Relative Arbitrage Opportunities in N Investors and Mean-Field Regimes

Tomoyuki Ichiba* Tianjiao Yang†

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Abstract

This paper analyzes the market behavior and optimal investment strategies to attain relative arbitrage both in the N investors and mean field regimes. An investor competes with a benchmark of market and peer investors, expecting to outperform the benchmark and minimizing the initial capital.

With market price of risk processes depending on the stock market and investors respectively, the minimal initial capital required is the optimal cost in the N-player games and mean field games. It can be characterized as the smallest nonnegative continuous solution of a Cauchy problem. The measure flow of wealth appears in the cost, while the joint flow of wealth and strategy is in state processes. We modify the extended mean field game with common noise and its notion of the uniqueness of Nash equilibrium. There is a unique equilibrium in N-player games and mean field games with mild conditions on the equity market.

1 Introduction

This paper discusses a differential game system of relative arbitrage problems where investors aim to outperform not only the market index but also peer investors.

The relative arbitrage problem is defined in stochastic portfolio theory, see Fernholz [9], in the sense that a strategy outperforms a benchmark portfolio at the end of a certain time span. It shows in [11] that relative arbitrage can exist in equity markets that resemble actual markets, and that the relative arbitrage results from market diversity, a condition that prevents the concentration of all the market capital into a single stock. Specific examples of market including the stabilized volatility model, in which relative arbitrage exists, are introduced in [10]. Our model arises from the pioneering work of Fernholz and Karatzas [7], which characterizes the best possible relative arbitrage with respect to the market portfolio, and derives nonanticipative investment strategies of the best arbitrage in a Markovian setting. The best arbitrage opportunity is further analyzed in [8] in a market with Knightian uncertainty. The smallest proportion of the initial market capitalization is described as the min-max value of a zero-sum stochastic game between the investor and the market. Further investigation of exploiting relative arbitrage opportunities has been done in [1, 12, 26, 27]. The papers [24] and [29] connect relative arbitrage with information theory and optimal transport problems.

However, most of the literature on relative arbitrage uses an absolute benchmark such as market portfolio. To the best of our knowledge, this is the first paper that discusses relative arbitrage with respect to a relative benchmark – matching the performances of a group of investors in a stochastic differential game. Our paper modifies the original relative arbitrage problem to provide a general structure of the market and optimal portfolios that allows the interaction among investors.

This paper first considers N investors in an equity market $\mathcal{M}$ over a time horizon $[0, T]$. We consider N is big so that the equity trading of this group as a whole contributes to the evolution of the market; whereas individuals among the group are too "small" to bring changes to the market. These investors interact with the market through a joint distribution of their wealth and strategies, particularly for example, through the total investments of this group to the assets. There are n stocks with prices-per-share driven by n independent Brownian motions

*Department of Statistics and Applied Probability, South Hall, University of California, Santa Barbara, CA 93106, USA (E-mail: ichiba@pstat.ucsb.edu) Part of research is supported by National Science Foundation grants DMS-1615229 and DMS-2008427.

†Department of Statistics and Applied Probability, South Hall, University of California, Santa Barbara, CA 93106, USA (E-mail: yang@pstat.ucsb.edu)

$W = (W_1, \ldots, W_n)$. The n-dimensional price process $\mathcal{X}^N = (X_1^N, \ldots, X_n^N)$ follows a nonlinear stochastic differential equation

$$d\mathcal{X}^N(t) = \mathcal{X}^N(t)\beta(t, \mathcal{X}^N(t), \nu_t^N)dt + \mathcal{X}^N(t)\sigma(t, \mathcal{X}^N(t), \nu_t^N)dW_t \tag{1}$$

in which its drift β and diffusion σ coefficients depend on the joint empirical measure

$$\nu_t^N := \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(V^\ell(t), \pi^\ell(t))} \tag{2}$$

of portfolio strategy and wealth of N investors. To emphasize the dependence of wealth on the initial capitalization and portfolio, we write $V^\ell = V^{v^\ell, \pi^\ell}$ for $\ell = 1, \ldots, N$. We show the market model is well-posed through a finite dynamical system.

To specify what we mean by relative arbitrage opportunities in this problem set-up, we first define a benchmark process $\mathcal{V}^N$ by the weighted average performance of the market and the investors

$$\mathcal{V}^N(t) = \delta \cdot X^N(t) + (1-\delta) \cdot \frac{1}{N}\sum_{\ell=1}^{N} V^\ell(t)/v^\ell, \quad 0 \leqslant t \leqslant T,$$

with a fixed weight $\delta \in [0, 1]$. An investor achieves the relative arbitrage if his/her terminal wealth can outperform this benchmark by c_ℓ, a constant personal index for the investor ℓ, given at time 0. Furthermore, $\mathbb{A}^N$ denotes all admissible, self-financing long-only portfolios for N investors.

The first question raised in this paper is: *What is the best strategy one can take, so that the arbitrage relative to the above benchmark can be attained?* Specifically, every investor we study aims to outperform the market and their competitors, starting with as little proportion of the benchmark capital as possible. Mathematically, given the other $(N-1)$ investors' portfolios $\pi^{-\ell} \in \mathbb{A}^{N-1}$, the objective of investor ℓ, $\ell = 1, \ldots, N$, is formulated as

$$u^\ell(T) = \inf\Big\{\omega^\ell \in (0,\infty) \,\Big|\, \exists \pi^\ell(\cdot) \in \mathbb{A} \text{ such that } v^\ell = \omega^\ell \mathcal{V}^N(0),\ V^{v^\ell, \pi^\ell}(T) > e^{c_\ell}\mathcal{V}^N(T)\Big\}$$

Since the interactions of a large group of investors are through stocks, portfolios and wealth, the next question arises is: *Is it possible for every investor to take the optimal strategy in the market $\mathcal{M}$?* We characterize the optimal wealth one can achieve by the unique Nash equilibrium of the finite population game. Here we denote the objective as $\tilde{u}^\ell_{T-t}$ to emphasize the time homogeneity in the coefficients of the market model, see Assumption 6. Using open loop or closed loop controls arrives at the same expression of $\tilde{u}^\ell_{T-t}$, which is the smallest nonnegative solution of a Cauchy problem

$$\begin{aligned} \frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y})}{\partial t} &= \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}), \quad (\tau, \mathbf{x}^N, \mathbf{y}) \in (0, T] \times (0,\infty)^n \times (0,\infty)^n, \\ \tilde{u}^\ell(0, \mathbf{x}^N, \mathbf{y}) &= e^{c_\ell}, \qquad\qquad (\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n, \end{aligned}$$

wherein $\mathcal{A}$ is an operator defined in (27), $\mathcal{Y}(t)$ is the empirical mean of ν_t^N, see (7). The resulting strategy we consider is

$$\pi_i^{\ell\star} = \mathbf{m}_i(t) + X_i^N(t)D_i\tilde{v}(t) + \tau_i(t)\sigma^{-1}(t)D_{p_i}\tilde{v}(t),$$

where

$$\tilde{v}(t) = \log \tilde{u}^\ell_{T-t} + \frac{1-\delta}{\delta X_t^N} \cdot \frac{1}{N}\sum_{\ell=1}^{N} V_t^\ell \log \tilde{u}^\ell_{T-t}.$$

It turns out that $\pi_i^{\ell\star}$ and $\tilde{u}^\ell_{T-t}$ are proportional to c_ℓ. We show the existence of relative arbitrage through modified portfolio generating functions and the Fichera drift.

After the discussion of N-player games of relative arbitrage, this paper applies the philosophy of mean field games from [4] and [15] to search for approximate Nash equilibrium when $N \to \infty$. It is expected to be more tractable than the N-player games and might give us more information of the finite investors situation. This approach of comparing N-player game and the corresponding mean field game is also discussed in [20], where the Merton problems with constant equilibrium strategies are studied. General results on limits of N-player game are first developed in [16] and [21]. The large population system in these papers are reformulated by [3] into the stochastic version to accommodate with the common noise. With the notion of weak MFG, [18] and [5] study mean field game with common noise in open loop equilibrium.

The relative arbitrage problem provide a new application and some modifications in mean field games. Because of the special problem set-up, there are two mean field measures evolve in different directions, while the uniqueness of Nash equilibrium depend on one of the measures. In particular, the objective relies on a weighted average of stock prices and the distribution of wealth

$$J^{\nu}(\pi) := \inf\Big\{\omega > 0 \,\Big|\, v = \omega\mathcal{V}(0),\, V^{v,\pi}(T) \geqslant e^{c} \cdot \mathcal{V}(T)\Big\},$$

where the mean field benchmark is given by

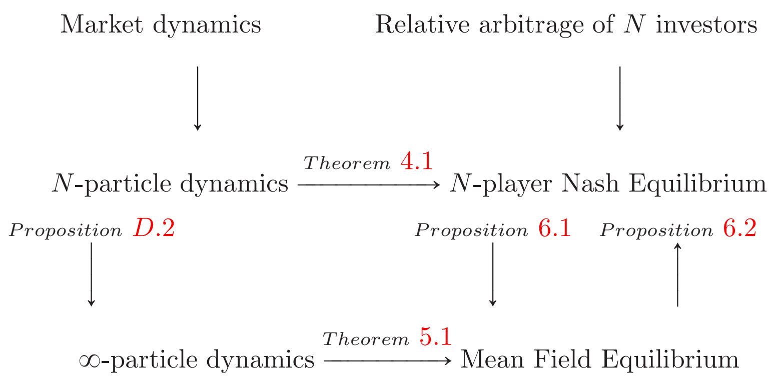
$$\mathcal{V}(T) = \delta \cdot X(T) + (1-\delta)\cdot m_T, \quad m := \mathbb{E}[V|\mathcal{F}^B].$$

On the other hand, the state processes depend on the conditional law of wealth and strategies $\nu := \mathrm{Law}(V, \pi|\mathcal{F}^B)$ with respect to the Brownian motion B. This yields the McKean-Vlasov SDEs of stock prices and a representative player's wealth

$$d\mathcal{X}_t = \beta(\mathcal{X}_t, \nu_t, m_t)dt + s(\mathcal{X}_t, \nu_t, m_t)dB_t, \quad X(0) = \mathbf{x};$$

$$dV_t = \pi(t)\beta(\mathcal{X}_t, \nu_t, m_t)dt + \pi(t)\sigma(\mathcal{X}_t, \nu_t, m_t)dB_t, \quad V(0) = \tilde{u}(T)\mathcal{V}(0).$$

A modified extended mean field game model is introduced. Both open and closed loop equilibrium are considered here regarding the well-posedness of mean field system and the approximation of games. We summarize these results in the following diagram.



Main Contributions

From the perspective of portfolio theory, we construct a general framework for multi-player portfolio optimization problem with no assumption on the equivalent martingale measure. We propose an interactive market model and introduce a relative arbitrage benchmark including peers and the market. The model is characterized as N-player games and mean field games in both open and closed loop, Markovian and non-Markovian case. Additionally, the portfolio generated functionals in SPT are defined accordingly in the multi-player settings. To our knowledge, this is the first paper to study Stochastic Portfolio Theory with market-investors interactions.

From the perspective of stochastic games, this paper contributes to mean field vs N-player game approach and its applications. Firstly, we establish a modified extended mean field game and accommodate a scheme to seek the mean field equilibrium: The infinite-player system involves two different fixed point conditions about the cost functional and the state processes, whereas only one of them is required to be unique. Secondly, we use a stochastic cost function instead of deterministic functions of states and controls, and demonstrate a Cauchy problem path to solve N-player and mean field games instead of the typical HJB or FBSDEs approaches.

Organization of this Paper

The organization of this paper is as follows. Section 2 introduces the market with N investors as a well-posed interacting particles system. Section 3 discusses the relative arbitrage problem and market price of risk processes. In Section 4, the existence and optimization of relative arbitrage is derived in N-player games. The functional generated portfolios in N-player set-up are constructed. Section 5 presents the problem under extended mean field games. We show in Section 6 that mean field game limit is indeed a nice approximation to the N-players game. Finally, we include theoretical supports of the model in Appendix.

2 The Market Model

We consider an equity market and focus on the market behavior and a group of investors in this market. The number of investors is large enough to affect the market. Nevertheless, there are other investors apart from this very group we consider.

2.1 Capitalizations

For a given time horizon $[0, T]$, an admissible market model $\mathcal{M}$ we use in this paper is consisted of a given n dimensional Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))'$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Filtration $\mathbb{F}$ represents the "flow of information" in the market, where $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leqslant t < \infty} = \{\sigma(\omega(s)); 0 < s < t\}$ with $\mathcal{F}(0) = \{\varnothing, \Omega\}$, mod $\mathbb{P}$. $W(\cdot)$ is adapted to the $\mathbb{P}$-augmentation of $\mathbb{F}$. All the local martingales and supermartingales are with respect to the filtration $\mathbb{F}$ if not written out specifically.

Thus, there are n risky assets (stocks) with prices-per-share $\mathcal{X}^N(\cdot) = (X_1^N(\cdot), \ldots, X_n^N(\cdot))$ driven by n independent Brownian motions as follows: for $t \in [0, T]$, $\omega \in \Omega$,

$$dX_i^N(t) = X_i^N(t)(\beta_i(t,\omega)dt + \sum_{k=1}^{n} \sigma_{ik}(t,\omega)dW_k(t)), \quad i = 1, \ldots, n, \tag{3}$$

or

$$X_i^N(t) = x_i^N \exp\left\{ \int_0^t (\beta_i(s,\omega) - \frac{1}{2}\sum_{k=1}^{n}(\sigma_{ik}(s,\omega))^2 dt + \sum_{k=1}^{n}\int_0^t \sigma_{ik}(s,\omega)dW_k(s) \right\},$$

with initial condition $X_i^N(0) = x_i^N$. We assume in this paper that $\dim(W(t)) = \dim(\mathcal{X}^N(t)) = n$, that is, we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$. The market $\mathcal{M}$ is hence a complete market. The dimension n is chosen to be large enough to avoid unnecessary dependencies among the stocks we define.

Here, $\beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_n(\cdot))' : [0, T] \times \Omega \to \mathbb{R}^n$ as the mean rates of return for n stocks and $\sigma(\cdot) = (\sigma_{ik}(\cdot))_{n \times n} : [0, T] \times \Omega \to \mathrm{GL}(n)$ as volatilities are assumed to be invertible, $\mathbb{F}$-progressively measurable in which $\mathrm{GL}(n)$ is the space of $n \times n$ invertible real matrices. Then $W(\cdot)$ is adapted to the $\mathbb{P}$-augmentation of the filtration $\mathbb{F}$. To satisfy the integrability condition, we assume

$$\sum_{i=1}^{n} \int_0^T \Big(|\beta_i(t,\omega)| + \alpha_{ii}(t,\omega)\Big)dt < \infty, \tag{4}$$

where $\alpha(\cdot) := \sigma(\cdot)\sigma'(\cdot)$, α_{ii} is the covariance process of X_i^N.

2.2 Wealth and Portfolios

In the above market model, there are N *small* investors, "small" is in the sense that each individual of these N investors has very little influence on the overall system. An investor ℓ uses the proportion $\pi_i^\ell(t)$ of current wealth $V^\ell(t)$ to invest in the stock i at each time t for $\ell = 1, \ldots, N$. The wealth process V^ℓ of an individual investor ℓ is

$$\frac{dV^\ell(t)}{V^\ell(t)} = \sum_{i=1}^{n} \pi_i^\ell(t)\frac{dX_i^N(t)}{X_i^N(t)}, \quad V^\ell(0) = v^\ell. \tag{5}$$

Since equity prices move according to the supply and demand for stock shares, we consider the average capital invested as a factor in the price processes.

Definition 2.1 (Investment strategy, long only portfolio and average capital invested). *(1) An $\mathcal{F}$-progressively measurable and adapted process $\pi^\ell : [0, \infty) \times \Omega \to \mathbb{R}^n$ is called an investment strategy if*

$$\int_0^T (|\pi^{\ell\prime}(t,\omega)\beta(t,\omega)| + \pi^{\ell\prime}(t,\omega)\alpha(t,\omega)\pi^\ell(t,\omega))dt < \infty, \quad T \in (0,\infty), \omega \in \Omega, \text{a.s.} \tag{6}$$

The strategy here is self-financing, since the wealth at any point of time is obtained by trading the initial wealth according to the strategy $\pi(\cdot)$.

(2) $\pi^\ell(\cdot) = (\pi_1^\ell(\cdot), \ldots, \pi_n^\ell(\cdot))'$ *is a long-only portfolio if it is a portfolio that takes values in the set*

$$\Delta_n := \{\pi = (\pi_1, ..., \pi_n) \in \mathbb{R}^n | \pi_1 \geqslant 0, \ldots, \pi_n \geqslant 0; \pi_1 + \ldots + \pi_n = 1\}.$$

An investment strategy that takes value in Δ_n *is called an admissible strategy, and we denote the admissible set as* $\mathbb{A}$. *If* $\pi^\ell \in \mathbb{A}$, *for all* $\ell = 1, \ldots, N$, *then* $(\pi^1, \ldots, \pi^N) \in \mathbb{A}^N$. *In the rest of the paper, we only consider strategies in the admissible set* $\mathbb{A}$.

(3) *Each investor* ℓ *uses the proportion* $\pi_i^\ell(t)$ *of current wealth* $V^\ell(t)$ *to invest in the ith stock at each time* t. *The average amount* $\mathcal{Y}_i(t)$ *invested by* N *players on stock* i *is assumed to satisfy*

$$\begin{aligned} \mathcal{Y}_i(t) = \frac{1}{N}\sum_{\ell=1}^N V^\ell(t)\pi_i^\ell(t) &= \int_0^t \gamma_i(r,\omega)dr + \int_0^t \sum_{k=1}^n \tau_{ik}(r,\omega)dW_k(r), \quad t \in (0,\infty) \\ \frac{1}{N}\sum_{\ell=1}^N V^\ell(0)\pi_i^\ell(0) &= y_{0,i}, \end{aligned} \tag{7}$$

where $\gamma(\cdot)$ *and* $\tau(\cdot)$ *follow*

$$\sum_{i=1}^n \int_0^T \Big(|\gamma_i(t,\omega)| + \psi_{ii}(t,\omega)\Big)dt < \infty \tag{8}$$

for every $T \in [0,\infty)$, $\psi(\cdot) := \tau(\cdot)\tau'(\cdot)$.

In fact, the average capitalization $\mathcal{Y}(t)$ is depending entirely upon $\mathcal{X}^N(t)$ and $\pi(t)$. The process in Definition 2.1(3) is defined for simplicity.

2.3 General finite dynamical system

The interaction between the players is of the mean field type, in that whenever an individual player has to make a decision, he or she sees the average of functions of the private states of the other players. Here we use mean field interaction particle models from statistical physics to describe the market - We model the N investors as N particles, for fixed N.

For any metric space $(\mathbb{X}, d)$, $\mathcal{P}(\mathbb{X})$ denotes the space of probability measures on $\mathbb{X}$ endowed with the topology of weak convergence. $\mathcal{P}_p(\mathbb{X})$ is the subspace of $\mathcal{P}(\mathbb{X})$ of the probability measures of order p. Then $\mu \in \mathcal{P}_p(\mathbb{X})$ holds $\int_{\mathbb{X}} d(x,x_0)^p\mu(dx) < \infty$, where $x_0 \in \mathbb{X}$ is an arbitrary reference point. For $p \geqslant 1$, μ, $\nu \in \mathcal{P}_p(\mathbb{X})$, The p-Wasserstein metric on $P_p(\mathbb{X})$ is defined by

$$W_p(\nu_1,\nu_2)^p = \inf_{\pi\in\Pi(\nu_1,\nu_2)} \int_{\mathbb{X}\times\mathbb{X}} d(x,y)^p\kappa(dx,dy),$$

where d is the underlying metric on the space. $\Pi(\nu_1,\nu_2)$ is the set of Borel probability measures π on $\mathbb{X}\times\mathbb{X}$ with first marginal ν_1 and second marginal ν_2. Precisely, $\kappa(A\times\mathbb{X}) = \nu_1(A)$ and $\kappa(\mathbb{X}\times A) = \nu_2(A)$ for every Borel set $A \subset \mathbb{X}$.

Also, denote by $C([0,T];\mathbb{R}^{d_0})$ the space of continuous functions from $[0,T]$ to $\mathbb{R}^{d_0}$. In this paper, we often take $\mathbb{X} = \mathbb{R}^{d_0}$ when considering a real-valued random variable or take $\mathbb{X}$ as the path space $\mathbb{X} = C([0,T];\mathbb{R}^{d_0})$ for a process, where a fixed number d_0 will be specified later. $\mathcal{P}_p(\mathbb{R}^{d_0})$ equipped with the Wasserstein distance $\mathcal{W}_p$ is a complete separable metric space, since $\mathbb{R}^{d_0}$ is complete and separable.

