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Optimal strategies for repeated games

### Permalink

<https://escholarship.org/uc/item/4w69h2xx>

### Journal

Advances in Applied Probability, 13(2)

### ISSN

0001-8678

### Authors

Finkelstein, Mark

Whitley, Robert

### Publication Date

1981-06-01

### DOI

10.2307/1426692

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## OPTIMAL STRATEGIES FOR REPEATED GAMES

MARK FINKELSTEIN\* AND  
ROBERT WHITLEY,\* *University of California, Irvine*

### Abstract

We extend the optimal strategy results of Kelly and Breiman and extend the class of random variables to which they apply from discrete to arbitrary random variables with expectations. Let  $F_n$  be the fortune obtained at the  $n$ th time period by using any given strategy and let  $F_n^*$  be the fortune obtained by using the Kelly–Breiman strategy. We show (Theorem 1(i)) that  $F_n/F_n^*$  is a supermartingale with  $E(F_n/F_n^*) \leq 1$  and, consequently,  $E(\lim F_n/F_n^*) \leq 1$ . This establishes one sense in which the Kelly–Breiman strategy is optimal. However, this criterion for ‘optimality’ is blunted by our result (Theorem 1(ii)) that  $E(F_n/F_n^*) = 1$  for many strategies differing from the Kelly–Breiman strategy. This ambiguity is resolved, to some extent, by our result (Theorem 2) that  $F_n^*/F_n$  is a submartingale with  $E(F_n^*/F_n) \geq 1$  and  $E(\lim F_n^*/F_n) \geq 1$ ; and  $E(F_n^*/F_n) = 1$  if and only if at each time period  $j$ ,  $1 \leq j \leq n$ , the strategies leading to  $F_n$  and  $F_n^*$  are ‘the same’.

KELLY CRITERION; OPTIMAL STRATEGY; FAVORABLE GAME; OPTIMAL GAMBLING SYSTEM; PORTFOLIO SELECTION; CAPITAL GROWTH MODEL

### 1. Introduction

Suppose a gambler is given the opportunity to bet a fixed fraction  $\gamma$  of his (infinitely divisible) capital on successive flips of a biased coin: on each flip, with probability  $p > \frac{1}{2}$  he wins an amount equal to his bet and with probability  $q = 1 - p$  he loses his bet. What is a good choice for  $\gamma$  and why is it good?

This question is subtle because the obvious answer has an obvious flaw. The obvious answer is for the gambler to choose  $\gamma = 1$  to maximize the expected value of his fortune. The obvious flaw is that he is then broke in  $n$  or fewer trials with probability  $1 - p^n$ , which tends to 1 as  $n$  tends to  $\infty$ .

A germinal answer was given by Kelly [10]: a gambler should choose  $\gamma = p - q$  so as to maximize the expected value of the *log* of his fortune. He shows that a gambler who chooses  $\gamma = p - q$  will ‘with probability 1 eventually get ahead and stay ahead of one using any other value of  $\gamma$ ’ ([10], p. 920).

In an important paper Breiman [4] generalizes and considers strategies other

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Received 23 April 1980; revision received 12 August 1980.

\* Postal address: Department of Mathematics, University of California, Irvine, CA 92717, U.S.A.

than fixed-fraction strategies and generalizes the random variable as follows: Let  $X$  be a random variable taking values in  $\{1, 2, \dots, s\} = I$ ,  $\mathcal{C}$  be a class  $\{A_1, A_2, \dots, A_r\}$  of subsets of  $I$  whose union is  $I$ , and  $o_1, o_2, \dots, o_r$  be positive numbers (odds). If for one round of betting a gambler bets fractional amounts  $\beta_1, \beta_2, \dots, \beta_r$  of his capital on the events  $\{X \in A_1\}, \dots, \{X \in A_r\}$ , then when  $X = i$  he gets a payoff of  $\sum \beta_j o_j$  summed over those  $j$  with  $i$  in  $A_j$ . In this setting Breiman discusses several 'optimal' properties of the fixed-fraction strategy which chooses  $\beta_1, \beta_2, \dots, \beta_r$  so as to maximize the expected value of the log of the fortune and then bets these fractions on each trial, leading to the fortune  $F_n^*$  at the conclusion of the  $n$ th trial. He shows that if  $F_n$  is a fortune resulting from the use of any strategy, then  $\lim F_n/F_n^*$  almost surely exists and  $E(\lim F_n/F_n^*) \leq 1$ . In what follows we shall be concerned solely with magnitude results, like this asymptotic magnitude result of Breiman's, but the reader should be aware that under additional hypotheses Breiman also shows that  $T(x)$ , the time required to have a fortune exceeding  $x$ , has an expectation which is asymptotically minimized by the above fixed-fraction strategy.

The problem of how to apportion capital between various random variables is exactly the problem of portfolio selection, and so it is correct to suppose that these results on optimal allocation of capital are of considerable interest to economists, as Kelly recognized ([10], p. 926). He also prophetically realized that economists, familiar with logarithmic utility, could easily misunderstand his result and think, incorrectly, that the choice of maximizing the expected value of the log of the fortune depended upon using logarithmic utility for money. For discussion see [15], p. 216 and [17]. An interesting concise discussion of the 'capital growth model of Kelly [10], Breiman [4], and Latané [11]' from an economic point of view can be found in [3].

A brief discussion of Kelly's proof will motivate his criterion and allow us to make an important conceptual distinction between his results and Breiman's. Suppose a gambler bets the fixed fraction  $\gamma$  of his capital at each toss of the  $p$ -coin. Kelly considers the exponential growth rate

$$G = \lim \log [(F_n/F_0)^{1/n}].$$

If our gambler has  $W$  wins and  $L$  losses in the first  $n$  trials,  $F_n = (1 + \gamma)^W (1 - \gamma)^L F_0$ , so

$$G = \lim \left( \frac{W}{n} \log(1 + \gamma) + \frac{L}{n} \log(1 - \gamma) \right) = p \log(1 + \gamma) + q \log(1 - \gamma),$$

by the law of large numbers. The growth rate  $G$  is maximized by  $\gamma = p - q$ , and if he uses another  $\gamma$  his  $G$  will be less and therefore eventually so will his fortune. A complication enters when we consider, as Kelly did not, strategies which are not fixed-fraction strategies. In that case we can have different

strategies with the same  $G$ , e.g., use  $\gamma = 1$  for the first 1000 trials and then use  $\gamma = p - q$ . This complication is intrinsic in the use of  $G$  and it has consequences which are quite serious for any application. For example, two strategies which at trial  $n$  give fortunes, respectively, of 1 and  $\exp(\sqrt{n})$ , both have  $G = 1$ ! It is obviously unsatisfactory to regard these two strategies with the same  $G$  as 'the same', but it is done because using  $G$  makes it easy to extend the Kelly results to more general situations which involve more general random variables; using  $G$  the argument is a simple one employing either the law of large numbers or techniques which 'rely heavily on those used to generalize the law of large numbers' ([15], p. 218). Breiman understood the problems created by using  $G$  and so he considered  $F_n/F_n^*$ , not  $(F_n/F_n^*)^{1/n}$ . This is mathematically more difficult, but the results are more useful.

