3D TOROIDAL FIELD MULTIPOLES FOR CURVED ACCELERATOR MAGNETS*

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Abstract

Curved magnets producing continuously rotating field multipoles along the length of the bend can provide strong and continuous transverse focusing, making them of interest for accelerator systems such as compact ion beam gantries and synchrotron light sources. Evaluating the utility of such rotating multipole systems requires an accurate description of field behavior for beam physics calculations. This paper presents a helical scalar potential solution in 3D toroidal harmonics relevant to boundary conditions on the surface of a torus. The resulting fields are evaluated for a curved helical quadrupole channel to illustrate field rotation and the effect of magnet curvature.

INTRODUCTION

Previous analysis and use of accelerator magnets producing helical fields have focused on straight, cylindrical designs [1]-[5]. There is interest in studying the extension of this concept to curved magnets [6], requiring expressions for allowable fields rotating around the bend of torus. Previous work on field description and measurement of curved magnets is specific to cases of axial symmetry where the assumption of no field variation along the bend is made [7]-[9].

In these references Schinzer *et al.* present a method for finding scalar potential solutions in local polar coordinates using approximate R-separation of the scalar Laplace equation (expansion in the aspect ratio of the torus). In this paper a similar approach was taken to obtain expressions relevant to rotating fields on the bend of a torus, with the difference being the use of toroidal coordinates and full R-separation in 3D toroidal harmonics.

TOROIDAL COORDINATES

The right hand toroidal system of η, ξ, ϕ is formed by the rotation of bipolar coordinates η and ξ about the vertical axis (Fig. 1). Surfaces of constant η are tori, making toroidal coordinates a natural choice for problems with boundary conditions on the surface of a torus. Scale factors for this system are

$$h_\eta = h_\xi = \frac{a}{\cosh\eta - \cos\xi}$$
$$h_\phi = \frac{a\sinh\eta}{\cosh\eta - \cos\xi} \quad (1)$$
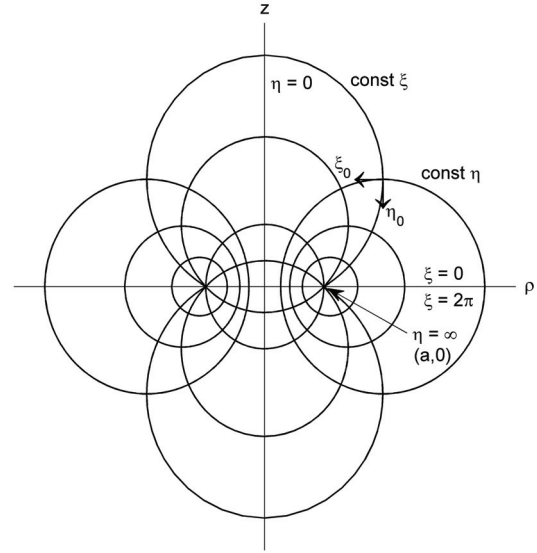
where $(\pm a, 0)$ are foci of the bipolar system.

Figure 1: Bipolar coordinates $0 < \eta < \infty$ and $0 \leq \xi \leq 2\pi$ forming toroidal coordinates when rotated about the z-axis.

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In a current free region the magnetic field can be expressed using only the gradient of a scalar potential $\vec{B} = -\nabla\psi$. The divergence of this field produces Laplace's equation for the scalar potential $\nabla \cdot \vec{B} = -\nabla^2\psi = 0$. In toroidal coordinates the scalar Laplace equation is

$$\nabla^2\psi = \frac{1}{h_\eta h_\xi h_\phi}\left[\frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\phi}{h_\eta}\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\phi}{h_\xi}\frac{\partial\psi}{\partial\xi}\right)\right.$$
$$\left. + \frac{\partial}{\partial\phi}\left(\frac{h_\eta h_\xi}{h_\phi}\frac{\partial\psi}{\partial\phi}\right)\right] = 0. \quad (2)$$

