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Nonequilibrium formulation of varying-temperature bit erasure

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Landauer's principle states that erasing a bit of information at fixed temperature T costs at least $k_B T \ln 2$ units of work. Here we investigate erasure at varying temperature, to which Landauer's result does not apply. We formulate bit erasure as a stochastic nonequilibrium process involving a compression of configuration space, with physical and logical states associated in a symmetric way. Erasure starts and ends at temperature T , but temperature can otherwise vary with time in an arbitrary way. Defined in this way, erasure is governed by a set of nonequilibrium fluctuation relations that show that varying-temperature erasure can be done with less work than $k_B T \ln 2$. As a result, erasure and the complementary process of bit randomization can be combined to form a work-producing engine cycle.

I Introduction

Landauer's principle states that resetting a one-bit memory at temperature T results in the emission of at least $k_B T \ln 2$ units of heat, and costs at least $k_B T \ln 2$ units of work [1, 2]. This principle has been verified experimentally, and places limits on the efficiency of irreversible computation [3--16].

However, Landauer's bound does not apply if temperature varies [17], which theory and simulation suggests is a characteristic of optimal bit-erasure protocols [18--20]. In this paper we investigate bit erasure at varying temperature. We use a Langevin particle in an external potential as a model of a one-bit memory, by associating physical states with logical states in a symmetric way. We define erasure in a fluctuating nonequilibrium setting, as a finite-time transformation from an initial double-well potential, in which the particle starts in thermal equilibrium, to a final single-well

potential.

Defined in this way, erasure at fixed temperature is governed by the Crooks [21] and Jarzynski [22] nonequilibrium fluctuation relations [23]. These relations allow us to calculate the efficiency of a cycle composed of bit erasure and its time-reverse, bit randomization, and to show that the mean work required to achieve erasure is bounded by a quantity that approaches $k_B T \ln 2$ as the potential barrier becomes large. This bound is a statement of the second law of thermodynamics, and is the analog of the Landauer bound for the nonequilibrium formulation of erasure that we consider.

We then allow temperature to vary during erasure, with the starting and ending temperatures equal to the value T used in the fixed-temperature erasure scheme. We show that erasure at varying temperature obeys the varying-temperature fluctuation relations of Ref. [24]. We use these relations to show that the work required to do erasure is not bounded by $k_B T \ln 2$,

and that erasure and randomization can operate as a work-producing engine.

In Section II we introduce the model memory used in this paper, a Langevin particle in an external potential. In Section III we consider erasure at fixed temperature, to which standard fluctuation relations apply. In Section IV we comment on the effect of ending erasure with a double-well form of the potential, which is standard, rather than the single-well form of the potential used in this paper. In Section V we carry out erasure at varying temperature, and analyze it using nonequilibrium fluctuation relations valid for varying-temperature protocols. We conclude in Section VI.

II Model and simulation details

We consider a particle at position x undergoing the Langevin dynamics

$$\dot{x} = -\partial_x U(\mathbf{c}, x) + \xi(t), \quad (1)$$

where $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\beta(t)^{-1}\delta(t-t')$. The temperature of the system is therefore $T(t) = \beta(t)^{-1}$, which can vary with time (henceforth we work in units in which $k_B = 1$). The potential $U(\mathbf{c}, x)$ is

$$U(\mathbf{c}, x) = \frac{k}{2} (|x - c_0| - c_1)^2, \quad (2)$$

parameterized by the coefficients $\mathbf{c} = (c_0, c_1)$ and the spring constant k . For $c_1 \neq 0$ this potential has a double-well form with minima at $x = c_0 \pm c_1$ and barrier height $kc_1^2/2$. For $c_1 = 0$ the potential has a single-well form with its minimum at $x = c_0$. Unless otherwise stated we set $k = 20$. We can consider the model to define a 1-bit memory if we associate positions $x \leq 0$

and $x > 0$ with logical states $s = 0$ and 1 , respectively¹.

The initial reciprocal temperature of the system is $\beta(0) = 1$. The initial potential has coefficients $\mathbf{c}(0) = (0, 1)$, which has the double-well form shown in panel A of Fig. 1(b). The particle starts in equilibrium with this potential, and so the distribution of its initial positions is that of the Boltzmann distribution $\rho_0(\mathbf{c}(0), x)$, where $\rho_0(\mathbf{c}(t), x) = Z^{-1}(\mathbf{c}(t))e^{-\beta(t)U(\mathbf{c}, x)}$ and $Z(\mathbf{c}(t)) = \int dx' e^{-\beta(t)U(\mathbf{c}, x')}$.