## 2. Definitions and lemmas

We shall consider situations with the property that at each time period a gambler can lose no more than the amount he invests, e.g., buying stock or betting on Las Vegas table games. Since there is a real limit to a gambler's liability, based on his total fortune, a broad interpretation of the phrase 'the amount he invests' will allow the inclusion of such situations as selling stock short or entering commodity futures contracts.

We suppose that there are a finite number of situations  $1, 2, \dots, N$  on which a gambler can bet various fractions of his (infinitely divisible) capital. The random variables  $X_1, X_2, \dots, X_N$  represent, respectively, the outcome of a unit bet on situations  $1, 2, \dots, N$ . Because the loss can be no more than the investment,  $X_k \geq -1$  for  $1 \leq k \leq N$ . (Breiman considers the amount returned to the gambler after he has given up his bet in order to play, a real example of this sequence of events being betting on the horses. Here the amount the gambler gets back is  $\geq 0$ , which corresponds to the amount he wins being  $\geq -1$ ). We further suppose, with no loss of applicability, that in all of what follows each  $X_k$  has an expectation, i.e., that  $E(|X_k|)$  is finite. These will be the only restrictions on the random variables, and so we are considering a substantially larger class than those discrete random variables Breiman considers.

We also suppose that the gambler can repeatedly reinvest and change the proportion of the capital bet on the situations. The outcome at time  $j$  corresponds to the random variables  $X_1^{(j)}, X_2^{(j)}, \dots, X_N^{(j)}$ . For each  $k$ ,  $1 \leq k \leq N$ , the results of repeated betting of one unit on the  $k$ th situation is a sequence  $X_k^{(1)}, X_k^{(2)}, \dots, X_k^{(m)}, \dots$  of independent random variables, each having the same distribution as  $X_k$ . In contrast to this independence, it is quite important for applications that  $X_1, X_2, \dots, X_N$  be allowed to be dependent.

A strategy for the game will be a sequence  $\gamma^{(1)}, \dots, \gamma^{(m)}, \dots$  of vectors,  $\gamma^{(m)} = (\gamma_1^{(m)}, \gamma_2^{(m)}, \dots, \gamma_N^{(m)})$  giving the fractional amount  $\gamma_k^{(m)}$  of the capital which at the  $m$ th bet is bet on the  $k$ th situation. Thus  $\gamma_k^{(m)} \geq 0, 1 \leq k \leq N$ , and  $\sum_{k=1}^N \gamma_k^{(m)} \leq 1$ . We allow the possibility that  $\gamma^{(m)}$  can depend, as a Borel-measurable function, on the past outcomes  $X_1^{(1)}, \dots, X_N^{(1)}, X_1^{(2)}, \dots, X_N^{(2)}, \dots, X_1^{(m-1)}, \dots, X_N^{(m-1)}$ . (Breiman includes the sure-thing bet  $X_0 \equiv 1$ , so that betting  $\gamma_0$  on  $X_0$  is the same thing as putting  $\gamma_0$  aside; in this way his  $\gamma$ 's always sum to 1. We shall not do this.)

Letting  $F_m$  be the fortune which is the result of  $m$  bets using  $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)}$ , and  $F_0$  be the initial fortune,

$$(1) \quad F_m = F_0 \prod_{j=1}^m \left[ 1 + \sum_{k=1}^N \gamma_k^{(j)} X_k^{(j)} \right].$$

To simplify the notation, let  $X^{(j)} = (X_1^{(j)}, \dots, X_N^{(j)})$  and denote the scalar product with  $\gamma^{(j)} = (\gamma_1^{(j)}, \gamma_2^{(j)}, \dots, \gamma_N^{(j)})$  by  $\gamma^{(j)} \cdot X^{(j)}$ , obtaining

$$(2) \quad F_m = F_0 \prod_{j=1}^m [1 + \gamma^{(j)} \cdot X^{(j)}].$$

A *fixed-fraction strategy* is a strategy  $\gamma^{(m)} = (\gamma_1, \gamma_2, \dots, \gamma_N)$  which bets the same amount  $\gamma_k$  on situation  $k$  for all  $m$ . The result of using  $\gamma = \gamma^{(j)}, 1 \leq j \leq m$ , for  $m$  bets is  $F_m = F_0 \prod_{j=1}^m (1 + \gamma \cdot X^{(j)})$ . We shall be particularly interested in 'the' fixed-fraction strategy  $\gamma^* = (\gamma_1^*, \gamma_2^*, \dots, \gamma_N^*)$  which maximizes  $E(\log(F_m))$ . In Lemma 3 we shall show that  $\gamma^*$  exists, and Lemma 1 shows in what sense it is unique. The strategy  $\gamma^*$  maximizes  $E(\log(F_m))$  if and only if it maximizes the function

$$(3) \quad \phi(\gamma) = \phi(\gamma_1, \dots, \gamma_N) = E(\log(1 + \sum \gamma_k X_k)) = E(\log(1 + \gamma \cdot X))$$

over the domain

$$(4) \quad D = \{(\gamma_1, \dots, \gamma_N) : \gamma_k \geq 0, 1 \leq k \leq N, \sum \gamma_k \leq 1\}.$$

*Lemma 1.*

- (i) The function  $\phi$  of (3) is concave.
- (ii) If  $\phi(a\alpha + (1-a)\beta) = a\phi(\alpha) + (1-a)\phi(\beta)$  for  $0 < a < 1$  with  $\phi(\alpha)$  and  $\phi(\beta)$  finite,  $\alpha \cdot X = \beta \cdot X$  almost surely. In particular, if  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_N)$  both maximize  $\phi$  over its domain  $D$ , then  $\sum \alpha_k X_k = \sum \beta_k X_k$  a.s.
- (iii) At  $\gamma = (\gamma_1, \dots, \gamma_N)$  in  $D$  with  $\sum \gamma_k < 1$ , the partial derivative  $\partial\phi/\partial\gamma_i$  exists and equals  $E(X_i/(1 + \gamma \cdot X))$ ,  $1 \leq i \leq N$ .

