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Abstract

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In one of the fundamental results of Arakelov's arithmetic intersection theory, Faltings and Hriljac (independently) proved the Hodge-index theorem for arithmetic surfaces by relating the intersection pairing to the negative of the Neron-Tate height pairing. More recently, Moriwaki and Yuan–Zhang generalized this to higher dimension. In this work, we extend these results to projective varieties over transcendence degree one function fields. The new challenge is dealing with non-constant but numerically trivial line bundles coming from the constant field via Chow's K/k-image functor.

As an application of the Hodge-index theorem to heights defined by intersections of adelic metrized line bundles, we also prove a rigidity theorem for the set height zero points of polarized algebraic dynamical systems over function fields. In the special case of a global field, this gives a rigidity theorem for preperiodic points, generalizing previous work of Mimar, Baker–DeMarco, and Yuan–Zhang.

To Dad.

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Chapter 1

Introduction

Intersection theory provides a robust and consistent framework to quantify how geometric objects meet each other, which can be thought of as a higher-dimensional version of how the zeros of a polynomial in one variable are counted with multiplicities. Classical intersection theory is well understood for projective varieties over algebraically closed fields, such as the complex numbers. Studying Diophantine equations, however, where only rational coefficients and solutions are allowed, requires arithmetic intersection theory, which is currently less developed. Here we prove the Hodge-index theorem, a fundamental result from classical intersection theory describing the signature of the symmetric bilinear intersection form, in the setting of function fields. Function fields are a classification of fields with many similarities to the rational numbers, and historically the study of each has helped inform the other.

As an important application, arithmetic intersections are used to define height functions, powerful gauges of the complexity of rational numbers, and thus the Hodge-index theorem informs our understanding of heights. Here we derive heights which scale consistently under forward iteration of dynamical systems. We then consider dynamical systems with two different functions defining how the position of each point evolves over time, and show that the sets of points under each which end up in periodic cycles either almost never overlap, or are exactly equal. We expect this to aid in understanding families of dynamical systems, and more broadly, it adds to the growing analogy between arithmetic dynamical systems and abelian varieties.

In order to better motivate the results of this work we begin in reverse order, starting with the dynamics applications of Chapter 4, and then proceeding to explain why their study requires the Hodge-index theorem proven in Chapter 3. This introduction is aimed at a more general mathematical audience, while the following chapters become more precise and technical.

1.1 Dynamical systems

A dynamical system is a geometric object X with a map $f : X \to X$ to itself, all defined over some field K (for example K could be $\mathbb{Q}$ or $\mathbb{C}$). What makes this setting interesting is that since f takes X back to itself, we can start with a point $x \in X$, map it to $f(x) \in X$, and then plug this result back into f. Thus, we can iterate f over and over, writing

$$f^n := \underbrace{f \circ \cdots \circ f}_{n},$$

and study how $f^n(x)$ progresses as we increase n. One can then ask questions such as: Which points in X are fixed by f? For which does the sequence $\{f^n(x)\}_{n\in\mathbb{N}}$ converge to a limit? For which does it diverge?

Even for seemingly simple examples these questions can become quite difficult. Consider, for example, the dynamical system $f_c : \mathbb{C} \to \mathbb{C}$ given by $x \mapsto x^2 + c$, for some parameter $c \in \mathbb{C}$. If we start at $x = 0$ and iterate f_c, for which $c \in \mathbb{C}$ does this remain bounded? The subset of $\mathbb{C}$ containing such c is called the Mandelbrot set, a set with infinitely complicated fractal boundary, which has become famous as an example of mathematical beauty and complexity after its introduction to the public in 1985 on the cover of *Scientific American* [16].

Two particular sets of interest are the set of periodic points, defined as points which eventually return to themselves:

$$\mathrm{Per}(f) := \{x \in X(\overline{K}) | f^n(x) = x \text{ for some } n \geq 1\},$$

and the set of preperiodic points, defined as points which become periodic:

$$\mathrm{Prep}(f) := \{x \in X(\overline{K}) | f^n(x) = f^m(x) \text{ for some } n > m \geq 0\}.$$

By convention, we set $f^0(x) := x$, so that the set of preperiodic points clearly contains all periodic points. Note that we make our definitions of periodic and preperiodic over the algebraic closure $\overline{K}$ of K, so that we don't need to worry about whether points showing up as solutions to algebraic equations exist in K or not.

While the study of dynamics over the complex numbers dates back to at least the early 1900s, the arithmetic dynamics perspective focused on here originated in the mid 1980s and early 1990s with ideas of Silverman, Zhang, and others [48, 38, 57]. We will study dynamical systems $f : X \to X$ defined over $\mathbb{Q}$ and other similar fields, where the integral and rational structure will be very important. As a technical condition which imposes additional structure, we also require a polarization, defined as a line bundle (i.e. vector bundle of rank one) L on X whose pullback f^*L is isomorphic to $L^{\otimes q}$ for some $q > 1$.

One of the primary ideas in arithmetic dynamics is that there are close parallels between arithmetic dynamical systems and abelian varieties. Even on a general variety which lacks the structure of an abelian group, the dynamical orbit $\mathrm{Orb}_f(x) := \{f^n(x)\}_{n\in\mathbb{N}}$ of a point $x \in X$ often behaves similarly to the group-theoretic orbit of a point $x \in A$ on an abelian variety

$$\mathrm{Orb}_A(x) := \left\{ \underbrace{x + x + \cdots + x}_{n} \right\}_{n\in\mathbb{N}} .$$

To start, observe that the multiplication by n map on an abelian variety is an example of a polarized algebraic dynamical system:

Example 1.1. *Let A be an abelian variety defined over any field K, and consider the endomorphism*

$$[n] : A \to A.$$

This is an algebraic dynamical system, and the set of peperiodic points, $\mathrm{Prep}([n])$ *is exactly the set of torsion points, $A(\overline{K})_{tors} = \{x \in A(\overline{K}) | [m]x = 0$ for some $m \in \mathbb{N}, m \neq 0\}$. It is in part for this reason that it is typically more natural to study the set of preperiodic points instead of the set of periodic points in dynamics.*

Recall the fact, which can be proven using the theorem of the cube [39], that for any line bundle L on A, we have

$$[n]^*L \xrightarrow{\sim} L^{\otimes\left(\frac{n(n+1)}{2}\right)} \otimes ([-1]^*L)^{\otimes\left(\frac{n(n+1)}{2}\right)}.$$

Now suppose L is a symmetric (meaning $[-1]^*L \xrightarrow{\sim} L$) and ample. Then L provides a polarization, as $[n]^*L \xrightarrow{\sim} L^{n^2}$.

Note: It is tempting to think that an antisymmetric line bundle (one for which $[-1]^*L \xrightarrow{\sim} L^{-1}$, and hence $[n]^*L \xrightarrow{\sim} L^n$) provides an additional example of a polarization, however antisymmetric line bundles are necessarily algebraically trivial, and thus not ample.

The above example motivates many of the results, questions, and conjectures in arithmetic dynamics. We give an additional example of non-abelian polarized arithmetic dynamical systems.

Example 1.2. *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a morphism of degree $d > 1$. The line bundle $\mathcal{O}(1)$, consisting of degree one homogeneous polynomials, is a polarization, as*

$$f^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}(d).$$

In general it is very difficult to describe the set of preperiodic points of f, however, in certain cases we can say more:

1. *(Square map) Suppose f is the square map, given in projective coordinates by*

$$(x_0 : \ldots : x_n) \mapsto (x_0^2 : \ldots : x_n^2).$$

Then $\mathrm{Prep}(f)$ *is the set of points in* $\mathbb{P}^n(\overline{K})$ *which can be written such that each* x_i *is either zero or a root of unity.*

2. *(Lattès map) Let E be an elliptic curve over K, and let $\pi : E \to \mathbb{P}^1$ be the projection map onto the x-coordinate. Any endomorphism $\phi : E \to E$ induces a map $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f \circ \pi = \pi \circ \phi$. Then*

$$\mathrm{Prep}(f) = \pi\left(E(\overline{K})_{tors}\right).$$

While it may be hard in general to exactly describe the set of preperiodic points, some things can be said. Just as it's known that $A(\overline{K})_{tors}$ is Zariski dense in A, Fakhruddin [17], Theorem 5.1, shows that $\mathrm{Prep}(f)$ is always Zariski dense in X.

We now introduce the main theorem in arithmetic dynamics proven here. The theorem should be viewed as a rigidity theorem for preperiodic points, essentially saying that if f and g are two polarized algebraic dynamical systems on the same variety X, their sets of preperiodic points are either identical or very different from each other.

Such a theorem is trivial the case of an abelian variety, as then the options for the group $\mathrm{End}(A)$ are well understood by the theory of complex multiplication, and all have the same set of preperiodic points, namely $A(\overline{K})_{tors}$. When X is not an abelian variety, much less is known about $\mathrm{End}(X)$, but this theorem says, roughly, that we still see only a limited set of distict options for $\mathrm{Prep}(f)$.

Here we state our theorem for global fields (which includes $\mathbb{Q}$ and extensions by adjoining roots of polynomials, as well as function fields of curves over finite fields), but we in fact prove a more general theorem on the rigidity of height zero points over any function field; see Theorem 4.7 for a precise statement of the full result. The similarities and differences between different kinds of fields are also discussed further in Section 1.4 of this introduction, with further comment on arithmetic dynamics over other fields in Chapter 5.

Theorem 1.3. *Let X be a projective variety over a global field K, and let f and g be two polarized algebraic dynamical systems on X. Suppose* $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is Zariski dense in X. Then*

$$\mathrm{Prep}(f) = \mathrm{Prep}(g).$$

Remark. *Note that Zariski dense is a much weaker notion than being dense with respect to the analytic topology (i.e. that defined by the usual distance metric on $\mathbb{C}$). For example, any infinite collection of points on a one dimensional variety is Zariski dense. More generally,* *being Zariski dense is equivalent to not being contained in a finite union of proper (lower dimensional) subvarieties.*

Such a theorem has been proven previously for rational functions on $\mathbb{P}^1(\overline{\mathbb{Q}})$ by Mimar [33], $\mathbb{P}^1(\mathbb{C})$ by Baker and DeMarco [4], and for polarized algebraic dynamical systems over number fields by Yuan and Zhang [55], and the contribution of this work is proving the result over function fields. This, as well as the main results of Chapter 3 have been recently submitted for publication; see [4].

1.2 Preperiodic points, heights, and metrized line bundles

One of the primary tools of this work, and of arithmetic dynamics in general, is the theory of heights. Height functions measure, in a heuristic sense, the arithmetic complexity of numbers. As a simple, but nonetheless useful example, we can define the so-called naïve height on $\mathbb{Q}$:

$$h_{\text{naïve}}\left(\frac{a}{b}\right) := \log\left(\max(|a|, |b|)\right),$$

where $a, b \in \mathbb{Z}$ have no common factor. Already we see three important observations:

1. $h_{\text{naïve}}$ is always non-negative.
2. The only rational numbers with height zero are ± 1.
3. Given any bound $B > 0$, there are only finitely many rational numbers with height less than B.

Generalizing this, let $M_{\mathbb{Q}}$ be the set of places, i.e. equivalence classes of absolute values on $\mathbb{Q}$. This consists of the usual absolute value $|\cdot|$ which simply changes the sign of negative numbers, often specified in this context by the notation $|\cdot|_\infty$, and all of the p-adic absolute values $|\cdot|_p$, which measure how divisible by p a rational number is. Let x be a point in the projective space $\mathbb{P}^n_{\mathbb{Q}}$ with projective coordinates $(x_0 : \ldots : x_n)$, and define the height of x to be

$$h(x) := \sum_{\nu \in M_{\mathbb{Q}}} \max\left(|x_0|_\nu, \ldots, |x_n|_\nu\right).$$

Now for any projective variety X, a very ample line bundle L on X can be used to define a projective embedding $\phi_L : X \hookrightarrow \mathbb{P}^n$, and then using the height above, we get a height function

$$h_L := h \circ \phi_L : X(\overline{K}) \to \mathbb{R}.$$

The process of defining this for any line bundle is called Weil's height machine; more details can be found [7]. The same definitions can be made over other fields K besides $\mathbb{Q}$ provided the set of places M_K still behaves similarly; this is returned to briefly in Section 1.4.

Given a height function h_L on a dynamical system polarized by L, we can define a dynamical canonical height function

$$h_f(x) := \lim_{n\to\infty} \frac{h(f^n(x))}{q^n}$$

where as before $f^*L \xrightarrow{\sim} L^{\otimes q}$. This has the important property that

$$h_f(f(x)) = qh_f(x).$$

This height depends on the polarization L as well as the endomorphism f, but some of its important properties do not. For example, we have the following lemma, which forms one of the fundamental connections between the theory of heights and dynamics. See also Lemma 4.8 for more general fields.

Lemma 1.4. *Let $f : X \to X$ be a polarized algebraic dynamical system defined over a global field K. Then*

$$\mathrm{Prep}(f) = \{x \in X(\overline{K}) | h_f(x) = 0\}.$$

Thus, we can prove Theorem 1.3 by showing that the height functions h_f and h_g agree, as then the lemma says f and g will have the same sets of preperiodic points.

Before being able to accomplish this, however, we require a further refinement of our height functions, and this is where the technical heart of the work enters, using Arakelov theory to define a more powerful theory of heights via arithmetic intersections.

In Chapter 2 we formally introduce the theory of adelic metrized line bundles and their intersections. For now, it suffices to say that an adelic metrized line bundle $\overline{L}$ is a line bundle L (in the usual sense) on a projective variety X, with the additional structure of a metric $||\cdot||_\nu$ measuring the sections s of L on each of the localizations X_ν of X. We allow ν to range over all places of M_K defined earlier.

This has two main advantages. One, we have a working arithmetic intersection theory of adelic metrized line bundles (for those unfamiliar with intersection theory, this gives a way to count, with the correct multiplicities, the intersections of the subvarieties cut out by the zeros of sections of these line bundles), generalizing the work of Arakelov [1, 2] and Gillet–Soulé [20, 21]. This can handle integral models, and thus say things about integral points in ways that classical intersection theory cannot.

