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THE EXTENDED RITZ METHOD APPLIED TO TRANSIENT, COUPLED THERMOELASTIC BOUNDARY-VALUE PROBLEMS

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CORRIGENDA

1. In the sixth line from the top on page 7, "$\underset{\sim}{u}$" and "kinematic" should be "$\underset{\sim}{u}$ and θ" and "displacement and temperature," respectively.

2. In the eleventh line from the bottom on page 14, "$_{2m}$" should be "δ_{2m}."

3. In the eighth line from the bottom on page 14, "$\{U^m\}$" should be "$\{u^m\}$."

4. In the second line of equation (2.7) on page 17, "[C]" should be "$[C]^T$."

5. In the fourth line of equation (2.7) on page 17, "$[C]^T$" should be "[C]."

6. In the second line of equation (2.16a) on page 22, "$\theta^2(t)$" should be "$\theta^{\alpha}(t)$."

7. In the fourth line from the bottom on page 26, "of interest" should be "(t_i, t_{i+1})."

8. In equation (3.11) on page 29, "cosy t" should be "cos yt."

9. In the fifth line from the bottom on page 31, "thermo-mechanical" should be "thermomechanical."

10. In equation (4.15) on page 36, and in equation (4.19d) on page 37, "$\sqrt{}(s+1+\delta)^2-4s$" should be "$\sqrt{(s+1+\delta)^2-4s}$."

11. In equations (4.20a), (4.20b) and (4.20c) on page 37, "$\sqrt{}(1+\delta+s)^2-4s$" should be "$\sqrt{(1+\delta+s)^2-4s}$."

12. In Figures 1 to 12 inclusive, the keys "—⊙—⊙—" and "—⊡—⊡—" should be "⊙" and "⊡," respectively.

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Summary

A method for obtaining approximate solutions to initial-boundary-value problems in the linear theory of coupled thermoelasticity is developed. This procedure is a direct variational method representing an extension of the Ritz method. As an illustration of the procedure, it is applied to a class of one-dimensional, transient problems involving weak thermal shocks. The problems considered are: (1) rapid heating of a half-space through a thermally conducting boundary layer, and (2) gradual heating of the boundary surface of a half-space. The solutions generated by the extended Ritz method are compared, for accuracy, to solutions obtained from a numerical inversion scheme for the Laplace transform based on Gaussian quadrature. These comparisons indicate that the variational procedure developed here can yield accurate results.

Introduction

The analytical treatment of the classical field theories, due primarily to their phenomenological foundation, customarily rests on the classification of the field variables into sets of conjugates (e.g., force-displacement, heat flow-temperature, current-electric potential, etc.) which influence only the members of the same set while exerting no influence on variables of other sets. Linearized classical field theories then employ such physically motivated constitutive relations as Ohm's law of electrical conduction, Fourier's law of heat conduction, Fick's law of mass diffusion, Darcy's law for fluid seepage through a porous medium, and Faraday's law of electromagnetic induction, among others, as field equations connecting the conjugate variables.

Physicists have long known that, while such simple divisions of phenomena often suffice to describe a great range of physical experience, the members of the different classes interact with each other, giving rise to various special field theories of energetico-mechanical, energetico-electrical, and electro-mechanical modes of behavior, for example. The work of Onsager [1], [2] pointed a way toward more systematic treatment of linearized coupled theories by demonstrating an argument for symmetry of the phenomenological coefficients that mark the generalizations of the uncoupled constitutive relations.

A more fundamental approach, which regards the Onsager relations and the linear phenomenological equations as suitable approximations to more general behavior, lies in the introduction of the principle of equipresence, which states that "a variable present as an independent variable in one constitutive equation should be present in all" [3]. Restrictions on the variables will then be a consequence of the application of the symmetry properties of the material in question, the principle of material objectivity, and the laws of thermodynamics [4]. From the physical point of view, of course, restrictions will be placed on the variables based on phenomenological intensities in given physical situations. From such a starting point the linearized coupled field theories emerge as special cases.

A point now appears to have been reached such that solution techniques lag behind the ability to derive consistent coupled field theories. The class of problems which admits closed form solutions is extremely small and does not include the complex shapes and composite construction so prevalent in engineering designs. This work represents an attempt to apply extensions of the type of direct variational methods recently being used in the stress analysis of solids (e.g. [5], [6]), transient heat conduction analysis [7], and flow through porous media [8]. This development will be restricted to the field of linear coupled thermoelasticity, although application to other coupled field phenomena, such as thermoelectricity, flow of a compressible fluid through a porous elastic medium, piezoelectric elasticity, etc., would be analogous.

The first section will introduce a variational principle characterizing the initial-boundary-value problem of linear coupled thermoelasticity. The second section will treat the development of the extended Ritz method, in a general form, for this theory. A check on the accuracy of the method will be obtained by comparison of several examples solved with recourse to a numerical inversion technique for the Laplace transform, as described in the third section. The approximate solutions to the class of problems, previously examined by Danilovskaya [9], [10], Mura [11], Sternberg and Chakravorty [12] (using dynamic, uncoupled theory), Muki and Breuer [13], and Boley and Tolins [14] (using dynamic, coupled theory), will be discussed in the fourth section.

1. The Variational Principle

The classical treatment of thermoelastic problems follows two different physically motivated paths [15]. If the effect of the straining of the solid on the temperature is assumed to be small, the temperature distribution is calculated from the Fourier heat conduction equation

$$\kappa \nabla^2 T = \frac{\partial T}{\partial t} , \tag{1.1}$$

where T is the temperature, t is the time, κ is the thermal diffusivity of the material, ∇^2 is the Laplacian operator, and $\partial/\partial t$ is the partial time derivative. The stress in the body is written as a function of the strain and temperature fields, through appropriate constitutive relations, and the stresses and displacements are then calculated from the equations of elasticity, with the temperature field serving as a forcing function.

On the other hand, certain problems may exist where the effect of the temperature distribution on the displacement field is assumed negligible, and the displacement may then be obtained from the Navier equations of motion [16]

$$(\lambda + \mu) \nabla (\nabla \cdot \underset{\sim}{u}) + \mu \nabla^2 \underset{\sim}{u} = \rho \frac{\partial^2 \underset{\sim}{u}}{\partial t^2} , \tag{1.2}$$

where $\underset{\sim}{u}$ is the vector displacement, ρ is the mass density, ∇ is the gradient operator, and λ and μ are the Lamé coefficients for the material. The entropy of the body, through an appropriate equation of state, is written as a function of the temperature and strain fields, and the distribution of temperature is determined from an amended heat conduction equation, the strain term then serving as the forcing function.

Under certain conditions the use of neither of these uncoupled approaches is justified. An adequate description of the interconversion of mechanical and thermal energy near the fronts of waves generated by rapid heating or loading may require the use of the fully coupled theory.

Analytical solutions of the linear coupled thermoelasticity equations have been obtained for a very few special cases which have lent themselves to attack by integral transform methods [13], [14], [17], [18]. The solutions are essentially one-dimensional, representing simple geometries and boundary conditions, and are valid over a limited range of either the time or a coupling parameter. Perturbation methods in terms of this coupling parameter have recently been introduced for more complex problems [19], but no comparisons were made between such methods and other numerical approaches to indicate greater utility for the perturbation techniques.

A powerful method for the generation of the governing differential equations of the coupled theory and for their approximate solution lies in the procedures of the variational calculus. The first variational development for linear coupled thermoelasticity appears to have been made by Biot [20]. The primary field variables were taken to be the displacement of a material point and a vector field variable called the entropy displacement. Three scalar invariants were introduced, representing the thermoelastic potential, the kinetic energy, and an energy dissipation functional, such that the combined variation yielded, as Euler equations, the field equations of equilibrium and transient heat conduction in terms of the primary variables. A Rayleigh-Ritz solution scheme was used to solve, approximately, the problems of pure heat conduction for a thin plate and thermoelastic damping of a vibrating rod. Biot [21], [22] and others [23], [24], [25], [26] have done considerable work in applying this variational technique to transient heat conduction problems, and Schapery [27] has extended the method to more general viscoelastic systems while neglecting the thermomechanical coupling.

Analogous to results obtained in isothermal elasticity, generalizations of Biot's work have recently been developed. Herrmann [28] derived a variational principle complementary to that of Biot and later [29] a mixed variational principle which yields, as Euler equations, a set of constitutive relations in addition to the equations of equilibrium and heat conduction. The principle also yields prescribed conditions on the heat flux vector, the temperature, the material displacement, and the traction vector as natural boundary conditions. Fu [30] and Ben-Amoz [31] have extended the mixed variational theorem by including, as Euler equations, the strain-displacement relations of geometrically linear elasticity. All of the authors have employed Biot's concept of an entropy displacement vector field and consider it conjugate to the temperature gradient. This is contrary to the usual notion, made firm by Coleman and Mizel [32], that the heat flux vector is conjugate to the temperature gradient in the linear theory.

Impetus for further development in this area is due to recent work by Gurtin [33], [34], who introduced new forms for the variational principles characterizing linear elastodynamics and transient heat conduction which incorporate the initial conditions explicitly into the variational statement. These ideas have previously been extended to construct variational principles of varying generality (explicitly incorporating initial conditions) appropriate for the linear coupled theory of thermoelasticity [35]. Of these, a particularly convenient variational principle for the approximate solution of initial-boundary-value problems takes the following form:

Let the vector displacement field $\underset{\sim}{u}(\underset{\sim}{x},t)$ and the scalar temperature field $\theta(\underset{\sim}{x},t)$ be a kinematically and thermally admissible state (i.e., let $\underset{\sim}{u}$ satisfy kinematic boundary conditions and let both have suitable smoothness properties). Then, for the functional Φ_t defined by

$$
\begin{aligned}
\Phi_t \{\underset{\sim}{u}, \theta\} = {} & \frac{1}{2} \int_V [g * \tau_{ij} * e_{ij}] (\underset{\sim}{x},t)\, dV \\
& - \frac{1}{2} \int_V \rho (\underset{\sim}{x}) [g * \eta * \theta] (\underset{\sim}{x},t)\, dV \\
& + \frac{1}{2} \int_V \rho (\underset{\sim}{x}) [u_i * u_i] (\underset{\sim}{x},t)\, dV \\
& + \frac{1}{2} \int_V \frac{1}{T_o} [g * g' * q_i * \vartheta_i] (\underset{\sim}{x},t)\, dV \\
& - \int_V [f_i * u_i] (\underset{\sim}{x},t)\, dV \\
& + \int_V \frac{1}{T_o} [g * h * \theta] (\underset{\sim}{x},t)\, dV \\
& - \int_{S_2} [g * \hat{T}_i * u_i] (\underset{\sim}{x},t)\, dS \\
& - \int_{S_2} \frac{1}{T_o} [g * g' * \hat{Q} * \theta] (\underset{\sim}{x},t)\, dS \quad , \qquad (1.3)
\end{aligned}
$$

$$\delta\, \Phi_t \{\underset{\sim}{u}, \theta\} = o \qquad (1.4)$$

if and only if $\underset{\sim}{u}(\underset{\sim}{x},t)$ and $\theta(\underset{\sim}{x},t)$ represent the solution to the initial-boundary-value problem of linear coupled thermoelasticity.

Here V denotes the volume of the region being considered and S its surface. It is convenient in this development to employ the standard indicial system in conjunction with a Cartesian reference frame. Repeated subscripts imply summation, Kronecker's delta is denoted by δ_{ij}, and differentiation with respect to space or time is indicated by subscripts preceded by commas or by superposed dots, respectively. Also $\underset{\sim}{u}(\underset{\sim}{x},t)$ represents the vector displacement field of a material point, $\theta(\underset{\sim}{x},t)$ is the scalar temperature field above a constant reference temperature T_o at which the system is assumed quiescent, $\underset{\sim}{x}$ is the spatial coordinate vector, and t is the time. The material density is $\rho(\underset{\sim}{x})$. The convolution of two functions of space and time ω_1 and ω_2 is defined by

$$[\omega_1 * \omega_2]\ (\underset{\sim}{x},t) = \int_o^t \omega_1\ (\underset{\sim}{x},t-\tau)\ \omega_2\ (\underset{\sim}{x},\tau)\ d\tau \tag{1.5}$$

and the functions g(t) and g'(t) are defined by

$$g(t) = t,\ g'(t) = 1. \tag{1.6}$$

The components of the stress tensor τ_{ij} and the heat flux vector q_i are defined through the constitutive equations

$$\tau_{ij}(\underset{\sim}{x},t) = C_{ijk\ell}(\underset{\sim}{x})\ e_{k\ell}(\underset{\sim}{x},t) - \beta_{ij}(\underset{\sim}{x})\ \theta\ (\underset{\sim}{x},t) \tag{1.7a}$$

and

$$q_i(\underset{\sim}{x},t) = -\ k_{ij}(\underset{\sim}{x})\ \vartheta_j(\underset{\sim}{x},t), \tag{1.7b}$$

where $e_{k\ell}$, ϑ_j, $C_{ijk\ell}$, β_{ij}, and k_{ij} are the components of the infinitesimal strain tensor, the thermal gradient vector, the isothermal elasticity tensor, the thermoelasticity tensor, and the thermal conductivity tensor, respectively. The material is assumed, for the present, to be generally anisotropic and nonhomogeneous. The entropy is defined by a linear relation with the current values of the deformation and temperature [36] and thus reduces to the caloric equation of state of thermostatics

$$\rho(\underset{\sim}{x})\ T_o\ \eta\ (\underset{\sim}{x},t) = \rho(\underset{\sim}{x})\ C_e(\underset{\sim}{x})\,\theta(\underset{\sim}{x},t)+\beta_{ij}(\underset{\sim}{x})\,T_o\ e_{ij}(\underset{\sim}{x},t), \tag{1.8}$$

where η is the specific entropy and C_e the specific heat for zero deformation. The strain and the thermal gradient are defined through the equations

$$e_{ij}(\underset{\sim}{x},t) = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \tag{1.9a}$$

and

$$\vartheta_i(\underset{\sim}{x},t) = \theta,i \quad . \tag{1.9b}$$

The body forces, internal heat generation, and initial conditions are combined in the functions

$$f_i(\underset{\sim}{x},t) = [g * F_i]\ (\underset{\sim}{x},t) + \rho(\underset{\sim}{x})\ [t\ v_i(\underset{\sim}{x}) + d_i(\underset{\sim}{x})\ ] \tag{1.10a}$$

and

$$h(\underset{\sim}{x},t) = [g' * H]\ (\underset{\sim}{x},t) + \rho(\underset{\sim}{x})\ T_o\ \eta_o(\underset{\sim}{x}), \tag{1.10b}$$

where $F_i(\underset{\sim}{x},t)$, $H(\underset{\sim}{x},t)$, $v_i(\underset{\sim}{x})$, $d_i(\underset{\sim}{x})$, and $\eta_o(\underset{\sim}{x})$ are the components of the body force vector, the internal heat generation, the prescribed initial velocities, the prescribed initial displacements, and the prescribed initial entropy, respectively. The natural boundary conditions are incorporated into the surface integrals over S_2, on which the traction vector $\hat{T}_i(\underset{\sim}{x},t)$ is prescribed, and over $\mathcal{S}_2$, on which the normal heat flux $\hat{Q}(\underset{\sim}{x},t)$ is prescribed. Admissible functions satisfy the kinematic boundary conditions over surface S_1 (complementary to S_2) and the temperature boundary conditions on $\mathcal{S}_1$ (complementary to $\mathcal{S}_2$), where $S_1 + S_2 = \mathcal{S}_1 + \mathcal{S}_2 = S$.

Taking the first variation of Φ_t results in the following set of Euler equations:

$$[g * (-\tau_{ij,j} - F_i) - \rho\,(tv_i + d_i) + \rho\,u_i]\,(\underset{\sim}{x},t) = o \tag{1.11a}$$

and

$$[g' * (-q_{i,i} + H) + \rho\,T_o\,\eta_o - \rho\,T_o\,\eta]\,(\underset{\sim}{x},t) = o, \tag{1.11b}$$

with the natural boundary conditions

$$\tau_{ij}\,n_j = \hat{T}_i \qquad \text{on } S_2 \tag{1.12a}$$

and

$$q_i\,n_i = \hat{Q} \qquad \text{on } \mathcal{S}_2\ . \tag{1.12b}$$

Equations (1.11a) and (1.11b) have been shown to be completely equivalent to the equations of motion and transient heat conduction [35]:

$$\tau_{ij,j} + F_i = \rho \ddot{u}_i \tag{1.13a}$$

and

$$q_{i,i} + \rho T_o \dot{\eta} = H . \tag{1.13b}$$

Thus it is seen that the variational principle stated above is equivalent to the conventional formulation of the linear theory of coupled thermoelasticity.

2. The Extended Ritz Method

The classical Rayleigh-Ritz method for the approximate solution of boundary value problems [37] rests on the equivalence between their differential equation characterization and the functional characterization of the variational calculus. In the latter, the differential equations are the Euler equations obtained by making the functional stationary. The Rayleigh method involves the construction of an approximating sequence based on the use of characteristic functions, valid for a given domain and for a given set of geometric boundary conditions, with arbitrary multipliers. The Ritz adaptation extends this notion to include approximating sequences based on any relatively complete (see [37]) set of admissible functions (i.e. functions which satisfy the appropriate smoothness requirements in the interior of the domain and satisfy geometric boundary conditions on the surface).

Both of the methods described above attempt the construction of the approximating sequence over the entire domain of interest by making each of the terms in the sequence satisfy the geometric boundary conditions. A more general procedure (called the finite element method in the literature) involves the construction of approximating sequences over geometrically simple subregions, insuring the continuity of the construction between adjacent subregions, and explicitly satisfying the geometric boundary conditions for the entire region. Since an easily manipulated set of complete [38] functions is the algebraic polynomials, this set has attracted the most application in the development of the extended Ritz method. The following work will also feature the use of the algebraic polynomials, and the approximating sequences will be constructed on the Cartesian product of space and time. The concept of geometrical boundary conditions will be extended to include boundary conditions and initial conditions on the primary variables of the variational functional. In this case (linear coupled thermoelasticity) the construction will be such that the approximating sequences for the displacement and temperature fields will be completely continuous in space and time. In addition, the velocity field, obtained by differentiation of the displacement approximating sequence with respect to time, will also be continuous.

The functional (1.3) can be placed in a form convenient for the application of the extended Ritz method to the approximate solution of initial-boundary-value problems in linear coupled thermoelasticity. By writing the stress and strain tensors in standard reduced (or vector) form [39], and utilizing a matrix representation, this functional, for a continuum divided into "M" subregions, becomes

$$\Phi_t \left\{\underset{\sim}{u}, \theta\right\} = \sum_{m=1}^{M} \left[\frac{1}{2} \int_{V_m} g * \left\{ \tau^m \right\}^T * \left\{ e^m \right\} dV_m \right.$$

$$- \frac{1}{2} \int_{V_m} \rho_m \, g * \eta^m * \theta^m \, dV_m$$

$$+ \frac{1}{2} \int_{V_m} \rho_m \left\{ u^m \right\}^T * \left\{ u^m \right\} dV_m \tag{2.1}$$

$$+ \frac{1}{2} \int_{V_m} \frac{1}{T_o} \, g * g' * \left\{ q^m \right\}^T * \left\{ \vartheta^m \right\} dV_m$$

$$- \int_{V_m} \left\{ u^m \right\}^T * \left\{ f^m \right\} dV_m$$

$$+ \int_{V_m} \frac{1}{T_o} \, g * \theta^m * h^m \, dV_m$$

$$- \int_{S_{2m}} g * \left\{ u^m \right\}^T * \left\{ p^m \right\} dS_m$$

$$- \int_{\mathcal{S}_{2m}} \frac{1}{T_o} \, g * g' * \theta^m * Q^m \, dS_m \Big] ,$$

where V_m denotes the volume of the "mth" subregion, S_{2m} the part of the surface of V_m on which the traction vector is prescribed, and $\;_{2m}$ the portion on which the normal heat flux is prescribed. The transpose of a matrix is denoted by the superscript T. The column vectors $\{\tau^m\}$, $\{e^m\}$, $\{U^m\}$, $\{q^m\}$, $\{\vartheta^m\}$, $\{f^m\}$, and $\{p^m\}$ respectively indicate, for the "mth" subregion, the matrix representation of the reduced stress tensor, the reduced strain tensor, the displacement vector, the heat flux vector, the thermal gradient vector, the vector representing body forces and initial conditions on the displacement and velocity fields, and the traction vector acting across the surface S_{2m}. The quantities η^m, θ^m, h^m, and Q^m are, for the "mth" subregion, the scalars entropy, temperature, internal heat generation together with initial conditions on the temperature and displacement gradients, and the normal heat flux across the surface $\mathcal{S}_{2m}$.

The constitutive relations, equations (1.7), for the "mth" subregion are

$$\left\{\tau^m\right\} = \left[H^m\right] \left\{e^m\right\} - \left\{B^m\right\} \theta^m \tag{2.2a}$$

and

$$\left\{q^m\right\} = - \left[R^m\right] \left\{\vartheta^m\right\} , \tag{2.2b}$$

where the square matrices $\left[H^m\right]$, $\left[R^m\right]$, and the column vector $\left\{B^m\right\}$ represent the reduced isothermal elasticity tensor, the thermal conductivity tensor, and the reduced thermoelasticity tensor. The equation of state (1.8) becomes

$$\rho_m \eta^m = \frac{\rho_m C_{em}}{T_o} \theta^m + \left\{B^m\right\}^T \left\{e^m\right\} , \tag{2.3}$$

where ρ_m, C_{em} are the mass density and the specific heat for zero deformation of the "mth" subregion.