Erasure trajectories are run for time t_f (note that the characteristic time for a free particle to diffuse a distance equal to the separation between potential wells is $t_0 = 2$ for $\beta(t) = 1$). We consider a memory-reset protocol $\mathbf{c}(t)$ similar to that used in Refs. [9, 25], namely

$$\mathbf{c}(t) = \begin{cases} (0, 1 - 2t/t_f) & 0 \leq t/t_f < 1/2 \\ (1 - 2t/t_f, 0) & 1/2 \leq t/t_f \leq 1, \end{cases} \quad (3)$$

which is plotted in Fig. 1(a). As shown in Fig. 1(b), this protocol merges the potential wells and slides the resulting single well to the left, until its minimum coincides with the left-hand minimum of the double-well form of the potential. At time $t = t_f$ the potential has the coefficients $\mathbf{c}(t_f) = (-1, 0)$ and the single-well form shown in panel B of Fig. 1(b). (We shall also consider the effect of switching back to the initial form of the potential, as shown in panel B' of Fig. 1(b).) At time t_f ,

¹ The association of the case $x = 0$ with state $s = 0$ but not state $s = 1$ means that we associate positions and logical states in a (very slightly) asymmetric manner. But since x is real-valued, this asymmetry has no practical significance.

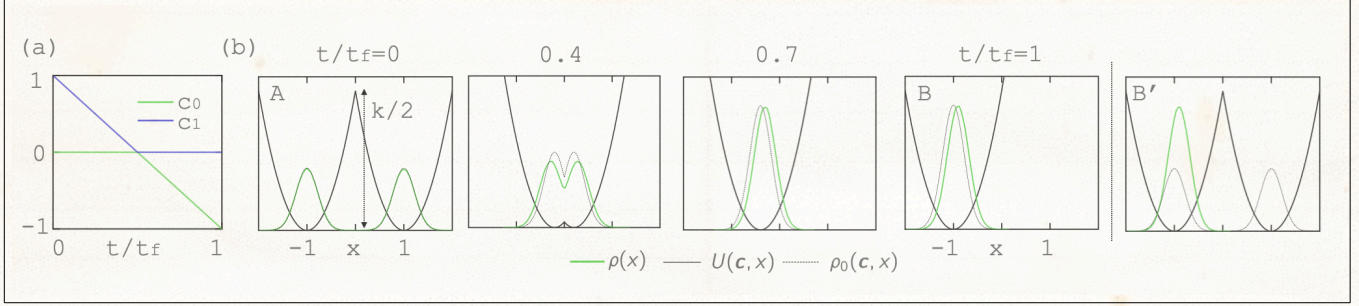


FIG. 1: (a) Erasure protocol (3). (b) Effect of the erasure protocol for trajectory time $t_f = 1$, showing potential (black), associated Boltzmann distribution (black dashed), and instantaneous particle-position distribution (green), calculated over 10^6 trajectories. Erasure ends with the form of the potential labeled B; we also discuss the effect of switching back to the double-well form B'.

temperature is returned to its initial value, $\beta(t_f) = 1$.

The free-energy change ΔF_e for the transformation from double- to single-well forms during erasure (A \rightarrow B in Fig. 1(b)) is $\Delta F_e = -\beta^{-1} \ln[Z(\mathbf{c}(t_f))/Z(\mathbf{c}(0))]$, giving

$$\beta \Delta F_e = \ln \left(2 - \operatorname{erfc} \sqrt{\beta k/2} \right), \quad (4)$$

where $\operatorname{erfc} x \equiv 2\pi^{-1/2} \int_x^\infty dt e^{-t^2}$ is the complementary error function. For $\beta = 1$ (the start and end temperature of our erasure protocol) and large spring constant k , we have

$$\Delta F_e \approx \ln 2 - (2\pi k)^{-1/2} e^{-k/2}, \quad (5)$$

which approaches $\ln 2$ as k diverges. For the value $k = 20$, ΔF_e is less than $\ln 2$ by about 4×10^{-6} .

We shall also consider the time-reverse of (3), $\mathbf{c}_r(t) = \mathbf{c}(t_f - t)$. This protocol effects randomization of a bit set initially to state 0, starting in equilibrium with the potential in the single-well form with $\mathbf{c}_r(0) = (-1, 0)$, shown in panel B of Fig. 1(b). The free-energy change for the randomization protocol is the negative of (4).