Second, the extra structure provided by the metric allows heights to be defined directly as intersections. In this way, we recover the theory of Weil heights, but also get new heights such as the Faltings height, which does not correspond to any Weil height [8]. These definitions also work to define not just heights of points, but heights of positive dimensional closed subvarieties as well. One can then compare the height of a variety with those of the points on it. We can now sketch the proof of Theorem 1.3 in four main steps.

1. Prove Lemma 1.4, showing that a point being preperiodic is the same as having dynamical height zero.

2. Show Theorem 4.2, which says that we can create dynamically equivariant metrics so that we can reproduce the dynamical canonical heights h_f and h_g with intersections of metrized line bundles $\overline{L}_f$ and $\overline{M}_g$ respectively.

3. By step (1) and the hypothesis of the theorem, X contains a dense set of height zero points with respect to both height functions h_f and h_g. Show that this means that X itself has height zero with respect to the height defined by the metrized line bundle $(\overline{L}_f + \overline{M}_g)$. This is a consequence of Zhang's successive minima, Proposition 2.8.

4. Use the arithmetic Hodge-index theorem, Theorem 3.1, to conclude from (3) that $\overline{L}_f$ and $\overline{M}_g$ are equal up to a factor which does not affect their height zero points, thus proving the theorem.

We explain what the Hodge-index theorem says, and why it is the crucial step of the proof in the following section.

1.3 The Hodge-index theorem

Let X be a projective surface defined over an algebraically closed field $K = \overline{K}$. Its Picard group $\mathrm{Pic}(X)$ consists of isomorphism classes of line bundles, or equivalently linear equivalence classes of divisors (i.e. formal sums of curves on X). The intersection pairing is a map

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{R},$$

defined for curves intersecting properly as the number of points of intersection, counted with multiplicities, and defined generally using rational equivalence and a moving lemma to move curves into such a position.

Call an element of $\mathrm{Pic}(X)$ *numerically trivial* if it has intersection number zero with all of $\mathrm{Pic}(X)$, and call two elements *numerically equivalent* if their difference is numerically trivial.

Then if we extend scalars to $\mathbb{R}$ and mod out by numerical equivalence, the intersection pairing becomes a symmetric bilinear form on the $\mathbb{R}$ vector space

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \Big/ \{\text{numerical equivalence}\}.$$

This is finite dimensional by the Néron-Severi theorem [40], thus by Sylvester's Law [28], the intersection pairing is defined by its signature, or index. The Hodge-index theorem says that this form is non-degenerate, and has signature $+1, -1, -1, \ldots, -1$. Often this is stated in a slightly different, but equivalent form, presented here as it generalizes more easily to higher dimension, and matches the form of the main theorem of this work.

Theorem 1.5 (Hodge-index for projective surfaces)**.** *Let L be an ample line bundle on X, and M any line bundle such that the intersection $M \cdot L = 0$. Then the self intersection*

$$M^2 \leq 0.$$

Further, $M^2 = 0$, if and only if M is numerically trivial.

The Hodge-index theorem, and other fundamental equations such as the Riemann-Roch theorem and Noether formula form the primary tools with which one studies intersections over projective schemes. In the mid-1970s, however, Arakelov developed an arithmetic intersection theory for arithmetic surfaces. Given a curve X defined over $\mathbb{Q}$, an integral model $\mathcal{X}$ is an arithmetic surface defined over $\mathbb{Z}$, which encodes arithmetic information, such as the localization at each prime, geometrically. Arakelov's motivation for developing such a theory, and indeed the potential that Faltings [18], and then Vojta [53] eventually realized, was that it could provide the necessary tools to prove Mordell's 1922 conjecture:

Theorem 1.6 (Mordell Conjecture, proven by Faltings in 1983)**.** *Let X be a smooth projective curve defined over a number field K. If the genus of X is greater than one, $X(K)$ is finite.*

In fact the specific idea for Arakelov's Theory comes from Paršhin's [42] 1966 proof of a version of the Mordell conjecture for function fields. Paršhin derives the main bounds on the number of rational points using intersection theory on the projective models which arise in the function field setting, and Arakelov set out to make these same arguments hold for number fields.

The difficulty when working with number fields is that the integral model $\mathcal{X}$ is no longer a projective scheme, so the classical intersection theory results do not apply. Thus the arithmetic Hodge-index theorem is an important step towards making Arakelov's intersection theory actually useful. The Hodge-index theorem for arithmetic surfaces was proven in 1984, independently by both Hriljac [25] and Faltings [19], who in the same paper also proved the arithmetic Riemann-Roch and Noether formulae.

Instead of using the Hodge-index theorem to prove properties about height functions defined by arithmetic intersections, as done in this work, Faltings and Hriljac show that the arithmetic intersection pairing is related to the Néron–Tate height, a height with previously known properties, and then derive the Hodge-index theorem from this fact. This result is re-proven in Chapter 3, and forms the base case of the proof of Theorem 3.1.

The results of this work generalize that of Faltings and Hriljac in several ways, with intermediate steps accomplished by multiple authors. Moriwaki [36] in 1996 generalized from arithmetic surfaces to arithmetic varieties in arbitrary dimension. In 2017, Yuan and Zhang proved a Hodge-index Theorem for intersections of adelic metrized line bundles (those introduced above) on varieties defined over number fields. Since an integral model induces an adelic metric (see Chapter 2 for details), this subsumes Moriwaki's work, and allows one to work in a more general, flexible setting. In their paper, Yuan and Zhang conjectured that a similar result should hold over function fields. This work restates their conjecture in a more natural way and then proves it.

We state here for comparison's sake with Theorem 1.5 above a version of this general Hodge-index theorem. The precise version can be found in Chapter 3. This theorem also appears in the author's paper [10], recently submitted for publication.

Theorem 1.7. *Let X be a projective variety with dimension n, defined over either a one-dimensional function field or a number field. Let $\overline{M}$ and $\overline{L}$ be adelic metrized line bundles on X, and let $\overline{L}$ be arithmetically positive. If $M \cdot L^{n-1} = 0$, then*

$$\overline{M}^2 \cdot \overline{L}^{n-1} \leq 0,$$

and equality holds if and only if the height function $h_{\overline{M}}$ defined by $\overline{M}$ is constant.

1.4 Number fields and function fields

There are several different classes of fields which show up in the results of this work as well as the related results discussed above. The primary groupings are:

- **Number Fields.** This class includes the rationals $\mathbb{Q}$, and all extensions of the rationals by adjoining roots of polynomials. For example $\mathbb{Q}(\sqrt[3]{2})$ or $\mathbb{Q}(i)$.
- **Function fields of transcendence degree one**. Let k be any field, called the field of constants, and let B be a smooth projective curve defined over k. A function field of transcendence degree one is the field of rational functions $K = k(B)$ on B. Examples

include $\mathbb{C}(t)$, (which is the function field of $\mathbb{P}^1_{\mathbb{C}}$), and $\mathbb{F}_p(x, y)$, where $y^2 = x^3 - 1$ (which is the function field of an elliptic curve defined over the finite field $\mathbb{F}_p$). We allow function fields of any characteristic. Transcendence degree one corresponds to the requirement that B have dimension 1; higher transcendence degree is discussed briefly in Chapter 5.

- **Global fields.** This grouping includes all number fields, as well as all function fields of curves defined over finite fields. For example $\mathbb{Q}(i)$ and $\mathbb{F}_p(x, y)$ above, but not $\mathbb{C}(t)$.

There's a loose yet pervasive analogy in number theory that results and conjectures which hold over number fields should have similar parallels over function fields, and understanding one setting can often yield new ideas and techniques in the other. The main basic reason of this analogy is the fact that both have one-dimensional coordinate rings ($\mathbb{Z}$ or $\mathcal{O}_K$ in the number field case, (rings like $\mathbb{C}[t]$ or $\mathbb{F}_q[x, y]$ in the function field examples above), and thus have sets of primes and sets of places M_K which behave similarly. In particular, we can define height functions similarly in both settings.

These coordinate rings are also one of the primary differences. For a number field K, the set of places (equivalence classes of absolute values) consists of non-Archimedean (also called *finite*) places, which correspond to primes and thus geometric points of $\operatorname{Spec} \mathcal{O}_K$, but also Archimedean (also called *infinite*) places, corresponding to extensions of the absolute value $|\cdot| = |\cdot|_\infty$ on $\mathbb{Q}$. Arakelov theory formally completes $\operatorname{Spec} \mathcal{O}_K$ to include these places when defining local intersection numbers, but there is no purely geometric way to complete $\operatorname{Spec} \mathcal{O}_K$ to a proper scheme which includes these places as closed points. For function fields, since the curve B is defined to be projective (thus proper) to begin with, there are no Archimedean places, and all places correspond to closed points of B.

It is this difference–the fact that places can be understood purely geometrically for function fields but not for number fields–that makes several results harder in the number field case. The Mordell Conjecture was proven by Manin [32] over function fields in 1963, but not until 1983 by Faltings [18] in the number field case. On the other hand Paršhin's 1966 proof for function fields manages to prove the existence of a bound on the height of the points on a curve of genus two or more (i.e. the *effective* Mordell conjecture), a result that is still only conjectural in the number field setting. In fact, the effective Mordell conjecture, and even the ABC conjecture, can be shown to depend on proving a certain canonical class inequality, called the Bogomolov-Miyaoka-Yau inequality, which is known for classical intersection theory, but still only conjectural for arithmetic intersections over number fields. See for example the appendix by Vojta in [29].

The existence of Archimedean places can also be an advantage. Consider, for example, the Bogomolov conjecture, proven for number fields by Ullmo [51] in the specific case of curves embedded in their Jacobian, and by Zhang [58] for general abelian varieties:

Conjecture 1.8 (Bogomolov Conjecture)**.** *Let K be a number field or a function field, and A an abelian variety defined over K. Let X be a proper subvariety of A. Then either there exists an $\epsilon > 0$ such that X does not contain a dense set of points with canonical height $\leq \epsilon$, or X is a translate of an abelian subvariety by a point of height zero.*

In the function field case, this is proven for all abelian varieties except for those with everywhere good reduction, primarily by Gubler [23], and with additional results summarized by Yamaki [54]. The main discrepancy is that A always has a uniformization as a non-trivial algebraic torus at the Archimedean places, but such a uniformization exists at non-Archimedean places only when A has bad reduction at that place. Thus the arguments of Zhang at the Archimedean place can be adapted to prove the same result in the function field case except when A has everywhere good reduction. In that case the conjecture is still open.

There is one other difficulty which arises for function fields but not for number fields. This stems from the fact that the curve B used to define a function field is itself defined over some smaller field. It is possible, then, that a variety X defined over K is actually defined over k as well. If this happens to be the case, the places of K, in essence, don't notice it. Thus, we must keep track of a whole class of varieties which slip through the arithmetic structure and don't behave as expected. In stating the Mordell Conjecture for function fields, for example, one must exclude so-called *isotrivial* curves, which are exactly curves who can be defined over k. Section 2.4 of Chapter 2 introduces Chow's K/k function field trace and image in order to quantify this behavior.

Thus, in comparing this work on the Hodge-index theorem to that of Yuan and Zhang [55], both settings present unique advantages and difficulties. The setting of adelic metrized line bundles, first introduced by Zhang [59], and as opposed to the arithmetic varieties used by Moriwaki, in part is designed to minimize the differences in notation between the two settings, and uses similarly defined metrics at both the Archimedean and non-Archimedean places. Once one begins working in this setting, however, there are separate complex geometry arguments required at the Archimedean places in the work of Yuan and Zhang which don't show up in the function fields proofs here. On the other hand, Yuan and Zhang do not need to separate out the objects which live over a field of constants at every step as done here, and thus have no need for the trace and image considerations which appear here throughout. We also note here that Theorem 3.1 here is stated slightly more naturally than the conjecture made by Yuan and Zhang, as their statement required a non-canonical isomorphism between the function field trace and image.

The above distinctions between function fields and number fields are of primary importance for the intersection theory of Chapter 3. Chapter 4, however, primarily uses the global field distinction when stating certain dynamics results. Recall property (3) of the naïve height noted above, that only finitely many rational numbers have height under any given bound. For more general heights, this is called the Northcott principle [41], but it is only true for heights on varieties defined over global fields. This is because for global fields, the residue fields (fields $\mathbb{Z}/p\mathbb{Z}$ in the number field case, and finite extensions of $\mathbb{F}_q$ in the global function field case) are finite. For a general function field, all of k will have height zero, thus if k is infinite, the Northcott principle must fail.

The Northcott principal is a simple but necessary step in proving that points of height zero are all preperiodic. Thus, Theorem 4.7 about points of height zero can be proven over any transcendence degree one function field, but to make it a statement about preperiodic points as in Theorem 1.3, one must work over a global field.

1.5 Outline of Paper and sketch of methods

Definitions and basic properties of adelic metrized line bundles and Chow's K/k-image and trace are recalled in Chapter 2. Additionally, this chapter includes technical lemmas, such as the existence of flat metrics, which will be needed throughout the paper.

Our main Hodge-Index theorem and its $\mathbb{R}$-linear variant are fully stated and proven in Chapter 3. We begin with the case of X being a curve. Decomposing adelic metrized line bundles into flat and vertical pieces, and addressing intersections of the vertical parts using the local Hodge-Index Theorem of [55], Theorem 2.1, we reduce to the case of flat metrics. Then, following the methods of Faltings [19] and Hriljac [25], we relate the intersection pairing to the Neŕon-Tate height pairing on the Jacobian variety of X, and complete the result for curves using properties of heights on the Jacobian.

Next we prove the inequality part of Theorem 3.1 by induction on the dimension of X, using a Bertini-type theorem of Seidenberg [45] to find sections which cut out nice subvarieties of X. Along the way we prove a Cauchy-Schwarz inequality for this intersection pairing. Theorem 3.2 and the equality part of Theorem 3.1 are then also proved by induction, where we again decompose into flat and vertical metrics and must show that the K/k-trace and image functors behave nicely when restricted to a subvariety. This is much harder than the inequality, however. For the inequality, we write each metrized line bundle as a limit of model metrics, and prove the result for model metrics, thus getting the same inequality on their limit. We can write the same limit in the equality case, but we cannot assume that the same equality hypothesis holds for the model metrics, and must argue by other means. Finally, Theorem 3.3 is easily deduced from Theorem 3.1 and its proof.