Substitution of the coordinate dependent scale factors (1) produces

$$\frac{k^2}{a^2}\left[\frac{\partial^2\psi}{\partial\eta^2} + \frac{\partial^2\psi}{\partial\xi^2} + \coth\eta\frac{\partial\psi}{\partial\eta}\right.$$
$$\left. -k^{-1}\left(\sinh\eta\frac{\partial\psi}{\partial\eta} + \sin\xi\frac{\partial\psi}{\partial\xi}\right) + \frac{1}{\sinh^2\eta}\frac{\partial^2\psi}{\partial\phi^2}\right] = 0,$$
$$(3)$$

where the simplification $k(\eta, \xi) = \cosh\eta - \cos\xi$ is made. The technique of R-separation is used, producing

$$\frac{k^{5/2}}{a^2}\left[\frac{\partial^2 u}{\partial\eta^2} + \frac{\partial^2 u}{\partial\xi^2} + \coth\eta\frac{\partial u}{\partial\eta} + \frac{1}{4}u + \frac{1}{\sinh^2\eta}\frac{\partial^2 u}{\partial\phi^2}\right] \quad (4)$$

which can be separated to obtain expressions for the substituted function $u = k^{-1/2}\psi$.

GENERAL HELICAL SOLUTION IN TOROIDAL HARMONICS

With u separable in each variable $u(\eta, \xi, \phi) = G(\eta)H(\xi)L(\phi)$, sinusoidal behavior in ξ and ϕ is assumed such that

$$L(\phi) = \sum_{m=0}^{\infty} [A_m \cos(m\phi) + B_m \sin(m\phi)] \quad (5)$$

$$H(\xi) = \sum_{n=0}^{\infty} [C_n \cos(n\xi) + D_n \sin(n\xi)]. \quad (6)$$

With these assumptions the separated "radial" equation for $G(\eta)$ can be obtained from (4) as

$$\frac{d^2G}{d\eta^2} + \coth\eta \frac{dG}{d\eta} + \left[\frac{1}{4} - n^2 - \frac{m^2}{\sinh^2\eta}\right]G = 0. \quad (7)$$

If a change of argument $Z = \cosh\eta$ is used, (7) can be reformulated into the associate Legendre differential equation

$$(1 - Z^2)\frac{d^2G}{dZ^2} - 2Z\frac{dG}{dZ} + \left[\nu(\nu+1) - \frac{m^2}{1-Z^2}\right]G = 0 \quad (8)$$

with degree $\nu = n - \frac{1}{2}$ and order m. General solutions for $G(\eta)$ are the associate Legendre polynomials P_ν^m and Q_ν^m with argument $Z = \cosh\eta$.

$$G = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[E_{n,m}P_{n-\frac{1}{2}}^m(\cosh\eta) + F_{n,m}Q_{n-\frac{1}{2}}^m(\cosh\eta)\right] \quad (9)$$

Combining the solutions of separable equations and the R-separation substitution, the general form of the helical potential is

$$\psi = k^{\frac{1}{2}}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[E_{n,m}P_{n-\frac{1}{2}}^m(Z) + F_{n,m}Q_{n-\frac{1}{2}}^m(Z)\right] \times$$
$$[C_n\cos(n\xi) + D_n\sin(n\xi)][A_m\cos(m\phi) + B_m\sin(m\phi)] \quad (10)$$

which is in agreement with similar separable solutions of Laplace's equation in toroidal coordinates (see [10] for example).