We integrate Eq. (1) using a first-order Euler scheme with timestep $\Delta t = 10^{-3}$, evaluated by electronic computing machine. At step $i = 1, 2, \dots, N = \lfloor t_f/\Delta t \rfloor$ of the simulation the time is $t_i = i\Delta t$, the position of the particle is x_i , and the values of the coefficients are \mathbf{c}_i . The work done in a single trajectory is

$$W = \sum_{i=0}^{N-1} [U(\mathbf{c}_{i+1}, x_i) - U(\mathbf{c}_i, x_i)], \quad (6)$$

and we shall consider averages $\langle W \rangle$ and distributions $P(W)$ taken over 10^6 independent trajectories. The probability of erasure is $p_e = \langle \Theta(-x_N) \rangle$, where $\Theta(-x) = 1$ if $x \leq 0$ and $\Theta(-x) = 0$ otherwise.

III Constant-temperature protocols

We first consider constant-temperature erasure and randomization protocols for which $\beta(t) = 1$ for all t . Fig. 2 shows the results of simulations carried out for a range of trajectory times t_f ; averages and distributions are calculated over 10^6 independent trajectories.

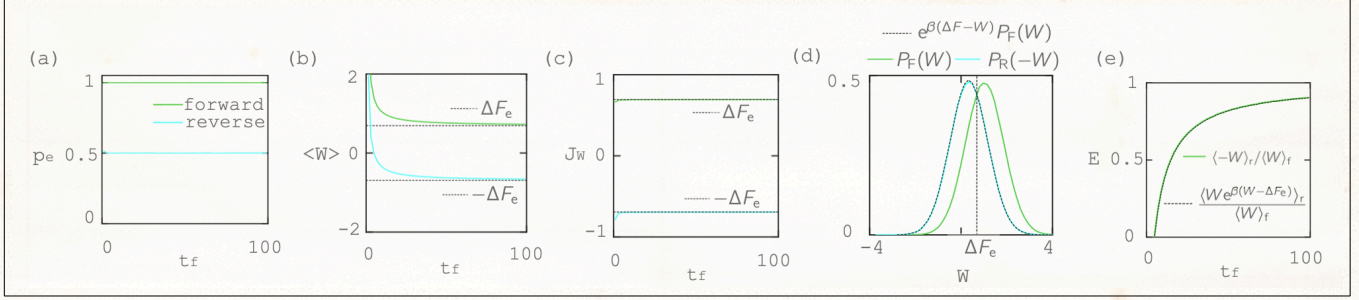


FIG. 2: Constant-temperature erasure protocol shown in Fig. 1 (green) and its time reverse, randomization (cyan), for various trajectory times t_f . (a) Probability of erasure. (b) Mean work done, which satisfies the second law of thermodynamics, Eq. (7). (c) Jarzynski free-energy estimator (9). (d) Crooks fluctuation relation (10) for trajectory length $t_f = 10$. (e) Efficiency of the forward-reverse erasure procedure, calculated using both forward and reverse protocols (green) and the forward protocol only (black dashed). The color scheme in (e) is distinct from that in the other panels.

In Fig. 1(a) we show the probability of erasure (the probability that the particle has coordinate $x < 0$ at time t_f), which is close to 1 and 1/2 for the erasure and randomization protocols, respectively. For the erasure protocol, 1 or 2 in a million trajectories results in a failed erasure (we will discuss the significance of this fact more in the following section). In panel (b) we show the mean work $\langle W \rangle$ under the two protocols, which obeys the second law of thermodynamics in the form [2]

$$\langle W \rangle \geq \Delta F. \quad (7)$$

Here ΔF is the free-energy change associated with the protocol, which is ΔF_e for erasure and $-\Delta F_e$ for randomization. The work values in panel (b) approach the asymptotes ΔF_e and $-\Delta F_e$ as $\sim 1/t_f$.

In Fig. 1(c) we show that the protocols obey the Jarzynski equality [22]

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}, \quad (8)$$

by plotting the free-energy estimator

$$J_W = -\beta^{-1} \ln \left(N_{\text{traj}}^{-1} \sum_{i=1}^{N_{\text{traj}}} e^{-\beta W_i} \right). \quad (9)$$

(Recall that $\beta = 1$ for the fixed-temperature erasure protocol.) In (8), the angle brackets denote an average over trajectories that start in thermal equilibrium and enact the specified protocol (with $\Delta F = \pm \Delta F_e$ for erasure and randomization, respectively). In (9), $N_{\text{traj}} = 10^6$ is the number of trajectories used, and W_i is the work value associated with trajectory i . Fig. 2(c) shows that (9) provides a good estimate of the free-energy change for erasure and randomization, for all but the smallest value of t_f .