*Proof.*

(i) The function  $f(x) = \log(1+x)$  is strictly concave on  $(-1, \infty)$ , and so for  $x = (x_1, \dots, x_N)$  a value of  $X, \alpha$  and  $\beta$  in  $D$ , and  $0 < a < 1$ ,

$$(5) \quad f(a\alpha \cdot x + (1-a)\beta \cdot x) \geq af(\alpha \cdot x) + (1-a)f(\beta \cdot x),$$

an inequality which also holds if either  $\alpha \cdot x$  or  $\beta \cdot x$  is  $-1$ . Integrating (5) with respect to the probability measure  $P$  of the space on which  $X$  is defined,

$$\begin{aligned} \phi(a\alpha + (1-a)\beta) &= \int f(a\alpha \cdot X + (1-a)\beta \cdot X) dP \\ &\cong \int (af(\alpha \cdot X) + (1-a)f(\beta \cdot X)) dP = a\phi(\alpha) + (1-a)\phi(\beta). \end{aligned}$$

(ii) Since  $\phi$  is concave the set where it attains its max is convex. If  $\phi(\alpha) = \phi(\beta)$  is a max, then for  $0 < a < 1$ ,  $\phi(a\alpha + (1-a)\beta) = a\phi(\alpha) + (1-a)\phi(\beta)$ , or

$$(6) \quad \int (f(a \cdot \alpha X + (1-a)\beta \cdot X) - af(\alpha \cdot X) - (1-a)f(\beta \cdot X)) dP = 0.$$

From (5) and (6),

$$(7) \quad f(a\alpha \cdot X + (1-a)\beta \cdot X) = af(\alpha \cdot X) + (1-a)f(\beta \cdot X) \quad \text{a.s.}$$

Because  $f$  is strictly concave,  $a\alpha \cdot X + (1-a)\beta \cdot X = \alpha \cdot X = \beta \cdot X$  at all values of  $X$  where  $f$  is finite. Both sides of (7) are  $-\infty$  only at values of  $X$  where  $\alpha \cdot X = \beta \cdot X = -1$ . In any case,  $\alpha \cdot X = \beta \cdot X$  almost surely.

(iii) Choose  $\varepsilon > 0$  so that  $\sum \gamma_k < 1 - \varepsilon$ . Then  $1/|1 + \gamma \cdot X| \leq 1/\varepsilon$ . The difference quotient for  $\partial\phi/\partial\gamma_i$  is

$$(8) \quad \int \frac{\log \left( 1 + \sum_{k \neq i} \gamma_k X_k + (\gamma_i + \Delta\gamma_i) X_i \right) - \log \left( 1 + \sum \gamma_k X_k \right)}{\Delta\gamma_i} dP.$$

Let  $x = (x_1, \dots, x_N)$  be a value of  $X$  and consider the function  $g(\gamma_i) = \log(1 + \sum_{k \neq i} \gamma_k x_k + \gamma_i x_i)$ . By the mean value theorem,

$$\left| \frac{g(\gamma_i + \Delta\gamma_i) - g(\gamma_i)}{\Delta\gamma_i} \right| = \frac{|x_i|}{|1 + \sum \gamma_k x_k + \xi_i x_i|},$$

$0 < \xi_i < \Delta\gamma_i$ . So the integrand in (8) is dominated by the  $L^1$  function  $|X_i|/\varepsilon$  for  $\Delta\gamma_i$  small. Result (iii) follows from the Lebesgue dominated convergence theorem.

Here is a simple example which conceptually illustrates a practical use of the Kelly-Breiman criterion: maximize  $E(\log F_m)$ .

*Example 1.* Define two random variables  $X_1$  and  $X_2$  by flipping a fair coin: if heads, then  $X_1 = 100$  and  $X_2 = -10$ , if tails, then  $X_1 = -1$  and  $X_2 = 1$ . The payoff from  $X_1$  is far superior to the payoff from  $X_2$ , but because  $X_1$  and  $X_2$  are (completely) correlated and have payoffs with opposite signs, the criterion

will mix both in order to smooth out the rate of capital growth. A simple calculation shows that  $\phi(\gamma) = E(\log(1 + \gamma_1 X_1 + \gamma_2 X_2))$  is maximized over  $D$  on the face  $\gamma_1 + \gamma_2 = 1$  at  $\gamma_1^* \cong 0.54$  and  $\gamma_2^* \cong 0.46$ . (Lemma 3 will discuss the basic problem created by maxima occurring at non-interior points of  $D$ .) The extent to which the criterion will sacrifice expectation is surprising:  $E(0.54X_1 + 0.46X_2) \cong 24.7$  vs.  $E(X_1) = 49.5$ .

A fascinating example of the use of this criterion, in which the underlying idea is the same as this example, is in hedging a warrant against its stock as described in [15], pp. 220–222.

*Example 2.* For  $\lambda > 0$ , let  $X$  have density  $\lambda e^{-\lambda(x+1)}$  for  $x \geq -1$ , and 0 for  $x < -1$ ; an exponential shifted to allow losses. We shall show that there is a unique  $\gamma^*$ ,  $0 \leq \gamma^* < 1$ , which maximizes  $\phi(\gamma) = E(\log(1 + \gamma X))$ ,  $0 \leq \gamma \leq 1$ ;  $\gamma^* = 0$  iff  $\lambda \geq 1$ .

By Lemma 1,

$$\phi'(\gamma) = - \int_{-1}^{\infty} \frac{x}{1 + \gamma x} \lambda e^{-\lambda(x+1)} dx.$$

Further,

$$\phi''(\gamma) = \int_{-1}^{\infty} \frac{-x^2}{(1 + \gamma x)^2} \lambda e^{-\lambda(x+1)} dx < 0,$$

so  $\phi$  is strictly concave on  $[0, 1)$ . Since  $\phi'(0) = E(X) = \lambda^{-1} - 1$ , the strict concavity of  $\phi$  shows that  $\gamma^* = 0$  iff  $\phi'(0) \leq 0$ .

