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Publication Date
2017

Peer reviewed|Thesis/dissertation

Theory of Real Bundles on the Projective Line

by

Daniel Appel

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2017

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Abstract

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by

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Doctor of Philosophy in Mathematics

University of California, Berkeley

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In this thesis we discuss the theory of vector bundles with real structure on the projective line. This extends classical work by Grothendieck classifying complex vector bundles on the projective line. In particular, we show that vector bundles with real structure can be classified in terms of the coroot lattice of $GL(n)$, similarly to the complex case. In addition, we provide a comparison of a certain K-group of sheaves on the moduli space of vector bundles to a K-group of sheaves on the moduli space of local systems, a kind of Langlands duality statement for real bundles, and give a uniformization of the moduli space.

Contents

Acknowledgments

I thank Professor David Nadler, without whom this thesis would have been impossible.

Chapter 1

Theory of Real Bundles

1.1 Moduli Spaces of Real Bundles

Introduction

The goal of this paper is to approach a special case of the Geometric Langlands Conjectures that bears on the representation theory of real groups. We will begin by recalling the general setup of Langlands-type conjectures in algebraic geometry. Given an algebraic curve C, and a reductive algebraic group G, one is interested in two different moduli spaces. On the one hand, one has $Bun(G, C)$, the moduli space of principal G bundles on C. On the other hand, there is the moduli space of local systems on C, $Loc(G, C)$. Both of these spaces can be described in terms of mapping spaces: $Bun(G, C)$ is $Hom(C, BG)$ and $Loc(G, C)$ is $Hom(C, BG^{disc})$, where G^{disc} is G equipped with the discrete topology (there is also a more sophisticated "de Rham" version of these conjectures, which we will not touch on).

The goal of Langlands type conjectures is to relate the geometry (and in particular sheaf theory) of these two kinds of moduli spaces. The first point in this project is that $Bun(G, C)$ and $Loc(G, C)$ are typically not directly related. Rather, one defines the Langlands dual group, $\check{G}$, as the group with root data defined by interchanging roots and coroots in the root data of G, and attempts to relate $Loc(\check{G}, C)$ with $Bun(G, C)$. Secondly, one has to be careful about the category of sheaves considered on each of these spaces. In general this is a complicated issue, but for $\mathbb{P}^1$ there is a known answer: there is a category defined in terms of coherent sheaves on $Loc(\check{G}, \mathbb{P}^1)$ that is equivalent to a category defined in terms of constructible sheaves on $Bun(G, \mathbb{P}^1)$. In this one sees immediately the connection with representation theory, since the category of coherent sheaves on $Loc(\check{G}, \mathbb{P}^1)$ is closely related to the category of finite dimensional representations of $\check{G}$. We will discuss precise versions of these theorems in Section 2.3. This theorem also has numerous generalizations and variations due to Deligne, Kazhdan, Lusztig, Bezrukavnikov, and others, which we will touch on and make use of in what follows.

Our goal is to get something like this theorem for real groups. The first question is then how to deal with real groups in the context of algebraic goemetry. The approach that we will use is descent theory, which describes real objects as complex objects equipped with additional structure. In this case, we have a complex reductive group G, together with an antiholomorphic involution $\rho : G \to G$, called the real structure (which should be thought of as complex conjugation) and we are interested in principal bundles over G also equipped with some kind of real structure. A natural choice would be to consider a principal bundle P together with an antiholomorphic isomorphism $P \to \hat{P}$, where $\hat{P}$ is the conjugate bundle (defined below). It turns out that the theory works better if we take into account that $\mathbb{P}^1$ also has a natural involution, α, which sends every point to the antipodal point. We then define real principal bundles to be complex principal bundles equipped with an isomorphism $\theta_P : P \to \alpha^* \bar{P}$ satisfying certain conditions, and similarly for local systems. One can then conjecture that the derived category of coherent sheaves on the moduli of real local systems for the dual group is equivalent to the derived category of constructible sheaves on the moduli of real bundles.

In fact this is not quite what we can prove. The first modification is that instead of considering principal bundles, we consider principal bundles equipped with a fixed Borel reduction at ∞. For the trivial bundle, this means we are considering real forms of the group G together with a fixed Borel, the same kind of structure considered by Vogan in his duality theory for real reductive groups. Indeed, a sufficiently strong form of the Langlands conjectures in our setting should imply Vogan's duality theorem (see [5]). Second, we are not able to get an equivalence of categories, but only of the Grothendieck groups of these categories.

In the next section, we will give more precise definitions of all the terms introduced above and state the main theorem. In the following sections, we will review the basics of Galois cohomology and descent that we will need, and begin to analyze the structure of the various moduli spaces. Once this is done we will give the proof of the main theorem and discuss some further directions. Afterwards, we will discuss some other results to do with moduli of real bundles, in particular we will show a uniformization statement for the moduli space of real bundles and also show that every real bundle reduces to a torus. Typically, the uniformization statement would come before the other material but we have chosen to delay it so as to make the proof easier.

Basic Definitions and Notation

The following are the basic objects we will refer to:

For G a reductive group, we have a moduli space Bun_G of G bundles on $\mathbb{P}^1$. G will either be a torus or $GL(n)$ in this paper. We will assume G comes equipped with a real structure, i.e. an antiholomorphic involution. If we want to refer to the underlying complex group we

will use the notation $G_{\mathbb{C}}$, and we will use $G_{\mathbb{R}}$ for the fixed points of the involution.

We also have the moduli space of ramified bundles, Bun'_G, which is the moduli space of G bundles equipped with Borel reductions at 0 and ∞.

The involution on G induces involutions on these spaces as follows: given a principal bundle, P, define the conjugate of P, namely $\hat{P}$, to have the same underlying space, but G acts by $(g, x) \to \bar{g}x$. Let α be the antipodal map on $\mathbb{P}^1$. Then let $\bar{P} = \alpha^*\hat{P}$. For Bun'_G, the Borel reduction of $\bar{P}$ at 0 (resp. ∞) is then the conjugate of the Borel reduction of P at ∞ (resp. 0). We define $Bun_{G,r}$ and $Bun'_{G,r}$ to be the fixed points of these actions. In the following section we will review what this means.

Define $K_{Loc}(X)$ to be the Grothendieck group of the category of constructible sheaves on X.

Given a reductive group with an antiholomorphic involution, the above defined $\mathbb{Z}/2\mathbb{Z}$ action on Bun_G induces an involution on $\check{G}$, the Langlands dual.

Given any group G with an involution, Loc_G is the moduli space of G local systems on $\mathbb{P}^1$. Loc'_G is the moduli space of local systems with Borel reductions at 0 and ∞. As above, the involution defines an involution of these spaces, and $Loc_{G,r}$ and $Loc'_{G,r}$ denote the fixed points. Note that our space $Loc'_{G,r}$ is not exactly the space that one expects to use in the most refined equivalence. Rather, one expects to look at the moduli space of local systems together with Borel reduction, real structure, and a monodromy, i.e. an element g such that $g\bar{g} \in U$, where U is the unipotent elements of B. On the level of K groups, however, this does not seem to make a difference.

$K_{Coh}(X)$ denotes the Grothedieck group of coherent sheaves on X. Where no confusion can arise we will simply use K for the appropriate Grothendieck group

The main theorem we want to prove is:

$$K_{Coh}(Loc'_{\check{G},r}) \cong K_{Loc}(Bun'_{G,r})$$

We will refer to statements of this type as duality theorems. The main theorem of this paper is that duality holds for tori and for $GL(n)$. In the case of $GL(n)$ we will freely use the identification of principal bundles with vector bundles.

1.2 Background

Before going on to the case on bundles with descent data, we review the classical results about bundles on $\mathbb{P}^1$.

Most classically, one can classify vector bundles on $\mathbb{P}^1$ using the following two facts:

2.1 Theorem Every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.

2.2 Theorem Every line bundle on $\mathbb{P}^1$ is $O(n)$ for some n.

It follows from these two facts that vector bundles on $\mathbb{P}^1$ can be classified entirely by a multiset of integers. In [7] Grothendieck provided a more general view of this fact that will be useful to us. Grothendieck views vector bundles within the context of principal G bundles for reductive group G– vector bundles are the special case where G is $GL(n)$. Any such G has a maximal torus T which is unique up to conjugation. It turns out that torus bundles are easy to classify:

2.3 Theorem let T be a connected torus. Then principal T bundles on $\mathbb{P}^1$ are classified by homomorphisms $\mathbb{G}_m \to T$. For T of dimension n, there is a rank n lattice of such homomorphisms.

Proof sketch: Recall that to give a T bundle on $\mathbb{P}^1$ one must give a T bundle on the northern hemisphere and southern hemisphere together with an isomorphism of these two bundles on the intersection of the two hemispheres, which is $\mathbb{G}_m$. It turns out that a T bundle on $\mathbb{A}^1$ is necessarily trivial (this is the hardest part in some sense, since it requires one to prove the existence of a section), so to give a T bundle one must give an automorphism of the trivial T bundle on $\mathbb{G}_m$, which is the same as a function $\mathbb{G}_m \to T$, because the automorphism group of any fiber of a T bundle is just T. Such a function is just a collection of n rational functions on $\mathbb{G}_m$ which are invertible everywhere, and the only such functions (up to conjugation by constants) are powers of t. So we get an n dimensional lattice of possible transition functions of the form $(t^{m_1}, ..., t^{m_n})$, which are all homomorphisms.