Chapter 4 proves the application of our result to polarized algebraic dynamical systems. We first describe and prove the existence of admissible metrics for a given polarized algebraic dynamical system, which generalize flat metrics. Next we show that intersecting with an admissible metrized line bundle can be used to define a height function on X which is zero exactly at the preperiodic points of the system. This transforms the rigidity statement on preperiodic points into a statement of the equality of two different height functions defined by intersections, which is proved using the Hodge-index Theorem.

Chapter 5 demonstrates a useful corollary of the main results proven here, and then discusses what can still be said about preperiodic points over larger fields where Lemma 1.4 does not hold, citing a number of related results and describing what is still not known. To finish, we discuss what the results of this work might look like over function fields of transcendence degree greater than one, and suggest how this might be done in future work.

Chapter 2

Preliminaries

Here we introduce the definitions, basic properties, and lemmas which will be needed throughout the paper. The core theory used in this paper is built on local intersection theory as developed by Gubler [22, 24], Chambert-Loir [11], Chambert-Loir—Thuillier [12], and Zhang [59]. More generally, one can find an introduction to Arakelov theory in [34, 30, 49].

2.1 Adelic metrized line bundles

Let K be a complete non-Archimedean field with non-trivial absolute value $|\cdot|$. Denote the valuation ring of K by

$$K^\circ := \{a \in K : |a| \leq 1\},$$

and its maximal ideal

$$K^{\circ\circ} := \{a \in K : |a| < 1\},$$

so that $\widetilde{K} := K^\circ/K^{\circ\circ}$ is the residue field.

Let X be a variety over K and denote by X^{an} its Berkovich analytification as in [6]. For $x \in X^{an}$, write $K(x)$ for the residue field of x. Given a line bundle L on X, it also has an analytification L^{an} as a line bundle on X^{an}.

Definition 2.1. (Metrized line bundle) *A continuous metric $||\cdot||$ on L consists of a $K(x)$-metric $||\cdot||_x$ on $L^{an}(x)$ for every $x \in X^{an}$, where this collection of metrics is continuous in the sense that for every rational section s of L, the map $X^{an} \to \mathbb{R}$ defined by $x \mapsto ||s(x)||_x$ is continuous away from the poles of s. We call L with a continuous metric a metrized line bundle and denote this by $\overline{L} = (L, ||\cdot||)$.*

An important example of a continuous metric is a *model metric*: Let $\mathcal{X}$ be a model of X over K°, i.e. a projective, flat, finitely presented, integral scheme over $\operatorname{Spec} K^\circ$ whose generic fiber

$\mathcal{X}_K$ is isomorphic to X, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ whose generic fiber $\mathcal{L}_K$ is isomorphic to L. Then we can define a continuous metric on L by specifying that for any trivialization $\mathcal{L}_\mathcal{U} \xrightarrow{\sim} \mathcal{O}_\mathcal{U}$ on an open set $\mathcal{U} \subset \mathcal{X}$ given by a rational section ℓ, we have $||\ell(x)||_x = 1$ for any x reducing to $\mathcal{U}_{\widetilde{K}}$ in the reduction $\widetilde{X}$ over $\widetilde{K}$. We now define several important properties and notations.

Definition 2.2. *Let $\overline{L} = (L, ||\cdot||)$ and $\overline{M}$ be metrized line bundles on X.*

1. *A model metric is* nef *if it is given by a relatively nef line bundle on $\mathcal{X}$.*
2. *Call both $\overline{L}$ and $||\cdot||$* nef *if $||\cdot||$ is equal to a uniform limit of nef model metrics.*
3. *$\overline{L}$ is* arithmetically positive *if it is nef and L is ample.*
4. *$\overline{L}$ is* integrable *if it can be written as $\overline{L} = \overline{L}_1 - \overline{L}_2$ with $\overline{L}_1$ and $\overline{L}_2$ nef.*
5. *$\overline{M}$ is $\overline{L}$-*bounded *if there exists a positive integer m such that $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are both nef.*
6. *$\overline{L}$ is* vertical *if it is integrable and $L \cong \mathcal{O}_X$*
7. *$\overline{L}$ is* constant *if it is isometric to the pull-back of a metrized line bundle on* $\operatorname{Spec} K$
8. *$\widehat{\mathcal{P}\mathrm{ic}}(X)$ is defined to be the category of integrable metrized line bundles, with morphisms given by isometries.*
9. *$\widehat{\mathrm{Pic}}(X)$ is defined to be the group of isometry classes of integrable metrized line bundles.*

Remark. *When we say a line bundle is relatively ample or nef, we always mean with respect to the structure morphism, here $\mathcal{X} \to \operatorname{Spec} K^\circ$. A concise discussion of the important aspects of relative amplitude and nefness can be found in [31], Chapter 1.7.*

We also have a local intersection theory for metrized line bundles on X. Let Z be a d-dimensional cycle on X, let $\overline{L}_0, \dots, \overline{L}_d$ be integral metrized line bundles on X, and $\ell_0, \dots, \ell_d$ sections of each respectively such that

$$\left(\bigcap_i |\mathrm{div}(\ell_i)|\right) \cap |Z| = \emptyset,$$

where $|Z|$ means the underlying topological space of the cycle Z. Then Z has a local height $\widehat{\mathrm{div}}(\ell_0) \cdots \widehat{\mathrm{div}}(\ell_d) \cdot [Z]$ with the following properties:

1. The local height is linear in $\widehat{\mathrm{div}}(\ell_i)$ and Z.

2. For fixed sections, it is continuous with respect to the metrics.

3. When $\overline{L}_i$ has a model metric given by $\mathcal{L}_i$, the height is given by classical intersections:

$$\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Z] = \mathrm{div}_{\mathcal{X}}(\ell_0)\cdots\mathrm{div}_{\mathcal{X}}(\ell_d)\cdot[\mathcal{Z}],$$

   where $\mathcal{Z}$ is the Zariski closure of Z in $\mathcal{X}$.

4. If the support of $\mathrm{div}(\ell_0)$ contains no component of Z, there is a measure $c_1(\overline{L}_1)\cdots c_1(\overline{L}_d)\delta_Z$ on X^{an} which allows the local height to be computed inductively as

$$\begin{aligned}\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Z] &= \widehat{\mathrm{div}}(\ell_1)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[\mathrm{div}(\ell_0)\cdot Z]\\ &\qquad - \int_{X^{an}} \log||\ell_0(x)||_x c_1(\overline{L}_1)\cdots c_1(\overline{L}_d)\delta_Z.\end{aligned}$$

   This notation is meant to suggest that $c_1(\overline{L}_i)$ should be thought of as the arithmetic version of the classical Chern form $c_1(L_i)$.

5. If $L_0|_{Z_j} \cong \mathcal{O}_{Z_j}$ and $c_1(L_1)\cdots c_1(L_d)\cdot[Z_j] = 0$ for every irreducible component Z_j of Z, then this pairing does not depend on the choice of sections, so we may simply write

$$\overline{L}_0\cdots\overline{L}_d\cdot Z = \widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[Z].$$

When $Z = X$, we typically omit Z in all of the above notation.

We now move to the global theory, which is built from the local theory at each place, discussing first models and then adelic metrized line bundles, which will be related similarly to the above. Return to the original setting, where k is any field, B is a smooth projective curve over k, $K = K(B)$ is its function field, and $\pi : X \to \mathrm{Spec}(K)$ is a normal, integral, projective variety.

Let $\mathcal{X}$ be a model for X, meaning that $\mathcal{X} \to B$ is integral, projective, and flat, and the generic fiber $\mathcal{X}_K$ is isomorphic to X. Given an integral subvariety $\mathcal{Y}$ of dimension $d+1$ in $\mathcal{X}$ and line bundles $\mathcal{L}_0, \ldots, \mathcal{L}_d$ on $\mathcal{X}$ each with a respective section $\ell_0, \ldots, \ell_d$ such that their common support has empty intersection with $\mathcal{Y}_K$, the arithmetic intersection pairing on $\mathrm{Pic}(\mathcal{X})$ is defined locally as

$$\mathcal{L}_0\cdots\mathcal{L}_d\cdot\mathcal{Y} := \widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[\mathcal{Y}] := \sum_\nu \left(\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[\mathcal{Y}]\right)_\nu,$$

where ν ranges over the closed points (places) of B, and

$$\left(\widehat{\mathrm{div}}(\ell_0)\cdots\widehat{\mathrm{div}}(\ell_d)\cdot[\mathcal{Y}]\right)_\nu$$

means the local intersection number after base-change to the complete field K_ν. As the notation suggests, this doesn't depend on the choice of sections. Again we typically drop $\mathcal{Y}$ in the notation if $\mathcal{Y} = X$, and when $\mathcal{X}$ is one dimensional, we call $\widehat{\deg}(\mathcal{L}) := \mathcal{L}_0 \cdot \mathcal{X}$ the arithmetic degree of $\mathcal{L}_0$.

Note importantly that this arithmetic intersection theory for $\mathcal{X} \to B$ is not the same as the classical intersection theory given by viewing $\mathcal{X}$ as a variety over the field k.

Given a line bundle L on X we call a line bundle $\mathcal{L}$ on $\mathcal{X}$ a model for L provided that $\mathcal{L}_K \cong L$. For each place ν of B, completing with respect to ν induces a model over K_ν° and a model metric $||\cdot||_{\mathcal{L},\nu}$ of $L_{K_\nu}^{an}$ on $X_\nu^{an} := X_{K_\nu}^{an}$. We can then define

Definition 2.3. *The collection $||\cdot||_{\mathcal{L},\mathbb{A}} = \{||\cdot||_{\mathcal{L},\nu}\}_\nu$ of continuous metrics for every place ν of B given by $(\mathcal{X}, \mathcal{L})$ is called a model adelic metric on L. More generally, an adelic metric $||\cdot||_{\mathbb{A}}$ on L is a collection of continuous metrics $||\cdot||_\nu$ of $L_{K_\nu}^{an}$ on X_ν^{an} for every place ν, which agrees with some model adelic metric at all but finitely many places. A line bundle on X with an adelic metric is called an adelic metrized line bundle, and is denoted $\overline{L} = (L, ||\cdot||_{\mathbb{A}})$.*

We extend our local definitions of properties of metrized line bundles to the global case.

Definition 2.4. *Let $\overline{L}$ be an adelic metrized line bundle.*

1. *$\overline{L}$ is* nef *if it is equal to a uniform limit of model metrics induced by nef line bundles on models of X.*

2. *$\overline{L}$ is* integrable *if it can be written as $\overline{L} = \overline{L}_1 - \overline{L}_2$, where each $\overline{L}_i$ is nef.*

3. *$\overline{L}$ is* arithmetically positive *if L is ample and $\overline{L} - \pi^*\overline{N}$ is nef for some adelic metrized line bundle $\overline{N}$ on $\operatorname{Spec} K$ with $\widehat{\deg}(\overline{N}) > 0$*

4. *$\overline{M}$ is* $\overline{L}$-bounded *if there exists a positive integer m such that $m\overline{L} + \overline{M}$ and $m\overline{L} - \overline{M}$ are both nef.*

5. *$\overline{L}$ is* vertical *if it is integrable and $L \cong \mathcal{O}_X$*

6. *$\overline{L}$ is* constant *if it is isometric to the pull-back of a metrized line bundle on $\operatorname{Spec} K$*

7. *$\widehat{\mathcal{P}\mathrm{ic}}(X)$ is defined to be the category of integrable metrized line bundles, with morphisms given by isometries.*

8. *$\widehat{\mathrm{Pic}}(X)$ is defined to be the group of isometry classes of integrable metrized line bundles.*

Remark. *We need not fix a model for X when making these definitions; it is not a problem to take a uniform limit of model metrics coming from different models for both X and L*

Remark. *In the definition of arithmetically positive, we've thus-far only defined the arithmetic degree in the model case, but every adelic metrized line bundle on* $\operatorname{Spec} K$ *has a model metric, so we may use that definition. The definition is also resolved in the following material.*

Remark. *To avoid confusion, note that the preceding definitions are specified globally, and are not always equivalent to requiring that the local property of the same name holds at every fiber. In fact, since relative amplitude (resp. nefness) holds if and only if the restriction to every fiber is ample (resp. nef), each property in the adelic setting implies that the corresponding property holds locally at every place, but the converse is false, as the differences and uniform limits at each place may not come from a global difference or uniform limit.*

Global intersections are defined similarly to the model case, except with the local metrics given explicitly by the adelic metric instead of induced by a model. Given a d-dimensional integral subvariety Z of X, integrable adelic metrized line bundles $\overline{L}_0, \dots, \overline{L}_d$ with respective sections $\ell_0, \dots, \ell_d$ with empty common intersection with Z, their intersection is

$$\overline{L}_0 \cdots \overline{L}_d \cdot Z := \widehat{\operatorname{div}}(\ell_0) \cdots \widehat{\operatorname{div}}(\ell_d) \cdot [Z] = \sum_\nu \widehat{\operatorname{div}}(\ell_0|_{X_\nu}) \cdots \widehat{\operatorname{div}}(\ell_d|_{X_\nu}) \cdot [Z|_{X_\nu}],$$

where again this is independent of the choice of sections. Summing the local induction formula at each place produces a global induction formula: letting ℓ_0 be a rational section of $\overline{L}_0$ whose support does not contain Z,

$$\overline{L}_0 \cdots \overline{L}_d \cdot Z = \overline{L}_1 \cdots \overline{L}_d \cdot (Z \cdot \operatorname{div}(\ell_0)) - \sum_\nu \int_{X_\nu^{an}} \log ||\ell_0(x)||_\nu c_1(\overline{L}_1, \nu) \cdots c_1(\overline{L}_d, \nu) \delta_Z|_{X_\nu}.$$

As before, we drop Z when $Z = X$, and when X is zero-dimensional, we call $\widehat{\deg}(\overline{L}_0) := \overline{L}_0 \cdot X$ the arithmetic degree of $\overline{L}_0$.

