Diagrams Benefit Symbolic Problem Solving

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Abstract

Problem presentation can influence students' understanding, choice of strategy, and accuracy. For example, the presence and type of external representation can alter performance. In this study, we examined the effect of diagrams on students' performance in a symbolic problem domain. Sixty-one seventh-grade students solved algebraic equations with or without an accompanying diagram. The presence of diagrams increased accuracy and use of informal strategies. Overall, the benefits of diagrams found for word problems generalized to symbolic problems.

Keywords: External Representation; Problem Solving; Algebra Equations

Problem Representation

If you had to teach children basic addition, what would be more helpful – a set of blocks they could touch and count, or a list of common arithmetic equations such as "2+3=5"?

External representations such as blocks and equations impact learning and performance (Belenky & Schalk, 2014). They exist as physical symbols or objects and impact how people internally represent problems, which in turn influences how they solve the problems (Koedinger, Alibali, & Nathan, 2008).

External representations vary in their concreteness. Concrete representations such as pictures, diagrams, and physical models are grounded in familiar experiences, connect with learners' prior knowledge, and have an identifiable perceptual correspondence with their referents (Fyfe, McNeil, Son, & Goldstone, 2014). However, they may contain extraneous perceptual details that distract learners from relevant information or inhibit transfer of knowledge to novel situations (Harp & Mayer, 1997; Kaminsky, Sloutsky, & Heckler, 2008). In contrast, symbolic representations such as formal equations and line graphs eliminate extraneous surface details, are more arbitrarily related to their referents, and represent the underlying structure of the referent more efficiently. Thus, they allow greater flexibility and generalizability to multiple contexts, but may appear as meaningless symbols to learners who lack conceptual understanding (Nathan, 2012).

Diagrams are one type of concrete, external representation thought to aid problem solving. In this study we focus on the effect of combining diagrams with a more symbolic representation, namely, algebraic equations.

Potential Benefits of Diagrams

Diagrams are schematic visual representations that express information via spatial relationships. Concrete details of the referent can be disregarded so that only the relevant problem features and quantitative relations are depicted. There are at least three reasons why diagrams might be helpful for solving symbolic problems such as algebraic equations.

First, diagrams may help highlight relevant information. For example, visual scanning is easier when information is presented spatially than when it is presented in a list of numbers or verbal statements (Larkin & Simon, 1987). Thus, including diagrams with equations may facilitate the cognitive process of searching the problem space and may help students extract relevant information.

Second, diagrams may decrease working memory load and support quantitative reasoning (Munez, Orrantia, & Rosales, 2013; Murata, 2008). For example, in a study with adults, functional magnetic resonance imaging results revealed that solving a word problem by constructing a mental diagram required fewer resources for controlling attention or retrieving procedural knowledge than solving the problem by constructing a mental equation (Lee et al., 2007). Thus, presenting diagrams may free up cognitive resources that are important for accurate problem solving, such as correctly selecting and implementing strategies.

Third, diagrams may scaffold algebraic reasoning by facilitating connections between concrete and symbolic representations (Koedinger & Terao, 2002; Lee et al., 2013). Specifically, diagrams may elicit students' intuitive, informal knowledge and strategies. Presenting diagrams with equations may allow students to connect this knowledge to formal, symbolic problem formats. Thus, a diagram benefit may be particularly apparent for students who are still developing familiarity with manipulating abstract symbols.

Evidence for a Diagram Benefit

In addition to theoretical reasons for a diagram benefit, there are several lines of work suggesting that diagrams can improve mathematical problem solving. Past research has focused on the benefits of diagrams for solving word problems.

The first line of evidence comes from research on individual differences in the spontaneous use of diagrams during word problem solving. 4^{th} to 6^{th} grade students who tend to use diagrams, whether by drawing diagrams on paper or by mental visualization, are more accurate solvers (e.g. Edens & Potter, 2008; Hegarty & Kozhevnikov, 1999). Similar results are reported for students with learning disabilities (Van Garderen & Montague, 2003).