In light of this fact, the fact that $GL(n)$ bundles are determined by a list of numbers follows from the fact that in some sense every $GL(n)$ bundle is really a T bundle for the maximal torus of $GL(n)$. More specifically,

2.4 Definition We say that a G bundle P has a reduction to H, where H is a subgroup of G, if there is an H bundle P' and an isomorphism $P' \times_G G/H \to P$.

Grothendieck's theorem is then

2.5 Theorem For connected reductive G with maximal torus T every principal bundle on $\mathbb{P}^1$ has a reduction to T.

This, together with a description of when two T bundles give the same G bundle, then gives a classification of G bundles. More specifically,

2.6 Theorem Let G be a connected reductive group with maximal torus T. Then, as

above, T bundles are precisely in bijection with $\Lambda = Hom(\mathbb{G}_m, T)$. The Weyl group W acts on T and hence on Λ The induction map $P \to P \times_G G/T$ gives a surjection from T bundles to G bundles, and for $\lambda, \lambda' \in \Lambda$, λ and λ' give the same G bundle if and only if there is $w \in W$ such that $w\lambda = \lambda'$. Thus principal G bundles are in bijection with Λ/W.

In the next section we give an overview of the proof of this statement.

Proof of Grothendieck's Theorem

There are many different ways to prove Grothendieck's theorem. We will sketch his proof reasonably closely. For an alternative, algorithmic proof, see [10].

Most of the work is to prove that every principal bundle has a reduction to a torus. This is done in three steps. First, one handles the special case of vector bundles, then one handles orthogonal bundles, and finally one handles bundles for a general group G.

First, one proves that every vector bundle reduces to a torus. Since direct sums of line bundles obviously reduce to a torus, it is enough to show that any vector bundle can be written as a direct sum of line bundles. This can be handled using a standard devisage argument as follows. Let V be a vector bundle. Any bundle has a rational section (see [11] II.5. Lemma 5.14), which gives an injection $O(n) \to V$ for some n. One then takes n to be as large as possible, and considers the exact sequence $0 \to O(n) \to V \to V/O(n) \to 0$. By replacing $O(n)$ with the preimage of the torsion part of $V/O(n)$, one can assume that $V/O(n)$ is a vector bundle of rank less than V, and hence is a direct sum of line bundles, $\oplus O(m_i)$. One then uses that extensions of vector bundles are classified by $Ext^1(\oplus O(m_i), O(n)) = \oplus Ext^1(O(m_i), O(n))$. It is then easy to calculate these ext groups and verify that all the bundle extensions they give are isomorphic to direct sums. For a detailed proof in English, see [12].

Second, one classifies orthogonal bundles. Here, the key point is that any vector bundle has at most one orthogonal structure. Let us sketch how this goes: Suppose V is a vector bundle with two nondegenerate symmetric forms $(-,-)_1$ and $(-,-)_2$. By the Riesz representation theorem there is a unique self-adjoint A (with respect to $(-,-)_1$) such that $(-,-)_2 = (A-,-)_1$. If we can find a self-adjoint square root u of A then we have $(-,-)_2 = (A-,-)_1 = (u*u-,-)_1 = (u-,u-)_1$ so that the two forms are equivalent. This is guaranteed by the fact that A can be decomposed into orthogonal Jordan blocks.

Given this, one only has to determine which vector bundles have orthogonal forms, and given the classification of vector bundles it is straightforward to see that a vector bundle has an orthogonal form if and only if it is isomorphic to its dual, in which case one can easily write down the orthogonal form and note that it reduces to a torus.

Finally, one has to show that principal bundles for arbitrary G have reduction to a torus.

This is probably the most interesting step and the one which is most directly relevant to the case of bundles with descent data. The strategy is to use the following result:

2.1.1 Proposition Let P be a G bundle, and ad the adjoint representation of G. Then P has a reduction to a torus if there is a section of $G \times_G ad$ which is regular semisimple in some fiber.

Then, one shows that such a section exists. Let's first see why the proposition is true. To begin with, one has:

2.1.2 Proposition P has a reduction to T if $P \times_G G/T$ has a section

Proof: Let P' be the fiber product of the projection $P \to P \times_G G/T$ with the section $\sigma : \mathbb{P}^1 \to P \times_G G/T$. Locally one can check that T, as a subgroup of the structure group G of P, preserves P', and acts transitively on its fibers, and so we get that P' is a T bundle. We then get a map $P' \times G \to P$ given by $(x, g) \to gx$, using G as the structure group, and one can check locally that this is an isomorphism.

This is then improved to:

2.1.3 Proposition P has a reduction to a torus if $P \times_G G/N$ has a section, where N is the normalizer of the torus.

Proof: One has a fiber sequence $P \times_G W \to P \times_G G/T \to P \times_G G/N$. But the first term of this sequence is a bundle over $\mathbb{P}^1$ with discrete fiber. Such bundles are always classified by $Hom(\pi_1(C), Aut(F))$ for C a curve and F the fiber, and in this case $\mathbb{P}^1$ is simply connected, so the fiber is trivial. This means there is no obstruction to lifting a section from $P \times_G G/N$ to $P \times_G G/T$.

Next, one connects this result with the representation ad:

2.1.4 Proposition if $P \times_G ad$ has a regular semisimple section, then $P \times G/N$ has a section

Proof: Suppose σ is a regular semisimple section of $P \times_G ad$. Let C be the set of Cartan subgroups of G. Since G acts transitively on Cartan subgroups, with kernel N, we have $C = G/N$. So to get a section of $P \times_G G/N$ it is sufficient to get a section of $P \times_G C$. But we can get such a section by assigning to each point the commutator of σ.

Finally, one can weaken the condition to having a section which is regular semisimple at a single point:

2.1.5 Proposition Let σ be a section of $P \times_G ad$ which is regular semisimple at a point. Then it is regular semisimple everywhere.

Proof: For σ a section of $P \times_G ad$ to be regular semisimple in some fiber is an algebraic condition on the characteristic polynomial of σ at that fiber. But the coefficients of the characteristic polynomial are algebraic functions on $\mathbb{P}^1$ and hence constant.

Putting these claims all together gives the proposition.

How, then, does one show that there is a section meeting the conditions of the proposition? The main idea is to use the fact that $P \times_G ad$ is an orthogonal bundle (via the Killing form), and we understand orthogonal bundles. In particular, we if we can find a regular semisimple element in the fiber of the positive part of this bundle at some point, then we can lift this element to a global section and be done:

Proof of reduction to a torus (sketch): We need to show that $P \times ad$ has a section which is regular semisimple at some point. Let F_x be the fiber at x, and s a regular semisimple element of F_x which is orthogonal to the nilpotent part of F_x. Let P_0 be the nonnegative part of $P \times ad$ and P_1 the positive part. Degree considerations show that these are Lie subbundles, and P_1 is nilpotent. By the classification of orthogonal bundles, P_0 is the orthogonal of P_1. Since s is orthogonal to the nilpotent part of P_x, it is orthogonal to the fiber $P_{1,x}$, and so is contained in the fiber $P_{0,x}$. But since this bundle is totally positive, we can lift s to a global section, as required.

What remains then is to determine the fibers of the map from Λ to principal G bundles, which we now know is surjective. For this it is enough to show that for λ, λ' not W conjugate, there is a representation of G such that the induced $GL(n)$ cocycles are not conjugate, which is relatively straightforward. For example, one can pick a faithful representation V of G and show that if two principal bundles are not related by an element of the Weyl group, their associated vector bundles with respect to V are not equivalent.

Real Orbits in the Finite Case

One result that we will show below gives the moduli space of vector bundles with real structure in terms of group orbits:

2.2.1 Theorem The moduli space of vector bundles with real structure is $L_\mathbb{R} \setminus L_G / I_+$, where L_G is the mapping space $Maps(\mathbb{G}_m, G)$, I_+ is the subgroup the mapping space $Maps(D, G)$ of maps X for which $X(0) \in B$, and $L_\mathbb{R}$ is the fixed point subgroup of L_G under the map $X \to \bar{X}^{-1}$ (here D is the formal disc around 0).

For a more detailed statement of this theorem and the relevant definitions see section 7.

Note that the map $X \to \bar{X}^{-1}$ is an antiholomorphic involution. Thus, the above double coset space is analogous to the double coset space $G(\mathbb{R}) \setminus G / B$ studied in the case of a finite dimensional reductive group. This space was studied by Richardson and Springer [9], among

others, and we now detail how our results can be seen as an analogue of theirs.

To begin with, Richardson and Springer study not $G(\mathbb{R}) \setminus G/B$ but $K \setminus G/B$ where K is the fixed point subgroup of a holomorphic involution on G. It turns out that the parameterization of these orbits is, however, the same as that for real orbits, due to a theorem of Matsuki. Before we state this theorem we need some definitions:

2.2.2 Definitions For θ a holomorphic involution, let τ be a Cartan involution commuting with θ and $\sigma = \tau\theta$ and assume the fixed point group G^τ is a maximal compact subgroup. Let $K = G^\theta$. Let $K_{\mathbb{R}} = K \cap G_{\mathbb{R}}$. For x equal to $\theta, \sigma,$, let T_x denote the space of tori fixed under θ, σ, or both, respectively.