It remains to show that for  $0 < \lambda < 1$  there is a unique maximizing point  $\gamma^*$  with  $0 < \gamma^* < 1$ . By a change of variable,  $\phi'(\gamma) = (\lambda/\gamma^2)e^a[g(a) - E1(a)]$ , where  $a = \lambda((1/\gamma) - 1)$ ,  $g(a) = e^{-a}/(a + \lambda)$ , and the exponential integral  $E1(a) = \int_a^{\infty} e^{-t}/t dt$ . Since  $E1(0) = \infty$  and  $g(0) = 1/\lambda$ ,  $\phi'(\gamma) < 0$  for  $\gamma$  close to 1; as  $\phi$  is strictly concave, there is thus a unique point  $\gamma^*$ ,  $0 < \gamma^* < 1$ , at which  $\phi$  is maximized.

For future reference we note that it is not obvious that  $\phi$  is continuous at 1; part of the computation involves an integration by parts and a change of variable to obtain  $\phi(\gamma) = \log(1 - \gamma) + e^a E1(a)$ ,  $a$  as above. The expansion  $E1(a) = e^{-a}(-\log a - \gamma_0 + o(a))$  ([1], p. 229), where  $\gamma_0 = 0.577 \dots$  is Euler's constant, shows that  $\lim_{\gamma \rightarrow 1} \phi(\gamma) = -\log(\lambda) - \gamma_0$ .

In Example 2  $\gamma^* = 0$  iff  $E(X) \leq 0$ , i.e., a gambler bets on  $X$  only if it has positive expectation. This is a special case of a more general result. Breiman [4], p. 65, calls a game *favorable* if there is a strategy such that the associated fortune  $F_n$  tends almost surely to  $\infty$  with  $n$ , and he shows that this condition is equivalent to  $\phi(\gamma^*)$  being positive ([4], Proposition 3, p. 68). Lemma 2 establishes the equivalence with the intuitive Condition (iv).

*Lemma 2. The following are equivalent:*

(i) *There is a strategy with the associated fortune*

$$F_n \rightarrow \infty \text{ a.s. as } n \rightarrow \infty.$$

(ii)  $F_n^* \rightarrow \infty$  a.s.

(iii)  $\phi(\gamma^*) > 0$ .

(iv)  $E(X_i) > 0$  for at least one  $i, 1 \leq i \leq N$ .

*Proof.*

(i) implies (ii). In Theorem 1 we show that  $F_n/F_n^*$  tends almost surely to a finite limit.

(ii) implies (iii). If  $\phi(\gamma^*) = 0$ , then  $F_n^* = F_0$  for all  $n$ .

(iii) implies (iv). If  $\gamma^* \cdot X = 0$ , then  $\phi(\gamma^*) = 0$ . So  $\gamma_i^* > 0$  for some  $i$  with  $X_i$  not identically 0. Fix all variables in  $\phi$  but  $\gamma_i$  and set  $\Psi(\gamma_i) = \phi(\gamma_1^*, \gamma_2^*, \dots, \gamma_{i-1}^*, \gamma_i, \gamma_{i+1}^*, \dots, \gamma_N^*)$ , a concave function which has a positive max at  $\gamma_i^*$ . If  $E(X_i) \leq 0$ , then  $\Psi'(0) \leq 0$ , and  $\Psi$  has a local max at 0 and so has a global max there because it is concave. Hence  $E(X_i) > 0$ .

(iv) implies (i). Define  $\Psi$  by setting all the variables but  $\gamma_i$  equal to 0 in  $\phi: \Psi(0, 0, \dots, 0, \gamma_i, 0, \dots, 0)$ . As  $E(X_i) > 0, \Psi'(0) > 0$  and so  $\Psi(\gamma_i) > 0$  for  $\gamma_i$  close to 0. Thus  $\phi(\gamma^*) > 0$ . Since  $\log(F_n^*/F_0) = \sum_1^n \log(1 + \gamma^* \cdot X^{(j)})$  and  $E((\log F_n^*/F_0)/n) = \phi(\gamma^*) > 0$ , the strong law of large numbers shows that  $F_n^*/F_0 \rightarrow \infty$  almost surely.

*Example 3.* Let  $X_1$  and  $X_2$  be the coordinates of a point distributed uniformly on  $[-1, b] \times [-1, b]$ . Then  $\phi(\gamma_1, \gamma_2) = E(\log(1 + \gamma_1 X_1 + \gamma_2 X_2))$  has a maximum at  $\gamma^* = (\frac{1}{2}, \frac{1}{2})$  if  $b \geq \log(16) - 1$ .

If  $(\gamma_1, \gamma_2)$  is a point where  $\phi$  attains its max, then so is  $(\gamma_2, \gamma_1)$  by symmetry. Since  $\phi$  is concave,  $(\frac{1}{2})(\phi(\gamma_1, \gamma_2) + \phi(\gamma_2, \gamma_1)) \leq \phi(\frac{1}{2}(\gamma_1 + \gamma_2), \frac{1}{2}(\gamma_1 + \gamma_2))$ , and we may look for the maximum of  $\phi$  along the diagonal  $(\gamma, \gamma), 0 \leq \gamma \leq \frac{1}{2}$ . Then

$$\frac{d}{d\gamma} \phi(\gamma, \gamma) = \frac{1}{(b+1)^2} \int_{-1}^b \int_{-1}^b \frac{x_1 + x_2}{(1 + \gamma(x_1 + x_2))} dx_1 dx_2.$$

The second derivative is  $< 0$ , and  $\phi'$  is decreasing. A direct but tedious integration and calculation shows that

$$\lim_{\gamma \rightarrow \frac{1}{2}^-} \frac{d}{d\gamma} \phi(\gamma, \gamma) = 2 \left( 1 - \frac{(\log 16)}{(b+1)} \right).$$

Hence  $\phi(\gamma, \gamma)$ , which can be shown to be continuous on  $[0, \frac{1}{2}]$ , increases up to its max at  $(\frac{1}{2}, \frac{1}{2})$  as long as  $b \geq \log(16) - 1 = 1.77 \dots$

It is interesting to compare this situation with betting on only one variable, say  $X_1$ . Then  $\phi_1(\gamma_1) = E(\log(1 + \gamma_1 X_1))$  is continuous on  $[0, 1]$ . Continuity on  $[0, 1)$  follows from Lemma 1 or inspection. Because of the singularity at



$\gamma_1 \cdot X_1 = -1$  in the integrand, continuity at 1 is not by inspection; one must compute  $\phi_1(1) = \log(1+b) - 1$  and show it is equal to the limit. (The situation in two variables, which we dismissed with a word, is not easier.)

