{X_n + Y_n} = (1 \pm \varepsilon)\varphi_\beta^{-1}\left(M K_0^{1/(\beta-1)}\Big((1 - 2n/M)^{1/2} - \frac{Z_0}{M}(1 - 2n/M)^{(d-1)/2}\Big)\right)$$

for $0 \leq n \leq m$. Define the function $\xi_\beta(s) = \frac{\beta s}{\beta s + (1-s)}$ which has a derivative bounded away from 0 on $(0,1)$. Since

$$\frac{Y_n}{\beta X_n + Y_n} = \frac{1}{1 + \frac{\beta}{Y_n/(X_n+Y_n)}} = \xi_\beta^{-1}\Big(\frac{Y_n}{X_n + Y_n}\Big),$$

and $\phi_\beta = \varphi_\beta \circ \xi_\beta$ we have asymptotically almost surely

$$\frac{Y_n}{\beta X_n + Y_n} = (1 \pm \varepsilon)\phi_\beta^{-1}\left(M K_0^{1/(\beta-1)}\Big((1 - 2n/M)^{1/2} - \frac{Z_0}{M}(1 - 2n/M)^{(d-1)/2}\Big)\right).$$

At the n-th step the conditional probability that in the $n+1$-th step we add a new red vertex is equal to

$$\frac{Y_n}{\beta X_n + Y_n}\frac{Z_n}{M-2n-1},$$

for M large enough. By Remark 2.4.12 for $\beta = 1$ asymptotically almost surely

$$|R_m - R_0| = (1\pm\epsilon)\sum_{n=0}^{m}\frac{Y_0}{X_0+Y_0}\frac{Z_0}{M}(1-2n/M)^{d/2-1} \to (1\pm\epsilon)\frac{Y_0}{X_0+Y_0}\frac{Z_0}{2}\int_{1-2m/M}^{1} t^{d/2-1}dt$$
$$= (1\pm\varepsilon)\frac{Y_0}{X_0+Y_0}\frac{Z_0}{d}\Big(1-(1-2m/M)^{d/2}\Big)\ .$$

Since $(1-2m/M)^{d/2}$ converges to 0 we can disregard this term, which proves (2.3.1).

For $\beta \neq 1$ we proceed as follows. By Remark 2.4.12 asymptotically almost surely we have

$$|R_m - R_0| = (1\pm\epsilon)\sum_{n=0}^{m}\phi_\beta^{-1}\left(MK_0^{1/(\beta-1)}\Big((1-\tfrac{2n}{M})^{1/2}-\tfrac{Z_0}{M}(1-\tfrac{2n}{M})^{(d-1)/2}\Big)\right)\tfrac{Z_0}{M}(1-\tfrac{2n}{M})^{d/2-1}.$$

The sum on the right hand side above converges to the integral

$$\frac{Z_0}{2}\int_{1-2m/M}^{1}\phi_\beta^{-1}\left(MK_0^{1/(\beta-1)}\Big(s^{1/2}-\frac{Z_0}{M}s^{(d-1)/2}\Big)\right)s^{d/2-1}ds$$
$$= \frac{Z_0}{d}\int_{(1-2m/M)^{d/2}}^{1}\phi_\beta^{-1}\left(MK_0^{1/(\beta-1)}\Big(t^{1/d}-\frac{Z_0}{M}t^{(d-1)/d}\Big)\right)dt,$$

which proves (2.3.2), except for the different lower bound in the integral. Note that for the convergence of Riemann sums to the corresponding integral we used the piecewise convexity and concavity of subintegral functions.

Now we come back to fixing the lower bound in the integral to get the on in (2.3.2). By Lemma 2.3.3 it suffices to prove that

$$\int_0^{(1-2m/M)^{d/2}}\frac{M^{1/\beta-1}}{K_0^{1/\beta}}\Big(t^{1/d}-\frac{Z_0}{M}t^{(d-1)/d}\Big)^{1/\beta-1}\wedge 1\ dt \le \delta_M\Big(\frac{M^{1/\beta-1}}{K_0^{1/\beta}}\wedge 1\Big), \tag{2.4.45}$$

for a sequence (δ_M), converging to 0 and depending only on β, d and L_M. When $\frac{M^{1/\beta-1}}{K_0^{1/\beta}} \ge 1$ then (2.4.45) holds as long as we take $\delta \ge (1-2m/M)^{d/2}$. When $\frac{M^{1/\beta-1}}{K_0^{1/\beta}} < 1$ then (2.4.45) holds for

$$\delta_M \ge \int_0^{(1-2m/M)^{d/2}} h(s)^{1/\beta-1}ds,$$

where $h(s) = s^{1/d}$, for $\beta < 1$ and $h(s) = s^{1/d} - s^{(d-1)/d}$, for $\beta > 1$ (in either case $h(s)^{1/\beta-1}$ is integrable in the neighborhood of 0).

To conclude the proof we need to give appropriate upper bounds on $(R_{\text{fin}} - R_m)/Z_0$, to replace R_m by R_{fin} above. Indeed it suffices to prove that $(R_{\text{fin}} - R_m)/Z_0$ and $(R_{\text{fin}} - R_m)K_0^{1/\beta}M^{1-1/\beta}Z_0^{-1}$ converge to 0 (and the convergence depends only on β, d and (L_M)).

Recall that the value of R can increase (always by 1) only if a new uncolored half-edge is matched, that is the value of Z decreases (always by d). Therefore $R_{\text{fin}} - R_m \leq Z_m$. If $n_1 \leq n_0$ (that is $m = n_1$), then by Theorem 2.6.1 we know that $Z_{n_1} \leq 2L_M$, and so it suffices to prove that both L_M/Z_0 and $L_M K_0^{1/\beta}M^{1-1/\beta}Z_0^{-1}$ converge to 0. The first convergence is follows from the assumption, and for the second on calculate

$$L_M\Big(\frac{K_0}{M}\Big)^{1/\beta}\frac{M}{Z_0} \leq L_M^{2-\kappa/\beta},$$

which converges to 0 for $\kappa > 2\beta$ large enough.

Now assume that $n_0 < n_1$. In the case $K_0 \geq M^{(1-\beta)/2}$, by Lemma 2.4.13, the sequence Y_n is decreasing so $n \geq n_0$, which means that no new red color vertex can be captured. Thus $R_m = R_{n_0} = R_{\text{fin}}$. If $K_0 < M^{(1-\beta)/2}$ then we will simply prove that $M - 2n_0$ converges to 0 when divided by either Z_0 or $Z_0 M^{1/\beta-1}K_0^{-1/\beta}$. This yields the claim as $R_{\text{fin}} - R_m = R_{\text{fin}} - R_{n_0} \leq M - 2n_0$. The first convergence follows trivially from the fact that $Z_0 \geq M/L_M$ and $M - 2n_0 \leq ML_M^{-\kappa}$, for κ large enough. The second convergence boils down to showing that

$$\frac{L_M\Big(\frac{M^{(1-\beta)/2}}{K_0}\Big)^{2/(1+\beta)}}{Z_0 M^{1/\beta-1}K_0^{-1/\beta}} = \frac{L_M}{Z_0}\Big(\frac{K_0}{M}\Big)^{\frac{1-\beta}{\beta(1+\beta)}}$$

converges to 0. Knowing that $Z_0 \geq M/L_M$, the inequality follows by using $K_0/M \leq L_M^{-\kappa}$ for $\beta \leq 1$ and $K_0 M^{\beta} \geq L_M^{\kappa}$ for $\beta > 1$, when κ is large enough. □

2.5 Dynamics on the torus

As mentioned in the introduction, the behavior of the competing infection process is extremely different when the underlying graph is a d dimensional torus, and not a random d-regular graph. The difference is stated in Theorem 2.1.6, which is proved in this section. The proof relies on the following shape theorem from [18], due to Cox and Durrett.

In the following theorem we consider the continuous time, rate 1, first passage percolation process on $\mathbb{Z}^2$ started from he origin. Let $\mathcal{S}_t$ be the set that the process occupies at time t, thickened by 1/2, that is $\mathcal{S}_t$ is the union of closed hipercubes of side length 1 centered at the points explored by the first passage percolation process at time t. The result was originally proven for more general distributions for edge weights, and holds in higher dimensions as well.

Theorem 2.5.1 (Cox, Durrett)**.** *There exists a non-trivial, convex set $A \subset \mathbb{R}^2$ which is symmetric around the origin, and such that for any $\delta > 0$*

$$\lim_{t\to\infty}\mathbb{P}\big((1-\delta)tA \subset \mathcal{S}_t \subset (1+\delta)tA\big) \to 1.$$

This theorem was generalized for every $d \geq 3$, see for example [45]. The theorem can be understood in the following way. For $x \in \mathbb{R}^d$, define $d(x) = \{\min t | x \in tA\}$. Since A is convex and symmetric, it is an easy exercise to show that $x \mapsto d(x)$ is a norm on $\mathbb{R}^d$, and thus $d(x, y) = d(y - x)$ is a metric on $\mathbb{R}^d$. Then Theorem 2.5.1 says that with probability converging to 1 as $t \to \infty$, the ball in the random first-passage percolation metric of radius t contains the d-metric ball of radius $(1 - \delta)t$ and is contained in the d-metric ball of radius $(1+\delta)t$. Furthermore, observe that changing the rate of the first-passage percolation process to β simply corresponds to scaling of the set A by a factor of β.

Assume that in Theorem 2.1.6 we start both processes simultaneously from two uniformly chosen vertices $\mathcal{B}_0 = \{x\}$ and $\mathcal{R}_0 = \{y\}$ (that is $B_0 = R_0 = 1$). Then by Theorem 2.5.1 it is easy to see that for any $t_0 > 0$ and $\epsilon > 0$, with probability converging to 1 as $n \to \infty$, every vertex $v \in \mathbb{T}(N, d)$ such that $d(x, v) < (t - \epsilon)\beta n$ and $d(y, v) > (t + \epsilon)\rho n$, for some $t > t_0$ satisfies $v \in \mathcal{B}_{\text{fin}}$, and every vertex $v \in \mathbb{T}(N, d)$ such that $d(x, v)/\beta > (t + \epsilon)n$ and $d(y, v)/\rho < (t-\epsilon)n$, for some $t > t_0$ satisfies $v \in \mathcal{R}_{\text{fin}}$. Also observe that for any $\delta > 0$ we can find $r > 0$ such that the probability that $d(x, y) < rn$ is less than δ. It follows that by scaling the torus by the factor $1/n$ (to a unit torus), and sending $n \to \infty$, Theorem 2.5.1 and the discussion in the previous paragraph imply that the pair of sets $(\mathcal{B}_{\text{fin}}/n, \mathcal{R}_{\text{fin}}/n)$ converge in Hausdorff metric to the Voronoi partition of the continuous unit torus in the metric d. More precisely, the limiting set for $\mathcal{B}_{\text{fin}}/n$ is a set of points v on the unit torus $\mathbb{R}^d/[0, 1]^d$ for which $d(x, v)/\beta < d(y, v)/\rho$, where x and y are two points chosen uniformly and independently on the torus. This in particular yields Theorem 2.1.6 in this special case, and the proof of the general case presented below is essentially the same.

Proof of Theorem 2.1.6. Fix $\epsilon > 0$. Assume first that $(\mathcal{B}_0, \mathcal{R}_0)$ are chosen uniformly at random of size (B_0, R_0). Then there exists $\delta' > 0$ such that, with probability at least $1-\epsilon/2$, the Euclidean distance between any pair of points in $\mathcal{B}_0 \cup \mathcal{R}_0$ is at least $\delta' n$. For $\delta'' > 0$ and every $x \in \mathcal{B}_0$ define the ball $B_x = \{v \in \mathbb{T}(N, d) : d(x, v) < n\beta\delta''\}$, and for every $y \in \mathcal{R}_0$ define $R_y = \{v \in \mathbb{T}(N, d) : d(y, v) < n\delta''\rho\}$. It is not hard to see that one can choose δ'' small enough so that all the sets B_x for $x \in \mathcal{B}_0$ and R_y, for $y \in \mathcal{R}_0$ are disjoint with probability at least $1-\epsilon/2$. Conditioned on this event, Theorem 2.5.1 yields that for n large enough, with probability at least $1 - \epsilon/2$ for $t = 3\delta'' n/4$, the set $\mathcal{B}_t$ contains all the balls $\frac{1}{2}B_x = \{v \in \mathbb{T}(N, d) : d(x, v) < n\beta\delta''/2\}$, $x \in \mathcal{B}_0$ and is contained in $\cup_{x\in\mathcal{B}_0} B_x$ (and the analogous claim holds for $\mathcal{R}_t$). As all sets B_x and R_y have size linear in N, the claim follows.

If on the other hand we select $(\mathcal{B}_0, \mathcal{R}_0)$ uniformly of size (B_0, R_0) with $\mathcal{B}_0$ center of size k_0, then simply apply the above argument with sets $B_x = \{v \in \mathbb{T}(N, d) : d(x, v) < n\beta\delta''\}$ defined for $x \in \mathcal{B}_0^0$.

□

2.6 The configuration model as an urn process

Finally we discuss the urn result we used extensively. The urn model discussed here is finite, i.e., its matrix has negative eigenvalues, which means that eventually there will be no balls left in the urn. The next theorem defines the urn model, and provides a concentration result for the values of S_n, Z_n throughout the process.

Theorem 2.6.1. *Assume $0 < a < b$ are positive integers. Let S_n and Z_n be the number of balls in the urn process with scheme $\begin{pmatrix} -a & 0 \\ b-a & -b \end{pmatrix}$ and let $M = S_0 + Z_0$. Denote the processes*

$$K_n = \frac{Z_n}{Z_0(1-an/M)^{b/a}}, \quad \text{and} \quad L_n = \frac{S_n}{(M-an) - Z_0(1-an/M)^{b/a}},$$

and the stopping time $\sigma_{Z_0,M}$ as the smallest integer n such that $Z_n = 0$. For fixed positive $\varepsilon < 1/2$ and t define $n_{1,t} = \lfloor M(1-tZ_0^{-a/b})/a \rfloor$ and $n_{2,t} = \lfloor M(1-t^{-1}Z_0^{-a/b})/a \rfloor$ and consider the event

$$\mathbf{K}_{Z_0,M,t,\varepsilon} = \{|K_n - 1| \le \varepsilon, \quad \text{for all } 0 \le n \le n_{1,t}\}.$$

Then

a) There exists a universal constant $C = C(a,b)$ (depending on a and b, but not on Z_0, M, ε and t) such that

$$\mathbb{P}(\mathbf{K}_{Z_0,M,t,\varepsilon}) \ge 1 - \frac{C}{t^{b/a}\varepsilon^2}, \text{ for all } t \ge \frac{CZ_0^{a/b}}{M\varepsilon} \tag{2.6.1}$$

and

$$\mathbb{P}(\sigma_{Z_0,M} \ge n_{2,t}) \le \frac{C}{t^{b/(2b-a)}}, \quad \text{for all } t \ge 1. \tag{2.6.2}$$

b) For any $\varepsilon > 0$ there is a positive sequence $(\lambda_{\varepsilon,M})_M$ such that $\lim_{M\to\infty} \lambda_{\epsilon,M} = 0$ and that for all $0 \le Z_0 \le M$

$$\mathbb{P}(|L_n - 1| \le \varepsilon, \text{ for all } 0 \le n < M/a) \ge 1 - \lambda_{\varepsilon,M}.$$

Proof. a) The condition on t in (2.6.1) ensures that when C is large enough, we have

$$M - an_{1,t} \ge \frac{C}{\epsilon} > 2a + 2b,$$

so we assume this condition throughout the proof of (2.6.1). Define the stopping time τ as the first time n such that $Z_n \le 2a + 2b$ and $K'_n = K_{n\wedge\tau}$. First observe that $K'_0 = 1$ and that

$Z_n + S_n = M - an$. Denote by $\mathcal{F}_n$ the σ-algebra generated by the first n draws. It is easy to check that

$$\begin{aligned}\mathbb{E}(K'_{n+1} - K'_n|\mathcal{F}_n) &= \left(\frac{Z_n}{Z_0(1-\frac{an+a}{M})^{b/a}} - \frac{Z_n}{Z_0(1-\frac{an}{M})^{b/a}} - \frac{bZ_n}{Z_0(1-\frac{an+a}{M})^{b/a}(M-an)}\right)\mathbf{1}_{\tau>n} \\ &= \frac{Z_n}{Z_0(1-\frac{an}{M})^{b/a}}\left(\left(1+\frac{a}{M-an-a}\right)^{b/a}\left(1-\frac{b}{M-an}\right) - 1\right)\mathbf{1}_{\tau>n}.\end{aligned} \tag{2.6.3}$$

Take a constant C_1 (which depends on a and b only) such that for any n satisfying $M - an \geq 2a + 2b$ we have

$$\left|\left(1+\frac{a}{M-an-a}\right)^{b/a} - 1 - \frac{b}{M-an-a}\right| \leq \frac{C_1}{(M-an-a)^2},$$

which after plugging in (2.6.3) gives

$$|\mathbb{E}(K'_{n+1} - K'_n|\mathcal{F}_n)| \leq \frac{Z_{n\wedge\tau}}{Z_0(1-\frac{a(n\wedge\tau)}{M})^{b/a}}\Big(\frac{b}{M-an-a} - \frac{b}{M-an} + \frac{C_1}{(M-an-a)^2}\Big).$$

This implies that for some constant $C_2 = C_2(a, b)$ and any n satisfying $M - an \geq 2a + 2b$ we have

$$|\mathbb{E}(K'_{n+1} - K'_n|\mathcal{F}_n)| \leq K'_n\frac{C_2}{(M-an)^2}. \tag{2.6.4}$$

Furthermore we have

$$\begin{aligned}\mathbb{E}((K'_{n+1} - K'_n)^2|\mathcal{F}_n) &\leq \Big(\frac{Z_{n\wedge\tau}}{Z_0(1-\frac{a(n\wedge\tau)+a}{M})^{b/a}} - \frac{Z_{n\wedge\tau}}{Z_0(1-\frac{a(n\wedge\tau)}{M})^{b/a}}\Big)^2 \\ &\quad + \Big(\frac{b}{Z_0(1-\frac{a(n\wedge\tau)+a}{M})^{b/a}}\Big)^2\frac{Z_{n\wedge\tau}}{M-a(n\wedge\tau)} \\ &\leq 2K'^2_n\left(\left(\Big(1+\frac{a}{M-a(n\wedge\tau)-a}\Big)^{b/a} - 1\right)^2 + \frac{b^2}{Z_{n\wedge\tau}(M-a(n\wedge\tau))}\right)\end{aligned}$$

