THE HONEST EMBEDDING DIMENSION OF A NUMERICAL SEMIGROUP.

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Abstract. Attached to a singular analytic curve germ in d-space is a numerical semigroup: a subset S of the non-negative integers which is closed under addition and whose complement is finite. Conversely, associated to any numerical semigroup S is a canonical mononial curve in e-space where e is the number of minimal generators of the semigroup. It may happen that $d < e = e(S)$ where S is the semigroup of the curve in d-space. Define the minimal (or 'honest') embedding of a numerical semigroup to be the smallest d such that S is realized by a curve in d-space. Problem: characterize the numerical semigroups having minimal embedding dimension d. The answer is known for the case $d = 2$ of planar curves and reviewed in an Appendix to this paper. The case $d = 3$ of the problem is open. Our main result is a characterization of the multiplicity 4 numerical semigroups whose minimal embedding dimension is 3. See figure 1. The motivation for this work came from thinking about Legendrian curve singularities.

A numerical semigroup is a subset $S \subset \mathbb{N}$ of the natural numbers $\mathbb{N}$ closed under addition and whose complement is finite. Invariantly attached to any curve singularity in d-space is a numerical semigroup. See section 1. For example, if $m < n$ are relatively prime integers then the planar curve $x = t^m, y = t^n + at^{n+1} + bt^{n+2} + \ldots$ has the semigroup $< m, n >$ attached to it regardless of the higher order terms $at^{n+1} + bt^{n+1} + \ldots$ etc. Here $< m, n >= \{km + \ell n : k, \ell \in \mathbb{N}\}$ denotes the semigroup generated by m and n, i.e. all sums of m and n.

Any numerical semigroup S has a finite set $\{n_1, n_2, \ldots, n_k\} \subset S$ of generators, meaning elements such that every element of S is a sum of the n_i. We write $S =< n_1, n_2, \ldots, n_k >$. Among all possible finite generating sets of S the one with the smallest cardinality is unique. The cardinality e of this minimal generating set $\{n_1, n_2, \ldots, n_e\}$ is called the embedding dimension of S. Underlying this 'embedding' terminology is a construction. Form the monomial curve $x_i = t^{n_i}, i = 1, 2, \ldots, e$ in e-space using the minimal generating set as exponents. The semigroup attached to the monomial curve is S. But many curves besides the monomial curve will have the same semigroup S attached to them. Some might even lie in a space of dimension $d < e$.

Definition 0.1. *The minimal embedding dimension of a numerical semigroup S is the minimal dimension d such that the semigroup is that of some analytic curve germ $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$*

Example 0.1. *The semigroup $S =< 4, 6, 13 >$ consists of all integers which are sums of $4, 6$ and 13, these being its minimal generating set. Its embedding dimension is 3. Its canonical curve is $x = t^4, y = t^6, z = t^{13}$. The plane curve $x = t^4, y = t^6 + t^7$ has this same semigroup. The generator 13 in the semigroup of this curve arises*

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as the order of polynomial $p(x,y) = y^2 - x^3$*, when pulled back to this plane curve :* $p(x,y) = 2t^{13} + t^{14}$*. So the minimal embedding dimension of* S *is* 2*.*

Problem. *Characterize those numerical semigroups whose minimal embedding dimension is* d*.*

This problem has been solved for the case $d = 2$ of plane curves. See Teissier [10] proposition 3.2.1, on page 132. We recall and clarify this proposition and sketch a proof based on the Puiseux characteristic in the Appendix at the end of the paper. The case $d = 3$ of space curves appears to be open. See Castellanos [2], particularly problem 2.4 and the examples in section 2 for perspective.

Teissier states his proposition 3.2.1 in a convoluted way so as to hold for all $d \le e$. In the appendix I make what sense I can of his proposition as it applies to $d > 2$. What he does is to provide a list of *sufficient* conditions amongst the e minimal generators of a semigroup in order for that semigroup to have minimal embedding dimension $d \le e$. Teissier's conditions imply that curves to which the semigroup is attached are complete intersections. All plane curve branches are complete intersections which allows his conditions to be necessary and sufficient when $d = 2$. Many space curve singularities fail to be complete intersections and consequently Teissier's conditions exclude many semigroups with minimial embedding dimension 3.