A second line of evidence for a diagram benefit comes from instructional practice. Countries such as Japan and Singapore have long incorporated diagrams into math instruction on a national level, and these countries typically perform at the top in international tests of mathematics achievement (Murata, 2008; Ng & Lee, 2009). For instance, 1st and 2nd graders in Singapore are introduced to a heuristic using horizontal bar diagram drawings to solve word problems (Ng & Lee, 2009). Students who construct a diagram representation are able to use informal arithmetic strategies to solve algebraic problems, thus making algebraic problems accessible earlier – beginning in 3rd grade, as opposed to the 7th grade in most US classrooms (Lee et al, 2013). In fact, 6th grade pre-algebra students in the US were able to learn and apply this diagram heuristic to solve algebra word problems (Koedinger & Terao, 2002). Extensive interventions that include practice generating diagrams have helped U.S. 3rd grade students improve their word problem solving success (Jitendra et al., 2007). However, it is unclear if benefits of the intervention are due to the use of diagrams or to receiving more general problem-solving strategy instruction.

A third, more direct line of research clarified this issue by experimentally manipulating whether students are given diagrams in conjunction with word problems. Munez, Orrantia, and Rosales (2013) found that presenting novel diagrams could enhance 9th graders' accuracy and response times on arithmetic word problems. Further, the improvement was greatest on more difficult problems. These findings are consistent with similar research on undergraduates (Lewis, 1989). However, the benefits of provided diagrams may be less robust in middle school students. Booth and Koedinger (2012) assessed 6th to 8th grade students on three algebraic problems differing in complexity. Each problem was presented in one of three formats: equation only, word, or word-with-diagram. The diagrams used were novel to students and tailored to each problem. While high-ability students of all grades performed equally with or without diagrams, low-ability students in the 7th and 8th grades were more accurate and made fewer conceptual errors on word problems with diagrams. However, these students benefited from diagrams only on the more complex double-reference problem where the unknown variable appeared twice. No diagram benefit was found on the simpler single-reference problem where the variable appeared only once. This suggests that grade, ability level and problem complexity may be key moderators of the potential diagram benefit. Although informative, this study did not include an equation-with-diagram condition, which would have revealed whether the diagram benefit was consistent across problem types or only for word problems.

In summary, diagrams generally aid in problem comprehension and solution of word problems. Little is known about the use of diagrams with more symbolic tasks such as equation solving. Extending the diagram research to a symbolic domain will test the generalizability of the diagram effect and also provide insight into how children interpret and solve symbolic problems.

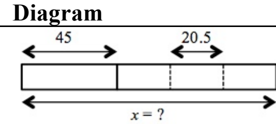
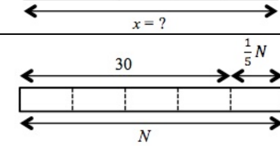
Current Study

The present study investigated the effectiveness of presenting diagrams alongside algebraic equations. Previous research suggests that diagrams can help students make sense of word problems. However, algebra equations are more abstract than word problems. Algebra equations not only require students to understand complex mathematical structure, but also require students to decode the symbolic language of algebra (Payne & Squibb, 1990).

Our primary research question was whether the presence of diagrams would influence algebraic equation-solving performance, including accuracy, type of errors made, and strategy use. We predicted that problem-solving accuracy would be higher if a diagram was provided than if it was not. We also hypothesized that diagrams could elicit students' intuitive knowledge of quantitative relations in the problem. Similar to findings in word problems (e.g. Koedinger & Nathan, 2004), this should reduce the frequency of conceptual errors, and increase both the usage of non-algebraic strategies and the accuracy of algebraic strategies.

Our secondary research question was whether the effect of diagrams would depend on student or problem characteristics. We explored problem complexity and students' general math ability as two factors that could influence the benefits of diagrams. We varied problem complexity by including both single-reference and more difficult double-reference problems, which differed in whether the variable appeared once or twice in the equation (Table 1). We explored the importance of general math ability by working with students drawn from advanced and regular mathematics classes.

Table 1: Examples of equations and diagrams

	Equation	Diagram
Single-Reference	(x-45) / 3 = 20.5	
Double-Reference	N + 1/5N = 30	

Note: These two equations were taken from Booth & Koedinger (2012). Other equations were adapted from these.

Method

Middle-school students participated in an experimenter-led classroom session. Using a within-subjects design, we manipulated the presence of diagrams during the equation solving assessment.

Participants

Participants were 62 seventh-grade students from four classes attending an independent private school. Students were tested 2 months into their first pre-algebra course. Two classes (34 students) were in an advanced math class, which covered the same breadth of content but in greater depth. Students were placed into the advanced class by their math teachers based on multiple components of their 6th grade performance, including standardized test scores and in-class grades. Students had experience with reading algebraic expressions and solving simple one step equations, but had not studied the equation forms used in the experiment. They did not have prior experience with the type of diagrams used in this study. We dropped the data of one student who did not attempt any of the assessment items. The final sample contained 61 students (33 male, mean age = 12.7 years).

