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SOLUTION OF THE INHOMOGENEOUS RAYLEIGH SCATTERING ATMOSPHERE

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ABSTRACT

The equation of radiative transfer for a Rayleigh scattering atmosphere including polarization is reduced to an azimuth-independent form. The integro-differential equations are then solved by approximating the integrals as sums and the differentials as finite differences. The resultant set of equations is then solved by the Feautrier technique. This method can solve inhomogeneous atmospheres by allowing for arbitrary variations with depth of the single scattering albedo and the source term. As an example, zero-phase geometric albedos and center-to-limb intensities are calculated for a planetary disk with a semi-infinite pressure-induced absorbing atmosphere.

Subject headings: atmospheres, planetary — polarization — radiative transfer

I. INTRODUCTION

Interest in calculating Rayleigh scattering atmospheres has recently been revived with the interpretation of limb darkening and the near-infrared spectrum of Uranus in terms of a deep molecular atmosphere (Belton and Spinrad 1973; Danielson and Wannier 1973; Belton, McElroy, and Price 1971). The strength of H_2 quadrupole lines implies as much as 1450 km-amagat of H_2 in the line of sight, and the limb darkening observed by *Stratoscope 2* has recently been interpreted in terms of a dense NH_3 cloud overlain with 500 km-amagat of H_2 (Danielson, Tomasko, and Savage 1972). In such an atmosphere, Rayleigh scattering would be an important—if not the dominant—source of albedo. A comparison of the near-infrared spectra of Neptune and Uranus suggests that Neptune has an atmosphere similar to that of Uranus (Wamsteker 1973). Observations of the degree of polarization at the poles of Jupiter (Gehrels, Herman, and Owen 1969) and photography of the polar regions of Saturn through methane-band filters (Owen 1969) again seem to indicate the presence of a deep molecular atmosphere with little scattering by aerosols. Belton and Price (1973) examined the geometric albedos and the expected limb darkening of the Uranian atmosphere on the assumption that it is semi-infinite. They were unable to compute the atmospheres directly, but were forced to rescale an inhomogeneous isotropic atmosphere in order to approximate the inhomogeneous Rayleigh atmosphere.

This paper presents a straightforward method for the calculation of a plane-parallel Rayleigh scattering atmosphere of any depth which is arbitrarily inhomogeneous. A cloud layer simulated by a Lambert surface may be inserted at any finite depth. The method yields the source function interior to the atmosphere and the emergent intensities at the quadrature points in μ taking full account of polarization and inhomogeneity. The technique employed is a modified version of the Feautrier (1964) method which is used extensively in stellar atmosphere calculations but which has not been applied before to the planetary problem.

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II. THE HOMOGENEOUS RAYLEIGH SCATTERING ATMOSPHERE

The equation of transfer for a Rayleigh scattering atmosphere including polarization has been formulated and is described in depth by Chandrasekhar (1950). Chandrasekhar's (1950) original solution is for the conservative, homogeneous atmosphere. More recent authors (Abhyankar and Fymat 1971) have tabulated the H-functions and related functions for varous nonconservative, homogeneous, semi-infinite atmospheres. All of these solutions have been restricted thus far to the homogeneous case. Another approach to the solution of the equation of transfer is through matrix operator theory (Plass, Kattawar, and Catchings 1973; Kattawar, Plass, and Catchings 1973) and its subsets such as the doubling technique (Hansen 1971). Plass *et al.* (1973) give an excellent review and description of this method. Matrix operator theory is able to deal with very complex scattering functions and is efficient in solving homogeneous atmospheres, but it is unable to construct smoothly inhomogeneous atmospheres.