The "multiplicity" m of a numerical semigroup is its smallest nonzero element. We have

$$me(S) \le e(S) \le m(S).$$

where $me(S), e(S)$, and $m(S)$ are the minimal emdedding dimension, embedding dimension and multiplicity of the semigroup S. (Refer to the first chapter of the book [3] for more standard terminology around numerical semigroups.) If we refine our problem according to multiplicity it becomes more tractable. For example, semigroups of multiplicity 2 or 3 have minimal embedding dimension less than or equal to 3 by the above inequality and so can be excluded as 'trivial' in the search for solutions to the problem for $d = 3$. In this paper we will solve:

Problem. *Describe all numerical semigroups whose minimal embedding dimension is* 3 *and whose multiplicity* 4*.*

We solve this problem completely below. See theorem 1.1 and figure 1. To give the readers a taste of the solution, represent an $m = 4, e = 4$ numerical semigroup in the form $S =< 4, n_1, n_2, n_3 >$ with $4 < n_1 < n_2 < n_3$. Since $e = 4$ the four integers $4, n_1, n_2, n_3$ must represent all 4 congruence classes mod 4. They also must satisfy the strict Kunz inequalities listed in equation (1) below. We will show that if n_3 is congruent to 2 mod 4 then S is not on our list. So for such S's we have $me(S) = 4$. This fact corresponds to the white top triangle of the Kuntz kite of figure 1. And will see that among those semigroups of the form $S =< 4, 6, n_2, n_3 >$ if $me(S) = 3$ then we must have $n_3 = n_2 + 2$.

The Kunz cone associated to $m = 4$ gives us a good language within which to solve the problem. We describe this cone after recalling how to attach a semigroup to a curve.

0.1. **Motivation: Legendrian semigroups.** A Legendrian curve is a space curve tangent to a contact distribution in 3-space. Legendrian curves are born from plane curve singularities by a process known variously as "prolonging", "Nash blow-up" or "forming the conormal variety" and which shares many properties with the

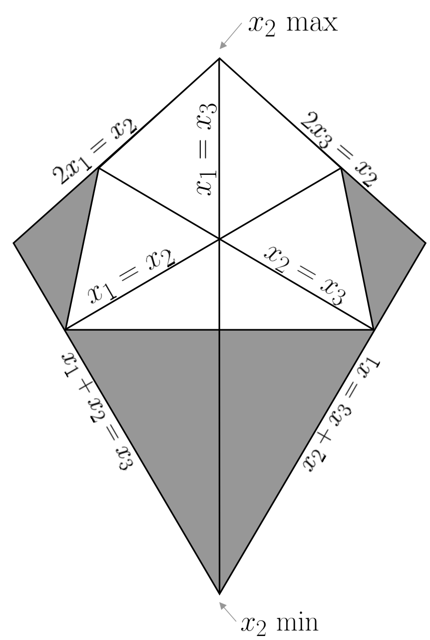

FIGURE 1. The Kunz cone for multiplicity 4 is the cone in $\mathbb{R}^3$ over the interior of this kite when placed on the plane $x_1 + x_2 + x_3 = 1$. See equations (1). The set of lattice points in and on the boundary of the cone for which $x_i \equiv i \pmod 4$ are in bijection with multiplicity 4 numerical semigroups. The set of such points lying in the interior of the cone sweep out the semigroups with embedding dimension 4. Those having minimal embedding dimension 3 correspond to the shaded region of the kite and the interiors of its edges. Those having minimal embedding 2 correspond to the left and right vertices of the kite. **Thanks to Emily O'Sullivan for the figure.**

classical blow-up of a singularity. The semigroup of a Legendrian curve has a close but poorly understood relationship with the semigroup of the planar curve which gave birth to it. See for example [5], [6], or [12]. The following questions motivate this work.

What is the set of semigroups which arise from Legendrian curves?

More importantly, is this really an interesting question? For example, does the semigroup of a singular Legendrian curve encode interesting unappreciated "contact topological" properties of the curve? If the answer to the last question ends up being "yes" then the answer to the second question would also be'yes'.