Matsuki then proves the following theorem:

2.2.3 Theorem Each K orbit on T_θ meets T_Γ in a unique $K_{\mathbb{R}}$ orbit. Each $G(\mathbb{R})$ orbit on T_σ meets T_Γ in a unique $K_{\mathbb{R}}$ orbit. Thus these two orbit sets are in bijection. (see [8])

Using this theorem, Richardson and Springer prove:

2.2.4 Theorem Let U be the fixed point group G^τ and C be $U \cap T$. Let $\mathcal{V}_\Gamma$ be the set of $u \in U$ such that $u\theta(u^{-1}) \in N_U(C)$, and V_Γ be the orbit set $C \setminus \mathcal{V}_\Gamma/G(\mathbb{K})$. Then the natural inclusions $V_\Gamma \to B \setminus G/K$ and $\to B \setminus G/G(\mathbb{R})$ are both bijections.(see [8] or [9]).

Thus, determining the orbits of the real group $G(\mathbb{R})$ is the same as determining the orbits of K. It is in this form that Richardson and Springer prove their theorem.

Richardson and Springer parameterize the orbits of K in the following way:

2.2.5 Theorem Let I be the set of twisted involutions, i.e. the $w \in W$ such that $\theta(w) = w^{-1}$. Then there is a map $k : G \to G$ given by $g \to g\theta(g)^{-1}$. Let(V) be the preimage of $N(T)$ under this map. Then the inclusion $T \setminus (V)/K \to B \setminus G/K$. Further, the map k restricted to $\mathcal{V}$ induces a map $T \setminus \mathcal{V}/K \to I$ and thus a map $B \setminus G/K \to I$. Since $N(T)$ acts on both spaces in such a way that T acts trivially we get a Weyl group action, and the map $B \setminus G/K \to I$ gives an injection of Weyl orbits.

The upshot of all this is that there is a map of real group orbits into twisted involutions which is an injection up to the Weyl group actions. Note that although the proof in general is complicated, in the special case of $GL(n)$ we can see explicitly that the real orbits are in bijection with twisted involutions in the Weyl group. This is because the real orbits $GL(n, \mathbb{R}) \setminus GL(n)/B$ parameterize Borel subgroups. On the other hand, we will see below that the Borel subgroups of inner forms of a reductive group G are parameterized by the union, over all twisted involutions w, of a group $H^1(T_w)$. However, for $GL(n)$ there are no nontrivial inner forms (this is Hilbert's Theorem 90, which is explained below), and $H^1(T_w)$

is always trivial (this is because $GL(n)$ has no tori with factors of type S^1, also explained below). So in this case we see immediately that the twisted involutions are in bijection with Borel subgroups up to real isomorphism. Let us compare with what we have in the case of the group L:

For L_G we know that $L_{\mathbb{R}} \setminus L_G / I_+$ parameterizes bundles with a real structure and a fixed flag at 0. The element $X \in L_G$ which corresponds to a given bundle V can be extracted as a certain transition map: V can be trivialized on $\mathbb{G}_m$ and on $D_0 \sqcup D_\infty$, the disjoint union of a formal disc around 0 and ∞, and the transition map results from identifying these two trivializations. This transition map has two components, X which is the transition in the neighborhood of the D_0 and $\bar{X}^{-1}$ which is the component of the transition map around ∞. This is described below in detail.

On the other hand, we also have that vector bundles with real structure can be described by a pair T where T is a cocycle giving a torus bundle and w where w is a twisted involution in the Weyl group which inverts T, together with a real structure on the bundle given by T, as described in the theorem 5.2.4. The proof of the main theorem shows that this data comes from the transition map giving the underlying torus bundle. Together, the pair (T, w) is precisely a twisted involution in the affine Weyl group. In the case of $GL(n)$, there is at most one real structure for any pair (T, w) because $GL(n)$ has no real tori with factor S^1. So there is an injection $L_{\mathbb{R}} \setminus L_G / I_+ \to I_{aff}$ where I_{aff} is the collection of twisted involutions in the Weyl group. Further, this map is realized by letting the two components of the cocycle $X, \bar{X}^{-1}$ come together, i.e. mutliplying them, so as to give the transition map for the underyling complex bundle. So we get a result exactly analogous to that of Richardson and Springer.

Geometric Satake

We begin with a review of what the Langlands conjectures say about vector bundles on $\mathbb{P}^1$ in the absence of a real structure. Here, the fundamental result is the Geometric Satake theorem. This theorem deals with the moduli space of principal G bundles on the "double disc", a space built from two copies of the formal neighborhood of a point in $\mathbb{A}^1$ by gluing the open complements of the points together.

2.3.1 Definition Let $\mathbb{D}$ denote the scheme defined as follows. Let D_1 and D_2 denote two copies of $Spec(k[[t]]$ and p_1, p_2 denote the closed point of D_1, D_2, an U denote $D_i \setminus p_i$, which we view as identical. Then $\mathbb{D}$ is $D_1 \sqcup_U D_2$, and $Bun_G(\mathbb{D})$ denote the moduli space of G bundles on $\mathbb{D}$.

The moduli space $Bun_G(\mathbb{D})$ has a nice uniformization:

2.3.2 Theorem $Bun_G(\mathbb{D})$ is $G[[t]] \setminus G((t))/G[[t]]$ (see [4])

The Geometric Satake Theorem deals with the category $Perv_G$ of perverse sheaves on Bun_G. In particular at its most basic level the theorem says:

2.3.3 Theorem There is a reductive group $\check{G}$ such that the category $Rep(\check{G})$ is equivalent to the category $Perv_G$ (see [2])

The group $\check{G}$ has a nice combinatorial description in the case of connected semisimple G: it is the unique connected semisimple group with root data equal to the coroot data of G and vice versa.

In fact the Geometric Satake Theorem gives quite a bit more than this, but first let's note how this fits into the general framework of Geometric Langlands. Recall that in general Geometric Langlands type statements are of the form:

Quasi-Conjecture Let G be a reductive group and C a complex curve. The category of constructible sheaves on $Bun_G(C)$ is equivalent to the category of quasi-coherent sheaves on $Loc_{\check{G}}(C)$.

Let's see how this compares to Geometric Satake. First, we note that $\mathbb{D}$ is not a complex curve. However, it turns out that the category of local systems on $Bun_G(\mathbb{D})$ can be reinterpreted this way:

2.3.4 Theorem The bounded derived category of constructible sheaves on $Bun_G(\mathbb{P}^1)$ which are 0 outside a finite type substack is equivalent to the bounded derived category of constructible sheaves on $Bun_G(\mathbb{D})$ (see [1])

We will come back later to why we might want to phrase the theorem over $\mathbb{D}$ instead of $\mathbb{P}^1$.

Next, let's take a look at the other side of the isomorphism: $Rep(\check{G})$. Recall that there is only one $\check{G}$-local system on $\mathbb{P}^1$, namely the trivial one, and it has automorphism group $\check{G}$. So as a stack the moduli space of $\check{G}$-local systems on $\mathbb{P}^1$ is just $B\check{G}$. A coherent sheaf on this space is just a coherent sheaf on a point together with descent data given by an action of $\check{G}$, i.e. a $\check{G}$ representation.

Finally, we note that $Perv_G$ is a heart of the derived category of constructible sheaves, so what the Geometric Satake Theorem gives us is really just the "degree 0" part of Geometric Langlands.

Let's return to the question of why we might work with $\mathbb{D}$ instead of $\mathbb{P}^1$. The reason is that this space has a greater deal of structure. We get an interesting product structure on $Perv(Bun_G(\mathbb{D}))$ given in the following way: Consider the diagram:

$$G_K/G_O \times G_K/G_O \xleftarrow{p} G_K \times G_K/G_O \xrightarrow{q} G_K \times_{G_O} G_K/G_O \xrightarrow{m} G_K/G_O$$

where G_K is $G(\mathbb{C}((z)))$ and G_O is $G(\mathbb{C}[[z]])$, the first two arrows are given by projection and the last by multiplication. Then for any L_0, L_1 defined on G_K/G_O, there is L defined on G_K/G_O such that $q^*(L) = p^*(H^{0,p}(L_0 \boxtimes^L L_1))$, where $H^{0,p}$ denotes 0-th cohomology with respect to the perverse heart. Then the convolution of L_0, L_1 is defined to be $Rm^*(L)$. Then Geometric Satake says:

2.3.5 Theorem The product defined above is symmetric monoidal and corresponds, under the (correct) equivalence, to the tensor product on $Rep(\check{G})$. (see [2])

In fact one of the main reasons that Geometric Satake is so much easier than other instances of Langlands-type conjectures is that this monoidal product characterizes $Rep(\check{G})$.

In fact the Geometric Satake theorem is not really the right analogue to the theorem we prove, since it deals with principal bundles rather than principal bundles with ramification, and it deals with the double disc instead of $\mathbb{P}^1$, and it deals with an abelian category instead of the derived category. All of these issues can be dealt with: Lafforgue addresses the fully derived case of $\mathbb{P}^1$ in [1], and Bezrukavnikov deals with the ramified case in [3]. See also [5], [6] for relevant alternate approaches to some of these results. However, there is one point that carries through all these cases: both the moduli space of principal bundles and the moduli space of local systems for the dual group come with natural stratifications, and the structure sheaves of these strata give sheaves on both sides indexed combinatorially in the same way. The important point is that the Satake equivalence DOES NOT match structure sheaves on each side in the obvious way. This is a main source of complications that we avoid dealing with in this paper. As a result, the bijection we give is not the maximally natural one. It would be desirable for future work to address this problem.