Using the same arguments as before and that $Z_k \leq M - ak$, we have that there is a constant $C_3 = C_3(a, b)$ such that whenever $M - an \geq 2a + 2b$ we have

$$\mathbb{E}((K'_{n+1} - K'_n)^2|\mathcal{F}_n) \leq K'^2_n\frac{C_3}{Z_{n\wedge\tau}(M-a(n\wedge\tau))} \leq K'_n\frac{C_3M^{b/a}}{Z_0(M-an)^{b/a+1}}. \tag{2.6.5}$$

Define $p_n = \frac{C_2}{(M-an)^2}$, $q_{\ell,k} = \prod_{i=\ell}^k(1+p_i)$ for $\ell \leq k$ and $q_{k+1,k} = 1$ and $r_n = \frac{C_3M^{b/a}}{Z_0(M-an)^{b/a+1}}$. Since $q_{\ell,k} \leq \exp\Big(\sum_{i=\ell}^k p_i\Big)$ we see that there is a constant $C_4 = C_4(a, b)$ such that $q_{\ell,k} \leq C_4$ for any n, and ℓ and k such that $M - a\ell, M - ak > 2a + 2b$. Then we have

$$\sum_{k=0}^{n_{1,t}-1} r_kq_{0,k-1} \leq C_4\frac{C_3M^{b/a}}{Z_0}\sum_{k=0}^{n_{1,t}-1}\frac{1}{(M-ak)^{b/a+1}} \leq \frac{C_5M^{b/a}}{Z_0(M-an_{1,t})^{b/a}} \leq \frac{C_5}{t^{b/a}},$$

for some $C_5 = C_5(a,b)$ Defining $I_0 = K'_0 = 1$ and $I_n = K'_n - \sum_{k=0}^{n-1} \mathbb{E}(K'_{k+1} - K'_k|\mathcal{F}_k)$, as in Lemma 2.4.1 we have that

$$\mathbb{E}((I_{n_{1,t}} - 1)^2) \leq \frac{C_5}{t^{b/a}},$$

and by a Doob's maximal theorem we have that for any $\delta > 0$

$$\mathbb{P}(\max_{0 \leq n \leq n_{1,t}} |I_n - 1| \geq \delta) \leq \delta^{-2}\mathbb{E}(\max_{0 \leq n \leq n_{1,t}} |I_n - 1|^2) \leq \frac{2C_5}{t^{b/a}\delta^2}.$$

Next note that for any $0 \leq n \leq n_{1,t}$ we have

$$\sum_{k=0}^{n-1} p_k q_{k+1,n-1} \leq \sum_{k=0}^{n_{1,t}-1} \frac{C_4 C_2}{(M-an)^2} \leq \frac{C_6}{M - an_{1,t}} \leq \frac{C_6 Z_0^{a/b}}{Mt},$$

for some $C_6 = C_6(a,b)$. Conditioned on the event that $|I_n - 1| \leq \delta,$ for all $0 \leq n \leq n_{1,t}$ by Lemma 2.4.1 we have for all $0 \leq n \leq n_{1,t}$ that

$$|K'_n - 1| \leq \delta + (1+\delta)\sum_{k=0}^{n-1} p_k q_{k+1,n-1} \leq \delta + (1+\delta)\frac{C_6 Z_0^{a/b}}{Mt}.$$

Fix $\epsilon > 0$ and take $t \geq \frac{CZ_0^{a/b}}{M\varepsilon}$ for sufficiently large C, so that the right hand side above is equal to ϵ for some $\delta \geq \varepsilon/2$. Thus with probability of at least $1 - \frac{8C_5}{t^{b/a}\varepsilon^2}$ we have that $|K'_n - 1| \leq \varepsilon$ for all $0 \leq n \leq n_{1,t}$. The claim in (2.6.1) is trivial if $t^{b/a} \leq C\varepsilon^{-2}$. Surely by increasing C if necessary we can conclude that for any $\varepsilon < 1/2$ and $t^{b/a} > C\varepsilon^{-2}$ we have $2(a+b)t^{-b/a} < 1 - \varepsilon$. Since assuming $\tau < n_{1,t}$ implies

$$K'_{n_{1,t}} = K_\tau \leq \frac{2(a+b)}{Z_0\Big(1 - \frac{an_{1,t}}{M}\Big)^{b/a}} \leq \frac{2(a+b)}{t^{b/a}} < 1 - \varepsilon,$$

we conclude that the condition that $|K'_n - 1| \leq \varepsilon$ for all $0 \leq n \leq n_{1,t}$ implies $\tau \geq n_{1,t}$ and thus also $K'_n = K_n$ for all $0 \leq n \leq n_{1,t}$. Thus

$$\mathbb{P}(\mathbf{K}_{Z_0,M,t,\varepsilon}) \geq 1 - \frac{C}{t^{b/a}\varepsilon^2},$$

for C large enough, which proves the inequality in (2.6.1).

To prove that $\mathbb{P}(\sigma_{Z_0,M} \geq n_{2,t}) \leq \frac{C}{t^{b/(2b-a)}}$, first observe that the proportion of Z-type balls $Z_n/(M-an)$ is a supermartingale if $M - an > a$. To check this calculate

$$\mathbb{E}\Big(\frac{Z_{n+1}}{M-a(n+1)}\Big|\mathcal{F}_n\Big) - \frac{Z_n}{M-an} = \frac{Z_n}{M-a(n+1)} - \frac{b}{M-a(n+1)}\frac{Z_n}{M-an} - \frac{Z_n}{M-an}$$
$$= -\frac{(b-a)Z_n}{(M-an)(M-a(n+1))}.$$

This implies that for $n_1 < n_{2,t}$ we have

$$\mathbb{E}(Z_{n_{2,t}}|\mathcal{F}_{n_1}) \le \frac{M - an_{2,t}}{M - an_1} Z_{n_1}. \tag{2.6.6}$$

Next define $s = t^{a/(2b-a)}$. Since $t \ge 1$, we have $n_{2,t} \ge n_{1,s_1}$. Using (2.6.6) for $n_1 = n_{1,s}$, we see that conditioned on the event $\mathbf{K}_{Z_0,M,s,1/2}$ we have

$$\mathbb{E}(Z_{n_{2,t}}|\mathbf{K}_{Z_0,M,s,1/2}) \le \frac{3}{2}\Big(1 - \frac{an_{2,t}}{M}\Big) Z_0^{a/b}\Big(1 - \frac{an_{1,s}}{M}\Big)^{b/a-1} Z_0^{a/b-1} \le \frac{3}{2} t^{-1} s^{b/a-1} \le \frac{3}{2} t^{-b/(2b-a)}. \tag{2.6.7}$$

Since $s \ge 1 \ge 2CZ_0^{a/b}M^{-1}$ for M large enough, by (2.6.1) we have $\mathbb{P}(\mathbf{K}^c_{Z_0,M,s,1/2}) \le \frac{4C}{s^{b/a}}$ and (2.6.7) with Markov inequality implies

$$\mathbb{P}(Z_{n_{2,t}} > 0) \le \mathbb{E}(Z_{n_{2,t}}|\mathbf{K}_{Z_0,M,s,1/2}) + \mathbb{P}(\mathbf{K}^c_{Z_0,M,s,1/2}) \le \frac{C}{t^{b/(2b-a)}},$$

for large enough C, which proves (2.6.2) (note that the proof works if $M - an_{2,t} \ge a$, which is something we can assume since the last drawn ball can not be of Z-type).

b) We start by writing

$$L_n = \frac{(M - an) - Z_n}{(M - an) - Z_0\Big(1 - \frac{an}{M}\Big)^{b/a}},$$

from where we easily get

$$L_n - 1 = \frac{Z_0\Big(1 - \frac{an}{M}\Big)^{b/a} - Z_n}{(M - an) - Z_0\Big(1 - \frac{an}{M}\Big)^{b/a}}, \quad \text{and} \quad \frac{|K_n - 1|}{|L_n - 1|} = \frac{M}{Z_0\Big(1 - \frac{an}{M}\Big)^{b/a-1}} - 1. \tag{2.6.8}$$

First consider all n such that

$$\frac{\log M}{Z_0^{a/b}} \le 1 - \frac{an}{M} \le \Big(\frac{M}{2Z_0}\Big)^{a/(b-a)}. \tag{2.6.9}$$

From the second relation in (2.6.8) it is easy to check that the event $\mathbf{K}_{Z_0,M,\log M,\varepsilon}$ implies that $|L_n - 1| \le \varepsilon$ for all n satisfying (2.6.9). For a given $\varepsilon > 0$ choose M large enough so that $\log M \ge \frac{C}{M^{1-a/b}\varepsilon} \ge \frac{CZ_0^{a/b}}{M\varepsilon}$, where C is the constant from part a). Then by (2.6.1) we have that

$$\mathbb{P}(|L_n - 1| \le \varepsilon, \text{ for all } n \text{ satisfying (2.6.9)}) \ge 1 - \frac{C}{(\log M)^{b/a}\varepsilon^2}. \tag{2.6.10}$$

Recall the definition of $n_{1,t}$ from the statement. Now conditioned on the event $\mathbf{K}_{Z_0,M,\log M,\varepsilon}$ we have that

$$\begin{aligned}
Z_{n_{1,\log M}} &\leq (1+\varepsilon)Z_0\Big(1-\frac{a(n_{1,\log M}+1)}{M}+\frac{a}{M}\Big)^{b/a}\\
&\leq (1+\varepsilon)2^{b/a-1}Z_0\left(\Big(1-\frac{a(n_{1,\log M}+1)}{M}\Big)^{b/a}+\Big(\frac{a}{M}\Big)^{b/a}\right)\\
&\leq (1+\varepsilon)2^{b/a-1}(\log M)^{b/a}+(1+\varepsilon)2^{b/a-1}Z_0\Big(\frac{a}{M}\Big)^{b/a}\\
&\leq 2^{b/a}(\log M)^{b/a}.
\end{aligned}$$

Thus, conditioned on $\mathbf{K}_{Z_0,M,\log M,\varepsilon/2}$, we have that for all n satisfying

$$\frac{1}{Z_0^{a/b}\log M}\leq 1-\frac{an}{M}\leq\frac{\log M}{Z_0^{a/b}}, \tag{2.6.11}$$

holds $0\leq Z_n\leq(2\log M)^{b/a}$, and from the first relation in (2.6.8) we get

$$-\frac{(2\log M)^{b/a}}{\frac{M}{Z_0^{a/b}\log M}-(\log M)^{b/a}}\leq L_n-1\leq\frac{(\log M)^{b/a}}{\frac{M}{Z_0^{a/b}\log M}-(\log M)^{b/a}}.$$

The denominator of both the left and the right hand side above is bounded from below by $M^{1-a/b}/\log M-(\log M)^{b/a}$. Since $\lim_{M\to\infty}\mathbb{P}(\mathbf{K}_{Z_0,M,\log M,\varepsilon})=1$, we conclude that for any $\varepsilon>0$ with probability converging to 1 we have that $|L_n-1|\leq\varepsilon$, for all n satisfying (2.6.11).

Next consider n such that

$$0<1-\frac{an}{M}\leq\frac{1}{Z_0^{a/b}\log M}. \tag{2.6.12}$$

By (2.6.2), with probability of at least $1-\frac{C}{(\log M)^{b/(2b-a)}}$ we have that $Z_n=0$ and $S_n=M-an$ for all n satisfying (2.6.12). Therefore, with probability converging to 1, for all such n, we have that

$$1\leq L_n=\frac{1}{1-\frac{Z_0}{M}\Big(1-\frac{an}{M}\Big)^{b/a-1}}\leq\frac{1}{1-\frac{Z_0^{a/b}}{M(\log M)^{b/a-1}}}\leq\frac{1}{1-M^{-1+a/b}(\log M)^{-b/a+1}},$$

which converges to 1 as $M\to\infty$. The above arguments show that for a fixed $\varepsilon>0$ with probability converging to 1 (uniformly in Z_0) as $M\to\infty$, we have that $|L_n-1|\leq\varepsilon$ for all n such that $0<1-\frac{an}{M}\leq(M/2Z_0)^{a/(b-a)}$. If $Z_0\leq M/2$ there is nothing left to prove, so assume $Z_0\geq M/2$.

Now consider the case when

$$\Big(\frac{M}{2Z_0}\Big)^{a/(b-a)}\leq 1-\frac{an}{M}\leq 1-\frac{\log M}{\sqrt{M}}. \tag{2.6.13}$$

By the second relation in (2.6.8) the condition that

$$|K_n - 1| \le \frac{\varepsilon}{2}\Big(MZ_0^{-1}\Big(1 - \frac{\log M}{\sqrt{M}}\Big)^{-b/a+1} - 1\Big) \tag{2.6.14}$$

holds for all n satisfying (2.6.13), implies that $|L_n - 1| \le \varepsilon$ for all n satisfying (2.6.13). Denote the right hand side of (2.6.14) by δ_M. Clearly, for a positive constant $C'' = C''(a,b)$ we have $\delta_M \ge C''M^{-1/2}\log M$. To calculate the probability of (2.6.14) holding for all n satisfying (2.6.13), we can apply (2.6.1) with $t = (MZ_0^{-a/b}/2)^{a/(b-a)}$ and $\varepsilon = \delta_M$. To justify this application we need to check that $\delta_M \le 1/2$, for M large enough. This follows easily from the fact that $\epsilon \le 1/2$ and $Z_0 \ge M/2$. Furthermore, using the fact that $\delta_M \ge C''M^{-1/2}\log M$, we have for M large enough $t \ge \frac{CZ_0^{a/b}}{M\delta_M}$. Since for the above t all $n_{1,t}$ satisfy (2.6.13), we get

$$\begin{aligned}\mathbb{P}(|L_n - 1| \le \varepsilon \text{ for all } n \text{ satisfying (2.6.13)}) &\ge 1 - \frac{C}{t^{b/a}\delta_M^2} \\ &\ge 1 - \frac{2^{b/(b-a)}CZ_0^{a/(b-a)}M}{C''^2M^{b/(b-a)}(\log M)^2} \ge 1 - \frac{2^{b/(b-a)}C}{C''^2(\log M)^2}.\end{aligned} \tag{2.6.15}$$

Since the right hand side converges to 1, we are left to consider the case when $0 \le n \le (\sqrt{M}\log M)/a$. To this end write

$$L_n = \frac{S_n}{\Big(1 - \frac{an}{M}\Big)\Big(S_0\Big(1 - \frac{an}{M}\Big)^{b/a-1} + M\Big(1 - \Big(1 - \frac{an}{M}\Big)^{b/a-1}\Big)\Big)}.$$

Consider the case when $S_0 \ge \sqrt{M}(\log M)^2$ and so $S_0(1-\varepsilon/2) \le S_n \le S_0(1+\varepsilon/2)$, for M large enough and all n such that $0 \le n \le (\sqrt{M}\log M)/a$. Furthermore, for such n the denominator satisfies

$$\begin{aligned}\Big(1 - \frac{\log M}{\sqrt{M}}\Big)^{b/a} S_0 \le \Big(1 - \frac{an}{M}\Big)\Big(S_0\Big(1 - \frac{an}{M}\Big)^{b/a-1} + M\Big(1 - \Big(1 - \frac{an}{M}\Big)^{b/a-1}\Big)\Big) \\ \le S_0 + M\frac{(b-a)n}{M} \le S_0 + \frac{b-a}{a}\sqrt{M}\log M,\end{aligned}$$

and so it is bounded by $S_0(1-\varepsilon/2)$ from below and by $S_0(1+\varepsilon/2)$ from above, for M large enough. This now implies the deterministic fact that $|L_n - 1| \le \varepsilon$ for all $0 \le n \le (\sqrt{M}\log M)/a$.

We are left to consider $S_0 \le \sqrt{M}(\log M)^2$, so from now on we assume this. The last two cases we consider are

$$0 \le n \le \sqrt{M}/(\log M)^3, \tag{2.6.16}$$

and

$$\sqrt{M}/(\log M)^3 \le n \le (\sqrt{M}\log M)/a. \tag{2.6.17}$$

Clearly, for M large enough, and all n which satisfy either (2.6.16) or (2.6.17) we have $S_n \leq 2\sqrt{M}(\log M)^2$. Thus the probability of drawing an S-type ball at the nth step for n satisfying (2.6.16) is bounded from above by $3(\log M)^2/\sqrt{M}$ and the expectation of the total number of S-type balls drawn in this time interval is less than $3/\log M$. By Markov inequality, with probability converging to 1, in this time interval we will draw only Z-type balls which implies that with probability converging to 1 as $M \to \infty$ we have for all n satisfying (2.6.16)

$$L_n = \frac{S_0 + (b-a)n}{\left(1 - \frac{an}{M}\right)\left(S_0\left(1 - \frac{an}{M}\right)^{b/a-1} + M\left(1 - \left(1 - \frac{an}{M}\right)^{b/a-1}\right)\right)}.$$

For M large enough and for all n satisfying (2.6.16) or (2.6.17) we have

$$\frac{1}{1+\varepsilon/2} \leq \left(1 - \frac{an}{M}\right)^{b/a} \leq \frac{1}{1-\varepsilon/2}, \text{ and } \frac{(b-a)n}{1+\varepsilon/2} \leq M\left(1 - \left(1 - \frac{an}{M}\right)^{b/a-1}\right) \leq \frac{(b-a)n}{1-\varepsilon/2}. \tag{2.6.18}$$

The last inequality follows from the fact that

$$\left|1 - \left(1 - \frac{an}{M}\right)^{b/a-1} - \frac{(b-a)n}{M}\right| \leq \frac{C'n^2}{M^2} \leq \frac{C' \log M}{a\sqrt{M}} \frac{n}{M},$$

for some constant $C' > 0$. From (2.6.18) it is clear that $|L_n - 1| \leq \varepsilon$ for all n satisfying (2.6.16).

Using similar arguments as above we see that the expected number of S-type balls drawn for ns satisfying (2.6.17) is no more than $3(\log M)^3$ for large M. Thus with high probability we have

$$S_0 + (b-a)n - (\log M)^4 \leq S_n \leq S_0 + (b-a)n.$$

The upper bound on $L_n - 1$ follows from the arguments above and for the lower bound we only need to modify the second inequality in (2.6.18) as

$$\frac{(b-a)n - (\log M)^4}{1+\varepsilon/2} \leq M\left(1 - \left(1 - \frac{an}{M}\right)^{b/a-1}\right) \leq \frac{(b-a)n - (\log M)^4}{1-\varepsilon/2},$$

which holds for M large enough and all n satisfying (2.6.17).