Definition 2.2 (Empirical measure in the finite N-particle system)**.** *Consider* $(V^\ell,\pi^\ell) \in C([0,T];\mathbb{R}_+) \times C([0,T];\mathbb{A})$ *that are* $\mathcal{F}$*-measurable random variables, for every investor* $\ell = 1,\ldots,N$. *We define empirical measures* $\nu^N \in \mathcal{P}^2(C([0,T],\mathbb{R}_+)\times C([0,T],\mathbb{A})) \cong \mathcal{P}^2(C([0,T],\mathbb{R}_+\times\mathbb{A}))$ *of the random vectors* $(V^\ell(t),\pi^\ell(t))$ *as*

$$\nu_t^N := \frac{1}{N}\sum_{\ell=1}^N \boldsymbol{\delta}_{(V^\ell(t),\pi^\ell(t))}, \quad \forall\ \ell = 1,\ldots,N,$$

where $\boldsymbol{\delta}_x$ *is the Dirac delta mass at* $x \in \mathbb{R}_+\times\mathbb{A}$. *Thus for any Borel set* $A \subset \mathbb{R}_+\times\mathbb{A}$,

$$\nu_t^N(A) = \frac{1}{N}\sum_{\ell=1}^N \boldsymbol{\delta}^A_{(V^\ell(t),\pi^\ell(t))} = \frac{1}{N}\cdot\#\{\ell \leqslant N : (V^\ell(t),\pi^\ell(t)) \in A\}.$$

The admissible strategies $\pi(t)$ might have different structures given the accessible information at time t.

Definition 2.3. *A control $\pi(t) \in \mathbb{A}$ is an* ***open loop control*** *if it is a function of time t and initial states v_0. It is called a* ***closed loop feedback control*** *if $\pi(t) \in \mathbb{A}$ is a function of time t and states of every controller $\mathbf{V}(t)$. It is specified by feedback functions $\phi^\ell : [0,T] \times \Omega \times \mathbb{R}^n_+ \to \mathbb{A}$, for $\ell = 1, \ldots, N$, to be evaluated along the path of the state process.*

Denote $\mathcal{X}_t^N = (X_1(t), \ldots, X_n(t))$, $\mathbf{V}_t = (V^1(t), \ldots, V^N(t))$. For a fixed N, with ν_t^N in definition 2.2 that generalizes $\mathcal{Y}(t)$, we can generalize the N-particle system as

$$d\mathcal{X}_t^N = \mathcal{X}_t^N \beta(t, \mathcal{X}_t^N, \nu_t^N)dt + \mathcal{X}_t^N \sigma(t, \mathcal{X}_t^N, \nu_t^N)dW_t; \quad \mathcal{X}_0^N = \mathbf{x}_0^N \tag{9}$$

and

$$dV_t^\ell = V_t^\ell \pi_t^\ell \beta(t, X_t^N, \nu_t^N)dt + V_t^\ell \pi_t^\ell \sigma(t, X_t^N, \nu_t^N)dW_t; \quad V_0^\ell = v^\ell. \tag{10}$$

A strong solution of the conditional Mckean-Vlasov system (9)-(10) is a triplet $(\mathcal{X}^N, \mathbf{V}, \nu^N)$, with $\mathcal{X}^N$ taking values in $C([0,T], \mathbb{R}^n_+)$, $\mathbf{V}$ in $C([0,T], \mathbb{R}^N_+)$, $\nu^N \in \mathcal{P}^2(C([0,T], \mathbb{R}_+) \times C([0,T], \mathbb{A})) \cong \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$. The following assumptions on the triplet ensure that the system is well-posed.

Assumption 1. *The initial wealth and strategies of the N players are i.i.d samples from ν_0^N the distribution of (v_0, π_0). The stock prices at time 0, x_0, has a finite second moment, $\mathbb{E}|x_0|^2 < \infty$, and is independent of Brownian motion $\{W_t\}$.*

Assumption 2. *The following Lipschitz conditions are satisfied with Borel measurable mappings β, σ from $[0,T] \times C([0,T], \mathbb{R}_+) \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$ to $\mathbb{R}^n$, i.e., there exists a constant $L \in (0, \infty)$, such that*

$$|x\beta(t,x,\nu) - x'\beta(t,x',\nu')| + |x\sigma(t,x,\nu) - x'\sigma(t,x',\nu')| \leqslant L[|x - x'| + \mathcal{W}_2(\nu, \nu')]$$

$$|v\beta(t,x,\nu) - v'\beta(t,x',\nu')| + |v\sigma(t,x,\nu) - v'\sigma(t,x',\nu')| \leqslant L[|(x,v) - (x',v')| + \mathcal{W}_2(\nu, \nu')]$$

and the growth conditions for a constant $C^G \in (0, \infty)$,

$$|x\beta(t,x,\nu)| + |x\sigma(t,x,\nu)| \leqslant C^G(1 + |x| + M_2(\nu)),$$

$$|v\beta(t,x,\nu)| + |v\sigma(t,x,\nu)| \leqslant C^G(1 + |x| + |v| + M_2(\nu)),$$

for every $t \in [0,T]$, $x \in \mathbb{R}^n_+$, $\nu \in \mathcal{P}_2(C([0,T], \mathbb{R}_+ \times \mathbb{A})$, where

$$M_2(\nu) = \left(\int_{C([0,T], \mathbb{R}_+ \times \mathbb{A})} |x|^2 d\nu(x) \right)^{1/2}; \quad \nu \in \mathcal{P}_2(C([0,T], \mathbb{R}_+ \times \mathbb{A})).$$

Assumption 3. *For a closed loop feedback control, we assume π^ℓ is Lipschitz on v, i.e., there exists a mapping $\phi^\ell : \mathbb{R}^n_+ \times \mathbb{R}^N_+ \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A})) \to \mathbb{A}$ such that $\pi_t^\ell = \phi^\ell(\mathbf{V}_t)$.*

$$|\phi^\ell(x,v,\nu) - \phi^\ell(x',v',\nu')| \leqslant L[|(x,v) - (x',v')| + \mathcal{W}_2(\nu, \nu')]$$

for every $x, x' \in \mathbb{R}^n_+$, $v, v' \in \mathbb{R}^n_+$, $\nu, \nu' \in \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$

Proposition 2.1. *Under Assumption 2 and 3, the N-particle SDE system (9)-(10) admits a unique strong solution, for each N.*

Proof. For simplicity, we discuss the time homogeneous case, whereas the inhomogeneous case can be proved in the same fashion. Rewrite the system as a $n + N$-dimension SDE system:

$$d\begin{pmatrix} \mathcal{X}_t^N \\ \mathbf{V}_t \end{pmatrix} = \begin{pmatrix} X_1^N(t)\beta_1(\mathcal{X}_t^N, \nu_t^N)dt + X_1(t)\sum_{k=1}^n \sigma_{1k}(\mathcal{X}_t^N, \nu_t^N)dW_k(t) \\ \cdots \\ X_n^N(t)\beta_n(\mathcal{X}_t^N, \nu_t^N)dt + X_n(t)\sum_{k=1}^n \sigma_{nk}(\mathcal{X}_t^N, \nu_t^N)dW_k(t) \\ V_t^1 \pi_t^{1\prime} \beta(\mathcal{X}_t^N, \nu_t^N)dt + V_t^1 \pi_t^{1\prime} \sigma(\mathcal{X}_t^N, \nu_t^N)dW_t \\ \cdots \\ V_t^N \pi_t^{N\prime} \beta(\mathcal{X}_t^N, \nu_t^N)dt + V_t^N \pi_t^{N\prime} \sigma(\mathcal{X}_t^N, \nu_t^N)dW_t \end{pmatrix} := f(\mathcal{X}_t^N, \mathbf{V}_t, \nu_t^N)dt + g(\mathcal{X}_t^N, \mathbf{V}_t, \nu_t^N)dW_t,$$

where $f(\mathcal{X}_t^N, \mathbf{V}_t, \nu_t) = (f_1(\cdot), \ldots, f_{n+N}(\cdot))$, $f_i(\cdot) = X_i^N(t)\beta_i(\cdot)$ for $i = 1, \ldots, n$, $f_j(\cdot) = \pi_t^{j-n}\beta(\cdot)$ for $j = n+1, \ldots, n+N$. Similiarly, $g(\mathcal{X}_t^N, \mathbf{V}_t, \nu_t) = (g_1(\cdot), \ldots, g_{n+N}(\cdot))$, $g_i(\cdot) = X_i^N(t)\sigma_i(\cdot)$ for $i = 1, \ldots, n$, $g_j = V_t^{j-n}\pi_t^{j-n}\sigma(\mathcal{X}_t^N, \nu_t)$ for $j = n+1, \ldots, n+N$.

Let us consider a closed loop strategy $\pi_t^\ell = \phi^\ell(\mathbf{V}_t)$. Open loop strategies case can be show in the same way. Define a mapping $L_N : \mathbb{R}_+^N \to \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$,

$$L_N(\mathbf{V}_t) = \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(V_t^\ell, \phi^\ell(\mathbf{V}_t))} = \nu_t^N.$$

Define $F : \mathbb{R}_+^{N+n} \to \mathbb{R}^{N+n}$, $G : \mathbb{R}_+^{N+n} \to \mathbb{R}^{N+n} \times \mathbb{R}^n$, with

$$F(\mathcal{X}_t, \mathbf{V}_t) = f(\mathcal{X}_t, \mathbf{V}_t, L_N(\mathbf{V}_t)); \quad G(\mathcal{X}_t, \mathbf{V}_t) = g(\mathcal{X}_t, \mathbf{V}_t, L_N(\mathbf{V}_t)).$$

Let $(x, v) = (x_1, \ldots, x_n, v^1, \ldots, v^N)$ and $(y, u) = (y_1, \ldots, y_n, u^1, \ldots, u^N)$ be two pairs of values of $(\mathcal{X}(\cdot), \mathbf{V}(\cdot))$. By the inequality $(a+b)^2 \leqslant 2(a^2 + b^2)$, uniformly boundedness and Lipschitz condition of ϕ^ℓ,

$$\begin{aligned}
&|F(x,v) - F(y,u)|^2 \\
\leqslant &\sum_{i=1}^{n} |x_i\beta_i(x, L_N(v)) - y_i\beta_i(y, L_N(u))|^2 + \sum_{\ell=1}^{N} |v^\ell\phi^\ell(v)\beta(x, L_N(v)) - u^\ell\phi^\ell(u)\beta(y, L_N(u))|^2 \\
\leqslant &2L^2[|x-y|^2 + |v-u|^2 + N\mathcal{W}_2^2(L_N(v), L_N(u)].
\end{aligned}$$

Denote the empirical measure induced by the joint distribution of random variable u and v by

$$\tilde{\pi} = \frac{1}{N}\sum_{\ell=1}^{N} \boldsymbol{\delta}_{(u^\ell, v^\ell)}.$$

It is a coupling of the function $L_N(v)$ and $L_N(u)$. By the definition of Wasserstein distance,

$$\mathcal{W}_2^2(L_N(v), L_N(u)) \leqslant \int_{\mathbb{R}^N \times \mathbb{R}^N} |v-u|^2 \tilde{\pi}(dv, du) \leqslant \frac{1}{N}|v-u|^2$$

We treat $G(\cdot)$ in the same fashion, and consequently,

$$|F(x,v) - F(y,u)|^2 \leqslant 4L^2[|x-y|^2 + |v-u|^2], \quad |G(x,v) - G(y,u)|^2 \leqslant 4L^2[|x-y|^2 + |v-u|^2].$$

Then according to the existence and uniqueness conditions of multi-dimensional SDEs, the system (9)-(10) admits a unique strong solution. □

3 Optimization of relative arbitrage in finite systems

3.1 Arbitrage relative to the market and investors

We first recall the definition of relative arbitrage in Stochastic Portfolio Theory.

Definition 3.1 (Relative Arbitrage)**.** *Given two investment strategies $\pi(\cdot)$ and $\rho(\cdot)$, with the same initial capital $V^\pi(0) = V^\rho(0) = 1$, we shall say that $\pi(\cdot)$ represents an arbitrage opportunity relative to $\rho(\cdot)$ over the time horizon $[0,T]$, with a given $T > 0$, if*

$$\mathbb{P}\big(V^\pi(T) \geqslant V^\rho(T)\big) = 1 \quad \textit{and} \quad \mathbb{P}\big(V^\pi(T) > V^\rho(T)\big) > 0,$$

The *market portfolio* $\mathbf{m}$ is used to describe the behavior of the market: By investing in proportion to the market weight of each stock,

$$\pi_i^{\mathbf{m}}(t) := \frac{X_i^N(t)}{X^N(t)}, \quad i = 1, \ldots, n, \quad t \geqslant 0, \tag{11}$$

it amounts to the ownership of the entire market - the total capitalization

$$X^N(t) = X_1^N(t) + \ldots + X_n^N(t),$$

for $t \geqslant 0$, since

$$\frac{dV^{\mathbf{m}}(t)}{V^{\mathbf{m}}(t)} = \sum_{i=1}^{n} \pi_i^{\mathbf{m}}(t) \cdot \frac{dX_i^N(t)}{X_i^N(t)} = \frac{dX^N(t)}{X^N(t)}, \quad t \geqslant 0; \quad V^{\mathbf{m}}(0) = X^N(0). \tag{12}$$

The performance of a portfolio is measured withh respect to the market portfolio and other factors. For example, asset managers improves not only absolute performance comparing to the market index, but also relative performance with respect to all collegial managers - they try to exploit strategies that achieve an arbitrage relative to market and peer investors. We next define the benchmark of the overall performance.

Definition 3.2 (Benchmark)**.** *Relative arbitrage benchmark $\mathcal{V}^N(T)$, which is the weighted average of performances of the market portfolio and the average portfolio of N investors, is defined as,*

$$\mathcal{V}^N(T) = \delta \cdot X^N(T) + (1-\delta) \cdot \frac{1}{N} \sum_{\ell=1}^{N} \frac{V^\ell(T)}{v^\ell}, \quad T \in (0, \infty), \tag{13}$$

with a given constant weight $0 < \delta < 1$.

We assume each investor measures the logarithmic ratio of their own wealth at time T to the benchmark in (13), and searches for a strategy with which the logarithmic ratio is above a personal level of preference almost surely. For $\ell = 1, \ldots, N$, we denote the investment preference of investor ℓ by c_ℓ, a real number given at $t = 0$. Note that c_ℓ is investor-specific constant, and so it might be different among individuals $\ell = 1, \ldots, N$. An arbitrary investor ℓ tries to achieve

$$\log \frac{V^\ell(T)}{\mathcal{V}^N(T)} > c_\ell, \quad \text{a.s.} \quad \text{or equivalently,} \quad V^\ell(T) \geqslant e^{c_\ell} \mathcal{V}^N(T), \quad \text{a.s.}. \tag{14}$$

Thus $\mathcal{V}^N(T)$ is the benchmark and an investor ℓ aims to match $e^{c_\ell}\mathcal{V}^N(T)$ based on their preferences.

Assumption 4. *Assume that the preferences of investors c_ℓ are statistically identical and independent samples from a common distribution $Law(c)$.*

Assumption 5. *We assume the existence of a market price of risk processes $\theta, \lambda : [0,\infty) \times \Omega \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A})) \to \mathbb{R}^n$, an $\mathbb{F}$-progressively measurable process such that for any $(t, \omega, \nu) \in [0,\infty) \times \Omega \times \mathcal{P}^2(C([0,T], \mathbb{R}_+ \times \mathbb{A}))$,*

$$\sigma(t,\omega,\nu)\theta(t,\omega,\nu) = \beta(t,\omega,\nu), \quad \tau(t,\omega,\nu)\lambda(t,\omega,\nu) = \gamma(t,\omega,\nu); \tag{15}$$

$$\mathbb{P}\Big(\int_0^T ||\theta(t,\omega,\nu)||^2 + ||\lambda(t,\omega,\nu)||^2 dt < \infty, \forall T \in (0,\infty) \Big) = 1.$$

The new price of risk process $\lambda(t)$ is from the fact that the market is simultaneously defined by the stocks and the investors. In future sections, We shall see $\lambda(t)$ is a key entity for more tractable and practical results in game formations. Next we define the deflator based on the market price of risk processes.

Definition 3.3. *We define a local martingale $L(t)$,*

$$dL(t) = \Theta(t)L(t)dW_t, \quad \textit{where} \quad \Theta(t) = \sqrt{||\theta(t)||^2 + ||\lambda(t)||^2}.$$

Equivalently,

$$L(t) := \exp\Big\{ -\int_0^t (\theta'(s) + \lambda'(s))dW(s) - \frac{1}{2}\int_0^t (||\theta(s)||^2 + ||\lambda(s)||^2)ds \Big\}, \quad 0 \leqslant t < \infty.$$

Thus under Assumption 5, the market is endowed with the existence of a local martingale L with $\mathbb{E}[L(T)] \leqslant 1$. We denote the discounted processes $\widehat{V}^\ell(\cdot) := V^\ell(\cdot)L(\cdot)$, and $\widehat{X}(\cdot) := X(\cdot)L(\cdot)$. $\widehat{V}^\ell(\cdot)$ admits

$$d\widehat{V}^\ell(t) = \widehat{V}^\ell(t)\pi^{\ell\prime}(t)\big(\beta(t) - \sigma(t)\Theta(t)\big)dt + \widehat{V}^\ell(t)\big(\pi^{\ell\prime}(t)\sigma(t) - \Theta'(t)\big)dW(t); \quad \widehat{V}^\ell(0) = \widehat{v}_\ell. \tag{16}$$

Proposition 3.1. *We have the following properties of c_ℓ and δ.*

1. *In [7] a special case is considered, when $c_\ell = c$ for every $\ell = 1, \ldots, N$, and $\delta = 1$;*

2. *If every investor achieves relative arbitrage opportunity in the sense of* (14), *then*

$$(1-\delta)\sum_{\ell=1}^{N}\frac{e^{c_\ell}}{v^\ell} < 1; \tag{17}$$

3. *Relative arbitrage is guaranteed, if* $(c_1,\ldots,c_N)$ *satisfies that*

$$c_\ell \leqslant \log\left(\frac{V^\ell(T)}{\min\{X^N(T),V^1(T),\ldots,V^N(T)\}}\right) \quad a.s.; \tag{18}$$

4. *When* $c_\ell \geqslant \log v_\ell - \log(\delta v + 1 - \delta)$, *if* $L(T)$ *is a martingale, then no arbitrage relative to the market and investors is possible.*

We already know from [7] and [8] that any $c_\ell \leqslant 0$ is a valid level of satisfaction. (17) in Proposition 3 tells us that c_ℓ can be a small positive number. Investors pursuing relative arbitrage should follow the condition (17) for c_ℓ.

Now, we shall answer the questions posed in the introduction - Given the portfolios

$$\pi^{-\ell}(\cdot) := (\pi^1(\cdot),\ldots,\pi^{\ell-1}(\cdot),\pi^{\ell+1}(\cdot),\ldots,\pi^N(\cdot)),$$

of all but investor ℓ, what is the best strategy to achieve relative arbitrage for investor $\ell = 1,\ldots,N$, and if there exists such optimal strategy, is it possible for all N investors to follow it? We first utilize an idea in the same vein of optimal relative arbitrage in [7], i.e., using the optimal strategy $\pi^{\ell\star}$, the investor ℓ will start with the least amount of the initial capital (or initial cost) relative to $\mathcal{V}^N(0)$, in order to match or exceed the benchmark $e^{c_\ell}\mathcal{V}^N(T)$ at the terminal time T, that is, given $\pi^{-\ell}(\cdot)$, each investor ℓ optimizes

$$u^\ell(T) = \inf\left\{\omega^\ell \in (0,\infty)\,\middle|\,\exists\pi^\ell(\cdot)\in\mathbb{A} \text{ such that } v^\ell = \omega^\ell\mathcal{V}^N(0),\ V^{v^\ell,\pi^\ell}(T) > e^{c_\ell}\cdot\mathcal{V}^N(T)\right\}. \tag{19}$$

To use martingale representation results in a complete market, we assume $\mathcal{F} = \mathcal{F}^{X^N,Y} = \mathcal{F}^W$, where $\mathcal{F}^{X^N,Y}$ is $\{\sigma(\mathcal{X}^N(s),\mathcal{Y}(s)); 0 < s < t\}$. The following proposition is essential to allow a PDE characterization of the objective $u^\ell(T)$. This result follows from the supermartingale property of $\widehat{V}^\ell(\cdot)$ and martingale representation theorem, see Appendix B for the details of the proof.

Proposition 3.2. *$u^\ell(T)$ in (19) can be derived as $e^{c_\ell}\mathcal{V}^N(T)$'s discounted expected values over $\mathbb{P}$.*

$$u^\ell(T) = \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)\big]/\mathcal{V}^N(0), \tag{20}$$

3.2 PDE characterization of the control problem

Starting from this section, we consider $\mathcal{X}^N(t)$ and $\mathcal{Y}(t)$ are time homogeneous processes.