1.3 Review of Descent and Galois Cohomology

In this section we review the few categorical notions we need to make sense of all the definitions above and in the proofs that follow. Of course all the definitions and theorems we give can be generalized but for simplicity of notation we stick to the case of $\mathbb{Z}/2\mathbb{Z}$ which is all that we will need.

We begin with the notion of fixed point sets of stacks:

3.1 Definition Let X be a stack with a $\mathbb{Z}/2\mathbb{Z}$ action. Then the quotient stack $X/\mathbb{Z}/2\mathbb{Z}$ is the functor which returns to a scheme A the groupoid of principal $\mathbb{Z}/2\mathbb{Z}$ bundles over A equipped with an equivariant map to X.

3.2 Proposition The functor $X \to X/\mathbb{Z}/2\mathbb{Z}$ is an equivalence of categories between $\mathbb{Z}/2\mathbb{Z}$-stacks and stacks over $B\mathbb{Z}/2\mathbb{Z}$

One can take this as the definition of the category of $\mathbb{Z}/2\mathbb{Z}$-stacks,

3.3 Definition The fixed points of a $\mathbb{Z}/2\mathbb{Z}$-stack X, denoted $X^{\mathbb{Z}/2\mathbb{Z}}$ is the mapping stack $Maps_{\mathbb{Z}/2\mathbb{Z}}(pt, X)$ where the point has the trivial action.

It is possible to give a more understandable description of what a fixed point stack classifies. First we need a definition:

3.4 Definition Let G be a groupoid with an action of $\mathbb{Z}/2\mathbb{Z}$. Let F be the functor associated with the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ For any object x, a fixed point structure on x is a map $f : x \to F(x)$ such that $f \circ F(f) = 1$. We call a pair (x, f) a fixed point object if f is a fixed point structure on x.

3.5 Proposition $X^{\mathbb{Z}/2\mathbb{Z}}(pt)$ is the groupoid of fixed point objects of $X(pt)$.

Proof: By definition, an object of $X^{\mathbb{Z}/2\mathbb{Z}}(pt)$ is a map from the groupoid of $\mathbb{Z}/2\mathbb{Z}$ bundles over a point to the groupoid of $\mathbb{Z}/2\mathbb{Z}$ bundles over a point equipped with a map to X. In other words, objects of the fixed point groupoid equip each principal bundle with a map to X in a functorial way. In the case of principal bundles over a point, we have exactly one, the trivial principal bundle, and it has one automorphism. So to the trivial bundle, we need to give an equivariant map to X, i.e. a pair of objects of X which are related by the $\mathbb{Z}/2\mathbb{Z}$ action. Call these objects x and $F(x)$. Then since the trivial bundle has one automorphism, we need maps $x \to F(x)$ and $F(x) \to x$, again related by $F(x)$. Call these maps f and $F(f)$. Finally, since the nontrivial automorphism squares to the identity, we get $f \circ F(f) = 1$. So we get exactly the data of a fixed point object.

The main technical tool we will use to analyze fixed point objects is Galois cohomology. This allows us to classify, for any given (x, f) a fixed point object, all the other fixed point structures on x.

3.6 Definition Let G be a group with a $\mathbb{Z}/2\mathbb{Z}$ action, and for $g \in G$ denote by $\bar{g}$ the action of the nontrivial element on g. The the cohomology $H^1(G)$ is defined to be the set of $g \in G$ such that $g\bar{g} = 1$, modulo the relation that for any $h \in G$, g $hg\bar{h}^{-1}$. The representatives of these equivalence classes are called cocycles, and this equivalence relation is called twisted conjugation.

The first main fact that we use is that cohomology classifies the fixed point structures on a given object:

3.7 Proposition: Let (x, f) be a fixed point object in G. As before denote by F the functor associated to the nontrivial element. Then there is a $\mathbb{Z}/2\mathbb{Z}$ action on $Aut(x)$, such that the isomorphism types of fixed point structures on x are in bijection with $H^1(Aut(x))$.

Proof: The nontrivial element acts on $\sigma \in Aut(x)$ by taking it to $\bar{\sigma} = f^{-1} \circ F(\sigma) \circ f$. Now we have a bijection $\phi : Aut(x) \to Hom(x, F(x))$ given by $\sigma \to F(\sigma) \circ f$. A calculation shows that σ represents a cocycle if and only if $\phi(\sigma)$ is a fixed point structure on x. Further, given $h \in Aut(x)$, another calculation shows that $h\sigma\bar{h}^{-1}$ corresponds under ϕ to $h\phi(\sigma)F(h)^{-1}$, so equivalence of cocycles matches with isomorphism of fixed point structures.

Note that since the trivial G bundle always has a real structure given by the identity map, a special case of this is that the real principal bundles with underlying bundle trivial are classified by $H^1(G)$

Thus cohomology gives us a calculational tool with which to get a handle on fixed points. What is more, there are a couple of fundamental results about cohomology that make it useful as a way of reducing complexity of fixed point calculations. The reduction step is given by the following:

3.8 Proposition H^1 is a functor from groups with $\mathbb{Z}/2\mathbb{Z}$ action to sets. Given an exact sequence of groups, $1 \to N \to G \to G/N \to 1$, we have that $H^1(N) \to H^1(G) \to H^1(G/N)$ is a fiber sequence is the sense that $H^1(N)$ surjects onto each nonempty fiber of $H^1(G) \to H^1(G/N)$.

The base case of inductive arguments will typically come from:

3.9 Proposition (A generalization of Hilbert's Theorem 90) Let $\mathbb{Z}/2\mathbb{Z}$ act antiholomorphically on a unipotent group U. Then $H^1(U)$ is trivial. Thus for $U \rtimes T$ a unipotent extension of T, $H^1(U \rtimes T) \to H^1(T)$ is injective.

Proof: For the first assertion, note that U is an extension of a smaller unipotent group by A^n. Thus by the previous proposition and induction, it is sufficient to show that $H^1(A^n)$ is trivial. There is only one real structure on a vector space, so we just have to check that H^1 vanishes for the standard complex conjugation. In this case, cocycles are v such that $v + \bar{v} = 0$, i.e. v is purely imaginary, and twisted conjugation by w sends v to $(w - \bar{w}) + v$. It is clear that all purely imaginary vectors are thus equivalent. The statement about unipotent extensions then follows from the vanishing statement and the previous proposition about exact sequences.

3.10 Proposition If $\mathbb{Z}/2\mathbb{Z}$ acts holomorphically, $H^1(U)$ is trivial, and $H^1(U \rtimes T) \to H^1(T)$ is injective.

Proof: This is exactly the same as the above until we get to the case of A^n. We still get $H^1(A^n)$ is trivial, but it takes a little more work. We need to check two cases: A^n decomposes into a direct sum of copies of A^1 on which the action is either trivial or given by multiplication by -1, and we need to check H^1 is trivial for both kinds of summands.. In the case of the trivial action, there are no cocyles ($2x \neq 0$ if $x \neq 0$), so H^1 is trivial. In the

case of the nontrivial action, every element is a cocyle, but the equivalence relation is given by $x \to 2y + x$ for arbitrary y, so all the cocycles are equivalent.

For future reference, we note one instance of the above theorems which is of special importance to us:

3.11 Definition Let G be a group with a real structure, i.e. an antiholomorphic involution. As above, this induces a real structure on $Aut(G)$, and $H^1(Aut(G))$ classifies real structures on the underlying complex group. For $w \in Aut(T)$ representing a cocyle, denote by T_w the corresponding group with real structure. This is called G twisted by w.

One sometimes useful point is that for w an inner automorphism, $Bun_{G,r}$ and $Bun_{G_w,r}$ are canonically isomorphic. In particular, let $\rho : P \to \hat{P}$ be a real structure on a G bundle. Let w act on $\hat{P}$ by the structure action of the principal bundle (i.e. as an element of the structure group, not as a principal bundle isomorphism). Then $w \circ \rho$ gives a real structure for P as a G_w bundle.

1.4 Moduli of Torus Bundles

In this section we will begin our approach to the duality statement by proving it for tori. In the first section, we will study moduli of bundles for general tori. The main point in this section is to reduce the duality statement for general tori to the special case of irreducible tori. In the second section, we will establish duality for irreducible tori by direct computation. Finally, we will give a different and more explicit description of the moduli spaces for general tori which is suitable for comparison to the description of the moduli space of vector bundles we will get in the next section. Thus we will get a duality statement for general tori, which we will later lift to a duality statement for vector bundles.

Reduction to Irreducible Bundles

In this section we shall show that duality for general tori reduces to the case of irreducible tori. The strategy is very straightforward: most of the definitions clearly preserve products.