□

Chapter 3

Tug-of-War

3.1 Statements of results

Setting and notation

All graphs that we consider in this chapter will be connected and of finite diameter (in graph metric), but we allow graphs to have an uncountable number of vertices (and vertices with uncountable degrees). However, for most of the main results we will need additional assumptions.

Definition 3.1.1. Let (V, d) be a compact length space, that is, a compact metric space such that for all $x, y \in V$ the distance $d(x, y)$ is the infimum of the lengths of rectifiable paths between x and y. For a fixed $\epsilon > 0$, the *ϵ-adjacency graph* is defined as a graph with the vertex set V, such that two vertices x and y are connected if and only if $d(x, y) \leq \epsilon$. When the value of ϵ is not important we will simply use the term adjacency graph.

A particular example of ϵ-adjacency graphs, already described in the previous subsection, corresponds to taking (V, d) to be a closure of a domain $\Omega \subset \mathbb{R}^d$ with C^1 boundary $\partial\Omega$ and the intrinsic metric in $\overline{\Omega}$. This means that $d(x, y)$ is equal to the infimum of the lengths of rectifiable paths contained in $\overline{\Omega}$, between points x and y. We will call these graphs *Euclidean ϵ-adjacency graphs*. Note that we will limit our attention to domains with C^1 boundary (meaning that for any $x \in \partial\Omega$ we can find open sets $U \subset \mathbb{R}^{n-1}$ and $V \subset \mathbb{R}^n$ containing 0 and x respectively, a C^1 function $\phi\colon U \to \mathbb{R}^{n-1}$ and an isometry of $\mathbb{R}^n$ which maps $(0, \phi(0))$ to x and the graph of ϕ onto $V \cap \partial\Omega$).

While the fact that $(\overline{\Omega}, d)$ is a metric space is fairly standard, at this point we need to argue that this space is compact. Actually one can see that the topology induced by this metric space is the same as the Euclidean topology, and we only need to show that it is finer. To end this assume that (x_n) is a sequence of points such that $\lim_n |x_n - x| = 0$, for some $x \in \overline{\Omega}$. If $x \in \Omega$ it is clear that $\lim_n d(x_n, x) = 0$. On the other hand if $x \in \partial\Omega$ then let y_n be the closest point on $\partial\Omega$ to x_n. It is clear that $d(x_n, y_n) = |x_n - y_n|$ converges to zero as $n \to \infty$ and $\lim_n |x - y_n| = 0$. To complete the argument simply consider the paths between

x and y_n contained in $\partial\Omega$, which are obtained by composing a C^1 parametrization of $\partial\Omega$ in a neighborhood of x and an affine function. The lengths of these paths converge to zero.

Another interesting class of graphs will be finite graphs with loops at each vertex. Intuitively, these graphs might be thought of as ϵ-adjacency graphs where (V, d) is a finite metric space with integer valued lengths and $\epsilon = 1$.

Note that for two vertices x and y we write $x \sim y$ if x and y are connected with an edge. The graph metric between vertices x and y will be denoted by $\mathrm{dist}(x, y)$ in order to distinguish it from the metric in Definition 3.1.1. The diameter of a graph G will be denoted by $\mathrm{diam}(G)$. We consider the supremum norm on the space of functions on V, that is $\|u\| = \max_x |u(x)|$. When V is compact length space, this norm makes $C(V, \mathbb{R})$, the space of continuous functions on V, a Banach space. We also use the notation $B(x, \epsilon) = \{y \in V : d(x, y) \le \epsilon\}$ (we want to emphasize that, in contrast with [52], the balls $B(x, \epsilon)$ are defined to be closed) .

We consider a version of the tug-of-war game in which the terminal set is empty, but in which the game is stopped after n steps. We say that this game has *horizon n*. At each step, if the token is at the vertex x Player II pays Player I value $f(x)$. If the final position of the token is a vertex y then at the end Player II pays Player I value $g(y)$. Here f and g are real bounded functions on the set of vertices called the running and the terminal payoff. Actually this game can be realized as the original stochastic tug-of-war game introduced in [52] played on the graph $G \times \{1, 2, \dots, n\}$ (the edges connecting vertices of the form (v, i) and $(w, i+1)$, where v and w are neighbors in G), for the running payoff $f(v, i) := f(v)$, the terminal set $V \times \{n\}$ and the terminal payoff $g(v, n) := g(v)$.

Define strategy of a player to be a function that, for each $1 \le k \le n$, at the k-th step maps the previous k positions and k coin tosses to a vertex of the graph which neighbors the current position of the token. For a Player I strategy $\mathcal{S}_{\mathrm{I}}$ and Player II strategy $\mathcal{S}_{\mathrm{II}}$ define $F_n(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})$ as the expected payoff in the game of horizon n, when Players I and II play according to strategies $\mathcal{S}_{\mathrm{I}}$ and $\mathcal{S}_{\mathrm{II}}$ respectively. Define the value for Player I as $u_{\mathrm{I},n} = \sup_{\mathcal{S}_{\mathrm{I}}} \inf_{\mathcal{S}_{\mathrm{II}}} F_n(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})$, and the value for Player II as $u_{\mathrm{II},n} = \inf_{\mathcal{S}_{\mathrm{II}}} \sup_{\mathcal{S}_{\mathrm{I}}} F_n(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})$. Note that we consider both $u_{\mathrm{I},n}$ and $u_{\mathrm{II},n}$ as functions of the initial position of the token. Intuitively $u_{\mathrm{I},n}(x)$ is the supremum of the values that Player I can ensure to earn and $u_{\mathrm{II},n}(x)$ is the infimum of the values Player II can ensure not to overpay, both in the game of horizon n that starts from $x \in V$. It is clear that $u_{\mathrm{I},0} = u_{\mathrm{II},0} = g$ and one can easily check that $u_{\mathrm{I},n} \le u_{\mathrm{II},n}$. In a game of horizon $n+1$ that starts at x, if Players I and II play according to the strategies $\mathcal{S}_{\mathrm{I}}$ and $\mathcal{S}_{\mathrm{II}}$ which, in the first step, push the token to x_{I} and x_{II} respectively, we have

$$F_{n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})(x) = f(x) + \frac{1}{2}\Big(F_{2,n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})(x_{\mathrm{I}}) + F_{2,n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})(x_{\mathrm{II}})\Big), \tag{3.1.1}$$

where $F_{2,n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})(y)$ is the expected payoff between steps 2 and $n+1$ conditioned on having position y in the second step of the game. It is easy to see that using the above notation $u_{\mathrm{I},n} = \sup_{\mathcal{S}_{\mathrm{I}}} \inf_{\mathcal{S}_{\mathrm{II}}} F_{2,n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})$ and $u_{\mathrm{II},n} = \inf_{\mathcal{S}_{\mathrm{II}}} \sup_{\mathcal{S}_{\mathrm{I}}} F_{2,n+1}(\mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}})$, where $\mathcal{S}_{\mathrm{I}}$ and $\mathcal{S}_{\mathrm{II}}$ are strategies for Player I and Player II in the game that lasts for $n+1$ steps. Now using induction in n and (3.1.1) one can check that $u_n := u_{\mathrm{I},n} = u_{\mathrm{II},n}$ for any n, and that the sequence (u_n) satisfies $u_0 = g$ and

$$u_{n+1}(x) = \frac{1}{2}\Big(\min_{y\sim x} u_n(y) + \max_{y\sim x} u_n(y)\Big) + f(x). \tag{3.1.2}$$

Furthermore the infima and the suprema in the definitions of $u_{\mathrm{I},n}$ and $u_{\mathrm{II},n}$ are achieved for the strategies that at step k of the game of horizon n pull the token to a neighbor that maximizes (minimizes) the value of u_{n-k} (such a neighbor exists for finite degree graphs, and also in the case of an ϵ-adjacency graph provided u_{n-k} is known a priori to be continuous).

In this paper we will mainly study the described game through the recursion (3.1.2).

Remark 3.1.2. In the case of ϵ-adjacency graphs we will normally assume that the terminal and the running payoff are continuous functions on V and heavily use the fact that the game values u_n are continuous functions. To justify this it is enough to show that, if u is a continuous function on V, then so are $\overline{u}^\epsilon(x) = \max_{B(x,\epsilon)} u$ and $\underline{u}_\epsilon(x) = \min_{B(x,\epsilon)} u$. For this, one only needs to observe that for any two points $x, y \in V$ such that $d(x, y) < \delta$, any point in $B(x, \epsilon)$ is within distance of δ from some point in $B(y, \epsilon)$, and vice versa. Now the (uniform) continuity of $\overline{u}^\epsilon$ and $\underline{u}_\epsilon$ follows from the uniform continuity of u, which holds by compactness of V.

For a game played on an arbitrary connected graph of finite diameter, if the sequence of game values (u_n) converges pointwise, the limit u is a solution to the equation

$$u(x) - \frac{1}{2}(\min_{y\sim x} u(y) + \max_{y\sim x} u(y)) = f(x). \tag{3.1.3}$$

The *discrete infinity Laplacian* $\Delta_\infty u$ is defined at a vertex x as the negative left hand side of the above equation. As mentioned before, the case of Euclidean ϵ-adjacency graphs is interesting because of the connection between the game values and the viscosity solutions of (3.1.5) defined in Definition 3.1.4. To observe this it is necessary to scale the payoff function by the factor of ϵ^2. Therefore in the case of ϵ-adjacency graphs we define the *ϵ-discrete Laplace operator* as

$$\Delta_\infty^\epsilon u(x) = \frac{(\min_{y\in B(x,\epsilon)} u(y) + \max_{y\in B(x,\epsilon)} u(y)) - 2u(x)}{\epsilon^2},$$

and consider the equation

$$-\Delta_\infty^\epsilon u = f. \tag{3.1.4}$$

Remark 3.1.3. Observe that, compared to the discrete infinity Laplacian, we removed a factor 2 from the denominator. This definition is more natural when considering the infinity Laplacian Δ_∞ described below. As a consequence we have that the pointwise limit u of the game values $(u_n)_n$ played on an ϵ-adjacency graph with the payoff function $\epsilon^2 f/2$ is a solution to (3.1.4).

We will consider the infinity Laplace equation on a connected domain Ω with C^1 boundary $\partial\Omega$, with vanishing Neumann boundary conditions

$$\begin{cases} -\Delta_\infty u = f & \text{in } \Omega, \\ \nabla_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.5}$$

Here ν (or more precisely $\nu(x)$) denotes the normal vector to $\partial\Omega$ at a point $x \in \partial\Omega$. The *infinity Laplacian* Δ_∞ is formally defined as the second derivative in the gradient direction, that is

$$\Delta_\infty u = |\nabla u|^{-2} \sum_{i,j} u_{x_i} u_{x_i x_j} u_{x_j}. \tag{3.1.6}$$

We will define the solutions of (3.1.5) and prove the existence in the following viscosity sense. First define the operators Δ_∞^+ and Δ_∞^- as follows. For a twice differentiable function u and a point x such that $\nabla u(x) \neq 0$ define $\Delta_\infty^+ u$ and $\Delta_\infty^- u$ to be given by (3.1.6), that is $\Delta_\infty^+ u(x) = \Delta_\infty^- u(x) = \Delta_\infty u(x)$. For x such that $\nabla u(x) = 0$ define $\Delta_\infty^+ u(x) = \max\{\sum_{i,j} u_{x_i x_j}(x) \mathbf{v}_i \mathbf{v}_j\}$ and $\Delta_\infty^- u(x) = \min\{\sum_{i,j} u_{x_i x_j}(x) \mathbf{v}_i \mathbf{v}_j\}$, where the maximum and the minimum are taken over all vectors $\mathbf{v} = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$ of Euclidean norm 1.

Definition 3.1.4. A continuous function $u \colon \overline{\Omega} \to \mathbb{R}$ is said to be a *(viscosity) subsolution* to (3.1.5) if for any function $\varphi \in C^\infty(\overline{\Omega})$ (infinitely differentiable function on an open set containing $\overline{\Omega}$) and a point $x_0 \in \overline{\Omega}$, such that $u - \varphi$ has a strict local maximum at x_0, we have either

a) $-\Delta_\infty^+ \varphi(x_0) \leq f(x_0)$ or

b) $x_0 \in \partial\Omega$ and $\nabla_{\nu(x_0)} \varphi(x_0) \leq 0$.

A continuous function $u \colon \overline{\Omega} \to \mathbb{R}$ is said to be a *(viscosity) supersolution* if $-u$ is a (viscosity) subsolution when f is replaced by $-f$ in (3.1.5). A continuous function u is said to be a *(viscosity) solution* to (3.1.5) if it is both a subsolution and a supersolution.

Note that the notion of (sub, super)solutions does not change if one replaces the condition $\varphi \in C^\infty(\overline{\Omega})$ with $\varphi \in C^2(\overline{\Omega})$.

Remark 3.1.5. The above definition, while having the advantage of being closed under taking limits of sequences of solutions, might be slightly unnatural because the condition in a) is sufficient for $x_0 \in \partial\Omega$ at which $u - \varphi$ has a strict local maximum. Following [19] we can define the *strong (viscosity) subsolution* as a continuous function u such that, for any $\varphi \in C^\infty(\overline{\Omega})$ and any $x_0 \in \overline{\Omega}$, at which $u - \varphi$ has a strict local maximum, we have

a') $-\Delta_\infty^+ \varphi(x_0) \leq f(x_0)$, if $x_0 \in \Omega$,

b') $\nabla_\nu \varphi(x_0) \leq 0$, if $x_0 \in \partial\Omega$.

Strong (viscosity) supersolutions and solutions are defined analogously. While it is clear that the requirements in this definition are stronger than those in Definition 3.1.4, it can be shown that, when Ω is a convex domain, any (sub, super)solution is also a strong (sub, super)solution. To show this assume that u is a viscosity subsolution to (3.1.6) in the sense of Definition 3.1.4 and let $x \in \partial\Omega$ and $\varphi \in C^\infty(\overline{\Omega})$ be such that $u - \varphi$ has a strict local maximum at x and $\nabla_\nu \varphi(x) > 0$. Without the loss of generality we can assume that $x = 0$ and that the normal vector to $\partial\Omega$ at 0 is $\nu = -\mathbf{e}_d$, where $\mathbf{e}_d$ is the d-th coordinate vector. Thus by the convexity, the domain Ω lies above the coordinate plane $x_d = 0$. Now define the function $\phi \in C^\infty(\overline{\Omega})$ as $\phi(y) = \varphi(y) + \alpha y_d - \beta(y_d)^2$, for positive α and β. Since $\nabla\phi(0) = \nabla\varphi(0) + \alpha\mathbf{e}_d$, for α small enough we still have $\nabla_\nu\phi(0) > 0$. Moreover the Hessian matrix of ϕ is the same as that of φ, with the exception of the (d, d)-entry which is decreased by 2β. Since $\nabla_\nu\phi(0) > 0$ and $\nabla_\nu\phi(0)$ does not depend on β, for β large enough we will have $-\Delta_\infty^+\phi(0) > f(0)$. Moreover since we can find an open set U such that $\varphi(y) \leq \phi(y)$ for all $y \in \overline{\Omega} \cap U$, we have that $u - \phi$ has again a strict local maximum at 0. Since it doesn't satisfy the conditions in Definition 3.1.4, this leads to a contradiction.

Statements of results

We want to study the values of games as their horizons tend to infinity. Clearly taking payoff function f to be of constant sign will make the game values diverge. Since increasing the payoff function by a constant c results in the increase of the value of the game of horizon n by nc, the most we can expect is that we can find a (necessarily unique) shift $f + c$ of the payoff function f for which the game values behave nicely. The first result in this direction is the following theorem which holds for all connected graphs of finite diameter.

Theorem 3.1.6. *For any connected graph $G = (V, E)$ of finite diameter and any bounded function $f : V \to \mathbb{R}$ there is a constant c_f, such that the following holds: For any bounded function $u_0 : V \to \mathbb{R}$, if (u_n) is the sequence of game values with the terminal and running payoffs u_0 and f respectively, then the sequence of functions $(u_n - nc_f)$ is bounded.*

We will call c_f from Theorem 3.1.6 the *Player I's long term advantage* for function f. For both adjacency graphs and finite graphs with loops we have convergence of the game values.

Theorem 3.1.7. *Let $G = (V, E)$ be either an adjacency graph, or a finite graph with a loop at each vertex. Let $f, u_0 : V \to \mathbb{R}$ be functions on the set of vertices, which are assumed to be continuous if G is an adjacency graph. Assume $c_f = 0$. In a game played on G with the terminal and running payoffs u_0 and f respectively, the sequence of game values (u_n) is uniformly convergent.*

The following theorem gives the correspondence between the tug-of-war games and the equation (3.1.3). While for adjacency graphs and finite graphs with loops this is a straightforward corollary of Theorem 3.1.7, this result also holds for all finite graphs, even when Theorem 3.1.7 may fail to hold (see Example 3.2.9).

Theorem 3.1.8. *Let $G = (V, E)$ be either an adjacency graph, or a finite graph. Let $f, u_0 \colon V \to \mathbb{R}$ be functions on the set of vertices, which are assumed to be continuous if G is an adjacency graph. Then the equation* (3.1.3) *has a solution u if and only if $c_f = 0$.*

Let V be a compact length space and a consider a function $f \in C(V)$. To emphasize the dependence on ϵ define $c_f(\epsilon)$ as the Player I's long term advantage for a game played on the ϵ-adjacency graph defined on V with the running payoff f. As already mentioned, to study the limiting case $\epsilon \downarrow 0$ for ϵ-adjacency graphs, we need to scale the running payoff function by a factor of ϵ^2, that is, the we take $\epsilon^2 f$ as the running payoff function. Note that the Player I's long term advantage corresponding to this game is equal to $\epsilon^2 c_f(\epsilon)$.

The first problem one encounters is the fact that $c_f(\epsilon)$ depends on the value of ϵ (see Example 3.3.1). The following theorem gives meaning to the notion of Player I's long term advantage in the continuous case.

Theorem 3.1.9. *For any compact length space V and any continuous function $f \colon V \to \mathbb{R}$ the limit $\lim_{\epsilon\downarrow 0} c_f(\epsilon)$ exists.*

We will denote the limit from the above theorem by $\bar{c}_f = \lim_{\epsilon\downarrow 0} c_f(\epsilon)$.