BOUNDARY CONDITIONS ON A TORUS

A surface of constant $\eta = \eta_0$ forms a torus. If the minor and major radius of a torus are respectively R_c and R_0, the constants producing this surface are

$$\eta_0 = \cosh^{-1}\epsilon^{-1} \quad (11)$$

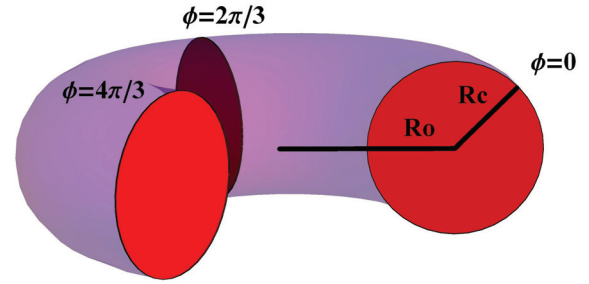$$a = R_c\sinh\eta_0 = R_0(1 - \epsilon^2)^{\frac{1}{2}}, \quad (12)$$

Figure 2: A partial torus of $\epsilon = 50/100$ is shown with bore cross sections at $\phi = 0$, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$ marked in red. The rotation of transverse fields through similar cross sections for a helical quadrupole channel is shown in Fig. 3.

where $\epsilon = R_c/R_0$ is the aspect ratio of the torus (Fig. 2). The potential inside and outside the torus' surface is given by

$$\psi^{in}(\eta_0 < \eta < \infty) = k^{\frac{1}{2}}A_{n,m}^{in}Q_{n-\frac{1}{2}}^m(\cosh\eta)\sin(n\xi - m\phi) \quad (13)$$

$$\psi^{out}(0 < \eta < \eta_0) = k^{\frac{1}{2}}A_{n,m}^{out}P_{n-\frac{1}{2}}^m(\cosh\eta)\sin(n\xi - m\phi), \quad (14)$$

where the Legendre polynomials are required to remain finite, and phase in ϕ and ξ is chosen such that $B_\xi(\eta, \xi, \phi = 0)$ is symmetric about the $\xi = 0$ axis.

Bore Fields

The fields inside the bore from $\vec{B} = -\nabla\psi^{in}$ (13) are,

$$B_\eta^{in} = -\frac{A^{in}}{a}k^{3/2}\left[\left(\frac{n - \frac{1}{2}}{\tanh\eta} + \frac{k^{-1}}{2}\sinh\eta\right) \times \right.$$
$$\left. Q_{n-1/2}^m(\cosh\eta) - \frac{m + n - \frac{1}{2}}{\sinh\eta}Q_{n-3/2}^m(\cosh\eta)\right] \times$$
$$\sin(n\xi - m\phi) \quad (15)$$

$$B_\xi^{in} = -\frac{A^{in}}{a}k^{3/2}Q_{n-1/2}^m(\cosh\eta)\left[n\cos(n\xi - m\phi)\right.$$
$$\left. + \frac{k^{-1}}{4}\left[\cos((n-1)\xi - m\phi) - \cos((n+1)\xi - m\phi)\right]\right] \quad (16)$$

$$B_\phi^{in} = \frac{mA^{in}}{a\sinh\eta}k^{3/2}Q_{n-1/2}^m(\cosh\eta)\cos(n\xi - m\phi), \quad (17)$$

with use made of the properties of associate Legendre polynomials found in [11]. For the evaluation of the Legendre polynomials using the DTOR algorithms see [12].
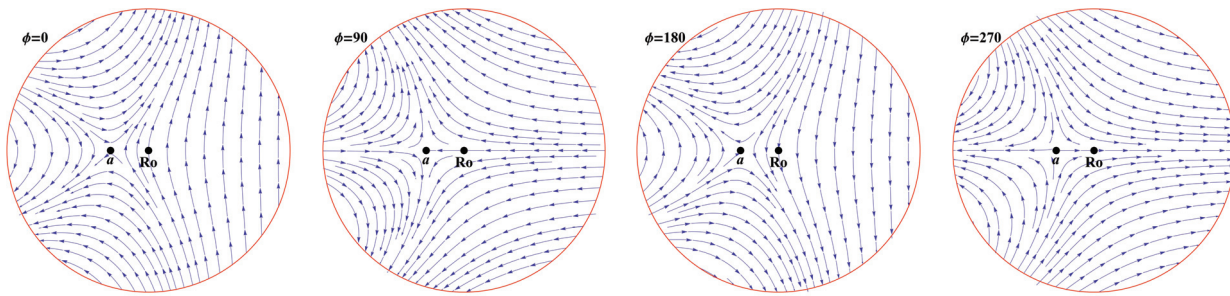
Figure 3: Transverse field rotation within a helical quadrupole channel ($n = 2$, $m = 1$) is shown for a torus of aspect ratio $\epsilon = 50/100$. The coordinate ϕ is the angle around the bend of the torus (see Fig. 2).