4.1.1 Proposition Let T_0 be a torus with a real structure, and $T_0 = T_1 \times T_2$. Then $Bun_{T_0,r} = Bun_{T_1,r} \times Bun_{T_2,r}$

Proof: Recall that $Bun_{T_i,r}$ is the fixed points of the $\mathbb{Z}/2\mathbb{Z}$ action on the complex moduli space, Bun_{T_i}, and fixed points for any given $\mathbb{Z}/2\mathbb{Z}$ -space are given by $Maps_{\mathbb{Z}/2\mathbb{Z}}(pt, X)$. Furthermore, complex T_i bundles are just given by $Maps(\mathbb{P}^1, BT_i)$. So using the universal property of fiber products we have $Bun_{T_0,r} =$

$$Maps_{\mathbb{Z}/2\mathbb{Z}}(pt, Maps(\mathbb{P}^1, BT_0)) =$$

$Maps_{\mathbb{Z}/2\mathbb{Z}}(pt, Maps(\mathbb{P}^1, BT_1) \times Maps_{\mathbb{Z}/2\mathbb{Z}}(pt, Maps(\mathbb{P}^1, BT_2)) =$

$Bun_{T_1,r} \times Bun_{T_2,r}$

as required.

We also have the corresponding statement for local systems:

4.1.2 Proposition Let T_0 be a torus with an involution, and $T_0 = T_1 \times T_2$. Then $Loc_{T_0,r} = Loc_{T_1,r} \times Loc_{T_2,r}$.

The proof of this statement is exactly the same as for the previous statment, except we replace BT_i with BT_i^{disc}, the classifying space for T_i with the discrete topology, which classifies local systems instead of principal bundles.

These two propositions immediately imply that duality for general tori can be reduced immediately to duality for irreducible tori:

4.1.3 Corollary Let $T_0 = T_1 \times T_2$ be a torus with real structure and $\check{T}_0 = \check{T}_1 \times \check{T}_2$ be the dual tori with involution. Then if we have $K_{Coh}(\check{T}_i) \cong K_{Loc}(T_i)$ for $i = 1, 2$, the same also is true for $i = 0$.

Proof: This follows immediately from the previous two propositions together with the fact that K converts product to tensor products.

Duality for Irreducible Tori

In this section, we will give an explicit computation of the relevant K groups and moduli spaces for irreducible tori, from which duality can be directly seen. This is essentially the only part of this paper where we use anything specific about the involution α. Recall that there are three isomorphism types of irreducible tori with real structure, which we name based on their fixed points (for a proof of this classification see the Appendix). They are:

Type $\mathbb{R}^\times$. The underlying complex torus is $\mathbb{G}_m$ and the complex conjugation is the standard one. The dual torus $\check{\mathbb{R}}^\times$ is $\mathbb{G}_m$ with the trivial involution.

Type S^1. The underlying complex torus is $\mathbb{G}_m$ and the complex conjugation is $z \to \bar{z}^{-1}$. The dual torus $\check{S}^1$ is $\mathbb{G}_m$ with involution given by $z \to z^{-1}$.

Type $\mathbb{C}^\times$. The underlying torus is $\mathbb{G}_m^2$ and the conjugation is given by $(z, w) \to (\bar{w}, \bar{z})$. The dual torus is $\mathbb{G}_m^2$ with involution given by $(z, w) \to (w, z)$.

We now go through the various moduli spaces case by case. To save some work we note now that the automorphism group of a torus bundle is always canonically identified with the

torus, so we don't have to separately calculate automorphism groups.

Type $\mathbb{R}^\times$: Let $p = 0$ be in $\mathbb{P}^1$. The complex moduli space of bundles is the disjoint union over all $n \in \mathbb{Z}$ of the moduli spaces of bundles isomorphic to $\mathcal{L}(np)$. Let C_n denote the component corresponding to bundles isomorphic to $\mathcal{L}(np)$. The $\mathbb{Z}/2\mathbb{Z}$ action sends $\mathcal{L}(np)$ to $\mathcal{L}(n\alpha(p))$, since the cocyle t^n becomes $(-1)^n t^n$ under the substitution $t \to -\bar{t}^{-1}$. A map $\phi : L(np) \to \mathcal{L}(n\alpha(p))$ is given by a polynomial at^{-n}, which has conjugate $\bar{a}(-1)^n t^n$, and so for ϕ to give a fixed point structure we require $a\bar{a}(-1)^n = 1$. This happens if and only if n is even, and in this case the set of fixed point structures is indexed by a such that $a\bar{a} = 1$, i.e. a is on the unit circle. Then twisted conjugation by an automorphism given by scaling by b sends $a \to b\bar{b}^{-1}a$, and it is clear that all a are conjugate in this way. So the moduli space in this case has exactly one real bundle for each even n, with automorphism group $\mathbb{R}^\times$, (recall that the automorphism group of a torus bundle is always identified with the torus), and the K group has thus two generators for each even number.

Dually, we have one complex local system, with maps given by scaling. A fixed point structure is given by a such that $a^2 = 1$, of which there are two up to conjugation, so we get two bundles with automorphism group $\mathbb{C}^\times$. So again we K group with generators indexed by two copies of $\mathbb{Z}$, and we get an isomorphism between the two K groups.

Type S^1: The complex moduli space is the same, but the cocyle t^n becomes $(-1)^n t^{-n}$, so the only bundle with a chance of having a fixed point structure is the trivial bundle. The fixed point structures on the trivial bundle are then given by a such that $a\bar{a}^{-1} = 1$, i.e. $a \in \mathbb{R}^\times$, and the twisted conjugation by b takes a to $b\bar{b}a$, so up to conjugation there are two fixed point structures given by -1 and 1, and they each have automorphisms given by S^1. So there are two generators of the K group.

Dually, the fixed point structures on the trivial local system are given by a such that $aa^{-1} = 1$, i.e. all of $\mathbb{C}^\times$, modulo $a \to b^2 a$, so there is one fixed point object with automorphisms μ_2 (the fixed points of $z \to z^{-1}$), so again there are two generators of K.

Finally, the type $\mathbb{C}^\times$ case. In this case the cocyle (t^n, t^m) becomes $(-1^m t^m, -1^n t^n)$, so there is a fixed point structure only if $n = m$. In this case, a fixed point structure is given by a map represented by a pair of constants (a, b) and we require that $-1^n a\bar{b} = 1$, i.e. b is freely determined by a. Finally, acting on the fixed point structure indexed by a by the automorphism indexed by (x, y) we see it is equivalent to the fixed point structure indexed by $x\bar{y}^{-1}a$, so clearly all the fixed point structures are conjugate. So we get for each n a single real bundle with automorphisms $\mathbb{C}^\times$. In this case the K group is free on a set of generators indexed by $\mathbb{Z}$.

Dually, we consider fixed point structures on the trivial bundle given by maps (a, b) such that $ab = 1$, i.e. we have one for each $a \in \mathbb{C}^\times$, and conjugating by (x, y) we see that the structure indexed by a is equivalent to the one indexed by $xy^{-1}a$, i.e. there is one fixed point

up to conjugation, and it has automorphisms $\mathbb{C}^\times$, so again the K group is free on a a set of generators indexed by $\mathbb{Z}$.

4.2.1 Corollary $K_{Coh}(Loc_{\check{T},r}) \cong K_{Loc}(Bun_{T,r})$ for a torus.

An Explicit Description of the Moduli Space of Real Torus Bundles

Now that we have duality for tori, we will give a more explicit description of the relevant moduli spaces. This new description will be in a form that can be directly compared to a similar one we will produce for moduli of vector bundles in the next section, and it is from this comparison statement that we will deduce duality for vector bundles.

Given a torus T, we have that the T-bundles are indexed by elements of Λ, the coroot lattice, in such a way that each $\lambda \in \Lambda$ corresponds to a component of Bun_T consisting of bundles isomorphic to the one given by the cocycle λ. For a given λ the corresponding bundle is formed by gluing the trivial bundle $Triv_0$ on $\mathbb{P}^1 \setminus 0$ to the trivial bundle $Triv_\infty$ on $\mathbb{P}^1 \setminus \infty$ by identifying, for each $x \in \mathbb{G}_m$, the fiber $Triv_{0,x}$ with $Triv_{\infty,x}$ using the map $\lambda(x)$ (for a more detailed discussion of how this works see the section on Uniformization). Call this component S_λ. The conjugation on T induces an involution on the set of components (equivalently, on Λ). We have the following description of Bun_T:

4.3.1 Proposition: Let $\mathbb{Z}/2\mathbb{Z}$ act on Λ as above, and let $\Lambda^{\mathbb{Z}/2\mathbb{Z}}$ be the invariants. For an invariant λ, we have S_λ gets an induced $\mathbb{Z}/2\mathbb{Z}$ action, and define $|\lambda|$ to be 0 if this component has a fixed point and 1 otherwise. Then the moduli space of real T bundles is:

$$\sqcup_{\lambda \in \Lambda^{\mathbb{Z}/2\mathbb{Z}},|\lambda|=0} H^1(T) \times pt/T_\mathbb{R}$$

(Recall that we have shown H^1 is trivial for tori of type $\mathbb{C}^\times$ or $\mathbb{R}^\times$ and $\mathbb{Z}/2\mathbb{Z}$ for tori of type S^1, and respects products.

Proof: Certainly every fixed point is in exactly one fixed stratum, so the fixed points are just the union of the fixed points of all fixed strata which contain some fixed point. Then for any given λ which is fixed and has $|\lambda| = 0$, there is a fixed point. Necessarily the complexification of this fixed point has automorphisms given by T, and hence the set of all fixed points in this component is classified by $H^1(T)$ as explained in the previous section, and each of these fixed points has automorphism group necessarily $T_\mathbb{R}$.