Theorem 3.1.10. *Let V be a compact length space, and (ϵ_n) a sequence of positive real numbers converging to zero. Any sequence (u_n) of continuous functions on V satisfying $-\Delta_\infty^{\epsilon_n} u_n = f - c_f(\epsilon_n)$ and such that 0 is in the range of u_n for all n, has a subsequence converging to a Lipshitz continuous function. Moreover the Lipshitz constant is bounded by a universal constant multiple of $\operatorname{diam}(V)\|f\|$.*

For Euclidean ϵ-adjacency graphs, the limits from Theorem 3.1.10 give us viscosity solutions of (3.1.5).

Theorem 3.1.11. *Let Ω be a domain of finite diameter with C^1 boundary $\partial\Omega$ and $f \colon \overline{\Omega} \to \mathbb{R}$ a continuous function, such that $\bar{c}_f = 0$. Then the equation* (3.1.5) *has a viscosity solution u which is Lipshitz continuous, with Lipshitz constant depending on Ω and the norm $\|f\|$.*

It is natural to expect the existence of viscosity solutions to (3.1.5) only for one shift of the function f. This is proven in the following theorem for convex domains Ω.

Theorem 3.1.12. *Let Ω be a convex domain of finite diameter with C^1 boundary $\partial\Omega$ and $f \colon \overline{\Omega} \to \mathbb{R}$ a continuous function. Then the equation* (3.1.5) *has a viscosity solution u if and only if $\bar{c}_f = 0$.*

Remark 3.1.13. Directly from Theorem 3.1.6 one can deduce that $c_{\lambda f} = \lambda c_f$ and $c_{f+\lambda} = c_f + \lambda$ for any $\lambda \in \mathbb{R}$. For compact length spaces, after taking an appropriate limit, we obtain the same properties for $\bar{c}_f$. Thus Theorem 3.1.8 tells us that, under its assumptions, for any function g on the vertex set (continuous in the case of adjacency graphs), there is a unique constant c, such that equation (3.1.3) can be solved when $f = g - c$. Theorems 3.1.11 and 3.1.12 tell us that any function $g \in C(\overline{\Omega})$ can be shifted to obtain a function $f \in C(\overline{\Omega})$ for which (3.1.5) can be solved, and that this shift is unique when Ω is convex.

Remark 3.1.14. In the case of finite graphs and for a fixed f (such that $c_f = 0$), the solutions to the equation (3.1.3) are not necessarily unique (even in the case of finite graphs with self loops). A counterexample and a discussion about the continuous case is given in Section 3.4.

3.2 The discrete case

Since the terminal payoff can be understood as the value of the game of horizon 0, we will not explicitly mention the terminal payoff when it is clear from the context.

Lemma 3.2.1. *Let $G = (V, E)$ be a connected graph of finite diameter and let f, g, u_0 and v_0 be bounded functions on V.*

(i) Let (u_n) and (v_n) be the sequences of values of games played with the running payoff f. If $u_0 \leq v_0$, then $u_n \leq v_n$ for all $n > 0$. Furthermore, if for some $c \in \mathbb{R}$ we have $v_0 = u_0 + c$, then $v_n = u_n + c$ for all $n > 0$.

(ii) Let (u_n^1) and (u_n^2) be sequences of values of games played with the terminal payoffs $u_0^1 = u_0^2 = u_0$ and the running payoffs f and g respectively. If $f \leq g$ then $u_n^1 \leq u_n^2$, for all n. Furthermore if for some $c \in \mathbb{R}$ we have $g = f + c$, then $u_n^2 = u_n^1 + nc$ for all $n > 0$.

Proof. All statements are easy to verify by induction on n using relation (3.1.2). □

Lemma 3.2.2. *For a connected graph $G = (V, E)$ of finite diameter and bounded functions $f, u_0 \colon V \to \mathbb{R}$, let $(u_n)_n$ be the sequence of values of games played on G with running payoff f. Then for all $n \geq 0$ we have*

$$\max u_n - \min u_n \leq (\max u_0 - \min u_0) + \operatorname{diam}(G)^2(\max f - \min f).$$

Proof. Consider the sequence of game values (v_n) played with the running payoff f and zero terminal payoff. From part (i) of Lemma 3.2.1 we get $v_n + \min u_0 \leq u_n \leq v_n + \max u_0$. This implies that

$$\max u_n - \min u_n \leq (\max v_n - \min v_n) + (\max u_0 - \min u_0).$$

From this it's clear that it is enough to prove the claim when $u_0 = 0$. Furthermore, by part (ii) of Lemma 3.2.1 it is enough to prove the claim for an arbitrary shift of the payoff function f, and therefore we assume that $\min f = 0$. This implies that $u_n \geq 0$, for $n \geq 0$. Now, for a fixed n, by Lemma 3.2.1 (i), playing the game of horizon $n - k$ with the running payoff f and the terminal payoffs u_k and 0, gives the game values u_n and u_{n-k} respectively, and

$$u_{n-k} \leq u_n. \tag{3.2.1}$$

Fix a vertex $z \in V$. For a vertex $y \in V$ pick a neighbor $z(y) \in V$ of y so that $\operatorname{dist}(z(y), z) = \operatorname{dist}(y, z) - 1$ (if y and z are neighbors then clearly $z(y) = z$). Let $\mathcal{S}^0_{\mathrm{II},k}$

be the optimal strategy for Player II in the game of horizon k. For any Player I strategy $\mathcal{S}_{\mathrm{I},k}$ for a game of horizon k, we have $F_k(\mathcal{S}_{\mathrm{I},k}, \mathcal{S}^0_{\mathrm{II},k}) \le u_k$. Now define the "pull towards z" strategy $\mathcal{S}_{\mathrm{II}}$ for a game of length n as follows. At any step of the game if the token is at the vertex $y \neq z$ and if z is not among the past positions of the token, then strategy $\mathcal{S}_{\mathrm{II}}$ takes the token to the vertex $z(y)$. If T is the first time at which the token is at the vertex z, at this point Player II starts playing using the strategy $\mathcal{S}^0_{\mathrm{II},n-T}$. If X_t is the position of the token at time t, then it can be easily checked that for $Y_t = (\mathrm{diam}(G) - \mathrm{dist}(X_t, z))^2 - t$, the process $Y_{t\wedge T}$ is a submartingale, with uniformly bounded differences. Moreover the stopping time T has a finite expectation since it is bounded from above by the first time that Player II has won $\mathrm{diam}(G)$ consecutive coin tosses (partition coin tosses into consecutive blocks of length $\mathrm{diam}(G)$ and notice that the order of the first block in which Player II wins all the coin tosses has exponential distribution with mean $2^{\mathrm{diam}(G)}$). Therefore applying the optional stopping theorem we get $\mathbb{E}(Y_T) \ge \mathbb{E}(Y_0)$, hence $\mathbb{E}(T) \le \mathrm{diam}(G)^2$. Now consider the game in which Player I plays optimally and Player II plays according to the above defined strategy $\mathcal{S}_{\mathrm{II}}$. Since each move in the optimal strategies depends only on the current position of the token, by the independence of the coin tosses, we have that conditioned on $T = k$, the expected payoff in steps $k+1$ to n is bounded from above by $u_{n-k}(z)$. Clearly the total payoff in the first k steps is bounded from above by $k \max f$. For $x \neq z$ the strategy $\mathcal{S}^0_{\mathrm{II}}$ is suboptimal and

$$u_n(x) \le \sum_{k=1}^{n} \mathbb{P}(T = k)(k \max f + u_{n-k}(z)) + \mathbb{P}(T \ge n) n \max f.$$

Since f is a non-negative function, so is u_n for any n. Using this with (3.2.1) we get

$$\begin{aligned} u_n(x) \le \sum_{k=1}^{n} \mathbb{P}(T = k) k \max f + u_n(z) + \mathbb{P}(T \ge n) n \max f \\ \le u_n(z) + \mathbb{E}(T) \max f. \end{aligned}$$

Since x and z are arbitrary and $\mathbb{E}(T) \le \mathrm{diam}(G)^2$ for all x and z, the claim follows. □

Proof of Theorem 3.1.6. As in the proof of Lemma 3.2.2, we can assume that $u_0 = 0$. Denote $M_k = \max u_k$ and $m_k = \min u_k$. By part (i) of Lemma 3.2.1, playing the game with the constant terminal payoff M_k gives the sequence of game values $(u_n + M_k)_n$. Comparing this to the game with the terminal payoff u_k we obtain $u_{n+k}(x) \le u_n(x) + M_k$. Taking maximum over all vertices x leads to the subaditivity of the sequence (M_n), that is $M_{n+k} \le M_n + M_k$. In the same way we can prove that the sequence (m_n) is superaditive. By Lemma 3.2.2 we can find a constant C so that $M_n - m_n \le C$ for any n, and thus we can define

$$c_f := \lim_n \frac{M_n}{n} = \inf_n \frac{M_n}{n} = \lim_n \frac{m_n}{n} = \sup_n \frac{m_n}{n}. \tag{3.2.2}$$

Then, for any $n \ge 0$ we have

$$nc_f \le M_n \le m_n + C \le nc_f + C,$$

and therefore, for any $x \in V$

$$|u_n(x) - nc_f| \leq \max\{|M_n - nc_f|, |m_n - nc_f|\} \leq C.$$

□

For an arbitrary graph $G = (V, E)$ and a function f on V, define the (non-linear) operator A_f acting on the space of functions on V, so that for each $x \in V$

$$A_f u(x) = \frac{1}{2}\Big(\max_{y \sim x} u(y) + \min_{y \sim x} u(y) \Big) + f(x).$$

Lemma 3.2.3. *Assume $G = (V, E)$ is either an adjacency graph or a finite graph. Let f, u and v be functions on V, which are also assumed to be continuous if G is an adjacency graph. Then we have*

$$\min(v - u) \leq \min(A_f v - A_f u) \leq \max(A_f v - A_f u) \leq \max(v - u), \tag{3.2.3}$$

and

$$\|A_f v - A_f u\| \leq \|v - u\|. \tag{3.2.4}$$

Moreover for $x \in V$ we have $A_f v(x) - A_f u(x) = \max(v - u)$ if and only if for any two neighbors y_1 and y_2 of x such that $u(y_1) = \min_{y \sim x} u(y)$, and $v(y_2) = \max_{y \sim x} v(y)$, we also have $v(y_1) = \min_{y \sim x} v(y)$, and $u(y_2) = \max_{y \sim x} u(y)$ and $v(y_i) - u(y_i) = \max(v - u)$, for $i \in \{1, 2\}$.

Proof. Fix a vertex $x \in V$ and note that $\max_{y \sim x} v(y) \leq \max_{y \sim x} u(y) + \max(v - u)$ and $\min_{y \sim x} v(y) \leq \min_{y \sim x} u(y) + \max(v - u)$. Adding these inequalities one obtains $A_f v(x) \leq A_f u(x) + \max(v - u)$. The inequality $\min(v - u) \leq \min(A_f v - A_f u)$ now follows by replacing u and v by $-u$ and $-v$ respectively, and (3.2.4) follows directly from (3.2.3). It is clear that the equality $A_f v(x) - A_f u(x) = \max(v - u)$ holds if and only if both

$$\max_{y \sim x} v(y) = \max_{y \sim x} u(y) + \max(v - u) \tag{3.2.5}$$

and

$$\min_{y \sim x} v(y) = \min_{y \sim x} u(y) + \max(v - u) \tag{3.2.6}$$

hold. It is obvious that the conditions in the statement are sufficient for (3.2.5) and (3.2.6) to hold, and it is only left to be proven that these conditions are also necessary. To end this assume that both (3.2.5) and (3.2.6) hold and take y_1 and y_2 to be arbitrary neighbors of x such that $u(y_1) = \min_{y \sim x} u(y)$ and $v(y_2) = \max_{y \sim x} v(y)$. Clearly we have

$$\min_{y \sim x} v(y) \leq v(y_1) \leq u(y_1) + \max(v - u) = \min_{y \sim x} u(y) + \max(v - u),$$

and moreover all the inequalities in the above expression must be equalities. This implies both $v(y_1) = u(y_1) + \max(v - u)$ and $v(y_1) = \min_{y \sim x} v(y)$. The claim for y_2 can be checked similarly. □

The following proposition proves Theorem 3.1.7 in the case of finite graphs with loops. For a sequence of game values $(u_n)_n$ with the running payoff f, define $M_n^f(u_0) = \max_{x\in V}(u_n(x) - u_{n-1}(x))$ and $m_n^f(u_0) = \min_{x\in V}(u_n(x) - u_{n-1}(x))$.

Proposition 3.2.4. *Under the assumptions of Theorem 3.1.7 the sequence of game values converges if it has a convergent subsequence.*

Proof. If $M_n^f(u_0) = -\delta < 0$ for some $n \geq 1$, we have $u_n \leq u_{n-1} - \delta$ and by applying part (i) of Lemma 3.2.1 we obtain $u_m \leq u_{m-1} - \delta$, for any $m \geq n$. This is a contradiction with the assumption that $c_f = 0$. Therefore $M_n^f(u_0) \geq 0$ and similarly $m_n^f(u_0) \leq 0$. By Lemma 3.2.3 the sequences $(M_n^f(u_0))_n$ and $(m_n^f(u_0))_n$ are bounded and non-increasing and non-decreasing respectively and therefore they converge.

Let w be the limit of a subsequence of $(u_n)_n$. Assume for the moment that $M_1^f(w) = m_1^f(w) = 0$, or equivalently $A_f w = w$. Now Lemma 3.2.3 implies

$$\|u_{n+1} - w\| = \|A_f u_n - A_f w\| \leq \|u_n - w\|.$$

Therefore $\|u_n - w\|$ is decreasing in n and, together with the fact that 0 is its accumulation point, this yields $\lim_n \|u_n - w\| = 0$. Therefore it is enough to prove that $M_1^f(w) = m_1^f(w) = 0$. The rest of the proof will be dedicated to showing $M_1^f(w) = 0$ (the claim $m_1^f(w) = 0$ following analogously).

First we prove that $(M_n^f(w))$ is a constant sequence. Assume this is not the case, that is for some k we have $M_{k+1}^f(w) < M_k^f(w)$. For any $n \geq 1$ the mapping $v \mapsto M_n^f(v)$ is continuous and therefore we can find a neighborhood $\mathcal{U}$ of w ($\mathcal{U} \subset C(V, \mathbb{R})$ in the adjacency case) and $\delta > 0$ such that $M_{k+1}^f(v) < M_k^f(v) - \delta$, for any $v \in \mathcal{U}$. Observing that $M_k^f(u_n) = M_{n+k}^f(u_0)$, and that $u_n \in \mathcal{U}$ for infinitely many positive integers n, we have $M_{\ell+1}^f(u_0) < M_\ell^f(u_0) - \delta$ for infinitely many positive integers ℓ. This is a contradiction with the fact that $(M_n^f(u_0))_n$ is a nonnegative decreasing sequence.

Now let $M = M_n^f(w) \geq 0$ and denote by (w_n) the sequence of game values with terminal and running payoffs $w_0 = w$ and f respectively. Define the compact sets $V_n = \{x \in V : w_{n+1}(x) = w_n(x) + M\}$ and $t_n = \min_{x\in V_n} w_n(x)$. Taking $x \in V_n$ we have $M = w_{n+1}(x) - w_n(x) = \max(w_n - w_{n-1})$ and thus we can apply Lemma 3.2.3 to find $y \sim x$ such that $w_n(y) = \min_{z\sim x} w_n(z)$ and $y \in V_{n-1}$. Because the graph G satisfies $x \sim x$ for any vertex x, we obtain

$$w_n(x) \geq w_n(y) = w_{n-1}(y) + M \geq t_{n-1} + M, \tag{3.2.7}$$

for any $x \in V_n$. Taking the minimum over $x \in V_n$ yields $t_n \geq t_{n-1} + M$. For $M > 0$ this is a contradiction with the boundedness of the sequence (w_n), which in turn follows from $c_f = 0$. Thus $M = 0$ which proves the statement. □

Remark 3.2.5. Note that in the case of finite graphs, the first inequality in (3.2.7) is the only place where loops were used.

The existence of accumulation points will follow from Lemma 3.2.7, which in turn will use the following lemma. Note that these two lemmas can replace the last paragraph in the proof of Lemma 3.2.4. However we will leave the proof of Lemma 3.2.4 as it is, since it gives a shorter proof of Theorem 3.1.7 for finite graphs with loops.

Lemma 3.2.6. *Under the assumption of Theorem 3.1.7 we have*

$$\lim_n (u_{n+1} - u_n) = 0.$$

Proof. We will prove that $\lim_n \max(u_{n+1} - u_n) = 0$. The claim then follows from the fact that $\lim_n \min(u_{n+1} - u_n) = 0$ which follows by replacing u_n by $-u_n$ and f by $-f$.

First assume that for some real numbers λ_1 and λ_2, a vertex $x \in V$ and a positive integer n we have $u_{n+1}(x) - u_n(x) \geq \lambda_1$ and $M_n^f(u_0) \leq \lambda_2$. Since $\max_{z\sim x} u_n(z) - \max_{z\sim x} u_{n-1}(z) \leq \lambda_2$, by (3.1.2) we see that

$$\min_{z\sim x} u_n(z) - \min_{z\sim x} u_{n-1}(z) \geq 2\lambda_1 - \lambda_2.$$

This implies that for a vertex $y \sim x$, such that $u_{n-1}(y) = \min_{z\sim x} u_{n-1}(z)$, we have

$$\min\{u_n(x), u_n(y)\} \geq \min_{z\sim x} u_n(z) \geq u_{n-1}(y) + 2\lambda_1 - \lambda_2. \tag{3.2.8}$$

We will inductively apply this simple argument to prove the statement.