The function  $\phi_1$  is differentiable on  $[0, 1)$  by inspection or Lemma 1. It is only differentiable at 1 in an extended sense; we can show that  $\phi'_1(1) = -\infty$  and that  $\lim_{\gamma \rightarrow 1} \phi'_1(\gamma) = -\infty$ .

The function  $\phi_1$  has a unique maximum at a point  $\gamma_1^*$ ,  $0 < \gamma_1^* < 1$ . From Lemma 2,  $\gamma_1^* = 0$  iff  $E(X) = (b-1)/2 \leq 0$ . The existence of  $\gamma_1^* < 1$  follows from  $\phi'_1(0) = (b-1)/2$  and  $\phi'_1(1) = -\infty$ , the uniqueness from strict concavity.

The surprising fact is that for one variable  $X_1$  a gambler does not bet all his fortune no matter what  $b$  is, but he does bet all his fortune on two independent copies of  $X_1$  for  $b$  large enough.

The reader who has carried out the calculations of Examples 2 and 3 knows that, because of the possible singularity on  $\sum \gamma_k = 1$ , it is not clear that  $\phi$  attains a maximum, and the differentiability of  $\phi$  on the boundary  $\sum \gamma_k = 1$  is even less clear.

Think of a continuous strictly increasing concave function  $f$  on  $[0, 1]$  and redefine it at 1 so that its value there is less than  $f(0)$ . If this redefined function were  $E(\log(1+\gamma X))$ , then  $X$  would be a most interesting game with no Kelly-Breiman optimal strategy: with unit fortune, if a gambler bet an amount less than 1 he could always do better by betting slightly more, but betting all would be worst.

One result of Lemma 3 is that there is an optimal  $\gamma^*$  so no game can have the property discussed in the paragraph above. Another result of Lemma 3 is that  $\phi$  is continuous, when finite. This is important because when we compute  $\gamma^*$ , a numerical calculation which will generally give  $\gamma^*$  to a certain number of decimals, we want to know that using this approximation to the exact  $\gamma^*$  will give close to optimal performance.

The other result is a substitute for differentiation when  $\gamma^*$  has  $\sum \gamma_k^* = 1$ , which allows us to derive the basic inequalities (9) and (10). Note that if all the random variables  $X_1, X_2, \dots, X_N$  are discrete, with a finite number of values, as they are in [4], then we can differentiate  $\phi$  at  $\gamma^*$ : for then if  $\gamma^* \cdot X$  equals  $-1$  it does so with positive probability and  $\phi(\gamma^*) = -\infty < \phi(0) = 0$ , contrary to  $\phi(\gamma^*)$  a maximum; thus  $\phi$  is actually defined on a neighborhood of  $\gamma^*$  (which may extend outside  $D$ ) and is differentiable as in Lemma 1. The problems which Lemma 3 resolves are those which arise from more general random variables.

*Lemma 3.*

(i) *The function  $\phi$  is continuous where finite, and attains a maximum at a point  $\gamma^*$  in  $D$ .*

(ii) Let  $K = \{k : \gamma_k^* \neq 0\}$ . For  $k$  in  $K$ ,  $E(X_k/(1 + \gamma^* \cdot X))$  is finite and non-negative. If  $k$  belongs to  $K$ , then for all  $m$

$$(9) \quad E(X_k/(1 + \gamma^* \cdot X)) \geq E(X_m/(1 + \gamma^* \cdot X)).$$

If both  $k$  and  $m$  belong to  $K$ ,

$$(10) \quad E(X_k/(1 + \gamma^* \cdot X)) = E(X_m/(1 + \gamma^* \cdot X)).$$

*Proof.* Suppose that  $\gamma^{(n)}$  converges to  $\gamma^{(0)}$  with  $\phi(\gamma^{(0)})$  finite. Denote the positive and negative parts of  $\log$  by  $\log^+$  and  $\log^-$ :  $\log^+(x) = \log x$  if  $x \geq 1$ ,  $\log^+(x) = 0$  if  $x \leq 1$  and  $\log^-(x) = -\log(x)$  if  $x \leq 1$ ,  $\log^-(x) = 0$  if  $x \geq 1$ . Since  $\log^+(1 + \gamma \cdot X) \leq |\gamma \cdot X| \leq \max |X_k|$ ,  $\phi$  is infinite only if it is  $-\infty$ . By the Lebesgue dominated convergence theorem  $\lim \int \log^+(1 + \gamma^{(n)} \cdot X) dP = \int \log^+(1 + \gamma^{(0)} \cdot X) dP$ , by Fatou's lemma  $\liminf \int \log^-(1 + \gamma^{(n)} \cdot X) dP \geq \int \log^-(1 + \gamma^{(0)} \cdot X) dP$ , and putting these two facts together,  $\limsup \phi(\gamma^{(n)}) \leq \phi(\gamma^{(0)})$ . Thus  $\phi$  is upper semicontinuous and therefore attains its maximum on the compact set  $\{\gamma \text{ in } D : \phi(\gamma) \geq 0\}$ .

For  $0 < a < 1$ ,  $\phi(a\gamma + (1 - a)\gamma^{(0)}) \geq a\phi(\gamma) + (1 - a)\phi(\gamma^{(0)})$ , and so  $\liminf_{a \rightarrow 0} \phi(a\gamma + (1 - a)\gamma^{(0)}) \geq \phi(\gamma^{(0)})$  if  $\phi(\gamma)$  is finite, i.e.,  $\phi$  is continuous along lines directed towards  $\gamma^{(0)}$  from points where  $\phi$  is finite.

Suppose that  $\gamma^{(1)}$  and  $\gamma^{(2)}$  are in  $D$  with  $\phi(\gamma^{(2)})$  finite, and let  $a \in [0, 1)$ . Now

$$\begin{aligned} 1 + a\gamma^{(1)} \cdot X + (1 - a)\gamma^{(2)} \cdot X &\geq 1 + a \sum \gamma_k^{(1)}(-1) + (1 - a)\gamma^{(2)} \cdot X \\ &\geq (1 - a) + (1 - a)\gamma^{(2)} \cdot X \\ &= (1 - a)(1 + \gamma^{(2)} \cdot X). \end{aligned}$$

Since  $\log^-(1 + z)$  is a decreasing function of  $z$ ,  $\log^-(1 + a\gamma^{(1)} \cdot X + (1 - a)\gamma^{(2)} \cdot X) \leq \log^-(1 - a)(1 + \gamma^{(2)} \cdot X)$  which equals  $-\log(1 - a) - \log(1 + \gamma^{(2)} \cdot X)$  when  $(1 - a)(1 + \gamma^{(2)} \cdot X) \leq 1$ , and equals 0 otherwise. Hence  $\int \log^-(1 + a\gamma^{(1)} \cdot X + (1 - a)\gamma^{(2)} \cdot X) dP \leq -\log(1 - a) + \int \log^-(1 + \gamma^{(2)} \cdot X) dP < \infty$ , and thus  $\phi(a\gamma^{(1)} + (1 - a)\gamma^{(2)})$  is finite for  $a \neq 1$ .