The proposed procedure for the solution of inhomogeneous Rayleigh scattering atmospheres is the reduction of the equation of transfer to a second-order differential equation in

$$j(\tau, \mu) = \tfrac{1}{2}[I(\tau, +\mu) + I(\tau, -\mu)],$$

where $0 \leq \mu \leq 1$. The differential equation is expressed as a difference equation over a suitable τ-grid, and the integrals are replaced by quadrature sums. The resulting equations form a block tridiagonal system which is then backsolved to give j at each grid point and at each quadrature point. The single scattering albedo may be arbitrarily inhomogeneous but must be defined at every grid point. A full description of the application of the standard Feautrier method (Auer 1971) to the specialized problem of Rayleigh scattering with polarization is given in the Appendix.

As an illustrative example, a set of zero-phase geometric albedos for a spherical planet with an inhomogeneous semi-infinite atmosphere is calculated. The choice of inhomogeneity in this case is dictated by the known presence of pressure-induced absorption processes and is chosen to match that of Belton and Price (1973). The form of the monochromatic single scattering albedo in an isothermal atmosphere with both linear and pressure-induced absorption processes can be expressed simply as $\varpi(\tau) = \varpi_0(1 + K\tau)^{-1/2}$ (Goldstein 1962; Belton and Price 1973). The value of K is determined by all the opacity sources, and ϖ_0 is the single scattering albedo in the linear limit ($\tau \rightarrow 0$). Since the atmosphere is semi-infinite, the τ-pressure or τ-mass relationship is unnecessary and the surface properties of the atmosphere are described fully by the two parameters ϖ_0 and K. The Rayleigh-scattered zero-phase geometric albedos are listed in table 1 as a function of ϖ_0 and K, and the interpolated albedo contours in the (ϖ_0, K)-plane are shown in figure 1. A comparison of table 1 for $\varpi_0 = 1.0$ with table 2 from Belton and Price (1973) shows that their albedos are consistently too high. The zero-phase backscattered intensities from center to limb are plotted in figure 2 for $\varpi_0 = 1.0$ and for selected values of K. The intensities in the region of the turnover from limb darkening to limb brightening are not monotonic, as is shown by Belton and Price (1973, fig. 1). Hence, the transition from limb darkening to limb brightening cannot be precisely defined except to say that the transition occurs as the geometric albedo drops from 0.3 to 0.15.

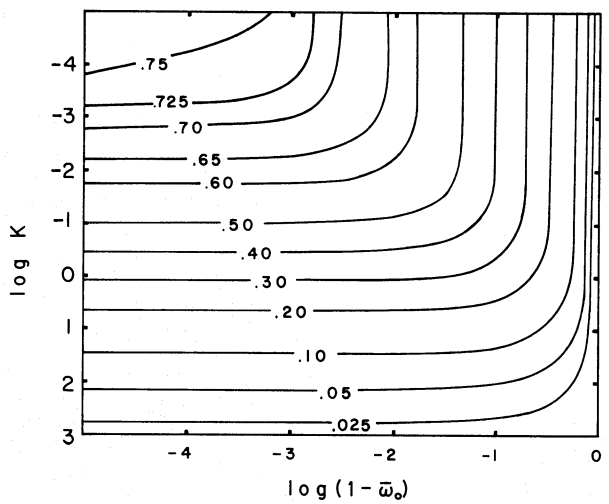

FIG. 1.—Zero-phase geometric albedo contours in the [log $(1 - \varpi_0)$, log K]-plane. The curves are labeled with the planetary albedo for a Rayleigh scattering atmosphere with single scattering albedo $\varpi(\tau) = \varpi_0(1 + K\tau)^{-1/2}$. The upper-left corner corresponds to a conservative atmosphere ($\varpi_0 = 1$, $K = 0$).