As a beginning step towards answering these questions we have classified the Legendrian semigroups of multiplicity 2, 3 and 4. We hope to present our findings in a companion paper. The first two cases (m = 2 and 3) are easy and well-known. Work on the last case ($m = 4$) naturally led into the problem solved here, since a Legendrian semigroup must have minimal embedding dimension 3 or less.

1. The semigroup of a curve

By a *curve* we mean a non-constant analytic map $c : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$. The notation $c : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ means that $c(0) = 0$. It also indicates that our true interest is the germ of the curve, meaning its restriction to any small neighborhood of 0. In order to attach a numerical semigroup to a curve we assume that the curve c is well-parameterized:

Definition 1.1. *A curve is well-parameterized if it is one-to-one as a map when restricted to a sufficiently small neighborhood of zero.*

If c is not well-parameterized then it can be replaced by another curve which is 'the same' as c and which is well-parameterized. See the final subsection of this section.

Recall the order of a power series converging in a neighborhood of $t = 0$. The order of $f(t) = \Sigma_{i \geq 0} a_i t^i$, written "$ord(f)$", is the smallest i such that $a_i \neq 0$. For example $ord(t^3 + 7t^5) = 3$. If $\mathbb{C}\{t\}$ denotes the space of power series which converge in some neighborhood of 0 then

$$ord : \mathbb{C}\{t\} \to \mathbb{N}.$$

The order is a valuation, i.e. a semigroup homomorphism: $ord(fg) = ord(f) + ord(g)$. Let $\mathcal{O}_0$ denote the ring of germs of analytic functions $f : (\mathbb{C}^d, 0) \to \mathbb{C}$ defined in a neighborhood of 0. Let $\mathbb{C}\{x_1, \ldots, x_d\}$ denote the ring of convergent power series in the coordinates x_i of $\mathbb{C}^d$, converging in some neighborhood (depending on the series) of $0 \in \mathbb{C}^d$. Pull-back defines a ring homomorphism

$$c^* : \mathbb{C}\{x_1, \ldots, x_d\} \to \mathbb{C}\{t\}$$

by sending $p \in \mathfrak{m}$ to $c^* p := p \circ c$. Thus $c^* \mathbb{C}\{x_1, \ldots, x_d\} \subset \mathbb{C}\{t\}$ is a subring. The semigroup of c is the collection of all integers of the form $n = ord(c^* p)$ which arise in this way. In symbols

Definition 1.2 (Semigroup of a curve.)**.** *The semigroup of the analytic well-parameterized curve c is the set of integers $S = ord(c^* \mathbb{C}\{x_1, \ldots, x_d\})$.*

To see that S is closed under addition use $ord(fg) = ord(f) + ord(g)$ and $c^*(pq) = (c^* p)(c^* q)$.

It may be helpful to write the pullback operation and order map out in coordinates. Write out both our curve and our function in coordinates:

$$c(t) = (x_0(t), x_1(t), \ldots, x_d(t))$$

and $p = p(x_1, \ldots, x_d)$ where the x_i are coordinates for $\mathbb{C}^d$. Then

$$f(t) = c^* p(t) = p(x_0(t), x_1(t), \ldots, x_d(t))$$

is an analytic function of t.

Remark 1. *We could replace $\mathcal{O}_0 = \mathbb{C}\{x_1, \ldots, x_d\}$ by the ring of polynomials in the x_i or the algebra of formal power series in the x_i or even germs of smooth functions at 0 and we would obtain the same semigroup S as $ord(c^* R)$. This is because c is well-parameterized and as a consequence the semigroup S is numerical and as such has a "conductor" , a number k such that all integers greater than or equal to k lie in S. Considerations of degrees then show that by restricting oneself to polynomials in the x_i of a fixed degree (roughly degree $(k/m) + 1$ where m is the multiplicity of the curve c) will suffice to realize all elements of S.*

To see that our semigroup S is "numerical", i.e. that the complement of S is finite, we must use that S is well-parameterized. Suppose, by way of contradiction, that the complement of S were infinite. Then the g.c.d of S would not be 1, but instead some integer $k > 1$. From this it follows that, roughly speaking, all the exponents arising in all power series $c^*p, p \in \mathfrak{m}$ would be divisible by k and from this that we could express $c(t) = \gamma(t^k)$ for some analytic curve γ. This implies that c is not well-parameterized. (We say 'roughly speaking' because we might need to reparameterize first: $c(t) = \gamma(\tau(t)^k)$ where $\tau'(0) \neq 0$.)

