Title
Notes on Quadrupole Focusing

Author
McMillan, Edwin M

Publication Date
1956-02-01

University of California

Ernest O. Lawrence Radiation Laboratory

Berkeley, California

UCRL-3333
Internal Distribution

UNIVERSITY OF CALIFORNIA

Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

NOTES ON QUADRUPOLE FOCUSING

Edwin M. McMillan

February 9, 1956

Printed for the U. S. Atomic Energy Commission

NOTES ON QUADRUPOLE FOCUSING

Edwin M. McMillan

Radiation Laboratory
University of California
Berkeley, California

February 9, 1956

A. General Considerations

1. Consider two planes perpendicular to the axis of the system, with coordinates x and y in these planes lying along the principal directions of the quadrupoles. Let x_1 and x_1' be the x displacement and slope of an orbit at the first plane, while x_2 and x_2' are the corresponding quantities at the second plane. Then the most general linear relations between these quantities are

$$x_2 = ax_1 + bx_1',$$

$$x_2' = cx_1 + dx_1'.$$

This pair of equations can be represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2. In all matrices encountered in this application, the determinant ad - bc = 1. This fact can be used either to simplify the calculations, or as a check on calculations made without specifically using this fact.

3. If the orbit goes in succession through two regions for which the matrices are known, the matrix for the whole system is the product of the individual matrices. The matrix for the region traversed first appears at the right in the product.

4. If all elements of a region with matrix (a, b, c, d) are physically inverted along the axis, as if by reflection in a plane midway between the planes described in (A. 1), the matrix representing the new situation is: $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$. A corollary of this is that if a region has a plane of symmetry, then a = d.

5. If a region with matrix (a, b, c, d) is followed by the same region inverted as in (A.4), the resultant matrix is

$$\begin{pmatrix} ad + bc & 2db \\ 2ac & ad + bc \end{pmatrix} .$$

(Note that ad + bc = 2ad - 1 = 2 bc + 1.)

6. Matrices for the y direction are obtained from those for the x direction by simply changing the sign of the magnetic field gradient throughout.

7. Significance of vanishing matrix elements: From the equations in (A.1), it is apparent that if the coefficient $a = 0$, parallel rays incident at the left are focused to a point at the second plane; if $d = 0$, a point source at the first plane gives a parallel beam at the right; if $b = 0$, a point source at the first plane gives a point focus at the second; if $c = 0$, an incident parallel beam gives an emergent parallel beam.

B. Analogy with Geometrical Optics

8. The matrix for a field-free region of length L is

$$\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} .$$

9. The matrix for a thin lens of focal length f is

$$\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} .$$

10. Make the following combination: Field-free region of length L_1, followed by region of matrix (a b c d), followed by field-free region of length L_2. The resulting matrix is

$$\begin{pmatrix} a + cL_2 & b + aL_1 + dL_2 + cL_1L_2 \\ c & d + cL_1 \end{pmatrix} .$$

If this combination is to give a point-to-point focus, then according to (A.7) the upper right element must vanish. Using (A.2), we can write this condition in the form

$$(L_1 + d/c)\;(L_2 + a/c) = 1/c^2.$$

A plot of L_1 against L_2 is thus a hyperbole. This can also be written

$$\frac{1}{L_1 + (d-1)/c} + \frac{1}{L_2 + (a-1)/c} = -c,$$

which has the form of the standard lens formula, if $-1/c$ is taken for the focal length and $(d-1)/c$, $(a-1)/c$ are taken for the distances of the principal planes inward from the ends of the lens. Note that if the coefficients a, b, c, d given for the thin lens in (B.9) are substituted into this, it gives

$$1/L_1 + 1/L_2 = 1/f.$$

If L_1 and L_2 satisfy the conditions given above, the linear magnification is equal to the upper left element in the matrix, $a + cL_2$. The ratio of angular spreads is the reciprocal of this. These are negative if the image is inverted.

11. If $L_1 = L_2 = L$ in the above,

$$L = -\frac{1}{c}\left[\frac{a+d}{2} + \sqrt{\left(\frac{a-d}{2}\right)^2 + 1}\right],$$

$$\text{linear magnification} = \frac{a-d}{2} - \sqrt{\left(\frac{a-d}{2}\right)^2 + 1}\,.$$

(The general solution has a a ± sign in front of the square roots, but since the + sign is to be chosen in most practical cases, it is written this way to avoid confusion.)

C. Basic Quadrupole Formulas

12. Let ρ be the radius of curvature of the particle in the field at a distance r from the axis, and let ℓ be the length of the region considered.