Assumption 6. *$\beta(t)$, $\sigma(t)$, $\gamma(t)$ and $\tau(t)$ are time-homogeneous, i.e.,*

$$X_i^N(t)\beta_i(t) = b_i(\mathcal{X}^N,\mathcal{Y}),\quad X_i^N(t)\sigma_{ik}(t) = s_{ik}(\mathcal{X}^N,\mathcal{Y}),\quad \sum_{k=1}^{n}s_{ik}(t)s_{jk}(t) = a_{ij}(\mathcal{X}^N,\mathcal{Y}),$$

$$\gamma_i(t) = \gamma_i(\mathcal{X}^N,\mathcal{Y}),\quad \tau_i(t) = \tau_i(\mathcal{X}^N,\mathcal{Y}).$$

where $b_i, s_{ik}, a_{ij}, \gamma_i, \tau_i : (0,\infty)^n \times (0,\infty)^n \to \mathbb{R}$ are Hölder continuous and $\mathcal{Y} := (\mathcal{Y}_1([0,T]),\ldots,\mathcal{Y}_n([0,T]))$ is the total trading volume defined in (7). Market price of risk is $\Theta(\mathbf{x}^N,\mathbf{y}) := \sigma^{-1}(\mathbf{x}^N,\mathbf{y})b(\mathbf{x}^N,\mathbf{y})$, for each $T\in(0,\infty)$.

We define $\tilde{u}^\ell : (0,\infty)\times(0,\infty)^n\times(0,\infty)^n \to (0,\infty)$ from the processes $(\mathcal{X}^N(\cdot),\mathcal{Y}(\cdot))$ starting at $(\mathbf{x},\mathbf{y})\in(0,\infty)^n\times(0,\infty)^n$, and write the terminal values

$$\tilde{u}^\ell(T) = \tilde{u}^\ell(T,\mathbf{x}^N,\mathbf{y});\quad \ell = 1,\ldots N. \tag{21}$$

3.2.1 Open loop and closed loop control problem

We use the notation D_i and D_{ij} for the partial and second partial derivative with respect to the ith or the ith and jth variables in $\mathcal{X}^N(t)$, respectively; D_p and D_{pq} for the first and second partial derivative in $\mathcal{Y}(t)$.

Assumption 7. *There exist two functions $H, I : \mathbb{R}^n_+ \to \mathbb{R}$ of class C^2, such that*

$$b(\mathbf{x}^N, \mathbf{y}) = a(\mathbf{x}^N, \mathbf{y}) DH(\mathbf{x}), \quad \gamma(\mathbf{x}^N, \mathbf{y}) = \psi(\mathbf{x}^N, \mathbf{y}) DI(\mathbf{y}),$$

i.e., $b_i(\cdot) = \sum_{j=1}^n a_{ij}(\cdot) D_j H(\cdot)$, $\gamma_i(\cdot) = \sum_{j=1}^n \psi_{ij}(\cdot) D_j I(\cdot)$ in component wise for $i = 1, \ldots, n$.

Hence the infinitesimal operator can be written as

$$\mathcal{L}f = \sum_{i=1}^n \sum_{j=1}^n a_{ij}(\mathbf{x}^N, \mathbf{y}) \big[\frac{1}{2} D_{ij} f + D_i f D_j H(\mathbf{x}^N, \mathbf{y})\big] + \sum_{p=1}^n \sum_{q=1}^n \psi_{pq}(\mathbf{x}^N, \mathbf{y}) \big[\frac{1}{2} D_{pq} f + D_p f D_q I(\mathbf{x}^N, \mathbf{y})\big],$$

and by the definition of $\theta(\cdot)$ and $\lambda(\cdot)$ in (15),

$$\theta(\mathbf{x}^N, \mathbf{y}) + \lambda(\mathbf{x}^N, \mathbf{y}) = s(\mathbf{x}^N, \mathbf{y}) DH(\mathbf{x}) + \tau(\mathbf{x}^N, \mathbf{y}) DI(\mathbf{y}). \tag{22}$$

Then it follows from Ito's lemma applying on $H(\cdot)$ and $I(\cdot)$ that

$$\begin{aligned} L(t) &= \exp\Big\{ - \int_0^t \theta'(s) + \lambda'(s) dW(s) - \frac{1}{2} \int_0^t ||\theta(s)||^2 + ||\lambda(s)||^2 ds \Big\} \\ &= \exp\Big\{ - H(\mathcal{X}^N(t)) - I(\mathcal{Y}(t)) + H(\mathbf{x}) + I(\mathbf{y}) - \int_0^t k(\mathcal{X}^N(s)) + \tilde{k}(\mathcal{Y}(s)) ds \Big\}, \end{aligned}$$

where

$$\begin{aligned} k(\mathbf{x}) &:= - \sum_{i=1}^n \sum_{j=1}^n \frac{a_{ij}}{2} [D^2_{ij} H(\mathbf{x}) + D_i H(\mathbf{x}) D_j H(\mathbf{x})], \\ \tilde{k}(\mathbf{y}) &:= - \sum_{i=1}^n \sum_{j=1}^n \frac{\psi_{pq}}{2} [D^2_{pq} I(\mathbf{y}) + D_p I(\mathbf{y}) D_q I(\mathbf{y})]. \end{aligned}$$

Denote

$$g^\ell(\mathbf{x}^N, \mathbf{y}, \pi) := e^{c_\ell} \mathcal{V}^N(0) e^{-H(\mathbf{x}) - I(\mathbf{y})}, \quad G^\ell(T, \mathbf{x}^N, \mathbf{y}) := \mathbb{E}^{\mathbb{P}}\big[g^\ell(\mathcal{X}^N(T), \mathcal{Y}(T)) e^{-\int_0^T k(\mathcal{X}^N(t)) + \tilde{k}(\mathcal{Y}(t)) dt}\big]. \tag{23}$$

Based on [14] Section 6.4, we have the following assumptions of make sure the solvability of the Cauchy problem.

Assumption 8. *Assume $\mathbb{E}^{\mathbb{P}}\big|g^\ell(\mathcal{X}^N(t), \mathcal{Y}(t)) e^{-\int_0^T k(\mathcal{X}^N(t)) + \tilde{k}(\mathcal{Y}(t)) dt}\big| < \infty$. The functions $b_i(\cdot), \sigma_{ik}(\cdot)$ are of class $C^1((0,\infty)^n \times (0,\infty)^n)$ and satisfy the linear growth condition*

$$||b(\mathbf{x}^N, \mathbf{y})|| + ||s(\mathbf{x}^N, \mathbf{y})|| \leqslant C(1 + ||\mathbf{x}|| + ||\mathbf{y}||), \quad (\mathbf{x}^N, \mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+.$$

$a_{ij}(\cdot)$ satisfy the nondegeneracy condition, i.e., if there exists a number $\epsilon > 0$ such that

$$a_{ij}(\mathbf{x}^N, \mathbf{y}) \geqslant \epsilon(||\mathbf{x}||^2 + ||\mathbf{y}||^2), (\mathbf{x}^N, \mathbf{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+.$$

$g^\ell(\cdot)$ is Hölder continuous, uniformly on compact subsets of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, $\ell = 1, \ldots, N$. $k(\cdot)$ and $\tilde{k}(\cdot)$ are continuous and lower bounded, $G^\ell(\cdot)$ is continuous on $(0,\infty) \times (0,\infty)^n \times (0,\infty)^n$, of class $C^2((0,\infty) \times (0,\infty)^n \times (0,\infty)^n)$.

Under Assumption 8, (20) becomes

$$\tilde{u}^\ell(T, \mathbf{x}^N, \mathbf{y}) = \frac{G^\ell(T, \mathbf{x}^N, \mathbf{y})}{g^\ell(\mathbf{x}^N, \mathbf{y})}, \tag{24}$$

where $\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}) \in C^2((0,\infty) \times (0,\infty)^n \times (0,\infty)^n)$ is bounded on $K \times (0,\infty)^n \times (0,\infty)^n$ for each compact $K \subset (0,\infty)$. By Feynman-Kac formula, the function $G^\ell(\cdot)$ solves

$$\frac{\partial G^\ell}{\partial \tau}(\tau, \mathbf{x}^N, \mathbf{y}) = \mathcal{L} G^\ell(\tau, \mathbf{x}^N, \mathbf{y}) - \big(k(\mathbf{x}) + \tilde{k}(\mathbf{y})\big) G^\ell(\tau, \mathbf{x}^N, \mathbf{y}), \quad t \in (0,\infty), (\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n,$$

$$G^\ell(0, \mathbf{x}^N, \mathbf{y}) = g(\mathbf{x}^N, \mathbf{y}), \ (\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n.$$

This yields a Cauchy problem

$$\frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y})}{\partial \tau} = \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}), \quad \tau \in (0,\infty), \ (\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n, \tag{25}$$

$$\tilde{u}^\ell(0, \mathbf{x}^N, \mathbf{y}) = e^{c_\ell}, \qquad (\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n, \tag{26}$$

where

$$\mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\mathbf{x}^N, \mathbf{y})\Big(D^2_{ij}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}) + 2\delta D_i\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}) \cdot [\mathcal{V}^N(0)]^{-1}\Big) + \frac{1}{2}\sum_{p=1}^n\sum_{q=1}^n \psi_{pq}(\mathbf{x}^N, \mathbf{y}) D^2_{pq}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}). \tag{27}$$

We emphasize that (25) is determined entirely from the volatility structure of $\mathcal{X}^N(\cdot)$ and $\mathcal{Y}(\cdot)$. Moreover, c_ℓ enters into the initial condition (26). Assumption 6 ensures that the Cauchy problem is solvable.

Remark 1. *If the market price of risk process depended solely on $\theta(\cdot)$ in Assumption 5, then the Cauchy problem $\frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y})}{\partial t} = \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y})$ involves a terminal term $\tilde{u}^\ell(T)$ which would largely increase the intractability.*

Theorem 3.1. *Under Assumption 6, the function $\tilde{u}^\ell : [0,\infty) \times (0,\infty)^n \times (0,\infty)^n \to (0,1]$ is the smallest nonnegative continuous function, of class C^2 on $(0,\infty) \times (0,\infty)^n$, that satisfies $\tilde{u}^\ell(0,\cdot) \equiv e^{c_\ell}$ and*

$$\frac{\partial \tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y})}{\partial t} \geqslant \mathcal{A}\tilde{u}^\ell(\tau, \mathbf{x}^N, \mathbf{y}), \tag{28}$$

where $\mathcal{A}(\cdot)$ follows (27).

3.3 Existence of Relative Arbitrage

The Cauchy problem (25)-(26) admits a trivial solution $\tilde{u}^\ell(\tau, \mathbf{x}, \mathbf{y}) \equiv e^{c_\ell}$. Meanwhile, We use portfolio generating functionals, as shown in Section 4, to construct relative arbitrage portfolios for a certain time span. This result indicates that $\tilde{u}(\tau, \mathbf{x}, \mathbf{y})$ could take values less than 1, that is, the uniqueness of Cauchy problem fails.

Through the Föllmer exit measure [13] we can relate the solution of Cauchy problem $u^\ell(\cdot)$ to the maximal probability of a supermartingale process staying in the interior of the positive orthant through $[0,T]$. Following the route suggested by [7] and [26], there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$, such that $\mathbb{P}$ is locally absolutely continuous with respect to $\mathbb{Q}$: $\mathbb{P} << \mathbb{Q}$, $\Lambda^\ell(T)$ is a $\mathbb{Q}$-martingale, and $d\mathbb{P} = \Lambda^\ell(T)d\mathbb{Q}$ holds on each $\mathcal{F}_T$, $T \in (0,\infty)$. We can characterize $\tilde{u}^\ell(t)$ by an auxiliary diffusion which takes values in the nonnegative orthant $[0,\infty)^{2n}/\{\mathbf{0}\}$.

Definition 3.4 (Auxiliary process and the Fichera drift)**.** *We define the following*

1. *The auxiliary process $\zeta^\ell = (\zeta^\ell_1, \ldots, \zeta^\ell_{2n})$ is defined as*

$$d\zeta^\ell_i(\cdot) = \hat{b}_i(\zeta(\cdot))dt + \hat{\sigma}_{ik}(\zeta(\cdot))dW_k, \quad \zeta^\ell_i(0) = \zeta^\ell_i, \quad i = 1, \ldots, 2n,$$

where

$$\hat{b}_i(\mathbf{x}^N, \mathbf{y}) = \begin{cases} \frac{\delta}{\mathcal{V}^N(0)}\sum_{j=1}^n a_{ij}(\mathbf{x}^N, \mathbf{y}) & \text{if } i = 1, \ldots, n, \\ 0 & \text{if } i = n+1, \ldots, 2n, \end{cases}$$

$$\hat{a}_{ij}(\mathbf{x}^N, \mathbf{y}) = \begin{cases} a_{ij}(\mathbf{x}^N, \mathbf{y}) & \text{if } i,j = 1, \ldots, n, \\ \psi_{ij}(\mathbf{x}^N, \mathbf{y}) & \text{if } i,j = n+1, \ldots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

2. *The Fichera drift is defined as*

$$f_i(\cdot) := \hat{b}_i(\mathbf{x}^N, \mathbf{y}) - \frac{1}{2}\sum_{j=1}^n D_j\hat{a}_{ij}(\mathbf{x}^N, \mathbf{y}),$$

$i = 1, \ldots, 2n$, $(\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$.

Assumption 9. *The system of $\zeta^\ell(\cdot)$ admits a unique-in-distribution weak solution with values in $[0,\infty)^n \times [0,\infty)^n/\{\mathbf{0}\}$.*

We set $\mathcal{T}^\ell := \{t \geqslant 0 | \zeta^\ell(t) \in \mathcal{O}^{2n}\}$ as the first hitting time of auxiliary process $\zeta^\ell(\cdot)$ to $\mathcal{O}^{2n}$, the boundary of $[0,\infty)^{2n}$.

Proposition 3.3. *With the nondegeneracy condition of a_{ij}, suppose that the functions $\hat{\sigma}_{ik}(\cdot)$ are continuously differentiable on $(0,\infty)^{2n}$; that the matrix $\hat{a}(\cdot)$ degenerates on $\mathcal{O}^{2n}$; and that the Fichera drifts for the process $\zeta^\ell(\cdot)$ can be extended by continuity on $[0,\infty)^{2n}$. For an investor ℓ, if $f_i(\cdot) \geqslant 0$ holds on each face of the orthant, then $\tilde{u}^\ell(\cdot,\cdot) \equiv 1$, and no arbitrage with respect to the market portfolio exists on any time-horizon. If $f_i(\cdot) < 0$ on each face $\{x_i = 0\}$, $i = 1,\ldots,n$ and $\{y_i = 0\}$, $i = n+1,\ldots,2n$ of the orthant, then $\tilde{u}^\ell(\cdot,\cdot) < 1$ and arbitrage with respect to the market portfolio exists, on every time-horizon $[0,T]$ with $T \in (0,\infty)$.*

Proof. With the nondegeneracy condition of covariance $(a_{ij})_{1\leqslant i,j\leqslant n}$, Theorem 2 in [7] suggests that

$$\tilde{u}^\ell(T,\mathbf{x}^N,\mathbf{y}) = \mathbb{Q}[\mathcal{T}^\ell > T], \quad (T,\mathbf{x}^N,\mathbf{y}) \in [0,\infty)\times[0,\infty)^n\times[0,\infty)^n.$$

For the first claim, we only need to show the probability $\mathbb{Q}[\mathcal{T}^\ell > T] \equiv 1$, for $(T,\mathbf{x}^N,\mathbf{y}) \in [0,\infty)\times[0,\infty)^n\times[0,\infty)^n$. Denote a bounded and connected C^3 boundary $G_R := \{z \in \mathbb{R}^{2n}, z_i < 0, ||z|| < R\}$, and R can be arbitrarily large. Then the claim follows from Theorem 9.4.1 (or Corollary 9.4.2) of [14], since

$$\sum_{i=1}^n \Big(\hat{b}_i(\mathbf{x}^N,\mathbf{y}) - \frac{1}{2}\sum_{j=1}^n D_j\hat{a}_{ij}(\mathbf{x}^N,\mathbf{y})\Big)\mathbf{n}_i \leqslant 0,$$

in which $\mathbf{n} = (\mathbf{n}_1,\ldots,\mathbf{n}_{2n})$ is the outward normal vector at $(\mathbf{x}^N,\mathbf{y})$ to $\mathcal{O}^{2n}$, the boundary $\mathcal{O}^{2n}$ is an obstacle from outside of G_R, i.e., $\mathcal{G} := B_R(0)/G_R$. The Fichera vector field points toward the domain interior at the boundary. Let $R\to\infty$, the boundary is not attainable almost surely for $(\mathbf{x}^N,\mathbf{y}) \in [0,\infty)^{2n}$.

If $f_i(\cdot) < 0$ on each face $\{z_i = 0\}$, $i = 1,\ldots,2n$, then

$$\sum_{i=1}^n \Big(\hat{b}_i(\mathbf{x}^N,\mathbf{y}) - \frac{1}{2}\sum_{j=1}^n D_j\hat{a}_{ij}(\mathbf{x}^N,\mathbf{y})\Big)\mathbf{n}_i \geqslant 0,$$

and the Fichera drift at $\mathcal{O}^{2n}$ points toward the exterior of $[0,\infty)^{2n}$. It is equivalent to show that $\mathbb{Q}[\mathcal{T}^\ell > T] < 1$, for $(T,\mathbf{x}^N,\mathbf{y}) \in [0,\infty)\times[0,\infty)^n\times[0,\infty)^n$, we only need to show $\mathbb{Q}[\mathcal{T}^\ell < T] > 0$, i.e., the boundary $\{z_i = 0\}$, $i = 1,\ldots,2n$, is attainable by $\zeta^\ell(\cdot)$.

From Chapter 11 and 13 in [14], every point in $\partial\mathcal{G}$ is a regular point, and thus

$$\lim_{z\to z_0, z\in\mathcal{G}} \mathbb{Q}_z(\tau^g < \infty, ||\zeta^\ell(\tau^g) - z_0|| < \delta) = 1,$$

where τ^g is the exit time from $\bar{\mathcal{G}}$. Therefore, if $z_0 \in \Sigma := \cup_{i=1}^{2n}\{z \in \mathbb{R}^{2n} : z_i = 0\} \cap \overline{\mathcal{G}}$, for a fixed δ such that $B_\delta^+(z_0) := \cap_{i=1}^{2n}\{z \in \mathbb{R}^{2n} : z_i > 0\} \cap B_\delta(z_0)$ is a proper subset of $\mathcal{G}$, we have

- If $||\zeta_i^\ell - z_0|| \leqslant \eta$,

$$\mathbb{Q}(\tau^g < \infty,\ \zeta^\ell(\tau^g) \in \Sigma) > 0$$

- If $||\zeta_i^\ell - z_0|| > \eta$,

$$\inf_{z\in A} \mathbb{Q}_z(\zeta^\ell(\tau^g) \in B_\delta(z_0), \tau^g < \infty) > \frac{1}{2},$$

where

$$A := \bigcap_{i=1}^{2n}\{z \in \mathbb{R}^{2n} : z_i > 0, ||z - z_0|| = \eta\}.$$

Now take $r \in A$ and a continuous sample path $\omega_\star$ such that $\omega_\star(0) = z_0$, $\omega_\star(\tau_\star) = r$, and $\omega_\star(s) \notin A$ for $0 \leqslant s < \tau_\star$, where $\tau_\star := \inf\{t > 0 : \zeta^\ell(t) \in A\}$. Consider an ϵ-neighborhood $N_{\epsilon,\omega_\star}$ of $\omega_\star \in C(\mathcal{G})$,

$$N_{\epsilon,\omega_\star} = \{\omega \in C(\mathcal{G}) :\ \omega(0) = \zeta_i^\ell,\ ||\omega - \omega_\star|| < \epsilon,\ \omega(\tau_\star) = r\} \subset \{\omega \in \Omega : \zeta^\ell(\tau_\star,\omega) \in A\},$$

then the support theorem in [28] shows that

$$\mathbb{Q}_{\zeta_i^\ell}(N_{\epsilon,\omega_\star}) > 0,$$

where $\phi : [0,\infty) \to \mathbb{R}^{2n}$ is continuously differentiable, and $||\cdot||_T^s$ is the supremum norm $||\omega_1 - \omega_2|| = \sup_{0\leqslant s\leqslant \tau_\star} |\omega_1 - \omega_2|$, ω_1, $\omega_2 \in C(\mathcal{G})$. Hence

$$\mathbb{Q}_{z_0}(N_{\epsilon,\omega_\star}) \leqslant \mathbb{Q}_{z_0}(\tau_\star < \infty,\ \zeta(\tau_\star) \in A).$$

Therefore

$$\begin{aligned}
\mathbb{Q}_{\zeta_i^\ell 0}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) &\geqslant \mathbb{Q}_{\zeta_i^\ell}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \\
&\geqslant \mathbb{E}_{\zeta_i^\ell}[\mathbb{Q}_{z_0}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star),\ \tau_\star < \infty)|\mathcal{F}_{\tau_\star}] \\
&= \mathbb{E}_{\zeta_i^\ell}[\mathbb{Q}_{\zeta^\ell(\tau_\star)}(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty)] \\
&\geqslant \mathbb{E}_{\zeta_i^\ell}[\inf_{z\in A} \mathbb{Q}_z(\zeta^\ell(\tau^g) \in \Sigma, \tau^g < \infty) \cdot \mathbf{1}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty)] \\
&\geqslant \frac{1}{2}\mathbb{Q}_{\zeta_i^\ell}(\zeta^\ell(\tau_\star) \in A,\ \tau_\star < \infty).
\end{aligned}$$

The equality in the above expressions is from the strong Markov property of $\zeta^\ell(\cdot)$.