The reader may wish to refer to section 5 of Arnol'd [1] for a beautiful perspective on the relation between a curve and its semigroup.

1.0.1. *Analytic equivalence.* Call two curves $c_1, c_2 : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ "analytically equivalent" if there are germs of analytic diffeomorphisms $\Psi : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ and $\psi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $c_2 = \Psi \circ c_1 \circ \phi$. In the singularity literature this equivalence relation is also called "RL equivalence". We write $c_1 \sim c_2$ to mean that the two curves are analytically equivalent. One easily verifies that any analytically equivalent curves share the same numerical semigroup. **For us, this is the main point of studying numerical semigroups.**

By way of orienting ourselves to the subject, it may help to state a few elementary results. Below, let S denote the semigroup of a curve c.

Fact. $m(S) = 1 \iff S = \mathbb{N} \iff c'(0) \neq 0 \iff c$ is equivalent to the line $t \mapsto (t, 0, \dots, 0)$.

Fact: $m(S) = 2 \iff S =< 2, 2k+1 >$ for some integer $k \iff c$ is equivalent to the curve christened as the "A_k singularity": (t^2, t^{2k+1}), or, if $d > 2$, $(t^2, t^{2k+1}, 0, \dots, 0)$.

Fact: If $m(S) = m > 1$ and n_1 is the smallest generator of S besides m then c is equivalent to a curve of the form $x_1(t) = t^m, x_2(t) = t^{n_1} + \dots$ where the " $\dots$" means term of order greater than n_1 and where $ord(x_i(t)) > n_1$ for $i > 2$, if $d > 2$.

1.1. **On well-parameterizing and real curves.** Consider curves $c, \gamma : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ as being "the same curve" if there is a non-constant analytic map $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that either $c = \gamma \circ \phi$ or $\gamma = c \circ \phi$ holds. Curves that are "the same" in this sense share the same image: $c(D) = \gamma(D')$ for $D, D' \subset \mathbb{C}$ sufficiently small appropriate neighborhoods of zero.

Given a curve c which is not well-parameterized we can always find another curve γ that is "the same" as c and which is well-parameterized. To give a quintessential example, take $d = 2$ and $c(t) = (x(t), y(t)) = (t^4, t^6)$. Then c is badly parameterized. The curve $\gamma(t) = (t^2, t^3)$ is "the same curve" as c since $c = \gamma \circ \phi$ with $\phi(t) = t^2$ and γ is well-parameterized.

Here is an alternative definition of "well-parameterized". A badly parameterized curve $c(t)$ can be factored as $c = \gamma \circ \phi$ where $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a reparameterization having $\phi'(0) = 0$. A "well-parameterized curve' is a curve that is not badly parameterized.

Our original definition of "well-parameterized" in terms of being one-to-one does not work for curves over $\mathbb{R}$ because of maps like $\phi(t) = t^3$ which are one-to-one over $\mathbb{R}$. If $\gamma(t) = (t^2, t^3)$ then $\gamma \circ \phi$ is still one-to-one when viewed as a real curve but we do not consider it to be well-parameterized.

1.2. **Kunz Cone.** The Kunz cone for classifying multiplicity m semigroups is a convex polyhedral cone in $\mathbb{R}^{m-1}$ which provides a direct and concrete way to parameterize all semigroups of multiplicity m. The semigroups arise as a subset of the lattice points in this cone. See [9] or [7] and references therein.

To describe the Kunz cone for multiplicity $m = 4$ we use 3 coordinates $(x_1, x_2, x_3) \in \mathbb{R}^3$. To obtain a semigroup associated to a point in this cone we insist that the x_i are positive integers greater than 4 and that

$$x_i \equiv i(mod 4).$$

Definition 1.3. *An Apery point in $\mathbb{R}^3$ is a point $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_i \in \mathbb{N}$, $x_i > 4$ and $x_i \equiv i$. Here, this and all subsequent congruences "$\equiv$" are congruences mod 4.*

The semigroup associated to an Apery point is

$$S =< 4, x_1, x_2, x_3 >$$

It is essential here that we do not impose the ordering $x_1 < x_2 < x_3$. It is also essential that the list $4, x_1, x_2, x_3$ need not be the list of minimal generators.