A similar description is available for the moduli of local systems:

4.3.2 Proposition The moduli space of local systems with fixed point structure is just $H^1(\check{T}) \times pt/(\check{T}^{\mathbb{Z}/2\mathbb{Z}})$

Proof: This is essentially trivial since there is only one stratum, which definitely has a fixed point structure. As above, the set of fixed points is classfied by H^1, and each fixed point has automorphisms identified with the invariant torus.

1.5   Moduli of Real Ramified Vector Bundles

We can now describe the moduli space of real vector bundles with ramification. Recall that this means, first of all, that we want to classify quadruples $\{V, F_0, F_1, \rho\}$ where V is a vector bundle, F_0, F_1 are flags in the fibers of V at 0 and ∞, respectively, and ρ is an isomorphism $V \to \alpha^* \hat{V}$ such that $\rho \circ \hat{\rho} = 1$ and $\rho(F_0) = \hat{F}_1$, up to isomorphism. In the remainder of this section we will only deal with vector bundles, so the structure group G is always $GL(n)$, the Weyl group is always S_n, etc.

Recall that $Bun'_{GL(n)}$ denotes the moduli space of ramified vector bundles, without real structure. In this section we will fix n once and for all, and drop the subscript, so Bun' denotes the moduli space of complex ramified vector bundles, and Bun'_r denotes the moduli space of real vector bundles. We know that there is a $\mathbb{Z}/2\mathbb{Z}$ action on Bun' and that Bun'_r is given by taking the fixed points of this action, and we will analyze Bun'_r using this description. The strategy we will take is based on three observations: First, taking fixed points is local, so we can chop up Bun' into strata and take fixed points on each stratum individually. Second, the existence of a fixed point in a given stratum is directly related to existence of fixed points in a related stratum of a space of torus bundles, and we already know how to analyze torus bundles. Finally, given a fixed point, the analysis of all fixed points in a given stratum is controlled by a calculation in Galois cohomology, which by Hilbert's Theorem 90 again essentially reduces to a calculation for a torus, which we have already done.

Combinatorics of Complex Strata

To begin with, let us recall the division of the moduli space of complex vector bundles into strata. Combinatorially, we know that these strata are indexed by the affine Weyl group:

5.1.1 Theorem The moduli space Bun' is a union of strata, one for each isomorphism type of bundle with ramification, and these strata are in bijection with the elements of the affine Weyl group (see [8])

Understanding how this bijection works geometrically will be very useful for us. Recall that set theoretically, we have that the affine Weyl group is the product of the coroot lattice and the Weyl group, and that the coroot lattice is in bijection with isomorphism types of T bundles for T a torus. So to understand the indexing of the strata geometrically, we will produce a functorial way of assigning to each ramified vector bundle a torus bundle and a permutation. The main idea is that the associated graded functor accomplishes this task

for filtered objects, and we can lift local filtrations to global filtrations in order to take the associated graded of a vector bundle with a flag. We approach this in stages, building up from isotypic bundles.

5.1.2 Lemma Suppose V is a vector bundle and F_0 is a flag in the fiber at 0 which is compatible with the degree filtration. Then functorially in the pair (V, F_0), there is a filtration of V which restricts in the fiber at 0 to the flag F_0.

Proof: First, suppose V is isomorphic to $O(k)^r$ for some k, r. Then F_0 induces a flag in $V(-k)$, which is trivial, and hence induces a filtration of $V(-k)$. Finally, this induces a filtration of V by twisting back.

In the case where V is not isotypic, we proceed by induction: suppose the flag F_0 begins with the line L, contained in the lowest degree piece of V. By the above this lifts to a line bundle V_0 with fiber at 0 equal to L, contained in the lowest degree piece of V. Modding out V_0 we get a by induction a filtration on the quotient with fiber at 0 equal to F_0/L. The preimage of this filtration then completes V_0 to a filtration of the entire bundle V with fiber F_0 at 0. QED

If the flag F_0 is not compatible with the degree filtration, we can force it to be compatible by replacing V with the associated graded bundle of V with respect to the degree filtration:

5.1.3 Lemma Let V be a vector bundle with F_0 a flag in the fiber at 0. Then F_0 induces a flag in the fiber of grV at 0 compatible with the canonical direct sum decomposition of grV, where grV is the associated graded with respect to the degree filtration.

Proof: For each k, let V_k be the $k-th$ piece of the degree filtration, and $V_{k,0}$ the fiber at 0. Then let $F_{0,k}$ be the image of $F_0 \cap V_{k,0}$ in $V_{k,0}/V_{k+1,0}$. Thus we have functorially a (redundant) filtration in the fiber at 0 of each summand of $gr(V)$. Now we shall construct F', a filtration of grV_0 from these pieces, inductively. Suppose we have defined F'_i and we want to define F'_{i+1}. There is exactly one k for which the image of $F_{i+1} \cap V_{k,0}$ in V_k/V_{k+1} is not equal to that of $F_i \cap V_{k,0}$. This image is then F'_{i+1}. QED

Together these two lemmas allow us to lift a filtration from the fiber at 0 and hence to define a torus bundle and a permutation:

5.1.4 Theorem-Definition There is a functor grF which takes a triple (V, F_0, F_1) as above and returns a torus bundle and a permutation

Proof: Given (V, F_0), construct a triple (V', F'_0, F'_1) such that the flags are compatible with the degree filtration, by taking associated graded with respect to the degree filtration. Then from F'_0 define a global filtration on V'. The associated graded of this filtration is a vector bundle with canonical direct sum decomposition, which determines a torus bundle.

The permutation may be defined by comparing the fiber at ∞ of this global filtration with F_1'.

As a consequence of this theorem, we get a coroot and a permutation assigned canonically to each stratum.

Comparison to Torus Bundles

We can now analyze the fixed points stratum by stratum. First, we need to determine which strata are fixed, and then for each stratum we can determine the fixed points.

5.2.1 Proposition Let (λ, π) be a pair of a coroot and a permutation indexing a stratum, as defined in the previous section. Call this stratum $S_{\lambda,\pi}$. For a given torus, T let S_λ denote the stratum in the moduli space of T bundles indexed by the coroot λ, and let T_π denote the diagonal torus of $GL(n)$ with its standard complex conjugation, twisted by π. Then $S_{\lambda,\pi}$ is fixed by the $\mathbb{Z}/2\mathbb{Z}$ action if and only if π represents an element of $H^1(W)$ and S_λ is a fixed stratum in the moduli of T_π bundles.

Proof: Since the (λ, π) is extracted functorially, the only way a stratum $S_{\lambda,\pi}$ can be fixed is if the associated (λ, π) is fixed, and again by functoriality, this can be calculated using any given representative of the isomorphism type indexed by the stratum $S_{\lambda,\pi}$. A convenient representative is the bundle $\oplus O(\lambda_i)$ with the flag at 0 given by the fiber of the direct sum decomposition and the flag at ∞ given by π of the fiber of the direct sum decomposition. We then see that if we conjugate, the permutation becomes π^{-1} and the coroot becomes $\pi\lambda$. So π represents a cocycle in $H^1(W)$ and λ goes to the coroot that it would go to under conjugation for T_π. QED.

Next, we must determine whether a fixed stratum $S_{\lambda,\pi}$ actually has any fixed points.

5.2.2 Proposition Let $S_{\lambda,\pi}$ be a fixed stratum, and S_λ the stratum indexed by λ in the moduli space of T_π bundles. Then $S_{\lambda,\pi}$ has a fixed point if and only if S_λ does.

Proof: To show this, it is sufficient to show that there are $\mathbb{Z}/2\mathbb{Z}$ equivariant maps $S_{\lambda,\pi} \to S_\lambda$ and $S_\lambda \to S_{\lambda,\pi}$. Then if there is a fixed point in one space its image will be a fixed point of the other space.

The map $S_{\lambda,\pi} \to S_\lambda$ is grF. Since this is a functor, and there is only one object of either groupoid (up to isomorphism), to check that grF is $\mathbb{Z}/2\mathbb{Z}$ equivariant it is sufficient to show this for a single object. For this we choose $\oplus O(\lambda_i)$ as above.

For the map in the other direction, we take the functor from T_π bundles to vector bundles given by induction. The ordering on the factors of T_π determines the flag at 0, and the conjugate ordering determines the flag at ∞. QED

Lastly, we need to determine, for each stratum bearing fixed points, what the fixed points actually are.

5.2.3 Proposition Let $S_{\lambda,\pi}$ be a stratum bearing a fixed point, and S_λ the corresponding torus stratum as above. Then there is a bijection b from fixed points of S_λ to fixed points of $S_{\lambda,\pi}$ such that for each fixed point p, $Aut(b(p))$ is a unipotent extension of $Aut(p)$

Proof: Let p_0 be a fixed point. We know from the proof of the previous proposition that it may be assumed to be induced from some torus bundle, q_0. Then there is a map $Aut(q_0) \to Aut(p_0)$. Since we already know that $Aut(p_0)$ is a unipotent extension of a torus, this map must induce an isomorphism $H^1(Aut(q_0)) \to H^1(Aut(p_0))$ by Hilbert's Theorem 90, and so we get a bijection on the fixed points. It is immediate from the definition of functor H^1 that this bijection just takes a torus bundle to the induced vector bundle. So there is a map $Aut(p) \to Aut(b(p))$ for each fixed point p in S_λ, and hence $Aut(b(p))$ is a unipotent extension of $Aut(p)$. QED

Putting all these pieces together we are now in a position to give a description of the entirety of Bun'_r in terms of torus bundles.