In conclusion, the process $\zeta^\ell(\cdot)$ attain the set $\cup_{i=1}^{2n}\{z_i = 0\}$ with positive probability, so $\tilde{u}^\ell(\cdot,\cdot) < 1$ when Fichera drift $f_i(\cdot) < 0$. □

Therefore investor ℓ can find relative arbitrage opportunities with a unique $\tilde{u}^\ell$, the minimal solution of (28) given $f_i(\cdot) < 0$ on each face of $\mathcal{O}^{2n}$.

4 N player game

As we have seen in the previous sections, the stock prices and investors' wealth are coupled. Variation of one investor's strategies contributes to the change of the trading volume of each stock, and thus the change of stock prices. Consequently, the wealth of others is affected by this investor. In addition, all the investors considered here are competitive. They attempt to not only behave better than the market index but also beat the performance of peers exploiting similar opportunities - everyone simultaneously wishes to optimize their initial wealth to achieve a relative arbitrage.

Investors interact with each other, adopt a plan of actions after analyzing other people's options, and finally, make decisions. This motivates us to model the investors as participants in a N-player game.

4.1 Construction of Nash equilibrium

The solution concept of this N-player game is Nash equilibrium. In this spirit, assuming that the others have already chosen their own strategies, a typical player computes the best response to all the other players, which amounts to the solution of an optimal control problem to minimize the expected cost $\tilde{u}^\ell$. Specifically, when investor ℓ assumes the wealth of other players are fixed, they wish to take the solution of (25) and (26) as their wealth to begin with so that

$$V^\ell(T) \geqslant e^{c_\ell}\mathcal{V}^N(T) = \delta \cdot e^{c_\ell} X^N(T) + (1-\delta)\cdot e^{c_\ell}\frac{1}{N}\sum_{\ell=1}^{N}\frac{V^\ell(T)}{v^\ell}.$$

We articulate the definition of Nash equilibrium in this problem.

Definition 4.1 (Nash Equilibrium). *A vector $\pi^{\ell *} = (\pi_i^{\ell *}, \ldots, \pi_n^{\ell *})$ of admissible strategies in Definition 2.1 is a Nash Equilibrium, if for all $\pi_i^\ell \in \mathbb{A}$ and $i = 1, \ldots, n$,*

$$J^\ell(\pi_i^{\ell *}, \pi_i^{-\ell *}) \leqslant J^\ell(\pi_i^\ell, \pi_i^{-\ell *}), \tag{29}$$

where the cost to investor ℓ yields

$$J^\ell(\pi) := \inf\left\{\omega^\ell > 0 \ \middle|V^{\omega^\ell e^{c_\ell}\mathcal{V}^N(0),\pi^\ell}(T) \geqslant e^{c_\ell}\mathcal{V}^N(T)\right\},$$

where $\pi(\cdot) = (\pi^1(\cdot), \ldots, \pi^N(\cdot))$. *Hence,*

$$\inf_{\pi^\ell \in \mathbb{A}} J^\ell(\pi) = u^\ell(T). \tag{30}$$

Since $v^\ell = \omega^\ell \mathcal{V}^N(0)$, the infimum is attained, and

$$J^\ell(\pi; 0, x_0) = e^{c_\ell} \frac{\mathcal{V}^N(T)}{\mathcal{V}^N(0)} \exp^{-1}\left\{ \int_0^T \pi_t^{\ell\prime}(\beta_t - \frac{1}{2}\alpha_t \pi_t^\ell) dt + \int_0^T \pi_t^{\ell\prime} \sigma_i(t) dW_t \right\} \leqslant \omega^\ell. \tag{31}$$

Each individual aims to minimize the relative amount of initial capital, beginning with which one can match or exceed the benchmark.

Definition 4.2. *With the same conditions in Definition 2.2, we define empirical measures of the random vectors* $\left(\frac{V^\ell(t)}{v^\ell}\right)_{\ell=1,\ldots,N} \in \mathbb{R}_+^N$, *given the initial measure* $\mu_0^N \in \mathcal{P}^2(\mathbb{R}_+)$,

$$\mu^N := \frac{1}{N} \sum_{\ell=1}^N \boldsymbol{\delta}_{(V^\ell / v^\ell)}.$$

Subsequently, we clarify the notion of unique Nash equilibrium we will apply in this paper. Investors pay more attention to the change of the wealth processes than the change of the strategies, since two different strategy processes may result in the same wealth at time T. Therefore we investigate the uniqueness in distribution of wealth, and we use the strong uniqueness here because it satisfies the nature of the investment goal in this paper.

Definition 4.3. *We say that the uniqueness holds for Nash equilibrium if any two solutions* μ_a^N, μ_b^N, *defined on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, *with the same initial law* $\mu_0^N \in \mathcal{P}^2(\mathbb{R}_+)$,

$$\mathbb{P}[\mu_a^N = \mu_b^N] = 1,$$

where μ^N *is the empirical distribution of wealth processes as in definition 4.2.*

We construct the fixed point condition on the control space. Suppose we start from a control π, then solve the equation of wealth processes (5) and trading volume (7) with the equation of optimal cost function (28). If the corresponding optimal strategy $\pi^\star$ agree with π, then the associated μ^N is the Nash equilibrium. We specify the pathodology below.

Searching Nash equilibrium in N-player game

1. Suppose we start with a given set of control processes $\pi := (\pi^1, \ldots, \pi^N)$. With the empirical distribution μ^N and ν^N, solve the N-particle system (9) and (10).
2. We get $J^\ell(\cdot)$ from μ^N and ν^N. Solve $\tilde{u}^\ell(T) := \inf_{\pi \in \mathbb{A}} J^\ell(\pi)$ and the corresponding optimal control $\pi^\star$. We find a function Φ so that $\pi^\star = \Phi(\pi)$.
3. If there exists $\hat{\pi}$, such that $\hat{\pi} = \Phi(\hat{\pi})$, then $\mu^{N\star} := \frac{1}{N}\sum_{\ell=1}^N \boldsymbol{\delta}_{(V^{v^\ell, \hat{\pi}^\ell}/v^\ell)}$ is the Nash equilibrium.

4.2 Open-loop and closed loop Nash equilibrium

We recall the information structure and the types of actions that players take in a game. It is an *open loop Nash equilibrium* if the admissible strategies satisfy the conditions of Definition 4.1, with the control $\pi^\ell(t)$ given by the form

$$\pi^\ell(t) = \phi^\ell(t, \mathbf{v}, W_{[0,t]}), \tag{32}$$

for every $t \geqslant 0$, $\mathbf{v} := (v^1, \ldots, v^N)$, $v^\ell = \tilde{u}^\ell(T)\mathcal{V}^N(0)$, $W_{[0,t]}$ is the path of the Wiener process between time 0 and time t deterministic functions $\phi^\ell : [0, T] \times \Omega \to \mathbb{A}$, $\ell = 1, \ldots, N$. Here, $\pi^{-\ell}$ is the process with the same trajectories as the $(\pi^{1\star}, \ldots, \pi^{\ell\star}, \ldots, \pi^{N*})$, even after player ℓ changes strategy from $\pi^{\ell*}$ to π^ℓ. Thus the strategies π^k for $k \neq \ell$ of the other players are not affected from the deviation of player ℓ.

However, in closed loop equilibria, the trajectory of the state of the system enters the strategies, then when ℓ change $\pi^{\ell*}(t)$ to $\pi^\ell(t)$, other players is likely to be affected. Players at time t have complete information of the states of all the other players at time t, or in other words we allow feedback strategies. As a special case in closed loop equilibria, a Markovian equilibrium is the admissible strategies profile $\pi^\star = (\pi^{1\star}, \ldots, \pi^{\ell\star}, \ldots, \pi^{N\star})$ of the form

$$\pi^\ell(s) = \phi^\ell(s, \mathbf{V}_s^{t,x}), \tag{33}$$

for each (t,x), where $\phi^\ell : [0,T] \times \Omega \times \mathbb{R}_+^n \to \mathbb{A}$, $\mathbf{V}_s^{t,x} := (V^1(s), \ldots, V^N(s))^{t,x}$, and $(V^\ell(s))_{t \leqslant s \leqslant T}$ is the unique solution of

$$\frac{dV^\ell(s)}{V^\ell(s)} = \sum_{i=1}^n \pi_i^\ell(s) \frac{dX_i^N(s)}{X_i^N(s)}, \quad V^\ell(t) = v_t^\ell, \quad t \leqslant s \leqslant T.$$

We have the following result of Nash equilibrium strategies.

Theorem 4.1. *Nash equilibrium is attained when the strategies yield*

$$\pi_i^{\ell\star} = \mathbf{m}_i(t) + X_i^N(t) D_i \bar{v}^N(t) + \tau_i(t)\sigma^{-1}(t) D_{p_i} \bar{v}^N(t). \tag{34}$$

for $\ell = 1, \ldots, N$*, where* $\bar{v}^N(t) = \log \tilde{u}^\ell(T-t, \mathcal{X}_{[0,t]}^N, \mathcal{Y}_{[0,t]}) + (\frac{1-\delta}{\delta X_t^N}) \frac{1}{N} \sum_{\ell=1}^N \frac{V^\ell(t)}{v^\ell} \log \tilde{u}^\ell(T-t, \mathcal{X}_{[0,t]}^N, \mathcal{Y}_{[0,t]})$*. The corresponding Nash equilibrium* $\mu^{N\star}$ *is unique in the sense of Definition 4.3 when the first exit time from the compact set* K *is greater than* T*, i.e.,* $\tau^K > T$ *where*

$$K = \left[0, \frac{\left(N - (1-\delta)\sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t)\right)^2}{N\delta |\sum_{\ell=1}^N e^{c_\ell} D_m \tilde{u}^\ell(T-t)|}\right], \quad \tau^K = \inf\{t \geqslant 0; X^N(t) \in K\}. \tag{35}$$

Proof. For a given choice of $\pi \in \mathbb{A}$, $\tilde{u}^\ell := \inf_{\pi \in \mathbb{A}} J^\ell(\pi)$ is uniquely determined by the smallest nonnegative solution of (28). For simplicity we denote $\tilde{u}^\ell(T-t, \mathcal{X}_{[0,t]}^N, \mathcal{Y}_{[0,t]})$. Assuming that all controls $\pi^k(\cdot)$, $k \neq \ell$ are chosen, player ℓ will choose the optimal strategy $\pi^\star$ that achieves $V^{\ell\star}(\cdot) = e^{c_\ell}\mathcal{V}^N(t)\tilde{u}^\ell(T-t)$. Suppose every player ℓ follows $V^{\ell\star}(\cdot)$, we have a fixed point problem that yields

$$V^{\ell\star}(t) = e^{c_\ell}\tilde{u}^\ell(T-t)\delta X^N(t)\left(1 + \frac{(1-\delta)\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)/v^\ell}{N - (1-\delta)\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)/v^\ell}\right), \tag{36}$$

equivalently,

$$\begin{aligned} \log V^{\ell\star}(t) =& \log \frac{\delta x_0 e^{c_\ell}\tilde{u}^\ell(T)}{1 - (1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T)/v^\ell} + \int_0^t \mathbf{m}_s'(\beta_s - \frac{1}{2}\alpha_s \mathbf{m}_s) ds + \int_0^t \mathbf{m}_t' \sigma_i(s) dW_s \\ &+ \log \tilde{u}^\ell(T-t) - \log\left(1 - (1-\delta)\frac{1}{N}\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)/v^\ell\right). \end{aligned} \tag{37}$$

With a fixed set of control processes π, we solve $\tilde{u}_{T-t}^\ell$, and expect that the $\pi^\star$ will coincide with the fixed π. Thus we can find the Nash equilibrium strategy by comparing $V^{\ell\star}$ in (37) and V^ℓ defined in (10). By Ito's formula on $\tilde{u}^\ell(\cdot)$ as a function of $\mathcal{X}_{[0,t]}^N$ and $\mathcal{Y}_{[0,t]}$, we obtain

$$d\tilde{u}^\ell(T-t) = (\mathcal{L}\tilde{u}^\ell - \frac{\partial \tilde{u}^\ell}{\partial(T-t)})(T-t)dt + \sum_{k=1}^n R_k^\ell(T-t, \mathcal{X}_{[0,t]}^N, \mathcal{Y}_{[0,t]}) dW_k(t),$$

where $\mathcal{L}$ is the infinitesimal generator of $(\mathbf{x}^N, \mathbf{y}) \in (0,\infty)^n \times (0,\infty)^n$, i.e.,

$$\begin{aligned} \mathcal{L}\tilde{u}^\ell(\tau) =& b(\mathbf{x}^N, \mathbf{y}) \cdot \partial_x \tilde{u}^\ell(\tau) + \gamma(\mathbf{x}^N, \mathbf{y}) \cdot \partial_y \tilde{u}^\ell(\tau) \\ &+ \frac{1}{2}\mathrm{tr}\left[a(\mathbf{x}^N, \mathbf{y}) \cdot \partial_{xx}^2 \tilde{u}^\ell(\tau) + \psi(\mathbf{x}^N, \mathbf{y}) \cdot \partial_{yy}^2 \tilde{u}^\ell(\tau) + (s\tau' + \tau s')(\mathbf{x}^N, \mathbf{y}) \cdot \partial_{xy}^2 \tilde{u}^\ell(\tau)\right] \end{aligned}$$

and

$$R_k^\ell(\tau, \mathbf{x}^N, \mathbf{y}) = \sum_{i=1}^n \sigma_{ik}(\mathbf{x}^N, \mathbf{y}) x_i D_i \tilde{u}^\ell(\tau) + \sum_{p=1}^n \tau_{pk}(\mathbf{x}^N, \mathbf{y}) D_p \tilde{u}^\ell(\tau).$$

Thus the volatility term in (37) is

$$\int_0^t \mathbf{m}_i'(s)\sigma_i(s) dW(s) + \int_0^t \frac{1}{\tilde{u}^\ell(T-s)} \sum_{k=1}^n R_k^\ell(T-s) dW_k(s) + \frac{(1-\delta)\mathcal{V}^N(t)}{N\delta X^N(t)} \int_0^t \sum_{\ell=1}^N \frac{e^{c_\ell}}{v^\ell} \sum_{k=1}^n R_k^\ell(T-s) dW_k(s).$$

By comparing the drift and volatility of (10) and (37), we arrive at (34).

Next, to investigate the uniqueness of Nash equilibrium, we consider a fixed point mapping $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ of the empirical mean of wealth m_t^N from (36),

$$\Phi(m_t^N) := \frac{\delta X^N(t) \sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(m_t^N)/v^\ell}{N - (1-\delta)\sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(m_t^N)/v^\ell}.$$

Denote D_m as the partial derivative with respect to m_t^N, then the derivative of $\Phi(m_t^N)$ is

$$\Phi'(m_t^N) = \frac{N\delta X^N(t) \sum_{\ell=1}^N e^{c_\ell} D_m \tilde{u}^\ell(T-t)/v^\ell}{\left(N - (1-\delta)\sum_{\ell=1}^N e^{c_\ell} \tilde{u}^\ell(T-t)/v^\ell\right)^2}.$$

We denote $A_t = N - (1-\delta)\sum_{\ell=1}^N e^{c_\ell}\tilde{u}^\ell(T-t)/v^\ell$. In the above derivative, $D_m\tilde{u}^\ell(T-t) = \sum_{p=1}^N D_p\tilde{u}^\ell(T-t)\phi^\ell$ for open loop controls, and $D_m\tilde{u}^\ell(m_t^N) = \sum_{p=1}^N D_p\tilde{u}^\ell(T-t)(\phi^\ell + \phi^{\ell'} V^\ell)$ when π is of a closed loop form. In addition $0 < \tilde{u}^\ell(T-t) \leqslant 1$, $c_\ell \leqslant \epsilon$ for a positive ϵ, $\partial\phi^\ell/\partial V^\ell$ is bounded by Lipschitz coefficient L under Assumption 3. Hence $|\Phi'(m_t^N)| < 1$ is satisfied when

$$0 \leqslant X^N(t) < \frac{A_t^2}{N\delta|\sum_{\ell=1}^N e^{c_\ell} D_m\tilde{u}^\ell(T-t)/v^\ell|}. \tag{38}$$

For simplicity, we set $D_t = N\delta|\sum_{\ell=1}^N e^{c_\ell} D_m\tilde{u}^\ell(T-t)/v^\ell|$, and $K := [0, \frac{A_t^2}{D_t})$. By mean value theorem, Φ is a contraction of m_t^N. The first exit time for the compact set K is $\tau^K = \inf\{t \geqslant 0; X^N(t) \in K\}$. If $\tau^K > T$ then Nash equilibrium generated by (34) is unique.

By (12), $\log X^N(t)$ is of the same distribution as $\log V^{\mathbf{m}}(t)$. Thus, $X^N(t)$ is a log-normal distribution where

$$\log X^N(t) \sim N\Big(\log x^N + \big(\mathbf{m}(t)\beta(t) - \frac{1}{2}\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)\big)t,\ \mathbf{m}'(t)\alpha(t)\mathbf{m}(t)t\Big).$$

As a result, with the solution $\tilde{u}$ of (28), the probability of attaining the unique Nash equilibrium is

$$P(X^N(t) \in K) = \mathcal{N}\left(\frac{\log\frac{A_t^2}{D_t x^N} - \big(\mathbf{m}(t)\beta(t) - \frac{1}{2}\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)\big)t}{\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)t}\right),$$

where $\mathcal{N}$ is the cumulative distribution function of a standard Gaussian distribution. □

The end of Section 3.2 suggests that optimal strategies are linearly dependent on e^{c_ℓ}, $\ell = 1, \ldots, N$. To illustrate, the investors pursuing relative arbitrage end up with the terminal wealth $V^\ell(T)$ proportional to e^{c_ℓ} if starting from a same initial wealth. However, at every time t, the information of every $V^\ell(t)$, $\ell = 1, \ldots, N$ is required to pinpoint the optimal strategy. Therefore, a mean field regime is discussed in the next chapter to resolve the complexity in N-player game.

As a special case when investment decisions are based upon the current market environment only, we consider the Markovian dynamics so that we write $u^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$. We can obtain the optimal strategies in a different way. This approach will be useful when we derive the mean field equilibriums in the next section.

Assumption 10. *In addtion to Assumption 6, we assume $\beta(t)$, $\sigma(t)$, $\gamma(t)$ and $\tau(t)$ take values in $\mathbb{R}_+^n \times \mathbb{R}_+^n$, and are Markovian, i.e.,*

$$X_i^N(t)\beta_i(t) = b_i(\mathcal{X}^N(t), \mathcal{Y}(t)), \quad X_i^N(t)\sigma_{ik}(t) = s_{ik}(\mathcal{X}^N(t), \mathcal{Y}(t)), \quad \sum_{k=1}^n s_{ik}(t)s_{jk}(t) = a_{ij}(\mathcal{X}^N(t), \mathcal{Y}(t)),$$

$$\gamma_i(t) = \gamma_i(\mathcal{X}^N(t), \mathcal{Y}(t)), \quad \tau_i(t) = \tau_i(\mathcal{X}^N(t), \mathcal{Y}(t)).$$

Proposition 4.1. *Under Assumption 10, when controls of a closed loop Markovian form (33), or an open loop $\phi(t, \mathbf{v}, W_t)$ are adopted, there is a Nash equilibrium $\pi^\star = (\pi^{1\star}, \ldots, \pi^{N\star})$, where for $\ell = 1, \ldots, N$, $\pi^{\ell\star}$ follows*

$$\pi^{\ell\star}(t) = \mathbf{m}_i(t) + X_i^N(t)D_i\tilde{v}^N(t) + \tau_i(t)\sigma^{-1}(t)D_{p_i}\tilde{v}^N(t),$$

where

$$\tilde{v}^N(t) = \log \tilde{u}^\ell(T-t, \mathbf{x}^N, \mathbf{y}) + \Big(\frac{1-\delta}{\delta X^N(t)}\Big)\frac{1}{N}\sum_{\ell=1}^{N}\frac{V^\ell(t)}{v^\ell}\log \tilde{u}^\ell(T-t, \mathbf{x}^N, \mathbf{y}), \tag{39}$$

and $\tilde{u}^\ell(t)$ *is the smallest nonnegative solution in* (28).