In Fig. 1(d) we show, for one value of t_f , that erasure and randomization protocols obey the Crooks work fluctuation relation [21]

$$P_F(W) e^{-\beta W} = e^{-\beta \Delta F} P_R(-W). \quad (10)$$

Here $P_F(W)$ is the probability distribution of W for erasure

('forward') protocols, while $P_R(-W)$ is the probability distribution of $-W$ for randomization ('reverse') protocols. The quantity $\Delta F = \Delta F_e$ refers to the free-energy change for the forward protocol.

Panels (b), (c), and (d) of Fig. 1 make increasingly detailed statements about work fluctuations for these protocols, illustrating Equations (7), (8), and (10), respectively. Eq. (8) can be obtained from Eq. (10) by integration, while Eq. (7) can be obtained from Eq. (8) by application of the Jensen inequality.

In Fig. 1(e) we show the efficiency $\langle -W \rangle_r / \langle W \rangle_f$ of the forward-reverse cycle, the ratio of the work extracted by randomization to that expended during erasure. This quantity is positive for trajectory lengths $t_f \gtrsim 1$, and approaches 1 for large t_f . In that limit, a logically irreversible procedure, the resetting and subsequent randomization of a bit, is done in a thermodynamically reversible way [2, 14]. The forward-reverse cycle could be also be considered a battery, with work stored during erasure and extracted during randomization.

Using (10) we can rewrite averages over the reverse process in terms of averages over the forward process, $\langle (\cdot) \rangle_r = \langle (\cdot) e^{\beta(W - \Delta F)} \rangle_f$, and so compute the efficiency of the erasure-randomization cycle by doing simulations of the forward process only. This fact is shown by the coincidence of the green and black lines in Fig. 1(e).

IV Consequences of the final form of the potential

We have defined the erasure process as one that starts with the double-well potential given by (2) with coefficients $\mathbf{c}(0) = (0, 1)$,

$$U_d(x) = U(\mathbf{c}(0), x) = \frac{k}{2}(|x| - 1)^2, \quad (11)$$

and ends with the single-well potential with coefficients $\mathbf{c}(t_f) = (-1, 0)$,

$$U_s(x) = U(\mathbf{c}(t_f), x) = \frac{k}{2}(x + 1)^2. \quad (12)$$

The free-energy change ΔF_e at temperature β^{-1} associated with this change is given by Eq. (4), which is very close to $\beta^{-1} \ln 2$ for the value $k = 20$ used in our simulations. The mean work required to effect this change is therefore constrained by the second law, Eq. (7), and is $\langle W \rangle \geq \Delta F_e$.

However, is conventional in the literature to formulate erasure problems that start and end with a double-well potential [1, 2, 8, 9]. The practical consequences of switching back to the double-well form at the end of the erasure protocol are relatively minor, as long as the trap spring constant k is large enough, but doing so introduces a conceptual complication: the free-energy change is now zero, and so the work required to do erasure is bounded only by zero, $\langle W \rangle \geq 0$.

To illustrate this point, imagine that we carry out the erasure protocol of Figs. 1 and 2 very slowly, so that the particle-position distribution remains in quasiequilibrium with the potential (similar considerations apply to nonequilibrium protocols that result in erasure with probability close to