5.2.4 Theorem For $w \in H^1(W)$ let T_w be the torus twisted by w and $Bun_{w,r}$ the moduli space of real torus bundles for T_w. Then there is a map $\phi : \sqcup_{w \in H^1(W)} Bun_{T_w,r} \to Bun'_r$ such that 1) ϕ is surjective on points 2) For each point, $\phi(p)$ has automorphism group a unipotent extension of $Aut(p)$, and 3) The induced map on K groups of local systems is an isomorphism.

Proof: The previous two propositions give parts 1 and 2. Part 3 follows immediately since unipotent extensions do not change the K group of local systems.

5.2.5 Corollary $K_{Loc}(Bun'_r)$ is the direct sum over $w \in H^1(S_n)$ of $K_{Loc}(Bun_{T_w,r})$, where S_n is the Weyl group of $GL(n)$ i.e. the symmetric group on n letters.

1.6   Spectral Ramified Real Bundles

We can perform a similar analysis of the space dual to the space of ramified real bundles. This analysis differs in two small ways. First, a simplification, is that as we shall see every fixed stratum of the complex moduli actually has a fixed point. Second, making things slightly more complicated, is that we are using holomorphic rather than antiholomorphic involutions, so we do not have Hilbert's Theorem 90 available. The result of this will be that we have some extra affine directions in the moduli space, which doesn't affect the final result in K theory. Recall the notation used above: Loc' denotes the moduli space of local systems (of fixed rank n) with a flag at 0 and ∞, and Loc'_r represents the fixed points of the $\mathbb{Z}/2\mathbb{Z}$ action. Recall that the strata of Loc' are indexed by elements of the Weyl group

6.1.1 Proposition A stratum of Loc' is fixed if and only if the corresponding $w \in W$ satisfies $w^2 = id$. In this case the stratum has a fixed point.

Proof: The involution switches the flags without changing the isomorphism type of the underlying local system, so the permutation indexing the stratum is inverted. Thus we must have $w = w^{-1}$. In this case, the map given by w gives a fixed point structure to the trivial bundle, which interchanges the standard Borel B with wB, hence equipping the trivial bundle with w and this choice of flags is a fixed point in the stratum labelled by w.

We will use S_w to denote the stratum of Loc' indexed by w and $S_{w,r}$ for its fixed points.

6.1.2 Proposition Let T_w be the diagonal torus with involution given by w. Then there is a map $\phi : Loc_{T_w,r} \to S_{w,r}$ which is surjective on components and such that for each component p, $\phi(p)$ is $pt/U \rtimes T$ for U unipotent, and p is pt/T

Proof: The map is simply given by induction, taking the flag at 0 to be the standard B and the flag at ∞ to be wB. Let 1_{T_w} be the trivial T_w local system. Then by functoriality we have a map $Aut(1_{T_w}) \to Aut(\phi(1_{T_w}))$, so the latter is a unipotent extension of the former. Then the components of $S_{w,r}$ are indexed by $H^1(U \rtimes T_w)$, which is isomorphic to $H^1(T_w)$, which indexes the components of $Bun_{T_w,r}$, and so we get a bijection on components. Once again functoriality of the automorphism groups then tells us that for a torus bundle P, the automorphism group of $\phi(P)$ is a unipotent extension of P.

6.1.3 Theorem The Grothendieck group of coherent sheaves on Loc'_r is just the direct sum of the Grothendieck groups of coherent sheaves on $Loc_{T_w,r}$ over all w such that $w^2 = 1$.

Proof: This follows immediately from the previous proposition, together with the fact that taking unipotent extensions does not affect the K theory.

6.1.4 Theorem We have duality for vector bundles, i.e. $K_{Loc}(Bun'_r) \cong K_{Coh}(Loc'_r)$

Proof This is immediate from 4.2.5, 5.1.4, and 3.2.1

1.7   Uniformization

The goal of this section is to give a presentation of the stacks $Bun_{G,r}$ and $Bun'_{G,r}$ in terms of loop groups, analogous to the usual presentation of Bun_G. Recall that Bun_G is usually presented as $L_{G,-}\backslash L_G/L_{G,+}$ where:

7.1 Definition L_G, the loop group of G, is the group of Laurent series with values in G

$L_{G,+}$ is the subgroup of L_G consisting of power series in G

$L_{G,-}$ is the subgroup consisting of series with values in G that extend to ∞,

where we think of series as functions on a subscheme of $\mathbb{P}^1$.

This presentation arises in the following way: first, one considers the space $X = \mathbb{A}^1 \sqcup D$, where D is the formal disc around 0, which has a faithfully flat map $X \to \mathbb{P}^1$. By faithfully flat descent, or by the Beauville-Laszlo theorem (see [13]) we have that a bundle on $\mathbb{P}^1$ is the same as a bundle B_1 on $\mathbb{A}^1$ together with a bundle B_2 on D and an identification $B_1 \to B_2$ defined on $\mathbb{A}^1 \times_{\mathbb{P}^1} D$, which we denote D^o. Next, one proves that vector bundles on $\mathbb{A}^1$ and D^1 are trivial, so that the only data really is given by the isomorphism, which is an element of L_G. Finally, one has to identify bundles that are isomorphic: the automorphisms of the trivial bundle on $\mathbb{A}^1$ are precisely $L_{G,+}$ and on D the automorphisms are $L_{G,-}$, so we mod out these two actions to give the presentation $L_{G,-} \backslash L_G / L_{G,+}$.

We now wish to do the same sort of thing for $Bun_{G,r}$. Clearly the key technical step is to prove triviality for a real bundle on an affine open subspace. Then, we simply need to pick a judicious covering of $\mathbb{P}^1$ by affine spaces.

To prove triviality, we will take an approach that gives us somewhat more. In particular, we will show that every G bundle with real structure on $\mathbb{P}^1$ has what is called a reduction to a torus. Then we will use the fact that torus bundles are easily classifiable to prove the triviality statement. Let us define what this means:

7.2 Definition Let P be a G bundle with a real structure. We say that P has a reduction to a torus if there exist 1) an inner form G_w of G, 2) an invariant torus T_w of G_w and 3) a principal T_w bundle P_t together with an isomorphism $P_t \times_G G_w/T_w \to P$, where we are implicitly using the identification of G_w bundles with G bundles to view P as a G_w bundle, and $\times_G$ means we mod out $P_t \times G_w/T_w$ by the "antidiagonal" G action $(p, g) \to (ph, h^{-1}g)$.

In the complex case, this is a classical theorem of Grothendieck.

We will delay the proof that every bundle has a torus reduction to the next section. The reader may also note that it follows from the classification statement given in the previous sections, but we will give a more direct proof.

We will now prove the proposition:

7.3 Proposition A G bundle with real structure is trivializable on $\mathbb{G}_m$.

Since we have reduced the bundle P to a torus, it is sufficient to show that torus bundles are trivial on $\mathbb{P}^1 \backslash \{0, \infty\}$.

7.4 Proposition A real torus bundle on $\mathbb{P}^1 \backslash \{0, \infty\}$ is trivial.

Proof: Of course, the underlying complex bundle will necessarily be trivial, so what we really need to show is that the trivial T bundle on $\mathbb{G}_m$ has only one real structure. Furthermore, since we know that T is a product of factors of the three types $\mathbb{R}^\times, S^1, \mathbb{C}^\times$, it is sufficient to prove the proposition for these three types of factors.

$\mathbb{R}^\times$. A map from the trivial bundle to itself is of the form at^m for m an integer, and we require $at^m \bar{a}(-\bar{t})^{-m} = (-1)^m a\bar{a}$ to be 1, in order for this map to give a real structure. This requires m to be even, and a to be on S^1. Then if $m = 2k$, and $a = e^{iz}$, conjugation by $e^{iz/2}t^k$ gives that this map is equivalent to the identity.

S^1. A map from the trivial bundle to itself is again given by at^m, but now the conjugation sends $a \to \bar{a}^{-1}$. So multiplying at^m by its conjugate we see that we require $(-1)^m a\bar{a}^{-1}t^{2m}$ to be 1. This is possible exactly when $m = 0$ and $a \in \mathbb{R}^\times$. If a is positive, we can conjugate by $a^{-\frac{1}{2}}$ to see that this structure is equivalent to 1. Otherwise, we conjugate by $(-a)^{-\frac{1}{2}}t$

$\mathbb{C}^\times$ A map from the trivial bundle to itself is given by (at^m, bt^n) and we require $(-1)^n a\bar{b}t^{m-n}$ to be 1. Conjugating by $(a^{-1}t^{-m}, 1)$ gives 1.

We are now in a position to give a uniformization of $Bun_{G,r}$.

7.5 Theorem The moduli space of bundles with real structure on $\mathbb{P}^1$ is given by $L_R \backslash L_G / L_{G,+}$ where L_R is the subgroup of L_G consisting of real loops, i.e. $\gamma(t)$ such that $\gamma(t) = \overline{\gamma((-\bar{t})^{-1})}$.