The Kunz cone is defined by the four inequalities

$$\text{Kunz cone} = \begin{cases} x_1 + x_2 & \geq x_3 \\ x_3 + x_2 & \geq x_1 \\ 2x_1 & \geq x_2 \\ 2x_3 & \geq x_2 \end{cases} \tag{1}$$

To understand how these inequalities arise, look at the first one. If $x_3 \geq x_1 + x_2$ then, since $1 + 2 \equiv 3(mod 4)$ we have that $x_3 = x_1 + x_2 + 4k$ for some integer $k \geq 0$. It follows that $x_3 \in< 4, x_1, x_2 >$ or that $< 4, x_1, x_2 >=< 4, x_1, x_2, x_3 >$. So all Apery points with $x_3 \geq x_1 + x_2$ yield the same semigroup, and, notably, a semigroup whose embedding dimension is 3 (or perhaps even 2) , not 4. Since we want a bijection between points and semigroups, we exclude all Apery points having $x_3 > x_1 + x_2$. We keep the single point $x_3 = x_1 + x_2$ so as to represent the semigroup $< 4, x_1, x_2 >$.

Proposition 1.1. *The Apery points in the Kunz cone for multiplicity 4 are in bijection with numerical semigroups of multiplicity 4. The points lying in the interior of the cone (i.e. all inequalities are strict) correspond to those semigroups whose embedding dimension is 4. The Apery points lying in the interior of the codimension 1 faces of the cone correspond to semigroups whose embedding dimension is 3 and those lying on the rays of the cone (codimension 2 faces) correspond to semigroups of embedding dimension 2.*

Now, return to our problem of characterizing the multiplicity 4 semigroups whose minimal embedding dimension is 3. All the points on the faces and rays of the Kunz cone have embedding dimension 3 or less, and so minimal embedding dimension 2 or 3. In this way we have reduced our problem to the interior of the Kunz cone, which is to say, to those semigroups whose embedding dimension is 4.

Theorem 1.1. *Let $S =< 4, x_1, x_2, x_3 >$ be a multiplicity 4 semigroup with embedding dimension 4, so that all the Kunz inequalities are strict. Then, S has minimal embedding dimension 3 if and only if*

- *a)* $max(x_1, x_2, x_3) \neq x_2$

- *b)* $max(x_1, x_2, x_3) > 2min(x_1, x_2, x_3)$

We have indicated these semigroups in figure XX. We are endebted here to the thesis of O'Sullivan [8] for pointers in how to depict the Kunz cone via a slice. Here we use the same slice $x_1 + x_2 + x_3 = 1$ that she used. See figure 1.

It is easier to understand the theorem and give its proof if, instead of stating it as above, we run through all 6 possible orderings of the x_i:
(1) $x_1 < x_2 < x_3$
(2) $x_2 < x_1 < x_3$
(3) $x_1 < x_3 < x_2$
(4) $x_3 < x_1 < x_2$
(5) $x_2 < x_3 < x_1$
(6) $x_3 < x_2 < x_1$
Then item (a) of the theorem says that the orderings (3) and (4) do not occur for any semigroup whose minimal embedding dimension is 3 and embedding dimension is 4. So, for example, the semigroup $< 4, 7, 9, 10 >$ can only arise as the semigroup of a curve in 4 dimensional space.

Item (b) of the theorem asserts that if the x_i satisfy one of the orderings (1), (2), or (5), (6) then the inequality of (b) is the necessary and sufficient conditions for $< 4, x_1, x_2, x_3 >$ to have minimal embedding dimension 3 and embedding dimension 4. For example, if our generators are in the order of condition (1) and satisfy $x_3 > 2x_1$ (and neccessarily $x_3 < x_1 + x_2$) then the semigroup $< 4, x_1, x_2, x_3 >$ has minimal embedding dimension 3 and embedding dimension 4. An example of such a semigroup is $< 4, 5, 10, 11 >$.