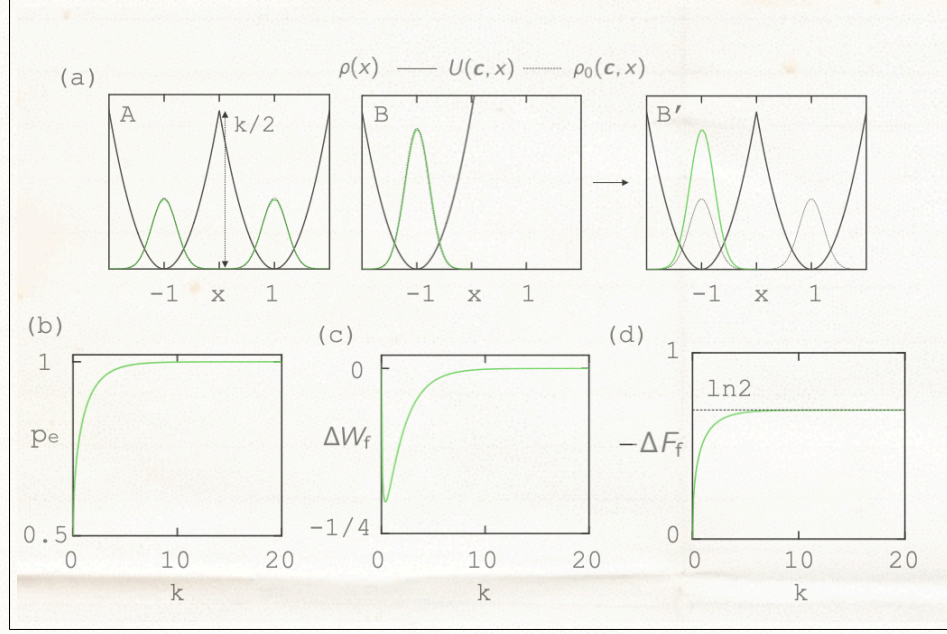


FIG. 3: Effect of switching back to a double-well potential after the erasure protocol of Fig. 1. (a) Slow transformation from A→B, followed by a sudden switch to B'. (b) Probability of erasure if the final-time particle-position distribution is in equilibrium with the single-well potential B. (c) Change of work upon switching to the double-well form B'. (d) Change of free energy (at $\beta=1$) upon switching from B to the double-well form B'.

unity). This scenario is sketched in Fig. 3(a), with panels A and B representing the start and end of the process. Following the transformation A→B, the particle-position distribution for $\beta=1$ is

$$\rho_s(x) = \sqrt{\frac{k}{2\pi}} e^{-\frac{k}{2}(x+1)^2}, \quad (13)$$

reflecting equilibrium with the single-well form of the potential, Eq. (12). The probability of erasure is the probability that the particle resides in the sector $x \leq 0$, which is

$$p_e = \int_{-\infty}^0 dx \rho_s(x) = 1 - \frac{1}{2} \operatorname{erfc}(\sqrt{k/2}). \quad (14)$$

Eq. (14) is plotted as a function of trap spring constant k in Fig. 3(b). The erasure probability p_e is less than unity by about 4×10^{-6} at $k=20$.

The mean work required to do this

erasure is $\langle W \rangle = \Delta F_e$ (if done infinitely slowly). We then switch suddenly to the double-well form of the potential, as shown in panel B'. What happens?

The difference in energy upon changing the single-well potential $U_s(x)$ to the double-well potential $U_d(x)$ is

$$\Delta U(x) \equiv U_d(x) - U_s(x) = \begin{cases} 0 & (x < 0) \\ -2kx & (x \geq 0) \end{cases}, \quad (15)$$

and so the mean change of work caused by the final-time switch in potential is

$$\begin{aligned} \Delta W_f &= \int_{-\infty}^{\infty} dx \Delta U(x) \rho_s(x) = -2k \int_0^{\infty} dx x \rho_s(x) \\ &= k \operatorname{erfc}(\sqrt{k/2}) - \sqrt{\frac{2k}{\pi}} e^{-k/2}. \end{aligned} \quad (16)$$

This quantity is plotted as a function of k in Fig. 3(c). For $k=20$, the mean change of work ΔW_f is negligible, and the work required to perform the process

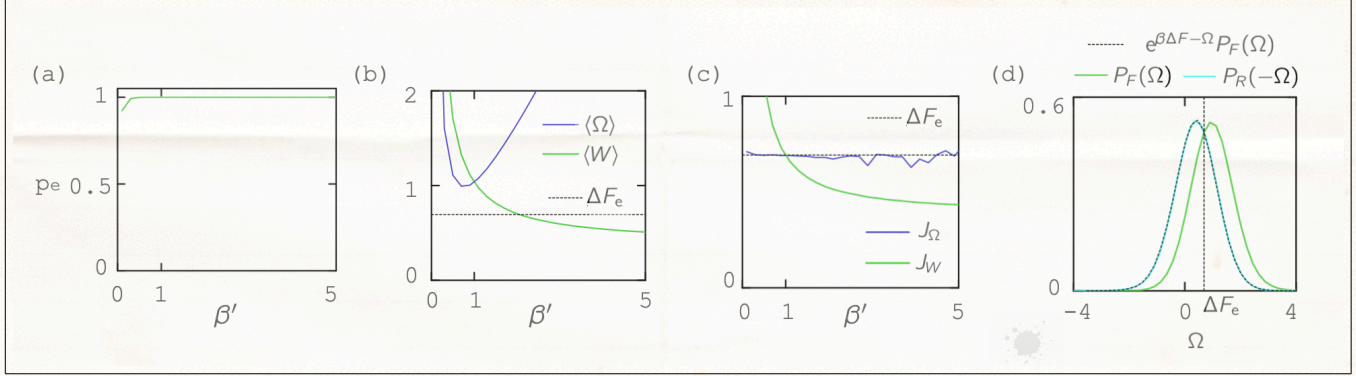


FIG. 4: Erasure process of Fig. 1(a) for trajectory time $t_f = 10$, now at varying temperature: $\beta(t) = \beta'$ for $0 < t < t_f$, with $\beta(0) = \beta(t_f) = 1$. (a) Probability of erasure as a function of β' . (b) The mean of Ω is bounded by ΔF_e , by Eq. (21), but the mean work $\langle W \rangle$ is not. (c) The integral fluctuation relation (18) and (d) the fluctuation relation (24) hold at varying temperature.