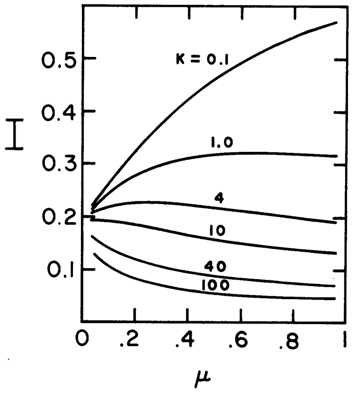

FIG. 2.—Center-to-limb intensities predicted for a planet at zero phase with a Rayleigh scattering atmosphere. The atmosphere is inhomogeneous with a single scattering albedo $\varpi(\tau) = (1 + K \cdot \tau)^{-1/2}$. The curves are labeled with their respective K values. The cosine angle of the emergent intensity is μ where $\mu = 0$ corresponds to the limb of the planet. The intensity I is in units of π incident flux.

III. NUMERICAL ASPECTS

The calculation time on a CDC 6400 computer for an atmosphere of N depth points and M quadrature points is approximately $0.0026NM^2$ seconds for the small values of M used in the calculation. If only mean intensities are desired, the computation time is reduced by 25 percent. It was found that increasing M from 4 to 5 gave no significant improvement in the value of the geometric albedo to four figures, and thus a 4-point Gaussian quadrature over the region [0, 1] was adopted. In this example the atmospheres are intended to be semi-infinite, and the placement of the lower boundary—which must be set at some finite depth—is highly critical. The calculated albedos were found to be insensitive to the lower boundary as long as it was set at a depth of $\tau = 10^3$ or deeper with a fully reflective surface ($\lambda_0 = 1$). This holds true for a wide variety of albedos including the conservative case ($\varpi \equiv 1$).

The largest identifiable source of error in the resultant intensities and thus in the geometric albedo comes from n, the number of grid points per decade of τ. The error in the representation of the second derivative by finite differences is proportional to $(\Delta\tau)^2$, where $\Delta\tau \propto 1/n$ is the grid spacing. This error propagates linearly through the solution of the difference equations and results in an error in the intensities which is proportional to $1/n^2$. For a given atmosphere calculated with two different n's, an estimate of the true intensities can be made by $I_{\text{true}} = (1 + c)I_2 - cI_1$, where $c = (n_2^2/n_1^2 - 1)^{-1}$. This correction procedure is very stable and, for test runs of several different n, predicts the same "true" geometric albedo within 0.02 percent for any combination of n_2 and n_1, whereas the original albedos differed by more than 1 percent. Thus it takes 28 seconds to calculate a geometric albedo by computing four atmospheres each of $N = 70$ and $N = 100$ with $M = 4$. Such a corrected albedo is accurate to better than 0.1 percent.