Design and Procedure

Students completed the experiment in their classrooms during their regular 50-minute math period. The experiment included four parts: introduction, diagram practice, representation translation, and equation solving. The within-subjects manipulation occurred only during equation solving. There were no time limits for any of the tasks. Calculators were not permitted on any task.

Diagram Introduction. The experimenter spent about 8 minutes with the entire class to describe diagrams as a special kind of picture that represents information about numbers and quantities. She explained four guidelines used to construct diagrams, highlighting important features (e.g. arrows, labels, dotted lines). She described how diagrams could represent each of the basic operations: addition, subtraction, multiplication, and division. For each operation, she presented and described an example diagram and asked students to copy her drawings on a worksheet. These example diagrams were much simpler than the diagrams students would encounter in the rest of the experiment. The intent was to develop basic knowledge of interpreting the diagrams as they were unfamiliar to the students. No references to equations were made.

Based on the diagram approach used in Singapore, diagrams were constructed according to these guidelines: (1) Quantities were represented by rectangular bars using solid borders; (2) Dotted vertical lines divided bars into equal portions; (3) Rectangular bars were labeled internally with variables; and (4) Horizontal arrows over the length of a bar indicated the quantity's magnitude and were labeled with known values or '?'. Diagrams were drawn to approximate the relative quantities in each problem, but were not of the same scale across problems.

Diagram practice. To provide brief exposure to problem-solving using diagrams in isolation, we asked students to work individually on four problems. Students had to solve for an unknown variable from a given diagram. The corresponding equation was not included. The first three problems were single-reference problems and the fourth was a double-reference problem. After 10 minutes of individual work, the experimenter announced the correct answer for each problem. Students checked their own work. The intent was to increase familiarity with the diagrams.

Representation-translation task. We constructed four items to measure how well students could translate between diagrams and equations. The first two problems required students to choose between two diagrams that described a given equation. Students received one point for circling the correct diagram. The next two problems required students to generate and write an equation describing a given diagram. Students received one point for writing a valid equation; expressions (e.g. $2x + 1$) were not valid. For both pairs of problems, a single-reference problem was presented first, followed by a double-reference problem. Students first read directions and a completed example before attempting each pair of problems. No feedback was provided. We included this task as a check of how well students were able to make the connection between diagrams and equations.

Equation-solving Assessment. To evaluate our primary research question, we designed eight algebra problems contrasting two factors, presentation format and problem complexity. Students saw four problems as equations and an isomorphic set of four problems as equations with accompanying diagrams, using different variable letters and constant values. For each presentation format, the first two problems were single-reference and the other two problems were double-reference. On the four equation-with-diagram problems, students also indicated if they had used the diagram to solve each problem by circling "yes" or "no" inside a small box below the diagram. This provided a measure of diagram use frequency.

We designed four counterbalanced forms of the equation-solving assessment to control for order of presentation (equation-first or equation-with-diagrams first), and version (which problems had diagrams). Problems were blocked by presentation format.

Coding

Students received one point for each correct answer. We coded students' errors and strategy use based on their written work, using the scheme outlined in Tables 2 and 3. These schemes were adapted from previous research on students' solution of algebraic equations and word problems (Koedinger et al., 2008). To establish inter-rater reliability, a second rater coded the written responses of 25% of the children. Inter-rater agreement was high (Cohen's κ = .92 for errors, κ = .85 for strategies).

Results

To evaluate the effect of counterbalanced forms, a 2 (order) x 2 (version) ANOVA on equation-solving assessment scores was done. This analysis revealed no significant effects of either factor or their interaction, F's < 1. Thus the subsequent analyses treated the four forms as equivalent.

Table 2: Errors made on Equation-solving Assessment

Error	Definition	Error frequency	
		Equation only	Equation-with-diagram
Conceptual	Student employs invalid or incomplete strategies.	34 (4.0)	21 (3.0) *
Arithmetic	Student makes a computational error, but solution is otherwise correct.	4 (1.3)	5 (1.4)
No Attempt	Student leaves problem blank or wrote some form of "I don't know".	13 (3.0)	12 (3.0)
No Work	Student write incorrect answer without any work shown.	12 (3.3)	15 (3.7)
Copy Slip	Student miscopies a value from the problem or from own work, but solution is otherwise correct.	2 (.01)	0 (-)

Note: Scores are percentages of all problems on the Equation-solving Assessment, presented as mean (with standard errors in parentheses). * denotes differences at p = .05 level of significance.