Define

$$\phi = \frac{\ell}{\sqrt{\rho r}} .$$

Then the matrix for a focusing section is

$$\begin{pmatrix} \cos\phi & \frac{\ell}{\phi}\sin\phi \\ -\frac{\phi}{\ell}\sin\phi & \cos\phi \end{pmatrix} .$$

For a defocusing section, replace ϕ by $i\phi$ to get

$$\begin{pmatrix} \cosh\phi & \frac{\ell}{\phi}\sinh\phi \\ \frac{\phi}{\ell}\sinh\phi & \cosh\phi \end{pmatrix} .$$

13. A focusing section of strength ϕ_1 followed by a defocusing section of strength ϕ_2, both having the same length ℓ, has the matrix

$$\begin{pmatrix} \cos\phi_1\cosh\phi_2 - \frac{\phi_1}{\phi_2}\sin\phi_1\sinh\phi_2 & \ell\left(\frac{1}{\phi_1}\sin\phi_1\cosh\phi_2 + \frac{1}{\phi_2}\cos\phi_1\sinh\phi_2\right) \\ \frac{1}{\ell}\left(-\phi_1\sin\phi_1\cosh\phi_2 + \phi_2\cos\phi_1\sinh\phi_2\right) & \cos\phi_1\cosh\phi_2 + \frac{\phi_2}{\phi_1}\sin\phi_1\sinh\phi_2 \end{pmatrix}$$

The corresponding matrix for the y-direction is derived from this by replacing ϕ by $i\phi$. This combination will be called a symmetrical doublet if $\phi_1 = \phi_2$. A useful approximation is obtained by letting $\phi_1 = \phi(1+\alpha)$, $\phi_2 = \phi(1-\alpha)$, and expanding to order ϕ^2 the coefficients of terms of order α. Then, for $\alpha << 1$ and ϕ not too large, the above matrix is represented approximately by

$$\begin{pmatrix} \cos\phi\cosh\phi - \sin\phi\sinh\phi - 4\phi^2\alpha & \frac{\ell}{\phi}(\sin\phi\cosh\phi + \cos\phi\sinh\phi \\ \frac{\phi}{\ell}(-\sin\phi\cosh\phi + \cos\phi\sinh\phi - 4\phi\alpha) & \cos\phi\cosh\phi + \sin\phi\sinh\phi - 4\phi^2\alpha \end{pmatrix}$$

This is exact if $\alpha = 0$. A still further approximation can be obtained by expanding the functions in all terms. If this is carried out to two orders above the leading term, taking account of the fact that in the cases we are interested in α is of order ϕ^4, we get

$$\begin{pmatrix} 1 - \phi^2 & 2\ell \\ \frac{2\phi^2}{\ell}\left(-\frac{1}{3}\phi^2 - 2\alpha\right) & 1 + \phi^2 \end{pmatrix}.$$

(Note that in these approximate formulas the relation ad - bc = 1 must not be used to compute elements given to higher order than the others.)

14. A <u>symmetrical triplet</u> is produced by placing in series a doublet and a reversed doublet of equal strength. Using (A.5) and the last form given in (C.13), we get

$$\begin{pmatrix} 1 & 4\ell \\ \frac{4\phi^2}{\ell}\left(-\frac{1}{3}\phi^2 + \frac{1}{3}\phi^4 - 2\alpha\right) & 1 \end{pmatrix}.$$

The more accurate matrix (exact for $\alpha = 0$) appears in its simplest form when functions of double angles are used:

$$\begin{pmatrix} \cos 2\phi \cosh 2\phi - 16\phi^2\alpha & \frac{\ell}{\phi}(\sinh 2\phi + \sin 2\phi \cosh 2\phi \\ \frac{\phi}{\ell}(\sinh 2\phi - \sin 2\phi \cosh 2\phi - 8\phi\alpha) & \cos 2\phi \cosh 2\phi - 16\phi^2\alpha \end{pmatrix},$$

These matrices apply to the case with the focusing section at the ends; the change $\phi \to i\phi$ takes one to the case with the focusing section in the middle.

D. Applications

In all cases given below, it will be required that the foci coincide for x and y displacements. In practice, astigmatic arrangements may sometimes be wanted, but this is a good starting point for examining the properties of quadrupole lenses. Arrangements for producing a parallel beam and for focusing between points at equal distances from the lens will be considered in some detail, and comparisons made between the doublet and triplet lenses. The comparisons will presumably hold in more general cases where the focal distances are unequal.

15. Production of parallel beam from a point source. Refer to the formulas in (B. 10) and note that a parallel beam on the right implies $L_2 = \infty$. This leads to

$$L_1 = -d/c.$$

With the source at the distance L_1, the ratio of the width of the outgoing beam to the angular spread of the incident beam is $-1/c$. If the source is displaced laterally by a unit distance, the outgoing beam is deflected by an angle c. This displacement has a further effect, since the principal planes are not necessarily at the ends of the lens section; this is a lateral displacement of the central ray (the ray passing through the center of the lens entrance) by an amount $a - 1/d$ at the lens exit.