TABLE 1

ZERO-PHASE GEOMETRIC ALBEDOS

	log K													
$\tilde{\omega}_o$	$-\infty$	-4.0	-3.0	-2.0	-1.4	-1.0	-0.4	0.0	+0.6	+1.0	+1.6	+2.0	+2.6	+3.0
1.0000	.7975	.7585	.7162	.6350	.5597	.4981	.3904	.3147	.2076	.1492	.0848	.0565	.0297	.0191
.9999	.7798	.7556	.7150	.6345	.5594	.4979	.3903	.3147	.2075	.1492	.0847	.0565	.0297	.0191
.9990	.7432	.7346	.7043	.6300	.5569	.4962	.3894	.3141	.2073	.1490	.0847	.0564	.0296	.0191
.9950	.6821	.6801	.6664	.6113	.5460	.4889	.3855	.3115	.2060	.1482	.0843	.0562	.0295	.0190
.9900	.6402	.6392	.6315	.5904	.5332	.4800	.3807	.3084	.2044	.1471	.0838	.0559	.0293	.0189
.9750	.5649	.5646	.5616	.5395	.4987	.4552	.3667	.2992	.1996	.1441	.0823	.0549	.0289	.0186
.9500	.4906	.4904	.4891	.4777	.4515	.4190	.3450	.2844	.1919	.1392	.0798	.0534	.0281	.0181
.9000	.3998	.3997	.3992	.3941	.3803	.3602	.3063	.2572	.1770	.1296	.0749	.0503	.0265	.0171
.8500	.3391	.3390	.3387	.3358	.3271	.3134	.2728	.2326	.1630	.1203	.0701	.0472	.0250	.0161
.8000	.2927	.2927	.2925	.2906	.2846	.2748	.2433	.2101	.1496	.1113	.0653	.0441	.0234	.0152
.7500	.2551	.2551	.2550	.2535	.2493	.2419	.2171	.1895	.1370	.1026	.0607	.0411	.0219	.0142
.7000	.2233	.2233	.2232	.2222	.2190	.2133	.1935	.1704	.1249	.0942	.0562	.0382	.0204	.0132
.6000	.1716	.1716	.1715	.1709	.1690	.1655	.1526	.1364	.1024	.0783	.0473	.0323	.0174	.0113
.5000	.1304	.1304	.1303	.1300	.1288	.1266	.1180	.1068	.0818	.0633	.0387	.0266	.0144	.0094
.4000	.0963	.0963	.0962	.0960	.0952	.0939	.0883	.0806	.0628	.0491	.0305	.0211	.0114	.0075
.3000	.0672	.0672	.0672	.0671	.0666	.0657	.0622	.0573	.0454	.0358	.0225	.0156	.0085	.0056
.2000	.0420	.0420	.0420	.0419	.0417	.0412	.0392	.0363	.0292	.0232	.0147	.0103	.0057	.0037
.1000	.0198	.0198	.0198	.0198	.0197	.0195	.0186	.0173	.0141	.0113	.0073	.0051	.0028	.0018

After the calculations were completed, Auer (private communication) pointed out that the system of linear equations defined by the difference equations can be readily resolved for a different (μ_0, F_0) and lower boundary condition without resolving the entire atmosphere. Application of this method would reduce the calculation time for geometric albedos.

IV. CONCLUSIONS

The modified Feautrier technique allows for a fast, simple solution of an inhomogeneous Rayleigh scattering atmosphere which yields the surface intensities and the run of mean intensity throughout the atmosphere. The results of this program can be applied, for example, to narrow-band zero-phase geometric-albedo observations. These albedos can be matched with those calculated in order to estimate the atmospheric parameters. These parameters (ϖ_0, K) in turn determine the flux deposition and hence the heating rate. The program is restricted to plane-parallel atmospheres and is at present limited to Rayleigh scattering but can be adapted to other scattering matrices or functions that are even functions of the cosine angle.

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APPENDIX

SOLUTION OF THE EQUATION OF RADIATIVE TRANSFER

For a semi-infinite plane-parallel atmosphere with a flux $\pi \boldsymbol{F}_0$ (normal to itself) incident on the atmosphere in the direction $(-\mu_0, \phi_0)$, the equation of transfer becomes

$$\mu \frac{d\boldsymbol{I}}{d\tau}(\tau, \mu, \phi) = \boldsymbol{I}(\tau, \mu, \phi) - \frac{\varpi(\tau)}{4\pi}\int_{-1}^{+1}\int_{0}^{2\pi} P(\mu, \phi; \mu', \phi')\cdot \boldsymbol{I}(\tau, \mu', \phi')d\mu' d\phi'$$

$$- \tfrac{1}{4}\varpi(\tau)\cdot P(\mu, \phi; -\mu_0, \phi_0)\cdot \boldsymbol{F}_0 \exp(-\tau/\mu_0),$$

where

$$\boldsymbol{I} = (I_l, I_r, U, V), \qquad \boldsymbol{F}_0 = (F_{0,l}, F_{0,r}, F_{0,u}, F_{0,v}),$$

and $\varpi(\tau)$ is the single scattering albedo.