$A \rightarrow B \rightarrow B'$ is essentially the same as that required to perform the process $A \rightarrow B$, namely ΔF_e .

However, the change of free energy upon making the change $B \rightarrow B'$ is significant. For temperature $\beta^{-1} = 1$, the free-energy change resulting from the final change of potential is, by the Zwanzig [26] or Jarzynski identities,

$$\Delta F_f = -\ln \int_{-\infty}^{\infty} dx e^{-\Delta U(x)} \rho_s(x) = -\Delta F_e. \quad (17)$$

The value ΔF_f is exactly the negative of the free-energy change Eq. (4) resulting from the transformation $A \rightarrow B$. The total change in free energy is therefore $\Delta F_{\text{tot}} = \Delta F_e + \Delta F_f = 0$. This is obvious, because we have started and ended with the same potential, and so the total change in free energy must be zero.

We then have $\langle W \rangle \geq 0$ by the second law of thermodynamics, and so the measured value of $\langle W \rangle \approx \Delta F_e$ is very far from the bound. This discrepancy results from the fact that the final-time change of work is related to the probability of non-erasure, which for large k is very small, while the final-time change in

free energy is an exponential average that applies large weight to very rare trajectories that exhibit non-erasure. It is difficult to determine this change of free-energy from numerical evaluation of the Jarzynski identity. The integral in (17) goes as $\sim \int_0^{\infty} e^{-k(x-1)^2/2}$, which is dominated by the contribution from the point $x_0 = 1$. For $k = 20$, the likelihood of realizing this value of x is $\rho_s(x_0) \approx 10^{-17}$, and so we would need to run more than 10^{17} trajectories in order to accurately measure $\Delta F_{\text{tot}} = 0$ (similar sampling issues have been noted elsewhere [23, 27--29]).

If we end with the transformation $B \rightarrow B'$ then we could preserve the bound $\langle W \rangle \geq \Delta F_s$ by ensuring that the final-time particle-position distribution $\rho(x(t_f))$ is strictly zero for $x > 0$. However, it is difficult to do this for finite-time protocols involving finite potential energy barriers. Conceptually, therefore, it is cleaner to formulate erasure as a compression of configuration space (e.g. the transformation $A \rightarrow B$ from a double- to a single-well potential) with

logical states identified with physical states in a symmetric way. Then the mean work expended in a nonequilibrium erasure process is rigorously bounded as $\langle W \rangle \geq \Delta F_e$, which is $\approx \ln 2$ for large enough spring constant². This formulation also makes it simpler to identify the origin of the "violation" of the bound $\langle W \rangle \geq \Delta F_e$ for varying-temperature protocols, as we discuss in the following section.

V Varying-temperature erasure

We now allow the erasure protocol to occur at a non-constant temperature $\beta^{-1}(t)$, with the constraint that $\beta(0) = \beta(t_f) = \beta = 1$ (note that β with no time argument or subscript label refers to the fixed reciprocal temperature, here chosen to be unity, at the start and end of the protocol). In this case the Jarzynski equality (8) is replaced by [24]

$$\langle e^{-\Omega} \rangle = e^{-\beta \Delta F}. \quad (18)$$

Here $\Omega \equiv \beta W + \beta Q - \Sigma$, where Q is the heat exchanged with the bath (the energy transferred to the system from the bath),

$$Q = \sum_{i=0}^{N-1} [U(\mathbf{c}_{i+1}, x_{i+1}) - U(\mathbf{c}_{i+1}, x_i)], \quad (19)$$

and $-\Sigma$ is the entropy change associated with the trajectory,

$$\Sigma = \sum_{i=0}^{N-1} \beta_{i+1} [U(\mathbf{c}_{i+1}, x_{i+1}) - U(\mathbf{c}_{i+1}, x_i)]. \quad (20)$$

² To model a two-state device we could consider the restoration of the double-well potential to constitute an additional step of the protocol, which could be done with negligible change of work.