Proof: Consider the faithfully flat cover X which is the disjoint union of $\mathbb{G}_m$ and $D^+ \sqcup D^-$, where D^+ is the formal neighborhood of 0 and D^- is the formal neighborhood of ∞. Note that both pieces of this cover are fixed by α. By faithfully flat descent, a bundle with a real structure is just a bundle with real structure on each piece, together with identification on $D^{+,o} \sqcup D^{-,o}$. But we know that bundles with real structure on $\mathbb{G}_m$ are all trivial, and this is obviously also the case for $D^+ \sqcup D^-$, so again the only data is the identification. Further, since the identification on $D^{-,o}$ is determined from that on $D^{+,o}$ by the real structure, the only data is a single loop in the punctured disc, i.e. an element of L_G. Then we need to mod out by automorphisms of the trivial bundle on each piece of the cover. For the disjoint union of two formal discs, an automorphism respecting real structure is freely determined by an automorphism of the trivial bundle on one disc, so we get $L_{G,+}$. On $\mathbb{G}_m$, an automorphism is precisely an element of L_R. So we get the desired presentation.

A simple modification of the above also gives a uniformization for the moduli of bundles with a real structure and a Borel reduction at 0. Namely, we define the Iwahori group I_+ to be the subgroup of $L_{G,+}$ consisting of functions which, when evaluated at 0, lie in the Borel B. This is then the group of automorphisms of the bundle defined on D and preserving the Borel, so we get

7.6 Theorem The moduli space of bundles with real structure and Borel reduction,

$Bun'_{G,r}$, is $L_R \backslash L_G / I_+$.

1.8 Reduction to a Torus for Real Principal Bundles

Let us fix notation: G will be a reductive group, T a torus in G, and $N(T)$ the normalizer of T. We will always take principal bundles to be on $\mathbb{P}^1$. For P a principal G-bundle, $Ad(P)$ is the bundle associated to P and the adjoint representation of G, i.e. $Ad(P) = P \times^G \mathfrak{g}$. This is a Lie algebra in vector bundles, which follows from the fact that the Lie bracket on $\mathfrak{g}$ is invariant under conjugation.

We begin with a discussion of Grothendieck's method for producing a reduction of structure for a principal bundle with group G to the normalizer of a torus, $N(T)$

8.1 Proposition 1(Grothendieck): Let P be a principal bundle with structure group G, a reductive group, and let s be a section of $Ad(P)$ such that the value of s at every point is regular semisimple. Then P has a reduction of structure to $N(T)$.

Proof: Let $\tilde{s}$ be the function that assigns to each point of $\mathbb{P}^1$ the commutator of the value of s at that point, inside the fiber at that point of $Ad(P)$. Because s is regular semisimple, this is a section of $P \times^G C$, where C is the set of Cartan subalgebras of $\mathfrak{g}$. As a G set, C is isomorphic to $G/N(T)$, so we get a section of $P \times^G G/N(T)$, which is the same as a reduction of structure.

Now it is easy to see that this can be souped up with real structures:

8.2 Proposition: Let P be a principal bundle with real structure, and assume $N(T)$ is invariant under conjugation. Then P has a reduction to $N(T)$ if $Ad(P)$ has an invariant regular semisimple section as above.

Proof: In the above proof, we simply note that an invariant s gives rise to an invariant $\tilde{s}$, which is the same as an invariant section of $P \times^G G/N(T)$.

Grothendieck gives us the following useful facts as well:

8.3 Proposition: Let s be a section of $Ad(P)$ which is regular semisimple at some point. Then it is regular semisimple at all points

8.4 Proposition: $Ad(P)_0$, the subbundle of $Ad(P)$ of nonnegative degree, is a Lie algebra subbundle, and the fiber of $Ad(P)_0$ contains regular semisimple elements

Now we can show that any principal bundle with real structure has a reduction of structure to an invariant normalizer of a torus:

8.5 Proposition: Let P be a principal G-bundle with real structure, and $N(T)$ real invariant. Then P has a reduction of structure to $N(T)$.

Proof: In light of Propositions 8.1 and 8.3 , it is sufficient to show that $Ad(P)$ has a real invariant section which is regular semisimple at some point. We have that $Ad(P)_0$ is a real Lie algebra subbundle, and we know that its fiber at some point x contains regular semisimple elements. Since the set of regular semisimple elements of $\mathfrak{g}$ is Zariski dense, the intersection of this locus with $Ad(P)_{0,x}$ is also Zariski dense. Then since $\Gamma(Ad(P)_0) \to Ad(P)_{0,x}$ is a surjection, the regular semisimple elements of the global section space is also Zariski dense in all global sections. But the real invariant sections form an $\mathbb{R}$ -vector subspace of global sections, and hence must intersect any Zariski dense subset nontrivially.

8.6 Theorem Let P be a principal G bundle with real structure. Then P has a reduction to a torus.

Proof: By the previous proposition, P has a reduction to $N(T)$ for some torus T. Inducing from $N(T)$ to W, we get a W bundle with a real structure. Since W is discrete and $\mathbb{P}^1$ is connected, this gives a well defined element w of $H^1(W)$. Twisting G, T by this element we get that P has a reduction to T_w whose induced W bundle is trivial, so that P has a reduction to T_w, as required.

1.9   Appendix

In this appendix, we show that every torus with real structure is a product of ones of the three types list above: $\mathbb{R}^\times, S^1, \mathbb{C}^\times$.

We know that every complex torus is just $\mathbb{G}_m^n$. So we are trying to classify real structures on this torus. This torus has a real structure given by complex conjugation, with automorphism group $SL(n, \mathbb{Z})$. So the general machinery set up in section 2 equates the collection of real tori with the collection of $g \in SL(n, \mathbb{Z})$ such that $g^2 = 1$, up to conjugation. Thus our goal becomes classifying such g, for all n.

This latter classification problem can be understood as that of classifying $A = \mathbb{Z}[g]/g^2 - 1$ modules with underlying abelian group a lattice. Our task is to show that all such modules are direct sums of those of the following three types:

$\mathbb{Z}^+$, with underlying group $\mathbb{Z}$ and g acting trivially.

$\mathbb{Z}^-$, with underlying group $\mathbb{Z}$ and g acting by -1

$\mathbb{Z}^2$ with underlying group $\mathbb{Z}^2$ and g acting by switching the factors.

We prove this by induction on the rank of the underlying group.

Suppose M is an A-module of rank n. Let v be an element of M. If $gv \neq -v$, then $w = v + gv$ is a nonzero invariant element. Otherwise, let $w = v$ Let $M_0 = \mathbb{Z}w$ be its span, so that M_0 is an A-module. Then M/M_0 has smaller rank, but it might not be free. Let M_0' then be the preimage of the torsion subgroup of M/M_0. We still have that the underlying group of M_0' must be invariant and free of rank 1, and now M/M_0' has free underlying group of rank less than that of M. Define M_2 to be this quotient, so by induction $M_2 = \oplus N_i$ where each N_i is of one of the three types. We now have that M is an extension of either $\mathbb{Z}^+$ or $\mathbb{Z}^-$ with a direct sum of modules of the desired form.

To finish the theorem, we just need to do some extension calculations. Direct calculation of Ext^1 shows that there are no extensions between $\mathbb{Z}^2$ and anything (it is a free module), or $\mathbb{Z}^+$ or $\mathbb{Z}^-$ with themselves. So we can split off any such components, and now we are left with an extension of $\mathbb{Z}^+$ by a sum of $\mathbb{Z}^-$'s, or the reverse.

First assume the submodule is $\mathbb{Z}^+$, with basis vector e_0. Let $e_1, ..., e_l$ map to a basis of the quotient, such that $ge_i = -e_i + a_i e_0$. If any a_i is 0, we can split that component off, so without loss of generality they are all nonzero. If a_i is even, $a_i = 2k$, then replacing e_i with $e_i' = e_i - ke_0$ we see that $ge_i' = -e_i'$, so we can split off the span of e_i' as a factor of type $\mathbb{Z}^-$. So we can assume all the a_i are odd, and define $a_i = 2k_i + 1$. Now define e_0', e_1' by $k_1 e_0 - e_1, -(k_1 + 1)e_0 + e_1$, so that g switches e_0', e_1'. Then define e_i' for all $i \neq 0, 1$ by $e_i - e_1 + (-k_i + k_1)e_0$, so that $ge_i' = -e_i'$. It thus follows that the module is a direct sum of a copy of $\mathbb{Z}^2$ and copies of $\mathbb{Z}^-$.

Alternatively, assume the submodule is $\mathbb{Z}^-$. As before, we can assume we have a basis $e_0, e_1, ..., e_l$ with $ge_0 = -e_0$ and $ge_i = a_i e_0 + e_i$. Again, if any $a_i = 2k_i$, we can replace e_i with $k_i e_0 + e_i$, and split off a factor of the form $\mathbb{Z}^+$, so without loss of generality $a_i = 2k_i + 1$ for all i. Then we define e_0', e_1' to be $2k_1 e_0 + e_1, e_0 + e_1$ and e_i' to be $e_i - e_1 + (k_i - k_1)e_0$, so that we see we get a sum of $\mathbb{Z}^2$ and $\mathbb{Z}^+$'s.

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