As argued in the proof of Proposition 3.2.4 the sequence $(M_n^f(u_0))$ is non-increasing and nonnegative and therefore converges to $M = \lim_n M_n^f(u_0) \geq 0$. For a fixed $\delta > 0$ let n_0 be an integer such that $M_n^f(u_0) \leq M + \delta$, for all $n \geq n_0$. For a given positive integer k, let x_0 be a point such that $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \geq M$. Then applying the reasoning that leads to (3.2.8) for $\lambda_1 = M$ and $\lambda_2 = M + \delta$, we can find a point x_1 such that

$$\min\{u_{n_0+k-1}(x_0), u_{n_0+k-1}(x_1)\} \geq u_{n_0+k-2}(x_1) + (M - \delta).$$

If $k \geq 3$ we can apply the same argument for functions u_{n_0+k-1}, u_{n_0+k-2} and u_{n_0+k-3}, point x_1, $\lambda_1 = M - \delta$ and $\lambda_2 = M + \delta$. Inductively repeating this reasoning we obtain a sequence of points (x_ℓ), $1 \leq \ell \leq k-1$ such that

$$\min\{u_{n_0+k-\ell}(x_{\ell-1}), u_{n_0+k-\ell}(x_\ell)\} \geq u_{n_0+k-\ell-1}(x_\ell) + M - (2^\ell - 1)\delta.$$

Summing the inequalities

$$u_{n_0+k-\ell}(x_{\ell-1}) \geq u_{n_0+k-\ell-1}(x_\ell) + M - (2^\ell - 1)\delta,$$

for $1 \leq \ell \leq k-1$ and $u_{n_0+k}(x_0) - u_{n_0+k-1}(x_0) \geq M$ leads to

$$u_{n_0+k}(x_0) \geq u_{n_0}(x_{k-1}) + kM - 2^k\delta, \tag{3.2.9}$$

for all $k \geq 1$. Taking $k(\delta)$ to be the smallest integer larger than $\frac{\log(M/\delta)}{\log 2}$ we obtain

$$u_{n_0+k(\delta)}(x_0) \geq u_{n_0}(x_{k(\delta)-1}) + M\Big(\frac{\log(M/\delta)}{\log 2} - 2\Big). \tag{3.2.10}$$

If $M > 0$ then $\lim_{\delta\downarrow 0} M\Big(\frac{\log(M/\delta)}{\log 2} - 2\Big) = \infty$, which by (3.2.10) implies that the sequence (u_n) is unbounded. This is a contradiction with the assumption that $c_f = 0$. □

Lemma 3.2.7. *The sequence of game values (u_n) for a game on an adjacency graph is an equicontinuous sequence of functions.*

The proof of this lemma uses an idea similar to the proof of Lemma 3.2.6. We obtain a contradiction by constructing a sequence of points along which the function values will be unbounded. Since the induction step is more complicated we put it into a separate lemma. First define the oscillation of a continuous function $v\colon V \to \mathbb{R}$ as $\operatorname{osc}(v,\delta) = \sup_{d(x,y)\le\delta} |v(x)-v(y)|$.

Lemma 3.2.8. *Let (u_n) be a sequence of game values played on an adjacency graph. Assume that for positive real numbers λ_1, λ_2, λ_3, $\rho < \epsilon$ and a positive integer n we have*

$$\operatorname{osc}(f,\rho) \le \lambda_1, \ \operatorname{osc}(u_n,\rho) \le \lambda_2, \ \textit{and} \ \ u_n \le u_{n+1} + \lambda_3. \tag{3.2.11}$$

Let (x, y) be a pair of points which satisfies $d(x, y) < \rho$ and $u_{n+1}(x) - u_{n+1}(y) \ge \delta$, for some $\delta > 0$. Then there are points (x_1, y_1) which satisfy $d(x_1, y_1) < \rho$, and the inequalities

$$u_n(x_1) - u_n(y_1) \ge 2\delta - 2\lambda_1 - \lambda_2, \tag{3.2.12}$$

and

$$u_{n+1}(y) - u_n(y_1) \ge 2\delta - 2\lambda_1 - \lambda_2 - \lambda_3. \tag{3.2.13}$$

Proof. If $\delta \le \lambda_1 + \lambda_2/2$ then consider the set $S = \{z : u_n(z) \le u_{n+1}(y) + \lambda_3\}$, which is nonempty by the last condition in (3.2.11). If S is equal to the whole space V, then (3.2.13) will be satisfied automatically, and take x_1 and y_1 to be any points such that $d(x_1, y_1) < \rho$ and $u_n(x_1) \ge u_n(y_1)$ (so that (3.2.12) is satisfied). Otherwise, since V is path connected we can choose points x_1 and y_1 so that $d(x_1, y_1) < \rho$ and $y_1 \in S$ (so that (3.2.13) is satisfied) and $x_1 \notin S$ (so that (3.2.12) is satisfied). In the rest of the proof we will assume that $\delta > \lambda_1 + \lambda_2/2$.

Choose points x_m, x_M, y_m, and y_M so that

$$\begin{aligned} u_n(x_m) &= \min\nolimits_{z\sim x} u_n(z), \quad u_n(x_M) = \max\nolimits_{z\sim x} u_n(z), \\ u_n(y_m) &= \min\nolimits_{z\sim y} u_n(z), \quad u_n(y_M) = \max\nolimits_{z\sim y} u_n(z). \end{aligned}$$

Take a point z_m such that $d(y, z_m) \le \epsilon - d(x, y)$ and $d(z_m, y_m) < \rho$, which surely exists, since $d(x, y) < \rho$ and $d(y, y_m) \le \epsilon$. By the triangle inequality this point satisfies $d(x, z_m) \le \epsilon$. Therefore we have $z_m \sim x$, $z_m \sim y$ and

$$d(y_m, \{z : z \sim x, z \sim y\}) < \rho. \tag{3.2.14}$$

Analogously we construct a point z_M such that $d(x_M, z_M) < \rho$ and $d(z_M, y) \le \epsilon$. Now we have

$$u_n(x_M) - u_n(y_M) \le u_n(x_M) - u_n(z_M) \le \lambda_2. \tag{3.2.15}$$

Next calculate

$$\begin{aligned}(u_n(x_M) - u_n(y_M)) + (\min_{z\sim x, z\sim y} u_n(z) - u_n(y_m))&\\ \geq (u_n(x_M) - u_n(y_M)) + (u_n(x_m) - u_n(y_m))&\\ = 2(u_{n+1}(x) - f(x) - u_{n+1}(y) + f(y))&\\ \geq 2\delta - 2\lambda_1.& \qquad (3.2.16)\end{aligned}$$

Plugging (3.2.15) into (3.2.16) we get

$$\min_{z\sim x, z\sim y} u_n(z) - u_n(y_m) \geq 2\delta - 2\lambda_1 - \lambda_2. \qquad (3.2.17)$$

Now define r as the supremum of the values $\tilde{r}$ such that for every $z_0 \in B(y_m, \tilde{r})$ we have $u_n(z_0) < \min_{z\sim x, z\sim y} u_n(z)$. By (3.2.14), (3.2.17) and the assumption on δ it follows that r is well defined and $0 < r < \rho$. Finally we take a point x_1 such that $d(y_m, x_1) = r$ with $u_n(x_1) = \min_{z\sim x, z\sim y} u_n(z)$ (which exists by the definition of r). By (3.2.17) we have

$$u_n(x_1) = \min_{z\sim x, z\sim y} u_n(z) \geq u_n(y_m) + 2\delta - 2\lambda_1 - \lambda_2.$$

Furthermore (3.2.17) also implies

$$u_{n+1}(y) \geq u_n(y) - \lambda_3 \geq \min_{z\sim x, z\sim y} u_n(z) - \lambda_3 \geq u_n(y_m) + 2\delta - 2\lambda_1 - \lambda_2 - \lambda_3.$$

This proves the claim with $y_1 = y_m$. □

Proof of Lemma 3.2.7. By part (ii) of Lemma 3.2.1 it is enough to prove the claim for an arbitrary shift of the payoff function f, so by the definition of c_f, we can assume that $c_f = 0$ (see Remark 3.1.13).

By Lemma 3.2.6, for a given $\lambda_3 > 0$ choose n_0 large enough so that $\|u_n - u_{n-1}\| \leq \lambda_3$, for all $n \geq n_0$. Assume that the sequence (u_n) is not equicontinuous. Then there is a $\delta_0 > 0$ such that for any $\rho > 0$ there are infinitely many integers k satisfying $\operatorname{osc}(u_{n_0+k}, \rho) \geq \delta_0$. Fix such a k and ρ and define $\delta_1 = \operatorname{osc}(u_{n_0+k}, \rho)$. Since $\|u_{n_0+k} - u_{n_0+\ell}\| \leq (k-\ell)\lambda_3$, for all $0 \leq \ell \leq k$ we get that

$$\operatorname{osc}(u_{n_0+\ell}, \rho) \leq \delta_1 + 2(k-\ell)\lambda_3. \qquad (3.2.18)$$

Now fix an arbitrary τ and let x_0 and y_0 be points that satisfy $d(x_0, y_0) < \rho$ and

$$u_{n_0+k}(x_0) - u_{n_0+k}(y_0) \geq \delta_1 - \tau. \qquad (3.2.19)$$

Applying Lemma 3.2.8 to the pair (x_0, y_0) with $\delta = \delta_1 - \tau$, $\lambda_2 = \delta_1 + 2\lambda_3$, $\lambda_1 = \operatorname{osc}(f, \rho)$ and λ_3 defined as before we obtain points x_1 and y_1 such that $d(x_1, y_1) < \rho$, and

$$u_{n_0+k-1}(x_1) - u_{n_0+k-1}(y_1) \geq \delta_1 - 2\tau - 2\lambda_1 - 2\lambda_3,$$

and

$$u_{n_0+k}(y_0) - u_{n_0+k-1}(y_1) \geq \delta_1 - 2\tau - 2\lambda_1 - 3\lambda_3.$$

Using (3.2.18) and applying the same arguments inductively, for λ_1, λ_3 and τ small enough, we obtain a sequence of points (x_ℓ), $1 \leq \ell \leq k$, which satisfy the inequalities

$$u_{n_0+k-\ell}(x_\ell) - u_{n_0+k-\ell}(y_\ell) \geq \delta_1 - a_\ell\tau - b_\ell\lambda_1 - c_\ell\lambda_3, \tag{3.2.20}$$

and

$$u_{n_0+k-\ell+1}(y_{\ell-1}) - u_{n_0+k-\ell}(y_\ell) \geq \delta_1 - a_\ell\tau - b_\ell\lambda_1 - (c_\ell+1)\lambda_3, \tag{3.2.21}$$

where the coefficients satisfy $a_1 = 2$, $b_1 = 2$, $c_1 = 2$ and

$$a_{\ell+1} = 2a_\ell, \; b_{\ell+1} = 2(b_\ell + 1), \; c_{\ell+1} = 2(c_\ell + \ell + 1).$$

This leads to $a_\ell = 2^\ell$, $b_\ell = 2^{\ell+1} - 2$ and $c_\ell = 2^{\ell+2} - 2\ell - 4$. Summing (3.2.21) with these values of coefficients for $1 \leq \ell \leq k$ we obtain

$$u_{n_0+k}(y_0) - u_{n_0}(y_k) \geq k\delta_1 - 2^k(2\tau + 4\lambda_1 + 8\lambda_3). \tag{3.2.22}$$

Now taking k to be the largest integer not larger than $\frac{\log(\delta_1/(2\tau+4\lambda_1+8\lambda_3))}{\log 2}$ and increasing the value of n_0 if necessary, leads to

$$u_{n_0+k}(y_0) - u_{n_0}(y_k) \geq \delta_1\Big(\frac{\log(\delta_1/(2\tau + 4\lambda_1 + 8\lambda_3))}{\log 2} - 2\Big). \tag{3.2.23}$$

Since the values of τ, λ_1 and λ_3 can be chosen arbitrarily small and δ_1 is bounded from below by δ_0, the right hand side of (3.2.23) can be arbitrarily large. This is a contradiction with the assumption that $c_f = 0$.

□

Proof of Theorem 3.1.7. By Theorem 3.1.6, (u_n) is a bounded sequence of functions In the case of finite graphs with loops the statement follows from Proposition 3.2.4. For the case of adjacency graphs, note that, by Lemma 3.2.7 (u_n) is also equicontinuous and, by the Arzela-Ascoli theorem, it has a convergent subsequence. Now the claim follows from Proposition 3.2.4. □

Proof of Theorem 3.1.8 in the case of adjacency graphs. If u is a solution to (3.1.3) then playing the game with terminal payoff $u_0 = u$ and running payoff f gives the constant sequence of game values $u_n = u$. In the other direction, it is clear that the limit in Theorem 3.1.7 is a solution to the equation (3.1.3). □

Example 3.2.9. Let G be a bipartite graph with partition of the vertex set into V_1 and V_2 (meaning $V_1 \cap V_2 = \emptyset$, $V = V_1 \cup V_2$ and all edges in the graph are connecting vertices in V_1 and V_2). Let f be a function on V having value 1 on V_1 and -1 on V_2. Then if $u_0 = 0$ it is easy to check from (3.1.2) that $u_n = f$ if n is odd and $u_n = 0$ if f is even, and therefore the sequence (u_n) does not converge. However $u = f/2$ is a solution to (3.1.3).

From the proof of Lemma 3.2.6 we can extract the following result about the speed of convergence.

Proposition 3.2.10. *There is a universal constant $C > 0$ such that, under the assumptions of Theorem 3.1.7, for $n \geq 2$ we have*

$$\|u_{n+1} - u_n\| \leq \frac{AC}{\log n}, \tag{3.2.24}$$

where $A = (\max u_0 - \min u_0) + \operatorname{diam}(G)^2(\max f - \min f)$.

Proof. Again, it is enough to prove the claim when $\|u_{n+1} - u_n\|$ is replaced by $M_n^f(u_0) = \max(u_n - u_{n-1})$. If $\max u_m < \min u_n$ for some $m < n$ then by Lemma 3.2.3 we have $u_{m+k(n-m)} - u_m \geq k(\min u_n - \max u_m)$ for all $k \geq 0$, which contradicts the boundedness of (u_n) (which in turn follows from the assumption $c_f = 0$). Similarly we get the contradiction when $\max u_m < \min u_n$ for some $n < m$. Therefore we have $\min u_n \leq \max u_m$ for all m and n and Lemma 3.2.2 implies that

$$\max u_{n+k} - \min u_n \leq 2A. \tag{3.2.25}$$

By Lemmas 3.2.3 and 3.2.6 we know that $(M_n^f(u_0))$ is a non-increasing sequence converging to 0. For given $r > \delta$ assume n and k are such that for all $n \leq m \leq n+k$ we have $r - \delta \leq M_m^f(u_0) \leq r$. Now (3.2.9) implies that

$$\max u_{n+k} - \min u_n \geq kr - k\delta - 2^k\delta.$$

Combining this with (3.2.25) we see that, if $K(r-\delta, r)$ is the number of indices m such that $r - \delta \leq M_m^f(u_0) \leq r$, then for all integers $0 \leq k \leq K(r-\delta, r)$ we have $kr - 2^{k+1}\delta \leq 2A$. Taking $\delta = r2^{-2A/r-2}$ we get that $K(r-\delta, r) < 1 + 2A/r$. Now let $r_0 > 0$ and define the sequence $r_{n+1} = r_n(1 - 2^{-2A/r_n - 2})$, which is clearly decreasing and converging to 0. By the above discussion we have

$$K(r_{n+1}, r_n) \leq \frac{2A}{r_n} + 1. \tag{3.2.26}$$

Furthermore define $N(\alpha, \beta) = \sum K(r_{n+1}, r_n)$, where the sum is taken over all indices n for which the interval $[r_{n+1}, r_n]$ intersects the interval $[\alpha, \beta]$. Defining $s_n = \log(2A/r_n)$ we have $s_{n+1} = s_n - \log\left(1 - 2^{-e^{s_n}-2}\right)$. Since the function $s \mapsto \log\left(1 - 2^{-e^{s}-2}\right)$ is negative and increasing, the number of indices n such that the interval $[s_n, s_{n+1}]$ intersects a given interval $[a, b]$ is no more than

$$\frac{b-a}{-\log\left(1 - 2^{-e^b-2}\right)} + 2 \leq (b-a)2^{e^b+2} + 2,$$

where we used the inequality $\log(1-x) \leq -x$, for $0 \leq x < 1$. This together with (3.2.26) implies

$$N(2Ae^{-b}, 2Ae^{-a}) \leq \left((b-a)2^{e^b+2} + 2\right)\left(e^b + 1\right).$$

Therefore we have

$$N(2Ae^{-t}, 2A) \le (4 + o(1))2^{e^t} e^t,$$

and since $M_1^f(u_0) \le 2A$, there are no more than $(4+o(1))2A2^{2A/r}r^{-1}$ indices n such that $M_n^f(u_0) \ge r$, which then easily implies the claim. □

Remark 3.2.11. From Lemma 3.2.1 (ii) it is clear that removing the assumption $c_f = 0$ from the statements of Lemma 3.2.6 and Proposition 3.2.10 yields $\lim_n(u_{n+1} - u_n) = c_f$ and $|\|u_{n+1} - u_n\| - c_f| \le \frac{AC}{\log n}$ respectively.

One of the obstacles to faster convergence is the fact that for each vertex x the locations where the maximum and the minimum values of u_n among its neighbors are attained depends on n. However, in the case of finite graphs with loops, these locations will eventually be "stabilized", if (for example) the limiting function is one-to-one. Therefore after a certain (and possibly very large) number of steps, we will essentially see a convergence of a certain Markov chain, which is exponentially fast. To prove this in the next theorem recall some basic facts about finite Markov chains. A time homogeneous Markov chain X on a finite state space is given by its transition probabilities $P(i,j) = \mathbb{P}(X_1 = j | X_0 = i)$. Denote the transition probabilities in k steps as $P^k(i,j) = \mathbb{P}(X_k = j | X_0 = i)$ (these are just entries of the kth power of the matrix $(P(i,j))_{ij}$). An *essential class* of a Markov chain is a maximal subset of the state space with the property that for any two elements i and j from this set there is an integer k such that $P^k(i,j) > 0$. An essential class is called *aperiodic* if it contains an element i such that the greatest common divisor of integers k satisfying $P^k(i,i) > 0$ is 1. The state space can be decomposed into several disjoint essential classes and a set of elements i which are not contained in any essential class and which necessarily satisfy $P^k(i,j) > 0$ for some integer k and some element j contained in an essential class. If all essential classes of a Markov chain are aperiodic then the distribution of (X_n) converges to a stationary distribution and, moreover this convergence is exponentially fast. This result is perhaps more standard when the chain is *irreducible* (the whole state space is one essential class). However the more general version we stated is a straightforward consequence of this special case after we observe that the restriction of a Markov chain to an aperiodic essential class is an irreducible Markov chain, and that for any element i not contained in any essential class, conditioned on $X_0 = i$, the time of the first entry to an essential class is stochastically dominated from above by a geometric random variable. For more on this topic see [46].