Let a point  $\gamma$  be a given at which  $\phi$  is finite. For  $1 \leq k \leq N$ , let  $e^{(k)}$  be the vector in  $D$  whose  $k$ th coordinate is 1,  $e_i^{(k)} = \delta_{ik}$ ,  $e^{(0)} = 0$ , and set  $\gamma^{(k)} = ae^{(k)} + (1 - a)\gamma$ ,  $a$  in  $(0, 1)$ , for  $0 \leq k \leq N$ . Note that  $D$  is the convex hull  $\text{co}(e^{(0)}, e^{(1)}, \dots, e^{(N)})$ . As we have seen above,  $\phi(\gamma^{(k)})$  is finite and so  $\phi$  is continuous on the line joining  $\gamma^{(k)}$  to  $\gamma$ . Given  $\epsilon > 0$ , by choosing  $a$  small enough we have  $|\phi(\gamma^{(k)}) - \phi(\gamma)| \leq \epsilon$  for  $0 \leq k \leq N$ . For any vector  $v$  in the convex hull  $U = \text{co}(\gamma^{(0)}, \gamma^{(1)}, \dots, \gamma^{(N)})$ ,  $v = \sum a_k \gamma^{(k)}$ ,  $a_k \geq 0$ ,  $\sum a_k = 1$ , we have  $\phi(v) \geq \sum a_k \phi(\gamma^{(k)}) \geq \phi(\gamma) - \epsilon$ . The convex hull  $U$  is easily seen to have an interior (relative to  $D$ ). Since  $\phi$  is upper semicontinuous,  $V = \{\alpha : \phi(\alpha) < \phi(\gamma) + \epsilon\}$  is open. Therefore for  $v$  in the neighborhood  $U^0 \cap V$ ,  $|\phi(v) - \phi(\gamma)| \leq \epsilon$  and  $\phi$  is continuous at  $\gamma$ .

For  $\gamma$  in  $D$  with  $\sum \gamma_i < 1$ , and  $0 \leq a \leq 1$ , define  $\Psi(a) = \phi(a\gamma + (1-a)\gamma^*)$ . We have seen that  $\Psi$  is continuous on  $[0, 1]$ . Given  $0 < \varepsilon < 1$ , for  $a$  in  $[\varepsilon, 1]$ ,  $1 + (a\gamma + (1-a)\gamma^*) \cdot X \geq \varepsilon(1 - \sum \gamma_i) > 0$ . As in Lemma 1,  $|(\gamma - \gamma^*) \cdot X / (1 + (a\gamma + (1-a)\gamma^*) \cdot X)|$  is bounded by the  $L^1$  function  $|(\gamma - \gamma^*) \cdot X / \varepsilon(1 - \sum \gamma_i)|$  and we may use the Lebesgue dominated convergence theorem to justify differentiating under the integral to obtain

$$\Psi'(a) = \int \frac{(\gamma - \gamma^*) \cdot X}{1 + (a\gamma + (1-a)\gamma^*) \cdot X} dP.$$

Since  $\varepsilon$  was arbitrary, this holds on  $(0, 1]$ .

If  $\sum \gamma_k^* < 1$ , then Lemma 1 establishes the fact that the expectations in (10) are 0. As in Lemma 2, if  $m$  is not in  $K$ , then fixing all the variables in  $\phi$  but  $\gamma_m$  and considering that concave function with a max at 0 shows that  $\partial\phi/\partial\gamma_m(\gamma^*) \leq 0$  and (9) follows.

Now suppose that  $\sum \gamma_k^* = 1$  and let  $x = (x_1, \dots, x_N)$  be some value of the random variable  $X$  in which not all the  $x_k = -1$  for  $k$  in  $K$ . Note that the event  $x_k = -1$  for all  $k$  in  $K$  has probability 0 since the integrand in the integral defining  $\phi$  is  $-\infty$  there and  $\phi(\gamma^*)$  is finite.

For  $0 \leq a < 1$  and  $\gamma$  in  $D^0$ , define

$$f(a) = \log(1 + (a\gamma + (1-a)\gamma^*) \cdot x).$$

The function  $f$  is finite because  $x$  was so chosen, and is differentiable with

$$f'(a) = \frac{(\gamma - \gamma^*) \cdot x}{1 + (a\gamma + (1-a)\gamma^*) \cdot x}.$$

Because  $f'' \leq 0$ ,  $f'(a)$  increases to  $(\gamma - \gamma^*) \cdot x / (1 + \gamma^* \cdot x)$  as  $a$  decreases to 0. For  $a \neq 0$ ,  $(\gamma - \gamma^*) \cdot X / (1 + (a\gamma + (1-a)\gamma^*) \cdot X)$  in  $L^1$  we may apply the B. Levi theorem to obtain

$$\lim_{a \downarrow 0} \Psi'(a) = \int \frac{(\gamma - \gamma^*) \cdot X}{1 + \gamma^* \cdot X} dP.$$

By the mean value theorem,  $(\Psi(a) - \Psi(0))/a = \Psi'(b)$ ,  $0 < b < a$ . Then, because  $\Psi'$  is increasing,  $\lim_{a \downarrow 0} \Psi'(a) = \lim_{a \downarrow 0} ((\Psi(a) - \Psi(0))/a)$ . Since  $\phi(\gamma^*)$  is a maximum the right-hand side is  $\leq 0$  and we obtain the basic

$$(11) \quad \int \frac{(\gamma - \gamma^*) \cdot X}{1 + \gamma^* \cdot X} dP \leq 0.$$

For  $0 < \varepsilon < 1$  and  $k$  in  $K$ , the choice  $\gamma_j = \gamma_j^*$  for  $j \neq k$  and  $\gamma_k = \gamma_k^*(1 - \varepsilon)$  in (11) gives  $-\int X_k / (1 + \gamma^* \cdot X) dP \leq 0$ , and  $\int X_k / (1 + \gamma^* \cdot X) dP$  is non-negative and therefore finite.