Proof. The Markov property of $\mathcal{V}^N(\cdot)$ gives

$$\frac{\mathbb{E}^{\mathbb{P}}\big[\mathcal{V}^N(T)L(T)|\mathcal{F}(t)\big]}{\mathcal{V}^N(t)L(t)} = \frac{\mathbb{E}^{\mathbb{P}}\big[\mathcal{V}^N(T-t)L(T-t)\big]}{\mathcal{V}^N(t)L(t)} = \tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t)),$$

where $\tilde{u}^\ell(\cdot)$ is the minimal nonnegative solution of (28). Again we use the property for $0 \leqslant t \leqslant T$ that $V^\ell(t) = \mathcal{V}^N(t)\tilde{u}^\ell(T-t, \mathcal{X}^N(t), \mathcal{Y}(t))$, the deflated wealth process

$$\hat{V}^\ell(t) := V^\ell(t)L(t) = \mathbb{E}^{\mathbb{P}}\big[\mathcal{V}^N(T)L(T)|\mathcal{F}_t\big]$$

is a martingale. As a result, the dt terms in $d\hat{V}^\ell(t)$ will vanish, namely,

$$\hat{V}^\ell(t) = \hat{V}^\ell(0) + \sum_{k=1}^{n}\int_0^t \hat{V}^\ell(s)B_k(T-s, \mathcal{X}(s), \mathcal{Y}(s))dW_k(s), \ \ 0 \leqslant t \leqslant T, \tag{40}$$

where

$$\begin{aligned} B_k(t,x,\pi) = &\sum_{i=1}^{n}\sigma_{ik}(\mathbf{x}^N,\mathbf{y})x_i D_i \log \tilde{u}^\ell(T-t,\mathbf{x}^N,\mathbf{y}) + \sum_{m=1}^{n}\tau_{mk}(\mathbf{x}^N,\mathbf{y})D_m \log \tilde{u}^\ell(T-t,\mathbf{x}^N,\mathbf{y}) \\ &+ \sum_{i=1}^{n}\frac{\delta X^N(t)}{\mathcal{V}^N(t)}\Big(\frac{x_i}{\sum_{i=1}^n x_i}\sigma_{ik}(t) - \Theta_k(\mathbf{x}^N,\mathbf{y})\Big) + \frac{(1-\delta)/N}{\mathcal{V}^N(t)}\sum_{i=1}^{n}\sum_{\ell=1}^{N}\Big(\frac{V^\ell(t)}{v^\ell}\pi_i^\ell\sigma_{ik}(t) - \frac{V^\ell(t)}{v^\ell}\Theta_k(\mathbf{x}^N,\mathbf{y})\Big). \end{aligned}$$

Thus we have the fixed point problem

$$\begin{aligned} \pi_i^{\ell\star}(t) = &X_i^N(t)D_i \log \tilde{u}^\ell(T-t,\mathbf{x}^N,\mathbf{y}) + \tau_i(\mathbf{x}^N,\mathbf{y})\sigma^{-1}(\mathbf{x}^N,\mathbf{y})D_k \log \tilde{u}^\ell(T-t,\mathbf{x}^N,\mathbf{y}) \\ &+ \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\mathbf{m}_i(t) + \frac{(1-\delta)}{N\mathcal{V}^N(t)}\sum_{\ell=1}^{N}\frac{V^{\ell\star}(t)}{v^\ell}\pi_i^{\ell\star}(t), \end{aligned} \tag{41}$$

where $V^{\ell\star}(t)$ is generated from $\pi^{\ell\star}(t)$.

Next, we check the consistency condition of $\pi^\star$ in (41) and π we start with. Define a map $\rho : \mathbb{A} \to \mathbb{A}$, we want to find a fixed point so that $\rho(\pi) = \pi$. By Brouwer's fixed-point theorem, since $\mathbb{A}$ is a compact convex set, there exists a fixed point for the mapping ρ. In Nash equilibrium, we assume that all players follow the strategy $\pi^\star$ - if we multiply both sides by V^ℓ and then summing over $\ell = 1, \ldots, n$ in (41), and after some computations we conclude

$$\pi_i^{\ell\star} = \mathbf{m}_i(t) + X_i(t)D_i\tilde{v}^N(t) + \tau_i(t)\sigma^{-1}(t)D_k\tilde{v}^N(t), \tag{42}$$

where $\tilde{v}^N(t)$ satisfies (39). □

4.3 Equilibrium with functional generated portfolios

I is the identity matrix of size n, and $\mathbf{1}$ is a n-dimension column of ones. Now we want to show $\mathcal{M}$ contains strong arbitrage opportunities relative to the performance benchmark, at least for sufficiently large real numbers $T > 0$. We illustrate this path by example 4.1. We employ the idea of functional generated portfolios [9] to seek optimal strategies. By doing so, we may reduce the intractability of the N-player game problem.

The market portfolio follows the dynamic

$$d\mathbf{m}_i(t) = \mathbf{m}_i(t)\Big[\gamma_i^m dt + \sum_{k=1}^{n}\tau_{ik}^m(t)dW_k(t)\Big], \ \ i = 1, \ldots, n. \tag{43}$$

Here $\tau^m(t)$ is the matrix with entries $\tau_{ik}^m(t) := \sigma_{ik}(t) - \sum_{j=1}^n \mathbf{m}_j(t)\sigma_{jk}(t)$, $\mathbf{e}_i$ is the ith unit vector in $\mathbb{R}^n$ and the vector $\gamma^m(t)$ is with the entries $\gamma_i^m(t) := (\mathbf{e}_i - \mathbf{m}(t))'(\beta(t) - \alpha(t)\mathbf{m}(t))$.

Theorem 4.2. *Let* $\mathbf{G}_1,\ \mathbf{G}_2 : U \to (0,\infty)$ *be positive* C^2 *functions defined on a neighborhood* U *of* $\mathbb{A}$ *such that for all* i, $x_i D_i \log \mathbf{G}_1(x)$, $x_i D_i \log \mathbf{G}_2(x)$ *are bounded on* $\mathbb{A}$. *For* $t \in [0,T]$, $\mathbf{G}_1$, $\mathbf{G}_2$ *generate the portfolio*

$$\pi^\ell(t) = \widetilde{G}_1(t) + \widetilde{G}_2(t) + \mathcal{R}(t) \tag{44}$$

where

$$\widetilde{G}_1(t) = (D_i \log \mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t))_n(I-\mathbf{1}\mathbf{m}(t));\ \widetilde{G}_2(t) = D\log \mathbf{G}_2(\mathcal{Y}(t))\tau(t)\sigma^{-1}(t);\ \mathcal{R}(t) = \frac{\delta X_i^N(t) + (1-\delta)\mathcal{Y}(t)}{\mathcal{V}^N(t)}.$$

The process

$$d\log \frac{V^\ell(t)}{e^{c_\ell}\mathcal{V}^N(t)} = d\log \mathbf{G}_2(\mathcal{Y}(t)) + d\log \mathbf{G}_1(\mathbf{m}(t)) + d\Xi_t, t \in [0,T], a.s. \tag{45}$$

is with a drift process $\Xi(\cdot)$ *such that a.s., for* $t \in [0,T]$,

$$\begin{aligned}\frac{d\Xi(t)}{dt} =& \widetilde{G}_1(t)\alpha(t)\mathbf{m}(t) + \widetilde{G}_2(t)\alpha(t)\pi^\ell(t) - \frac{1}{2}\Big(||\widetilde{G}_1(t)\sigma||^2 + ||\widetilde{G}_2(t)\sigma||^2 - ||\pi^{\ell\prime}\sigma||^2\Big)\\ &+ \frac{1}{2\mathbf{G}_1(\mathbf{m}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t)) + \frac{1}{2\mathbf{G}_2(\mathcal{Y}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_2(\mathcal{Y}(t))\psi_{ij}(t),\end{aligned}$$

More importantly, the notion of optimal strategies (42) can be treated through Theorem 4.2. Let $\mathbf{G}_1,\ \mathbf{G}_2 := U \to (0,\infty)$ be positive C^2 functions defined on a neighborhood U of $\mathbb{A}$ such that for all i, $x_i D_i \log \mathbf{G}_1(x)$, $x_i D_i \log \mathbf{G}_2(x)$ are bounded on $\mathbb{A}$. We write $\tilde{u}^\ell(t,\mathbf{x},\mathbf{y}) = w^\ell(t,(\mathbf{m}_i)_{i=1,\dots,n},(\mathcal{Y}_i)_{i=1,\dots,n})$, then by taking derivatives of $\mathcal{X}^N(t)$, $\mathcal{Y}(t)$, it follows

$$X_i^N(t)D_i\log\tilde{u}^\ell(T-t,\mathcal{X}^N(t),\mathcal{Y}(t)) = \Big[D_i\log\mathbf{G}_1(\mathbf{m}(t)) - \sum_{i=1}^n D_i\log\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t)\Big]\mathbf{m}_i(t);$$

$$\tau_i(\mathbf{x}^N,\mathbf{y})\sigma^{-1}(\mathbf{x}^N,\mathbf{y})D_{p_i}\log\tilde{u}^\ell(T-t,\mathcal{X}^N(t),\mathcal{Y}(t)) = (D_i\log\mathbf{G}_2(\mathcal{Y}(t)))_n\tau(t)\sigma^{-1}(t)\mathbf{e}_i.$$

Furthermore, we can use portfolio generating functions to find conditions on investment strategies by $\sum_{i=1}^n \pi_i(t) = 1$, $t \in [0,T]$. We get

$$\frac{1-\delta}{N\delta X^N(t)}\sum_{\ell=1}^N V_t^\ell w_t^\ell = w_t^\ell,$$

where $w_t^\ell := X_i(t)D_i\log\tilde{u}^\ell(t) + \tau_i(t)\sigma^{-1}(t)D_{p_i}\log\tilde{u}^\ell(t)$. Hence $\sum_{\ell=1}^N V^\ell(t)w^\ell(t) = 0$ or $\delta X^N(t) = (1-\delta)\frac{1}{N}\sum_{\ell=1}^N V^\ell(t)$. The latter indicates that the market is exactly consisted of the N investors we considered. If $w^\ell(t) = 0$, then every investor is the same, and their strategy follows the market portfolio. If $w^\ell(t) \neq 0$, then $\mathbf{1}'\widetilde{G}_2(t) = 0$, and

$$\sum_{j=1}^n\sum_{i=1}^n D_i\log\mathbf{G}_2(t)(\tau(t)\sigma^{-1}(t))_{ji} = 0. \tag{46}$$

Example 4.1. *Suppose that* $\mathcal{M}$ *is nondegenerate, weakly diverse in [0,T], and has bounded variance, see Appendix A for the definitions. We assume for* $t \in [0,T]$, *there exists constants* $c_0, N_c, M_\pi > 0$ *such that*

$$\widehat{V}^\ell(t)/\widehat{V}^\ell(0) \geqslant c_0 X^N(t)L(t);$$

$$\mathcal{Y}_i(t)/\mathbf{m}_i(t) \leqslant N_c, \quad i = 1,\dots,n;$$

$$|\sum_{i=1}^n \gamma_i(t)| \leqslant M_\pi.$$

Consider the function $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are defined by*

$$\mathbf{G}_1(x) = \prod_{i=1}^n x_i, \quad \mathbf{G}_2(x) = 1 - \frac{1}{2}\sum_{i=1}^n x_i^2.$$

$\mathbf{G}_1$ *and* $\mathbf{G}_2$ *generate the portfolio*

$$\pi_i^\ell(t) = 1 - \Big(n + \frac{\delta X^N(t)}{\mathcal{V}^N(t)}\Big)\mathbf{m}_i(t) + \frac{(1-\delta)\mathcal{Y}_i(t)}{\mathcal{V}^N(t)} + \Big(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\Big)'\tau(t)\sigma^{-1}(t)\mathbf{e}_i, \quad i = 1, \dots, n. \tag{47}$$

Then π^ℓ *strictly dominates* $\mathcal{V}^N(t)$ *in* (13) *if*

$$T \geqslant \frac{nN^2 - 2n^2 - 2}{-2\epsilon n + 2M_\pi n^2 - \frac{2n^2 M_0}{1-\frac{N^2}{2n}} + Mn^2\Big(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda^2_{\max}(\tau)}{\lambda^2_{\min}(\sigma^{-1})}\Big)}.$$

The notations of constants and details of the proof can be found in Appendix C.

5 Mean Field Games

We have observed that it is unlikely to get a tractable equilibrium from N-player game, especially when N is large. In this section, we study relative arbitrage for the infinite limit population of players. With propagation of chaos results provided, a player in a large game limit should feel the presence of other players through the statistical distribution of states and actions. Then they make decisions through a modified objective involves mean field as $N \to \infty$. For this reason, we expect MFG framework to be more tractable than N-player games.

5.1 Formulation of Extended Mean Field Games

We formulate the model on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ which support Brownian motion B, a n-dimensional common noise B, equally distributed as W. The systemic effect of random noises towards the market might be different when we consider a finite or infinite group of investors interacting with the market. B is adapted to the $\mathbb{P}$-augmentation of $\mathbb{F}$ and can explain the limit random noises in the market $\mathcal{M}$ when $N \to \infty$. The admissible strategies $\pi(\cdot) \in \mathbb{A}^{MF}$ follow similiar conditions as (6) and is $\mathcal{F}^B$-progressively measurable.

In general, the stock prices and state processes depend on the joint distribution of (V^ℓ, π_i^ℓ), $\ell = 1, \dots, N$, while the cost function is related to the empirical distribution of the private states. With a given initial condition $\mu_0 \in \mathcal{P}^2(C([0,T];\mathbb{R}_+))$ as a degenerate distribution of value 1, we define the conditional law of $V(t)/v_0$ given $\mathcal{F}^B$ as

$$\mu_t := Law(\frac{V(t)}{v_0}|\mathcal{F}_t^B), \tag{48}$$

and the conditional law of $(V(t), \pi(t))$ given $\mathcal{F}^B$, with a given initial condition $\nu_0 \in \mathcal{P}^2(C([0,T];\mathbb{R}_+ \times \mathbb{A}))$, is

$$\nu_t := Law(V(t), \pi(t)|\mathcal{F}_t^B).$$

Assumption 11. *Assume* $\mathbf{x} \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^n_+)$, *and* $\mathbb{E}[\sup_{0\leqslant t\leqslant T}||(V(t), X(t))||^2] \leqslant \infty$.

Under Assumption 11, the mean field game model contains McKean-Vlasov SDEs of stock prices and wealth

$$d\mathcal{X}(t) = \mathcal{X}(t)\beta(\mathcal{X}(t), \nu_t)dt + \mathcal{X}(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad X_0 = \mathbf{x};$$

$$dV(t)/V(t) = \sum_{i=1}^n \pi_i(t)dX_i(t)/X_i(t). \tag{49}$$

From Proposition D.1-D.3, we show that the above McKean-Vlasov problem admits a unique solution, where $\nu_t := \text{Law}(V(t), \pi(t)|\mathcal{F}_t^B)$. Furthermore, the weak limit of ν^N in Definition 2.2 is exactly ν_t, $V^\ell(t)$ is asymptotically identical independent copies given the common noise B when $\ell = 1, \dots, N$, $N \to \infty$. Hence we consider a representative player which is randomly selected from the infinite number of investors in mean field set-up. Small deviations of a single player would not influence the entire system given the common noise B.

The player competes with the market and the entire group with respect to the benchmark

$$\mathcal{V}(T) = \delta \cdot X(T) + (1-\delta) \cdot \mu_T,$$

and they try to minimize the relative amount of initial capital. The objective is

$$J^{\mu,\nu}(\pi; 0, x_0) := \inf\Big\{\omega > 0 \;\big|V^{\omega e^c \mathcal{V}(0), \pi}(T) \geqslant e^c \mathcal{V}(T)\Big\}. \tag{50}$$

5.2 Mean Field Equilibrium

Specifically, if the mean field interaction is through the expected investments of an investor on assets - the conditional expectation of the product of wealth and controls, a representative player's wealth is

$$dZ_t = d\mathbb{E}(V(t)\pi(t)|\mathcal{F}_t^B) = \gamma(\mathcal{X}(t), Z_t)dt + \tau(\mathcal{X}(t), Z_t)dB_t, \quad Z_0 = z_0, \tag{51}$$

with

$$dX_i(t) = X_i(t)\beta(\mathcal{X}(t), Z_t))dt + X_i(t)\sigma(\mathcal{X}(t), Z_t)dB_t, \quad X_i(0) = \mathbf{x}_{i,0}.$$

Mean field equilibrium appears as a fixed point of best response function.

Definition 5.1. *(Mean Field Equilibrium) Let $\pi^\star(\cdot) \in \mathbb{A}^{MF}$ be an admissible strategy, then it gives mean field equilibrium (MFE) if $J^{\mu,\nu}$ in (50) satisfies*

$$J^{\mu,\nu}(\pi^\star) = \inf_{\pi \in \mathbb{A}^{MF}} J^{\mu,\nu}(\pi).$$

In particular, $\hat{\mathbb{A}} = \arg\inf_{\pi\in\mathbb{A}} J^{\mu,\nu}(\pi)$ denotes the set of optimal controls. In the control problem, the flow of measure $(m_T, Z(T))$ is frozen conditional on the common noise. $(m_T, Z(T))$ is an equilibrium if there exists $\pi^\star \in \hat{\mathbb{A}}$ such that the fixed point of the mean field measure exists, i.e., $m_T = \mathbb{E}[V_T^\star|\mathcal{F}_T^B]$; $Z(T) = \mathbb{E}[Z_T^\star|\mathcal{F}_T^B]$.

Definition 5.2. *We say that uniqueness holds for the MFG equilibrium if any two solutions μ^a, μ^b, defined on filtered probabilistic set-ups $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with the same initial law $\mu_0 \in \mathcal{P}^2(\mathbb{R}_+)$,*

$$\mathbb{P}[\mu^a = \mu^b] = 1,$$

where μ is the distribution of wealth processes as in (48).

When $\mathbb{F} = \mathbb{F}^{X,Z} = \mathbb{F}^B$, the representative agent's optimal initial proportion to achieve relative arbitrage can be characterized as

$$u(T) := \inf_{\pi\in\mathbb{A}^{MF}} J^{\mu,\nu}(\pi) = \mathbb{E}[e^c\mathcal{V}(T)L(T)]/\mathcal{V}(0), \tag{52}$$

and it solves a single Cauchy problem as opposed to the N-dimensional PDEs system in N-player game,

$$\frac{\partial \tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m)}{\partial \tau} \geqslant \mathcal{A}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m), \quad \tilde{u}(0, \mathbf{x}, \mathbf{z}, m) = e^c, \tag{53}$$

$$\text{where} \quad \mathcal{A}\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m) = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_{ij}(\mathbf{x}, \mathbf{z})\left(D_{ij}^2\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m) + \frac{2\delta D_i\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m)}{\mathcal{V}(0)}\right)$$
$$+ \frac{1}{2}\sum_{p=1}^n\sum_{q=1}^n \psi_{pq}(\mathbf{x}, \mathbf{z})\left(D_{pq}^2\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m)\right) + \mathcal{L}_m\tilde{u}(\tau, \mathbf{x}, \mathbf{z}, m),$$

for $\tau \in (0, \infty)$, $(\mathbf{x}^N, \mathbf{y}) \in (0, \infty)^n \times (0, \infty)^n$.

Note that $Z^\star = \mathbb{E}[V^\star\pi^\star|\mathcal{F}^B]$ is not expected to be unique. Moreover, since the diffusion process of $Z(T)$ is given by Definition 2.1(3) and (51), we consider the fixed point over the control space when it comes to $Z(T) = \mathbb{E}[Z_T^\star|\mathcal{F}_T^B]$. The steps of searching equilibrium for extended mean field game with joint measure of state and control is formulated in [4]. The paper [6] manifests an example of extended mean field games with application in price anarchy. They use two different measures as law of the state processes and the law of control. The equilibrium approaching steps we introduce is different in that a modified version of extended mean field game is discussed, where the state processes and cost functional depend on different measures, and uniqueness of Nash equilibrium is specified here. In the following, we show the steps to attain a unique equilibrium in open loop or closed loop Markovian form.

Steps of Solving Mean Field Game

(i) Start with a fixed ϕ such that $\pi = (\pi(t))_{0\leqslant t\leqslant T} = \phi(\mathbf{v}, B_{[0,T]})$ or $\phi(V(t))$, the open loop and feedback function respectively, and solve

$$dV(t) = \pi(t)\beta(\mathcal{X}(t), Z_t)dt + \pi(t)\sigma(\mathcal{X}(t), Z_t)dB_t, \quad V(0) = \tilde{u}(T)\mathcal{V}(0) := v_0,$$

$$dX_i(t) = X_i(t)\beta_i(\mathcal{X}(t), Z_t)dt + X_i(t)\sum_{k=1}^{n}\sigma_{ik}(\mathcal{X}(t), Z_t)dB_k(t), \quad i = 1, \dots, n,$$

where $Z_t = \mathbb{E}[V(t)\pi(t)|\mathcal{F}_t^B]$ for $0 \leqslant t \leqslant T$.