The equation of transfer for Rayleigh scattering has been derived by Chandrasekhar (1950, pp. 35, ff.), and we shall use his notation with only minor changes. While all four Stokes parameters are needed to describe the radiation field, the Rayleigh phase matrix is immediately reducible with respect to V. The reduced phase matrix is then given by

$$P(\mu, \phi; \mu', \phi') = P^{(0)}(\mu; \mu') + P^{(1)}(\mu, \phi; \mu', \phi') + P^{(2)}(\mu, \phi; \mu', \phi')$$

$$= \frac{3}{4}\begin{bmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 & 0 \\ \mu'^2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \tfrac{3}{4}(1-\mu^2)^{1/2}(1-\mu'^2)^{1/2}\begin{bmatrix} 4\mu\mu'\cos(\phi'-\phi) & 0 & 2\mu\sin(\phi'-\phi) \\ 0 & 0 & 0 \\ -4\mu'\sin(\phi'-\phi) & 0 & 2\cos(\phi'-\phi) \end{bmatrix}$$

$$+ \frac{3}{4}\begin{bmatrix} \mu^2\mu'^2\cos 2(\phi'-\phi) & -\mu^2\cos 2(\phi'-\phi) & \mu^2\mu'\sin 2(\phi'-\phi) \\ -\mu'^2\cos 2(\phi'-\phi) & \cos 2(\phi'-\phi) & -\mu'\sin 2(\phi'-\phi) \\ -2\mu\mu'^2\sin 2(\phi'-\phi) & 2\mu\sin 2(\phi'-\phi) & 2\mu\mu'\cos 2(\phi'-\phi) \end{bmatrix},$$

where the $P^{(n)}$ are periodic in $n(\phi' - \phi)$.

The intensity I can be expanded $I = I^{(0)} + I^{(1)} + I^{(2)} + \cdots$ such that each $I^{(n)}(\tau, \mu, \phi)$ is periodic in $n\cdot\phi$. Since the phase matrix only couples terms of like periodicity in $0\cdot\phi$, $1\cdot\phi$, and $2\cdot\phi$, the equation of transfer separates into three equations,

$$\mu\frac{d\boldsymbol{I}^{(n)}}{d\tau} = \boldsymbol{I}^{(n)} - \frac{\varpi}{4\pi}\int_{-1}^{+1}\int_0^{2\pi} P^{(n)}\cdot\boldsymbol{I}^{(n)}d\phi' d\mu' - \frac{\varpi}{4}\cdot P^{(n)}\cdot\boldsymbol{F}_0 e^{-\tau/\mu_0}, \qquad n = 0, 1, 2;$$

and the higher-order terms become trivial ($I^{(n)} \equiv 0$ for $n \geq 3$). The azimuth-independent phase matrix $P^{(0)}$ is reducible with respect to U, and the transfer equation for $\boldsymbol{I}^{(0)} = (I_l^{(0)}, I_r^{(0)})$ becomes

$$\mu\frac{d\boldsymbol{I}^{(0)}}{d\tau}(\tau, \mu) = \boldsymbol{I}^{(0)}(\tau, \mu) - \frac{\varpi(\tau)}{2}\int_{-1}^{+1} P^{(0)}(\mu; \mu')\cdot\boldsymbol{I}^{(0)}(\tau, \mu')d\mu' - \frac{\varpi(\tau)}{4}P^{(0)}(\mu; -\mu_0)\cdot\boldsymbol{F}_0\exp(-\tau/\mu_0),$$

where

$$P^{(0)}(\mu; \mu') = \frac{3}{4}\begin{bmatrix} 2(1-\mu^2)(1-\mu'^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{bmatrix}$$

and

$$\boldsymbol{F}_0 = (F_{0,l}, F_{0,r}).$$

The azimuth dependence of $\boldsymbol{I}^{(1)}$ and $\boldsymbol{I}^{(2)}$ can be derived explicitly by substituting into the equation of transfer:

$$\boldsymbol{I}^{(n)}(\tau, \mu, \phi) = P^{(n)}(\mu, \phi; -\mu_0, \phi_0)\cdot\boldsymbol{F}_0\cdot\chi^{(n)}(\tau, \mu), \qquad n = 1, 2.$$