We evaluated students' performance on the equation-solving assessment using a repeated-measures ANOVA with *presentation format* (equation or equation-with-diagram) and *problem complexity* (single- or double-reference) as within-subjects variables and math proficiency (regular or advanced class) as the between-subjects variable.

As shown in Figure 1, students provided more correct answers when solving equations with provided diagrams (48%) than without (36%), $F(1,59) = 11.6$, p=.001, η_p^2=.16. Students also solved fewer double-reference equations correctly (32%) than single-reference equations (50%), $F(1,59) = 17.6$, $p < .001$, $\eta_p^2 = .23$. Students in advanced classes solved more problems correctly (56%) than students in regular classes (22%), $F(1,60) = 24.28$, $p < .001$, $\eta_p^2 = .29$. None of the two-way or three-way interactions between presentation format, problem complexity and math ability were significant (F's < 1). Thus, the positive effect of diagrams was consistent across simple and complex problems and across students with low and high ability.

Eleven students failed to indicate whether they used a diagram for a specific problem at least once. The remaining 50 students reported using the diagram on a majority of problems (60% of equation-with-diagram problems). Students reported using diagrams more often on double-reference problems (72%) than on single-reference problems (49%), $t(49) = 3.34$, $p < .01$. However, reported diagram use was not correlated with accuracy on equation-with-diagram problems (r = -.045, p > .75).

Table 2 presents the frequencies of each type of error, as a percentage of all problems on the equation-solving assessment. Overall, conceptual errors were the most common. Students made fewer conceptual errors on equation-with-diagram problems, compared to equation-only problems, $t(33) = 3.51$, $p < .001$.

Next consider the strategies students used to solve the equations (see Table 3). Overall, students used an unwind strategy twice as often as an algebra strategy (35% vs. 15% of problems), $t(60) = 4.03$, $p < .001$. Though not reliable, the presence of diagrams marginally increased use of the unwind strategy, $t(60) = 1.69$, $p < .1$. We also evaluated the effectiveness of each strategy by considering the percentage of problems where each strategy led to a correct answer. Overall, accuracy of the strategies was moderate. The presence of diagrams appeared to increase the success of these strategies, although overall frequencies are too small for meaningful statistical analyses.

Finally, consider student success on the representation-translation task. Students were moderately successful, (M = 2.44 out of 4, SD = 1.35), and their representation-translation score was positively correlated with their overall equation-solving accuracy (r = .492, $p < .001$) To assess if representation translation ability moderated the diagram benefit, we calculated a diagram effect score for each student (equation-with-diagram score – equation score) and found a small but insignificant correlation between representation translation score and diagram effect score (r = .21, p > 0.1).

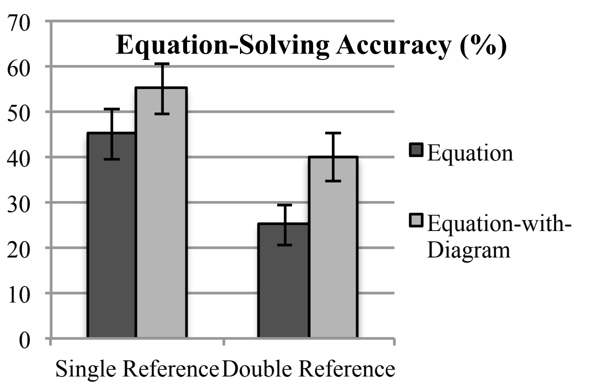


Figure 1: Percentage correct on Equation-solving Assessment by presentation format and problem complexity. Error bars are standard errors.