CURVED HELICAL "QUADRUPOLE" FOCUSING CHANNELS

The $n=2$ toroidal multipoles resemble quadrupole fields twisting along the length of the torus with m total rotations. This is illustrated in Fig. 3 for a curved quadrupole channel with one full period around the torus ($m=1$). The aspect ratio ϵ and field period length $2\pi R_0/m$ of the magnet determine the deviation of the helical toroidal field multipoles from traditional cylindrical multipoles. As the aspect ratio of the torus approaches zero and field period length approaches infinity, the fields approach those produced by a straight non-twisting quadrupole (see Fig. 4).

$\epsilon_1 = R_c/R_0 = 50/100$ $\epsilon_2 = R_c/R_0 = 50/250$

$\epsilon_3 = R_c/R_0 = 50/5000$ Normalized vertical field gradient along midplane

Figure 4: Axially symmetric quadrupole-like fields (n=2, m=0) are shown for a fixed bore radius R_c and increasing major radius R_0. As the aspect ratio of the torus $\epsilon = R_c/R_0$ tends to zero, the fields are seen approaching those of a straight cylindrical quadrupole.

CONCLUSION

A 3D scalar potential obtained through R-separation of Laplace's equation in toroidal coordinates was presented. This potential can be used to describe magnetic fields rotating helically along the length of a curved magnet in terms of toroidal harmonics. The deviation of these toroidal harmonics from straight cylindrical multipoles is influenced by the period of field rotation and the aspect ratio of the magnet (torus). This deviation tends to zero in the limit of no rotation and a aspect ratio of zero. A curved helical quadrupole channel was used to illustrate this behavior.

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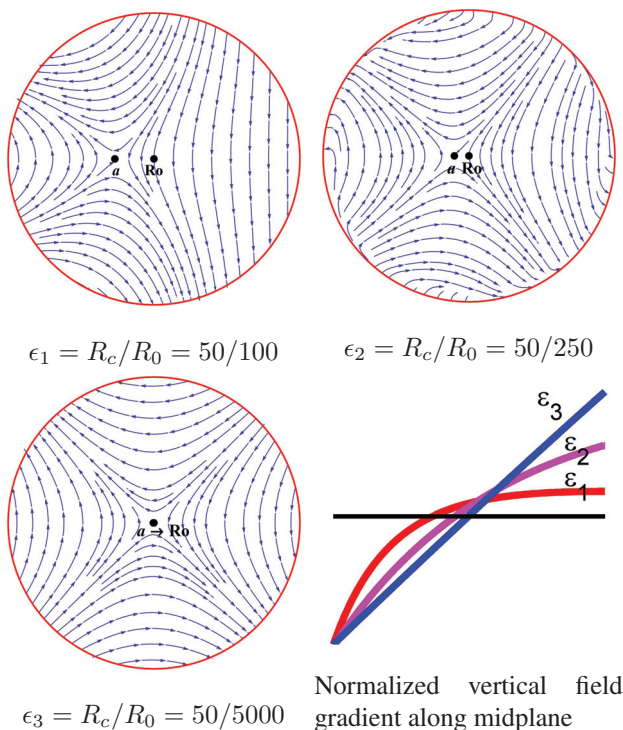[12] J. Segura and A.Gil, *Computer Physics Communications*, vol. 124, pp. 104-122, 2000.