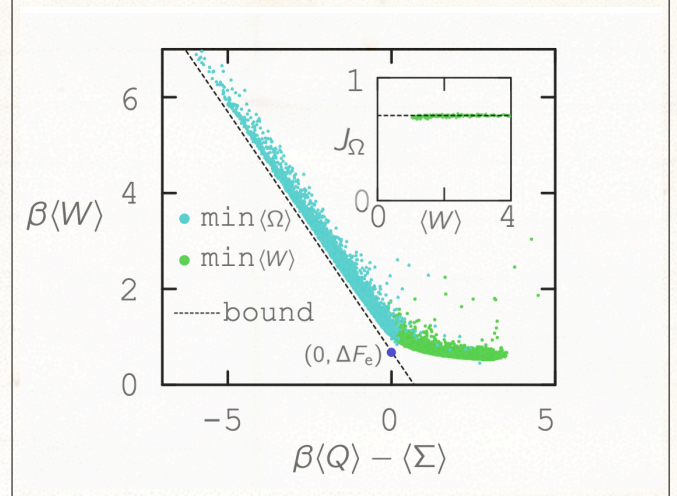


FIG. 5: Averaged values of heat, work, and entropy production for a large number of erasure protocols with non-constant temperature, plotted so that the bound (22) is shown as a black dashed line. The blue dot is the equivalent of the conventional Landauer bound. Inset: Estimator (23) for a subset of trajectories from the main figure, illustrating the exact result (18). The dotted black line is the free-energy change ΔF_e .

In the above expressions the label $i = 1, 2, \dots, N = \lfloor t_f / \Delta t \rfloor$ refers to the discrete simulation step. The angle brackets in (18) denote an average over trajectories that start in equilibrium at temperature β^{-1} , end at the same temperature (not necessarily in equilibrium), and otherwise involve an arbitrary variation of temperature and other control parameters. The quantity ΔF is the free-energy change resulting from the protocol, evaluated (as in (8)) at temperature β^{-1} .

The Jensen inequality applied to (18) yields the second law of thermodynamics in the form

$$\langle \Omega \rangle \geq \beta \Delta F, \quad (21)$$

the statement that the total entropy production must be non-negative. This

can be written

$$\langle W \rangle \geq \Delta F + \beta^{-1} \langle \Sigma \rangle - \langle Q \rangle. \quad (22)$$

Given that the combination $\beta^{-1} \langle \Sigma \rangle - \langle Q \rangle$ can be positive or negative, the statement (22) indicates that if temperature can vary then the work required to do erasure is no longer bounded by $\Delta F = \Delta F_e$.

In Fig. 4 we illustrate these relations numerically. We perform the erasure protocol of Figs. 1 and 2 for trajectory length $t_f = 10$, but now allow temperature to vary. We set $\beta(t) = \beta'$ for $0 < t < t_f$ (with $\beta(0) = \beta(t_f) = 1$). We consider a range of β' either side of 1.

In Fig. 4(a) we show the erasure probability resulting from this protocol, as a function of β' . As β' becomes large enough, all of 10^6 trajectories achieve erasure.

In Fig. 4(b) we show that $\langle \Omega \rangle$ is bounded by ΔF_e , but the mean work $\langle W \rangle$ is not: for large enough β' we observe $\langle W \rangle < \Delta F_e$. This is so because the second law applied to varying-temperature processes, Eq. (22), does not constrain mean work to be greater than the free-energy change for the process. We have ruled out other possibilities for this effect: with our definition of erasure we have $\Delta F_e \approx \ln 2$, for large enough k , rather than $\Delta F_e = 0$. We have also shown that $p_e \approx 1$, and so the information entropy change upon enacting the process is essentially $\ln 2$.

In Fig. 4(c) we show that the free-energy estimator

$$J_\Omega = -\beta^{-1} \ln \left(N_{\text{traj}}^{-1} \sum_{i=1}^{N_{\text{traj}}} e^{-\Omega_i} \right) \quad (23)$$

returns an approximation of the

free-energy change ΔF_e for erasure (the estimator (9) applies only to a fixed-temperature protocol). The statistical error in J_Ω is smallest in the region $\beta' \lesssim 1$, where Ω and its fluctuations are smallest.

In Fig. 4(c) we show, for the value $\beta' = 0.7$, the validity of the fluctuation relation associated with (18), namely [24]

$$P_F(\Omega) e^{-\Omega} = e^{-\beta \Delta F} P_R(-\Omega). \quad (24)$$

Here $P_F(\Omega)$ denotes the probability distribution of Ω under the erasure protocol, and $P_R(-\Omega)$ the distribution of $-\Omega$ under the time-reversed randomization protocol.