Definition 2.5. *An adelic metrized line bundle* $\overline{M}$ *on* X *of dimension* n *is called* numerically trivial *if for any* $\overline{L}_1, \dots, \overline{L}_n \in \widehat{\operatorname{Pic}}(X)$*,*

$$\overline{M} \cdot \overline{L}_1 \cdots \overline{L}_n = 0.$$

Call two adelic metrized line bundles numerically equivalent *if their difference is numerically trivial.*

As an important example, observe that $\pi^* \widehat{\operatorname{Pic}}^0(K) + \widehat{\operatorname{Pic}}_k^{im}(X)$ is numerically trivial; Theorem 3.1 says that this is the entire numerically trivial subgroup of $\widehat{\operatorname{Pic}}(X)$.

2.2 Heights of points and subvarieties

An important application of the intersection theory of adelic metrized line bundles is to define height functions.

Definition 2.6. *Let $\overline{N} \in \widehat{\mathrm{Pic}}(X)$. We define the height of a point $x \in X(\overline{K})$ by*

$$h_{\overline{N}}(x) := \frac{1}{[K(x) : K]} \overline{N} \cdot \tilde{x},$$

where $\tilde{x}$ is the image of x in X via $X_{K(x)} \to X_K = X$.

In addition to the height of a point, we can use $\overline{N}$ to define the height and the essential minimum of a subvariety:

Definition 2.7. *Let $d = \dim Y$. The height of Y with respect to $\overline{N}$ is defined to be*

$$h_{\overline{N}}(Y) := \frac{\left(\overline{N}|_Y\right)^{d+1}}{(d+1)\left(N|_Y\right)^d}$$

and the essential minimum of Y with respect to $\overline{N}$ is

$$\lambda_1(Y, \overline{N}) := \sup_{\substack{U \subset Y \\ open}} \left(\inf_{x \in U(\overline{K})} h_{\overline{N}|_Y}(x). \right)$$

By the successive minima of Zhang [59], Theorem 1.1, and proven in the function field setting by Gubler [23], Theorem 4.1, we can state the following.

Proposition 2.8. *When $\overline{N}$ is nef,*

$$\lambda_1(Y, \overline{N}) \geq h_{\overline{N}}(Y) \geq 0.$$

2.3 Flat metrics

Adelic metrized line bundles with flat metrics form an especially nice class of adelic metrized line bundles. We will often be able to split a metrized line bundle into a bundle with a flat metric plus a vertical bundle, and then work with each of these separately, as flatness will tell us that these have trivial intersection.

Definition 2.9. *Let X be a projective variety over a complete field K, and let $\overline{L}$ be a metrized line bundle on X. Then $\overline{L}$ is flat if for any morphism $f : C \to X$ of a projective curve over K into X, we have $c_1(f^*\overline{L}) = 0$ on the Berkovich analytification C^{an}. If now X is a projective variety over a global field and $\overline{L}$ an adelic metrized line bundle on X, call $\overline{L}$ flat provided it is flat at every place.*

Note that if $\overline{L}$ is flat, L must be numerically trivial, as

$$\deg(L|_C) = \int_{C^{an}} c_1(\overline{L}|_C) = 0.$$

Additionally, we define admissible metrics, a notion which we will generalize in Section 4.

Definition 2.10. *Given an abelian variety A over K and a metrized line bundle $\overline{L}$ on A, call $\overline{L}$ admissible if $[2]^*\overline{L} \cong 2\overline{L}$.*

These two definitions will be related in the proof of the following lemma

Lemma 2.11. *Let L be a numerically trivial line bundle on a projective, normal variety X over a global function field K. Then L has a flat metric, which is unique up to constant multiple.*

When X is a curve this lemma has a much simpler proof using linear algebra; see for example [25], Theorem 1.3).

Proof. First suppose A is an abelian variety. Replacing L by a power if necessary, we may assume L is algebraically trivial, in which case we have an isomorphism $\phi : [2]^*L \cong 2L$. Take any metric $||\cdot||_1$ on L. Then Tate's limiting argument defines an admissible metric on L as the limit of

$$||\cdot||_n := \phi^*[2]^*||\cdot||_{n-1}^{\frac{1}{2}}$$

as $n \to \infty$. [59], Theorem 2.2, shows that this limit converges uniformly to an admissible adelic metric $||\cdot||_0$ on L, and that this is the unique admissible metric on L up to constant multiples.

Now let $C \to A$ be a smooth projective curve in A. After a translation, we can fix a point $x_0 \in C(K)$ which maps to $0 \in A$. By the universal property of the Jacobian, $C \to A$ factors through the Jacobian map $C \to \mathrm{Jac}(C)$ taking $x_0 \to 0$, and the pullback of $(L, ||\cdot||_0)$ to $\mathrm{Jac}(C)$ is also admissible. Then by Remark 3.14 of [24], $c_1(L, ||\cdot||_0) = 0$, and hence A has a flat metric. By taking the tensor product of this metric with the inverse of any other flat metric on L, uniqueness up to constant multiple is reduced to showing that $||1||$ is constant for any flat metric on $\mathcal{O}_X$. Any two points on X are connected by a curve; let D be its normalization. Then $||1||$ is constant by local Hodge-Index Theorem in dimension one at each place.

Now choose a point $x_0 \in X(K)$ and recall the Albanase map $i : X \to \mathrm{Alb}(X)$ taking x to 0. L corresponds to a point $\xi \in \mathrm{Pic}_{X/K}(K) = \mathrm{Alb}(X)^\vee$. By definition, L is (isomorphic to) the Poincare bundle P on $\mathrm{Alb}(X) \times \mathrm{Alb}(X)^\vee$ restricted to $\mathrm{Alb}(X) \times \{\xi\}$, then pulled back through

$$i \times \mathrm{id} : X \times \mathrm{Alb}(X)^\vee \to \mathrm{Alb}(X) \times \mathrm{Alb}(X)^\vee.$$

$P|_{\mathrm{Alb}(X)\times\{\xi\}}$ is algebraically trivial, and hence has a flat metric. But the pullback of a flat metric is also flat, so this defines a flat metric for L.

□

The main use of flat metrics is the following lemma:

Lemma 2.12. *Let K be a complete non-archimedean field, and $X \to \operatorname{Spec} K$ a geometrically connected, normal, projective variety of dimension n, with a flat metrized line bundle $\overline{M}$. Then given any integrable metrized line bundles $\overline{L}_1, \ldots, L_{n-1}$ on X,*

$$c_1(\overline{M})c_1(\overline{L}_1)\cdots c_1(\overline{L}_{n-1}) = 0.$$

This is proved in [24], and also follows directly from the definition of a flat metric and the induction formula for local intersections. In particular, it implies that the intersection of a flat metrized line bundle with a vertical metrized line bundle is zero.

2.4 Chow's K/k-trace and image

Proofs of the following can be found in [27] and [15]. Let A be an abelian variety defined over K. The K/k-image $\big(\mathrm{Im}_{K/k}(A), \lambda\big)$ consists of an abelian variety $\mathrm{Im}(A)$ over k and a surjective morphism

$$\lambda : A \to \mathrm{Im}_{K/k}(A)_K$$

with the following universal property: If V is an abelian variety defined over k, and $\phi : A \to V_K$ a morphism, then ϕ factors through λ. Provided the fields K and k are clear, we will usually drop the K/k subscript and just write $\mathrm{Im}(A)$.

The K/k-trace is $\big(\mathrm{Tr}_{K/k}(A), \tau\big)$ where $\mathrm{Tr}_{K/k}(A)$ is an abelian variety over k, and

$$\tau : \mathrm{Tr}_{K/k}(A)_K \to A$$

is universal among all morphisms from k-abelian varieties to A. Again we will often drop the K/k when the fields are unambiguous. The image can be thought of as the largest quotient of A that can be defined over k and the trace the largest abelian subvariety that can be defined over k. This heuristic is literally true in characteristic zero, but in positive characteristic additional care is required for the trace; see [15], Section 6 for details.

These constructions are dual to each other in the sense that

$$\mathrm{Tr}(A^\vee) = \mathrm{Im}(A)^\vee,$$

and the image and trace are isogenous via the composition $\lambda \circ \tau$ (descended to the k-varieties).

Given a morphism of abelian varieties $f : A \to B$, we get morphisms $f_{\mathrm{Tr}} : \mathrm{Tr}(A) \to \mathrm{Tr}(B)$ and $f_{\mathrm{Im}} : \mathrm{Im}(A) \to \mathrm{Im}(B)$ commuting with τ and λ. By slight abuse of notation, given a morphism $f : X \to Y$ of (not necessarily abelian) varieties, the Albanese functor gives us a morphism of abelian varieties $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$, and we also denote by f_{Tr} and f_{Im} the descent of this morphism to the trace and image.

Chapter 3

Proof of Hodge-Index Theorem

3.1 Statement of results

Assume K is large enough so that $X(K)$ is non-empty, and then we may fix an Albanese variety and morphism $i : X \to \mathrm{Alb}(X)$. We will need to differentiate line bundles which come from the constant field k, and so we define a map $j : \mathrm{Pic}^0\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X))\right)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ given as the composition

$$\begin{aligned}\mathrm{Pic}^0\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X))\right)_{\mathbb{Q}} &\hookrightarrow \mathrm{Pic}\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X))\right)_{\mathbb{Q}} \\ &\to \mathrm{Pic}\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X)) \times_k B\right)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X))_K\right)_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}},\end{aligned}$$

where $\mathrm{Im}_{K/k}$ is Chow's K/k-image functor, the second map is the pullback of projection onto the first factor, and the last map is the pullback of the composition of the K/k-image and Albanese maps, $\lambda_{K/k} \circ i$. To shorten notation, define

$$\widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}} := j\left(\mathrm{Pic}^0\left(\mathrm{Im}_{K/k}(\mathrm{Alb}(X))\right)_{\mathbb{Q}}\right)$$

to be the image of the map in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$. We can now state our main theorem:

Theorem 3.1. *(Arithmetic Hodge-Index Theorem for function fields) Let $\overline{M}$ be an integrable adelic $\mathbb{Q}$-line bundle on X and $\overline{L}_1, \ldots, \overline{L}_{n-1}$ nef adelic $\mathbb{Q}$-line bundles on X. Suppose that $M \cdot L_1 \ldots L_{n-1} = 0$ and that each L_i is big. Then*

$$\overline{M}^2 \cdot \overline{L}_1 \ldots \overline{L}_{n-1} \leq 0.$$

If every $\overline{L}_i$ is arithmetically positive, and $\overline{M}$ is $\overline{L}_i$-bounded for every i, then

$$\overline{M}^2 \cdot \overline{L}_1 \ldots \overline{L}_{n-1} = 0$$

if and only if

$$\overline{M} \in \pi^* \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}.$$

Note the important case that when k is finite, $\widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$ is zero.

Call a metrized line bundle $\overline{M}$ on X numerically trivial if

$$\overline{M} \cdot \overline{M}_1 \cdots \overline{M}_n = 0$$

for every choice of metrized line bundles $\overline{M}_1, \ldots, \overline{M}_n$. The classical Hodge-index theorem says that the only divisors on a surface with zero self intersection are the numerically trivial divisors. We show that that is nearly, but not quite the case here:

Theorem 3.2. *The following three subgroups of* $\widehat{\mathrm{Pic}}(X)$ *are equal:*

1. *The numerically trivial elements of* $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$.
2. *The set of* $M \in \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ *such that the height function* $h_{\overline{M}}$ *is zero on* $X(\overline{K})$.
3. $\pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$, *where* $\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}}$ *is defined to be the elements of* $\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ *with arithmetic degree zero.*

Define $\mathrm{Pic}^\tau(X)$ to be the group of isomorphism classes of numerically trivial line bundles on X. We also state an $\mathbb{R}$-linear version of this theorem, which more closely resembles the classical Hodge-Index Theorem on the signature of the intersection pairing:

Theorem 3.3. *Let* $M \in \mathrm{Pic}^\tau(X)_{\mathbb{R}}$, *and let* $L_1, \ldots, L_{n-1} \in \mathrm{Pic}(X)_{\mathbb{Q}}$ *be nef. Pick any adelic metrics on* $L_1, \ldots, L_{n-1}$ *and any flat adelic metric on* M, *and then*

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} := \overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}$$

is a well-defined quadratic form, independent of the choice of metrics, and

$$\langle M, M\rangle_{L_1,\ldots,L_{n-1}} \leq 0.$$

Further, if every L_i *is ample, then equality holds if and only if* $M \in \mathrm{Pic}_k^{im}(X)_{\mathbb{R}}$.

These results are proven over the next three sections, with the bulk of the work going into proving Theorem 3.1, and then Theorems 3.2 and 3.3 following as corollaries.

3.2 Curves

To begin, assume $\dim(X) = 1$. Let $M \in \widehat{\mathrm{Pic}}(X)$ (as opposed to $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$). Then the theorem discusses the self-intersection $\overline{M}^2$ when $\deg M = 0$. By Lemma 2.11, M has a flat metric $\overline{M}_0 = (M, ||\cdot||_0)$.

Let $\overline{N}$ be the vertical line bundle defined by

$$\overline{M} = \overline{M}_0 + \overline{N}.$$

Since $\overline{M}_0$ is flat, $\overline{M}_0 \cdot \overline{N} = 0$ so that

$$\overline{M}^2 = \overline{M}_0^2 + \overline{N}^2 = \overline{M}_0^2 + \sum_\nu \overline{N}_\nu^2,$$

where $\overline{N}_v$ is the restriction of of $\overline{N}$ to $X_\nu := X \otimes_K K_\nu$ for each place ν of K (i.e closed point of B). Now $\overline{N}_\nu^2 \leq 0$ with equality if and only if $\overline{N}_\nu$ is constant by the local hodge index theorem, [55] Theorem 2.1 Hence

$$\sum_\nu \overline{N}_\nu^2 \leq 0$$

with equality if and only if $\overline{N} \in \pi^* \widehat{\mathrm{Pic}}(K)$.