For  $k$  in  $K$  and any  $m$ , and  $0 < \varepsilon < \gamma_k^*$ , the choice  $\gamma_j = \gamma_j^*$  for  $j$  neither  $k$  nor

$m, \gamma_n = \gamma_n^* - \varepsilon, \gamma_m = \gamma_m^* + \varepsilon$  in (11) gives (9). Finally, (10) follows from (9) by symmetry.

### 3. Theorems

If  $F_n$  is a gambler's fortune at the  $n$ th time period, obtained by using some strategy  $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(n)}, \dots$ , and  $F_n^*$  is the fortune obtained by using the fixed fraction strategy  $\gamma^*, \gamma^*, \dots, \gamma^*, \dots$ , then Breiman concludes ([4], Theorem 2, p. 72) that  $\lim F_n/F_n^*$  almost surely exists and  $E(\lim F_n/F_n^*) \leq 1$ . We extend these results in Theorem 1 below. Two comments are in order.

First, the advantage in using  $\gamma^*$ , as indicated by the fact that  $E(\lim F_n/F_n^*) \leq 1$ , does not require passage to the limit. This was noted by Durham, for the case of two branching processes, in the proof of Theorem 1 of [6], p. 571. In fact, given that the limiting result is true and given a finite strategy  $\gamma^{(1)}, \dots, \gamma^{(m)}$ , extend it by setting  $\gamma^{(j)} = \gamma^*$  for  $j > m$ ; then  $F_n/F_n^* = F_m/F_m^*$  for  $n \geq m$  and  $E(F_m/F_m^*) \leq 1$ . This should reassure the careful investor who wonders whether a strategy good in the long run may not be inferior in any practical number of trials—disregard of this point leads to an overevaluation of games of the type which produces the St. Petersburg paradox.

Second, by our analysis of  $\phi$ , we are able to show in Theorem 1 that the presence of the expectation in  $E(\lim F_n/F_n^*) \leq 1$  raises serious problems in any superficial attempt to use this as an indication of the superiority of  $\gamma^*$ .

*Theorem 1. Let  $F_n$  be the fortune obtained by using a strategy  $\gamma = \gamma^{(1)}, \dots, \gamma^{(n)}$  for  $n$  repeated investment periods, and let  $F_n^*$  be the fortune obtained by using the fixed fraction strategy  $\gamma^*$ . Then*

(i)  $F_n/F_n^*$  is a supermartingale with  $E(F_n/F_n^*) \leq 1$ . Consequently,  $\lim F_n/F_n^*$  exists almost surely as a finite number and  $E(\lim F_n/F_n^*) \leq 1$ .

(ii) Suppose that  $\gamma$  bets only on those  $X_k$  with  $\gamma_k^* > 0$ , i.e., that  $\gamma_l^{(j)} = 0$  if  $l \notin K$ ,  $1 \leq l \leq N$  and all  $j$ . If  $\sum \gamma_k^* = 1$ , then further suppose that  $\sum \gamma_k^{(j)} = 1$  for all  $j$ . Then  $F_n/F_n^*$  is a martingale with  $E(F_n/F_n^*) = 1$ .

*Proof.* Let  $m$  be given and let  $\mathcal{C}_m$  be the sigma-algebra generated by  $X_k^{(j)}$ ,  $1 \leq k \leq N, 1 \leq j \leq m$ . Then

$$E\left(\frac{F_{m+1}}{F_{m+1}^*} \middle| \mathcal{C}_m\right) = E\left(\frac{1 + \gamma^{(m+1)} \cdot X^{(m+1)}}{1 + \gamma^* \cdot X^{(m+1)}} \cdot \frac{F_m}{F_m^*} \middle| \mathcal{C}_m\right).$$

Since  $F_m/F_m^*$  is  $\mathcal{C}_m$ -measurable, this equals

$$\frac{F_m}{F_m^*} E\left(\frac{1 + \gamma^{(m+1)} \cdot X^{(m+1)}}{1 + \gamma^* \cdot X^{(m+1)}} \middle| \mathcal{C}_m\right).$$

Because  $\gamma^{(m+1)}$  is a strategy depending on the past values of the  $X_k^{(j)}$ , it too is

$\mathcal{C}_m$ -measurable. That, together with the independence of  $X^{(m+1)}$  and the previous  $X^{(i)}$ , shows this equals

$$\frac{F_m}{F_m^*} E\left(\frac{1}{1+\gamma^* \cdot X}\right) + \sum_{k=1}^N \gamma_k^{(m+1)} E\left(\frac{X_k^{(m+1)}}{1+\gamma^* \cdot X}\right).$$

Since  $X^{(m+1)}$  has the same distribution as  $X$ , this can be written

$$(12) \quad \frac{F_m}{F_m^*} \left[ E\left(\frac{1}{1+\gamma^* \cdot X}\right) + \sum_{k=1}^N \gamma_k^{(m+1)} E\left(\frac{X_k}{1+\gamma^* \cdot X}\right) \right] = A, \text{ say.}$$

By Lemma 3,  $E(X_k/(1+\gamma^* \cdot X))$  equals a constant  $c$  for  $k$  in  $K$  and is  $\leq c$  for all  $k$ ; this constant  $c=0$  if  $\sum \gamma_k^* < 1$ . Hence

$$A \leq \frac{F_m}{F_n^*} \left[ E\left(\frac{1}{1+\gamma^* \cdot X}\right) + \sum \gamma_k^* \cdot c \right] = \frac{F_m}{F_m^*} E\left[\frac{1+\gamma^* \cdot X}{1+\gamma^* \cdot X}\right] = \frac{F_m}{F_m^*}.$$

We have shown that  $F_n/F_n^*$  is a positive supermartingale, with  $E(F_n/F_n^*) \leq E(F_{n-1}/F_{n-1}^*) \leq \dots \leq E(F_0/F_0) = 1$ , and so by the supermartingale convergence theorem it converges almost surely to a finite limit. Using Fatou's lemma,  $E(\lim F_n/F_n^*) \leq \liminf E(F_n/F_n^*) \leq 1$ .

An examination of (12) shows that under the conditions of (ii),  $E(F_{m+1}/F_{m+1}^* | \mathcal{C}_m) = F_m/F_m^*$ , and (ii) follows.