2. PROOF

Let $n_1 < n_2 < n_3$ be the list x_1, x_2, x_3 *permuted so as to be in numerical order.* (For example, in case (5) $n_1 = x_2, n_2 = x_3, n_3 = x_1$.) Then, after a local diffeomorphism and reparameterization we can put our curve into the form

$$x = t^4, y = t^{n_1} + at^{n_1+s_1} + \dots, z = t^{n_2} + \dots \tag{2}$$

Here $s_1 > 0$ and the $\dots$ in both expansions means terms of order higher than the smallest written down, so, in the case of z, of order higher than n_2. When written in this form we have $ord(x(t)) = 4, ord(y(t)) = n_1, ord(z(t)) = n_2$. Now $ord(x^k(ay+bz)) \in < 4, n_1, n_2 >$ so that, in order to realize the largest generator n_3 of S we require polynomials which contain terms quadratic in y, z. Since $ord(ay^2 + byz + cz^2) \geq 2n_1$ we require

$$n_3 \geq 2n_1 \tag{3}$$

if S is to be the semigroup of a space curve.

We use inequality (3) to eliminate orderings (3) and (4). In both these cases $n_3 = x_2$. If we are case (3) then $n_1 = x_1$, but the strict Kunz inequality requires that $x_2 < 2x_1$ or $n_3 < 2n_1$ violating inequality (3). Similarly in case (4) we have $n_1 = x_3$ and the strict Kunz inequality asserts again that $n_3 < 2n_1$, violating inequality (3).

To finish off the proof we exhibit space curves for the other four orderings (1), (2) and (5) , (6) which realizes the given semigroup. In these cases either n_1 or n_2 is x_2 so that

$$n_1 + n_2 \equiv n_3$$

(Recall that all congruences "≡" are congruences mod 4.) The strict Kunz inequalities then read

$$n_3 < n_1 + n_2 = ord(y(t)z(t)).$$

We are left with the fact that if we can find a polynomial whose order is n_3 then we must have that

$$max\{n_2, 2n_1\} < n_3 < n_1 + n_2.$$

We are to show that any n_3 in this range and congruent to 1 or 3 as appropriate, can be realized as the order of a polynomial pulled back to the curve. We can set

$$n_3 = 2n_1 + s \text{ for } 1 \leq s < (n_2 - n_1)$$

and further fix the normal form (2) of the curve so that

$$x = t^4, y = t^{n_1} + t^{n_1+s}, z = t^{n_2}.$$

Then

$$y^2 = t^{2n_1} + 2t^{2n_1+s} + t^{2n_1+2s}$$

If $n_1 = x_2$ then $2n_1 \equiv 0$ so that $2n_1 = 4k$. Then $y^2 - x^k = 2t^{2n_1+s} + \dots$ so that $ord(y^2 - x^k) = n_3$. This takes care of cases (2) and (5). In the remaining two cases, (1) and (6), $2n_1 \equiv 2$ and $n_2 \equiv 2$ so that so that $2n_1 = n_2 + 4k$ which is the order of $x^k z$. Then $y^2 - x^k z = 2t^{2n_1+s} + \dots$ and $ord(y^2 - x^k z) = n_3$. This takes care of all cases and completes the proof.

QED

3. Appendix. Teissier deciphered.

We describe necessary and sufficient conditions for a semigroup S to have minimal embedding dimension $me(S) = 2$. It is a rewording and clarification of proposition 3.2.1 on p. 132-3 of Teissier [10]. We sketch a proof based on the Puiseux characteristic of a plane curve and a recursion relation between this characteristic and the semigroup of the curve. To do this, we will recall the Puiseux characteristic.

If $me(S) > 2$ then one implication in Teissier's proposition still holds. This direction gives a sufficient condition on a semigroup S to have $me(S) = d$ for $d \leq e$. We describe what we have been able to understand of this part of the proposition.