The equation for the scalar functions $\chi^{(n)}(\tau, \mu)$ reduces to

$$\mu\frac{d\chi^{(n)}}{d\tau}(\tau, \mu) = \chi^{(n)}(\tau, \mu) - \frac{\varpi(\tau)}{2}\int_{-1}^{+1}\chi^{(n)}(\tau, \mu')g^{(n)}(\mu')d\mu' - \frac{\varpi(\tau)}{4}e^{-\tau/\mu_0}, \qquad n = 1, 2,$$

where

$$g^{(1)}(\mu') = \tfrac{3}{4}(1-\mu'^2)(1+2\mu'^2), \qquad g^{(2)}(\mu') = \tfrac{3}{8}(1+\mu'^2)^2.$$

The problem has now been reduced to solving three transfer equations which are independent of azimuth.

These equations can be readily solved via the Feautrier method as given by Auer (1971) by defining

$$\boldsymbol{j}(\tau, \mu) \equiv \tfrac{1}{2}[\boldsymbol{I}^{(0)}(\tau, +\mu) + \boldsymbol{I}^{(0)}(\tau, -\mu)]\,, \qquad 0 \leq \mu \leq 1\,,$$

and expressing the equation of transfer as a second-order differential equation with a first-order equation for use at the boundaries:

$$\mu^2 \frac{d^2\boldsymbol{j}}{d\tau^2} = \boldsymbol{j}(\tau, \mu) - \frac{\varpi(\tau)}{2}\int_0^1 P^{(0)}(\mu; \mu')\cdot\boldsymbol{j}(\tau, \mu')d\mu' - \frac{\varpi(\tau)}{4} P^{(0)}(\mu; -\mu_0)\cdot\boldsymbol{F}_0 \exp(-\tau/\mu_0)\,,$$

$$\mu \frac{d\boldsymbol{j}}{d\tau} = \boldsymbol{I}^{(0)}(\tau, +\mu) - \boldsymbol{j}(\tau, +\mu) = \boldsymbol{j}(\tau, +\mu) - \boldsymbol{I}^{(0)}(\tau, -\mu)\,.$$

We treat the $\chi^{(n)}$ functions similarly:

$$j^{(n)}(\tau, \mu) \equiv \tfrac{1}{2}[\chi^{(n)}(\tau, +\mu) + \chi^{(n)}(\tau, -\mu)]\,; \qquad n = 1, 2\,; \qquad 0 \leq \mu \leq 1\,;$$

$$\mu^2 \frac{d^2 j^{(n)}}{d\tau^2} = j^{(n)}(\tau, \mu) - \frac{\varpi(\tau)}{2}\int_0^1 j^{(n)}(\tau, \mu')\cdot g^{(n)}(\mu')d\mu' - \frac{\varpi(\tau)}{4}\exp(-\tau/\mu_0)\,;$$

$$\mu \frac{dj^{(n)}}{d\tau}(\tau, \mu) = \chi^{(n)}(\tau, +\mu) - j^{(n)}(\tau, \mu)$$

$$= j^{(n)}(\tau, \mu) - \chi^{(n)}(\tau, -\mu)\,.$$

Since the incident flux is included in the source term, the upper boundary condition becomes

$$\boldsymbol{I}(\tau = 0, -\mu, \phi) \equiv 0\,,$$

$$\mu \left.\frac{d\boldsymbol{j}}{d\tau}\right|_{\tau=0} = \boldsymbol{j}(0, \mu)\,,$$

$$\mu \left.\frac{dj^{(n)}}{d\tau}\right|_{\tau=0} = j^{(n)}(0, \mu)\,, \qquad 0 \leq \mu \leq 1\,.$$