16. Use of symmetrical triplets for parallel beam from a point source. Start with the approximate matrix given in (C. 14), and set up the condition that d/c is unchanged when ϕ is changed to $i\phi$. This leads to the requirement

$$a = \frac{1}{6}\,\phi^4.$$

With this requirement satisfied, the matrix is very simple, and is the same for both x and y directions:

$$\begin{pmatrix} 1 & 4\ell \\ -4\phi^4/3\ell & 1 \end{pmatrix}.$$

The length L_1 is equal $3\ell/4\,\phi^4$, and the coefficient $-1/c$ for computing the ratios of deflections and angles has the same value.

How good an approximation is this? We can compare with better calculations made using the second form given in (C. 13). At $\phi = 0.3$ radian agreement is within the accuracy of a 6-inch slide rule; at $\phi = 0.6$ radian, the approximation gives $\alpha = 0.022$, $L_1 = 5.8\,\ell$, while the better calculation gives $\alpha = 0.016$, $L_1 = 4.3\,\ell$. The coefficient $-1/c$ is equal to $7.6\,\ell$ and $5.9\,\ell$, $a - 1/d$ is equal to -1.1 and 0.6, for the cases with the focusing section in the end and the middle, respectively. If $\phi = \pi/4$, the approximation is rather bad; an exact calculation shows that the foci coincide at the edge of the lens for $\alpha = 0$, with $-1/c$ equal to $2.8\,\ell$ and $1.3\,\ell$.

17. Use of doublets for parallel beam from a point source. Using the last form in (C. 13), we find that the condition for coinciding foci is again that $\alpha = \phi^4/6$. The matrix is then

$$\begin{pmatrix} 1 - \phi^2 & 2\,\ell \\ -(2\,\phi^4/3\,\ell)(1 + \phi^2) & 1 + \phi^2 \end{pmatrix}.$$

The length $L_1 = 3\,\ell/2\,\phi^4$. The coefficients $-1/c$ differ for the x and y directions, but their product is equal to $L_1^{\,2}$ to the order for which this calculation is good.

A better calculation for $\phi = 0.6$ radian gives better agreement than with the triplet; L_1 comes out to be 3% below the approximate value, while the coefficients $-1/c$ are both correct to within 4%. At $\phi = \pi/4$ the approximate value is still rather good; the correct values are given below followed by the approximate ones in parentheses:

$\alpha = 0.056$ (0.064); $L_1 = 3.7\,\ell$ (3.9 ℓ); $-1/c = 2.6\ell$ and 8.2ℓ (2.4ℓ and 10.1ℓ), for cases with the focusing section in the ends and in the middle, respectively.

18. Focusing between points at equal distances from a symmetrical triplet. This situation can be thought of as two equal doublets back to back each having a point source outside and a parallel beam at the point of contact.

Therefore the results of (D.17) can be used directly in this case. The linear magnification = -1.

19. Focusing between points at equal distances from a symmetrical doublet. Symmetry considerations show that α should be zero in this case, and that the foci will automatically coincide for the x and y displacements. To find the focal distance, use (B.11) and the first form in (C.13), giving

$$L = \frac{\ell}{\phi} \quad \frac{\cos\phi\cosh\phi + \sqrt{\sin^2\phi\sinh^2\phi + 1}}{\sin\phi\cosh\phi - \cos\phi\sinh\phi},$$

or approximately $L = 3\ell/\phi^4$. To this approximation, the linear magnification is $-(1 + \phi^2)$ when the focusing section is toward the source, and $-(1 - \phi^2)$ when it is away from the source; the ratios of angular spreads are the reciprocals of these.

20. Angular aperture. This is the angle of a ray from the source that just touches the boundary of the usable region in the magnetic field. Let the radius of this boundary be r; then the angular aperture for a source distance L_1 can be written as (r/L_1) times some function of ϕ. In all the cases discussed above, the product of the two functions of ϕ for the x and y directions is equal to $1 - \phi^2$ in an approximation good to this order in ϕ. More exact values can be computed without too much trouble, and some numerical examples will be given later.

21. Comparison of triplet and doublet. Collecting formulas given earlier, to the simplest approximation, we have the following:

$$\phi = \ell/\sqrt{r\rho},$$

$$L_1 \sim k\ell/\phi^4 = k\, r^2 \rho^2/\ell^3.$$

Product of x and y angular apertures,

$$\sim (r/L_1)^2 (1 - \phi^2) \sim (\ell^6/k^2 r^2 \rho^4)(1 - \ell^2/r\rho).$$

The numerical coefficient is $k = 3/4$ for production of a parallel beam by a triplet, $3/2$ for the same case with a doublet, and twice these values for the case of point-to-point focusing at equal distances by a triplet and doublet, respectively. The comparison of doublet to triplet can now be made under several sets of assumed circumstances. The ratios of quantities in the doublet and triplet arrangements do not depend on whether one is considering point-to-parallel or point-to-point focusing, to this approximation.