Discussion

External representations such as diagrams generally support learning and problem solving. However, incorporating diagrams with symbolic problems has largely gone unstudied. As predicted, we found a clear diagram benefit. Presenting diagrams alongside algebra equations enhanced students' accuracy and reduced the frequency of conceptual errors. The diagram benefit was independent of problem complexity or students' math proficiency. In contrast, a previous study on word problem-solving found that

Table 3: Strategies used on Equation-solving Assessment

Strategy	Definition	% Used (% Correct) Equation only	Equation-with-diagram
Algebra	Student uses algebraic manipulations to derive solution. A sub-equation or simplified algebraic expression is written.	17 (51)	15 (62)
Unwind	Student works backward using arithmetic strategies to derive solution.	32 (51)	38 (69)
Guess and Check	Student substitutes different value(s) for the variable.	5 (46)	4 (67)
Other	Student uses other strategies, or strategy is ambiguous	15 (19)	8 (32)
Answer Only	Answer is provided without any working.	19 (24)	24 (27)
Not Attempted	Student leaves problem blank or wrote some form of "I don't know"	13 (-)	12 (-)

diagrams were most helpful for more difficult problems and for students with lower ability (Booth & Koedinger, 2012; Lewis, 1989). Other research has found that concrete representations might even reduce performance on complex problems. For instance, college students are more accurate solving double-reference problems in equation format than in word problem format, even though they benefit from the concreteness of word problems on simpler single-reference problems (Koedinger et al, 2008). Why did diagrams provide a clearer benefit in this study than in previous work with word problems?

One possible explanation is that students in our sample had greater diagram familiarity and understanding than students in previous studies, due to the experimenter-led diagram introduction. We also used a consistent diagram type for all problems, unlike in Booth & Koedinger's (2012) study. However, students had not seen these diagrams before the experiment, and we did not teach them how to relate diagrams to equations. The assessment problems used were also much harder than the examples we used in the introduction. It is unlikely that familiarity played a major role for helping students to benefit from diagrams.

Another explanation is the fact that algebra equations are generally more difficult than equivalent word problems. Even high school students may make persistent errors in understanding and solving algebra equations, although they are more accurate on word problems (Koedinger & Nathan, 2004). When all problems are difficult for most students, diagrams can aid performance across problem complexity and students' math proficiency. Thus, even the simpler single-reference equation problems used in this study were likely challenging enough that students benefited from an alternate concrete representation. Students also indicated more frequent use of diagrams on more complex double-reference problems, suggesting that they believed diagrams might be helpful on those problems.

Our results also support some explanations suggested in the literature. First, diagrams may influence internal representation. Our finding of a diagram benefit is consistent with various cognitive models of problem solving. These models generally posit that constructing appropriate mental models of a problem is key to successful problem solving (Johnson-Laird, 1983; Koedinger & Nathan, 2004). Our results match their predictions of increased accuracy and a trend toward use of more informal strategies. By providing students with a pre-constructed diagram, we may have removed some of the difficulty of constructing an internal representation of the problem,. By providing an additional *external* representation on paper, we may also have reduced students' working memory and attention demands by offloading some cognitive processing onto perceptual processing (e.g. Larkin & Simon, 1987).

Second, diagrams may facilitate informal reasoning. Consistent with previous research, we found an indication that students in the current study used more non-algebraic strategies when concrete diagrams were present (Koedinger & Nathan, 2004). Similar to how adding a concrete story context can improve performance on arithmetic problems by activating real-world knowledge of common operations and quantitative relations (Carraher, Carraher, & Schliemann, 1985), adding a concrete diagram might improve children's performance on algebra equations by activating informal strategies that do not rely on newer algebraic strategies that students are in the process of learning.

Despite the positive contributions of the current study, we are unable to provide detailed accounts of how students used the diagrams. In order to develop a mechanistic account of diagrammatic reasoning, future research should investigate specific processes that problem-solvers engage in when using a diagram. Do students iterate between representations or fixate on the more concrete one? Tracking participants' eye movements may reveal diagram elements that are particularly helpful, distracting, or ignored.

Future research could also investigate potential trait by treatment interactions, such as the influence of quantitative reasoning, visual-spatial ability, or representation translation skill on students' use of diagrams. For instance, students of different math ability may benefit from diagrams due to different reasons. Higher-ability students, who are more likely to spontaneously generate useful diagrams on their own, may benefit because diagrams more closely matched their own internal representation of the problem. However, low-ability students may benefit because diagrams highlight important information they would have otherwise ignored or misinterpreted. Understanding how different students utilize multiple representations can help educators personalize instruction.

In summary, the current study extends previous research of a diagram benefit in problem solving to a symbolic domain. Providing novel diagrams enhanced students' accuracy on difficult algebra equation problems independent of the problem and student characteristics studied. Concrete external representations may be more powerful than previously leveraged, especially when combined with symbolic problems.

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