In Fig. 5 we show that the bound (22) is obeyed for a large number of protocols, for trajectories of length $t_f = 10$. Following Ref. [30] we expressed the time-dependent protocol $(\beta(t), \mathbf{c}(t))$ using a neural network, and trained the neural network by genetic algorithm to minimize either $\langle \Omega \rangle$ (cyan dots) or $\langle W \rangle$ (green dots). During training, we collected the values of $\langle \Omega \rangle$ and $\langle W \rangle$ generated by each protocol encountered, and plotted them as shown. On this figure the bound (22) is a straight line of slope -1 that passes through the point $(0, \Delta F_e)$. This point, Eq. (7), is the analog of the constant-temperature Landauer bound for the nonequilibrium formulation of erasure studied here.

The bound (22) is descriptive (i.e. tight) in the left-hand portion of the figure, and less so in the right-hand portion. By contrast, the relations (18) and (24) are exact, and provide more detailed statements than the bound (22) about the relationship between work and entropy production for

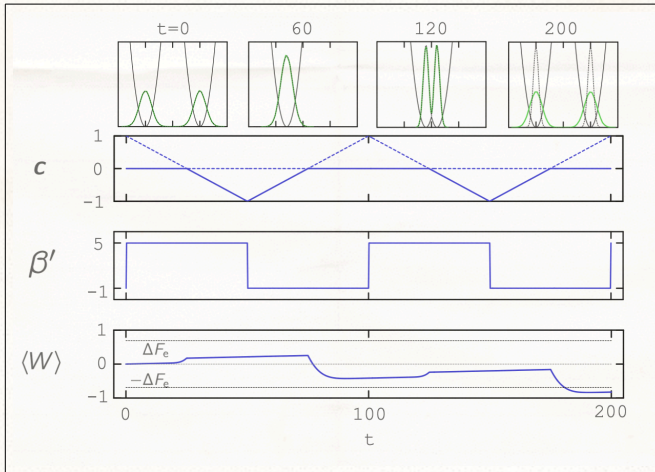


FIG. 6: Two cycles of erasure and randomization, running the erasure protocol of Fig. 1(a) and its time reverse at different temperatures. Under this cycle, work can be extracted. Averages are calculated over 10^5 independent trajectories. Snapshots at the top have the same format as Fig. 1(b).

varying-temperature protocols. In the inset of Fig. 5 we show that Eq. (18) holds for these protocols³.

We note that the bound (22) is equivalent to that given in Ref. [17]: from the first law of thermodynamics we have $W + Q = \Delta U$, the change of internal energy of the system, and so (22) can be written $\langle \Delta U \rangle - \Delta F \geq \beta^{-1} \langle \Sigma \rangle$. Noting that $\Delta F = \langle \Delta U \rangle - \beta^{-1} \langle \Delta S \rangle$, where ΔS is the change in entropy of the system, we can write (22) as $\langle \Sigma \rangle \leq \langle \Delta S \rangle$. This is the bound resulting from the non-negativity of Eq. (1) of Ref. [17] (noting that, in that paper, Q is heat transferred from the system to the bath, and so is the negative of the heat Q considered here).

One consequence of the inequality (22) is that a varying-temperature

erasure-randomization cycle can be run as a work-producing engine. As shown in Fig. 3(b), erasure carried out at a suitably large value of β' can be done with less work than ΔF_e . If we randomize the memory at reciprocal temperature $\beta' = 1$, we can extract work up to an amount ΔF_e . We should therefore be able to extract work from the erasure-randomization cycle, if performed slowly enough. In Fig. 6 we confirm this prediction numerically. We perform a varying-temperature cycle of erasure, randomization, erasure, and randomization, with each component of the cycle lasting for time $t = 50$. As shown in the bottom panel, work is extracted.

VI Conclusions

We have investigated bit erasure done at varying temperature, to which Landauer's result does not apply. We have formulated erasure as a stochastic nonequilibrium process involving a compression of configuration space. Erasure starts and ends at temperature T , but temperature can otherwise vary with time in an arbitrary way. Defined in this way, erasure is described by a set of nonequilibrium fluctuation relations that place limits on the efficiency of the process and its reverse, bit randomization. Under a varying-temperature protocol, erasure and randomization can be operated as a work-producing cycle.

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³ Note that $\Omega = \beta W$ when temperature is fixed, in which case (18) and (24) reduce to the standard relations (8) and (10), respectively.

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