Over each subregion the scalar temperature field and the vector displacement field are approximated by algebraic polynomials in the space coordinates with time dependent coefficients of the form

$$\left\{u^m\left(\underset{\sim}{x}, t\right)\right\} = \left[a_1^m \left(\underset{\sim}{x}\right)\right] \left\{\alpha_1^m\left(t\right)\right\} \tag{2.4a}$$

and

$$\theta^m(\underset{\sim}{x}, t) = \left\langle a_2^m(\underset{\sim}{x})\right\rangle \left\{\alpha_2^m(t) \right\} . \tag{2.4b}$$

The rectangular matrix $[a_1^m]$ and the transposed column vector $\langle a_2^m \rangle$ contain the algebraic polynomials and the column vectors $\{\alpha_1^m\}$ and $\{\alpha_2^m\}$ are the generalized coordinates. These generalized coordinates are to be converted into physical coordinates (nodal point values) through the transformation matrices $[\varphi_1^{\ m}]$ and $[\varphi_2^{\ m}]$ by the equations

$$\left\{\alpha_1^{\ m}(t)\right\} = \left[\varphi_1^{\ m}\right] \left\{u(t)\right\} \tag{2.5a}$$

and

$$\left\{\alpha_2^{\ m}(t)\right\} = \left[\varphi_2^{\ m}\right] \left\{\theta(t)\right\} , \tag{2.5b}$$

where the column vectors $\{u(t)\}$ and $\{\theta(t)\}$ represent the physical coordinates for the entire domain. The choice of the physical coordinates (nodal point values) must be such that the temperature and displacement field construction is continuous between subregions. (In practice, the subregions are chosen to be simple geometric shapes, such as triangles or quadrilaterals in two space dimensions and tetrahedrons or hexahedrons in three space dimensions; the physical coordinates are usually chosen to be the values of the field variables at the vertices of these simple shapes). The vectors $\{e^m\}$ and $\{\vartheta^m\}$ are obtained by appropriate space differentiation of (2.4) in accordance with the strain-displacement and temperature-thermal gradient relations (1.9). The results are

$$\left\{e^m(\underset{\sim}{x},t)\right\} = \left[\mathcal{b}_1^{\ m}(\underset{\sim}{x})\right]\left[\varphi_1^{\ m}\right]\left\{u(t)\right\} \tag{2.6a}$$

and

$$\left\{\vartheta^m(\underset{\sim}{x},t)\right\} = \left[\mathcal{b}_2^{\ m}(\underset{\sim}{x})\right]\left[\varphi_2^{\ m}\right]\left\{\theta(t)\right\} , \tag{2.6b}$$

where the rectangular matrices $\left[\mathcal{b}_1^{\ m}\right]$ and $\left[\mathcal{b}_2^{\ m}\right]$ are composed of algebraic polynomials derived by the space differentiation of the matrices $\left[a_1^{\ m}\right]$ and $\left\langle a_2^{\ m}\right\rangle$ respectively. The variational functional, equation (2.1), can then be written in the form

$$
\begin{aligned}
\Phi_t \left\{\underset{\sim}{u}, \theta\right\} = & \frac{1}{2} g * \left\{u(t)\right\}^T \left[K_1\right] * \left\{u(t)\right\} \\
& - \frac{1}{2} g * \left\{\theta(t)\right\}^T \left[C\right] * \left\{u(t)\right\} \\
& - \frac{1}{2} g * \left\{\theta(t)\right\}^T \left[M_2\right] * \left\{\theta(t)\right\} \\
& - \frac{1}{2} g * \left\{u(t)\right\}^T \left[C\right]^T * \left\{\theta(t)\right\} \\
& + \frac{1}{2} \left\{u(t)\right\}^T \left[M_1\right] * \left\{u(t)\right\} \\
& - \frac{1}{2} g * g' * \left\{\theta(t)\right\}^T \left[K_2\right] * \left\{\theta(t)\right\} \\
& - g * \left\{u(t)\right\}^T * \left\{F(t)\right\} \\
& + g * g' * \left\{\theta(t)\right\}^T * \left\{Q(t)\right\} \\
& - \left\{u(t)\right\}^T \left[M_1\right] * \left\{u(o)\right\} \\
& - t \left\{u(t)\right\}^T \left[M_1\right] * \left\{\dot{u}(o)\right\} \\
& + g * \left\{\theta(t)\right\}^T \left[M_2\right] * \left\{\theta(o)\right\} \\
& + g * \left\{\theta(t)\right\}^T \left[C\right]^T * \left\{u(o)\right\} ,
\end{aligned}
\tag{2.7}
$$

where

$$\left[K_1\right] = \sum_{m=1}^{M} \left[\varphi_1^{\,m}\right]^T \int_{V_m} \left[\mathcal{b}_1^{\,m}(\underset{\sim}{x})\right]^T \left[H^m\right] \left[\mathcal{b}_1^{\,m}(\underset{\sim}{x})\right] dV_m \left[\varphi_1^{\,m}\right] , \tag{2.8a}$$

$$\left[M_1\right] = \sum_{m=1}^{M} \left[\varphi_1^{\,m}\right]^T \int_{V_m} \left[\, a_1^{\,m}(\underset{\sim}{x})\right]^T \rho_m \left[\, a_1^{\,m}(\underset{\sim}{x})\right] dV_m \left[\varphi_1^{\,m}\right] , \tag{2.8b}$$

$$\left[K_2\right] = \frac{1}{T_o} \sum_{m=1}^{M} \left[\varphi_2^{\,m}\right]^T \int_{V_m} \left[\mathcal{b}_2^{\,m}(\underset{\sim}{x})\right]^T \left[R^m\right] \left[\mathcal{b}_2^{\,m}(\underset{\sim}{x})\right] dV_m \left[\varphi_2^{\,m}\right] , \tag{2.8c}$$

$$\left[M_2\right] = \frac{1}{T_o} \sum_{m=1}^{M} \left[\varphi_2^{\,m}\right] \int_{V_m} \left\{a_2^{\,m}(\underset{\sim}{x})\right\} \rho_m \, C_{em} \left\langle a_2^{\,m}(\underset{\sim}{x})\right\rangle dV_m \left[\varphi_2^{\,m}\right], \tag{2.8d}$$

and

$$\left[C\right] = \sum_{m=1}^{M} \left[\varphi_2^{\,m}\right]^T \int_{V_m} \left\{a_2^{\,m}(\underset{\sim}{x})\right\} \left\langle B^m \right\rangle \left[\mathcal{b}_1^{\,m}(\underset{\sim}{x})\right] dV_m \left[\varphi_1^{\,m}\right] . \tag{2.8e}$$

The vectors $\{F(t)\}$, $\{Q(t)\}$, $\{u(o)\}$, $\{\dot{u}(o)\}$, and $\{\theta(o)\}$ represent, respectively, the prescribed volumetric body forces and surface tractions, volumetric internal heat generation and surface heat flux, initial displacement distribution, initial velocity distribution, and initial temperature distribution. Taking the first variation of the functional (2.7) and utilizing the lemmas developed in [35] yields the governing matrix equations of the system:

$$g * \left[K_1\right]\left\{u(t)\right\} - g * \left[C\right]\left\{\theta(t)\right\} + \left[M_1\right]\left\{u(t)\right\}$$

$$= g * \left\{F(t)\right\} + \left[M_1\right] \left\{u(o)\right\} + t\left[M_1\right]\left\{\dot{u}(o)\right\} \tag{2.9a}$$

and

$$g' * \left[K_2\right]\left\{\theta(t)\right\} + \left[C\right]^T\left\{u(t)\right\} + \left[M_2\right]\left\{\theta(t)\right\}$$

$$= g' * \left\{Q(t)\right\} + \left[M_2\right]\left\{\theta(o)\right\} + \left[C\right]^T\left\{u(o)\right\} . \qquad (2.9b)$$

The results of the spatial approximation in the extended Ritz method have led to the system of equations (2.9). The remaining construction of the time approximation can be carried out in at least two ways. A mode superposition method could be employed, involving the calculation of approximate characteristic values for the system. Since the number of degrees of freedom in the extended Ritz solution technique may be in the hundreds, or even thousands, and since the types of excitations anticipated to be of importance in the linear coupled theory of thermoelasticity are likely to affect the higher frequencies of the system, the mode superposition method is felt in this case to be of limited value. Instead, a step forward integration method in time, analogous to the construction of the space approximation previously described, will be used.

For the step forward integration method, the time interval of consideration will be denoted (o,t). At the time $\tau=o$ (or $\tau=t_i$), all information about the system is assumed to be known, and the solution is sought at time $\tau=t$ (or $\tau=t_{i+1}$). The values of the displacements, velocities, and temperatures computed for time $\tau=o$ (or $\tau=t_i$) will serve as initial conditions for the solution procedure at time $\tau=t$ (or $\tau=t_{i+1}$). It is seen from this that the degree of continuity in the construction of the approximating sequence in time must be such that the displacement field and its first time derivative are continuous, while the temperature field is continuous. To accomplish this expand the nodal point displacement and temperature vectors as

$$\{u(\tau)\} = \{A_0\} + \tau \{A_1\} + \frac{1}{2} \tau^2 \{A_2\} \tag{2.10a}$$

and

$$\{\theta(\tau)\} = \{B_0\} + \tau \{B_1\}, \qquad (0 \leqq \tau \leqq t) \tag{2.10b}$$

where $\{A_0\}$, $\{A_1\}$, $\{A_2\}$, $\{B_0\}$, and $\{B_1\}$ are constant vectors. Equations (2.10) can easily be written in terms of the nodal point values of the displacement, velocity, and temperature vectors at the end points of the time interval, so as to insure proper continuity. Then

$$\left\{u(\tau)\right\} = \frac{(t^2-\tau^2)}{t^2} \left\{u(0)\right\} + \frac{\tau^2}{t^2} \left\{u(t)\right\} + \frac{\tau}{t} (t-\tau) \left\{\dot{u}(0)\right\} \tag{2.11a}$$

and

$$\left\{\theta(\tau)\right\} = \frac{(t-\tau)}{t} \left\{\theta(0)\right\} + \frac{\tau}{t} \left\{\theta(t)\right\}, \quad (0 \leqq \tau \leqq t). \tag{2.11b}$$

After the displacement and temperature fields have been calculated at time t, the velocity field at time t is computed from (2.11a) by time differentiation, giving

$$\left\{\dot{u}(t)\right\} = \frac{2}{t} \left\{u(t)\right\} - \frac{2}{t} \left\{u(0)\right\} - \left\{\dot{u}(0)\right\} . \tag{2.12}$$

The time dependence of the body forces, applied surface tractions, internal heat generation, and applied surface heat flux can be integrated explicitly and removed from the problem; however, since the motivation here is to automate the procedure as much as possible, the system excitations are assumed to vary linearly in the time interval. Then

$$\{F(\tau)\} = \frac{(t-\tau)}{t} \{F(0)\} + \frac{\tau}{t} \{F(t)\} \tag{2.13a}$$

and

$$\{Q(\tau)\} = \frac{(t-\tau)}{t} \{Q(0)\} + \frac{\tau}{t} \{Q(t)\} . \tag{2.13b}$$

The necessary convolution evaluations then become

$$g * \{u(t)\} = \frac{t^2}{12} \{u(t)\} + \frac{5}{12} t^2 \{u(0)\} + \frac{1}{12} t^3 \{\dot{u}(0)\} , \tag{2.14a}$$

$$g * \{\theta(t)\} = \frac{t^2}{6} \{\theta(t)\} + \frac{t^2}{3} \{\theta(0)\} , \tag{2.14b}$$

$$g * \{F(t)\} = \frac{t^2}{6} \{F(t)\} + \frac{t^2}{3} \{F(0)\} , \tag{2.14c}$$

$$g' * \{\theta(t)\} = \frac{t}{2} \{\theta(t)\} + \frac{t}{2} \{\theta(0)\} , \tag{2.14d}$$

and

$$g' * \{Q(t)\} = \frac{t}{2} \{Q(t)\} + \frac{t}{2} \{Q(0)\} . \tag{2.14e}$$

Using these results, together with equations (2.9), a system of governing matrix equations for the temperature and displacement fields is obtained at time t (or t_{i+1}):

$$\left[[M_1] + \frac{t^2}{12} [K_1] \right] \{u(t)\} - \frac{t^2}{6} [C] \{\theta(t)\} = \frac{t^2}{6} \{F(t)\} + \frac{t^2}{3} \{F(0)\}$$

$$+ \left[[M_1] - \frac{5t^2}{12} [K_1] \right] \{u(0)\} + t \left[[M_1] - \frac{t^2}{12} [K_1] \right] \{\dot{u}(0)\} + \frac{t^2}{3} [C] \{\theta(0)\} \tag{2.15a}$$

and

$$\left[\left[M_2 \right] + \frac{t}{2} \left[K_2 \right] \right] \left\{ \theta(t) \right\} + \left[C \right]^T \left\{ u(t) \right\} = \frac{t}{2} \left\{ Q(t) \right\} + \frac{t}{2} \left\{ Q(0) \right\}$$

$$+ \left[\left[M_2 \right] - \frac{t}{2} \left[K_2 \right] \right] \left\{ \theta(0) \right\} + \left[C \right]^T \left\{ u(0) \right\} . \tag{2.15b}$$

The remaining condition on the construction of approximate temperature and displacement fields over the continuum is the requirement that the fields satisfy the geometric boundary conditions of the variational principle - in this case, prescribed boundary displacements and temperatures. In order to achieve this, the matrix equations (2.15) are written in partitioned form, where a superscript "α" denotes an unmodified nodal point and "β" denotes a nodal point to be used in satisfying the boundary conditions cited above. Then, in a symbolic notation,

$$\left[\begin{array}{c|c} K_1^{\alpha\alpha} & K_1^{\alpha\beta} \\ \hline K_1^{\alpha\beta T} & K_1^{\beta\beta} \end{array} \right] \left\{ \begin{array}{c} u^{\alpha}(t) \\ \hline u^{\beta}(t) \end{array} \right\} + \frac{12}{t^2} \left[\begin{array}{c|c} M_1^{\alpha\alpha} & M_1^{\alpha\beta} \\ \hline M_1^{\alpha\beta T} & M_1^{\beta\beta} \end{array} \right] \left\{ \begin{array}{c} u^{\alpha}(t) \\ \hline u^{\beta}(t) \end{array} \right\}$$

$$- 2 \left[\begin{array}{c|c} C^{\alpha\alpha} & C^{\alpha\beta} \\ \hline C^{\beta\alpha} & C^{\beta\beta} \end{array} \right] \left\{ \begin{array}{c} \theta^{2}(t) \\ \hline \theta^{\beta}(t) \end{array} \right\} = 2 \left\{ \begin{array}{c} F^{\alpha}(t) \\ \hline F^{\beta}(t) \end{array} \right\} + 4 \left\{ \begin{array}{c} F^{\alpha}(0) \\ \hline F^{\beta}(0) \end{array} \right\} \tag{2.16a}$$

$$- 5 \left[\begin{array}{c|c} K_1^{\alpha\alpha} & K_1^{\alpha\beta} \\ \hline K_1^{\alpha\beta T} & K_1^{\beta\beta} \end{array} \right] \left\{ \begin{array}{c} u^{\alpha}(0) \\ \hline u^{\beta}(0) \end{array} \right\} + \frac{12}{t^2} \left[\begin{array}{c|c} M_1^{\alpha\alpha} & M_1^{\alpha\beta} \\ \hline M_1^{\alpha\beta T} & M_1^{\beta\beta} \end{array} \right] \left\{ \begin{array}{c} u^{\alpha}(0) \\ \hline u^{\beta}(0) \end{array} \right\}$$

$$
+\,4\begin{bmatrix} C^{\alpha\alpha} & C^{\alpha\beta} \\ C^{\beta\alpha} & C^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \theta^{\alpha}(0) \\ \theta^{\beta}(0) \end{Bmatrix} - t\begin{bmatrix} K_1{}^{\alpha\alpha} & K_1{}^{\alpha\beta} \\ K_1{}^{\alpha\beta T} & K_1{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \dot{u}^{\alpha}(0) \\ \dot{u}^{\beta}(0) \end{Bmatrix}
+\frac{12}{t}\begin{bmatrix} M_1{}^{\alpha\alpha} & M_1{}^{\alpha\beta} \\ M_1{}^{\alpha\beta T} & M_1{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \dot{u}^{\alpha}(0) \\ \dot{u}^{\beta}(0) \end{Bmatrix}
$$

and

$$
\begin{bmatrix} M_2{}^{\alpha\alpha} & M_2{}^{\alpha\beta} \\ M_2{}^{\alpha\beta T} & M_2{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \theta^{\alpha}(t) \\ \theta^{\beta}(t) \end{Bmatrix} + \frac{t}{2}\begin{bmatrix} K_2{}^{\alpha\alpha} & K_2{}^{\alpha\beta} \\ K_2{}^{\alpha\beta T} & K_2{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \theta^{\alpha}(t) \\ \theta^{\beta}(t) \end{Bmatrix}
$$

$$
+\begin{bmatrix} C^{\alpha\alpha T} & C^{\alpha\beta T} \\ C^{\beta\alpha T} & C^{\beta\beta T} \end{bmatrix}\begin{Bmatrix} u^{\alpha}(t) \\ u^{\beta}(t) \end{Bmatrix} = \frac{t}{2}\begin{Bmatrix} Q^{\alpha}(t) \\ Q^{\beta}(t) \end{Bmatrix} + \frac{t}{2}\begin{Bmatrix} Q^{\alpha}(0) \\ Q^{\beta}(0) \end{Bmatrix} \tag{2.16b}
$$

$$
+\begin{bmatrix} M_2{}^{\alpha\alpha} & M_2{}^{\alpha\beta} \\ M_2{}^{\alpha\beta T} & M_2{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \theta^{\alpha}(0) \\ \theta^{\beta}(0) \end{Bmatrix} - \frac{t}{2}\begin{bmatrix} K_2{}^{\alpha\alpha} & K_2{}^{\alpha\beta} \\ K_2{}^{\alpha\beta T} & K_2{}^{\beta\beta} \end{bmatrix}\begin{Bmatrix} \theta^{\alpha}(0) \\ \theta^{\beta}(0) \end{Bmatrix}
$$

$$
\begin{bmatrix} C^{\alpha\alpha T} & C^{\alpha\beta T} \\ C^{\beta\alpha T} & C^{\beta\beta T} \end{bmatrix}\begin{Bmatrix} u^{\alpha}(0) \\ u^{\beta}(0) \end{Bmatrix} .
$$

In the usual manner employed in finite element methods, the modification is accomplished by substituting the values of the prescribed boundary temperatures and displacements in the upper block of equations and rewriting the lower block of equations so that these prescribed values are computed trivially.

Details of the actual computational procedure by which this is accomplished may be found, for example, in [40]. It should be noted that this technique does not yield the surface tractions (heat flux) at the boundary nodal points where the displacements (temperature) are prescribed unless the information destroyed when rewriting the lower block of equations in trivial form is either saved in an auxiliary manner or reproduced. Substituting the prescribed boundary values and expanding (2.16) yields

$$
\begin{aligned}
&\left[\left[K_1^{\alpha\alpha}\right] + \frac{12}{t^2}\left[M_1^{\alpha\alpha}\right] \right]\left\{u^{\alpha}(t)\right\} - 2\left[C^{\alpha\alpha}\right]\left\{\theta^{\alpha}(t)\right\} = 2\left\{F^{\alpha}(t)\right\} + 4\left\{F^{\alpha}(0)\right\} \\
&- \left[5\left[K_1^{\alpha\alpha}\right] - \frac{12}{t^2}\left[M_1^{\alpha\alpha}\right]\right]\left\{u^{\alpha}(0)\right\} - t\left[\left[K_1^{\alpha\alpha}\right] - \frac{12}{t^2}\left[M_1^{\alpha\alpha}\right]\right]\left\{\dot{u}^{\alpha}(0)\right\} \\
&+ 4\left[C^{\alpha\alpha}\right]\left\{\theta^{\alpha}(0)\right\} - \left[K_1^{\alpha\beta}\right]\left\{\left\{u^{\beta}(t)\right\} + 5\left\{u^{\beta}(0)\right\} + t\left\{\dot{u}^{\beta}(0)\right\}\right\} \qquad (2.17a) \\
&- \frac{12}{t^2}\left[M_1^{\alpha\beta}\right]\left\{\left\{u^{\beta}(t)\right\} - \left\{u^{\beta}(0)\right\} - t\left\{\dot{u}^{\beta}(0)\right\}\right\} \\
&+ 2\left[C^{\alpha\beta}\right]\left\{\left\{\theta^{\beta}(t)\right\} + 2\left\{\theta^{\beta}(0)\right\}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left[\left[M_2^{\alpha\alpha}\right] + \frac{t}{2}\left[K_2^{\alpha\alpha}\right]\right]\left\{\theta^{\alpha}(t)\right\} + \left[C^{\alpha\alpha}\right]^T\left\{u^{\alpha}(t)\right\} = \frac{t}{2}\left\{Q^{\alpha}(t)\right\} + \frac{t}{2}\left\{Q^{\alpha}(0)\right\} \\
&+ \left[\left[M_2^{\alpha\alpha}\right] - \frac{t}{2}\left[K_2^{\alpha\alpha}\right]\right]\left\{\theta^{\alpha}(0)\right\} + \left[C^{\alpha\alpha}\right]\left\{u^{\alpha}(0)\right\} \\
& \qquad (2.17b) \\
&- \left[M_2^{\alpha\beta}\right]\left\{\left\{\theta^{\beta}(t) - \left\{\theta^{\beta}(0)\right\}\right\} - \frac{t}{2}\left[K_2^{\alpha\beta}\right]\left\{\left\{\theta^{\beta}(t)\right\} + \left\{\theta^{\beta}(0)\right\}\right\} \\
&- \left[C^{\alpha\beta}\right]^T\left\{\left\{u^{\beta}(t)\right\} - \left\{u^{\beta}(0)\right\}\right\} .
\end{aligned}
$$

It is convenient, at this point, to define new quantities to represent the unknowns to be calculated at time t (or t_{i+1}). These new quantities are defined as follows:

$$\left\{u^a\right\} = \frac{1}{6}\left\{u^{\alpha}(t)\right\} + \frac{5}{6}\left\{u^{\alpha}(0)\right\} + \frac{t}{6}\left\{\dot{u}^{\alpha}(0)\right\} \tag{2.18a}$$

and

$$\left\{\theta^a\right\} = \frac{1}{2}\left\{\theta^{\alpha}(t)\right\} + \frac{1}{2}\left\{\theta^{\alpha}(0)\right\} \tag{2.18b}$$

so that

$$\left\{u^{\alpha}(t)\right\} = 6\left\{u^a\right\} - 5\left\{u^{\alpha}(0)\right\} - t\left\{\dot{u}^{\alpha}(0)\right\} \tag{2.19a}$$

and

$$\left\{\theta^{\alpha}(t)\right\} = 2\left\{\theta^a\right\} - \left\{\theta^{\alpha}(0)\right\} \quad . \tag{2.19b}$$

Putting these results into (2.17) gives

$$\left[\left[K_1^{\alpha\alpha}\right] + \frac{12}{t^2}\left[M_1^{\alpha\alpha}\right]\right]\left\{u^a\right\} - \frac{2}{3}\left[C^{\alpha\alpha}\right]\left\{\theta^a\right\} = \frac{1}{3}\left\{F^{\alpha}(t)\right\} + \frac{2}{3}\left\{F^{\alpha}(0)\right\}$$

$$+ \frac{12}{t^2}\left[M_1^{\alpha\alpha}\right]\left\{u^{\alpha}(0)\right\} + \frac{1}{3}\left[C^{\alpha\alpha}\right]\left\{\theta^{\alpha}(0)\right\} + \frac{4}{t}\left[M_1^{\alpha\alpha}\right]\left\{\dot{u}^{\alpha}(0)\right\} \tag{2.20a}$$

$$- \left[K_1^{\alpha\beta}\right]\left\{\frac{1}{3}\left\{u^{\beta}(t)\right\} + \frac{2}{3}\left\{u^{\beta}(0)\right\}\right\} + \left[C^{\alpha\beta}\right]\left\{\frac{1}{3}\left\{\theta^{\beta}(t)\right\} + \frac{2}{3}\left\{\theta^{\beta}(0)\right\}\right\}$$

and

$$\left[\left[M_2^{\alpha\alpha}\right] + \frac{t}{2}\left[K_2^{\alpha\alpha}\right] \right] \left\{\theta^a\right\} + 3\left[C^{\alpha\alpha}\right]^T \left\{u^a\right\} = \frac{t}{4}\left\{Q^{\alpha}(t)\right\} + \frac{t}{4}\left\{Q^{\alpha}(0)\right\}$$

$$+ \left[M_2^{\alpha\alpha}\right]\left\{\theta^{\alpha}(0)\right\} + 3\left[C^{\alpha\alpha}\right]^T\left\{u^{\alpha}(0)\right\} + \frac{t}{2}\left[C^{\alpha\alpha}\right]^T\left\{\dot{u}^{\alpha}(0)\right\} \tag{2.20b}$$

$$- \frac{t}{4}\left[K_2^{\alpha\beta}\right]\left\{\left\{\theta^{\beta}(t)\right\} + \left\{\theta^{\beta}(0)\right\}\right\} - \frac{1}{2}\left[M_2^{\alpha\beta}\right]\left\{\left\{\theta^{\beta}(t)\right\} - \left\{\theta^{\beta}(0)\right\}\right\}$$

$$- \frac{1}{2}\left[C^{\alpha\beta}\right]\left\{\left\{u^{\beta}(t)\right\} - \left\{u^{\beta}(0)\right\}\right\},$$

where it has been assumed that the velocity of a nodal point whose displacement is prescribed is constant during the time interval of interest. Equations (2.20) represent the coupled system of governing matrix equations for linear thermoelasticity which result from the application of the extended Ritz method previously described.