Proposition 3.2.12. *Let G be a finite graph with a loop at each vertex, f a function on the set of vertices and (u_n) a sequence of game values played with running payoff f. Assuming $c_f = 0$, let u be the limit of the sequence (u_n) and assume that for each vertex $x \in V$ there are unique neighbors y_m and y_M of x, such that $u(y_m) = \min_{y\sim x} u(y)$ and $u(y_M) = \max_{y\sim x} u(y)$. Then there are constants $C > 0$ and $0 < \alpha < 1$ (depending on G, f and u_0) such that $\|u_n - u\| \le C\alpha^n$.*

Proof. Let $A = (a_{xy})$ be the matrix such that $a_{xy} = 1/2$ if either $u(y) = \max_{z\sim x} u(z)$ or $u(y) = \min_{z\sim x} u(z)$ and 0 otherwise. The Markov process X_k on the vertex set, with the transition matrix A, has the property that all essential classes are aperiodic. To see this fix an essential class $I \subset V$ let x be a vertex such that $u(x) = \max_I u$, and observe that $a_{xx} = 1/2$. Therefore the distribution of X_k converges exponentially fast to a stationary distribution.

Since $u = \lim_n u_n$, there is an n_0 such that for $n \geq n_0$ and any vertex x the unique neighbors of x where u attains the value $\max_{z\sim x} u(z)$ ($\min_{z\sim x} u(z)$) and where u_n attains the value $\max_{z\sim x} u_n(z)$ ($\min_{z\sim x} u_n(z)$) are equal. Writing functions as column vectors, this means that $u_{n+1} = Au_n + f$ for $n \geq n_0$. Thus, defining $v_n = u_{n+1} - u_n$, for $n \geq n_0$ we have

$$v_{n+1} = u_{n+2} - u_{n+1} = Au_{n+1} - Au_n = Av_n.$$

This means that for any $k \geq 0$ we have $v_{n_0+k}(x) = \mathbb{E}_x(v_{n_0}(X_k))$. Therefore the sequence of functions $(v_{n_0+k})_k$ converges exponentially fast. Since we necessarily have $\lim_n v_n = 0$ the claim follows from $\|u_n - u\| \leq \sum_{k=n}^{\infty} \|v_k\|$. □

Our next goal is to prove Theorem 3.1.8 for all finite graphs. Recall the (nonlinear) operator A_f from Lemma 3.2.3. For a real number $c \in \mathbb{R}$, and a function u define $D_f(u,c) = \|A_{f-c}u - u\|$. To prove the existence of a solution it is enough to prove that D_f has a minimum value equal to 0. First we use a compactness argument to prove that it really has a minimum. For the rest of this section all the graphs will be arbitrary connected finite graphs.

Lemma 3.2.13. *Let G be a finite connected graph, and f and u functions on V. Then*

$$\max u - \min u \leq 2^{\mathrm{diam}(G)+1}(\|f\| + D_f(u,0)). \tag{3.2.27}$$

Proof. Assume that the function u attains its minimum and maximum at vertices x_m and x_M respectively. Let $x_m = y_0, y_1, \ldots, y_{k-1}, y_k = x_M$ be a path connecting x_m and x_M with $k \leq \mathrm{diam}(G)$. Observe that

$$A_f u(y_i) \geq \frac{u(x_m) + u(y_{i+1})}{2} + f(y_i),$$

for $i = 0, \ldots, k-1$. Estimating the left hand side of the above equations by $A_f u \leq u + D_f(u,0)$ we get

$$u(y_{i+1}) \leq 2u(y_i) + 2D_f(u,0) - 2f(y_i) - u(x_m),$$

for $i = 0, \ldots, k-1$. Multiplying the i-th inequality by 2^{k-1-i}, for $i = 0, \ldots k-1$ and adding them we obtain

$$u(x_M) - u(x_m) \leq (2^{k+1} - 2)(D_f(u,0) - \min f),$$

which implies the claim. □

Lemma 3.2.14. *Under the assumptions of Lemma 3.2.13 the function $D_f(u,c)$ has a minimum value.*

Proof. Since D_f is a continuous function, we only need to prove that $\tau := \inf D_f = \inf_{\mathcal{U}\times I} D_f$, where the right hand side is the infimum of the values of D_f over $\mathcal{U} \times I$ for a bounded set of functions $\mathcal{U}$ and a bounded interval I. First assume that c is a constant large enough so that $f + c$ has all values larger than $\tau + 1$. If x_m is a vertex where a function u attains its minimum, we have

$$\frac{1}{2}(\max_{y\sim x_m} u(y) + \min_{y\sim x_m} u(y)) + f(x) + c \geq u(x_m) + \tau + 1.$$

This implies that $D_f(u, -c) \geq \tau + 1$ for any function u. Similarly for sufficiently large c we have that $D_f(u, c) \geq \tau + 1$ for any function u. Therefore there is a bounded interval I such that the infimum of values of $D(u, c)$ over all functions u and $c \notin I$ is strictly bigger than τ.

Furthermore by Lemma 3.2.13 we can find a constant K such that for any $c \in I$ we have that $\max u - \min u \geq K$ implies $D_f(u, c) \geq \tau + 1$. Also since $D_f(u+\lambda, c) = D_f(u, c)$ for any $\lambda \in \mathbb{R}$, we have that $\tau = \inf_{\mathcal{U}\times I} D_f$ where $\mathcal{U}$ is the set of functions such that $\min u = 0$ and $\max u \leq K$. Since the set $\mathcal{U}$ is bounded the claim follows. □

Proof of Theorem 3.1.8 in the case of finite graphs. Assuming the existence of a solution the argument proceeds as in the proof of the adjacency case. By the same argument, to show the other direction, it is enough to prove that there is a constant c for which there is a solution to (3.1.3), when the right hand side f is replaced by $f - c$, since then we necessarily have $c = 0$. In other words it is enough to show that $\min D_f = 0$. By Lemma 3.2.14 this minimum is achieved and denote it by $m = \min D_f$. Assume that $m > 0$. Fix a pair (u, c) where the minimum is achieved and define $S^+_{u,c} := \{x : A_{f-c}u(x) - u(x) = m\}$, $S^-_{u,c} := \{x : A_{f-c}u(x) - u(x) = -m\}$ and $S_{u,c} := S^+_{u,c} \cup S^-_{u,c}$. By definition $S_{u,c} \neq \emptyset$. If $S^+_{u,c} = \emptyset$ then there is a $\delta > 0$ small enough so that $A_{f-c+\delta}u - u < m$, and of course $A_{f-c+\delta}u - u > -m$. This implies that $D_f(u, c - \delta) < m$, which is a contradiction with the assumption that $m = \min D$. Therefore $S^+_{u,c} \neq \emptyset$, and similarly $S^-_{u,c} \neq \emptyset$.

Call a set $S_r \subset S^+_{u,c}$ *removable* for function u, if both of the following two conditions hold:

(i) For every $x \in S^+_{u,c}$ there is a $y \notin S_r$ so that $y \sim x$ and $u(y) = \min_{z\sim x} u(z)$.

(ii) There are no $x \in S^+_{u,c}$ and $y \in S_r$ so that $y \sim x$ and $u(y) = \max_{z\sim x} u(z)$.

By increasing values of the function u on S_r we can remove this set from $S^+_{u,c}$. More precisely, define the function $\tilde{u}_\delta$ so that $\tilde{u}_\delta(x) = u(x)$ for $x \notin S_r$ and $\tilde{u}_\delta(x) = u(x) + \delta$ for $x \in S_r$. Since the graph G is finite, for δ small enough and all points $x \notin S^+_{u,c}$, we have $A_{f-c}\tilde{u}_\delta(x) - \tilde{u}_\delta(x) < m$. Furthermore, by the above two conditions, if δ small enough, for any point $x \in S^+_{u,c}$ we have $A_{f-c}\tilde{u}_\delta(x) = A_{f-c}u(x)$. On the other hand for $x \in S_r$ we have

$$A_{f-c}\tilde{u}_\delta(x) - \tilde{u}_\delta(x) = m - \delta,$$

and therefore $S^+_{\tilde{u}_\delta,c} = S^+_{u,c} \backslash S_r$. Moreover $S^-_{\tilde{u}_\delta,c} \subset S^-_{u,c}$ is obvious.

Similarly we can define removable sets S^-_r contained in $S^-_{u,c}$ so that there are no $x \in S^-_{u,c}$ and $y \in S_r$ such that $u(y) = \min_{z\sim x} u(z)$ and that for every $x \in S^-_{u,c}$ there is a $y \notin S_r$ such that $u(y) = \max_{z\sim x} u(z)$. This set can be removed from $S^-_{u,c}$ be decreasing the value of u on this set. Note that the removable sets in $S^+_{u,c}$ and $S^-_{u,c}$ can be removed simultaneously as described above. Thus if a pair (u, c) minimizes the value of D_f, and $\tilde{u}$ is obtained from u by removing removable sets in $S^+_{u,c}$ and $S^-_{u,c}$, then the pair $(\tilde{u}, c)$ also minimizes the value of D_f, and moreover $S_{\tilde{u},c} \subset S_{u,c}$.

Call a function u *tight* (for f) if there is $c \in \mathbb{R}$ such that the pair (u, c) minimizes D_f, and so that the set $S_{u,c}$ is of smallest cardinality, among all minimizers of D_f. By the discussion above, tight functions have no non-empty removable sets. For a tight function u define $v = A_{f-c}u$. By Lemma 3.2.3 we have that $D_f(v, c) = \|A_{f-c}v - v\| \leq m$ and because $m = \min D_f$ we have $D_f(v, c) = m$.

Now observe that it is enough to prove that for any tight function u and $v = A_{f-c}u$, the set $S^+_{v,c}\backslash S^+_{u,c}$ is removable for function v. To see this first note that by symmetry the set $S^-_{v,c}\backslash S^-_{u,c}$ is also removable for v. Let v_1 be a function obtained by removing all these vertices as described above. In particular we have $v_1(x) = v(x) = u(x) + m$ for $x \in S^+_{u,c}$ and $v_1(x) = v(x) = u(x) - m$ for $x \in S^-_{u,c}$. The function v_1 then satisfies $S^+_{v_1,c} \subseteq S^+_{u,c}$ and $S^-_{v_1,c} \subseteq S^-_{u,c}$. By tightness of u it follows that $S^+_{v_1,c} = S^+_{u,c}$ and $S^-_{v_1,c} = S^-_{u,c}$ and thus the function v_1 is also tight. Now we can repeat this argument to obtain a sequence of tight functions (v_k) such that $S^+_{v_k,c} = S^+_{u,c}$, $S^-_{v_k,c} = S^-_{u,c}$, $v_k(x) = u(x) + km$ for $x \in S^+_{u,c}$ and $v_k(x) = u(x) - km$ for $x \in S^-_{u,c}$. Since $D(v_k, c) = m$ for all k and $\lim_k(\max v_k - \min v_k) = \infty$, Lemma 3.2.13 gives a contradiction with the assumption that $m > 0$.

Thus it is only left to prove that for any tight function u and $v = A_{f-c}u$ the set $S^+_{v,c}\backslash S^+_{u,c}$ is removable for function v. For this we need to check the conditions (i) and (ii) from the definition of the removable sets. Take a vertex $x \in S^+_{v,c}$ and note that since $A_{f-c}v(x) - v(x) = \max(v - u)$ and $v = A_{f-c}u$, by Lemma 3.2.3 for a $y_1 \sim x$ such that $u(y_1) = \min_{z\sim x} u(z)$, we have $v(y_1) = \min_{z\sim x} v(z)$ and $y_1 \in S^+_{u,c}$ which checks the first assumption. Furthermore by Lemma 3.2.3 for any y_2 such that $v(y_2) = \max_{z\sim x} v(z)$ we have $y_2 \in S^+_{u,c}$ which also checks the second assumption in the definition of removable sets. $\square$

Next we present two examples for which we explicitly calculate the value of the Player I's long term advantage c_f.

Example 3.2.15. If G is a complete graph with loops at each vertex and f a function on the set of vertices, then $c_f = (\max f + \min f)/2$. To see this, use the fact that c_f defined as above satisfies $c_{f+\lambda} = c_f + \lambda$ for any $\lambda \in \mathbb{R}$, and that $c_f = 0$ implies that $u = f$ solves (3.1.3).

When G is a complete graph without loops the situation becomes more complicated. If the function f attains both the maximum and the minimum values at more than one vertex then again we have $c_f = (\max f + \min f)/2$, and again in the case $\max f + \min f = 0$ the function $u = f$ satisfies equation (3.1.3).

If the maximum and the minimum values of f are attained at unique vertices then

$$c_f = \frac{\max f + \min f}{3} + \frac{\max_2 f + \min_2 f}{6}, \tag{3.2.28}$$

where $\max_2 f$ and $\min_2 f$ denote the second largest and the second smallest values of the function f respectively. To prove this assume the expression in (3.2.28) is equal to zero, and let x_M and x_m be the vertices where f attains the maximum and the minimum value. Then define a function u so that $u(x_M) = (2\max f + \max_2 f)/3$, $u(x_m) = (2\min f + \min_2 f)/3$ and $u(x) = f(x)$, for $x \notin \{x_m, x_M\}$. Now using the fact that $c_f = 0$ and that u attains its maximum and minimum values only at x_M and x_m respectively, it can be checked that u solves (3.1.3).

Finally in the case when the maximum value of the function f is attained at a unique vertex x_M and the minimum at more than one vertex we have $c_f = (2\max f + \max_2 f + 3\min f)/6$. When this expression is equal to zero, one solution u of the equation (3.1.3) is given by $u(x) = f(x)$ for $x \neq x_M$ and $u(x_M) = (2\max f + \max_2 f)/3$. Similarly when the maximum of f is attained at more than one vertex and minimum at a unique vertex we have $c_f = (2\min f + \min_2 f + 3\max f)/6$.

Example 3.2.16. Consider a linear graph of length n with loops at every vertex, that is take $V = \{1, \dots, n\}$ and connect two vertices if they are at Euclidean distance 0 or 1. Let f be a non-decreasing function on the set of vertices, that is $f(i) \leq f(i+1)$, for $1 \leq i \leq n-1$. By induction and (3.1.2), running the game with the vanishing terminal payoff and the running payoff f gives sequence of game values (u_n), each of which is a non-decreasing function. Representing functions u_n as column vectors, we have $u_{n+1} = Au_n + f$, where $A = (a_{ij})$ is a matrix with $a_{11} = a_{nn} = 1/2$, $a_{ij} = 1/2$, if $|i-j| = 1$ and $a_{ij} = 0$ for all other values of i and j. Therefore $v_n = u_{n+1} - u_n$ satisfies $v_{n+1} = Av_n$. Using this we see that $v_n(x) = \mathbb{E}_x(f(X_n))$, where X_n is the simple random walk on the graph with the vertex set $\{1, \dots, n\}$ where i and j are connected with an edge if $|i-j| = 1$ and with loops at 1 and n. The stationary distribution of the random walk (X_n) is uniform on $\{1, \dots, n\}$, and this is the limit of the distributions of X_n as n tends to infinity. From here it is clear that $\lim_n v_n(x) = \Big(\sum_{i=1}^n f(i)\Big)/n$ and c_f is equal to the average of the values of function f.

The condition that f is monotone is necessary. Consider for example the linear graph with loops and three vertices and the function $(f(1), f(2), f(3)) = (-1, 2, -1)$. Then by (3.1.2) we have $u_0 = 0$, $u_1 = f$ and $u_2 = f + 1/2$ which implies that $u_{n+1} = u_n + 1/2$ for all $n \geq 1$ and by Lemma 3.2.1 (i) we have $c_f = 1/2$.

3.3 The continuous case

The main goal of this section is to study the game values on Euclidean ϵ-adjacency graphs, as defined in Section 3.1, to obtain the existence of viscosity solutions to the equation (3.1.5). One of the main concerns will be the dependence of the game values and limits, obtained in the previous section, on values of step sizes ϵ. The following example shows that the issue starts already with the Player I's long term advantage $c_f(\epsilon)$ (recall that $c_f(\epsilon)$ was defined as the Player I's long term advantage for a game played on an ϵ-adjacency graph with the running payoff f).

Example 3.3.1. This example shows that, in general, for Euclidean ϵ-adjacency graphs on a domain Ω and a continuous function $f\colon \overline{\Omega} \to \mathbb{R}$, the value of $c_f(\epsilon)$ depends on ϵ. First observe a trivial fact that for any Ω of diameter $\mathrm{diam}(\Omega)$ and f we have $c_f(\mathrm{diam}(\Omega)) = (\max f + \min f)/2$. Next let $\Omega = (0,1)$ and let f be a piecewise linear function that is linear on the intervals $[0,1/2]$ and $[1/2,1]$ and has values $f(0) = f(1/2) = 1$ and $f(1) = -1$. By the above observation we have $c_f(1) = 0$. However notice that from (3.1.2) it is clear that, for any ϵ, playing the game with step size ϵ, the vanishing terminal payoff and the running payoff f, the game values will be non-increasing functions on $[0,1]$. Therefore in the game of step size $1/2$ the game values u_n at points 0, $1/2$ and 1 are equal to the game values played on the linear graph with three vertices and loops on each vertex, with terminal payoff zero and running payoff equal to 1, 1 and -1 at the leftmost, central and the rightmost vertex respectively. Using Example 3.2.16 and going back to the game on $[0,1]$ this implies that $c_f(1/2) = 1/3$.

For a more comprehensive example, construct a monotone function f on $[0,1]$ such that no value of $(2^n+1)^{-1}\sum_{k=0}^{2^n} f(k2^{-n})$ is attained for two distinct integers n. By the above reasoning and Example 3.2.16 the value of $c_f(\epsilon)$ varies for arbitrarily small values of ϵ.

For the remainder of this paper, all the graphs are assumed to be ϵ-adjacency graphs and the dependence on ϵ will be explicitly specified.

Theorem 3.1.9 settles the issue raised in the above example. We will need several technical lemmas for the proof of Theorem 3.1.9. The main ingredient of the proof is a comparison between the values of discrete infinity Laplacian with different step sizes from Lemma 3.3.4. The idea for (as well as one part of) this lemma came from [3].