3.1. **Divisor and factor vector of an increasing list.** Consider an increasing list of positive integers of length $e = g + 1$:

$$\vec{b} = [b_0; b_1, ..., b_g]; b_i < b_{i+1}$$

Associated to this vector we have another integer vector which we will call its "divisor vector":

$$\vec{e} = [e_0, e_1, e_2, \dots, e_g]$$

defined by the iteration scheme

$$e_i = \text{g.c.d.}(b_0, ..., b_i) \qquad , e_0 = b_0 \tag{4}$$

Note that $e_i = \text{g.c.d.}(e_{i+1}, b_i)$ and that all the e_i are factors of b_0. (Some of them may be 1.)

The list e_i is non-increasing and satisfies $e_i | e_{i-1}$. It follows that for each $i > 0$ there is an integer $n_i \geq 1$ given by

$$n_i = e_{i-1}/e_i.$$

The list of n's supply another integer vector

$$\vec{n} = [n_1, \dots, n_g],$$

now of length g, associated to $\vec{b}$. We call $\vec{n}$ the "factor vector". Note that

$$n_g = e_{g-1},$$

$$e_i = n_{i+1} n_{i+2} \dots n_g.$$

In particular

$$b_0 = n_1 n_2 \dots n_g.$$

3.2. **Planar semigroups lie on the boundary of the Kunz cone.** Set

$$m = b_0$$

in order to denote the multiplicity of the curve corresponding to the semigroup S. Since each $n_i \geq 2$ we have that $m \geq 2^g$ or $log_2 m \geq g$. Now, as long as $m \geq 6$ we have that $m > log_2 m + 2$ so $m - 1 > g + 1$. Whem $m = 3, 5$ we have $g = 1$ or $e = 2$ and so the planar semigroups lie on the 1-dimensional faces of the Kunz cone. In the case $m = 4$ we just dealt with we can have $g = 2$ and so $e = 3 = m - 1$ and the planar semigroups lie on the faces of the Kunz cone. It follows that in all situations in which $m > 2$ the planar semigroups lie on the boundary of the Kunz cone.

3.3. **Conditions for a semigroup to be that of a plane curve.**

Proposition 3.1. *Let $S =< b_0, b_1, \dots, b_g >$ be a numerical semigroup given by its minimal generators b_i listed in order. Write $\vec{e} = [e_0, e_1, \dots, e_g]$ for the divisor vector of $[b_0, b_1, \dots, b_g]$ and $\vec{n} = [n_1, n_2, \dots, n_g]$ for its factor vector ($n_i = e_{i-1}/e_i$).*

S has minimal embedding dimension 2 if and only if the following three conditions hold for its associated three integer vectors
(1): the divisor vector $\vec{e}$ is strictly decreasing and ends with 1.
(2): for $i = 1, \dots, g$ we have $n_i b_i \in < b_0, b_1, \dots, b_{i-1} >$
(3): for $i = 1, \dots, g - 1$ we have $n_i b_i < b_{i+1}$

We sketch a proof of the proposition at the very end of this appendix. We do so by translating the proposition, and the planar semigroups, into the languague of Puiseux characteristics. We recall that language in time for the proof at the end.

Example 3.1. *If $S =< b_0, b_1, \dots >$ and b_0, b_1 are relatively prime then $e_1 = 1$. If condition (1) holds then there cannot be an e_2, so it must be that $g = 1$. Thus $S =< b_0, b_1 >$ if S comes from a planar semigroup whose first two generators are relatively prime. These are represented by the curves discussed in the first paragraph of this paper.*

Corollary 3.1. *If $m = b_0$ is the multiplicity of a semigroup $S =< b_0, b_1, \dots, b_g >$ given by its minimal generating set b_i and if S is planar, i.e. has minimal embedding dimension 2, then the embedding dimension $g+1$ of S is less than or equal to 1 plus the number of prime factors of m counted with multiplicity.*

PROOF OF COROLLARY. The longest we can make the divisor vector $[e_0, e_1, \dots, e_g]$ and keep it strictly decreasing beginning with $m = e_0$ is 1 plus the number of prime factors of m. We do this by omitting one factor at a time from $m$so that $e_i = e_{i-1}/p_i$ where the p_i exhaust the prime factors, taken with multiplicity, of m. QED