(ii) For each arbitrary stochastic process $m = (m_t)_{0\leqslant t\leqslant T}$ on $\mathbb{R}_+$ adapted to the filtration generated by the random measure B, solve

$$\inf_{\pi\in\mathbb{A}^{MF}} J^{m,Z}(\pi) = u(T) = \mathbb{E}\big[e^c\big(\delta X(T) + (1-\delta)m_t\big)L(T)\big]\,/\,\mathcal{V}(0),$$

using $X(T)$ from step (i). The corresponding $\phi^\star := \arg\inf_{\pi\in\mathbb{A}^{MF}} J^{\mu,\nu}(\pi) = \arg\inf_{\pi\in\mathbb{A}^{MF}} J^{m,Z}(\pi)$. Define the mapping $\phi^\star = \Phi(\phi)$.

(iii) If there exists $\hat{\phi}$ such that $\hat{\phi}^\star = \Phi(\hat{\phi})$, find m so that for all $0 \leqslant t \leqslant T$, $m_t = \mathbb{E}[V_t^\star|\mathcal{F}_t^B]$, where $V^\star$ is the optimal path with $\phi^\star$ as a minimizer of $J^{m,Z}(\phi)$.

Here the fixed point is formulated on both the control space and the flows of measure.

Theorem 5.1. *Under Assumption 2,3, 10 and 11, there exists a unique Mean Field Equilibrium $\mu^\star$.The corresponding Nash equilibrium $\mu^\star$ is unique in the sense of Definition 4.3 when the first exit time from the compact set $\tilde{K}$ is greater than T, i.e., $\tilde{\tau}^K > T$ where*

$$\tilde{K} = \left[0, \frac{\big(1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t)|\mathcal{F}_t^B]\big)^2}{\delta\big|\mathbb{E}[e^c D_m\tilde{u}(T-t)|\mathcal{F}_t^B]\big|}\right], \quad \tilde{\tau}^K = \inf\{t \geqslant 0; X(t) \in \tilde{K}\}, \tag{54}$$

Proof. For simplicity we denote $\tilde{u}(T-t, \mathbf{x}, \mathbf{z}, m) = \tilde{u}(T-t)$. We fix the process m solve the optimal control problem for $V^\star$. Suppose every player follows $V^\star(t) = \mathcal{V}^\star(t)\tilde{u}(T-t)$, we solve a fixed point problem which yields

$$V^\star(t) = \frac{e^c\delta X_t^\star\tilde{u}_{T-t}}{1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t)/v_0|\mathcal{F}_t^B]}. \tag{55}$$

As in Theorem 4.1, after comparing $\log V^\star(t)$ in (49) and (55), this yields

$$\pi_i^\star(t) = \mathbf{m}_i(t) + X_i(t)D_i\tilde{v}(t) + \tau_i(t)\sigma^{-1}(t)D_k\tilde{v}(t),$$

where $\tilde{v}(t) = \log\tilde{u}_{T-t} + \frac{1-\delta}{\delta X_t}\mathbb{E}[\frac{V(t)}{v_0}\log\tilde{u}_{T-t}|\mathcal{F}_t^B]$, and $\tilde{u}_{T-t}$ is the smallest nonnegative solution in (53).

We can further derive the expression of $\pi^\star$ when $\tilde{u}$ is Markovian. We restrict m_t in the form of $\mathbb{E}(V|\mathcal{F}_t^B)$, for each i. From now on, we use *vol* to represent the volatility of a process, as we are not given the explicit form of m_t. By Ito's formula we have

$$\hat{V}(t) = \hat{V}(0) + \sum_{k=1}^{n}\int_0^t \hat{V}(s)B_k(T-s, \mathcal{X}(s), \mathcal{Z}(s))dW_k(s), \ 0 \leqslant t \leqslant T, \tag{56}$$

where

$$\begin{aligned} B_k(\tau, x, z) = &\sum_{i=1}^{n}\sigma_{ik}(\mathbf{x},\mathbf{z})x_iD_i\log\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{z},m) + \tau_m(\mathbf{x},\mathbf{z})D_m\log\tilde{u}^\ell(\tau,\mathbf{x},\mathbf{z},m) \\ &+ \sum_{i=1}^{n}\frac{\delta X(t)/x_0}{\mathcal{V}(t)}\left(\frac{x_i}{\sum_{i=1}^n x_i}\sigma_{ik}(t) - \Theta_k(\mathbf{x},\mathbf{z})\right) + \frac{(1-\delta)}{\mathcal{V}(t)}vol(dL_tm_t). \end{aligned}$$

By comparing (16) and (56), the strategy used for $V^\star$ should be

$$\begin{aligned} \pi_i^\star(t) = &X_i^\star(t)D_i\log\tilde{u}(T-t) + \tau_i(\mathbf{x},\mathbf{z})\sigma^{-1}(\mathbf{x},\mathbf{z})D_k\log\tilde{u}(T-t) \\ &+ \frac{\delta X^\star(t)}{\mathcal{V}^\star(t)}\mathbf{m}_i(t) + \frac{(1-\delta)}{\mathcal{V}^\star(t)}vol(dL_tm_t)\sigma^{-1}. \end{aligned} \tag{57}$$

The derivation of $\pi^\star$ ensures that it generates a wealth process $V^\star$, thus $\pi^\star \in \mathbb{A}^{MF}$.

Next, we show the equilibrium is unique. Denote $\Phi(m_t) := \mathbb{E}[V(t)|B]$, it is equivalent to show that there is the unique fixed point mapping $\Phi(m_t) = m_t$. We have

$$m_t = \Phi(m_t) = \frac{\delta X^\star(t)\mathbb{E}[e^c \tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]}{1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]},$$

$$\Phi'(m_t) = \frac{\delta X^\star(t)\mathbb{E}[e^c D_m\tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]}{\big(1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]\big)^2},$$

First, $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, since from Appendix D we have

$$\mathbb{E}[||V^{\nu_1} - V(t)^{\nu_2}||_t^2|\mathcal{F}_t^B] \leqslant (2t+2)L^2\mathbb{E}[\int_0^t \mathcal{W}_2^2(\nu_1,\nu_2)dr], \tag{58}$$

for any fixed $m_t \in \mathbb{R}_+$, $\Phi(m_t) = \mathbb{E}[V(t)|\mathcal{F}_t^B] \in \mathbb{R}_+$. Furthermore, we set $\tilde{A}_t = 1-(1-\delta)\mathbb{E}[e^c\tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]$, $\tilde{D}_t = \delta\big|\mathbb{E}[e^c D_m\tilde{u}(T-t,m)/v_0|\mathcal{F}_t^B]\big|$. By mean value theorem, Φ is a contraction of m_t if $\tilde{\tau}^K > T$, where $\tilde{\tau}^K = \inf\{t \geqslant 0; X(t) \in \tilde{K}\}$, $\tilde{K} := [0, \frac{\tilde{A}_t^2}{\tilde{D}_t})$. As a result, the mean field equilibrium generated by (57) is unique when the first exit time from $\tilde{K}$ is less than T. The probability of attaining the unique mean field equilibrium is

$$P(X(t) \in \tilde{K}) = \mathcal{N}\Bigg(\frac{\log\frac{A_t^2}{D_t x} - \big(\mathbf{m}(t)\beta(t) - \frac{1}{2}\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)\big)t}{\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)t}\Bigg),$$

where $\mathcal{N}$ is the cumulative distribution function of a standard Gaussian distribution, x is initial value of total capitalization $X(\cdot)$. □

It is clear from (57) that the mean field strategy actually depends on $(\mathcal{X}(t), Z(t), m_t)$, which means the optimal strategies are driven by stock prices, trading volumes and relative arbitrage benchmark. Similarly to N-player game, π is independent of preference c, meaning that the representative player's preference level c is not a crucial factor when determining strategies.

5.3 Example

Next, we encompass a simplified class of market models to shed some light on mean field regimes, where the models exhibit selected characteristics of real equity markets and provide a tractable mean field equilibrium. The class of models is inspired by the volatility-stabilized markets introduced in [10].

Example 5.1. *In real markets, the smaller stocks tend to have greater volatility than the larger stocks. Meanwhile, the higher the trading volume of a stock, the larger the volatility of the trading. The parameters β, σ, γ, τ in $\mathcal{M}$ are set to the following specific forms which agree with these market behaviors. For $1 \leqslant i,j \leqslant n$, with infinite number of investors,*

$$\beta_i(t) = (1+\zeta)\frac{Z_i(t)}{2\mathbf{m}_i(t)}, \quad a_{ij} = X_i(t)\delta_{ij};$$

$$\gamma_i(t) = \beta_i(t); \quad \psi_{ij}(t) = Z_i(t)\delta_{ij}.$$

We can check that the Fichera drift $f_i(\cdot) < 0$. Similiarly to Proposition 3.3, we can get $\tilde{u}(\cdot) < 1$.

By Theorem 5.1, the optimal strategy $\pi_i^{\ell\star}$ of investor ℓ in a mean field game is

$$\begin{aligned}\pi_i^\star(t) =& X_i(t)D_i\log\tilde{u}^\star_{T-t} + \tau_i(t)\sigma_i^{-1}(t)D_{p_i}\log\tilde{u}^\star_{T-t} + vol(m_t)\sigma_i^{-1}(t)D_m\log\tilde{u}^\star_{T-t}\\ &+ \frac{\delta X(t)\mathbf{m}_i(t)}{\delta X(t)+(1-\delta)m_t} + vol(dL_t m_t)\sigma^{-1}(t)\frac{1-\delta}{\delta X(t)+(1-\delta)m_t}.\end{aligned}$$

We denote p_t as the conditional density of $V(t)$ given B_t, which follows

$$dp_t = -(1+\zeta)\partial_v\big[V(t)\sum_{i=1}^n \pi_i(t)\frac{z_i(t)}{2\mathbf{m}_i(t)}p_t\big]dt - V(t)\sum_{i=1}^n \pi_i(t)(X_i(t))^{\frac{1}{2}}\partial_v p_t dB_t.$$

Next, plug $\pi_i^\star(t)$ into the equation of p_t, and let $m_t = \int vp_t(v)dv$, i.e., the consistency condition, we get

$$dm_t = -(1+\zeta)\int m_t\big(X_i^{-\frac{1}{2}}(t)z_i^{-\frac{1}{2}}(t)\partial_v vol(m_t)\frac{1}{m_t} - X_i^{-\frac{1}{2}}(t)z_i^{-\frac{1}{2}}(t)\frac{1}{m_t^2}vol(m_t)\partial_v m_t$$
$$- \frac{2\delta(1-\delta)X(t)z_i(t)}{(2\delta X(t)+2(1-\delta)m_t)^2}\partial_v m_t\big)dvdt$$
$$-2\int m_t\big(\partial_v vol(m_t)\frac{1}{m_t} - \frac{1}{m_t^2}vol(m_t)\partial_v m_t - \frac{\delta(1-\delta)X_i^{\frac{3}{2}}(t)z_i^{\frac{1}{2}}(t)}{(\delta X(t)+(1-\delta)m_t)^2}\partial_v m_t\big)dvdB_t$$
$$-2\int m_t L_t\Big(\frac{(1-\delta)(\Theta_t\partial_v m_t + \partial_v vol(m_t))}{\delta X(t)+(1-\delta)m_t} + \frac{(1-\delta)^2(\Theta_t m_t + vol(m_t))}{(\delta X(t)+(1-\delta)m_t)^2}\partial_v m_t\Big)dvdt,$$

where

$$vol(m_t) = \int \frac{\delta(1-\delta)X_i^{\frac{3}{2}}(t)z_i^{\frac{1}{2}}(t)}{(\delta X(t)+(1-\delta)m_t)^2}\partial_v m_t dv\Big[\frac{16\int\frac{1}{m_t}\partial_v m_t dv}{3(3-4\int\frac{1}{m_t}\partial_v m_t dv)} + \frac{4}{3}\Big].$$

With the explicit m_t expression, we can obtain closed form solution of $\pi^\star(t)$ in terms of $\mathcal{X}(t)$, $Z(t)$, $\tilde{u}_{T-t}$.

If $\delta = 0$, meaning a investor intend to achieve relative arbitrage with respect to peer competitors, and

$$\frac{\partial\tilde{u}(\tau,\mathbf{x},\mathbf{z})}{\partial\tau} = \sum_{i=1}^{2} X_i(\tau)D_{ii}^2\tilde{u}(x,z) + \sum_{p=1}^{2} Z_p(\tau)D_{pp}^2\tilde{u}(x,z). \tag{59}$$

We separate the variables τ, x_1, x_2, z_1, and z_2. We denote $S^k(x)$ as the function for x_1, x_2, z_1, and z_2, when $k = 1,2,3,4$, repsectively. $T(\tau) = e^{c+\xi\tau}$. A general solution can be found as

$$S^k(x) = x_k^{\frac{1}{2}}[c_1 J_1(2(-C_k x_k)^{\frac{1}{2}}) + c_2 Y_1(2(-C_k x_k)^{\frac{1}{2}})],$$

where $\sum_{k=1}^4 C_k = \xi$, J_1 and Y_1 are order 1 Bessel function of first and second kind, respectively. It concludes $\tilde{u} = e^{c+\xi\tau}\prod_{k=1}^4 S^k(x_k)$, which is the smallest amount of initial capital proportion that a generic investor need to outperform the others, given it is the minimal nonnegative solution of (59). Thus we can get an explicit equilibrium if we have the information on $\mathcal{X}(t)$ and $Z(t)$ and initial condition $\mathbf{x}$, z_0. A numerical scheme for the relative arbitrage problem will be considered elsewhere.

6 The relationship of N-player game and mean field game

In this last section we justify if mean field game is an appropriate generalization of N-player relative arbitrage problem.

6.1 From finite games to mean field games

We conclude in the following proposition that the MFE we obtain agrees with the limit of the finite equilibrium, and that the optimal arbitrage in the sense of (19) strongly converges to optimal arbitrage in the mean field game setting (52).

Proposition 6.1. *Suppose $(\beta,\sigma,\gamma,\tau)(t,\mathbf{x},\mathbf{z})$ take values in $\mathbb{R}^n\times GL(n)\times\mathbb{R}^n\times GL(n)$ is bounded and continuous in every variable. If $\pi(\cdot)$ is Markovian, and $\min\{\tau^K,\tilde{\tau}^K\} > T$, then $u(T) = \lim_{N\to\infty} u^\ell(T)$ a.s, for $T\in(0,\infty)$.*

Proof. It follows from Appendix D, $\mathbb{P}\circ(\mathcal{X}^N,\mathbf{V},\nu^N,W)$ is tight on the space $C([0,T];\mathbb{R}_+^n)\times C([0,T];\mathbb{R}_+^N)\times\mathcal{P}^2(C([0,T];\mathbb{R}_+\times\mathbb{A}))\times C([0,T];\mathbb{R}^n)$ and the weak limit exists. We proved in Proposition D.2 that the equilibrium μ is the weak limit of μ^N conditional on B. What left here to show is that the optimal cost in finite game converges to the mean field optimal cost, since $\tilde{u}$ and π are both bounded. By using the Markovian property of $\pi(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$, we would have

$$\tilde{u}^\ell(T-t) := \frac{\mathbb{E}^{\mathbb{P}}\big[e^{c_\ell}\mathcal{V}^N(T-t)L(T-t)\big]}{\mathcal{V}^N(t)L(t)}.$$

Then by the bounded convergence theorem and Proposition D.2, the deflator $L(\mathcal{X}(t),Z(t)) = \lim_{N\to\infty} L(\mathcal{X}^N(t),\mathcal{Y}(t))$ a.s., and $\mathcal{V}(T) = \lim_{N\to\infty}\mathcal{V}^N(T)$ in the weak sense. c_ℓ is i.i.d samples from $Law(c)$.

Therefore as $N \to \infty$,

$$u^\ell(T) := \inf_{\pi\in\mathbb{A}} J^\ell(\pi^{\ell\star}) \to \inf_{\pi\in\mathbb{A}^{MF}} J^{\mu,\nu}(\pi^\star) = u(T)$$

almost surely, and $u(T-t)$ is the weak limit of $u^\ell(T-t)$ when $t > 0$. □

6.2 From mean field games to finite games

We show here that MFE can be used to construct an approximate Nash equilibrium for the N-player game. Since we derive strong equilibrium in both N-player and mean field game, μ^N and μ are measurable with respect to the information generated by W and B, respectively.

From (57), the mean field control in general is of the form

$$\pi_i^\star(t) = \phi(X_i(t), \mathcal{X}(t), \mu_t, \nu_t, \tilde{u}_{T-t}). \tag{60}$$

Definition 6.1. *For $\epsilon_N \geqslant 0$, an open-loop ϵ_N-equilibrium is a tuple of admissible controls*

$$\phi^N := (\phi^{N,1}(t), \dots, \phi^{N,N}(t))_{0\leqslant t\leqslant T}, \quad \phi^{N,\ell}(t) \in \mathbb{A} \subset \Delta^n,$$

for every ℓ, such that

$$J^\ell(\phi^N) \leqslant \inf_{p\in\mathbb{A}} J^\ell(p, \phi^{N,-\ell}) + \epsilon_N,$$

where $p \in \mathbb{A}$ is an open loop control, and ϕ is of the form in (60). An closed-loop ϵ_N-equilibrium is a tuple ϕ^N such that

$$J^\ell(\phi^N) \leqslant \inf_{p\in\mathbb{A}} J^\ell(p^N) + \epsilon_N,$$

where each component in ϕ^N is defined in (60); $p^N := (p(U_{[0,t]}), \phi^{N,-\ell}(U_{[0,t]}))$, in which U_t is the N-vector of wealth processes generated by this strategy, $p : [0,T] \times C([0,T]; \mathbb{R}_+^N) \to \mathbb{A}$ is of the form $(p(t, U_{[0,t]}))_{0\leqslant t\leqslant T}$, $\phi^{N,-\ell}$ is defined in (60). For any $\ell = 1, \dots, N$, $\phi^{N,\ell}$ and p are $\mathbb{F}$-progressively measurable functions.

Proposition 6.2. *Under Assumption 2, 3, and 12, assume $\tilde{u}_{T-t}$ is Lipschitz in $(\mathcal{X}(t), \mu_t, \nu_t)$, there exists a sequence of positive real numbers $(\epsilon_N)_{N\geqslant 1}$ converging to 0, such that any admissible strategy $\pi^\ell = (\pi_t^\ell)_{t\in[0,T]}$ for the first player*

$$J^{N,\ell}(p^1, \pi^{2\star}, \dots, \pi^{N,\star}) \geqslant J - \epsilon_N, \quad \ell = 1, \dots, N.$$

Proof. We look into the approximate open and closed loop Nash equilibrium. Without loss of generality, by the symmetry of the game, we focus on player 1. For a fixed number of players N, each player utilizes the optimal strategy $\pi^\star$ from the associated mean field game, i.e., the strategy set is $\pi^N := (\pi_t^\star, \dots, \pi_t^\star)$ as in (60). The rest part of the proof is mainly adapted to the general method on [4]. We articulate the different part from general method: when π^N deviates to (p, π^{-1}), the state processes are $V^1(t)$ and $V^\ell(t)$, $\ell \neq 1$, and the empirical measures are

$$\mu_t^N = \frac{1}{N}\left(\boldsymbol{\delta}_{V^1(t)} + \sum_{\ell=2}^N \boldsymbol{\delta}_{V^{\star\ell}(t)}\right), \quad \nu_t^N = \frac{1}{N}\left(\boldsymbol{\delta}_{(V^1(t),p)} + \sum_{\ell=2}^N \boldsymbol{\delta}_{(V^{\star\ell}(t),\pi^\star)}\right).$$

We can show $(\mu_t^N, \nu_t^N) \to (\mu_t, \nu_t)$ in open loop, and $(U_t^1, \mu_t^N, \nu_t^N) \to (V_t^p, \mu_t, \nu_t)$ in closed loop in the similiar vein of Proposition D.1 and D.2.

$\mu_0^N \stackrel{d}{=} \mu_0$, $\mathbf{x}^N \stackrel{d}{=} \mathbf{x}$, By Ito's isometry and L_2 convergence we get

$$\lim_{N\to\infty} J^N((p, \pi^{-1}); 0, \mathbf{x}^N) = J^{\mu,\nu}(p; 0, \mathbf{x}).$$

□

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Appendices

A Market dynamics and conditions

This section recalls some properties of the market which are used to show the existence of relative arbitrage.