Lemma 3.3.2. *If f and u are continuous functions on a compact length space V such that $-\Delta_\infty^\epsilon u \le f$ then $c_f(\epsilon) \ge 0$. Similarly $-\Delta_\infty^\epsilon u \ge f$ implies $c_f(\epsilon) \le 0$.*

Proof. The second claim follows by replacing u and f by $-u$ and $-f$ respectively, so it is enough to prove the first one. The condition $-\Delta_\infty^\epsilon u \le f$ can be rewritten as

$$\frac{1}{2}\Big(\max_{z\sim x} u(z) + \min_{z\sim x} u(z)\Big) + \frac{\epsilon^2}{2} f(x) \ge u(x).$$

Thus the game value u_1 of the first step of the game, played with the terminal payoff $u_0 = u$, running payoff $\epsilon^2 f/2$ and step sizes ϵ satisfies $u_1 \ge u_0$. By Lemma 3.2.3 we have $u_{n+1} \ge u_n$ for any n, hence $c_f(\epsilon) \ge 0$ is clear. □

Lemma 3.3.3. *Mapping $\epsilon \mapsto c_f(\epsilon)$ is continuous on $\mathbb{R}^+$.*

Proof. For a given ϵ let $u_\epsilon \in C(V)$ be a solution of $-\Delta_\infty^\epsilon u_\epsilon = f - c_f(\epsilon)$, which exists by Theorem 3.1.8 and Remarks 3.1.3 and 3.1.13. Since for any $u \in C(V)$, it holds that $\epsilon \mapsto -\Delta_\infty^\epsilon u$ is a continuous function from $\mathbb{R}^+$ to $(C(V), \|\cdot\|_\infty)$, so for a fixed ϵ and any $\delta > 0$ we can find $\eta > 0$ such that $|-\Delta_\infty^{\epsilon_1} u_\epsilon| \le f - c_f(\epsilon) + \delta$ whenever $|\epsilon_1 - \epsilon| \le \eta$. Now by applying Lemma 3.3.2 we see that for such ϵ_1 we have $|c_f(\epsilon_1) - c_f(\epsilon)| \le \delta$, which gives the continuity. □

As mentioned above, the main part of the proof of Theorem 3.1.9 is contained in the following lemma. For Euclidean ϵ-adjacency graphs, the first inequality in (3.3.1) already appeared as Lemma 4.1 in [3]. However since their definition of the discrete Laplacian was somewhat different close to the boundary $\partial\Omega$, their estimates held only away from $\partial\Omega$. This issue does not appear in our case and their proof goes verbatim. For reader's convenience we repeat their proof of the first inequality in (3.3.1).

For a function $u\colon V \to \mathbb{R}$ we first define $\overline{u}^\epsilon = \max_{z\in B(x,\epsilon)} u(z)$ and $\underline{u}_\epsilon = \min_{z\in B(x,\epsilon)} u(z)$. Furthermore define $T_\epsilon^+ u(x) = \overline{u}^\epsilon(x) - u(x)$ and $T_\epsilon^- u(x) = u(x) - \underline{u}_\epsilon(x)$ (this corresponds to ϵS_ϵ^+ and ϵS_ϵ^- in [3]). Now we can write $-\Delta_\infty^\epsilon u = (T_\epsilon^- u - T_\epsilon^+ u)/\epsilon^2$.

Lemma 3.3.4. *Suppose that $u \in C(V)$ satisfies $-\Delta_\infty^\epsilon u \le f_1$ for some $f_1 \in C(V)$. Then we have*

$$-\Delta_\infty^{2\epsilon}\overline{u}^\epsilon \le \overline{f_1}^{2\epsilon} \text{ and } -\Delta_\infty^{\epsilon}\overline{u}^\epsilon \le \overline{f_1}^{\epsilon}. \tag{3.3.1}$$

If, in addition, we have $-\Delta_\infty^{2\epsilon} u \le f_2$ for some $f_2 \in C(V)$ then

$$-\Delta_\infty^{3\epsilon}\overline{u}^\epsilon \le (8\overline{f_2}^{2\epsilon} + \overline{f_1}^{\epsilon})/9. \tag{3.3.2}$$

Proof. In the proof we will repeatedly use the following arguments. If $z_0, z_1 \in V$ are such that $z_1 \in B(z_0, \delta)$ and $v(z_1) = \overline{v}^\delta(z_0)$ then we have

$$T_\delta^+ v(z_0) = v(z_1) - v(z_0) \le T_\delta^- v(z_1). \tag{3.3.3}$$

Furthermore the assumption $-\Delta_\infty^\delta v \le f$ implies that

$$T_\delta^+ v(z_0) \le T_\delta^- v(z_1) \le T_\delta^+ v(z_1) + \delta^2 f(z_1). \tag{3.3.4}$$

Denote points $y_1 \in B(x,\epsilon)$, $y_2 \in B(y_1,\epsilon)$, $z_M \in B(x, 2\epsilon)$ and $z_m \in B(x, 2\epsilon)$ so that $u(y_1) = \overline{u}^\epsilon(x)$, $u(y_2) = \overline{u}^\epsilon(y_1)$, $u(z_M) = \overline{u}^{2\epsilon}(x)$ and $u(z_m) = \underline{u}_{2\epsilon}(x)$. We calculate

$$\begin{aligned}
T_{2\epsilon}^+\overline{u}^\epsilon(x) &= \overline{u}^{3\epsilon}(x) - \overline{u}^\epsilon(x) \\
&= (\overline{u}^{3\epsilon}(x) - u(y_2)) + (u(y_2) - u(y_1)) \\
&\ge T_\epsilon^+ u(y_2) + T_\epsilon^+ u(y_1) \\
&\ge 2T_\epsilon^+ u(y_1) - \epsilon^2 f_1(y_2) \\
&\ge 2T_\epsilon^+ u(x) - \epsilon^2 (f_1(y_2) + 2f_1(y_1)) \\
&\ge T_\epsilon^+ u(x) + T_\epsilon^- u(x) - \epsilon^2 (f_1(y_2) + 2f_1(y_1) + f_1(x)).
\end{aligned} \tag{3.3.5}$$

In the first inequality we used the fact that $B(y_2,\epsilon) \subset B(x, 3\epsilon)$ and in the second inequality we used (3.3.4) with $z_0 = y_1$, $z_1 = y_2$ and $\delta = \epsilon$. In the next line we again used (3.3.4) with $z_0 = x$, $z_1 = y_1$ and $\delta = \epsilon$, and in the last line the assumption $-\Delta_\infty^\epsilon u \le f_1$.

Furthermore we have

$$\begin{aligned}
T_{2\epsilon}^-\overline{u}^\epsilon(x) &= \overline{u}^\epsilon(x) - \min_{B(x,2\epsilon)} \overline{u}^\epsilon \\
&\le (\overline{u}^\epsilon(x) - u(x)) + (u(x) - \underline{u}_\epsilon(x)) \\
&= T_\epsilon^+ u(x) + T_\epsilon^- u(x).
\end{aligned} \tag{3.3.6}$$

The inequality above follows from the fact that for every $z \in B(x, 2\epsilon)$ we have $\max_{B(z,\epsilon)} u \geq \min_{B(x,\epsilon)} u$. Now the first inequality in (3.3.1) is obtained by subtracting (3.3.5) from (3.3.6).

For the second inequality in (3.3.1) note that

$$T_\epsilon^+ \overline{u}^\epsilon(x) = \overline{u}^{2\epsilon}(x) - u(y_1) \geq T_\epsilon^+ u(y_1),$$

and

$$T_\epsilon^- \overline{u}^\epsilon(x) = \overline{u}^\epsilon(x) - \min_{B(x,\epsilon)} \overline{u}^\epsilon \leq u(y_1) - u(x) \leq T_\epsilon^- u(y_1).$$

Subtracting the above inequalities it follows that $-\Delta_\infty^\epsilon \overline{u}^\epsilon(x) \leq -\Delta_\infty^\epsilon u(y_1) \leq f_1(y_1) \leq \overline{f}_1^\epsilon(x)$.

We prove the inequality (3.3.2) similarly. First we calculate

$$\begin{aligned}
T_{3\epsilon}^+ \overline{u}^\epsilon(x) &= \overline{u}^{4\epsilon}(x) - \overline{u}^\epsilon(x) \\
&= (\overline{u}^{4\epsilon}(x) - u(z_M)) + (u(z_M) - u(y_2)) + (u(y_2) - u(y_1)) \\
&\geq T_{2\epsilon}^+ u(z_M) + T_\epsilon^+ u(y_1) \\
&\geq T_{2\epsilon}^+ u(z_M) + T_\epsilon^- u(y_1) - \epsilon^2 f_1(y_1) \\
&\geq T_{2\epsilon}^+ u(z_M) + T_\epsilon^+ u(x) - \epsilon^2 f_1(y_1).
\end{aligned}$$

In the third line we used the fact that $y_2 \in B(x, 2\epsilon)$ which implies that $u(y_2) \leq u(z_M)$, in the fourth line the assumption and in the last line (3.3.3).

Using similar arguments again we have

$$\begin{aligned}
T_{3\epsilon}^- \overline{u}^\epsilon(x) &= \overline{u}^\epsilon(x) - \min_{B(x,3\epsilon)} \overline{u}^\epsilon \\
&\leq \overline{u}^\epsilon(x) - \underline{u}_{2\epsilon}(x) \\
&= (u(y_1) - u(x)) + (u(x) - u(z_m)) \\
&= T_\epsilon^+ u(x) + T_{2\epsilon}^- u(x) \\
&\leq T_\epsilon^+ u(x) + T_{2\epsilon}^+ u(x) + (2\epsilon)^2 f_2(x) \\
&\leq T_\epsilon^+ u(x) + T_{2\epsilon}^- u(z_M) + (2\epsilon)^2 f_2(x).
\end{aligned}$$

Now subtracting the above calculations and dividing by $(3\epsilon)^2$, we obtain

$$-\Delta_\infty^{3\epsilon} \overline{u}^\epsilon(x) \leq (T_{2\epsilon}^- u(z_M) - T_{2\epsilon}^+ u(z_M))/(9\epsilon^2) + 4f_2(x)/9 + f_1(y_1)/9,$$

from where the fact follows directly. □

Recall the notation $\operatorname{osc}(f, \delta) = \sup_{d(x,y) \leq \delta} |f(x) - f(y)|$.

Proposition 3.3.5. *For any $f \in C(V)$, $\epsilon > 0$ and any positive integer n we have*

$$\max\{|c_f(\epsilon 2^{-n}) - c_f(\epsilon)|, |c_f(\epsilon 3^{-n}) - c_f(\epsilon)|\} \leq \operatorname{osc}(f, 2\epsilon). \quad (3.3.7)$$

Proof. First note that the functions $\overline{f}^r$ and $\underline{f}_r$ differ from f by at most $\operatorname{osc}(f,r)$ at any point, which easily implies

$$\max\{|c_f(\rho) - c_{\overline{f}^r}(\rho)|, |c_f(\rho) - c_{\underline{f}_r}(\rho)|\} \leq \operatorname{osc}(f,r), \tag{3.3.8}$$

for any ρ.

Taking u to be a continuous function such that $-\Delta_\infty^\delta u = f - c_f(\delta)$, by Lemma 3.3.4 we have $-\Delta_\infty^{2\delta}\overline{u}^\delta \leq \overline{f}^{2\delta} - c_f(\delta)$ and therefore also $-\Delta_\infty^{3\delta}\overline{u}^{2\delta} \leq \overline{f}^{4\delta} - c_f(\delta)$. By Lemma 3.3.2 these inequalities and their symmetric counterparts imply that

$$c_{\underline{f}_{2\delta}}(2\delta) \leq c_f(\delta) \leq c_{\overline{f}^{2\delta}}(2\delta) \quad \text{and} \quad c_{\underline{f}_{4\delta}}(3\delta) \leq c_f(\delta) \leq c_{\overline{f}^{4\delta}}(3\delta).$$

Applying these estimates inductively to $\delta = \epsilon 2^{-n}, \ldots \epsilon/2$ and $\delta = \epsilon 3^{-n}, \ldots \epsilon/3$ respectively, we see that

$$c_{\underline{f}_{2\epsilon}}(\epsilon) \leq c_f(\epsilon 2^{-n}) \leq c_{\overline{f}^{2\epsilon}}(\epsilon) \quad \text{and} \quad c_{\underline{f}_{2\epsilon}}(\epsilon) \leq c_f(\epsilon 3^{-n}) \leq c_{\overline{f}^{2\epsilon}}(\epsilon).$$

Using (3.3.8) with $r = 2\epsilon$ and $\rho = 2\epsilon$, these inequalities imply (3.3.7).

□

Proof of Theorem 3.1.9. Since $\min f \leq c_f(\epsilon) \leq \max f$, there are accumulation points of $c_f(\epsilon)$ as $\epsilon \downarrow 0$, and we only need to prove that there is only one. Suppose that there are two such accumulation points $c_1 < c_2$ and denote $\delta = c_2 - c_1$. Let I_1 and I_2 be disjoint open intervals of length $\delta/2$, centered around c_1 and c_2 respectively. Let ϵ_0 be a positive real number such that $\operatorname{osc}(f, \epsilon_0) \leq \delta/4$, and consider the open sets J_1 and J_2 defined as $J_i = c_f^{-1}(I_i) \cap (0, \epsilon_0/2)$. First note that the set $\{2^{-m}3^n : m,n \in \mathbb{Z}^+\}$ is dense in $\mathbb{R}^+$. This follows from the fact that $\{n \log 3 - m \log 2 : m,n \in \mathbb{Z}^+\}$ is dense in $\mathbb{R}$, which in turn follows from the fact that $\log 3/\log 2$ is an irrational number. Take an arbitrary $t \in J_1$ and, since $\{s/t : s \in J_2\}$ is an open set in $\mathbb{R}^+$, we can find non-negative integers m_0 and n_0 such that $3^{n_0}2^{-m_0}t \in J_2$. Therefore

$$|c_f(t) - c_f(3^{n_0}2^{-m_0}t)| > \delta/2.$$

However this gives a contradiction, since both t and $3^{n_0}2^{-m_0}t$ lie in the interval $(0, \epsilon_0/2)$, and so by Proposition 3.3.5 we have

$$\max\{|c_f(2^{-m_0}t) - c_f(t)|, |c_f(3^{n_0}2^{-m_0}t) - c_f(2^{-m_0}t)|\} \leq \operatorname{osc}(f, \epsilon_0) \leq \delta/4.$$

□

Proposition 3.3.6. *For a sequence (ϵ_n) converging to zero, let (u_n) be a sequence of continuous functions on a compact length space V, satisfying $-\Delta_\infty^{\epsilon_n} u_n = f - c(\epsilon_n)$. Then (u_n) is an equicontinuous sequence and for all n large enough we have*

$$\max u_n - \min u_n \leq 6 \operatorname{diam}(V)^2 \|f\|.$$

Furthermore, any subsequential limit of the sequence (u_n) is Lipshitz continuous, with the Lipshitz constant $5 \operatorname{diam}(V)\|f\|$.

Proof. It is enough to prove that for n large enough and any $x \in \overline{V}$, we have that

$$T^+_{\epsilon_n} u_n(x) \le 5\,\mathrm{diam}(V)\|f\|\epsilon_n. \tag{3.3.9}$$

This is because, for any two points $x, y \in V$ and n such that $\epsilon_n < d(x,y)$ there are points $x = x_0, x_1, \dots, x_k, x_{k+1} = y$ in V such that $d(x_i, x_{i+1}) < \epsilon_n$ and $k = \lfloor d(x,y)/\epsilon_n \rfloor$. Assuming that (3.3.9) holds we have that

$$u_n(y) - u_n(x) = \sum_{i=0}^{k} (u_n(x_{i+1}) - u_n(x_i)) \le \sum_{i=0}^{k} T^+_{\epsilon_n} u_n(x_i) \le K d(x,y) + K\epsilon_n, \tag{3.3.10}$$

where $K = 5\,\mathrm{diam}(V)\|f\|$. On the other hand for $\epsilon_n \ge d(x,y)$ we have $u_n(y) - u_n(x) \le K\epsilon_n$. These two facts then easily imply the equicontinuity, the required bound on $\max u_n - \min u_n$, for n large enough and the Lipshitz continuity of subsequential limits.

The rest of the proof will be devoted to establishing the bound in (3.3.9). We will use the "marching argument" of Armstrong and Smart from Lemma 3.9 in [3]. First by (3.3.4) if $y \in B(x, \epsilon_n)$ is such that $u_n(y) = \overline{u_n}^{\epsilon_n}(x)$ then using the fact that $\min f \le c_f(\epsilon_n) \le \max f$ we have

$$T^+_{\epsilon_n} u_n(x) \le T^+_{\epsilon_n} u_n(y) + \epsilon_n^2 \|f - c_f(\epsilon_n)\| \le T^+_{\epsilon_n} u_n(y) + 2\epsilon_n^2 \|f\|. \tag{3.3.11}$$

For a fixed n let $x_0 \in \overline{V}$ be a point where the value of $T^+_{\epsilon_n} u_n$ is maximized (it's a continuous function so it can be maximized) and let $M_n = T^+_{\epsilon_n} u_n(x_0)$ be the maximal value. Using the same argument as in (3.3.10) and the fact that V is bounded we have

$$u_n(y) - u_n(x) \le \Big(\frac{d(x,y)}{\epsilon_n} + 1\Big) M_n. \tag{3.3.12}$$

Then for any k let $x_{k+1} \in B(x_k, \epsilon_n)$ be such that $u_n(x_{k+1}) = \overline{u_n}^{\epsilon_n}(x_k)$. By (3.3.11) we have that $T^+_{\epsilon_n} u_n(x_{k+1}) \ge T^+_{\epsilon_n} u_n(x_k) - 2{\epsilon_n}^2 \|f\|$ and thus $T^+_{\epsilon_n} u_n(x_k) \ge T^+_{\epsilon_n} u_n(x_0) - 2k\epsilon_n^2 \|f\|$ which implies that for any $m \ge 1$

$$u_n(x_m) - u_n(x_0) = \sum_{k=0}^{m-1} T^+_{\epsilon_n} u_n(x_k) \ge m T^+_{\epsilon_n} u_n(x_0) - m^2 \epsilon_n^2 \|f\|.$$

Combining this with (3.3.12) we obtain that

$$m M_n - m^2 \epsilon_n^2 \|f\| \le \Big(\frac{\mathrm{diam}(V)}{\epsilon_n} + 1\Big) M_n,$$

which gives

$$M_n \le \frac{m^2 \epsilon_n^2 \|f\|}{m - 1 - \mathrm{diam}(V)/\epsilon_n}.$$

Plugging in $m = \lfloor 2\,\mathrm{diam}(V)/\epsilon_n + 2 \rfloor$ proves (3.3.9) for ϵ_n small enough. ∎

Proof of Theorem 3.1.10. The claim follows directly from Proposition 3.3.6 using the Arzela-Ascoli theorem. □

Finally Theorem 3.3.9 below proves Theorem 3.1.11. However we will first need to state an auxiliary result which appeared as Lemma 4.2 in [3]. For $x \in \mathbb{R}^d$ and $\epsilon > 0$, define $\mathbf{B}(x,\epsilon)$ as the closed ball around x of Euclidean radius ϵ. Also define the discrete infinity Laplacian $\widetilde{\Delta}^\epsilon_\infty$ on the whole $\mathbb{R}^d$ as

$$\widetilde{\Delta}^\epsilon_\infty v(x) = \frac{1}{\epsilon^2}\Big(\max_{\mathbf{B}(x,\epsilon)} v + \min_{\mathbf{B}(x,\epsilon)} v - 2v(x)\Big).$$

The first part of the following lemma is the content of Lemma 4.2 in [3], while the second part is contained in its proof (see (4.5) in [3]).