The lower boundary was chosen to be a Lambert surface of reflectivity λ_0 at depth τ_N which reflects a fraction (λ_0) of the flux incident upon it uniformly in all directions. Hence the reflected intensity I_N^+ is independent of μ and ϕ:

$$I_N^+ = \frac{\lambda_0}{1 + \lambda_0}\left\{\mu_0 F_0 \exp(-\tau_N/\mu_0) + 4\int_0^1 [j_l(\tau_N, \mu) + j_r(\tau_N, \mu)]\mu d\mu\right\},$$

and the reflected light is natural:

$$I_l^{(0)}(\tau_N, +\mu) = I_r^{(0)}(\tau_N, +\mu) = \tfrac{1}{2} I_N^+\,,$$

$$\boldsymbol{I}^{(1)}(\tau_N, +\mu, \phi) = \boldsymbol{I}^{(2)}(\tau_N, +\mu, \phi) \equiv 0\,.$$

The first-order lower boundary conditions become

$$\mu \left.\frac{d\boldsymbol{j}}{d\tau}\right|_{\tau=\tau_N} = \tfrac{1}{2}\cdot\boldsymbol{I}_N^+ - \boldsymbol{j}(\tau_N, \mu)\,, \qquad \text{where } \boldsymbol{I}_N^+ = (I_N^+, I_N^+)\,,$$

and

$$\mu \left.\frac{dj^{(n)}}{d\tau}\right|_{\tau=\tau_N} = -j^{(n)}(\tau_N, \mu) \qquad \text{for } n = 1, 2\,.$$

The equations are solved for $\boldsymbol{j}$, $j^{(1)}$, and $j^{(2)}$ by the standard Feautrier method described by Auer (1971) with second-order boundary conditions (Auer 1967). The integrals over μ are approximated by a sum over M quadrature points μ_k with weights a_k. The $\boldsymbol{j}(\tau)$ are treated as $2M$-length vectors,

$$\boldsymbol{j}(\tau) = [j_l(\tau, \mu_1), j_l(\tau, \mu_2), \ldots, j_l(\tau, \mu_M), j_r(\tau, \mu_1), \ldots, j_r(\tau, \mu_M)]\,,$$

and the $j^{(n)}(\tau)$, as M-length vectors. The value of μ_0 and the run of ϖ with τ are specified. It is assumed in the calculation that $\boldsymbol{F}_0 = (\tfrac{1}{2}, \tfrac{1}{2}, 0, 0)$ which corresponds to natural light of unit astrophysical flux.

The program solves for all the j-values; thus, both the mean intensity at each grid point throughout the atmosphere,

$$J(\tau_i)/F_0 = \sum_{k=1}^{M} [j_l(\tau_i, \mu_k) + j_r(\tau_i, \mu_k)] \cdot a_k ,$$

and the 180° backscattered intensities

$$I_{180}(\mu_0)/F_0 \equiv I(\tau = 0, \mu_0, \phi_0 + \pi)/F_0$$
$$= 2 \cdot [j_l(0, \mu_0) + j_r(0, \mu_0)] + \tfrac{3}{4}[4\mu_0^2(1 - \mu_0^2) \cdot j^{(1)}(0, \mu_0) + (1 - \mu_0^2)^2 \cdot j^{(2)}(0, \mu_0)] ,$$

can be easily evaluated. These latter intensities correspond to the observed zero-phase intensities and may be used to calculate a zero-phase geometric albedo, p, by computing the atmosphere for $\mu_0 = \mu_k(k = 1, M)$:

$$p = \frac{2}{F_0} \int_0^1 I(0, \mu = \mu_0, \phi = \phi_0 + \pi)\mu d\mu$$
$$= \frac{2}{F_0} \sum_{k=1}^{M} I_{180}(\mu_k = \mu_0)\mu_k \cdot a_k .$$

The amount of flux deposited in the atmosphere as a function of optical depth can be computed at every grid point from

$$\frac{dF}{d\tau} = [1 - \varpi(\tau)] \cdot [4 \cdot J(\tau) + F_0 \exp(-\tau/\mu_0)] .$$

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