Next, we consider $\overline{M}_0^2$. The necessary result is essentially Faltings' [19] and Hriljac's [25] work on the Hodge-index theorem for Arakelov divisors on curves. Faltings proves this by establishing the arithmetic Riemann-Roch theorem, while Hriljac instead uses Neron's local height functions. Since this is the fundamental base case for induction to higher dimensions, we provide a proof here as well, using a different method.

Lemma 3.4. *Let $\overline{D}$ and $\overline{E}$ be flat metrized line bundles on X. Then*

$$\overline{D} \cdot \overline{E} = -\langle D, E \rangle_{NT},$$

where $\langle \quad , \quad \rangle_{NT}$ is the Néron-Tate height pairing on $\mathrm{Pic}^0(X)$, *identified with the Jacobian (Albanese) variety $J(X)$.*

Proof. We first note that the statement makes sense, since as noted earlier, if $\overline{D}$ is flat, D is numerically trivial, and thus when X is a curve, must have degree zero.

We begin with well-known facts about Jacobians of curves. Let $g = \dim J(X)$ be the genus of X. Assume $g \geq 1$, otherwise $\mathrm{Pic}^0(X)$ is trivial. We allow $g = 1$, though the proof could be made simpler in that case. Enlarge K if necessary so that $X(k)$ contains a point e. Then e defines an embedding $i : X \hookrightarrow J(X)$ sending x to the divisor $x - e$. Using the group law we also get a map $i_{g-1} : X^{g-1} \to J(X)$ given by

$$(x_1, \ldots, x_{g-1}) \mapsto x_1 + \cdots + x_{g-1} - ge.$$

The image of this map is a divisor θ_e, and the Theta divisor

$$\Theta := \theta_e + [-1]^* \theta_e$$

is ample, symmetric, and independent of e.

The pullback $i^* : \mathrm{Pic}^0(J) \to \mathrm{Pic}^0(X)$ is an isomorphism, and it's inverse $(i^*)^{-1}$ is the principal polarization given by $-\phi_\theta$, where $\phi_\theta : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(J)$ is defined by

$$x \mapsto T_x^*\theta - \theta.$$

ϕ_Θ, definied similarly, is also a principal polarization, and $\phi_\Theta = 2\phi_\theta$.

With this notation, the Néron-Tate height pairing is defined as follows. Let h_Θ be a Weil height associated to Θ. We define the Néron-Tate canonical height $\widehat{h}_\Theta$ by Tate's limiting argument:

$$\widehat{h}_\Theta(x) := \lim_{n\to\infty} \frac{h_\Theta([n]x)}{n^2}.$$

Since h_Θ is well defined up to a bounded function, this limit is well defined independent of the choice of Weil height. By construction it is a quadratic form, and so we define the Néron-Tate pairing

$$\langle\ ,\ \rangle_{NT} : \mathrm{Pic}^0(X(\overline{K})) \times \mathrm{Pic}^0(X(\overline{K})) \to \mathbb{R}$$

by

$$\langle D, E\rangle_{NT} := \frac{1}{2}\left(\widehat{h}_\Theta(D+E) - \widehat{h}_\Theta(D) - \widehat{h}_\Theta(E)\right).$$

We now begin to prove the result. Let $L = L_E := -\phi_\theta(\mathcal{O}(E))$, the unique extension of $\mathcal{O}(E)$ to an algebraically trivial line bundle on $J(X)$.

Since L is algebraically trivial, it is anti-symmetric, i.e. $[-1]^*L \xrightarrow{\sim} L^{-1}$. By pulling back the multiplication by m map on J to $\mathrm{Pic}^0(J)$, we see that

$$[m]^*L \xrightarrow{\sim} L^{\otimes m}$$

for every $m \in \mathbb{Z}$, and we can define a canonical height function $\widehat{h}_L : J(\overline{K}) \to \mathbb{R}$ by

$$\widehat{h}_L(x) := \lim_{n\to\infty} \frac{h_L([n]x)}{n}.$$

where h_L is any Weil height coming from L. By construction this height is additive, so that

$$\widehat{h}_L(x_1 + x_2) = \widehat{h}_L(x_1) + \widehat{h}_L(x_2)$$

for all $x_1, x_2 \in J(\overline{K})$. We note that this limiting argument due to Tate is the same as that used to define the Néron-Tate height, however in that case L is required to be ample and symmetric, and then $[m]^*L \xrightarrow{\sim} L^{m^2}$ instead of L^m. For more on heights coming from both symmetric and antisymmetric line bundles, see Chapter 3 of [46].

We use this height as an intermediate step to break the symmetry between $\overline{D}$ and $\overline{E}$.

Proposition 3.5. *Let x_D be the point on J corresponding to $i(D)$. Then*

$$\overline{D} \cdot \overline{E} = \widehat{h}_{L_E}(x_D).$$

Proof. Let $\overline{L}$ be the line bundle L_E on J given a flat and admissible metric such that $[2]^*\overline{L} \xrightarrow{\sim} \overline{L}$. Such a metric exists and is unique up to constant multiple by Tate's limiting argument, as seen in the proof of Lemma 2.11. The pullback of a flat metric is also flat, which makes $i^*\overline{L}$ a flat extension of E. Since the intersection pairing does not depend on the choice of flat metric,

$$\overline{D} \cdot \overline{E} = \overline{D} \cdot i^*\overline{L}.$$

Then if we write $D = \sum a_j x_j$ as a sum of points on X, with $a_j \in \mathbb{Z}$, the flatness of $i^*\overline{L}$ allows us to write

$$\begin{aligned} \overline{D} \cdot i^*\overline{L} &= \sum a_j h_{i^*\overline{L}}(x_j) \\ &= \sum a_j h_{\overline{L}}(i(x_j)) \\ &= \sum a_j \widehat{h}_L(i(x_j)) \\ &= \widehat{h}_L\left(\sum a_j i(a_j)\right) = \widehat{h}_L(x_D) \end{aligned}$$

$\square$

We are half way to proving the result; it remains to show

$$\widehat{h}_L(x_D) = -\langle D, E \rangle_{NT}.$$

To this end, recall that $L = -\phi_\theta(\mathcal{O}(E))$, hence $2L = \Theta - T_E^*\Theta$, and then

$$\widehat{h}_\Theta - (T_E)_*\widehat{h}_\Theta$$

is a Weil height on J for the line bundle $2L$. Thus,

$$2\widehat{h}_L(D) = \widehat{h}_\Theta - (T_E)_*\widehat{h}_\Theta + O(1) = \widehat{h}_\Theta(D) - \widehat{h}_\Theta(D+E) + O(1).$$

Rearranging the definition of the Néron-Tate height pairing,

$$\widehat{h}_\Theta(D) - \widehat{h}_\Theta(D+E) = -\langle D, E \rangle_{NT} - \widehat{h}_\Theta(E).$$

Treat E as fixed, so that as D varies, $\widehat{h}_\Theta(E)$ may be absorbed into the $O(1)$ term, and combine this with the previous to get

$$\widehat{h}_\Theta(D) = -\langle D, E \rangle_{NT} + O(1).$$

Finally, since both sides are linear in D, the $O(1)$ term must be identically zero, proving the result.

$\square$

Since the Néron-Tate height is non-negative, this proves the inequality part Theorem 3.1, and Theorem 3.2 and the equality part of Theorem 3.1 are obtained by describing when the Néron-Tate pairing is degenerate.

By the Shioda-Tate Theorem [47], explained explicitly in this context in [52], the zeros of the Néron-Tate height are exactly the k-points of the K/k-trace of $\mathrm{Jac}(X) = \mathrm{Alb}(X)$, embedded via

$$\mathrm{Tr}(\mathrm{Alb}(X))(k) = \mathrm{Tr}(\mathrm{Alb}(X))_K(K) \xrightarrow{\tau} \mathrm{Jac}(X)(K) \hookrightarrow \mathrm{Pic}(X) \to \widehat{\mathrm{Pic}}(X).$$

We verify that this is the same as the map j in our theorem using the following diagram:

$$\begin{tikzcd}
\mathrm{Jac}(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] & \cdots \\
\mathrm{Pic}^0(\mathrm{Im}(\mathrm{Jac}(X))) \arrow[r] \arrow[u, "\lambda^*"] & \mathrm{Pic}(\mathrm{Im}(\mathrm{Jac}(X))) \arrow[r] \arrow[u, "\lambda^*"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Pic}(\mathcal{X}) \arrow[r] & \widehat{\mathrm{Pic}}(X) \\
\cdots \arrow[r] & \mathrm{Pic}\left(\mathrm{Im}(\mathrm{Jac}(X)) \times_k B\right) \arrow[r] & \widehat{\mathrm{Pic}}(\mathrm{Im}(\mathrm{Jac}(X))) \arrow[u, "\lambda^*"]
\end{tikzcd}$$

Since the Jacobian is self-dual, $\mathrm{Tr}(\mathrm{Jac}(X)) \cong \mathrm{Im}(\mathrm{Jac}(X))^\vee$ and $\lambda^\vee = \tau$, so that

$$\begin{tikzcd}
 & \mathrm{Jac}(X) \\
\mathrm{Tr}(\mathrm{Jac}(X)) \arrow[ur, "\tau"] \arrow[r, "\sim"] & \mathrm{Pic}^0(\mathrm{Im}(\mathrm{Jac}(X))) \arrow[u, "\lambda^*"]
\end{tikzcd}$$

commutes. As this also extends $\mathbb{R}$-linearly, this completes the proof of Theorem 3.3 in dimension 1, and Theorems 3.1 and 3.2 when $\overline{M}_0$ is flat.

If $\overline{M}$ is vertical, then a section corresponds to a sum of components of fibers of $\mathcal{X} \to B$, and $h_{\overline{M}}(x)$ is the intersection of this sum with the closure of x in $\mathcal{X}$. This is only constant for all $x \in X(\overline{K})$ if and only if the sum consists only of constant multiples of whole fibers $\mathcal{X}_\nu$, in which case $\overline{M} = \pi^* N$ for some $N \in \widehat{\mathrm{Pic}}(K)$, and then the height is the arithmetic degree of N.

Finally, we consider heights $h_{\overline{M}}$ where $\deg M \neq 0$. Suppose $\deg M > 0$. Then M is ample, and by scaling the height we may assume M is very ample so that it gives a $\deg M$ embedding into projective space. The height $h_{\overline{M}}$ differs only by a bounded function from the naïve height defined by this embedding. Repeatedly projecting down from a point, we get a degree $\deg M$ covering $X \to \mathbb{P}^1_K$. Thus since the height on $\mathbb{P}^1$ is unbounded, $h_{\overline{M}}$ is also unbounded. Finally, replacing $\overline{M}$ with $-\overline{M}$ covers the last remaining case where $\deg M < 0$.

3.3 Inequality

We now prove the inequality part of Theorem 3.1 by induction on $n = \dim X$, and get a version of the Cauchy-Schwarz inequality as a corollary. As in [55], we may assume that each $\overline{L_i}$ is arithmetically positive (instead of just big) by a limiting argument. Additionally, by approximation, we may assume that $\overline{M}$ and each $\overline{L}_i$ are induced by models $\mathcal{X}, \mathcal{M}, \mathcal{L}_i$ of X, M, L_i respectively. We must allow the possibility that $\mathcal{X}$ has isolated singularities, but we will assume $\mathcal{X}$ is normal, as we may simply replace it with its normalization if not.

First we prove the inequality

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} \leq 0.$$

Replace $\overline{L}_{n-1}$ by a positive power if necessary, so that $\mathcal{L}_{n-1}$ may be assumed to be very ample over k. For the moment base change to $\overline{k}$ to guarantee that we are working over an infinite field, and then Seidenberg's Bertini-type theorem [45], Theorem 7, tells us that almost all sections of $\mathcal{L}_{n-1}$ cut out normal, integral subvarieties. We choose such a section s, cutting out a horizontal subvariety $\mathcal{Y}$. This section is defined over some finite extension of k; replace k by that finite extension and now continue working over k. This finite base change doesn't affect the intersection numbers nor the group $\widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$, as the trace is compatible with base changes of the constant field, see [15], Theorem 6.8. Then

$$\mathcal{M}^2 \cdot \mathcal{L}_1 \cdots \mathcal{L}_{n-1} = \mathcal{M}|_{\mathcal{Y}}^2 \cdot \mathcal{L}_1|_{\mathcal{Y}} \cdots \mathcal{L}_{n-2}|_{\mathcal{Y}} \leq 0,$$

where the inequality follows from the induction hypothesis. As a corollary, we have the following Cauchy-Schwarz inequality:

Corollary 3.6. *Let $\overline{M}, \overline{M}'$ be two integral adelic line bundles on X, and let $\overline{L}_1, \ldots, \overline{L}_{n-1}$ be nef adelic line bundles on X such that*

$$M \cdot L_1 \cdots L_{n-1} = M' \cdot L_1 \cdots L_{n-1} = 0.$$

Then

$$\left(\overline{M} \cdot \overline{M}' \cdot \overline{L}_1 \cdots \overline{L}_{n-1}\right)^2 \leq \left(\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}\right) \left(\overline{M}'^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}\right).$$

Proof. This follows from the inequality part of the Hodge-Index theorem proven above, and from the standard proof of the Cauchy-Schwarz inequality using the (negative semi-definite) inner product

$$\langle M, M' \rangle_{\overline{L}_1,\ldots,\overline{L}_{n-1}} := \overline{M} \cdot \overline{M}' \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

□

3.4 Equality

Proceeding now to the equality part of Theorem 3.1, we add the assumptions that each $\overline{L}_i$ is arithmetically positive, that $\overline{M}$ is $\overline{L}_i$-bounded for all i, and that

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0.$$

Note that as a consequence of the Cauchy-Schwarz inequality above, the set of metrized line bundles $\overline{M}$ satisfying these properties forms a group.