3. Laplace Transform Inversion

In an effort to obtain a check on the numerical accuracy of the solutions generated by the extended Ritz method, some problems, amenable to solution by integral transform methods, were considered. A numerical inversion procedure for the Laplace transform (based on the work of Hurwitz and Zweifel [41], and Schmittroth [42]) was used to obtain the solutions, and is described below. The numerical integration procedure uses Gaussian quadrature formulae, which minimize the error in the calculation for a given number of integration points.

The complex inversion integral is written in the form

$$F(\underset{\sim}{x},t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(\underset{\sim}{x},s)e^{st}\, ds, \tag{3.1}$$

where s is the complex transform parameter and γ, the real part of the integration path, is chosen such that f is a regular analytic function of s in the half-plane $Re(s) > \gamma$. Introducing the definition for the integration variable

$$s = \gamma + iy \tag{3.2a}$$

so that

$$ds = idy, \tag{3.2b}$$

equation (3.1) may be written

$$F(\underset{\sim}{x},t) = \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} f(\underset{\sim}{x},\gamma+iy)\ e^{iyt}\ dy, \tag{3.3}$$

where the spatial point of interest $\underset{\sim}{x}$ and the real part of the integration variable are treated as parameters. Equation (3.3) can be written in a form convenient for numerical quadrature by using a procedure described in detail by Bradford [43]. Then

$$F(\underset{\sim}{x},t) = \frac{2e^{\gamma t}}{\pi} \int_0^{\infty} \mathrm{Re}\left\{f(\underset{\sim}{x},\gamma+iy)\right\} \cos yt \; dy \; . \tag{3.4}$$

The procedure is essentially as follows. The definition of the Laplace transform can be used to show that $f(\underset{\sim}{x},\bar{s})=\overline{f(\underset{\sim}{x},s)}$, where a superposed bar denotes the complex conjugate. If s is a complex variable defined by $s = \sigma + iy$, then

$$f(\underset{\sim}{x},s) = \int_0^{\infty} F(\underset{\sim}{x},t)e^{-st}\,dt$$

$$= \int_0^{\infty} F(\underset{\sim}{x},t)e^{-\sigma t}\,[\cos yt - i \sin yt]\,dt \tag{3.5}$$

and

$$f(\underset{\sim}{x},\bar{s}) = \int_0^{\infty} F(\underset{\sim}{x},t)e^{-\sigma t}\,[\cos yt + i \sin yt]\,dt, \tag{3.6}$$

so that

$$f(\underset{\sim}{x},\bar{s}) = \overline{f(\underset{\sim}{x},s)} \; . \tag{3.7}$$

Now the function $f(\underset{\sim}{x},s)$ can be written in terms of its real and imaginary parts as

$$f(\underset{\sim}{x},s) = u(\underset{\sim}{x},\sigma,y) + iv(\underset{\sim}{x},\sigma,y) \tag{3.8}$$

and equation (3.7) implies that

$$u(\underset{\sim}{x},\sigma,-y) + iv(\underset{\sim}{x},\sigma,-y) = u(\underset{\sim}{x},\sigma,y) - iv(\underset{\sim}{x},\sigma,y) \tag{3.9}$$

so that

$$u(\underset{\sim}{x},\sigma,-y) = u(\underset{\sim}{x},\sigma,y) \tag{3.10a}$$

and

$$v(\underset{\sim}{x}, \sigma, -y) = -v(\underset{\sim}{x}, \sigma, y) \quad . \tag{3.10b}$$

Then the real part of the function $f(\underset{\sim}{x}, s)$ is seen to be an even function of y while the imaginary part is odd. With this in mind, (3.3) can be written

$$F(\underset{\sim}{x}, t) = \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} f(\underset{\sim}{x}, \gamma + iy) \; [\cos y \; t + i \sin yt] \; dy \tag{3.11}$$

and, with the aid of (3.8) and (3.10), this becomes

$$\begin{aligned} F(\underset{\sim}{x}, t) = & \frac{e^{\gamma t}}{\pi} \int_{0}^{\infty} u(\underset{\sim}{x}, \gamma, y) \cos yt \; dy \\ & - \frac{e^{\gamma t}}{\pi} \int_{0}^{\infty} v(\underset{\sim}{x}, \gamma, y) \sin yt \; dy \; . \end{aligned} \tag{3.12}$$

Now, using the property of $F(\underset{\sim}{x}, t)$ that $F(\underset{\sim}{x}, t) = 0$ for $t < 0$, and putting $t = -t$ in (3.12)

$$\int_{0}^{\infty} u(\underset{\sim}{x}, \gamma, y) \cos yt \; dy = - \int_{0}^{\infty} v(\underset{\sim}{x}, \gamma, y) \sin yt \; dy \tag{3.13}$$

and therefore

$$\begin{aligned} F(\underset{\sim}{x}, t) &= \frac{2e^{\gamma t}}{\pi} \int_{0}^{\infty} u(\underset{\sim}{x}, \gamma, y) \cos yt \; dy \\ &= \frac{2e^{\gamma t}}{\pi} \int_{0}^{\infty} \mathrm{Re} \left\{ f(\underset{\sim}{x}, \gamma + iy) \right\} \cos yt \; dy. \end{aligned} \tag{3.14}$$

The integral (3.4) or (3.14) is now in a form convenient for the application of Gaussian quadrature formulae. The procedure to be following involves the mapping of the first quarter-cycle of cos yt ($0 \leq y \leq \pi/2t$) and each succeeding half-cycle $[\pi(2\,n-1)/2t \leq y \leq \pi\,(2\,n+1)/2t]$ onto the basic integration interval for the Gaussian quadrature scheme $(-1 \leq z \leq +1)$. The abscissae z_i for Gaussian quadrature are related to the integration variable y in (3.14) by

$$y_i = \pi/4t\ (1 + z_i) \tag{3.15a}$$

for the first quarter-cycle and

$$y_i = \pi/2t\ (2n - 2 + z_i) \tag{3.15b}$$

elsewhere. The arithmetic sum of two successive integrations is checked, in absolute value, against a predetermined error in order to test convergence. The complex arithmetic feature of the Fortran IV language is used to make the integrand evaluations and to find the real part of the result.

The results of the application of this technique to such transforms as the wave transform (e^{-sx}/s) and the diffusion transform $(e^{-\sqrt{s}x}/s)$ indicated sufficient accuracy to warrant use of this method to calculate "exact" or "closed-form" solutions. Comparisons to solutions obtained through use of the extended Ritz method are discussed in the next section.

4. Numerical Results

The first analytical solution to an initial-boundary-value problem in dynamic uncoupled thermoelasticity appears to have been obtained by Danilovskaya [9]. The problem concerned a linear elastic half-space subjected to a uniform sudden temperature change on its bounding plane, the plane being assumed traction free. The temperature distribution in the half-space was calculated from the classical heat conduction equation, neglecting thermo-mechanical coupling, and then used as a forcing function for the dynamic equations of motion. The thermal stresses were determined from an essentially one-dimensional kinematic theory, assuming lateral displacements completely restrained. Danilovskaya later [10] extended her results to account for boundary layer thermal conductance along the bounding plane. Sternberg and Chakravorty [12] determined analytical expressions for the displacements in the first Danilovskaya problem and developed the solution for the half-space subjected to ramp-type heating of the bounding plane, including both displacements and stresses. Recently Tsui [44] has superposed a mechanical effect on the Danilovskaya solution by prescribing the surface velocity of the boundary. All of the solutions mentioned above were obtained by means of Laplace transform techniques.

Investigations into the effect of including thermo-mechanical coupling, as well as inertia, have been made by many authors, such as Paria [17], Hetnarski [45], [46], [47], [48], Boley and Weiner [15], Deresiewicz [49], Chadwick [50], and others [51], [52]. Specific application of the fully coupled theory to the first Danilovskaya problem has been made by Boley and

Tolins [14] and Muki and Breuer [13]. Dunn [18] has investigated the long time mechanical mode response for this problem using asymptotic methods.

To illustrate the construction of approximate solutions to coupled linear thermoelastic initial-boundary-value problems, the original Danilovskaya problem and its modification to include boundary layer thermal conductance and ramp heating of the surface will be studied. Coupled solutions to these modified problems, to our knowledge, have not heretofore been obtained.

Consider the elastic half-space ($x > 0$) with the surface plane $x = 0$ assumed free of tractions for all time. The solid is assumed to be mechanically constrained and thermally insulated so that displacements of the form

$$u_x = u_x\,(x,t)\,;\; u_y = u_z = 0 \tag{4.1a}$$

and temperatures of the form

$$T = T\,(x,t) \tag{4.1b}$$

occur. The bounding plane $x = 0$ will be assumed to be exposed to two types of temperature environments: (1) sudden exposure to a high ambient temperature T_∞ through a boundary layer of finite thermal conductance, and (2) exposure to a linear temperature rise during a finite time interval, after which the temperature is held constant. In both of these cases the traveling discontinuity in the normal stress, present in the original Danilovskaya solution and also in the coupled solutions [13] and [14], are no longer effected, except as the limiting solution for an infinite boundary layer conductance or a zero boundary temperature rise time.

The coupled thermoelastic differential equations are

$$(\lambda+2\mu)\frac{\partial^2 u_x}{\partial x^2} = \rho\frac{\partial^2 u_x}{\partial t^2} + \alpha\,(3\lambda+2\mu)\frac{\partial(T-T_0)}{\partial x} \tag{4.2a}$$

and

$$k\frac{\partial^2 T}{\partial x^2} = \rho\, c_v \frac{\partial T}{\partial t} + \alpha\,(3\lambda+2\mu)\, T_0 \frac{\partial^2 u_x}{\partial x\,\partial t}\ , \tag{4.2b}$$

where λ, μ are the isothermal Lamé constants, ρ the density, α the linear coefficient of thermal expansion, T_0 the reference temperature, k the thermal conductivity, and c_v the specific heat at constant volume. The initial conditions are taken to be

$$u_x(x,0) = \frac{\partial u_x}{\partial t}(x,0) = 0 \tag{4.3a}$$

and

$$T(x,0) = T_0\ . \tag{4.3b}$$

The boundary condition on the normal stress is

$$\sigma_{xx}(0,t) = 0 \tag{4.4a}$$

and, on the temperature, either

$$k\frac{\partial T}{\partial x} = h\,(T - T_\infty),\ (t \geqq 0) \tag{4.4b}$$

for the case of boundary layer conductance, or

$$T(0,t) = \begin{cases} (T_1 - T_0)\ t/t_0, & 0 \leqq t \leqq t_o \\ T_1\ , & t_0 \leqq t \end{cases} \tag{4.4c}$$

for the case of ramp surface heating in the absence of a boundary layer. In equations (4.4) h denotes the boundary layer conductance, T_∞ the ambient temperature, t_0 the boundary temperature rise time, and T_1 the final surface temperature. These boundary conditions are to be supplemented by regularity conditions at infinity.

Introducing the dimensionless variables

$$\xi = ax/\varkappa \; ; \; \tau = a^2 t/\varkappa ; \quad \sigma = \sigma_{xx}/\beta T_0 ; \; \theta = (T-T_0)/T_0 ;$$
$$u = a(\lambda+2\mu)\, u_x/\varkappa\beta T_0 ; \tag{4.5}$$

where

$$\varkappa = k/\rho c_v ; \quad a^2 = (\lambda+2\mu)/\rho ; \quad \beta = \alpha(3\lambda+2\mu) ; \tag{4.6}$$

equations (4.2) can be written

$$\frac{\partial^2 u}{\partial \xi^2} = \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial \theta}{\partial \xi} \tag{4.7a}$$

and

$$\frac{\partial^2 \theta}{\partial \xi^2} = \frac{\partial \theta}{\partial \tau} + \delta \frac{\partial^2 u}{\partial \xi \, \partial \tau} \; . \tag{4.7b}$$

The thermomechanical coupling parameter δ is defined by the relationship

$$\delta = \beta^2 T_0/\rho c_v(\lambda+2\mu) . \tag{4.8}$$

The initial conditions become

$$u(\xi,0) = \frac{\partial u}{\partial \tau}(\xi,0) = \theta(\xi,0) = 0 \tag{4.9}$$

and the boundary conditions are

$$\sigma(0,\tau) = 0, \text{ for all } \tau \tag{4.10a}$$

and either

$$\frac{\partial\theta}{\partial\xi} = H(\theta - 1), \quad (\tau \geqq 0), \; (\xi = 0) \tag{4.10b}$$

or

$$\theta(0,\tau) = \begin{cases} \dfrac{\tau}{\tau_0}, & 0 \leqq \tau \leqq \tau_0 \\ 1, & \tau_0 \leqq \tau , \end{cases} \tag{4.10c}$$

where

$$\tau_0 = \frac{a^2 t_0}{\varkappa}.$$

For the problems to be considered here, the quantities $(T_1 - T_0)/T_0$ and $(T_\infty - T_0)/T_0$ have been set equal to unity, for convenience, in equations (4.10b) and (4.10c). In addition, a non-dimensional boundary layer thermal conductance,

$$H = \varkappa\, h/\, a\, k \tag{4.11}$$

is introduced in (4.10b).

The governing equations (4.7) can formally be solved through use of the Laplace transform. Applying the transform to (4.7) and using (4.9) gives

$$\frac{d^2\bar{u}}{d\xi^2} - s^2\,\bar{u} - \frac{d\bar{\theta}}{d\xi} = 0 \tag{4.12a}$$

and

$$\frac{d^2\bar{\theta}}{d\xi^2} - s\,\bar{\theta} - s\delta\,\frac{d\bar{u}}{d\xi} = 0 \;, \tag{4.12b}$$

where a superposed bar denotes a transformed function and s the transform parameter. Combining the two equations gives

$$\frac{d^4\bar{\theta}}{d\xi^4} - s\,(s+1+\delta)\,\frac{d^2\bar{\theta}}{d\xi^2} + s^3\,\bar{\theta} = 0 \;, \tag{4.13}$$

whose solution is, after taking the regularity conditions into account,

$$\bar{\theta}(\xi,s) = Ae^{-\lambda_1 \xi} + Be^{-\lambda_2 \xi} \;, \tag{4.14}$$

where

$$\lambda_{1,2} = \left\{ s(s+1+\delta) \pm s\sqrt{(s+1+\delta)^2-4s} \right\}^{1/2} / \sqrt{2} \;. \tag{4.15}$$

Throughout, that branch of the square root function must be chosen which yields a positive real root.

The boundary condition (4.10a) can be cast in terms of the transformed dimensionless temperature by using the stress-strain-temperature relation

$$\bar{\sigma} = \frac{d\bar{u}}{d\xi} - \bar{\theta} \tag{4.16}$$

and (4.12b). Then

$$\frac{d^2\bar{\theta}}{d\xi^2} - s(1+\delta)\,\bar{\theta} = \delta\, s\, \bar{\sigma} \;. \tag{4.17}$$

Solving for the constants A and B in (4.14) from the boundary conditions on the normal traction

$$\frac{d^2\bar{\theta}}{d\xi^2} - s(1+\delta)\,\bar{\theta} = 0, \quad (\xi=0) \tag{4.18a}$$

and on the surface heat flux

$$\frac{d\bar{\theta}}{d\xi} = H\left(\bar{\theta} - \frac{1}{s}\right) \quad , \ (\xi = 0) \tag{4.18b}$$

yields the following expressions for the temperature, normal stress, and displacement in the half-space:

$$\bar{\theta}(\xi,s) = H\left\{m_1 e^{-\lambda_2 \xi} - m_2 e^{-\lambda_1 \xi}\right\} \Big/ \left\{m_1(H+\lambda_2) - m_2(H+\lambda_1)\right\} \tag{4.19a}$$

$$\bar{\sigma}(\xi,s) = 2H\left\{e^{-\lambda_1 \xi} - e^{-\lambda_2 \xi}\right\} \Big/ \left\{m_1(H+\lambda_2) - m_2(H+\lambda_1)\right\} , \tag{4.19b}$$

and

$$\bar{u}(\xi,s) = 2H\left\{\lambda_2 e^{-\lambda_2 \xi} - \lambda_1 e^{-\lambda_1 \xi}\right\} \Big/ s^2 \left\{m_1(H+\lambda_2) - m_2(H+\lambda_1)\right\}, \tag{4.19c}$$

where

$$m_{1,2} = s-1-\delta \pm \sqrt{(s+1+\delta)^2 - 4s} \ . \tag{4.19d}$$

A similar set of calculations can be made for the ramp heating problem to yield

$$\bar{\theta}(\xi,s) = R(s)\left\{m_1 e^{-\lambda_2 \xi} - m_2 e^{-\lambda_1 \xi}\right\} \Big/ 2s^2 \sqrt{(1+\delta+s)^2 - 4s} \ , \tag{4.20a}$$

$$\bar{\sigma}(\xi,s) = R(s)\left\{e^{-\lambda_1 \xi} - e^{-\lambda_2 \xi}\right\} \Big/ s \sqrt{(1+\delta+s)^2 - 4s} \ , \tag{4.20b}$$

and

$$\bar{u}(\xi,s) = R(s)\left\{\lambda_2 e^{-\lambda_2 \xi} - \lambda_1 e^{-\lambda_1 \xi}\right\} \Big/ s^3 \sqrt{(1+\delta+s)^2 - 4s} \ , \tag{4.20c}$$

where

$$R(s) = \frac{\varkappa}{a^2 t_o}\left\{1 - e^{-s a^2 t_o/\varkappa}\right\} . \tag{4.20d}$$

The numerical inversions of the transforms shown in (4.19) and (4.20) were carried out on the CDC 6400 computer, using the procedure described in Section 3. The results of these calculations, and those obtained from the extended Ritz method described in Section 2, are shown in Figures 1-12. Three values of the coupling parameter δ were used - $\delta = 0$ (corresponding to the uncoupled theory), $\delta = 0.36$, and $\delta = 1.00$. Two values of H (0.5 and 5.0) were used in the study of the second Danilovskaya problem and two values of τ_0 (0.25 and 1.0) were used in the study of the Sternberg-Chakravorty problem.

To lend physical interpretation to these numbers, consider a fairly typical structural plastic having a specific gravity of 1.2 and a specific heat at constant volume of 0.25 cal/g ^{o}K. Let the reference temperature be 300 ^{o}K. Choose a bulk modulus of 2×10^{5} lb/in^{2} or about 1.4×10^{10} dynes/cm^{2}. Assume that the shear modulus can be neglected with respect to the bulk modulus at this temperature. Then the thermomechanical coupling constant becomes solely a function of the linear coefficient of thermal expansion. For values of α ranging from 7×10^{-5} to 4×10^{-4} cm/cm/^{o}K, δ takes on values ranging from approximately 0.04 to over 1.0. Since Dillon [53] has reported an α of 8×10^{-4} in/in/^{o}C for a plastic material having properties similar to those described above, values of the coupling parameter such as those investigated here seem to be reasonable. Also, for such a material, the values of H and τ_0 used as parameters in this study are equivalent to boundary layer heat transfer coefficients on the order of 2×10^{8} - 2×10^{9} Btu/ft^{2} hr ^{o}F and surface temperature rise times on the order of 10^{-13} seconds.

Figures 1-3 depict the temperature, displacement, and stress at $\xi = 1.0$, as a function of time, for the second Danilovskaya problem with $H = 0.5$, while Figures 4-6 portray these same quantities for $H = 5.0$. Similarly Figures 7-9 and 10-12 show the results of the Sternberg-Chakravorty problem for $\tau_0 = 0.25$ and $\tau_0 = 1.0$, respectively. On these figures the solid lines represent the "exact" solution, here meaning the Laplace transform inversion technique described in Section 3. Figures 13 and 14 indicate the spatial variation of the coupling effect on the temperature distribution. It is seen, from Figures 1-12, that the approximate solution based on the extended Ritz method developed here compares well to that obtained from the numerical inversion of the Laplace transform. This indicates that the extended Ritz method can yield accurate results.

An interesting feature of the extended Ritz solution is the effect, on the solution, of discontinuities in the time derivatives of the displacement and normal stress. This effect takes the form of a damped, oscillating disturbance, extending away from the point of discontinuity. Such disturbances are the result of the lack of smoothness in the temperature prescribed on the surface $x = 0$. Smoother and more physically meaningful inputs would eliminate the discontinuities and, therefore, the oscillations in the approximate solutions.