Lemma 3.3.7 (Lemma 4.2 and (4.5) from [3])**.** *For any open set U, function $\varphi \in C^3(U)$ and $\epsilon_0 > 0$ there is a constant $C > 0$, depending only on φ, such that the following holds.*

(i) For any point $x \in U$ that satisfies $\mathbf{B}(x, 2\epsilon_0) \subseteq U$ and $\nabla\varphi(x) \neq 0$ we have

$$-\Delta_\infty \varphi(x) \leq -\Delta^\epsilon_\infty \varphi(x) + C(1 + |\nabla\varphi(x)|^{-1})\epsilon,$$

for all $0 < \epsilon \leq \epsilon_0$.

(ii) For any $0 < \epsilon \leq \epsilon_0$ if $\mathbf{v} = \nabla\varphi(x)/|\nabla\varphi(x)|$ and $\mathbf{w} \in \mathbf{B}(0,1)$ is such that $\varphi(x + \epsilon\mathbf{w}) = \max_{\mathbf{B}(x,\epsilon)} \varphi$, then $|\mathbf{v} - \mathbf{w}| \leq C|\nabla\varphi(x)|^{-1}\epsilon$.

Next we give an auxiliary Lemma needed for the proof of Theorem 3.3.9. First define a cone in $\mathbb{R}^d$ with vertex x, direction $\mathbf{v} \in \mathbb{R}^d$, $|\mathbf{v}| = 1$, angle $2\arcsin\alpha$ and radius r as

$$\mathbf{C}(x, \mathbf{v}, \alpha, r) = \Big\{\lambda\mathbf{w} \in \mathbb{R}^d : 0 \leq \lambda \leq r, |\mathbf{w}| = 1, \mathbf{w}\cdot\mathbf{v} \geq 1 - \alpha\Big\}.$$

Lemma 3.3.8. *Let Ω be a domain with C^1 boundary $\partial\Omega$, let $x_0 \in \partial\Omega$. Assume $\varphi \in C^\infty(\overline{\Omega})$ is a smooth function on the closure of Ω and $\nabla_\nu\varphi(x_0) > 0$. Then we can find positive α and r and an open set U containing x_0 such that $\mathbf{C}(x, -\nabla\varphi(x)/|\nabla\varphi(x)|, \alpha, r) \subset \overline{\Omega}$, for all $x \in U \cap \overline{\Omega}$.*

Proof. Denote $\mathbf{v} = \nabla\varphi(x_0)/|\nabla\varphi(x_0)|$. By the continuity of $\nabla\varphi$ it is enough to prove that for some α and r and U we have $\mathbf{C}_x = \mathbf{C}(x, -\mathbf{v}, \alpha, r) \subset \overline{\Omega}$, for all $x \in U \cap \overline{\Omega}$. Moreover define the reverse cone $\mathbf{C}'_x = \mathbf{C}(x, \mathbf{v}, \alpha, r)$. It is clear that we can find α and r satisfying $\mathbf{C}_{x_0} \subset \overline{\Omega}$. Moreover, since the boundary $\partial\Omega$ is C^1, it is easy to see that, by decreasing α and r if necessary, we can assume that $\mathbf{C}_x \subset \overline{\Omega}$ and $\mathbf{C}'_x \subset \overline{\Omega^c}$, for all $x \in \partial\Omega$ with $|x - x_0| < 2r$. Then if $x \in \Omega$ is such that $|x - x_0| < r$ and $\mathbf{C}_x \not\subset \overline{\Omega}$ we can find a point $y \in \partial\Omega \cap \mathbf{C}_x$. Then the fact that $x \in \mathbf{C}'_y \cap \Omega$ and $|y - x_0| < 2r$ leads to contradiction. □

Theorem 3.3.9. *Let $\Omega \subset \mathbb{R}^d$ be a domain with C^1 boundary, (ϵ_n) a sequence of positive real numbers converging to zero and (u_n) a sequence of continuous functions on $\overline{\Omega}$ satisfying $-\Delta_\infty^{\epsilon_n} u_n = f - c_f(\epsilon_n)$. Any limit u of a subsequence of (u_n) is a viscosity solution to*

$$\begin{cases} -\Delta_\infty u = f - \overline{c}_f & \text{in } \Omega, \\ \nabla_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. We denote the subsequence again by (u_n). To prove that u is a solution to (3.1.5) we will check the assumptions from Definition 3.1.4 for local maxima. The conditions for local minima follow by replacing u and f by $-u$ and $-f$. Let $\varphi \in C^\infty(\overline{\Omega})$ be a smooth function and $x_0 \in \overline{\Omega}$ be a point at which $u - \varphi$ has a strict local maximum. We will prove the claim for $x_0 \in \partial\Omega$. For the case $x_0 \in \Omega$ see either of the two proofs of Theorem 2.11 in [3] (in Sections 4 and 5).

Assume that $\nabla_\nu \varphi(x_0) > 0$. For k large enough we can find points x_k so that $\lim_k x_k = x_0$ and such that $u_k - \varphi$ has a local maximum at x_k. We can assume that for all k we have $|\nabla\varphi(x_k)| > c$, for some $c > 0$. Denote $\mathbf{v}_k = -\nabla\varphi(x_k)/|\nabla\varphi(x_k)|$ and for a given k large enough, a point $\overline{x}_k \in \mathbf{B}(x_k, \epsilon_k)$ such that $\varphi(\overline{x}_k) = \min_{\mathbf{B}(x_k,\epsilon_k)} \varphi$. By Lemma 3.3.7 (ii) and Lemma 3.3.8 we can find α and r such that for k large enough we necessarily have $\overline{x}_k \in \mathbf{C}(x_k, \mathbf{v}_k, \alpha, r) \subset \overline{\Omega}$. For such k this readily implies that

$$-\widetilde{\Delta}_\infty^{\epsilon_k} \varphi(x_k) \leq -\Delta_\infty^{\epsilon_k} \varphi(x_k). \tag{3.3.13}$$

Lemma 3.3.7 further yields that there is a constant C such that, for k large enough

$$-\Delta_\infty \varphi(x_k) \leq -\widetilde{\Delta}_\infty^{\epsilon_k} \varphi(x_k) + C(1 + c^{-1})\epsilon_k. \tag{3.3.14}$$

Plugging (3.3.13) into (3.3.14) we obtain

$$-\Delta_\infty \varphi(x_k) \leq -\Delta_\infty^{\epsilon_k} \varphi(x_k) + C(1 + c^{-1})\epsilon_k. \tag{3.3.15}$$

Since $u_k - \varphi$ has a local maximum x_k we have that

$$-\Delta_\infty^{\epsilon_k} \varphi(x_k) \leq -\Delta_\infty^{\epsilon_k} u_k(x_k) = f(x_k) - c_f(\epsilon_k).$$

Inserting this into (3.3.15) and taking the limit as k tends to infinity then implies that $-\Delta_\infty \varphi(x_0) \leq f(x_0) - \overline{c}_f$. □

Proof of Theorem 3.1.11. The claim follows directly from Theorems 3.1.10 and 3.3.9. □

To prove Theorem 3.1.12 we will use Theorem 2.2 from [4]. A general assumption in [4] is that the boundary $\partial\Omega$ is decomposed into two disjoint parts, $\Gamma_D \neq \emptyset$ on which Dirichlet boundary conditions are given and Γ_N on which vanishing Neumann boundary conditions are given. While the assumption $\Gamma_D \neq \emptyset$ is crucial for their existence result (Theorem 2.4 in [4]) this assumption is not used in Theorem 2.2 from [4]. In the case when $\Gamma_D = \emptyset$ their result can be stated as follows.

Theorem 3.3.10 (Case $\Gamma_D \neq \emptyset$ of Theorem 2.2 in [4])**.** *Let Ω be a convex domain and $f, u\colon \Omega \to \mathbb{R}$ continuous functions such that u is a viscosity subsolution to the equation* (3.1.5)*. Then for any $\epsilon > 0$ it holds that $-\Delta_\infty^\epsilon \overline{u}^\epsilon \leq \overline{f}^{2\epsilon}$ on $\overline{\Omega}$.*

Proof of Theorem 3.1.12. By Theorem 3.1.11 the equation (3.1.5) has a solution when $\overline{c}_f = 0$. Now assume that u is a viscosity solution to (3.1.5). By Theorem 3.3.10 we have that $-\Delta_\infty^\epsilon \overline{u}^\epsilon \leq f + \operatorname{osc}(f, 2\epsilon)$. Now Lemma 3.3.2 implies that $c_f(\epsilon) \geq -\operatorname{osc}(f, 2\epsilon)$. Similarly one can obtain $c_f(\epsilon) \leq \operatorname{osc}(f, 2\epsilon)$ and the claim follows by taking the limit as $\epsilon \downarrow 0$. □

Remark 3.3.11. In the one dimensional case (say $\Omega = [0, r]$) the viscosity solutions to the equation (3.1.5) are standard solutions to the equation $-u'' = f$ where $u'(0) = u'(r) = 0$. It is clear that in this case $\overline{c}_f = \frac{1}{r}\int_0^r f(x)dx$. The fact that $\overline{c}_f$ is a linear functional of f relies heavily on the fact that the infinity Laplacian in one dimension is a linear operator. For higher dimensional domains $f \mapsto \overline{c}_f$ is in general not a linear functional. Next we show that a two dimensional disc is an example of a domain on which $\overline{c}_f$ is a nonlinear functional of f (essentially the same argument can be applied for balls in any dimension higher than one).

Take Ω to be a two dimensional disc of radius r centered at the origin, and assume that on $C(\overline{\Omega})$ the mapping $f \mapsto \overline{c}_f$ is a linear functional. Since it is clearly a positive functional and $\overline{c}_1 = 1$ by Riesz representation theorem it is of the form $\overline{c}_f = \int_{\overline{\Omega}} f d\mu$, for some probability measure μ on $\overline{\Omega}$. Let f be a radially symmetric function on $\overline{\Omega}$, that is $f(x) = g(|x|)$, where $g\colon [0, r] \to \mathbb{R}$ is a continuous function. Let $u_n\colon \overline{\Omega} \to \mathbb{R}$ and $v_n\colon [0, r] \to \mathbb{R}$ be the sequences of game values with the running payoff f played on $\overline{\Omega}$ and the running payoff g played on $[0, r]$ respectively, both games played with vanishing terminal payoff and step size ϵ. Using induction and (3.1.2) one can see that for any n we have $u_n(x) = v_n(|x|)$. Thus, using the expression for $\overline{c}_f$ in the one dimensional case we have that μ is necessarily of the form $\mu(dx, dy) = \frac{dxdy}{r\pi\sqrt{x^2+y^2}}$, that is μ is a radially symmetric measure which assigns equal measure to any annulus of given width.

Next let U_1, U_2, U_3 and U_4 be disjoint discs with centers at $(0, r/2)$, $(r/2, 0)$, $(0, -r/2)$, $(-r/2, 0)$ and radii $r/4$. By Ω_1 denote the smallest disc with the center $(0, r/2)$ which contains Ω. Let $f_1\colon \overline{\Omega_1} \to [0, 1]$ be a function with support in U_1 and which values are radially symmetric around $(0, r/2)$, decreasing with the distance from $(0, r/2)$ and such that $f_1(x) = 1$ whenever the distance between x and $(0, r/2)$ is no more than $r/4 - \delta_1$. Playing the game on $\overline{\Omega}_1$ with running payoff f_1 leads to

$$\overline{c}_{f_1} \geq \frac{2}{3r}\int_0^{r/4-\delta_1} 1dt = \frac{1}{6} - \frac{2\delta_1}{3r}. \tag{3.3.16}$$

Let w_n^1 and w_n be the sequences of game values with the running payoff f_1 played on $\overline{\Omega}_1$ and the running payoff $f_1|_{\overline{\Omega}}$ played on $\overline{\Omega}$ respectively, both games played with vanishing terminal payoff and step size ϵ. It is clear that w_n^1 are radially symmetric functions on $\overline{\Omega}_1$ with values decreasing with the distance from $(0, r/2)$. Using this and the induction on n one can see that $w_n^1(x) \leq w_n(x)$ for any $x \in \overline{\Omega}$. Therefore it holds that $\overline{c}_{f_1} \leq \overline{c}_{f_1|_{\overline{\Omega}}}$, which together with

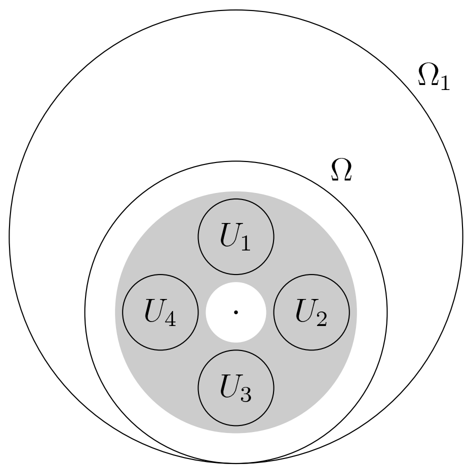

Figure 3.1: Example of functions from Remark 3.3.11. Functions f_1, f_2, f_3 and f_4 have supports in discs U_1, U_2, U_3 and U_4 respectively. The shaded area is the support of the function g.

(3.3.16) implies

$$\overline{c}_{f_1|_{\overline{\Omega}}} \geq \frac{1}{6} - \frac{2\delta_1}{3r}.$$

Take f_2, f_3 and f_4 to be equal to the function f_1 rotated clockwise for $\pi/2$, π and $3\pi/4$ respectively and $f = \sum_{i=1}^{4} f_i|_{\overline{\Omega}}$. By symmetry and the assumed linearity, the function $f\colon \Omega \to \mathbb{R}$ satisfies

$$\overline{c}_f \geq \frac{2}{3} - \frac{8\delta_1}{3r}. \tag{3.3.17}$$

Now take $g\colon \overline{\Omega} \to [0,1]$ to be a radially symmetric function on $\overline{\Omega}$, such that $g(x) > 0$ if and only if $r/4 - \delta_2 \leq |x| \leq 3r/4 + \delta_2$ and such that $f \leq g$. Again by the assumption we have

$$\overline{c}_g \leq \frac{1}{r}\int_{r/4-\delta_2}^{3r/4+\delta_2} 1dt = \frac{1}{2} + \frac{2\delta_2}{r}. \tag{3.3.18}$$

Since $\overline{c}_f \leq \overline{c}_g$, for small δ_1 and δ_2, inequalities (3.3.17) and (3.3.18) give a contradiction with the assumed linearity of the mapping $f \mapsto \overline{c}_f$. See Figure 3.1.

3.4 Uniqueness discussion

As Figure 3.2 illustrates, in the graph case, once we are given f, we do not always have a unique solution to

$$u(x) - \frac{1}{2}(\min_{y\sim x} u(y) + \max_{y\sim x} u(y)) = f(x),$$

even in the case of a finite graph with self loops. When the running payoff at each vertex is f, the corresponding "optimal play" will make $u(x_k)$ plus the cumulative running payoff a martingale (x_k is the position of the token at the kth step). Under such play, the players may spend all of their time going back and forth between the two vertices in one of the black-white pairs in Figure 3.2. (In the case of the second function shown, the optimal move choices are not unique, and the players may move from one black-white pair to another.) The basic idea is that as the players are competing within one black-white pair, neither player has a strong incentive to try to move the game play to another black-white pair.

A continuum analog of this construction appeared in Section 5.3 of [52], where it was used to show non-uniqueness of solutions to $\Delta_\infty u = g$ with zero boundary conditions. (In this case, g was zero except for one "positive bump" and one symmetric "negative bump". Game players would tug back and forth between the two bumps, but neither had the ability to make the the game end without giving up significant value.) It is possible that this example could be adapted to give an analogous counterexample to uniqueness in our setting (i.e., one could have two opposite-sign pairs of bumps separated by some distance on a larger domain with free boundary conditions — and players pulling back and forth within one pair of bumps would never have sufficient incentive to move to the other pair of bumps — and thus distinct functions such as those shown in Figure 3.2 could be obtained). However, the construction seems rather technical and we will not attempt it here.

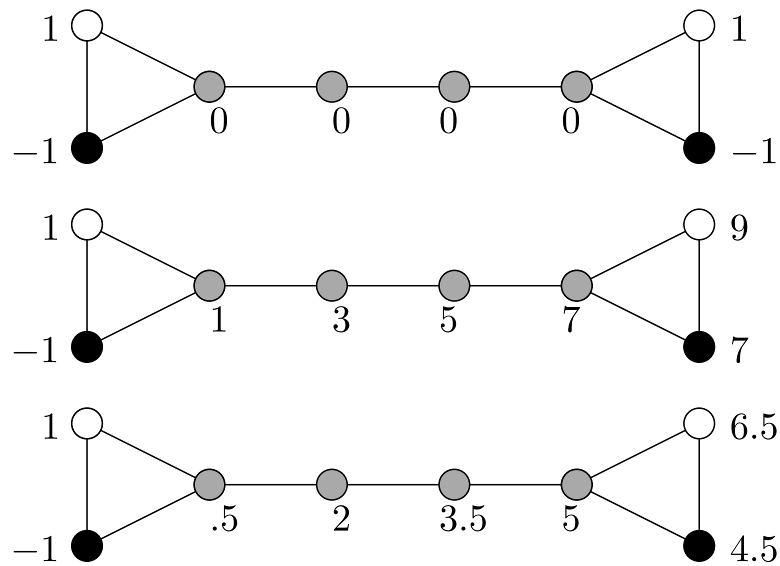

Figure 3.2: The difference equation (3.1.3) does not always have a unique (up to additive constant) solution. On each of the three copies of the same graph shown above (assume self-loops are included, though not drawn), we define f to be the function which is -1 on black vertices, 1 on white vertices, and 0 on gray vertices. Then each of the three functions illustrated by numbers above solves (3.1.3). In each case, the value at a gray vertex is the average of the largest and smallest neighboring values. The value at a black (white) vertex is 1 less (more) than the average of the largest and smallest neighboring values.

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