By Lemma 3.7 of [55] (this uses the fact that $\overline{L}_i$ is arithmetically positive), M is numerically trivial on X. Thus it has a flat metric; let $\overline{M}_0 = (M, ||\cdot||)$ be flat. Then, similar to the curve case, $\overline{N} := \overline{M} - \overline{M}_0$ is vertical, and

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \overline{M}_0^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} + \overline{N}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1}.$$

The inequality part of the hodge index theorem guarantees that both terms on the right are zero, and then by the local hodge index theorem at every place occurring in N, we have $N \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. Hence we are reduced to proving the statement in the flat metric case $\overline{M} = \overline{M}_0$.

Lemma 3.7. *Under the above conditions, and additionally assuming that $\overline{M}$ is flat,*

$$\overline{M}|_Y^2 \cdot \overline{L}_1|_Y \cdots \overline{L}_{n-2}|_Y = 0$$

for any closed integral subvariety Y of codimension one in X.

Proof. Possibly replacing $\overline{L}_{n-1}$ by a positive power, we can find a non-zero section s of $\overline{L}_{n-1}$ vanishing on Y. Write $\mathrm{div}(s) = \sum_{i=1}^t a_i Y_i$, where the a_is are positive integers, Y_is are distinct integral subvarieties, and $Y_1 = Y$. Then by the induction formula of Chambert-Loir [11],

$$\overline{M}^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \sum_{i=1}^t a_i \overline{M}|_{Y_i}^2 \cdot \overline{L}_1|_{Y_i} \cdots \overline{L}_{n-2}|_{Y_i} - \sum_v \int_{X_v^{an}} \log ||s||_v c_1(\overline{M})^2 c_1(\overline{L}_1) \cdots c_1(\overline{L}_{n-2}).$$

Since $\overline{M}$ is flat, all the integrals are zero, and since by the Hodge-Index Theorem inequality each term of the first sum is non-positive, the claim follows. □

Taking a general hyperplane section Y of some very ample line bundle on X, Seidenberg's Bertini theorem [45] tells us Y is normal, and then the above lemma tells us via the induction hypothesis that

$$\overline{M}|_Y \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(Y)_{\mathbb{Q}}$$

Write $\overline{M}|_Y = \overline{M}_1|_Y + \overline{M}_2|_Y$, with $\overline{M}_1|_Y \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$ and $\overline{M}_2|_Y \in \widehat{\mathrm{Pic}}_k^{im}(Y)_{\mathbb{Q}}$, for some $\overline{M}_1, \overline{M}_2 \in \widehat{\mathrm{Pic}}(X)$. To justify this notation, note that (replacing M by a positive integer multiple if necessary), $M_1|_Y = \mathcal{O}_Y$, so that we may specify $M_1 = \mathcal{O}_X$ and give $\overline{M}_1$ the same constant metric as $M_1|_Y$, and define $\overline{M}_2 = \overline{M} - \overline{M}_1$. Since M is numerically trivial, again replacing $\overline{M}$ by a positive integer multiple if necessary, we may further assume M is algebraically trivial, and then that $M_1, M_2 \in \mathrm{Pic}^0(X)$.

Forgetting the metric structure, the map j defined earlier gives us

$$\begin{array}{ccccc}
\mathrm{Pic}^0(\mathrm{Im}(\mathrm{Alb}(X)))_{\mathbb{Q}} & \longrightarrow & \mathrm{Pic}^0(\mathrm{Im}(\mathrm{Alb}(X)) \times_k K)_{\mathbb{Q}} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
\mathrm{Pic}^0(\mathrm{Im}(\mathrm{Alb}(Y)))_{\mathbb{Q}} & \longrightarrow & \mathrm{Pic}^0(\mathrm{Im}(\mathrm{Alb}(Y)) \times_k K)_{\mathbb{Q}} & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \mathrm{Pic}^0(\mathrm{Alb}(X))_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Pic}^0(X)_{\mathbb{Q}} \\
 & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{Pic}^0(\mathrm{Alb}(Y))_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{Pic}^0(Y)_{\mathbb{Q}}
\end{array}$$

where the vertical maps come from the pullback of $Y \hookrightarrow X$ and its descent to the K/k-image. We show via the following lemma that $M_2|_Y$ lifts to an element of $\widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$.

Lemma 3.8. *Let $f : A \to B$ be a morphism of abelian varieties defined over K. In the commutative diagram*

$$\begin{array}{ccc}
\mathrm{Tr}(A)(k)_{\mathbb{Q}} & \xrightarrow{\tau_A} & A(K)_{\mathbb{Q}} \\
\downarrow{\scriptstyle f_{\mathrm{Tr}}} & & \downarrow{\scriptstyle f} \\
\mathrm{Tr}(B)(k)_{\mathbb{Q}} & \xrightarrow{\tau_B} & B(K)_{\mathbb{Q}}
\end{array}$$

$(f \circ \tau_A)(\mathrm{Tr}(A)(k)_{\mathbb{Q}})$ *is equal to* $f(A(K)_{\mathbb{Q}}) \cap \tau_B(\mathrm{Tr}(B)(k)_{\mathbb{Q}})$.

By the duality of the K/k-image and trace, we may let A be the abelian variety $\mathcal{P}ic^0(\mathrm{Alb}(X)) = \mathrm{Alb}(X)^\vee$ and $B = \mathrm{Alb}(Y)^\vee$ so that this lemma proves the existence of a lift of $M_2|_Y$. We now prove the lemma:

Proof. To shorten notation, we will drop writing the map τ_A and consider $\mathrm{Tr}(A)(k)$ directly as a subgroup of $A(K)$ (and similarly for B). First reduce to the case where f is surjective: let B' be the image of f, an abelian subvariety of B. By Poincaré reducibility, B is isogenous to $B' \times B''$, for some abelian variety B''. Then $\mathrm{Tr}(B)$ is isogenous to $\mathrm{Tr}(B') \times \mathrm{Tr}(B'')$, and the intersection of $\mathrm{Tr}(B')(k) \times \mathrm{Tr}(B'')(k)$ with $B'(K)$ is just $\mathrm{Tr}(B')(k)$.

Now assume f is surjective. We can find abelian subvarieties $A' \subset A$ and $B' \subset B$, and abelian varieties A'', B'' such that A is isogenous to $A' \times A''$, $\mathrm{Tr}(A') = \mathrm{Tr}(A)$, and $\mathrm{Tr}(A'') = 0$, and similarly for B. Then f induces a surjection $A' \twoheadrightarrow B'$, but A' is isogenous to $\mathrm{Tr}(A')_K$, and B' is isogenous to $\mathrm{Tr}(B')_K$, so we get a surjection $\mathrm{Tr}(A)(k) \twoheadrightarrow \mathrm{Tr}(B)(k)$, proving the lemma. □

Hence we may lift $\overline{M}_2|_Y$ to an element $\overline{M}'_2 \in \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$, and we must have

$$\overline{M}_2 - \overline{M}'_2 \in \ker\left(\widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(Y)\right).$$

Since $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ has finite kernel, replacing M with a positive integer multiple, we may assume $M_2 - M'_2 = \mathcal{O}_X$, since it must be zero in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$. Additionally, by the Cauchy-Schwarz inequality 3.6,

$$\left(\overline{M}_2 - \overline{M}'_2\right)^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = \left(\overline{M} - \overline{M}_1 - \overline{M}'_2\right)^2 \cdot \overline{L}_1 \cdots \overline{L}_{n-1} = 0,$$

so that by the local Hodge-Index Theorem the metric must be constant at each place and $\overline{M}_2 - \overline{M}'_2 \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}}$. This means that

$$\overline{M} = \left(\overline{M}_1 + \overline{M}_2 - \overline{M}'_2\right) + \overline{M}'_2 \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}.$$

This proves that when $\overline{M}$ is $\overline{L}_i$ bounded and $\overline{L}_i$ is arithmetically positive for all i, then

$$\overline{M}^2 \cdot \overline{L}_1 \ldots \overline{L}_{n-1} = 0.$$

if and only if $\overline{M} \in \pi^*\widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$.

But in fact we've shown something more general, namely that if Y is a general hyperplane section of X, the preimage under $\widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{Pic}}(Y)$ of $\pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(Y)_{\mathbb{Q}}$ is exactly $\pi^*\widehat{\mathrm{Pic}}^0(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}$. Thus, by cutting down X by general hyperplane sections to a curve, we also prove Theorem 3.2.

Finally, we note that the above arguments also prove the equality part of Theorem 3.3: Given $M \in \mathrm{Pic}^{\tau}(X)_{\mathbb{R}}$ and L_i nef, M has a ($\mathbb{R}$-linear sum of) flat metric $\overline{M}$ as proven above, and each L_i can be extended to a nef adelic metrized bundle $\overline{L}_i$. Lemma 3.7 works just the same in this $\mathbb{R}$-linear setting, and then by induction, we can assume $\overline{M}|_Y \in \widehat{\mathrm{Pic}}_k^{im}(Y)_{\mathbb{R}}$. Lemma 3.8 is also the same in the $\mathbb{R}$-linear instead of $\mathbb{Q}$-linear setting, so that $\overline{M} \in \widehat{\mathrm{Pic}}_k^{im}(Y)_{\mathbb{R}}$ as desired.

Chapter 4

Algebraic Dynamical Systems

We work in the same setting as the previous chapter, where K is the function field of a smooth projective curve B over k, and X is a projective variety over K. Suppose (X, f, L) and (X, g, M) are two polarized dynamical systems on X, so that f and g are endomorphisms of X, and L and M are ample line bundles such that $f^*L \cong qL$ and $g^*M \cong rM$ for some $q, r > 1$.

Remark. *If X is not normal, we may replace X by its normalization $\psi : X' \to X$, replace f by the normalization $f' : X' \to X'$ of $f \circ \psi$, and replace L by $L' = \psi^* L$ to get a new polarized algebraic dynamical system (X', f', L') with $\operatorname{Prep}(f') = \psi^{-1} \operatorname{Prep}(f)$, and similarly for (X, g, M). Hence from here on out we assume without loss of generality that X is normal.*

Our main goal in this section is to prove a comparison theorem for the points with dynamical height 0 under f and g, with an important corollary comparing the preperiodic points of f and g when k is a finite field. We begin with general properties of polarized algebraic dynamical systems, then define the particular arithmetic dynamical heights involved before stating the theorem.

4.1 An f^*-splitting of the Néron-Severi sequence

f^* preserves the exact sequence

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \operatorname{NS}(X) \to 0$$

defining the Néron-Severi group $\operatorname{NS}(X)$, and the Néron-Severi Theorem tells us that $NS(X)$ is a finitely generated $\mathbb{Z}$-module. For arbitrary k, the $\mathbb{Z}$-module $\operatorname{Pic}^0(X)$ need not be finitely generated, but by the Lang–Néron Theorem,

$$\operatorname{Pic}^0(X) \Big/ \operatorname{Tr}_{K/k} \operatorname{Pic}^0(X) \cong \operatorname{Pic}^0(X) \Big/ \operatorname{Pic}^0\left(\operatorname{Im}_{K/k}(\operatorname{Alb}(X))\right)$$

is a finitely generated $\mathbb{Z}$-module. Note that the inclusion $\mathrm{Pic}^0(\mathrm{Im}(\mathrm{Alb}(X))) \to \mathrm{Pic}^0(X)$ is simply the map j defined in the introduction, with the metric structure dropped. To shorten our notation, define

$$\mathrm{Pic}^0_{tr}(X) := \mathrm{Pic}^0(X) \Big/ \mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X),$$

$$\mathrm{Pic}_{tr}(X) := \mathrm{Pic}(X) \Big/ \mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X),$$

so that we have an exact sequence of finite-dimensional $\mathbb{C}$-vector spaces

$$0 \to \mathrm{Pic}^0_{tr}(X)_{\mathbb{C}} \to \mathrm{Pic}_{tr}(X)_{\mathbb{C}} \to \mathrm{NS}(X)_{\mathbb{C}} \to 0,$$

which is also an exact sequence of f^*-modules.

Lemma 4.1. *The operator f^* is semisimple on $\mathrm{Pic}^0_{tr}(X)_{\mathbb{C}}$ with eigenvalues of absolute value $q^{1/2}$, and is semisimple on $NS(X)$ with eigenvalues of absolute value q.*

Proof. As usual let $n = \dim X$. By the classical Hodge-index theorem [50], Exposé XIII, Corollary 7.4, we can decompose $\mathrm{NS}(X)_{\mathbb{R}}$ as

$$\mathrm{NS}(X)_{\mathbb{R}} := \mathbb{R}L \oplus P(X), \quad P(X) := \left\{\xi \in \mathrm{NS}(X)_{\mathbb{R}} : \xi \cdot L^{n-1} = 0\right\}$$

and define a negative definite pairing on $P(X)$ by

$$\langle \xi_q, \xi_2 \rangle := \xi_1 \cdot \xi_2 \cdot L^{n-2}.$$

The projection formula for intersection numbers applied to L^n gives us $\deg f = q^n$, and then applied to this pairing, we have

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q^2 \langle \xi_1, \xi_2 \rangle.$$

Hence $\frac{1}{q} f^*$ is orthogonal with respect to the negative of this pairing, and $\frac{1}{q} f^*$ is diagonalizable on $NS(X)_{\mathbb{C}}$ with eigenvalues all of absolute value 1.

On $\mathrm{Pic}^0(X)_{\mathbb{R}}$ we can define a pairing as follows: for $\xi_1, \xi_2 \in \mathrm{Pic}^0(X)_{\mathbb{R}}$, let $\overline{\xi}_1$, $\overline{\xi}_2$ be flat metrized extensions, and let $\overline{L}$ be any integrable adelic line bundle extending L. Then define

$$\langle \xi_1, \xi_2 \rangle := \overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}.$$

By Theorem 5.19 of [55], this is well defined regardless of the choices of metrics, and Theorem 3.1 establishes that this pairing is also negative definite. Since $\mathrm{Tr}_{K/k}\,\mathrm{Pic}^0(X)$ is numerically trivial, this pairing descends to $\mathrm{Pic}^0_{tr}(X)_{\mathbb{R}}$.