Another point of interest is the spatial distributions of temperature shown in Figures 13 and 14. The effect of strong coupling, in both the second Danilovskaya problem and the Sternberg-Chakravorty problem, is to accelerate thermal diffusion ahead of the wave-front and decelerate it behind the wave-front. The cause is most likely the interconversion of thermal and mechanical energy taking place predominantly at or near the wave front. In fact, the numerical results obtained indicate that a reverse temperature gradient is generated ahead of the wave for strong coupling. Also, the figures show that the deviation from the uncoupled solution becomes more pronounced as the wave progresses into the half-space. As has been pointed out by Dunn [18], and as can be inferred from Figures 13 and 14, the temperature distributions behind the wave-front asymptotically approach the uncoupled solution with increasing time.

In summary, it should be pointed out that the real value of the extended Ritz method lies in its ability to generate approximate solutions to more complex problems, involving inhomogeneous, anisotropic materials, in one, two, or three space dimensions, under a wide range of loading conditions, and comprising irregular geometries. In such cases, the application of integral transformation methods would be impractical.

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Figure 12. Normal stress at $\xi = 1.0$ as a function of time for the Sternberg-Chakravorty problem with $\tau_o = 1.0$, indicating the effect of thermomechanical coupling.

Figure 13. Temperature for $\delta = 1.0$ as a function of time for the second Danilovskaya problem with $H = 5.0$, indicating the spatial effect of thermomechanical coupling.

Figure 14. Temperature for $\delta = 1.0$ as a function of time for the Sternberg-Chakravorty problem with $\tau_o = 0.25$, indicating the spatial effect of thermomechanical coupling.

Appendix A

The simplest type of coupled theory would involve a purely one-dimensional situation. Two such essentially one-dimensional developments will be treated here.

First, consider the displacement and temperature fields to be restricted to the case where

$$u_1(\underset{\sim}{x},t) = u_1(x_1,t) \equiv u(x,t) \quad ,$$

$$u_2(\underset{\sim}{x},t) = u_3(\underset{\sim}{x},t) \equiv 0 \quad , \qquad (A.1)$$

and

$$\theta(\underset{\sim}{x},t) = T(\underset{\sim}{x},t) - T_0 \equiv \theta(x,t) \quad .$$

Then, assuming isotropic behavior, the stress tensor in the "mth" subregion is given by

$$\begin{Bmatrix} \tau_{xx}^{m} \\ \tau_{yy}^{m} \\ \tau_{zz}^{m} \end{Bmatrix} = e_{xx}^{m} \begin{Bmatrix} \lambda_m + 2\lambda_m \\ \lambda_m \\ \lambda_m \end{Bmatrix} - \theta^{m} \begin{Bmatrix} \alpha_m(3\lambda_m + 2\mu_m) \\ \alpha_m(3\lambda_m + 2\mu_m) \\ \alpha_m(3\lambda_m + 2\mu_m) \end{Bmatrix} , \qquad (A.2)$$

where λ_m, μ_m are the two Lamé constants for the material of subregion "m", α_m is the linear coefficient of thermal expansion, $\theta^m(x,t)$ is the temperature increase above the reference temperature T_o, and $e_{xx}^m(x,t)$ is the non-zero strain component in the rectangular cartesian coordinate system. The remaining components of the strain tensor, and the stress components τ_{xy}^m, τ_{yz}^m, and τ_{xz}^m, vanish due to (A.1). Similarly, the non-zero component of the heat flux vector is given by

$$q_x^{\,m} = - k_m \vartheta_x^{\,m} \quad , \tag{A.3}$$

where k_m is the coefficient of thermal conduction for the material of subregion "m". The entropy is given by the equation of state

$$\rho_m \eta^m = \frac{\rho_m C_{em}}{T_0} \theta^m + \alpha_m (3\lambda_m + 2\mu_m) e_{xx}^{\,m} \; , \tag{A.4}$$

where ρ_m is the density and C_{em} the specific heat at constant volume for the material of subregion "m".

Consider the displacement and temperature in the "mth" subregion to be approximated by

$$u^m(x,t) = \langle 1, x \rangle \begin{Bmatrix} \alpha_{11}^{\,m}(t) \\ \alpha_{12}^{\,m}(t) \end{Bmatrix} \tag{A.5a}$$

and

$$\theta^m(x,t) = \langle 1, x \rangle \begin{Bmatrix} \alpha_{21}^{\,m}(t) \\ \alpha_{22}^{\,m}(t) \end{Bmatrix} . \tag{A.5b}$$

The transformation from the generalized coordinates $\underset{\sim}{\alpha}$ to the nodal point coordinates is

$$\left[\varphi_1^{\,m} \right] = \left[\varphi_2^{\,m} \right] = \frac{1}{x_j - x_i} \begin{bmatrix} x_j & -x_i \\ -1 & 1 \end{bmatrix} , \tag{A.6}$$

where x_i and x_j denote the coordinates of the end points of subregion "m". From (A.5),

$$\left[a_1^m(x)\right] = \left\langle a_2^m(x)\right\rangle = \left\langle 1, x\right\rangle \tag{A.7a}$$

and

$$\left[b_1^m(x)\right] = \left\langle b_2^m(x)\right\rangle = \left\langle 0, 1\right\rangle . \tag{A.7b}$$

Carrying out the matrix operations indicated in the development of the extended Ritz method, it can be deduced that

$$\left[K_1^m\right] = \begin{bmatrix} (\lambda_m+2\mu_m)/\ell_m & -(\lambda_m+2\mu_m)/\ell_m \\ -(\lambda_m+2\mu_m)/\ell_m & (\lambda_m+2\mu_m)/\ell_m \end{bmatrix} , \tag{A.8a}$$

$$\left[M_1^m\right] = \begin{bmatrix} \rho_m\ell_m/3 & \rho_m\ell_m/6 \\ \rho_m\ell_m/6 & \rho_m\ell_m/3 \end{bmatrix} , \tag{A.8b}$$

$$\left[K_2^m\right] = \begin{bmatrix} k_m/\ell_m T_o & -k_m/\ell_m T_o \\ -k_m/\ell_m T_o & k_m/\ell_m T_o \end{bmatrix} , \tag{A.8c}$$

$$\left[M_2^m\right] = \begin{bmatrix} \rho_m C_{em}\ell_m/3\ T_o & \rho_m C_{em}\ell_m/6\ T_o \\ \rho_m C_{em}\ell_m/6\ T_o & \rho_m C_{em}\ell_m/3\ T_o \end{bmatrix} , \tag{A.8d}$$

and

$$\left[C^m\right] = \begin{bmatrix} \alpha_m(3\lambda_m+2\mu_m)/2 & \alpha_m(3\lambda_m+2\mu_m)/2 \\ -\alpha_m(3\lambda_m+2\mu_m)/2 & -\alpha_m(3\lambda_m+2\mu_m)/2 \end{bmatrix} , \tag{A.8e}$$

where $\ell_m = x_j - x_i$ for the "mth" subregion.

Secondly, consider the problem of axisymmetric plane strain with radial heat flow. Such a geometry is of considerable interest to the analyst concerned with the structural and thermal response of spacecraft and missiles exposed to extreme environments. Then

$$u_1(\underset{\sim}{x},t) \equiv u_r(r,t) ,$$

$$u_2(\underset{\sim}{x},t) = u_3(\underset{\sim}{x},t) \equiv 0 , \tag{A.9}$$

and

$$\theta(\underset{\sim}{x},t) = T(\underset{\sim}{x},t) - T_0 \equiv \theta(r,t) .$$

The constitutive matrices for isotropic media are

$$\left[H^m\right] = \begin{bmatrix} \lambda_m+2\mu_m & \lambda_m \\ \lambda_m & \lambda_m+2\mu_m \\ \lambda_m & \lambda_m \end{bmatrix} , \tag{A.10a}$$

$$\left\{B^m\right\}^T = \left\langle \alpha_m(3\lambda_m+2\mu_m),\ \alpha_m(3\lambda_m+2\mu_m),\ \alpha_m(3\lambda_m+2\mu_m) \right\rangle , \tag{A.10b}$$

and

$$\left[R^m\right] = k_m . \tag{A.10c}$$

Also

$$\left[a_1^m(r)\right] = \left\langle a_2^m(r) \right\rangle = \left\langle 1,\ r \right\rangle \tag{A.11a}$$

and

$$\left[b_1^m(r)\right] = \begin{bmatrix} 0 & 1 \\ 1/r & 1 \end{bmatrix} , \tag{A.11b}$$

$$\left[b_2^m(r)\right] = \left\langle 0,\ 1 \right\rangle . \tag{A.11c}$$

Carrying out the matrix operations once more yields

$$\left[K_1^{\,m}\right] = \begin{bmatrix} \dfrac{(\lambda_m+2\mu_m)\,\ell n(r_j/r_i)}{(1-r_i/r_j)^2} - 2(\lambda_m+\mu_m) & \dfrac{(\lambda_m+2\mu_m)\,\ell n(r_j/r_i)}{(1-r_i/r_j)(1-r_j/r_i)} \\ \dfrac{(\lambda_m+2\mu_m)\,\ell n(r_j/r_i)}{(1-r_i/r_j)(1-r_j/r_i)} & \dfrac{(\lambda_m+2\mu_m)\,\ell n(r_j/r_i)}{(1-r_j/r_i)^2} + 2(\lambda_m+\mu_m) \end{bmatrix}, \qquad \text{(A.12a)}$$

$$\left[M_1^{\,m}\right] = \begin{bmatrix} \dfrac{\rho_m}{12}(r_j-r_i)(r_j+3r_i) & \dfrac{\rho_m}{12}(r_j-r_i)(r_j+r_i) \\ \dfrac{\rho_m}{12}(r_j-r_i)(r_j+r_i) & \dfrac{\rho_m}{12}(r_j-r_i)(3r_j+r_i) \end{bmatrix}, \qquad \text{(A.12b)}$$

$$\left[K_1^{\,m}\right] = \begin{bmatrix} \dfrac{k_m}{2T_0}\dfrac{(r_j+r_i)}{(r_j-r_i)} & -\dfrac{k_m}{2T_0}\dfrac{(r_j+r_i)}{(r_j-r_i)} \\ -\dfrac{k_m}{2T_0}\dfrac{(r_j+r_i)}{(r_j-r_i)} & \dfrac{k_m}{2T_0}\dfrac{(r_j+r_i)}{(r_j-r_i)} \end{bmatrix}, \qquad \text{(A.12c)}$$

$$\left[M_2^{\,m}\right] = \begin{bmatrix} \dfrac{\rho_m C_{em}}{12\ T_0}(r_j-r_i)(r_j+3r_i) & \dfrac{\rho_m C_{em}}{12\ T_0}(r_j-r_i)(r_j+r_i) \\ \dfrac{\rho_m C_{em}}{12\ T_0}(r_j-r_i)(r_j+r_i) & \dfrac{\rho_m C_{em}}{12\ T_0}(r_j-r_i)(3r_j+r_i) \end{bmatrix}, \qquad \text{(A.12d)}$$

and

$$\left[C^m\right] = \begin{bmatrix} \dfrac{\alpha_m(3\lambda_m+2\mu_m)}{6}(4r_i-r_j) & \dfrac{\alpha_m(3\lambda_m+2\mu_m)}{6}(2r_i+r_j) \\ -\dfrac{\alpha_m(3\lambda_m+2\mu_m)}{6}(r_i+2r_j) & -\dfrac{\alpha_m(3\lambda_m+2\mu_m)}{6}(4r_j-r_i) \end{bmatrix}, \qquad \text{(A.12e)}$$

where r_i and r_j are the end points of the "mth" subregion.

Appendix B

Computer Program Description

IDENTIFICATION

COUPLED LINEAR ONE-DIMENSIONAL THERMOELASTICITY

Programmed - R.E. Nickell

University of California, October 1966

PURPOSE

The purpose of this computer program is to determine temperatures, displacements, velocities, and stresses in either axisymmetric solids subjected to plane strain and radial heat flow or solids in double plane strain and uniaxial heat flow. The effects of displacement, traction, temperature, or heat flow boundary conditions are included, along with internal heat generation and volumetric body forces.

INPUT DATA

The first step in the analysis is to select a finite element representation of the solid. Elements and nodal points are then numbered in two sequences, each starting with one. The following group of punched cards numerically define the problem.

A. IDENTIFICATION CARD - (12A6)

Columns 1-72 of this card contain information to be printed with the results.

B. CONTROL CARD - (4I5,3F10.0,3I5)

Columns 1-5 Number of nodal points (150 maximum)

6-10 Number of elements (150 maximum)

11-15 Number of different materials (12 maximum)

16-20 Number of time increments (no limit)

21-30 Initial time increment (must be non-zero)

31-40 Acceleration of solid

41-50 Reference temperature (must be non-zero)

51-55 Print interval for results

56-60 Free control parameter (leave = 0)

61-65 Control parameter (if = 1 plane solid,

if = 0 axisymmetric solid)

C. MATERIAL PROPERTY INFORMATION (7F10.0)

Columns 1-10 Lamé constant λ

11-20 Lamé constant μ

21-30 Coefficient of thermal expansion

31-40 Coefficient of thermal conductivity

41-50 Specific heat at constant volume per unit volume

51-60 Material density

61-70 Internal heat generation per unit volume

D. NODAL POINT CARDS (2I5,4F10.0)

One card for each nodal point with the following information:

Columns 1-5 Nodal point number

6-10 Boundary condition code

11-20 X-ordinate or R-ordinate

21-30 Displacement/Traction

31-40 Velocity

41-50 Temperature/Heat flux

If the number in column 10 is

0 the traction is specified and the heat flux is specified.

1 the displacement is specified and the heat flux is specified.

2 the traction is specified and the temperature is specified.

3 the displacement is specified and the temperature is specified.

Nodal point cards must be in numerical sequence. If cards are omitted, the omitted nodal points are generated at equal intervals between the defined nodal points. Tractions, heat fluxes, and velocities are set equal to zero at generated nodal points.

E. ELEMENT CARDS (4I5)

One card for each element

Columns 1-5 Element number

6-10 Nodal point I

11-15 Nodal point J

16-20 Material identification number

Element cards must be in element number sequence. If element cards are omitted, the omitted information is obtained by incrementing by one the previous element number, I, and J. The material identification number for the generated cards is set equal to the value on the previous card. The last element card must always be supplied.

F. LOGICAL SWITCH CARD (L5)

If all boundary conditions, tractions, and heat fluxes are constant in time put F in column 1 (FIXED BOUNDARY CONDITIONS); if the boundary conditions are time-dependent put T in column 1 (TIME-DEPENDENT BOUNDARY CONDITIONS). If the logical card is false (F), this is the final input card. If the logical card is true (T), cards are required for each time increment (after the first) to define the time increment (F10.0), the nodal point number, the traction (displacement), and heat flux (temperature) for each nodal point that is time-dependent (I10,2F10.0), and a blank card.

OUTPUT INFORMATION

The following information is developed and printed by the computer program:

1. Reprint of input data,
2. Nodal point displacements, temperatures, and velocities, and
3. Stresses at the center of each element.

Appendix C

Computer Program Description

IDENTIFICATION

INVERSE LAPLACE TRANSFORM

Programmed - Robert E. Nickell

University of California, January 1967

PURPOSE

The purpose of these computer programs is to determine, by an approximate numerical integration procedure, inverse Laplace transforms associated with initial-boundary-value problems of linear coupled thermoelasticity. Eight-point Gaussian quadrature is used and the programs are flexible enough to be easily converted to higher order quadrature formulae, if needed.

INPUT DATA

A. IDENTIFICATION CARD - (12A6)

Columns 1 to 72 of this card contain information to be printed with the results.

B. CONTROL CARD - (2I5,4F10.0) or (2I5,5F10.0)

Columns 1-5 Number of integration cycles felt to be necessary for convergence by the user

6-10 Number of time points to be evaluated for a coordinate position

11-20 Real part of the integration variable

21-30 Coordinate value (in this case, x)

31-40 Value of thermomechanical coupling parameter

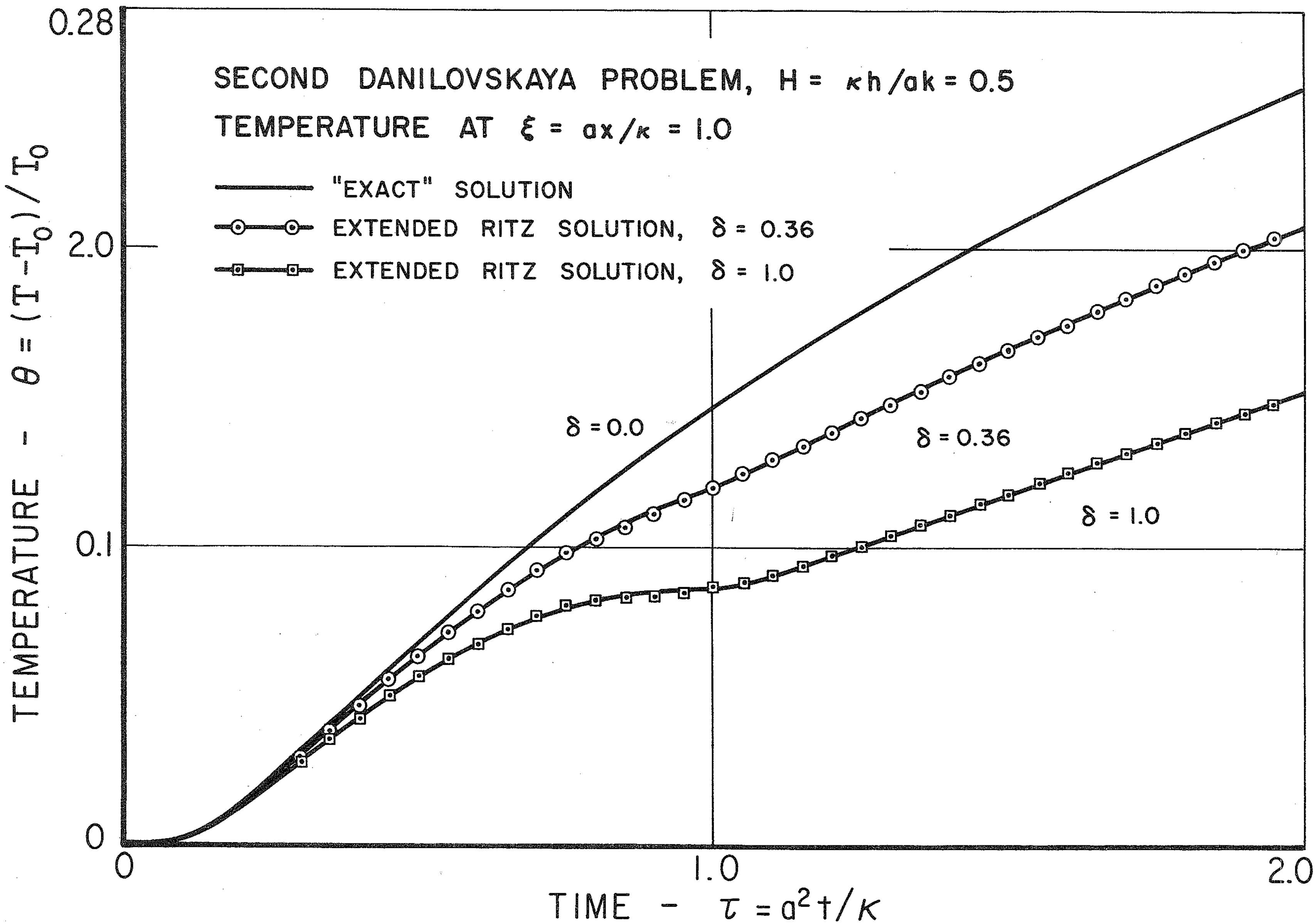
41-50 Error bound (for the second program, columns 41-50 contain the dimensionless boundary layer conductance and 51-60 the error bound)

C. TIME CARDS - (F10.0)

Column 1-10 Value of the time at which the inversion is to be carried out

OUTPUT INFORMATION

These programs print the input data and the displacement, temperature, and stress for each value of time and space specified. An error message, indicating that the number of integration cycles was not sufficient to achieve convergence, will be printed for cases in which this occurs.



SECOND DANILOVSKAYA PROBLEM, H = κh/ak = 0.5
TEMPERATURE AT ξ = ax/κ = 1.0
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION, δ = 1.0
δ = 0.0
δ = 0.36
δ = 1.0
0.28
2.0
0.1
0
0
1.0
2.0
TEMPERATURE - θ = (T-T0)/T0
TIME - τ = a2t/κ

FIGURE 1.