Definition A.1 (Non-degeneracy and bounded variance)**.** *A market is a family $\mathcal{M} = \{X_1, \dots, X_n\}$ of n stocks, each of which is defined as in (3), such that the matrix $\alpha(t)$ is nonsingular for every $t \in [0, \infty)$, a.s. The market $\mathcal{M}$ is called nondegenerate if there exists a number $\epsilon > 0$ such that for $x \in \mathbb{R}^n$*

$$\mathbb{P}(x\alpha(t)x^T \geqslant \epsilon||x||^2, \forall t \in [0, \infty)) = 1,$$

The market $\mathcal{M}$ has bounded variance from above, if there exists a number $M > 0$ such that for $x \in \mathbb{R}^n$

$$\mathbb{P}(x\alpha(t)x^T \leqslant M||x||^2, \forall t \in [0, \infty)) = 1,$$

Remark 2. *Let π be a portfolio in a nondegenerate market. Then there exists an $\epsilon > 0$ such that for $i = 1, \dots, n$,*

$$\tau^{\pi}_{ii}(t) \geqslant \epsilon(1 - \pi_{\max}(t))^2, \forall t \in [0, \infty) \tag{61}$$

almost surely. Indeed, this is directly from definition A.1, and $\tau^{\pi}_{ii}(t) = \alpha_{ii}(t) - 2\alpha_{i\pi}(t) + \alpha_{\pi\pi}(t)$, where $\alpha_{\pi\pi}(t) = \pi'(t)\alpha(t)\pi(t)$. Details of the proof can be found in [9].

Intuitively, no single company can ever be allowed to dominate the entire market in terms of relative capitalization.

Definition A.2 (Diversity of market)**.** *The model $\mathcal{M}$ of (3), (4) is diverse on the time-horizon $[0, T]$, with $T > 0$ a given real number, if there exists a number $\eta \in (0, 1)$ such that*

$$\max_{1 \leqslant i \leqslant n} \mathbf{m}_i := \mathbf{m}_{(1)} < 1 - \eta, \forall\ 0 \leqslant t \leqslant T \tag{62}$$

almost surely and $\mathcal{M}$ is weakly diverse if there exists a number $\eta \in (0, 1)$ such that

$$\frac{1}{T}\int_0^T \mathbf{m}_{(1)}(t)dt < 1 - \eta, \forall\ 0 \leqslant t \leqslant T \tag{63}$$

almost surely.

Proof of Proposition 3.1. For (2), Since everyone follows $\frac{V^\ell(T)}{v^\ell} \geqslant \frac{e^{c_\ell}}{v^\ell}\mathcal{V}^N(T)$, we sum up this expression for $\ell = 1, \ldots, N$ to get an inequality of $\sum_{k=1}^N V^k(t)/N$. (17) follows immediately. Next, (3) can be easily derived from Definition 3.1 that

$$c_\ell \leqslant \log\left(\frac{V^\ell(T)}{\delta X^N(t) + (1-\delta)\frac{1}{N}\sum_{\ell=1}^N \frac{V^\ell(T)}{v^\ell}}\right).$$

□

B Relative arbitrage and Cauchy problem

Proof of Proposition 3.2. From Ito's formula, discounted process $\widehat{V}^\ell(\cdot)$ admits

$$d\widehat{V}^\ell(t) = \widehat{V}^\ell(t)\pi^{\ell\prime}(t)\big(\beta(t) - \sigma(t)\Theta(t)\big)dt + \widehat{V}^\ell(t)\big(\pi^{\ell\prime}(t)\sigma(t) - \Theta'(t)\big)dW(t); \quad \widehat{V}^\ell(0) = \widehat{v}_\ell,$$

and $\widehat{V}^\ell(\cdot)$ is a supermartingale. For this reason, we get from (19) that for an arbitrary ω^ℓ,

$$\omega^\ell\mathcal{V}^N(0) \geqslant \mathbb{E}\big[\widehat{V}^\ell\big] \geqslant \mathbb{E}\left[\widehat{X}(T)\delta e^{c_\ell} + L(T)(1-\delta)e^{c_\ell}\frac{1}{N}\sum_{\ell=1}^N \frac{V^\ell(T)}{v^\ell}\right] := p^\ell.$$

Hence, $u^\ell(T) \geqslant p^\ell$.

To prove the opposite direction $u^\ell(T) \leqslant p^\ell$, we use martingale representation theorem (Theorem 4.3.4, [23]) to find

$$U^\ell(t) := \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)|\mathcal{F}_t\big] = \int_0^t \tilde{p}'(s)dW_s + p^\ell, \quad 0 \leqslant t \leqslant T, \tag{64}$$

where $\tilde{p} : [0,T]\times\Omega \to \mathbb{R}^k$ is $\mathbb{F}$-progressively measurable and almost surely square integrable. Next, construct a wealth process $V_*(\cdot) = U^\ell(\cdot)/L(\cdot)$, it satisfies $V_*(0) = p^\ell$, $V_*(T) = e^{c_\ell}\mathcal{V}^N(T)$. If we plug a trading strategy

$$h_*(\cdot) = \frac{1}{L(\cdot)V^\ell(\cdot)}\alpha^{-1}(\cdot)\sigma(\cdot)[\tilde{p}(\cdot) + U^\ell(\cdot)\Theta(\cdot)],$$

into (16), further calculations imply $V_*(\cdot) \equiv V^{p,h_*}(\cdot) \geqslant 0$ a.s. $V^{p,h_*}(\cdot)$ is the wealth process from $h_*(\cdot)$. Therefore, $h_*(\cdot) \in \mathbb{A}$ with exact replication property $V^{p,h_*}(T) = e^{c_\ell}\mathcal{V}^N(T)$ a.s. Consequently, $p^\ell \geqslant u^\ell(T)$ for

$$\frac{p^\ell}{\mathcal{V}^N(0)} \in \big\{\omega > 0 | \exists \pi^\ell \in \mathbb{A}, \text{given } \pi^{-\ell}(\cdot) \in \mathbb{A}^{N-1}, \text{ s.t.} V^{\omega\mathcal{V}^N(0),\pi^\ell} \geqslant e^{c_\ell}\mathcal{V}^N(T)\big\}.$$

Thus, we proved $u^\ell(T) = \mathbb{E}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)\big]/\mathcal{V}^N(0)$. □

Proof of Theorem 3.1. Suppose a solution of (28) and (26) is $\tilde{w}^\ell : C^2((0,\infty)\times(0,\infty)^n\times(0,\infty)^n) \to (0,\infty)$. Define $\tilde{N}(t) := \tilde{w}^\ell(T-t, \mathcal{X}^N_{[0,t]}, \mathcal{Y}_{[0,t]})e^{c_\ell}\mathcal{V}^N(t)L(t)$, $0 \leqslant t \leqslant T$.

By calculating $d\tilde{N}(t)/\tilde{N}(t)$ and using the inequality (28), we get that the dt terms in $d\tilde{N}(t)/\tilde{N}(t)$ is always no greater than 0. $\tilde{N}(t)$ is a local supermartingale. And since $\tilde{N}(t) = \tilde{w}^\ell(T-t, \mathcal{X}_{[0,t]}, \mathcal{Y}_{[0,t]})e^{c_\ell}\mathcal{V}^N(t)L(t) \geqslant 0$, $\tilde{N}(t)$ is a supermartingale.

Hence $\tilde{N}(0) = \tilde{w}^\ell(T, \mathbf{x}, \mathbf{y})\mathcal{V}^N(0) \geqslant \mathbb{E}^{\mathbb{P}}[\tilde{N}(t)] = \mathbb{E}^{\mathbb{P}}[e^{c_\ell}\mathcal{V}^N(T)L(T)]$ holds for every $(T,\mathbf{x},\mathbf{y}) \in (0,\infty)\times(0,\infty)^n\times(0,\infty)^n$. Then $\tilde{w}^\ell(T,\mathbf{x},\mathbf{y}) \geqslant \mathbb{E}^{\mathbb{P}}\big[e^{c_\ell}\mathcal{V}^N(T)L(T)\big]/\mathcal{V}^N(0) = \tilde{u}^\ell(T,\mathbf{x},\mathbf{y})$. □

C Function Generated Portfolios

Proof of Theorem 4.2. By Ito's lemma,

$$d\log\frac{V^\ell(t)}{e^{c_\ell}\mathcal{V}^N(t)} = \big[\pi^{\ell\prime}(t)(\beta(t) - \frac{\alpha(t)}{2}\pi^\ell(t)) - \mathcal{R}'(t)\beta(t) + \frac{1}{2}||\mathcal{R}'(t)\sigma(t)||^2\big]dt + \big[\pi^{\ell\prime}(t) - \mathcal{R}'(t)\big]\sigma(t)dW(t). \tag{65}$$

Since $\mathbf{G}_1$ and $\mathbf{G}_2$ are twice continuously differentiable function, it follows

$$\begin{aligned}
D_{ij}\log\mathbf{G}_1(\mathbf{m}(t)) &= \frac{D_{ij}\mathbf{G}_1(\mathbf{m}(t))}{\mathbf{G}_1(\mathbf{m}(t))} - D_i\log\mathbf{G}_1(\mathbf{m}(t))D_j\log\mathbf{G}_1(\mathbf{m}(t)),\\
D_{ij}\log\mathbf{G}_2(\mathcal{Y}(t)) &= \frac{D_{ij}\mathbf{G}_2(\mathcal{Y}(t))}{\mathbf{G}_2(\mathcal{Y}(t))} - D_i\log\mathbf{G}_2(\mathcal{Y}(t))D_j\log\mathbf{G}_2(\mathcal{Y}(t))
\end{aligned} \tag{66}$$

Then using (66) and Ito's lemma, the right hand side of (45) becomes

$$\begin{aligned}
d\log\mathbf{G}_1(\mathbf{m}(t)) + d\log\mathbf{G}_2(\mathcal{Y}(t)) &= \sum_{i=1}^n D_i\log\mathbf{G}_1(\mathbf{m}(t))d\mathbf{m}_i(t)\\
&+ \frac{1}{2\mathbf{G}_1(\mathbf{m}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))dt\\
&- \frac{1}{2}\sum_{i,j=1}^n D_i\log\mathbf{G}_1(\mathbf{m}(t))D_j\log\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))dt\\
&+ \sum_{i=1}^n D_i\log\mathbf{G}_2(\mathcal{Y}(t))d\mathcal{Y}_i(t) + \frac{1}{2\mathbf{G}_2(\mathcal{Y}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_2(\mathcal{Y}(t))\psi_{ij}(t)dt\\
&- \frac{1}{2}\sum_{i,j=1}^n D_i\log\mathbf{G}_2(\mathcal{Y}(t))D_j\log\mathbf{G}_2(\mathcal{Y}(t))\psi_{ij}(t)dt,
\end{aligned} \tag{67}$$

The local martingale part of (65) and (67) are the same, and this leads to

$$\pi^{\ell\prime}(t) = \left[(D_i\log\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t))_n'(I - \mathbf{1}m'(t)) + (D_i\log\mathbf{G}_2(\mathcal{Y}(t)))_n\tau(t)\sigma^{-1}(t)\right] + \mathcal{R}(t),$$

for $t \in [0, T]$, and for each k. Substitute this result into (65),

$$\begin{aligned}
d\log\frac{V^\ell}{e^{c_\ell}\mathcal{V}(t)} =& d\log\mathbf{G}_2(\mathcal{Y}(t)) + d\log\mathbf{G}_1(\mathbf{m}(t))\\
&- \Big\{\widetilde{G}_1(t)(-\alpha(t)\mathbf{m}(t)) + \widetilde{G}_2(t)(-\alpha(t)\pi^\ell(t)) + \frac{1}{2}\big(||\widetilde{G}_1(t)\sigma||^2 + ||\widetilde{G}_2(t)\sigma||^2 - ||\pi^{\ell'}\sigma||^2\big)\\
&- \frac{1}{2\mathbf{G}_1(\mathbf{m}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_1(\mathbf{m}(t))\mathbf{m}_i(t)\mathbf{m}_j(t)(\sum_{k=1}^n \tau_{ik}^m(t)\tau_{jk}^m(t))\\
&- \frac{1}{2\mathbf{G}_2(\mathcal{Y}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{G}_2(\mathcal{Y}(t))\psi_{ij}(t)\Big\}dt.
\end{aligned}$$

□

Lemma C.1. *A matrix A is semi-definite if and only if $(xAy')^2 \leqslant (xAx')(yAy')$ for all x, y in $\mathbb{R}^n$. The equality holds if and only if xA and yA are linearly dependent.*

Lemma C.2. *If $A = (a_{ij})$ is positive semi-definite matrix, then there is an index k such that $a_{kk} \geqslant a_{ij}$, for any i and j. In other words, the largest entry of the matrix A appears on the diagonal.*

We show here the derivation in Example 4.1.

Proof of Example 4.1. Let $\mathcal{M}$ be a market without dividends. Suppose that $\mathcal{M}$ is nondegerate and has bounded variance. Suppose $\mathcal{M}$ is weakly diverse in [0,T]. Consider the function $\mathbf{G}_1$ and $\mathbf{G}_2$ are defined as in example 4.1. $\sqrt{\prod_{i=1}^n \mathbf{m}_i} \leqslant \frac{\sum_{i=1}^n \mathbf{m}_i}{n} \leqslant \sqrt{\frac{\sum_{i=1}^n \mathbf{m}_i^2}{n}}$ implies that

$$0 \leqslant \mathbf{G}_1(\mathbf{m}) \leqslant \frac{1}{n^2}, \quad 1 - \frac{N^2}{2} \leqslant \mathbf{G}_2(\mathcal{Y}(t)) \leqslant 1 - \frac{N^2}{2n}$$

then

$$1 - \frac{N^2}{2} \leqslant \log\mathbf{G}_1(\mathbf{m}) + \log\mathbf{G}_2(\mathcal{Y}(t)) \leqslant \frac{1}{n^2} + 1 - \frac{N^2}{2n}.$$

The portfolio (47) generated by $\mathbf{G}_1$ and $\mathbf{G}_2$ implies

$$\pi_i^\ell > \max\{0, 1-(n+\delta X^N(t)/\mathcal{V}^N(t))(1-\eta) + \left(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\right)'\tau(t)\sigma^{-1}(t)\mathbf{e}_i\}; \tag{68}$$

$$\pi_i^\ell < \min\{1 + \frac{1-\delta}{N}(V^\ell(t))'_N(\pi_i^\ell(t))_N/\mathcal{V}^N(t) + \left(\frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\right)'\tau(t)\sigma^{-1}(t)\mathbf{e}_i, 1\}. \tag{69}$$

Denote $\max_{i=1,\dots,n}\mathbf{m}_i = \mathbf{m}_{(1)}$, $\min_{i=1,\dots,n}\mathbf{m}_i = \mathbf{m}_{(n)}$, $\max_{i=1,\dots,n}\pi_i = \pi_{(1)}$, $\min_{i=1,\dots,n}\pi_i = \pi_{(n)}$, and the eigenvalues of $\alpha(t)$: $\max_{i=1,\dots,n}\lambda_i = \lambda_{(1)}$, $\min_{i=1,\dots,n}\lambda_i = \lambda_{(n)}$. $\mathbf{m}_{\max} := (\mathbf{m}_{(1)}, \mathbf{m}_{(1)}, \dots, \mathbf{m}_{(1)})$.

We'll use the following results to simplify $\Xi(T)$:

(i) $\mathcal{M}$ is nondegenerate, weakly diverse and has bounded variation;

(ii) $\frac{1}{n} \leqslant \sum_{i=1}^n \mathbf{m}_i^2 \leqslant 1$ implies that $0 \leqslant ||(\mathbf{1}-n\mathbf{m})|| \leqslant \sqrt{n(n-1)}$;
$\sum_{i=1}^n(\pi_i^\ell)^2 \leqslant 1$ implies $||\sum_{\ell=1}^N \pi^{\ell\prime}(t)\tau\sigma^{-1}||_2 \leqslant ||\sum_{\ell=1}^N \pi^{\ell\prime}(t)||_2 \cdot ||\tau||_2 \cdot ||\sigma^{-1}||_2 \leqslant N\sqrt{\frac{\lambda_{max}(\psi)}{\lambda_{min}(\alpha)}}$, where the norm for τ and σ^{-1} is matrix induced norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\sqrt{\text{Trace}(AA')} = ||A||_F \leqslant \sqrt{n}||A||_2$, where $||\cdot||_2$ is the matrix induced norm. $\text{Trace}(\tau\tau^{\pi\prime}) = \sum_{i=1}^n\sum_{\ell=1}^N \tau_{ii}^\ell \geqslant n\epsilon\sum_{\ell=1}^N(1-\pi_{(1)}^\ell)^2$, then $||\tau||_2 \geqslant \epsilon\sum_{\ell=1}^N(1-\pi_{(1)}^\ell)^2$;

(iii) $|\beta_i|$ and $|\alpha_{ij}|$ for any i and j is bounded by lemma C.2, thus we could easily get $\mathcal{Y}(t)\tau(t)\sigma^{-1}(t)\beta(t) > M_0$; By lemma C.1, $\mathbf{e}_i'\alpha(t)\mathbf{m}(t) \leqslant (\mathbf{e}_i'\alpha(t)\mathbf{e}_i)(\mathbf{m}'(t)\alpha(t)\mathbf{m}(t)) \leqslant MM'||\mathbf{m}(t)||^2 \leqslant MM'$, where $\mathbf{e}_i'\alpha(t)\mathbf{e}_i \leqslant M||\mathbf{e}_i||^2$, $\mathbf{m}'(t)\alpha(t)\mathbf{m}(t) \leqslant M'||\mathbf{m}(t)||^2$.

$$\begin{aligned}
\Xi(T) &= \int_0^T \left\{(\mathbf{e}_i - \mathbf{m}(t))'\alpha(t)\mathbf{m}(t) + \frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\big(\gamma^\pi(t) - \tau\sigma^{-1}(t)\beta(t)\big) + d^\ell(t)\frac{\alpha(t)}{2}d^{\ell\prime}(t)\right\}dt \\
&\overset{(i)}{\leqslant} \int_0^T \Bigg\{\mathbf{e}_i'\alpha(t)\mathbf{m}_{(1)}(t)\mathbf{1} - \epsilon||\mathbf{m}||^2 + \frac{1}{\mathbf{G}_2(\mathcal{Y}(t))}\sum_{\ell=1}^N \pi^{m\prime}(t)\big(\gamma^\pi(t) - \tau\sigma^{-1}(t)\beta(t)\big) \\
&\quad + \frac{M}{2}\Big[||\mathbf{1}-n\mathbf{m}||^2 + \frac{1}{\mathbf{G}_2(\mathcal{Y}(t))}||\sum_{\ell=1}^N \pi^{\ell\prime}(t)\tau^\ell\sigma^{-1}||^2\Big] - \frac{\epsilon}{2}||\pi^\ell||^2\Bigg\}dt \\
&\overset{(ii,iii)}{\leqslant} T\Big[MM' - \frac{\epsilon}{n} + \frac{M_\pi}{1-\frac{N^2}{2}} - \frac{M_0}{1-\frac{N^2}{2n}} + \frac{M}{2}\Big(n(n-1) + \frac{N^2}{(1-\frac{N^2}{2})^2}\frac{\lambda_{(1)}^2(\tau)}{\lambda_{(n)}^2(\sigma^{-1})}\Big)\Big] - \frac{\epsilon}{2}\int_0^T \max_i|\pi_i^\ell|^2 dt
\end{aligned}$$

where

$$d^\ell(t) := \mathbf{1} - n\mathbf{m} - \pi^\ell(t) + \frac{-\mathcal{Y}(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\tau\sigma^{-1},$$

$$\max_i|\pi_i^\ell|^2 > \big[\max\{0, 1-(n+c_\ell\delta\frac{X^N(t)}{\mathcal{V}(t)})(1-\eta) - \frac{\mathcal{Y}_i(t)}{\mathbf{G}_2(\mathcal{Y}(t))}\tau(t)\sigma^{-1}(t)\mathbf{e}_i\}\big]^2.$$

Hence, for $t \in [0,T]$,

$$\begin{aligned}
\log\frac{V^\ell}{e^{c_\ell}\mathcal{V}^N(t)} &= \log\mathbf{G}_2(\mathcal{Y}(t)) + \log\mathbf{G}_1(\mathbf{m}(t)) + \Xi_t \\
&\leqslant 1 + \frac{1}{n^2} - \frac{N^2}{2n} + T\Big[MM' - \frac{\epsilon}{n} + \frac{M_\pi}{1-\frac{N^2}{2}} - \frac{M_0}{1-\frac{N^2}{2n}} + \frac{M}{2}\Big(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda_{\max}^2(\tau)}{\lambda_{\min}^2(\sigma^{-1})}\Big)\Big].
\end{aligned}$$

Then π strictly dominates the weighted average $\mathcal{V}^N(t)$ if

$$T \geqslant \frac{nN^2 - 2n^2 - 2}{-2\epsilon n + 2M_\pi n^2 - \frac{2n^2M_0}{1-\frac{N^2}{2n}} + Mn^2\Big(n(n-1) + \frac{N^2}{1-\frac{N^2}{2}}\frac{\lambda_{\max}^2(\tau)}{\lambda_{\min}^2(\sigma^{-1})}\Big)}.$$

□

D Limiting behavior of finite games vs mean field games

Differentiating from the usual Mckean-Vlasov SDEs of the form that the coefficients of the diffusion depend on the distribution of the solution itself, we here consider the joint distribution of the state processes and the control, and show the propagation of chaos holds.