(a) Same total length and diameter of quadrupole magnet, same focal distance. Let the product of the angular apertures = A. Then

$$\frac{\rho \text{ doublet}}{\rho \text{ triplet}} \sim 2, \quad \frac{\phi \text{ doublet}}{\phi \text{ triplet}} \sim \sqrt{2},$$

$$\frac{\text{A doublet}}{\text{A triplet}} \sim \frac{1 - 2\,\phi_t^2}{1 - \phi_t^2}.$$

Thus the triplet takes twice as great a magnetic field as the doublet, while the doublet has a smaller aperture. In case ϕ_t is large enough that the aperture difference is important, and if the available field is great enough to achieve the triplet case, the triplet is better; however, the latter may not prevail in practice.

(b) Same diameter and field strength of magnet, same focal distance.

$$\frac{\text{total length of doublet}}{\text{total lenght of triplet}} \sim (1/4)^{1/3}, \quad \frac{\phi_d}{\phi_t} \sim 2^{1/3},$$

$$\frac{A_d}{A_t} \sim \frac{1 - 2^{2/3}\,\phi_t^2}{1 - \phi_t^2}.$$

Again the triplet gives a larger aperture, but the triplet requires a magnet 1.59 times as long as the doublet.

(c) Same diameter, total length, and field strength of magnet.

$$\frac{L_d}{L_t} \sim 1/4, \quad \frac{\phi_d}{\phi_t} \sim 2,$$

$$\frac{A_d}{A_t} \sim 1/4 \quad \frac{1 - 4\,\phi^2}{\text{[illegible]}}$$

The doublet has a larger aperture than the triplet by a factor of 16 when ϕ_t is small; however, if ϕ_t becomes too large, the triplet will eventually have the larger aperture. The formula given here makes the apertures equal at $\phi_t = 0.49$ radian, $\phi_d = 0.98$ radian, but the approximations used become rather bad for such large values of ϕ.

22. Better calculation for (D.21c). Let M = total length of magnet. A few values computed using the second form in (C.13) and exact formulas for the orbit amplitudes are:

$\frac{M}{\sqrt{r\,\rho}}$	ϕ_t	ϕ_d	$\frac{L_t}{M}$	$\frac{L_d}{M}$	$\left(\frac{M}{r}\right)^2 A_t$	$\left(\frac{M}{r}\right)^2 A_d$	A_d/A_t
1.57	$\pi/8$	$\pi/4$	15.3	4.2	0.0036	0.032	8.7
3.14	$\pi/4$	$\pi/2$	0.92	0.32	0.50	1.02	2.04
4.70	$3\pi/8$	$3\pi/4$	0.133	0.002	7.0	2.9	0.41

This shows that the doublet has a larger aperture than the triplet at least up to $\phi_t = 0.79$, $\phi_d = 1.57$, in the case where both doublet and triplet have the same total length and the same field strength. For larger values of ϕ the focal point comes uncomfortably close to the magnet for some uses.

The conclusion is that in many applications (particularly where equality of the x and y magnifications is not important) the doublet will do a better job than the triplet arrangement.

23. Effect of Gaps. Gaps between focusing and defocusing sections tend to increase the strength of quadrupole arrangements. As an illustration, consider the case of a symmetrical doublet with a field-free gap of length G between the two sections. The matrix elements (for the case with the focusing section at the left) are:

$$a = \cos\phi\cosh\phi - \sin\phi\sinh\phi - \frac{L\phi}{\ell}\sin\phi\cosh\phi,$$

$$b = \frac{\ell}{\phi}\left(\sin\phi\cosh\phi + \cos\phi\sinh\phi + \frac{L\phi}{\ell}\cos\phi\cosh\phi\right),$$

$$c = \frac{\phi}{\ell}\left(-\sin\phi\cosh\phi + \cos\phi\sinh\phi - \frac{L\phi}{\ell}\sin\phi\sinh\phi\right),$$

$$d = \cos\phi\cosh\phi + \sin\phi\sinh\phi + \frac{L\phi}{\ell}\cos\phi\sinh\phi\ .$$

A numerical calculation with $\phi = \pi/4$, $G = [illegible]\ell/\phi = 4.6$ [illegible] case of point-to-point focus, gives the following result: The focal distance L_1 is 0.3 of that for the corresponding case with $G = 0$, while the aperture A is increased by a factor of 3.6.