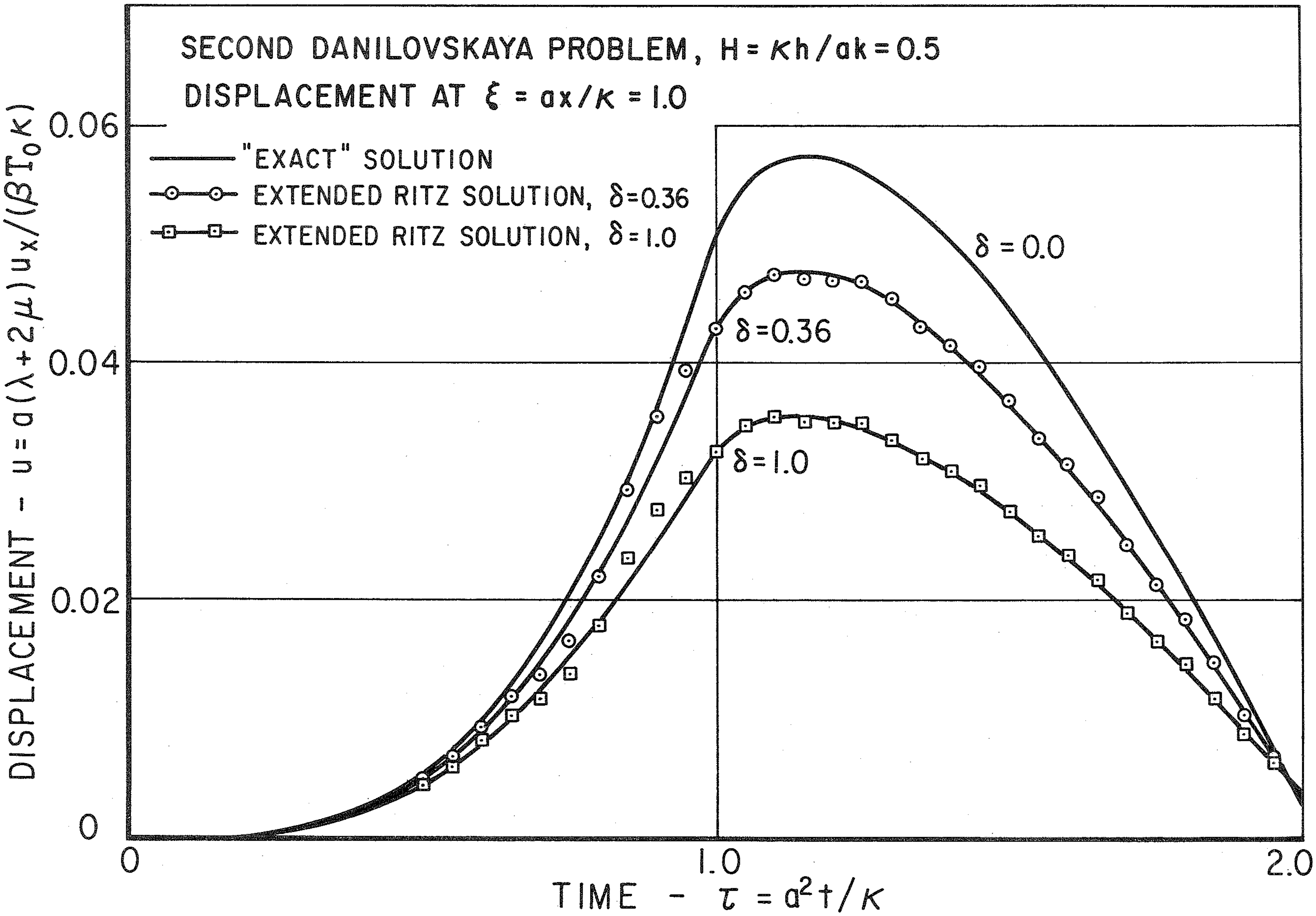
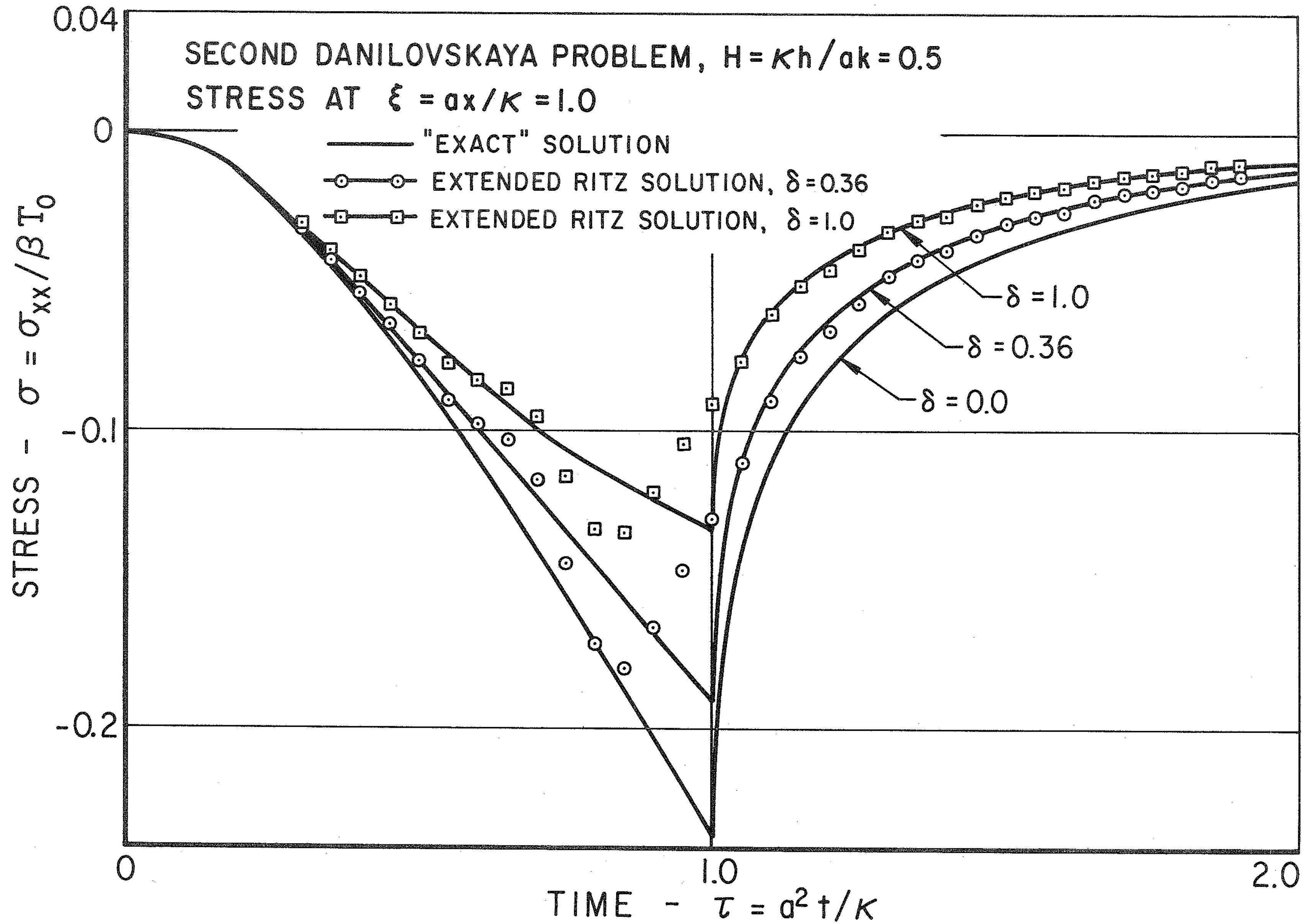
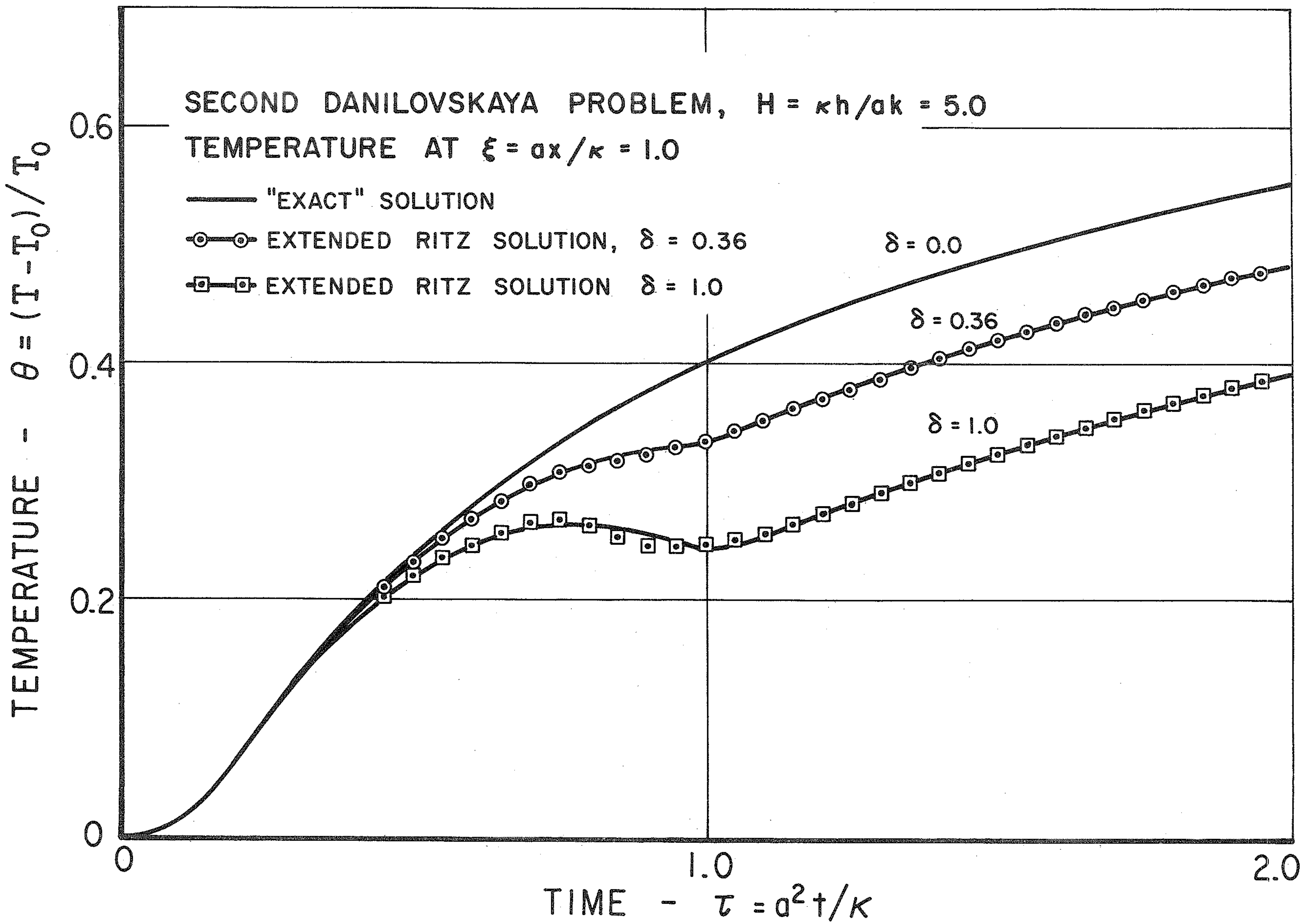

SECOND DANILOVSKAYA PROBLEM, $H = \kappa h / ak = 0.5$
DISPLACEMENT AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.0$
$\delta = 0.0$
$\delta = 0.36$
$\delta = 1.0$
0.06
0.04
0.02
0
1.0
2.0
DISPLACEMENT - $u = a(\lambda + 2\mu) u_x / (\beta T_0 \kappa)$
TIME - $\tau = a^2 t / \kappa$

FIGURE 2.



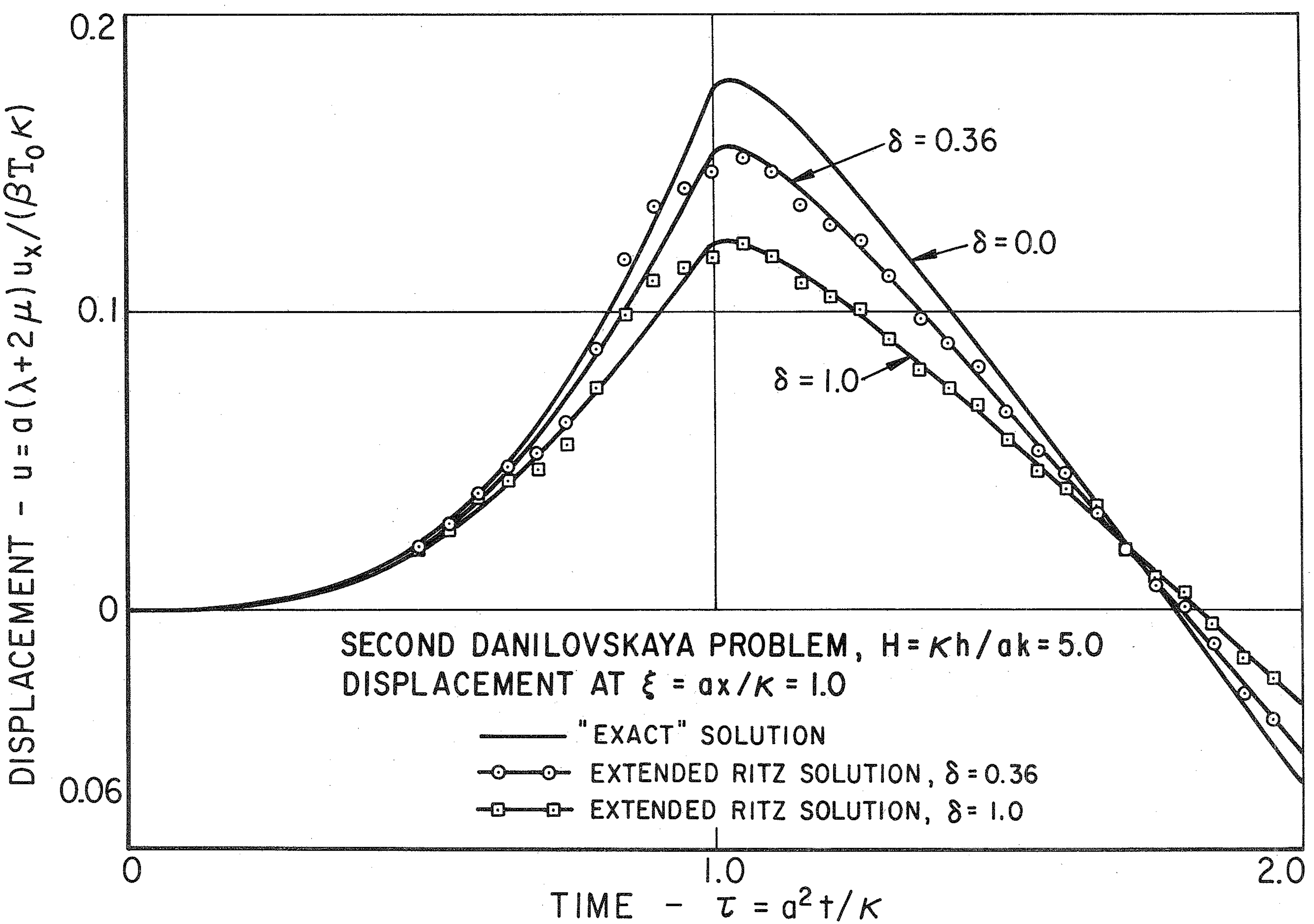
SECOND DANILOVSKAYA PROBLEM, $H = \kappa h / ak = 0.5$
STRESS AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.0$
$\delta = 1.0$
$\delta = 0.36$
$\delta = 0.0$
0.04
0
-0.1
-0.2
STRESS - $\sigma = \sigma_{xx}/\beta T_0$
0
1.0
2.0
TIME - $\tau = a^2 t/\kappa$

FIGURE 3.



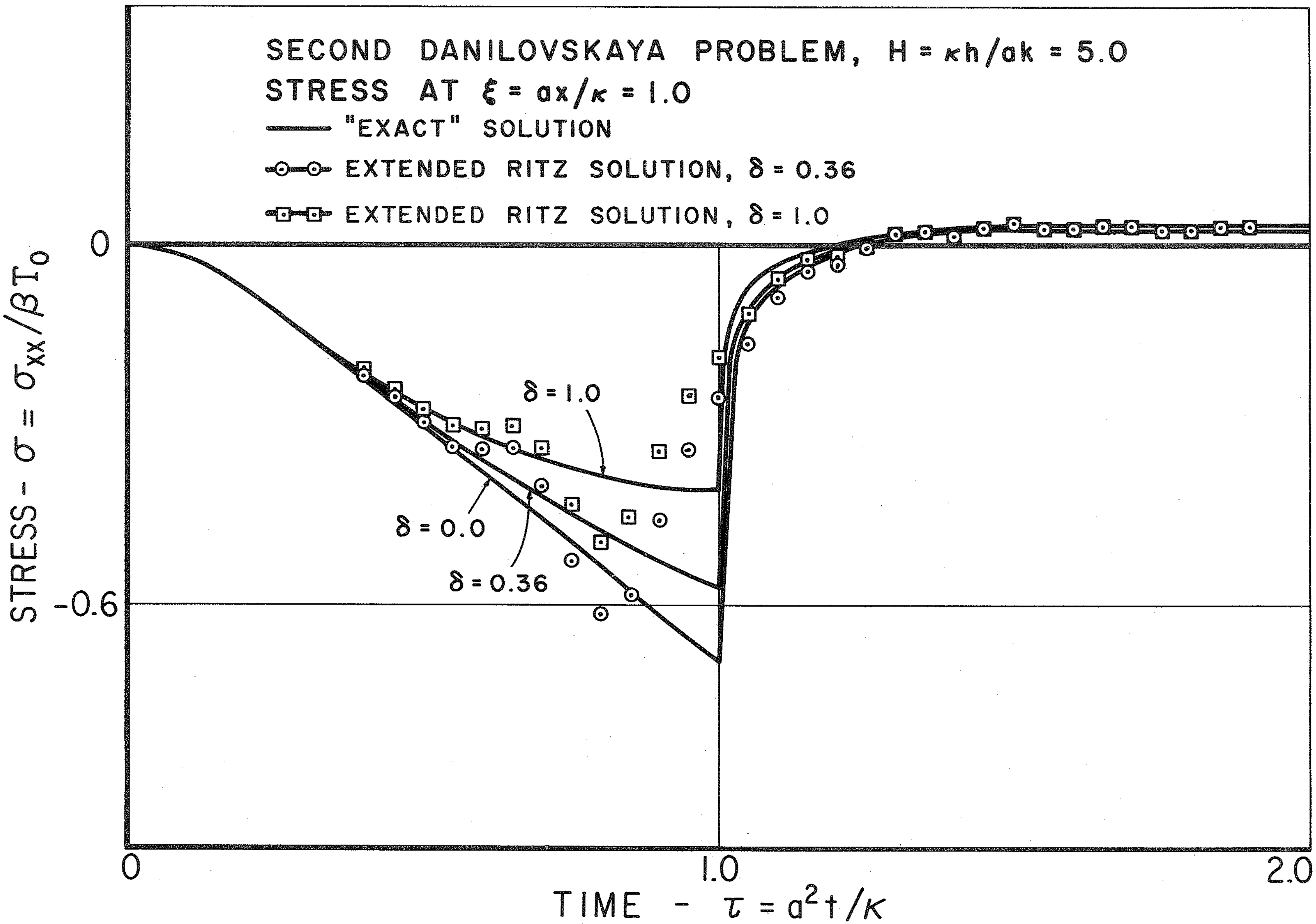
SECOND DANILOVSKAYA PROBLEM, H = κh/ak = 5.0
TEMPERATURE AT ξ = ax/κ = 1.0
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION δ = 1.0
δ = 0.0
δ = 0.36
δ = 1.0
0.6
0.4
0.2
0
0
1.0
2.0
TEMPERATURE - θ = (T - T0)/T0
TIME - τ = a2t/κ

FIGURE 4.



0.2
0.1
0
0.06
0
1.0
2.0
DISPLACEMENT - u = a(λ+2μ) u_x/(βT_0 κ)
TIME - τ = a²t/κ
δ = 0.36
δ = 0.0
δ = 1.0
SECOND DANILOVSKAYA PROBLEM, H = κh/ak = 5.0
DISPLACEMENT AT ξ = ax/κ = 1.0
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION, δ = 1.0

FIGURE 5.



SECOND DANILOVSKAYA PROBLEM, H = κh/ak = 5.0
STRESS AT ξ = ax/κ = 1.0
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION, δ = 1.0
δ = 1.0
δ = 0.0
δ = 0.36
0
-0.6
STRESS - σ = σxx/βT0
0
1.0
2.0
TIME - τ = a²t/κ

FIGURE 6.