In this section we attempt to show that in the limit $N \to \infty$, a vector of stock prices $\mathcal{X}(t) := (X_1(t), \dots, X_n(t))$ and the wealth of a representative player will satisfy Mckean-Vlasov SDEs. Namely,

$$d\mathcal{X}(t) = \mathcal{X}(t)\beta(\mathcal{X}(t), \nu_t)dt + \mathcal{X}(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad \mathcal{X}_0 = \mathbf{x}; \tag{70}$$

$$dV(t) = \pi(t)\beta(\mathcal{X}(t), \nu_t)dt + \pi(t)\sigma(\mathcal{X}(t), \nu_t)dB_t, \quad V(0) = v_0, \tag{71}$$

where $B_t = (B_1, \dots, B_n)$ is n-dimensional Brownian motion, $\nu := Law(V, \pi|\mathcal{F}_t^B)$. v_0 is with the same law as v^ℓ, and it is supported on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Remark 3. *In this section, we analyze a Mckean-Vlasov system with initial states given, but we shall see in mean field games sections that it is given in the form as* $v_0 = \tilde{u}(T)\mathcal{V}(0)$.

The following proposition shows that ν^N has a weak limit $\nu \in \mathcal{P}^2(C([0,T]; \mathbb{R}^n \times \mathbb{A}))$ with $\mathcal{W}_2$ distance. We denote $\mathcal{C}^{n,N} = C([0,T]; \mathbb{R}^n \times \mathbb{A})$ for simplicity.

Proposition D.1. *Under Assumption 2, 3, and 11, there exists a unique strong solution of the Mckean-Vlasov system* (70)-(71).

Proof. Define the truncated supremum norm $||x||_t$ and the truncated Wasserstein distance on $\mathcal{P}^2(\mathcal{C}^{n,N})$ as in [19]. $||x||_t^2 := \sup_{0 \leqslant s \leqslant t} |x_s|$,

$$d_t^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathcal{C}^n \times \mathcal{C}^n} ||x - y||_t^2 \pi(dx, dy).$$

Define a map $\Phi : \mathcal{P}^2(\mathcal{C}^{n,N}) \to \mathcal{P}^2(\mathcal{C}^{n,N})$ so that $\Phi(\nu) = \text{Law}(V^\nu, \pi^\nu|\mathcal{F}^B)$. Fix ν, solve (70) and (71). Since solutions of (70) and (71) are equivalent to fixed points of Φ, we begin by proving that Φ is a contraction mapping in a complete space $\mathcal{C}^{n,N}$.

We take two arbitrary measures $\nu^a, \nu^b \in \mathcal{P}^2(\mathcal{C}^{n,N})$, and denote the wealth involving measure ν as V^ν, and stock price vector involving ν as $\mathcal{X}^\nu$. By Cauchy-Schwartz and Jensen's inequality, Lipschitz conditions in Assumption 2 and 3, it follows

$$\begin{aligned}
&\mathbb{E}[||(V^{\nu^a}, \mathcal{X}^{\nu^a}) - (V^{\nu^b}, \mathcal{X}^{\nu^b})||_t^2 |\mathcal{F}_t^B] \\
\leqslant& 2t\mathbb{E}\left[\int_0^t |V^{\nu^a}(r)\pi^{\nu^a}(r)\beta(\mathbf{x}, \nu^a) - V^{\nu^b}(r)\pi^{\nu^b}(r)\beta(\mathbf{x}, \nu^b)|^2 + |b(\mathbf{x}, \nu^a) - b(\mathbf{x}, \nu^a)|^2 dr \middle| \mathcal{F}_t^B\right] \\
&+ 2\mathbb{E}\left[\sup_{0 \leqslant s \leqslant t} |\int_0^s V^{\nu^a}(r)\pi^{\nu^a}(r)\sigma(\mathbf{x}, \nu^a) - V^{\nu^b}(r)\pi^{\nu^b}(r)\sigma(\mathbf{x}, \nu^b)dB_r|^2 + \sup_{0 \leqslant s \leqslant t} |\int_0^s \mathcal{X}_r^{\nu^a}\sigma(\mathbf{x}, \nu^a) - \mathcal{X}_r^{\nu^b}\sigma(\mathbf{x}, \nu^b)dB_r|^2 \middle| \mathcal{F}_t^B\right] \\
\leqslant& (2t+2)L^2\mathbb{E}\left[\int_0^t (|V_r^{\nu^a} - V_r^{\nu^b}|^2 + |\mathcal{X}_r^{\nu^a} - \mathcal{X}_r^{\nu^b}|^2 + \mathcal{W}_2^2(\nu_r^a, \nu_r^b))dr \middle| \mathcal{F}_t^B\right]
\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}[||V^{\nu^a} - V^{\nu^b}||_t^2 |\mathcal{F}_t^B] \leqslant \mathbb{E}[||(V^{\nu^a}, \mathcal{X}^{\nu^a}) - (V^{\nu^b}, \mathcal{X}^{\nu^b})||_t^2 |\mathcal{F}_t^B] \leqslant (2t+2)L^2\mathbb{E}[\int_0^t \mathcal{W}_2^2(\nu_r^a, \nu_r^b)dr]. \tag{72}$$

If π is open loop control, i.e., $\pi(t) = \phi(v_0, \nu_t, B_{[0,T]})$,

$$\mathbb{E}[||\pi^{\nu^a} - \pi^{\nu^b}||_t^2 |\mathcal{F}_t^B] \leqslant 2L^2\mathbb{E}[|v_0^{\nu^a} - v_0^{\nu^b}|^2 + |B_{[0,t]}^{\nu^a} - B_{[0,t]}^{\nu^b}|^2 + \mathcal{W}_2^2(\nu^a, \nu^b)|\mathcal{F}_t^B] \leqslant 2L^2 E[\mathcal{W}_2^2(\nu_t^a, \nu_t^b)]. \tag{73}$$

If π is closed loop Markovian, i.e., $\pi(t) = \phi(t, V(t), \nu_t)$,

$$\mathbb{E}[||\pi^{\nu^a} - \pi^{\nu^b}||_t^2 |\mathcal{F}_t^B] \leqslant 2L^2\mathbb{E}[||V^{\nu^a} - V^{\nu^b}||_t^2 + \mathcal{W}_2^2(\nu_t^a, \nu_t^b)] \leqslant 4(t+1)L^4\mathbb{E}[\int_0^t \mathcal{W}_2^2(\nu_r^a, \nu_r^b)dr]. \tag{74}$$

Then the coupling of $\Phi(\nu_1), \Phi(\nu_2)$ gives the following inequality

$$\begin{aligned} d_t^2(\Phi(\nu^a), \Phi(\nu^b)) &\leqslant \mathbb{E}[||(V^{\nu^a}, \pi^{\nu^a}) - (V^{\nu^b}, \pi^{\nu^b})||_T^2 | \mathcal{F}_T^B] \\ &\leqslant C_T \mathbb{E}[\int_0^T \mathcal{W}_2(\nu_r^a, \nu_r^b) dr] \\ &\leqslant \int_0^T d_r^2(\nu_r^a, \nu_r^b) dr, \end{aligned} \tag{75}$$

where $C_T = 2(T+2)L^2 + 4(T+1)L^4$ for closed loop Markovian controls, and $C_T = 2(T+2)L^2$ for open loop controls.

Following Picard iteration scheme, choose an arbitrary $\nu^0 \in \mathcal{P}^2(\mathcal{C}^{n,N})$,

$$\nu^{\ell+1} = \Phi(\nu^\ell),$$

Φ has been proved as an contraction mapping when $0 < T < \frac{1}{2L^2} - 2$, $L < \frac{1}{2}$ and thus Φ has a unique fixed point and ν^ℓ converges to ν by the contraction mapping principle. For $T > \frac{1}{2L^2} - 2$, we prove the above argument for $[T, 2T]$, $[2T, 3T]$, etc. ☐

Subsequently, we show in the following proposition that MFE strategies coincide with the limit of optimal empirical measure in the weak sense.

Proposition D.2. *There exists limits for measure flows $\nu^N \in \mathcal{P}^2(\mathcal{C}^{n,N})$, $\mu^N \in \mathcal{P}^2(C([0,T];\mathbb{R}_+))$, i.e., the limits $\nu_t = \lim_{N\to\infty} \nu_t^N$, $\mu_t = \lim_{N\to\infty} \mu_t^N$ exist in the weak sense for $t \in [0,T]$ with respect to the 2-Wasserstein distance.*

Proof. Let (U^ℓ) be the solution of closed loop Markovian dynamics $\phi^\ell : [0,T] \times \mathbb{R}_+ \to \mathbb{A}$,

$$dU^\ell(t) = U^\ell(t)\phi^\ell(t, U^\ell(t))\beta(\mathcal{X}_t, \nu_t)dt + U^\ell(t)\phi^\ell(t, U^\ell(t))\sigma(\mathcal{X}_t, \nu_t)dB_t, \quad U^\ell(0) = v^\ell,$$

or of open loop dynamics

$$dU^\ell(t) = U^\ell(t)\phi^\ell(v^\ell, B_{[0,T]})\beta(X_t, \nu_t)dt + U^\ell(t)\phi^\ell(v^\ell, B_{[0,T]})\sigma(X_t, \nu_t)dB_t, \quad U^\ell(0) = v^\ell.$$

for $\ell = 1, \ldots, N$. The initial states v^ℓ are i.i.d copies of v. We assume the initial value of $U^\ell(0)$ is of the same law with $V^\ell(0)$.

$$\mathbb{E}[||(V^\ell, \phi^\ell(V^\ell)) - (U^\ell, \phi^\ell(U^\ell))||_t^2] \leqslant C_T \mathbb{E}\Big[\int_0^t \mathcal{W}_2^2(\nu_r^N, \nu_r)dr\Big] \leqslant C_T \mathbb{E}\Big[\int_0^t d_r^2(\nu^n, \nu)dr\Big] \tag{76}$$

for $t \in [0,T]$, C_T is defined in Proposition D.1. For simplicity, let us discuss in the case of closed loop dynamics, the result of which can be generalized to open loop dynamics.

$\tilde{\nu}^N$ are the empirical measure of N i.i.d samples U^ℓ. We follow the coupling arguments in [4], the empirical measure of (V^ℓ, U^ℓ) is a coupling of the N-player empirical measure ν^N defined in Definition 2.2 and $\tilde{\nu}^N$.

$$d_t^2(\nu^N, \tilde{\nu}^N) \leqslant \frac{1}{N}\sum_{\ell=1}^N ||(V^\ell, \phi^\ell(V^\ell)) - (U^\ell, \phi^\ell(U^\ell))||_t^2, \quad \text{a.s..} \tag{77}$$

By the triangle inequality and (76), (77),

$$\mathbb{E}[d_t^2(\nu^N, \nu)dr] \leqslant 2\mathbb{E}[d_t^2(\tilde{\nu}^N, \nu)] + 2C_T\mathbb{E}[\int_0^t d_r^2(\nu^N, \nu)dr],$$

and then by Gronwall's inequality and set $t = T$, it follows

$$\mathbb{E}[\mathcal{W}_2^2(\nu^N, \nu)] \leqslant 2e^{2C_T T}\mathbb{E}[\mathcal{W}_2^2(\tilde{\nu}^N, \nu)].$$

Since (U^ℓ, π^ℓ), $\ell = 1, \ldots, N$ is independent given the noise B, use conditional law of large numbers (Theorem 3.5 in [22]),

$$P\big(\lim_{n\to\infty}\sum_{\ell=1}^N f(U^\ell, \pi^\ell) - \mathbb{E}[f(U^\ell, \pi^\ell)|\mathcal{F}_t^B]) = 1, \quad \text{for every} f \in C_b(\mathbb{R}^n)\big)$$

We then use Theorem 6.6 in [25], which states that on a separable metric space, $\nu^N \to \nu$ weakly.

$$\lim_{N\to\infty}\int_{\mathbb{R}^N} d(x,x_0)^2\nu^N(dx) = \int_{\mathbb{R}^N} d(x,x_0)^2\nu(dx) \quad \text{a.s.},$$

which lead us to

$$\mathbb{E}[\mathcal{W}_2^2(\tilde{\nu}^N,\nu)] \to 0.$$

Therefore $\mathbb{E}[\mathcal{W}_2^2(\nu^N,\nu)] \to 0$. We can use similiar methods to derive $\mathbb{E}[\mathcal{W}_2^2(\mu^N,\mu)] \to 0$. □

Assumption 12. *There are the following bounds on β and σ:*

$$\int_s^t |\beta_i(r,\omega)|dr \leqslant \eta(\omega,\nu)|t-s|^{\frac{1+\beta}{\alpha}},$$

$$\int_s^t |\sigma_{ij}^2(r,\omega)|dr \leqslant \xi(\omega,\nu)|t-s|^{\frac{1+\beta}{\alpha}},$$

where $t,s\in[0,T]$, α and β are positive constants, and η,ξ being $\mathcal{F}$-measurable random variables with values in $(0,\infty)\times\Omega\times\mathcal{C}^{n,N}$ such that there is $\epsilon>0$ with $\mathbb{E}[\eta(\omega,\nu)^2]<\infty$, $\mathbb{E}[\xi(\omega,\nu)^2]<\infty$.

Proposition D.3. *If Assumption 12 holds, then there exist n dimensional continuous process $\mathcal{X}$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $\mathcal{X}(t)=\lim_{N\to\infty}\mathcal{X}^N(t)$ exists a.s. for all $t\in[0,T]$.*

Proof. First we show that $\{\mathbb{P}^{\mathcal{X}^N}\}$ is tight. By [17], a sequence of measures μ^N on $\mathcal{P}^2(C([0,T];\mathbb{R}_+))$ is tight if and only if

- there exist positive constants M_x and γ such that $\mathbb{E}\{|\mathbf{x}^N|^\gamma\}\leqslant M_x$ for every $N=1,2,\ldots$,
- there exist positive constants M_k and δ_1, δ_2 such that $\mathbb{E}\{|\mathcal{X}^N(t)-\mathcal{X}^N(s)|^{\delta_1}\}\leqslant M_k|t-s|^{1+\delta_2}$ for every $N=1,2,\ldots$, $t,s\in[0,T]$.

Apparently, the first condition holds. Then,

$$|\mathcal{X}^N(t)-\mathcal{X}^N(s)|^\alpha \leqslant N^{\alpha/2}(|X_1^N(t)-X_1^N(s)|^\alpha+\ldots+|X_n^N(t)-X_n^N(s)|^\alpha),$$

$$|X_i^N(t)-X_i^N(s)|^\alpha = |\int_s^t X_i^N(r)\beta_i(r)dr+\sum_{k=1}^n\int_s^t X_i^N(r)\sigma_{ik}(r)dW_k(r)|^\alpha \leqslant (n+1)^\alpha(\eta(\omega,\nu)^\alpha|t-s|^{1+\beta}+\sum_{k=1}^n|\int_s^t\sigma_{ik}(r)dW_k(r)|^\alpha).$$

Then let $\alpha=2$ in Assumption 12, by Ito's isometry,

$$\begin{aligned}\mathbb{E}[|\mathcal{X}^N(t)-\mathcal{X}^N(s)|^2] &\leqslant N^{\alpha/2}(n+1)^\alpha\Big(\mathbb{E}[\eta(\omega,\nu)^2]h^{1+\beta}+\sum_{i=1}^n\sum_{k=1}^n\mathbb{E}\big[\int_s^t|\sigma_{ik}(r)|^2dW_k(r)\big]\Big)\\ &\leqslant N(n+1)^2\Big(\mathbb{E}[\eta(\omega,\nu)^2]+\mathbb{E}[\xi(\omega,\nu)^2]\Big)h^{1+\beta},\end{aligned}$$

where $\mathbb{E}[\eta(\omega,\nu)^2]+\mathbb{E}[\xi(\omega,\nu)^2]<\infty$,$h\in(0,T]$. Thus the second condition follows.

By Prokhorov theorem [2], tightness implies relative compactness, which means here that each subsequence of $\mathcal{X}^N$ contains a further subsequence converging weakly on the space $C([0,T];\mathbb{R}_+^n)$. As a result, a subsequence exists such that $\mathcal{X}(t)=\lim_{N\to\infty}\mathcal{X}^N(t)$ a.s.. Then if every finite dimensional distribution of $\{\mathbb{P}^{\mathcal{X}^N}\}$ converges, then the limit of $\{\mathbb{P}^{\mathcal{X}^N}\}$ is unique and hence $\{\mathbb{P}^{\mathcal{X}^N}\}$ converges weakly to $\mathbb{P}$ as $N\to\infty$. □

Proposition D.4. *Under Assumption 2, $\mathcal{X}(t)=\lim_{N\to\infty}\mathcal{X}^N(t)$ exists in the weak sense, and the limit $X(t)$ match the solution of the Mckean-Vlasov SDE*

$$dX_i(t) = X_i(t)\beta_i(\mathcal{X}(t),\nu_t)dt + X_i(t)\sigma_i(\mathcal{X}(t),\nu_t)dB_t$$

Proof. Since $\nu_t = \lim_{N\to\infty} \nu_t^N$, it is equivalent to show that the drift and volatility of ν_t matches the weak limit of that of ν_t^N, i.e.,

$$\beta(\mathcal{X}(t), \nu_t) = \lim_{N\to\infty} \beta(\mathcal{X}^N(t), \nu^N(t)), \quad \sigma(\mathcal{X}(t), \nu_t) = \lim_{N\to\infty} \sigma(\mathcal{X}^N(t), \nu^N(t)).$$

in the weak sense.

By Lebesgue dominated convergence theorem

$$\begin{aligned} \|\int_0^t \beta(\mathcal{X}_s^N, \nu_s^N) - \beta(\mathcal{X}_s, \nu_s) ds\|_{L^2}^2 &\leqslant \int_0^t \|\beta(\mathcal{X}_s^N, \nu_s^N) - \beta(\mathcal{X}_s, \nu_s)\|_{L^2}^2 ds \\ &\leqslant L\mathbb{E}[\int_0^t |\mathcal{X}_s^N - \mathcal{X}_s|^2 ds + \int_0^t \mathcal{W}_2(\nu_s^N, \nu_s)^2 ds] \end{aligned}$$

By Ito's isometry and Assumption 2, we have

$$\begin{aligned} \|\int_0^t \sigma(\mathcal{X}_s^N, \nu_s^N) dW_s - \int_0^t \sigma(\mathcal{X}_s, \nu_s) dB_s\|_{L^2}^2 &= \mathbb{E}[\int_0^t |\sigma(\mathcal{X}_s^N, \nu_s^N) - \sigma(\mathcal{X}_s, \nu_s)|^2 ds] \\ &\leqslant L\mathbb{E}[\int_0^t |\mathcal{X}_s^N - \mathcal{X}_s|^2 + \mathcal{W}_2^2(\nu_s^N, \nu_s) ds]. \end{aligned}$$

Hence it follows from the fact that $\mathcal{X}(t) = \lim_{N\to\infty} \mathcal{X}^N(t)$ a.s., ν_t weakly convergent to $\nu^N(t)$ with $\mathcal{W}_2$, we get $\mathcal{X}(t)$ satisfies

$$d\mathcal{X}(t) = \mathcal{X}(t)\beta(\mathcal{X}(t), \nu_t)dt + \mathcal{X}(t)\sigma(\mathcal{X}(t), \nu_t)dB_t$$

□

Finally, under Assumption 6 we conclude that when $N \to \infty$, the limiting system is driven by X_t and $\nu_t := \text{Law}(V(t), \pi(t))$. The stock market follows

$$d\mathcal{X}_t = \mathcal{X}(t)\beta(\mathcal{X}_t, \nu_t|\mathcal{F}_t^B))dt + \mathcal{X}(t)\sigma(\mathcal{X}_t, \nu_t|\mathcal{F}_t^B)dB_t, \quad X_0 = \mathbf{x},$$

and a generic player's wealth is

$$dV(t) = \pi(t)\beta(\mathcal{X}(t), \nu_t|\mathcal{F}_t^B))dt + \pi(t)\sigma(\mathcal{X}(t), \nu_t|\mathcal{F}_t^B)dB_t, \quad V(0) = v_0. \tag{78}$$

With the notations in Definition 2.1 (3), if we consider the mean $Z(t)$ of the measure $\text{Law}(V(t), \pi(t)|\mathcal{F}_t^B)$, we can get $Z(t) = \lim_{N\to\infty} Y(t)$ exists in the weak sense, and the limit $Z(t)$ match the solution of the Mckean-Vlasov SDE

$$dZ(t) = \gamma(\mathcal{X}_t, Z(t))dt + \tau(\mathcal{X}_t, Z(t))dB_t. \tag{79}$$

This is used primarily in Section 5 and 6.