The requirement in Theorem 1(ii) that if  $\sum \gamma_n^* = 1$  then we must have  $\sum \gamma_k^{(j)} = 1$ , for all  $j$ , in order to be sure to get a martingale, is made clear by a one-variable example, Let  $X \equiv 2$ . Then  $\phi(\gamma) = \log(1+2\gamma)$  is maximal at  $\gamma^* = 1$ . The gambler will do worse betting any amount less than 1, even though he still bets on the same random variable as  $\gamma^*$  does, the key observation being  $\phi'(1) > 0$ . In general, the 'partial derivatives'  $E(X_k/(1+\gamma^* \cdot X))$ ,  $k$  in  $K$ , may be positive if  $\sum \gamma_k^* = 1$ , whereas they are all 0 if  $\sum \gamma_k^* < 1$ .

The surprising result of Theorem 1 is the broad conditions in (ii) under which  $E(F_n/F_n^*) = 1$ . To see what the surprise is, we shall superficially interpret Theorem 1(i): since 'on the average', and 'for large  $n$ ',  $F_n/F_n^* \leq 1$ , the gambler 'does better' with  $F_n^*$  than with  $F_n$ . But then Theorem 1(ii) tells us that if the gambler simply bets on the same variables as  $\gamma^*$  does, *but in any proportions at all*, and if  $\gamma^*$  bets all so does he, then  $E(F_n/F_n^*) = 1$ . So with the same intuitive interpretation as above, 'on the average' the gambler does the same with  $F_n$  as with  $F_n^*$ , so it really does not matter which strategy he uses! But we know that it does matter. For example, in a repeated biased-coin toss, if he plays a fixed fraction strategy betting an amount  $\gamma \neq \gamma^* = p - q$ , then almost surely  $F_n/F_n^* \rightarrow 0$ . Yet we have  $E(F_n/F_n^*) = 1$ . In general it will not help to look at  $\lim F_n/F_n^*$ . For example, if on the first flip of the coin he bets all his fortune, and from then on he bets  $p - q$ ,  $F_n/F_n^* > 1$  with probability  $p$ .

Theorem 2 will help our understanding of this situation by showing that  $F_n^*$  is the only denominator with  $E(F_n/F_n^*) \leq 1$  for all  $F_n$ ; in fact  $E(F_n^*/F_n) > 1$  if  $F_n$  does not come from a strategy equivalent to using  $\gamma^*$  repeatedly. The suspicious reader will note that this characterization of the sense in which  $\gamma^*$  is optimal contains an expectation. Anyone attempting to state intuitively the result of Theorem 2(i) in the form ' $F_n^*$  is better than  $F_n$  because, on the average,  $F_n^*/F_n \geq 1$ ', should also be willing to apply the same interpretation to Theorem 1(ii) and conclude that often, ' $F_n/F_n^* = 1$  on the average and so  $F_n$  and  $F_n^*$  are often the same after all'.

*Theorem 2.*

(i)  $F_n^*/F_n$  is a submartingale with  $E(F_n^*/F_n) \geq 1$ .  $\lim F_n^*/F_n$  almost surely exists as an extended real number and  $E(\lim F_n^*/F_n) \geq 1$ .

(ii)  $E(F_n^*/F_n) = 1$  iff  $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(n)}$  are all equivalent to  $\gamma^*$ , i.e., if  $\gamma^{(i)} \cdot X = \gamma^* \cdot X$  almost surely for almost all values of  $X^{(1)}, \dots, X^{(i)}$  (of which  $\gamma^{(i)}$  is a function) for  $1 \leq j \leq n$ .

*Proof.* By Theorem 1, the non-negative  $\lim F_n/F_n^*$  almost surely exists, and so  $\lim F_n^*/F_n$  almost surely exists as an extended real number.

As in Theorem 1,

$$E\left(\frac{F_{m+1}^*}{F_{m+1}} \mid \mathcal{C}_m\right) = \frac{F_m^*}{F_m} E\left(\frac{1 + \gamma^* \cdot X^{(m+1)}}{1 + \gamma^{(m+1)} \cdot X^{(m+1)}} \mid \mathcal{C}_m\right).$$

Suppose that  $(X^{(1)}, \dots, X^{(m)})$  takes on the value  $\omega$  in  $R^{mN}$ , at which point  $\gamma^{(m+1)}$  takes on the value  $\gamma^{(m+1)}(\omega)$ . Then

$$(13) \quad E\left(\frac{1 + \gamma^* \cdot X^{(m+1)}}{1 + \gamma^{(m+1)} \cdot X^{(m+1)}} \mid_{(X^{(1)}, \dots, X^{(m)}) = \omega}\right) = E\left(\frac{1 + \gamma^* \cdot X^{(m+1)}}{1 + \gamma^{(m+1)}(\omega) \cdot X^{(m+1)}}\right) = B, \quad \text{say,}$$

because  $X^{(m+1)}$  is independent of the values of  $(X^{(1)}, \dots, X^{(m)})$  ([5], Corollary 4.38, p. 80). By Jensen's inequality,

$$(14) \quad B \geq \exp(E(\log(1 + \gamma^* \cdot X^{(m+1)})) - E(\log(1 + \gamma^{(m+1)}(\omega) \cdot X^{(m+1)}))) = C, \quad \text{say.}$$

By Lemma 1,  $C > 1$  unless  $\gamma^* \cdot X = \gamma^{(m+1)}(\omega) \cdot X$  almost surely, in which case  $C = 1$ . Thus

$$(15) \quad E\left(\frac{F_{m+1}^*}{F_{m+1}} \mid \mathcal{C}_m\right) \geq \frac{F_m^*}{F_m}$$

with equality holding iff  $\gamma^* \cdot X = \gamma^{(m+1)}(\omega) \cdot X$  almost surely for almost all values  $\omega$  in the range of  $(X^{(1)}, \dots, X^{(m)})$ . If  $E(F_{m+1}^*/F_{m+1}) = 1$ , then  $1 = E(E(F_{m+1}^*/F_{m+1} \mid \mathcal{C}_m)) \geq E(F_m^*/F_m) \geq \dots \geq E(F_0/F_0) = 1$ . By (15), equality holds in (15) and therefore  $\gamma^* \cdot X = \gamma^{(m+1)}(\omega) \cdot X$  for almost all  $\omega$  in the range of  $(X^{(1)}, \dots, X^{(m)})$ . Continue for  $m, m-1, \dots, 1$ , to obtain (ii).

Let  $Y = \lim F_n^*/F_n$ . By Theorem 1,  $P(Y=0) = 0$  and  $E(1/Y) \leq 1$ . The function  $g(x) = 1/x$  is convex in  $(0, \infty)$  and Jensen's inequality applies, to obtain  $1 \geq E(1/Y) \geq 1/E(Y)$  which completes the proof.

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