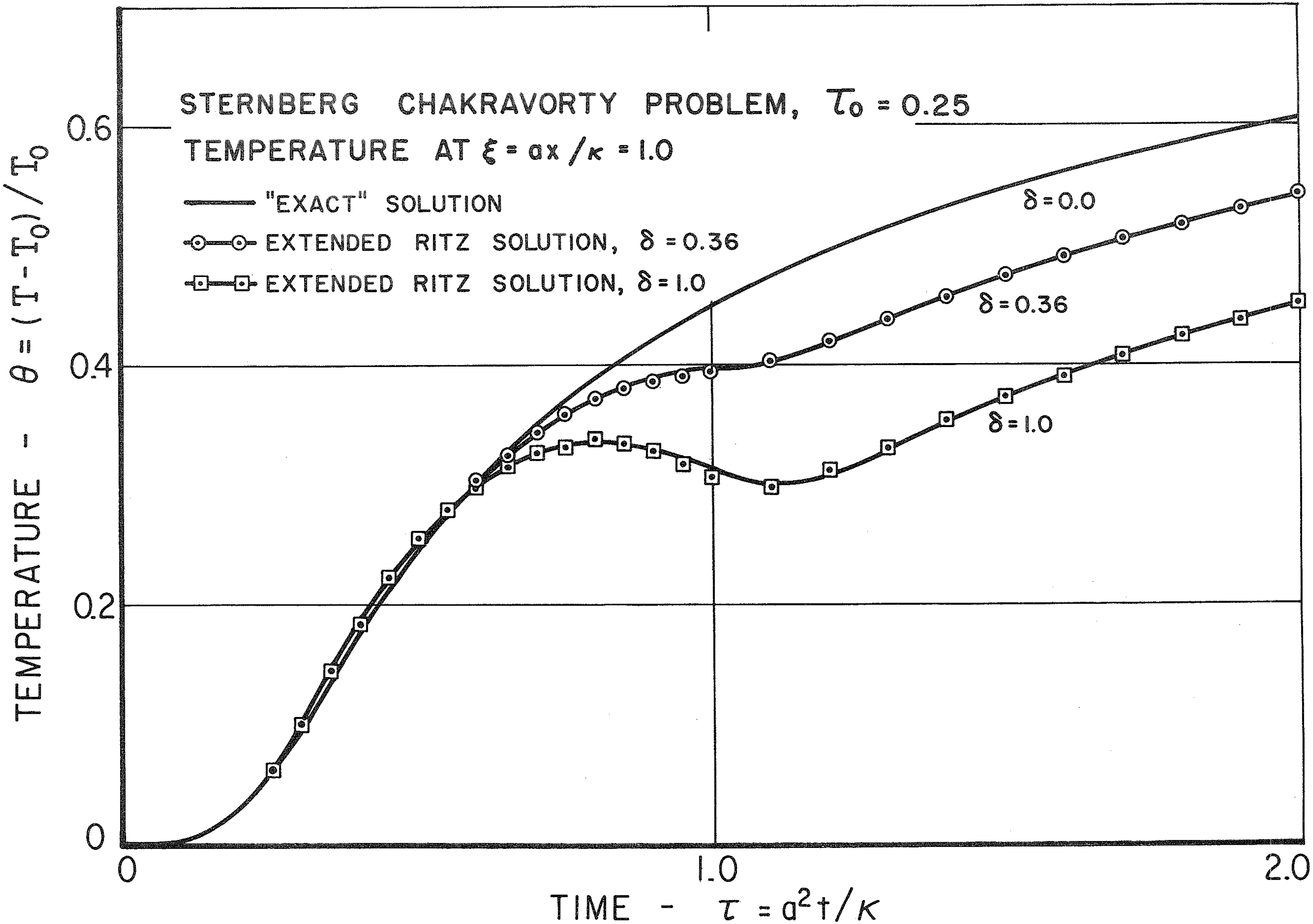
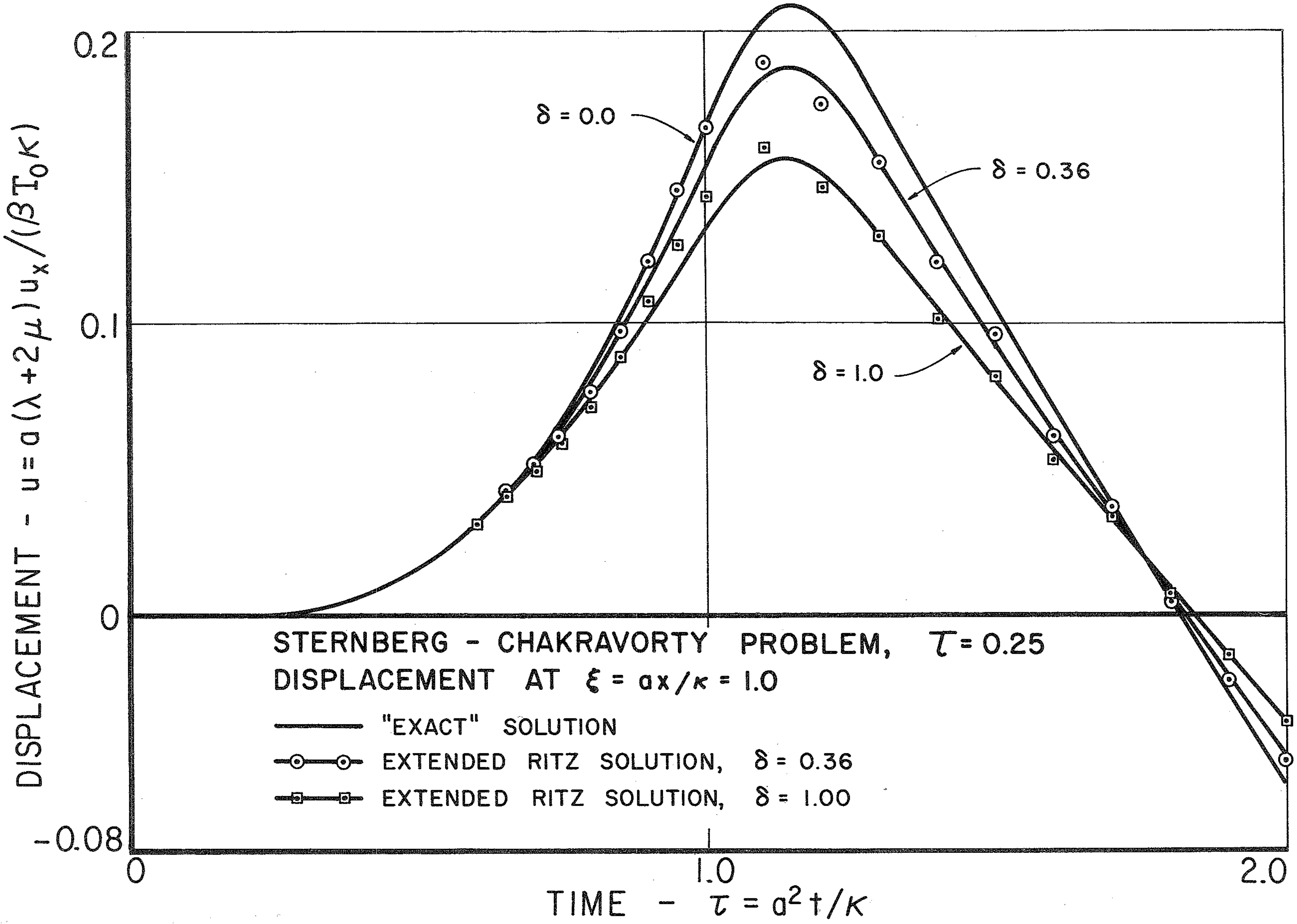
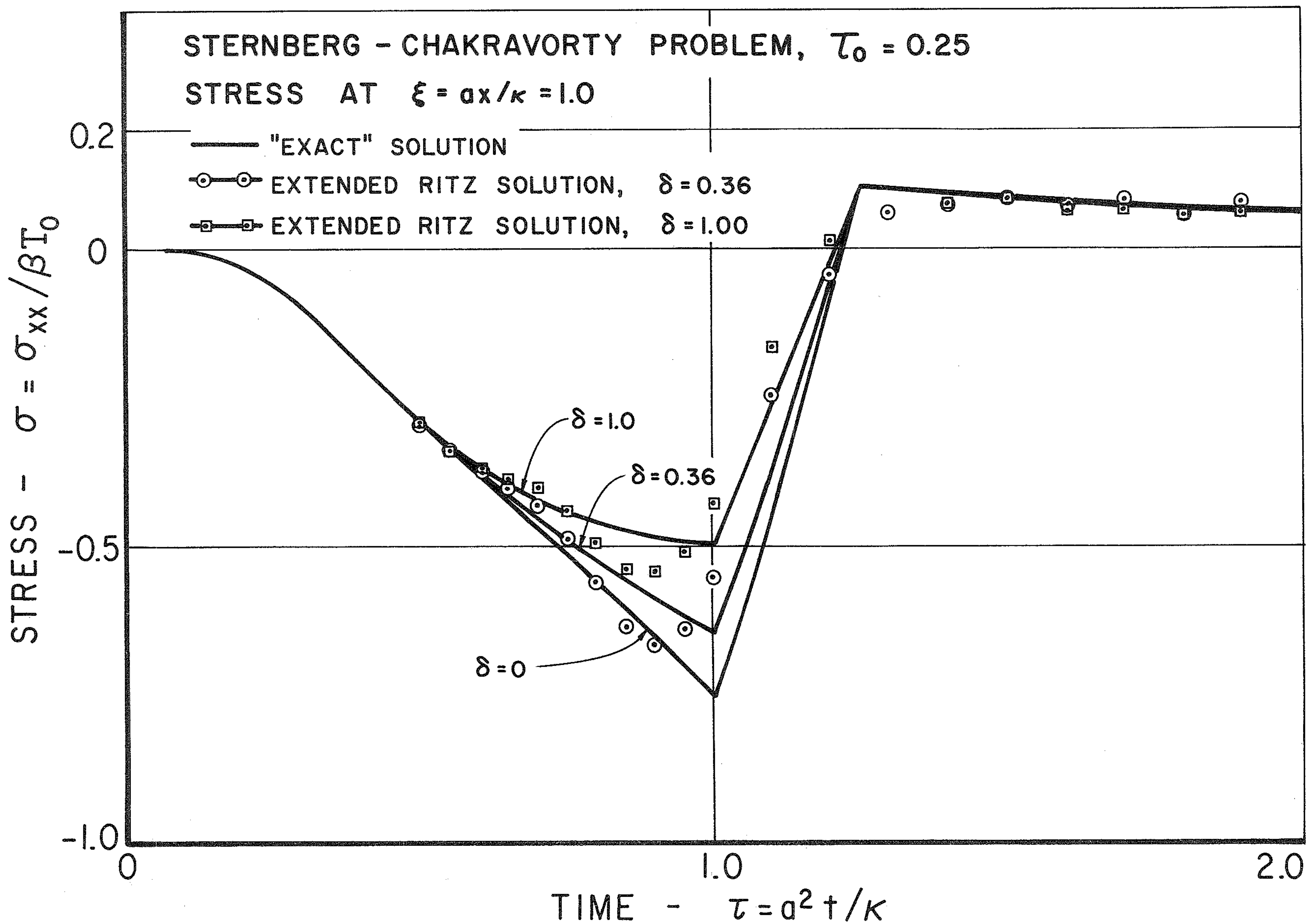

STERNBERG CHAKRAVORTY PROBLEM, $\tau_0 = 0.25$
TEMPERATURE AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.0$
$\delta = 0.0$
$\delta = 0.36$
$\delta = 1.0$
TEMPERATURE - $\theta = (T - T_0)/T_0$
0
0.2
0.4
0.6
0
1.0
2.0
TIME - $\tau = a^2 t/\kappa$

FIGURE 7.



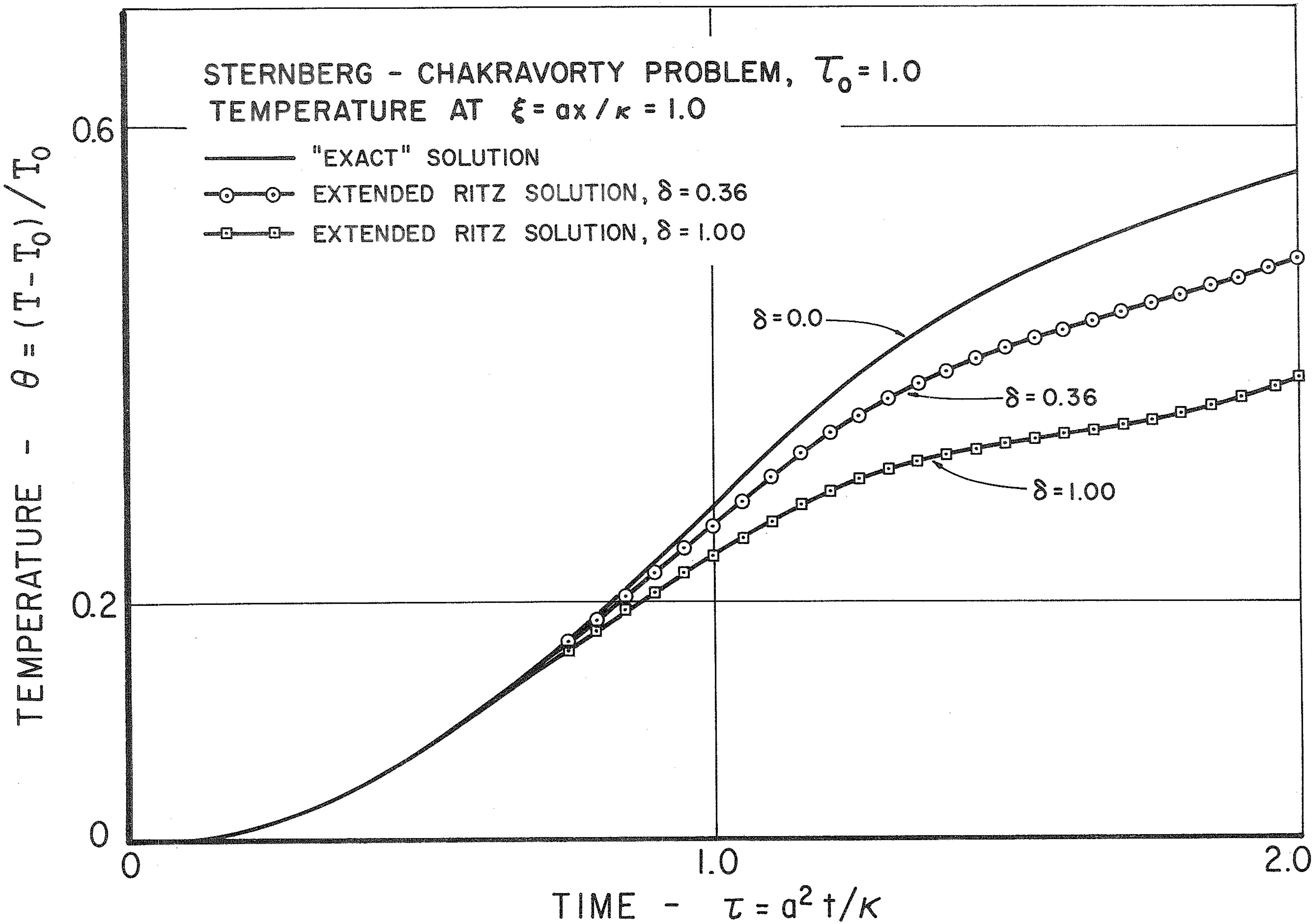
0.2
0.1
0
-0.08
0
1.0
2.0
DISPLACEMENT - u = a(λ + 2μ) u_x/(β T_0 κ)
TIME - τ = a²t/κ
δ = 0.0
δ = 0.36
δ = 1.0
STERNBERG - CHAKRAVORTY PROBLEM, τ = 0.25
DISPLACEMENT AT ξ = ax/κ = 1.0
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION, δ = 1.00

FIGURE 8.



STERNBERG - CHAKRAVORTY PROBLEM, $\tau_0 = 0.25$
STRESS AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.00$
$\delta = 1.0$
$\delta = 0.36$
$\delta = 0$
0.2
0
-0.5
-1.0
0
1.0
2.0
STRESS - $\sigma = \sigma_{xx}/\beta T_0$
TIME - $\tau = a^2 t/\kappa$

FIGURE 9.



STERNBERG - CHAKRAVORTY PROBLEM, $\tau_0 = 1.0$
TEMPERATURE AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.00$
$\delta = 0.0$
$\delta = 0.36$
$\delta = 1.00$
0.6
0.2
0
0
1.0
2.0
TEMPERATURE - $\theta = (T - T_0)/T_0$
TIME - $\tau = a^2 t/\kappa$

FIGURE 10.

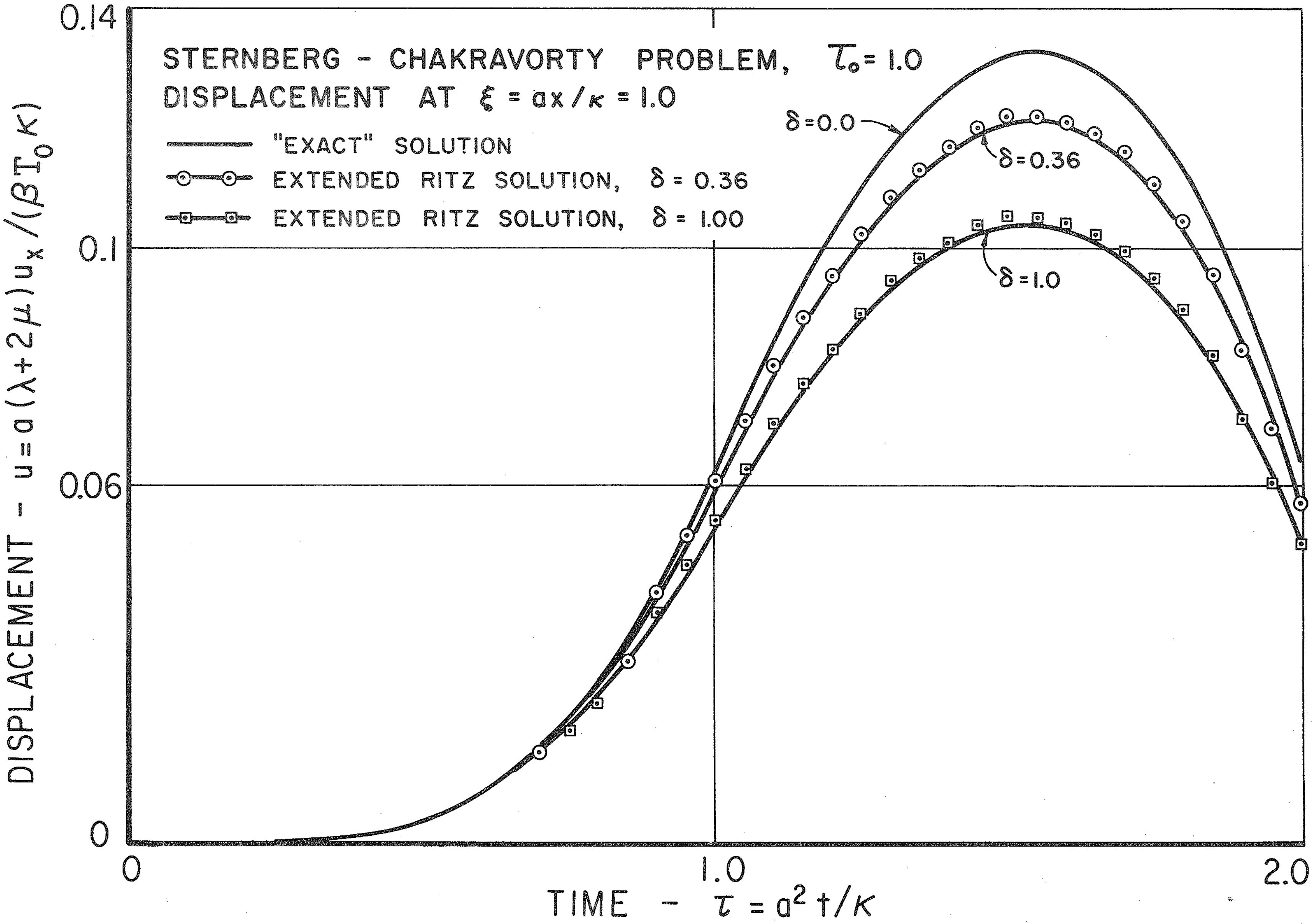

STERNBERG - CHAKRAVORTY PROBLEM, $\tau_o = 1.0$
DISPLACEMENT AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, $\delta = 0.36$
EXTENDED RITZ SOLUTION, $\delta = 1.00$
$\delta = 0.0$
$\delta = 0.36$
$\delta = 1.0$
0.14
0.1
0.06
0
0
1.0
2.0
DISPLACEMENT - $u = a(\lambda + 2\mu) u_x / (\beta T_0 \kappa)$
TIME - $\tau = a^2 t/\kappa$

FIGURE 11.

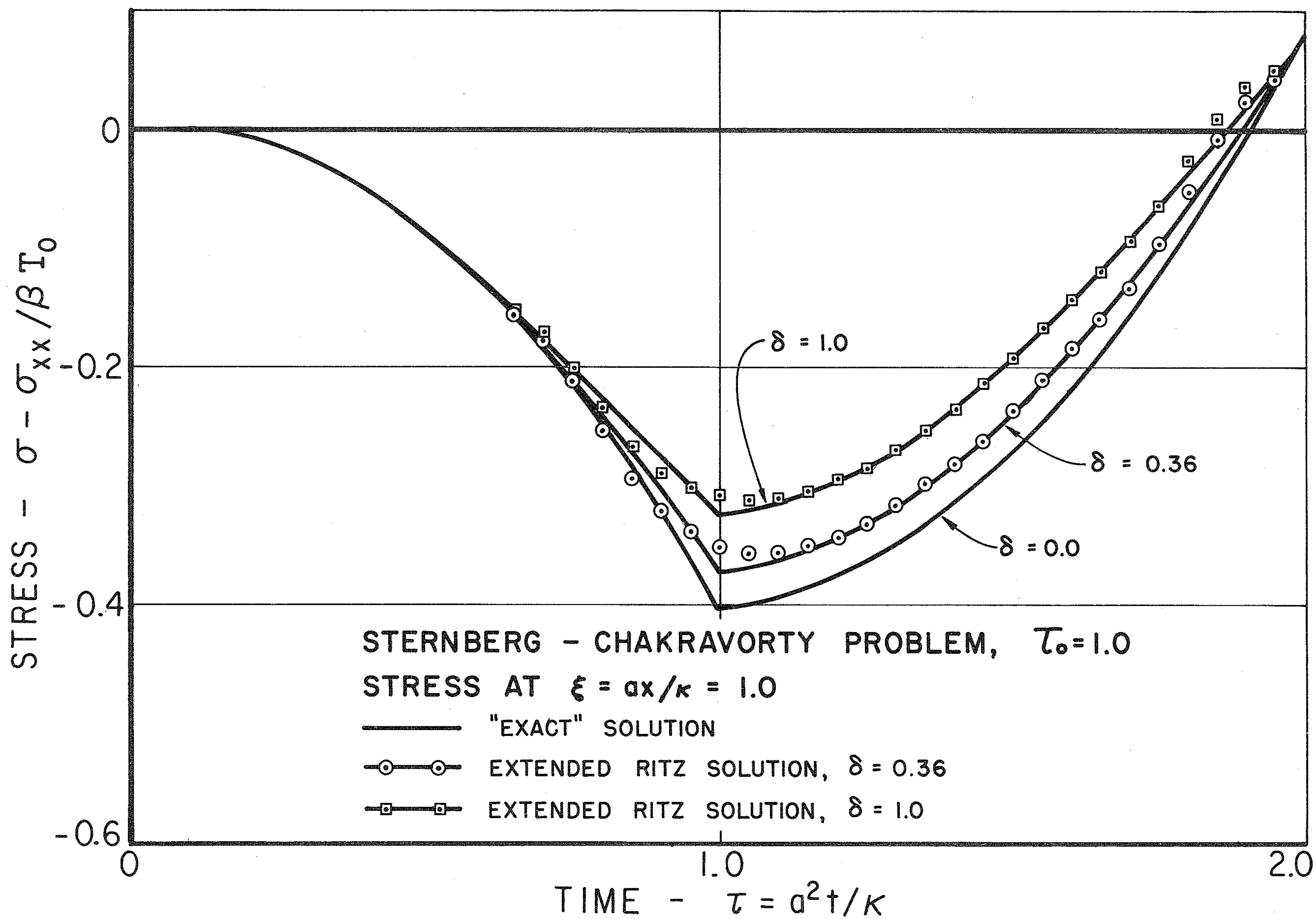

STERNBERG - CHAKRAVORTY PROBLEM, τ_o=1.0
STRESS AT $\xi = ax/\kappa = 1.0$
"EXACT" SOLUTION
EXTENDED RITZ SOLUTION, δ = 0.36
EXTENDED RITZ SOLUTION, δ = 1.0
δ = 1.0
δ = 0.36
δ = 0.0
STRESS - $\sigma - \sigma_{xx}/\beta T_0$
0
-0.2
-0.4
-0.6
0
1.0
2.0
TIME - $\tau = a^2 t/\kappa$

FIGURE 12.