Again applying the projection formula,

$$(f^*\overline{\xi}_1) \cdot (f^*\overline{\xi}_2) \cdot (f^*\overline{L})^{n-1} = q^n(\overline{\xi}_1 \cdot \overline{\xi}_2 \cdot \overline{L}^{n-1}),$$

Since $f^*\overline{\xi}_i$ is still flat, and since we may replace $f^*\overline{L}$ by $q\overline{L}$ because the pairing is independent of the choice of metric on L, we have

$$\langle f^*\xi_1, f^*\xi_2 \rangle = q\langle \xi_1, \xi_2 \rangle.$$

Hence, $q^{-\frac{1}{2}} f^*$ is orthogonal on $\mathrm{Pic}^0_{tr}(X)_{\mathbb{R}}$ with respect to the negative of this pairing, making it diagonalizable with eigenvalues of absolute value 1 as a transformation on $\mathrm{Pic}^0_{tr}(X)_{\mathbb{C}}$.

□

By the theorem,

$$0 \to \mathrm{Pic}^0_{tr}(X)_{\mathbb{C}} \to \mathrm{Pic}_{tr}(X)_{\mathbb{C}} \to \mathrm{NS}(X)_{\mathbb{C}} \to 0$$

has a unique splitting as f^*-modules by a section

$$\ell_f : \mathrm{NS}(X)_{\mathbb{C}} \to \mathrm{Pic}_{tr}(X)_{\mathbb{C}}.$$

Let $P, Q \in \mathbb{Q}[T]$ be the minimal polynomials of f^* on $\mathrm{Pic}^0_{tr}(X)_{\mathbb{Q}}$ and $\mathrm{NS}(X)_{\mathbb{Q}}$ respectively. Because the eigenvalues of f^* are different on $\mathrm{Pic}^0_{tr}(X)_{\mathbb{Q}}$ and $\mathrm{NS}(X)_{\mathbb{Q}}$, the product $R = PQ$ must be irreducible, and therefore is the minimal polynomial of f^* on $\mathrm{Pic}_{tr}(X)_{\mathbb{Q}}$. Define

$$\mathrm{Pic}_{tr,f}(X)_{\mathbb{Q}} := \ker Q(f^*)|_{\mathrm{Pic}_{tr}(X)_{\mathbb{Q}}}$$

and then this splitting can be given over $\mathbb{Q}$ as

$$\ell_f : \mathrm{NS}(X)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}_{tr,f}(X)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}_{tr}(X)_{\mathbb{Q}}$$

4.2 Admissible metrics

Theorem 4.2. *The projection $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ has a unique section $M \mapsto \overline{M}_f$ as f^*-modules, satisfying:*

1. *If $M \in \mathrm{Pic}^0(X)_{\mathbb{Q}}$ then $\overline{M}_f$ is flat.*
2. *If $M \in \mathrm{Pic}_f(X)_{\mathbb{Q}}$ is ample then $\overline{M}_f$ is nef.*

Adelic metrized line bundles of the form $\overline{M}_f$ are called f-admissible.

Proof. Define $\widehat{\mathrm{Pic}}(X)'$ to be the group of adelic line bundles on X with continuous (but not necessarily integrable) metrics. This contains $\widehat{\mathrm{Pic}}(X)$. We will show that the projection $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ has a unique section, and then that properties 1 and 2 of the theorem hold for this section. Since $\mathrm{Pic}^0(X)_{\mathbb{Q}}$ and the ample elements of $\mathrm{Pic}_f(X)_{\mathbb{Q}}$ generate $\mathrm{Pic}(X)_{\mathbb{Q}}$, the section does in fact produce integrable metrics, proving the theorem.

The kernel of the projection $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}}$ is

$$D(X) = \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} \bigoplus_v C(X_v^{an}),$$

where $C(X_v^{an})$ is the ring of continuous $\mathbb{R}$-valued functions on X_v^{an}, via the association $||\cdot||_v \to -\log ||1||_v$. Recall that $R = PQ$ was defined to be the minimal polynomial of f^* on $\mathrm{Pic}(X)_{\mathbb{Q}}$ and now consider the action of $R(f^*)$ on $D(X)$.

Lemma 4.3. *$R(f^*)$ is invertible on $D(X)$.*

Proof. f^* acts as the identity on $\widehat{\mathrm{Pic}}(X)$, hence $R(f^*)$ acts as $R(1)$, and this is not zero because the roots of R all have absolute value q or $q^{\frac{1}{2}}$. So it suffices to show that $R(f^*)$ is invertible on $C(X)_{\mathbb{C}} := (\bigoplus_v C(X_v^{an})) \otimes_{\mathbb{R}} \mathbb{C}$. Factor R over $\mathbb{C}$ as

$$R(T) = a \prod_i \left(1 - \frac{T}{\lambda_i}\right),$$

where $a \neq 0$, and by lemma 4.1, $|\lambda_i|$ is either $q^{\frac{1}{2}}$ or q. $R(f^*)$ is invertible provided each term $1 - f^*/\lambda_i$ is, and each term has inverse

$$\left(1 - \frac{f^*}{\lambda_i}\right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{f^*}{\lambda_i}\right)^k,$$

provided this series converges absolutely with respect to the operator norm, which is defined with respect to the supremum norm $||\cdot||_{sup}$ on $C(X_v^{an})_{\mathbb{C}}$ for every place v. f^* doesn't change the supremum norm, so the operator norm of f^* is 1, and

$$\left\| \left(\frac{f^*}{\lambda_i}\right)^k \right\| = \frac{1}{|\lambda_i|^k} \leq q^{-\frac{k}{2}},$$

so the series converges absolutely. □

Corollary 4.4. *The exact sequence*

$$0 \to D(X) \to \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \to \mathrm{Pic}(X)_{\mathbb{Q}} \to 0$$

has a unique f^*-equivariant splitting.

Proof. Define

$$E(X) := \ker\left(R(f^*) : \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} \to \widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}\right).$$

Since $R(f^*)$ kills all of $\mathrm{Pic}(X)_{\mathbb{Q}}$, this gives an f^*-invariant decomposition

$$\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}} = D(X) \bigoplus E(X)$$

such that the projection onto $\mathrm{Pic}(X)$ gives an isomorphism $E(X) \xrightarrow{\sim} \mathrm{Pic}(X)_{\mathbb{Q}}$, whose inverse is the desired splitting.

We can write this down even more explicitly. For $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$, let $\overline{M}$ be any choice of metric in $\widehat{\mathrm{Pic}}(X)'_{\mathbb{Q}}$. Then define

$$\overline{M}_f := \overline{M} - R(f^*)|_{D(X)}^{-1} R(f^*)\overline{M}.$$

□

It now remains to show that this splitting satisfies (1) and (2). To start, suppose M is in $\mathrm{Pic}^0(X)_{\mathbb{Q}}$. Let $x_0 \in \mathrm{Prep}(f)$, then after replacing f by an iterate and K by a finite extension if necessary, we may assume that x_0 is a fixed point. Let $i : X \to \mathrm{Alb}(X)$ be the Albanese map taking $x_0 \mapsto 0$, then f^* and i^* induce the following commutative diagram, where $f' := (f^*)^\vee$:

$$\begin{tikzcd}
\mathrm{Pic}^0(\mathrm{Alb}(X)) \arrow[r, shift left, "i^*"] \arrow[d, "(f')^*"'] & \mathrm{Pic}^0(X) \arrow[l, shift left, "\sim"] \arrow[d, "f^*"] \\
\mathrm{Pic}^0(\mathrm{Alb}(X)) \arrow[r, shift left, "i^*"] \arrow[d, "M \mapsto \overline{M}_{f'}"'] & \mathrm{Pic}^0(X) \arrow[l, shift left, "\sim"] \arrow[d, "M \mapsto \overline{M}_f"] \\
\widehat{\mathrm{Pic}}(\mathrm{Alb}(X))' \arrow[r, "i^*"] & \widehat{\mathrm{Pic}}(X)'
\end{tikzcd}$$

Because this commutes, it suffices to show (1) for abelian varieties, as i^* takes $M_{f'}$ to $\overline{M}_f$, and the pullback of a flat metric is also flat. Now $[2]^*M = 2M$, and since $[2]$ commutes with f',

$$[2]^*\overline{M}_{f'} = 2\overline{M}_{f'},$$

so that as in the proof of Lemma 2.11, $\overline{M}_{f'}$, and hence also $\overline{M}_f$ is flat.

Finally, we show that (2) also holds. This is proven in the arithmetic setting (i.e. when X is defined over a number field) in [55], Theorem 4.9, Step 4, however the proof is a purely numerical argument on weighted sums of adelic line bundles, and applies identically in our geometric setting. Note that Lemma 5.7, on which the proof relies and which states that arithmetic ampleness is an open condition, is a more well-know result in the geometric setting, proven in [31], Proposition 1.3.7, for example.

□

The above section also descends to a section $\mathrm{Pic}_{tr}(X) \to \widehat{\mathrm{Pic}}(X)/\widehat{\mathrm{Pic}}_k^{im}(X)$, as by construction every element of $\widehat{\mathrm{Pic}}_k^{im}(X)$ has a flat metric. Thus, we have an f^*-equivariant linear map

$$\widehat{\ell}_f : \mathrm{NS}(X)_{\mathbb{Q}} \to (\widehat{\mathrm{Pic}}(X)/\widehat{\mathrm{Pic}}_k^{im}(X))_{\mathbb{Q}}$$

given by the composition of the section developed in Theorem 4.2 and the map just preceding it.

4.3 Rigidity of height zero points and preperiodic points

Heights given by f-admissible metrized line bundles have particularly nice properties and correspond to the dynamical canonical heights defined by Call-Silverman [9].

Proposition 4.5. *Let $M \in \mathrm{Pic}(X)_{\mathbb{Q}}$. Then:*

1. *If $f^*M = \lambda M$ for some $\lambda \in \mathbb{Q}$, then $f^*\overline{M}_f = \lambda M_f$ in $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$, and $h_{\overline{M}_f}(f(\cdot)) = \lambda h_{\overline{M}_f}(\cdot)$.*

2. *For $x \in \mathrm{Prep}(f)$, $\overline{M}_f|_x$ is trivial in $\widehat{\mathrm{Pic}}(x)_{\mathbb{Q}}$, and in particular $h_{\overline{M}_f}$ is zero on $\mathrm{Prep}(f)$.*

Further, if M is ample and $f^*M = \lambda M$ for some $\lambda > 1$ (in particular, if $M = L$), then:

3. *$h_{\overline{M}_f}(x) \geq 0$ for all $x \in X(K)$.*

Proof. (1) is clear from the statement of Theorem 4.2. For (2), let $f^m(x) = f^n(x)$ for some $m > n \geq 0$. Consider the linear map of finite dimensional vector spaces

$$(f^*)^m - (f^*)^n : \mathrm{Pic}_{tr}(X)_{\mathbb{Q}} \to \mathrm{Pic}_{tr}(X)_{\mathbb{Q}}$$

Since f^* has eigenvalues with absolute value $q, q^{1/2} > 1$, this is an isomorphism, hence surjective. Then for some $N \in \mathrm{Pic}(X)_{\mathbb{Q}}$ we can write

$$h_{\overline{M}_f}(x) = h_{(f^*)^m \overline{N}_f - (f^*)^n \overline{N}_f}(x) = (f^*)^m \overline{N}_f|_x - (f^*)^n \overline{N}_f|_x = \overline{M}_f|_{f^m(x)} - \overline{M}_f|_{f^n(x)} = 0.$$

When M is ample and $f^*M = \lambda M$, let h_M be the Weil height (sometimes called naïve height) coming from M, and define

$$\hat{h}_{\overline{M}_f} := \lim_{n\to\infty} \frac{h_M(f^n(x))}{n}.$$

Call and Silverman [9] show that this gives a well-defined canonical height function which agrees with our $h_{\overline{M}_f}$ defined via intersections. (3) is then clear as it holds for all canonical heights. □

We can say more when K is a global function field, i.e. when k is finite.

Proposition 4.6. *Suppose k is a finite field. If $M \in \mathrm{Pic}\, X$ is ample and $f^*M = \lambda M$ for some $\lambda > 1$, then $h_{\overline{M}_f}(x) = 0$ if and only if $x \in \mathrm{Prep}(f)$.*

Proof. Suppose $h_{\overline{M}_f}(x) = \hat{h}_{\overline{M}_f}(x) = 0$ for some $x \in X(\overline{K})$. Consider the set $S = \{f^n(x)\}_{n\geq 0}$ of forward iterates of x. Since f is defined over K, we have

$$[K(f^n(x)) : K] \leq [K(x) : K]$$

and $h_{\overline{M}_f}(f^n(x)) = 0$ for all $n \geq 0$. Hence S is a set of bounded height and bounded degree, and must be finite by the Northcott property for global function fields. This means the forward orbit of x is finite and x is preperiodic. □

We can now state and prove our main theorem of this section.

Theorem 4.7. *Let (f, L) and (g, M) be two polarized algebraic dynamical systems on X. Define $Z_f := \{x \in X(\overline{K}) | h_{\overline{L}_f}(x) = 0\}$ to be the set of height zero points with respect to $\overline{L}_f$, and Z_g the set of height zero points with respect to $\overline{M}_g$, and let Z be the Zariski closure of $Z_f \cap Z_g$ in X. Then*

$$Z_f \cap Z(\overline{K}) = Z_g \cap Z(\overline{K}).$$

When k is finite, $Z_f = \mathrm{Prep}(f)$ and $Z_g = \mathrm{Prep}(g)$, so Theorem 1.3 stated in the introduction follows as an immediate consequence. If k is not finite, it is still true that $Z_f \supseteq \mathrm{Prep}(f)$, but there may be height zero points with infinite forward orbit. See Chapter 5 for further discussion.

Proof. We begin by proving a simpler lemma, justifying the notation that Z_f does not depend on the polarization L.