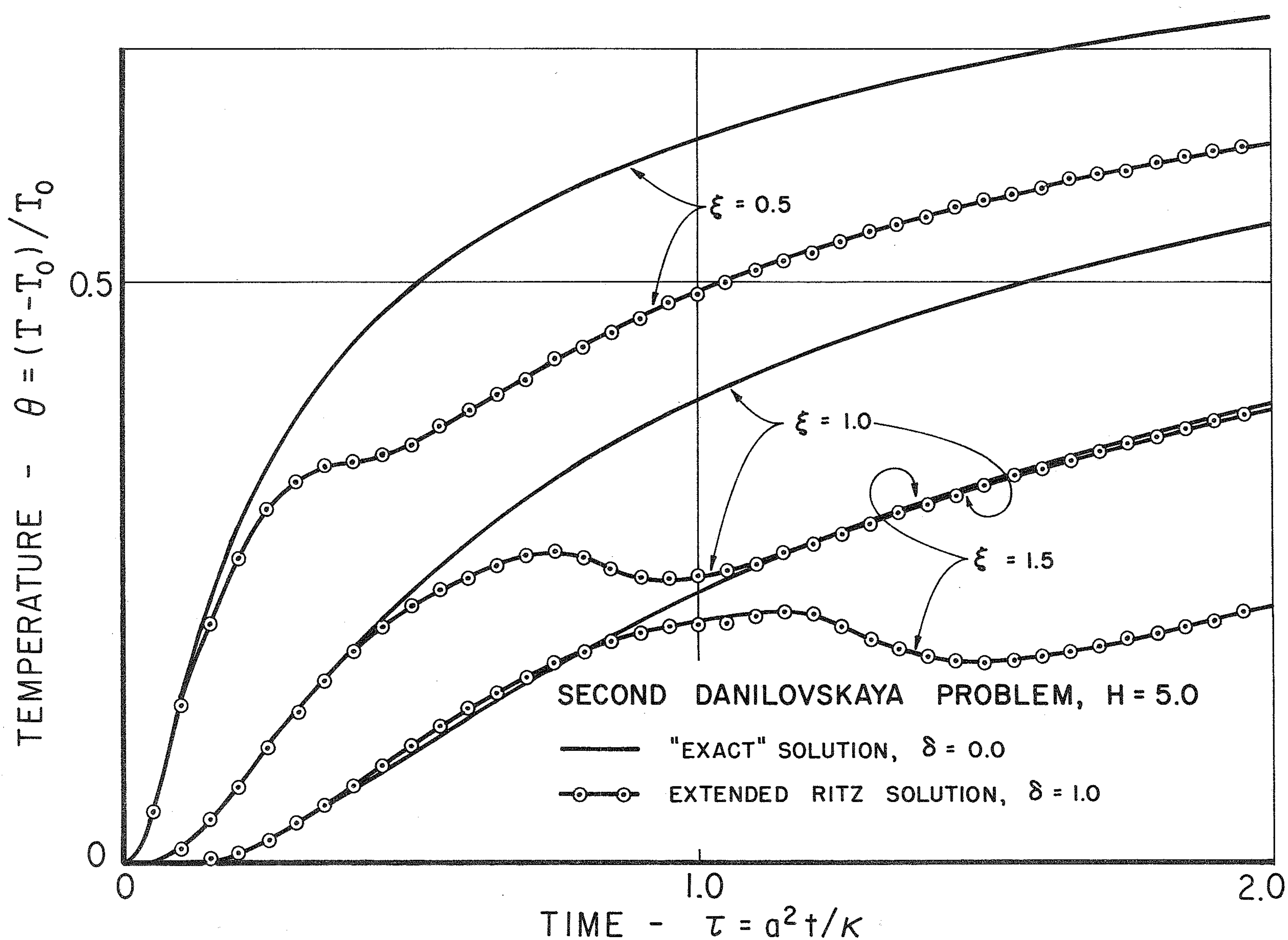
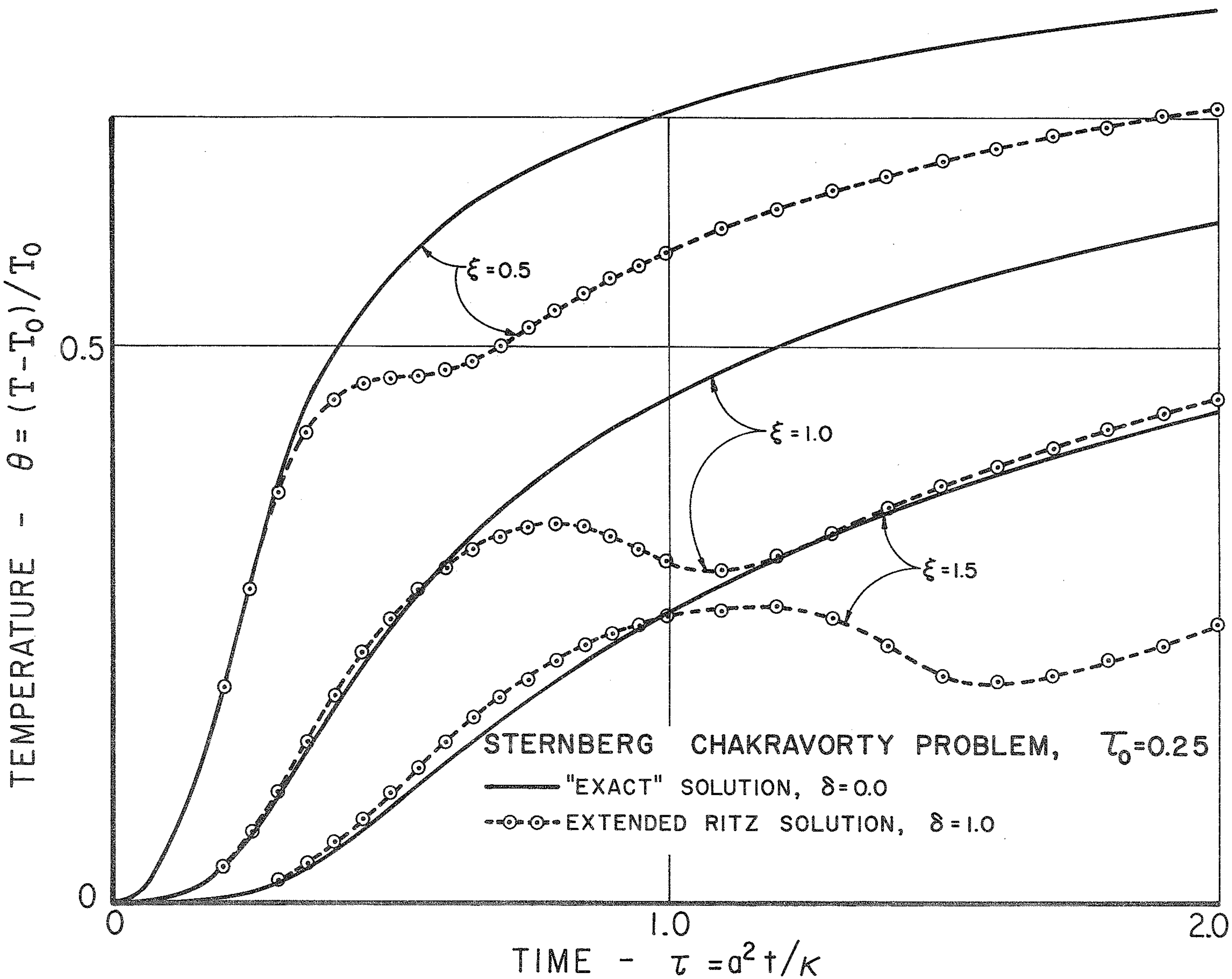

ξ = 0.5
ξ = 1.0
ξ = 1.5
SECOND DANILOVSKAYA PROBLEM, H = 5.0
"EXACT" SOLUTION, δ = 0.0
EXTENDED RITZ SOLUTION, δ = 1.0
0
0.5
0
1.0
2.0
TIME - τ = a²t/κ
TEMPERATURE - θ = (T-T₀)/T₀

FIGURE 13.



ξ = 0.5
ξ = 1.0
ξ = 1.5
STERNBERG CHAKRAVORTY PROBLEM, τ₀=0.25
"EXACT" SOLUTION, δ=0.0
EXTENDED RITZ SOLUTION, δ=1.0
TEMPERATURE - θ = (T-T₀)/T₀
0.5
0
0
1.0
2.0
TIME - τ = a²t/κ