Lemma 4.8. *Let $f : X \to X$, and let L and M be two ample line bundles which polarize f. Then*

$$\left\{x \in X(\overline{K}) | h_{\overline{L}_f}(x) = 0\right\} = \left\{x \in X(\overline{K}) | h_{\overline{M}_f}(x) = 0\right\},$$

and we unambiguously call both sets Z_f.

Proof. Since L is ample, there exists a constant $c > 0$ such that $cL - M$ is also ample. Then by Proposition 4.5, the canonical heights $h_{\overline{M}_f}$ and $h_{\overline{cL}_f} = ch_{\overline{L}_f}$ are related by

$$0 \leq h_{\overline{M}_f}(x) \leq ch_{\overline{L}_f}(x)$$

for all $x \in X(\overline{K})$. Thus

$$\left\{x \in X(\overline{K}) | h_{\overline{L}_f}(x) = 0\right\} \subseteq \left\{x \in X(\overline{K}) | h_{\overline{M}_f}(x) = 0\right\}.$$

By symmetry, we also have containment in the other direction. □

We now prove the theorem.

Let Y be the normalization of an irreducible component of Z and say $\dim Y = d$. Let ξ be the image of L in $\mathrm{NS}(X)$. ξ has two different lifts $\ell_f(\xi)$ and $\ell_g(\xi)$ to $\mathrm{Pic}_{tr}(X)_{\mathbb{Q}}$; let L_f and L_g be representatives in $\mathrm{Pic}(X)_{\mathbb{Q}}$ of these classes in $\mathrm{Pic}_{tr}(X)_{\mathbb{Q}}$. Since L is one such choice of representative for $\ell_f(\xi)$ and ampleness is preserved by numerical equivalency, L_f and L_g must both be ample.

By Theorem 4.2, L_f and L_g have f- and respectively g-admissible metrics, which we call $\overline{L}_f$ and $\overline{L}_g$. Both are nef. Their sum $\overline{N} := \overline{L}_f + \overline{L}_g$ is also nef, and defines a height function $h_{\overline{N}}$, which does not depend on the choice of representatives.

By Lemma 4.8 and the premise that $Z_f \cap Z_g \cap Z(\overline{K})$ is dense in Z, Y has a dense set of points which have height zero under $h_{\overline{N}}$. By the successive minima (Proposition 2.8),

$$\lambda_1(Y, \overline{N}) = h_{\overline{N}}(Y) = 0.$$

Rewriting the height of Y in terms of intersections,

$$0 = \left(\overline{L}_f|_Y + \overline{L}_g|_Y\right)^{d+1} = \sum_{i=0}^{d+1} \binom{d+1}{i} \left(\overline{L}_f|_Y\right)^i \cdot \left(\overline{L}_g|_Y\right)^{d+1-i}.$$

Since both $\overline{L}_f$ and $\overline{L}_g$ are nef, every term in the sum on the right is non-negative, hence all must be zero. Then

$$\left(\overline{L}_f|_Y - \overline{L}_g|_Y\right)^2 \cdot \left(\overline{L}_f|_Y + \overline{L}_g|_Y\right)^{d-1} = 0,$$

as well. Because $L_f - L_g$ is zero in the Néron-Severi group, and thus numerically trivial we also have,

$$(L_f|_Y - L_g|_Y) \cdot (L_f|_Y + L_g|_Y)^{d-1} = 0.$$

Additionally, $(\overline{L}_f - \overline{L}_g)$ is clearly $(\overline{L}_f + \overline{L}_g)$-bounded, and we are nearly in the right setting to apply Theorem 3.1, except that $(\overline{L}_f + \overline{L}_g)$ is nef, but not necessarily arithmetically positive.

To fix this, we simply adjust the metric by a small positive factor: let $\overline{C} \in \widehat{\mathrm{Pic}}(K)$ with $\widehat{\deg}(\overline{C}) > 0$. Replace the pair $\left(\overline{L}_f - \overline{L}_g, \overline{L}_f + \overline{L}_g\right)$ by $\left(\overline{L}_f + \overline{L}_g, \overline{L}_f + \overline{L}_g + \overline{\pi}^* C\right)$. Since $L_f - L_g$ is numerically trivial, the metric on $\overline{L}_f - \overline{L}_g$ is flat, so adding $\overline{\pi}^* C$, which is vertical, does not change the intersection number. All the conditions of the theorem are now satisfied, so that the theorem tells us

$$\left(\overline{L}_f - \overline{L}_g\right) \in \widehat{\mathrm{Pic}}(K)_{\mathbb{Q}} + \widehat{\mathrm{Pic}}_k^{im}(X)_{\mathbb{Q}}.$$

We therefore conclude by Theorem 3.2 that

$$h_{\overline{L}_f} - h_{\overline{L}_g}$$

is a constant height function on Y. Since these two heights both take value zero on a dense set in Z, they must be equal on Y. Thus these heights define the same sets of height zero points, and then by Lemma 4.8, Z_f and Z_g agree on Y, and hence on all of Z.

□

Chapter 5

Corollaries, Questions, and Future Work

5.1 Rigidity of preperiodic points over global function fields

We first summarize some basic consequences of Theorem 4.7 when K is a global function field, particularly in the case when $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ is dense in X.

Lemma 5.1. *Let K be a global function field, and let f and g be two polarized algebraic dynamical systems on a projective variety X. Then the following are equivalent:*

1. $\mathrm{Prep}(f) = \mathrm{Prep}(g)$.
2. $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ *is dense in X.*
3. $\mathrm{Prep}(f) \subset \mathrm{Prep}(g)$.
4. $g(\mathrm{Prep}(f)) \subset \mathrm{Prep}(f)$.

Proof. The equivalence of (1) and (2) is an immediate consequence of Theorem 4.7 and the fact that over a global function field, all dynamical height zero points are preperiodic. Clearly (1) implies (4). By Fakhruddin [17], $\mathrm{Prep}(f)$ is always dense in X, hence (3) implies (2). We now show (4) implies (3).

Stratify $\mathrm{Prep}(f)$ by degree, writing

$$\mathrm{Prep}(f) = \bigcup_{d \geq 0} \mathrm{Prep}(f, d),$$

where

$$\mathrm{Prep}(f,d) := \{x \in \mathrm{Prep}(f) | [K(x) : K] \leq d\}.$$

Since each $\mathrm{Prep}(f,d)$ has height zero and bounded degree, it is finite. Now (4) says that g fixes $\mathrm{Prep}(f)$, but since g is defined over K, it fixes each $\mathrm{Prep}(f,d)$ as well. Thus every point of $\mathrm{Prep}(f)$ has finite forward orbit under g. □

This lemma suggests two related questions which we do not answer here.

1. When is $\mathrm{Prep}(f)$ equal to $\mathrm{Prep}(g)$?
2. If $\mathrm{Prep}(f) = \mathrm{Prep}(g)$, how closely related must f and g be?

In the case of $f : \mathbb{P}^1 \to \mathbb{P}^1$, Mimar [33] gives a variety of partial answers to these questions, with the general takeaway being that if f and g have the same preperiodic points, their Julia sets must also be very similar. Even a partial classification of this sort is likely to be very hard in general, however, as the behavior of the Julia set may be much more complicated in higher dimension, and $\mathrm{Prep}(f) \cap \mathrm{Prep}(g)$ may be dense on a positive dimensional subvariety which is not all of X. But it may be possible and productive to study specific families of dynamical systems.

5.2 Preperiodic points over larger fields

Theorem 3.1, and most of the proof of Theorem 4.7, hold over all transcendence degree one function fields, not just global function fields. But because the Northcott principal fails when k is not a finite field or the algebraic closure of a finite field, we cannot equate height zero points with preperiodic points over arbitrary k, and thus Theorem 4.7 is a statement about height zero points and not preperiodic points. Can anything still be said about preperiodic points in this general case?

Fakhruddin's result, that $\mathrm{Prep}(f)$ is dense in X, is still true for any k. We also always have $\mathrm{Prep}(f) \subset Z_f$, but not necessarily an inclusion in the other direction. To start, we note that these sets not being equal is not just a hypothetical possibility.

Let k be any field, let K be the function field of a curve over k, and let A be an abelian variety defined over K. As always, the preperiodic points of A with respect to the endomorphism $[n]$, for $n \geq 2$, will be exactly the torsion points $A(\overline{K})_{tors}$. Now suppose A has a non-trivial K/k-trace. In general $\mathrm{Tr}_{K/k}(A)_K \to A$ may not be an embedding, but it is an injection on K points, so we have a natural subgroup

$$\mathrm{Tr}_{K/k}(A)(k) \hookrightarrow A(K).$$

Now all of $\mathrm{Tr}_{K/k}(A)(\overline{k})$ has canonical height zero, and for general k this will consist of more than just torsion points. Thus

$$\mathrm{Prep}([n]) \subsetneq Z_{[n]}.$$

Not all hope is lost, however. It turns out that this is essentially the only thing that can make these sets not equal. The Lang-Néron Theorem [26] says that for any function field K with constant field k, the quotient $A(K)/\,\mathrm{Tr}(A)(k)$ is a finitely generated group. Additionally, the induced canonical height

$$h_{\mathbb{R}} : \left(A(\overline{K})/\,\mathrm{Tr}(A)(\overline{k})\right)_{\mathbb{R}} \to \mathbb{R}$$

is positive definite. Thus the height zero points are exactly the torsion cosets of $\mathrm{Tr}(A)(\overline{k})$.

Similar results exist for general (non-abelian) varieties. First consider $X = \mathbb{P}^1$. Then polarized algebraic dynamical systems correspond to rational functions $f \in K(X)$. For simplicity, let $K = k(T)$, where k is any field. We compare two different dynamical systems:

1. $$f : \mathbb{P}^1_K \to \mathbb{P}^1_K \text{ given by } X \mapsto X^2.$$
   The pre-periodic points are 0, ∞, and all roots of unity in k, but all points of $\mathbb{P}^1(\overline{k}) \hookrightarrow \mathbb{P}^1(\overline{K})$ have dynamical height h_f equal to zero. If k is large, these sets are far from equal.

2. $$g : \mathbb{P}^1_K \to \mathbb{P}^1_K \text{ given by } X \mapsto TX^2$$
   The only preperiodic points are 0 and ∞. Now, however, by just considering the T-adic component in the limit defining the canonical height, we see that these are also the only points with dynamical height h_g equal to zero.

There is no notion of a K/k-trace for arbitrary projective varieties (recall that while Chapters 3 and 4 discuss arbitrary varieties, they only ever take the K/k-image of the albanese variety), but one can instead look at whether the endomorphism can be defined over k or not. The above examples illustrate the general behavior, that in dimension one, the only obstruction to the equality of preperiodic points and dynamical height zero points is the endomorphism descending to k. A theorem of Baker [3] makes this precise, first proven by Benedetto [5] in the case of polynomials.

Theorem 5.2. *Let $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be a rational function of degree ≥ 2, and suppose that f is not isotrivial, in the sense that there exists no finite extension K' of K and Möbius transformation $M \in \mathrm{PGL}_2(K')$ such that*

$$f' := M^{-1} \circ f \circ M$$

is defined over k. Then

$$\mathrm{Prep}(f) = Z_f.$$

Thus Theorem 4.7 proven here gives a rigidity result for preperiodic points under non-isotrivial maps on $\mathbb{P}^1$ over any transcendence degree one function field, not just global function fields.

Chatzidakis and Hrushovski [13, 14] prove a theorem generalizing both Baker's result and the Lang Néron Theorem. Note that this result doesn't require a polarization, and that they define a more general notion of dynamical height which reduces to that defined here when a system is polarized. For the results above, we did not explicitly mention a polarization only because non-trivial maps on abelian varieties and on projective space always have natural polarizations; see Chapter 1.

Theorem 5.3. *Let K be any function field and let k be its field of constants. Let $f : X \to X$ be an algebraic dynamical system defined over K, and assume f does not constructibly descend to k (a model-theoretic notion generalizing isotriviality). Then for every point $x \in X(\overline{K})$ with dynamical height zero there exists a proper Zariski closed subset $Y_x \subsetneq X$ such that the orbit of x is contained in Y_x.*

It would be interesting to see how this result could be used to still derive some kind of rigidity result for dynamical systems over function fields with any constant field, or to see if there are interesting families of dynamical systems for which all height zero points are still preperiodic.

5.3 Function fields of larger transcendence degree

All results thus far have been over function fields of transcendence degree one. What happens if one allows higher transcendence degree, i.e. fields K which are the function field of a higher dimensional projective variety B over k? If one extends the results here to all fields finitely generated over $\mathbb{Q}$, then by the Lefschetz principle they would hold over $\mathbb{C}$ and any other algebraically closed field of characteristic zero, and thus one could study dynamics over these fields by arithmetic methods.

Moriwaki, in [35, 37], accomplishes this by imposing additional polarization structures on the variety B defining K to produce well-defined $\mathbb{R}$-valued height functions, and manages to prove both a Northcott finiteness theorem for these heights, and recover a proof of Raynaud's Theorem [43, 44] over $\mathbb{C}$. Moriwaki doesn't explicitly prove a Hodge-index theorem for arithmetic intersections with respect to the polarizations he defines, but the additional structure would make such a result in this setting too weak to apply to arithmetic dynamics questions as done in Chapter 4 of this work.

In unpublished work, Yuan and Zhang [56] take a slightly different approach, developing a theory of vector-valued heights which applies to finitely generated fields K over any base field k. They then use this theory to prove a Hodge-index theorem and corresponding dynamics result for fields finitely generated over $\mathbb{Q}$, and state that the same method yields results for fields finitely generated over $\mathbb{F}_p$ as well. The theory needed to develop vector-valued heights (where the heights are valued in the $\mathbb{Q}$-vector space $\widehat{\mathrm{Pic}}(K)$) is quite technical, and requires further discussion to even define the the inequality and equality of the appropriate Hodge-index theorem. Despite these difficulties however, it seems likely that their work and the results of this work (particularly considerations for when the K/k-trace of $\mathrm{Pic}(X)_{\mathbb{Q}}$ is non-trivial) could be combined to yield a Hodge-index theorem and a rigidity statement for dynamical systems over finitely generated fields over an arbitrary base. The author intends to develop this further in future work.

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