FIGURE 14.

```
$IBFT C  MAIN2   DECK
C
C**** *     FULLY COUPLED THERMOELASTICITY-EXTENDED RITZ SOLUTION-
C**** *     TEMPERATURE APPROXIMATED OVER SPACE AND TIME LINEARLY-
C**** *     DISPLACEMENT APPROXIMATED LINEARLY IN SPACE AND QUADRATICALLY
C**** *     IN TIME-AXISYMMETRIC PLANE STRAIN, RADIAL HEAT FLOW-
C**** *     OR- SEMI INFINITE SOLID, ONE DIMENSIONAL HEAT FLOW.
C**** *     TIME DEPENDENT LOADS, FLUXES, AND BOUNDARY CONDITIONS
C
      COMMON/ELDATA/
     1 EE(8,12),R(150),TO,ANGFQ,KAT
      COMMON/ZYMARG/
     1 NUMK,MBAND,A(300,4),B(300)
      COMMON/MODQ/
     1 NUMNP,K1(150,2),K2(150,2),M1(150,2),M2(150,2),C(150,3),F(150),
     2 Q(150)
      COMMON/STRESQ/
     1 NUMEL,IX(150,3),T(150),UR(150)
      COMMON/FORMST/
     1 ST(2,2),TC(2,2),DA(2,2),HC(2,2),XM(2,2),PF(2),PQ(2)
      DIMENSION
     1 HED(12),KODE(150),UDOT(150),URR(150),TT(150)
      REAL K1,K2,M1,M2
      LOGICAL BCALTR
      LBAND=2
      NBAND=3
      MBAND=4
      REWIND 8
      REWIND 9
    5 CONTINUE
C
C**** *     READ AND PRINT OF CONTROL INFORMATION
C
      READ (5,1000) HED,NUMNP,NUMEL,NUMMAT,NDT,DT,ANGFQ,TO,INTER,N1,KAT
      IF (KAT.EQ.0) GO TO 30
      WRITE (6, 2050)
      GO TO 35
   30 WRITE (6, 2055)
   35 CONTINUE
      WRITE (6,2000) HED,NUMNP,NUMEL,NUMMAT,NDT,DT,ANGFQ,TO,INTER
      DO 25 M=1, NUMMAT
      READ (5,1005) (EE(I,M), I=1,7)
   25 WRITE (6,2005) M, (EE(I,M), I=1,7)
C
C**** *  READ OR GENERATE NODAL POINT INFORMATION
C
      IF (KAT.EQ.0) WRITE (6,2010)
      IF (KAT.NE.0) WRITE (6,2011)
      L=0
   50 READ (5,1015) N,KODE(N),R(N),UR(N),UDOT(N),T(N)
      NNL=L+1
      DIFF=N-L
      DR=(R(N)-R(L))/DIFF
   75 L=L+1
```

```
      IF (N-L) 55,60,65
   65 KODE(L)=0
      IF (KODE(NNL-1).EQ.KODE(N)) KODE(L)=KODE(N)
      R(L)=R(L-1)+DR
      UR(L)=0.0
      T(L)=0.0
      UDOT(L)=0.0
      GO TO 75
   60 WRITE (6,2015) (K,KODE(K),R(K),UR(K),UDOT(K),T(K), K=NNL,N)
      IF (NUMNP-N) 55,100,50
   55 WRITE (6,2100) N
      CALL EXIT
  100 CONTINUE
C
C**** *     READ AND PRINT OF ELEMENT PROPERTIES
C
      WRITE (6,2025)
      N=0
  225 READ (5,1020) M, (IX(M,I), I=1,3)
  205 N=N+1
      IF (M-N) 210,215,220
  220 IX(N,1)=IX(N-1,1)+1
      IX(N,2)=IX(N-1,2)+1
      IX(N,3)=IX(N-1,3)
  215 WRITE (6,2020) N, (IX(N,I), I=1,3)
      IF (M-N) 210,230,205
  230 IF (NUMEL-N) 200,200,225
  210 WRITE (6,2030) N
      CALL EXIT
  200 CONTINUE
      NUMK=2*NUMNP
C
C**** *     INITIALIZE
C
      DO 240 N=1, NUMNP
      F(N)=0.0
      Q(N)=0.0
      C(N,3)=0.0
      DO 240 M=1, 2
      K1(N,M)=0.0
      K2(N,M)=0.0
      M1(N,M)=0.0
      M2(N,M)=0.0
  240 C(N,M)=0.0
C
C**** *     FORM TOTAL MATRIX QUANTITIES
C
      DO 275 N=1,NUMEL
      II=IX(N,1)
      JJ=IX(N,2)
      KK=IX(N,3)
      CALL FORM (II,JJ,KK)
      F(II)=F(II)+PF(1)
      F(JJ)=F(JJ)+PF(2)
      Q(II)=Q(II)+PQ(1)
```

```
      Q(JJ)=Q(JJ)+PQ(2)
      DO 250 I=1,2
      K1(II,I)=K1(II,I)+ST(1,I)
      K2(II,I)=K2(II,I)+TC(1,I)
      M1(II,I)=M1(II,I)+XM(1,I)
      M2(II,I)=M2(II,I)+HC(1,I)
      C(II,I+1)=C(II,I+1)+DA(1,I)
  250 C(JJ,I)=C(JJ,I)+DA(2,I)
      K1(JJ,1)=K1(JJ,1)+ST(2,2)
      K2(JJ,1)=K2(JJ,1)+TC(2,2)
      M1(JJ,1)=M1(JJ,1)+XM(2,2)
      M2(JJ,1)=M2(JJ,1)+HC(2,2)
  275 CONTINUE
C
C**** *     WRITE TIME INDEPENDENT INFORMATION ON TAPE
C
      WRITE (8) ((K1(I,J), K2(I,J), M1(I,J), M2(I,J), J=1, LBAND),
     1 (C(I,J), J=1, NBAND), UR(I), T(I), F(I), Q(I), I=1, NUMNP)
      REWIND 8
      TIME=0.0
      LL=0
      READ (5,1010) BCALTR
C
C**** *     BEGIN TRANSIENT CALCULATIONS
C
      DO 600 LNDT=1, NDT
      IF (TIME.EQ.0.0) GO TO 605
      IF (.NOT.BCALTR) GO TO 610
      READ (5,1025) DT
      READ  (8) ((K1(I,J), K2(I,J), M1(I,J), M2(I,J), J=1, LBAND),
     1 (C(I,J), J=1, NBAND), UR(I), T(I), F(I), Q(I), I=1, NUMNP)
      REWIND 8
  605 DT2=0.5*DT
      DT3=12.0/(DT*DT)
C
C**** *     MODIFY FOR DISPLACEMENT AND TEMPERATURE BOUNDARY CONDITIONS-
C**** *     ADD CONCENTRATED FORCES AND HEAT FLUXES
C
      DO 300 N=1,NUMNP
      IF (.NOT.BCALTR) GO TO 325
      IF (TIME.EQ.0.0) GO TO 345
      IF (N.EQ.1) READ (5,1030) NN,DF,TQ
  335 IF (N-NN) 350,320,330
  330 READ (5,1030) NN,DF,TQ
      GO TO 335
  320 UR(N)=DF
      T(N)=TQ
      GO TO 350
  325 URR(N)=UR(N)
      TT(N)=T(N)
      GO TO 350
  345 URR(N)=0.0
      TT(N)=0.0
      IF (KODE(N).EQ.2.OR.KODE(N).EQ.3) TT(N)=TO
  350 F(N)=F(N)+UR(N)/3.0+2.0*URR(N)/3.0
```

```
      Q(N)=Q(N)+0.5*(T(N)+TT(N))
      IF (KODE(N).EQ.0) GO TO 340
      IF (KODE(N).NE.1.AND.KODE(N).NE.3) GO TO 295
      U=UR(N)/3.0+2.0*URR(N)/3.0
      V=0.5*(UR(N)-URR(N))
      K=N-1
      IF (K.LE.0) GO TO 285
      F(K)=F(K)-K1(K,2)*U
      Q(K)=Q(K)-C(N,1)*V
      K1(K,2)=0.0
      M1(K,2)=0.0
      C(N,1)=0.0
  285 K=N+1
      IF (K.GT.NUMNP) GO TO 290
      F(K)=F(K)-K1(N,2)*U
      Q(K)=Q(K)-C(N,3)*V
      K1(N,2)=0.0
      M1(N,2)=0.0
      C(N,3)=0.0
  290 F(N)=UR(N)
      Q(N)=Q(N)-C(N,2)*V
      K1(N,1)=1.0
      M1(N,1)=0.0
      C(N,2)=0.0
  295 IF (KODE(N).NE.2.AND.KODE(N).NE.3) GO TO 340
      U=0.5*(T(N)+TT(N)-2.0*TO)
      V=0.5*(T(N)-TT(N))
      W=(T(N)-TO+2.0*(TT(N)-TO))/3.0
      K=N-1
      IF (K.LE.0) GO TO 310
      F(K)=F(K)+C(K,3)*W
      Q(K)=Q(K)-K2(K,2)*U-M2(K,2)*V
      K2(K,2)=0.0
      M2(K,2)=0.0
      C(K,3)=0.0
  310 K=N+1
      IF (K.GT.NUMNP) GO TO 315
      F(K)=F(K)+C(K,1)*W
      Q(K)=Q(K)-K2(N,2)*U-M2(N,2)*V
      K2(N,2)=0.0
      M2(N,2)=0.0
      C(K,1)=0.0
  315 Q(N)=T(N)-TO
      F(N)=F(N)+C(N,2)*W
      K2(N,1)=1.0
      M2(N,1)=0.0
      C(N,2)=0.0
  340 URR(N)=UR(N)
      TT(N)=T(N)
      IF (TIME.EQ.0.0) T(N)=TO
      IF (TIME.EQ.0.0) UR(N)=0.0
  300 CONTINUE
C
C**** *     FORM EFFECTIVE LEFT HAND SIDE
C
```

```
      DO 400 N=1, NUMNP
      II=2*N-1
      JJ=II+1
      A(II,1)=K1(N,1)+DT3*M1(N,1)
      A(II,2)=-2.0*C(N,2)/3.0
      A(II,3)=K1(N,2)+DT3*M1(N,2)
      A(II,4)=-2.0*C(N,3)/3.0
      A(JJ,1)=-(M2(N,1)+DT2*K2(N,1))*2.0/9.0
      A(JJ,2)=-2.0*C(N+1,1)/3.0
      A(JJ,3)=-(M2(N,2)+DT2*K2(N,2))*2.0/9.0
  400 A(JJ,4)=0.0
C
C**** *     PUT A-MATRIX IN TRIANGULAR FORM
C
      CALL SYMSOL (1)
C
C**** *     CALCULATE UNKNOWNS AT THE END OF EACH TIME STEP
C
  610 TIME=TIMF+DT
      LL=LL+1
C
C**** *     CALCULATE EFFECTIVE LOAD MATRIX FOR THE TIME INCREMENT
C
      IF (TIME.EQ.DT)  GO TO 550
      IF (.NOT.BCALTR) GO TO 550
      READ  (9)  (UR(I),T(I), I=1, NUMNP)
      REWIND 9
  550 DO 500 N=1, NUMNP
      II=2*N-1
      JJ=II+1
      B(II)=F(N)+M1(N,1)*(DT3*UR(N)+4.0/DT*UDOT(N))+C(N,2)*(T(N)-TO)/3.0
      B(JJ)=-DT*Q(N)/9.0-C(N,2)*(2.0*UR(N)/3.0+DT*UDOT(N)/9.0)-M2(N,1)*
     1 2.0/9.0*(T(N)-TO)
      IF (N.NE.1)
     1 B(II)=B(II)+M1(N-1,2)*(DT3*UR(N-1)+4.0/DT*UDOT(N-1))+C(N,1)*(T(N-1
     2 )-TO)/3.0
      IF (N.NE.NUMNP)
     1 B(II)=B(II)+M1(N,2)*(DT3*UR(N+1)+4.0/DT*UDOT(N+1))+C(N,3)*(T(N+1)-
     2 TO)/3.0
      IF (N.NE.1)
     1 B(JJ)=B(JJ)-C(N-1,3)*(2.0*UR(N-1)/3.0+DT/9.0*UDOT(N-1))-M2(N-1,2)*
     2 (T(N-1)-TO)*2.0/9.0
      IF (N.NE.NUMNP)
     1 B(JJ)=B(JJ)-C(N+1,1)*(2.0*UR(N+1)/3.0+DT*UDOT(N+1)/9.0)-M2(N,2)*(
     2 T(N+1)-TO)*2.0/9.0
  500 CONTINUE
C
C**** *     SOLVE FOR UNKNOWNS
C
      CALL SYMSOL (2)
      DO 650 N=1, NUMNP
      II=2*N-1
      JJ=II+1
      TEMP=6.0*B(II)-5.0*UR(N)-DT*UDOT(N)
      IF (KODE(N).EQ.1.OR.KODE(N).EQ.3)
```

```
     1 TEMP=B(II)
       T(N)=2.0*B(JJ)-T(N)+2.0*TO
       IF (KODE(N).EQ.2.OR.KODE(N).EQ.3)
     1 T(N)=TO+B(JJ)
       UDOT(N)=2.0/DT*(TEMP-UR(N))-UDOT(N)
  650  UR(N)=TEMP
       IF (.NOT.BCALTR) GO TO 675
       WRITE (9) (UR(I),T(I), I=1, NUMNP)
       REWIND 9
C
C**** *     PRINT DISPLACEMENTS AND TEMPERATURES
C
  675  IF (LL.LT.INTER) GO TO 600
       IF (KAT.EQ.0) WRITE (6,2040) TIME
       IF (KAT.NE.0) WRITE (6,2041) TIME
       LL=0
       DO 725 N=1, NUMNP
  725  WRITE (6,2035) N,R(N),UR(N),T(N),UDOT(N)
       CALL STRESS
  600  CONTINUE
       WRITE (6,2045)
       GO TO 5
 1000  FORMAT (12A6/4I5,3F10.0,3I5)
 1005  FORMAT (7F10.0)
 1010  FORMAT (L5)
 1015  FORMAT (2I5,4F10.0)
 1020  FORMAT (4I5)
 1025  FORMAT (F10.0)
 1030  FORMAT (I10,2F10.0)
 2000  FORMAT (1H0  12A6/
     1    30H0 NUMBER OF NODAL POINTS------ I3/
     2    30H0 NUMBER OF ELEMENTS---------- I3/
     3    30H0 NUMBER OF DIFF. MATERIALS--- I3/
     4    30H0 NUMBER OF TIME STEPS-------- I3/
     5    30H0 TIME STEP INCREMENT--------- E12.4/
     6    30H0 ANGULAR VELOCITY------------ E12.4/
     7    30H0 REFERENCE TEMPERATURE------- F12.2/
     8    30H0 CYCLE PRINT INTERVAL-------- I3//)
 2005  FORMAT (16H0MATERIAL NUMBER15H    LAMBDA      15H       MU        15H
     1      ALPHA      15H  CONDUCTIVITY 15H SPECIFIC HEAT 15H    DENSITY
     2    15HHEAT GENERATION/(1I16,7E15.5))
 2010  FORMAT (102H1NODAL POINT  B.C. TYPE   R-ORDINATE R-LOAD OR DISPLAC
     1 EMENT       VELOCITY      TEMPERATURE OR HEAT FLUX//)
 2011  FORMAT (102H1NODAL POINT  B.C. TYPE   X-ORDINATE X-LOAD OR DISPLAC
     1 EMENT       VELOCITY      TEMPERATURE OR HEAT FLUX//)
 2015  FORMAT (2I12,1F12.2,1E24.7,1E18.7,1E24.7)
 2020  FORMAT (1I13, 2I6, 1I13)
 2025  FORMAT (40H1     ELEMENT      I      J      MATERIAL   )
 2030  FORMAT (22H0ELEMENT CARD ERROR N=I5)
 2035  FORMAT (I5,1F13.2,3E17.5)
 2040  FORMAT (16H1            TIME=E12.5//69H    NP    R-ORDINATE    R-DISPLA
     1 CEMENT       TEMPERATURE          VELOCITY//)
 2041  FORMAT (16H1            TIME=E12.5//69H    NP    X-ORDINATE    X-DISPLA
     1 CEMENT       TEMPERATURE          VELOCITY//)
 2045  FORMAT (15H1END OF PROBLEM)
```

```
2050   FORMAT (27H1ONE DIMENSIONAL PLANE BODY )
2055   FORMAT (24H1AXISYMMETRIC SOLID BODY )
2100   FORMAT (26H0NODAL POINT CARD ERROR N=I5)
       END
```

```
$IBFT C  ODEP     DECK,LIST,REF
C
C**** *      ONE-DIMENSIONAL ELEMENT SUBROUTINE
C
      SUBROUTINE FORM (II,JJ,KK)
      COMMON/ELDATA/
     1 EE(8,12),R(150),TO,ANGFQ,KAT
      COMMON/FORMST/
     1 ST(2,2),TC(2,2),DA(2,2),HC(2,2),XM(2,2),PF(2),PQ(2)
      RI=R(II)
      RJ=R(JJ)
      RIJ=RJ-RI
      XLAM=EE(1,KK)
      XMU=EE(2,KK)
      BETA=EE(3,KK)*(3.0*XLAM+2.0*XMU)
      CON=EE(4,KK)/TO
      DEN=EE(6,KK)
      SPHT=EE(5,KK)/TO
      QX=EE(7,KK)/TO
      FANG=EE(6,KK)*ANGFQ
      IF (KAT.EQ.0) FANG=FANG*ANGFQ
      DO 50 I=1,2
      PF(I)=0.0
      PQ(I)=0.0
      DO 50 J=1,2
      ST(I,J)=0.0
      TC(I,J)=0.0
      DA(I,J)=0.0
      HC(I,J)=0.0
   50 XM(I,J)=0.0
      IF (KAT.EQ.0) GO TO 100
      ST(1,1)=(XLAM+2.0*XMU)/RIJ
      ST(1,2)=-ST(1,1)
      ST(2,1)=ST(1,2)
      ST(2,2)=ST(1,1)
      TC(1,1)=CON/RIJ
      TC(1,2)=-TC(1,1)
      TC(2,1)=TC(1,2)
      TC(2,2)=TC(1,1)
      DA(1,1)=-0.5*BETA
      DA(1,2)=DA(1,1)
      DA(2,1)=0.5*BETA
      DA(2,2)=DA(2,1)
      HC(1,1)=SPHT*RIJ/3.0
      HC(1,2)=0.5*HC(1,1)
      HC(2,1)=HC(1,2)
      HC(2,2)=HC(1,1)
      XM(1,1)=DEN*RIJ/3.0
      XM(1,2)=0.5*XM(1,1)
      XM(2,1)=XM(1,2)
      XM(2,2)=XM(1,1)
      PF(1)=0.5*FANG*RIJ
      PF(2)=PF(1)
      PQ(1)=0.5*QX*RIJ
```

```
      PQ(2)=PQ(1)
      GO TO 200
100   XI=0.0
      IF (RI.NE.0.0) XI=ALOG(RJ/RI)
      XJ=2.0*(XLAM+XMU)
      XK=XLAM+2.0*XMU
      XK=XI*XK/(RIJ*RIJ)
      ST(1,1)=RJ*RJ*XK-XJ
      ST(1,2)=-RI*RJ*XK
      ST(2,1)=ST(1,2)
      ST(2,2)=RI*RI*XK+XJ
      TC(1,1)=0.5*CON*(RI+RJ)/RIJ
      TC(1,2)=-TC(1,1)
      TC(2,1)=TC(1,2)
      TC(2,2)=TC(1,1)
      DA(1,1)=BETA*(RJ-4.0*RI)/6.0
      DA(1,2)=-BETA*(RJ+2.0*RI)/6.0
      DA(2,1)=BETA*(2.0*RJ+RI)/6.0
      DA(2,2)=BETA*(4.0*RJ-RI)/6.0
      HC(1,1)=SPHT*RIJ*(RJ+3.0*RI)/12.0
      HC(1,2)=SPHT*RIJ*(RJ+RI)/12.0
      HC(2,1)=HC(1,2)
      HC(2,2)=SPHT*RIJ*(3.0*RJ+RI)/12.0
      XM(1,1)=DEN*RIJ*(RJ+3.0*RI)/12.0
      XM(1,2)=DEN*RIJ*(RI+RJ)/12.0
      XM(2,1)=XM(1,2)
      XM(2,2)=DEN*RIJ*(3.0*RJ+RI)/12.0
      PF(1)=FANG*RIJ*(RJ+2.0*RI)/6.0
      PF(2)=FANG*RIJ*(2.0*RJ+RI)/6.0
      PQ(1)=QX*RIJ*(RJ+2.0*RI)/6.0
      PQ(2)=QX*RIJ*(2.0*RJ+RI)/6.0
200   RETURN
      END
```

```
$IBFT C  SYMS    DECK,LIST,REF
        SUBROUTINE SYMSOL (KKK)
        COMMON/SYMARG/ NN,MM,A(300,4),B(300)
        GO TO (1000,2000), KKK
C
C**** *      REDUCE MATRIX
C
 1000   DO 280 N=1,NN
        DO 260 L=2,MM
        C=A(N,L)/A(N,1)
        I=N+L-1
        IF (NN.LT.I) GO TO 260
        J=0
        DO 250 K=L,MM
        J=J+1
  250   A(I,J)=A(I,J)-C*A(N,K)
  260   A(N,L)=C
  280   CONTINUE
        GO TO 500
C
C**** *      REDUCE VECTOR
C
 2000   DO 290 N=1,NN
        DO 285 L=2,MM
        I=N+L-1
        IF (NN.LT.I) GO TO 290
  285   B(I)=B(I)-A(N,L)*B(N)
  290   B(N)=B(N)/A(N,1)
C
C**** *      BACK SUBSTITUTION
C
        N=NN
  300   N=N-1
        IF (N.EQ.0) GO TO 500
        DO 400 K=2,MM
        L=N+K-1
        IF (NN.LT.L) GO TO 400
        B(N)=B(N)-A(N,K)*B(L)
  400   CONTINUE
        GO TO 300
C
C**** *      RETURN
C
  500   RETURN
        END
```

```
$IBFT C  STRS    DECK,LIST,REF
C
C**** *      SUBROUTINE TO CALCULATE THE STRESSES
C
      SUBROUTINE STRESS
      COMMON/ELDATA/
     1 EE(8,12),R(150),TO,ANGFQ,KAT
      COMMON/STRESQ/
     1 NUMEL,IX(150,3),T(150),UR(150)
      REAL LAM,MU
      IF (KAT.EQ.0) WRITE (6,2000)
      IF (KAT.NE.0) WRITE (6,2001)
      DO 100 N=1, NUMEL
      I=IX(N,1)
      J=IX(N,2)
      MTYPE=IX(N,3)
      LAM=EE(1,MTYPE)
      MU=EE(2,MTYPE)
      AL=EE(3,MTYPE)
      AL=AL*(3.0*LAM+2.0*MU)
      DEL=R(J)-R(I)
      RM=0.5*(R(J)+R(I))
      ERR=(UR(J)-UR(I))/DEL
      ETT=0.0
      IF (KAT.EQ.0)
     1 ETT=ERR-(R(I)/RM*UR(J)-R(J)/RM*UR(I))
      TM=T(J)/DEL*(RM-R(I))+T(I)/DEL*(R(J)-RM)
      TRR=LAM*(ERR+ETT)+2.0*MU*ERR-AL*(TM-TO)
      TTT=LAM*(ERR+ETT)+2.0*MU*ETT-AL*(TM-TO)
      TZZ=LAM*(ERR+ETT)-AL*(TM-TO)
      SIG=0.0
      ERG=0.0
      IF (AL.NE.0.0)
     1 SIG=TRR/(AL*TO)
      IF (AL.NE.0.0)
     1 ERG=ERR*(LAM+2.0*MU)/(AL*TO)
  100 WRITE (6,2005) N,RM,TRR,TTT,TZZ,SIG,ERG
      RETURN
 2000 FORMAT (58H1 ELEMENT   R-ORDINATE    R-STRESS     T-STRESS     Z-STR
     1 ESS//)
 2001 FORMAT (58H1 ELEMENT   X-ORDINATE    X-STRESS     Y-STRESS     Z-STR
     1 ESS//)
 2005 FORMAT (I10,0P1F14.2,1P3E12.3,0P2E15.4)
      END
```

```
$IBFT C  INLT    DECK
C
C**** *    THIS CODE COMPUTES THE INVERSE LAPLACE TRANSFORM NUMERICALLY
C**** *    THROUGH THE USE OF GAUSSIAN QUADRATURE FORMULAE.
C
      DIMENSION
     1 HED(12),XI(8),WI(8),A(3)
      DATA PI/3.1415926/
      DATA XI/-0.96028986,-0.79666648,-0.52553241,-0.18343464,+0.1834346
     1 4,+0.52553241,+0.79666648,+0.96028986/,WI/+0.10122854,+0.22238103,
     2 0.31370665,0.36268378,0.36268378,0.31370665,0.22238103,0.10122854/
      MCONT=8
      ISTOP=0
C**** *     NCONT=NUMBER OF INTEGRATION CYCLES TO BE CONSIDERED
C**** *     MCONT=ORDER OF THE GAUSSIAN QUADRATURE FOR EACH HALF WAVE
C**** *     GAM=REAL PART OF THE INTEGRATION VARIABLE
C**** *     DELTA=ELASTIC,THERMOMECHANICAL COUPLING CONSTANT
C**** *     X=SPACE POINT OF INTEREST
C**** *     T=TIME POINT OF INTEREST
C
C**** *    READ AND PRINT OF INPUT DATA
C
    5 READ (5,1000) HED,NCONT,NT,GAM,X,DELTA,ERROR
      WRITE (6,2000) HED,NCONT,MCONT,NT,GAM,DELTA,ERROR
      DO 500 I=1,NT
      READ (5,1100) T
      SUML=0.0
      SUMM=0.0
      SUMN=0.0
      DO 100 N=1,NCONT,2
      XL=2*N-3
      XM=2*N-1
      XN=2*N+1
      IF(N.EQ.1) XL=0.0
      A1=XL*PI/(2.0*T)
      A2=XM*PI/(2.0*T)
      A3=XN*PI/(2.0*T)
      SUML1=0.0
      SUML2=0.0
      SUMM1=0.0
      SUMM2=0.0
      SUMN1=0.0
      SUMN2=0.0
      DO 200 M=1, MCONT
      Y1=0.5*(A1+A2+XI(M)*(A2-A1))
      Y2=0.5*(A2+A3+XI(M)*(A3-A2))
      CALL RFN (X,Y1,GAM,DELTA,A)
      SUML1=SUML1+WI(M)*A(1)*COS(Y1*T)*(A2-A1)/2.0
      SUMM1=SUMM1+WI(M)*A(2)*COS(Y1*T)*(A2-A1)/2.0
      SUMN1=SUMN1+WI(M)*A(3)*COS(Y1*T)*(A2-A1)/2.0
      CALL RFN (X,Y2,GAM,DELTA,A)
      SUML2=SUML2+WI(M)*A(1)*COS(Y2*T)*(A3-A2)/2.0
      SUMM2=SUMM2+WI(M)*A(2)*COS(Y2*T)*(A3-A2)/2.0
  200 SUMN2=SUMN2+WI(M)*A(3)*COS(Y2*T)*(A3-A2)/2.0
      ADDL=SUML1+SUML2
```

```
      ADDM=SUMM1+SUMM2
      ADDN=SUMN1+SUMN2
      ADD1=ABS(ADDL)
      ADD2=ABS(ADDM)
      ADD3=ABS(ADDN)
      ADD=ADD1+ADD2+ADD3
      IF (ADD.LE.ERROR) GO TO 300
      SUML=SUML+2.0*EXP(GAM*T)/PI*ADDL
      SUMM=SUMM+2.0*EXP(GAM*T)/PI*ADDM
  100 SUMN=SUMN+2.0*EXP(GAM*T)/PI*ADDN
      ISTOP=1
      WRITE (6,2300) ISTOP
  300 WRITE (6.2100) X,T,SUML,SUMM,SUMN
  500 CONTINUE
      WRITE (6,2200)
      GO TO 5
 1000 FORMAT (12A6/2I5,4F10.0)
 1100 FORMAT (F10.0)
 2000 FORMAT (1H1   12A6/
     1   40HONUMBER OF INTEGRATION CYCLES-----------=I10/
     2   40HOORDER OF GAUSSIAN INTEGRATION----------=I10/
     3   40HONUMBER OF TIMES TO BE EVALUATED--------=I10/
     4   40HOREAL PART OF INTEGRATION VARIABLE-----=F5.2/
     5   40HOTHERMOMECHANICAL COUPLING CONSTANT----=E12.4/
     6   40HOINTEGRATION ERROR BOUND---------------=E12.4///)
 2100 FORMAT (10X,2HX=F5.2,5X,2HT=F5.2,5X,12HTEMPERATURE=E15.6,5X,7HSTRE
     1SS=E15.6,5X,13HDISPLACEMENT=E15.6)
 2200 FORMAT (15H1END OF PROBLEM)
 2300 FORMAT (5X,6HISTOP=I5//)
      END
```

```
$IBFT C  REFN     DECK
C
C**** *      COMPUTES TEMPERATURE,STRESS,AND DISPLACEMENT INTEGRANDS FOR
C**** *      FIRST DANILOVSKAYA PROBLEM (COUPLED)
C
      SUBROUTINE RFN (X,Y,GAM,DELTA,A)
      DIMENSION A(3)
      COMPLEX P,P1,P2,P3,PM1,PM2,PP,PQ,DEN,PE1,PE2,PT1,PT2,PT3
      CON=SQRT(2.0)
      P=CMPLX(GAM,Y)
      P1=P+1.0+DELTA
      P2=P1*P1-4.0*P
      P3=CSQRT(P2)
      PM1=CSQRT(P*P1+P*P3)/CON
      PM2=CSQRT(P*P1-P*P3)/CON
      PP=P-1.0-DELTA+P3
      PQ=P-1.0-DELTA-P3
      PE1=CEXP(-PM1*X)
      PE2=CEXP(-PM2*X)
      DEN=P3
      PT1=(PP*PE2-PQ*PE1)/(2.0*P*DEN)
      PT2=(PE1-PE2)/DEN
      PT3=(PM2*PE2-PM1*PE1)/(P*P*DEN)
      A(1)=REAL(PT1)
      A(2)=REAL(PT2)
      A(3)=REAL(PT3)
      RETURN
      END
```

```
$IBFT C  REFN    DECK
C
C**** *     COMPUTES TEMPERATURE,STRESS,AND DISPLACEMENT INTEGRANDS FOR
C**** *     THE RAMP TEMPERATURE INPUT (COUPLED)
C
      SUBROUTINE RFN (X,Y,GAM,DELTA,A)
      DIMENSION A(3)
      COMPLEX P,P1,P2,P3,PM1,PM2,PP,PQ,DEN,PE1,PE2,PT1,PT2,PT3
      CON=SQRT(2.0)
      P=CMPLX(GAM,Y)
      P1=P+1.0+DELTA
      P2=P1*P1-4.0*P
      P3=CSQRT(P2)
      PM1=CSQRT(P*P1+P*P3)/CON
      PM2=CSQRT(P*P1-P*P3)/CON
      PP=P-1.0-DELTA+P3
      PQ=P-1.0-DELTA-P3
      PE1=CEXP(-PM1*X)
      PE2=CEXP(-PM2*X)
      DEN=P*P3
      PT1=(PP*PE2-PQ*PE1)/(2.0*P*DEN)
      PT2=(PE1-PE2)/DEN
      PT3=(PM2*PE2-PM1*PE1)/(P*P*DEN)
      A(1)=REAL(PT1)
      A(2)=REAL(PT2)
      A(3)=REAL(PT3)
      RETURN
      END
```

```
$IBFT C  INLT    DECK
C
C**** *     THIS CODE COMPUTES THE INVERSE LAPLACE TRANSFORM NUMERICALLY
C**** *    THROUGH THE USE OF GAUSSIAN QUADRATURE FORMULAE.
C
      DIMENSION
     1 HED(12),XI(8),WI(8),A(3)
      DATA PI/3.1415926/
      DATA XI/-0.96028986,-0.79666648,-0.52553241,-0.18343464,+0.1834346
     1 4,+0.52553241,+0.79666648,+0.96028986/,WI/+0.10122854,+0.22238103,
     2 0.31370665,0.36268378,0.36268378,0.31370665,0.22238103,0.10122854/
      MCONT=8
      ISTOP=0
C**** *     NCONT=NUMBER OF INTEGRATION CYCLES TO BE CONSIDERED
C**** *     MCONT=ORDER OF THE GAUSSIAN QUADRATURE FOR EACH HALF WAVE
C**** *     GAM=REAL PART OF THE INTEGRATION VARIABLE
C**** *     DELTA=ELASTIC,THERMOMECHANICAL COUPLING CONSTANT
C**** *     X=SPACE POINT OF INTEREST
C**** *     T=TIME POINT OF INTEREST
C**** *     H=DIMENSIONLESS BOUNDARY LAYER CONDUCTANCE
C
C**** *    READ AND PRINT OF INPUT DATA
C
    5 READ (5,1000) HED,NCONT,NT,GAM,X,DELTA,H,ERROR
      WRITE (6,2000) HED,NCONT,MCONT,NT,GAM,DELTA,H,ERROR
      DO 500 I=1,NT
      READ (5,1100) T
      SUML=0.0
      SUMM=0.0
      SUMN=0.0
      DO 100 N=1,NCONT,2
      XL=2*N-3
      XM=2*N-1
      XN=2*N+1
      IF(N.EQ.1) XL=0.0
      A1=XL*PI/(2.0*T)
      A2=XM*PI/(2.0*T)
      A3=XN*PI/(2.0*T)
      SUML1=0.0
      SUML2=0.0
      SUMM1=0.0
      SUMM2=0.0
      SUMN1=0.0
      SUMN2=0.0
      DO 200 M=1, MCONT
      Y1=0.5*(A1+A2+XI(M)*(A2-A1))
      Y2=0.5*(A2+A3+XI(M)*(A3-A2))
      CALL RFN (X,Y1,GAM,DELTA,A,H)
      SUML1=SUML1+WI(M)*A(1)*COS(Y1*T)*(A2-A1)/2.0
      SUMM1=SUMM1+WI(M)*A(2)*COS(Y1*T)*(A2-A1)/2.0
      SUMN1=SUMN1+WI(M)*A(3)*COS(Y1*T)*(A2-A1)/2.0
      CALL RFN (X,Y2,GAM,DELTA,A,H)
      SUML2=SUML2+WI(M)*A(1)*COS(Y2*T)*(A3-A2)/2.0
      SUMM2=SUMM2+WI(M)*A(2)*COS(Y2*T)*(A3-A2)/2.0
```

```
  200   SUMN2=SUMN2+WI(M)*A(3)*COS(Y2*T)*(A3-A2)/2.0
        ADDL=SUML1+SUML2
        ADDM=SUMM1+SUMM2
        ADDN=SUMN1+SUMN2
        ADD1=ABS(ADDL)
        ADD2=ABS(ADDM)
        ADD3=ABS(ADDN)
        ADD=ADD1+ADD2+ADD3
        IF (ADD.LE.ERROR) GO TO 300
        SUML=SUML+2.0*EXP(GAM*T)/PI*ADDL
        SUMM=SUMM+2.0*EXP(GAM*T)/PI*ADDM
  100   SUMN=SUMN+2.0*EXP(GAM*T)/PI*ADDN
        ISTOP=1
        WRITE (6,2300) ISTOP
  300   WRITE (6,2100) X,T,SUML,SUMM,SUMN
  500   CONTINUE
        WRITE (6,2200)
        GO TO 5
 1000   FORMAT (12A6/2I5,5F10.0)
 1100   FORMAT (F10.0)
 2000   FORMAT (1H1   12A6/
     1     40HONUMBER OF INTEGRATION CYCLES----------=I10/
     2     40HOORDER OF GAUSSIAN INTEGRATION---------=I10/
     3     40HONUMBER OF TIMES TO BE EVALUATED-------=I10/
     4     40HOREAL PART OF INTEGRATION VARIABLE-----=F5.2/
     5     40HOTHERMOMECHANICAL COUPLING CONSTANT----=E12.4/
     6     40HOBOUNDARY LAYER CONDUCTANCE------------=E12.4/
     7     40HOINTEGRATION ERROR BOUND---------------=E12.4///)
 2100   FORMAT (10X,2HX=F5.2,5X,2HT=F5.2,5X,12HTEMPERATURE=E15.6,5X,7HSTRE
     1 SS=E15.6,5X,13HDISPLACEMENT=E15.6)
 2200   FORMAT (15H1END OF PROBLEM)
 2300   FORMAT (5X,6HISTOP=I5//)
        END
```

```
$IBFT C   REFN     DECK
C
C**** *       COMPUTES TEMPERATURE,STRESS,AND DISPLACEMENT INTEGRANDS FOR
C**** *       SECOND DANILOVSKAYA PROBLEM (COUPLED)
C
      SUBROUTINE RFN (X,Y,GAM,DELTA,A,H)
      DIMENSION A(3)
      COMPLEX P,P1,P2,P3,PM1,PM2,PP,PQ,H1,H2,DEN,PE1,PE2,PT1,PT2,PT3
      CON=SQRT(2.0)
      P=CMPLX(GAM,Y)
      P1=P+1.0+DELTA
      P2=P1*P1-4.0*P
      P3=CSQRT(P2)
      PM1=CSQRT(P*P1+P*P3)/CON
      PM2=CSQRT(P*P1-P*P3)/CON
      PP=P-1.0-DELTA+P3
      PQ=P-1.0-DELTA-P3
      PE1=CEXP(-PM1*X)
      PE2=CEXP(-PM2*X)
      H1=H+PM1
      H2=H+PM2
      DEN=H2*PP-H1*PQ
      PT1=H*(PE2*PP-PE1*PQ)/(DEN*P)
      PT2=2.0*H*(PE1-PE2)/DEN
      PT3=2.0*H*(PM2*PE2-PM1*PE1)/(P*P*DEN)
      A(1)=REAL(PT1)
      A(2)=REAL(PT2)
      A(3)=REAL(PT3)
      RETURN
      END
```