Remark 3.1. *Condition (1) is extraneous in that it is implied by condition (2) and the assumption that the* b_i *form a minimal set of generators. For, if (2) holds, then the non-increasing list* e_i *has to be strictly decreasing. To see this we argue by contraposition. If* e_i *is not strictly decreasing then there is an* i *such that* $e_{i-1} = e_i$ *in which case* $n_i = 1$*. But then (2) says that* $b_i \in < b_1, \ldots, b_{i-1} >$ *which implies that the generator* b_i *can be omitted from our list of generators and we still have a generating set for* S*.*

3.4. **An aside: self-duality and planarity.** The conductor c of a numerical semigroup S is the smallest element of S such that all integers greater than c lie in S. The set of gaps of a numerical semigroup is the finite set $\mathbb{N} \setminus S$. A numerical semigroup is called "self-dual" if the cardinality of the set of its gaps is half of its conductor c. It is a well-known fact that planar semigroups are self-dual. Here is a notable example, described as "3.2.3 Remark" on p. 133 of Teissier, of a self-dual semigroup which is not that of a planar curve.

Example 3.2 (Teissier)**.** *Set* $S =< 9, 21, 22 >$*. A session with GAP yields that the conductor of* S *is 78 and the number of gaps is 39 so that* S *is self-dual. The divisor list of* S *is* $\vec{e} = [9, 3, 1]$ *and the factor list is* $\vec{n} = [3, 3]$*. We have* $(b_0, b_1, b_2) = (9, 21, 22)$*. Since* $n_1 b_1 = 63 > b_2 = 22$ *this semigroup fails condition of (3) of the proposition so cannot be the semigroup of a planar curve.*

3.5. **Teissier's proposition for non-planar curves.** Teissier wrote his proposition so that one direction of its implications holds for any minimal embedding dimension d in place of $d = 2$. I describe the statement as I understand it. Take it with a bit of scepticism, since I do not understand his proof.

Suppose that a semigroup S is given by its minimal generators b_i as before. Drop condition (1) of the proposition. Suppose that condition (2) holds. (Teissier labels this condition (1)). And suppose that condition (3) holds, but not for all $g-1$ listed indices, but instead for $\ell = g + 1 - d$ of these indices. Then there exists an analytic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ having S as its semigroup.

He says nothing about the converse direction of the implication. It is generally false as examples with $e = me$ and b_0, b_1 relatively prime show.

Within the proof on page 133 Teissier says that condition (2) implies that the canonical curve $x_i = t^{b_i}$ is a complete intersection. Is it a necessary and sufficient condition to be a complete intersection? I do not know.

3.6. **Strategy of proof of the Proposition.** We show how to go from a plane curve to its Puiseux characteristic. We describe a transformation taking us from the Puiseux characteristic to the semigroup of the curve. It is straightforward to check that the resulting semigroup satisfies the proposition. Conversely, starting from the semigroup we can reverse the Puiseux-to-semigroup transformation to recover the Puiseux characteristic from the semigroup, provided the semigroup satisfies the conditions of the proposition. And given the Puiseux characteristic it is immediate to write down a plane curve with this Puiseux characteristic. In order to implement this strategy we begin with reviewing the Puiseux characteristic of a plane curve.

3.7. **The Puiseux characteristic of a plane curve.** The Puiseux characteristic of an analytic plane curve germ $c : (\mathbb{C}, 0) \to \mathbb{C}^2$ is an increasing vector $\vec{\lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_g]$ of positive integers encoding the key exponents which arise in the power series expanison of $c(t) = (x(t), y(t))$. A reparameterization and change of coordinates puts the curve into the form

$$x = t^m, \quad y = a_n t^n + a_{n+1} t^{n+1} + a_{n+2} t^{n+2} + \cdots, \tag{5}$$

where $1 < m < n$ and $a_n \neq 0$. Then

$$\lambda_0 = m.$$

We can assume that n is not a multiple of m in the expansion of the curve. For if $n = km$ then the diffeomorphism $(x, y) \mapsto (x, y - a_n x^k)$ kills the term $a_n t^n$. We can kill all powers of t^m arising in the power series of $y(t)$ by this same trick. With this in mind, write $supp(y) = A$ for the set of all exponents occuring in the power series of y. Thus

$$A = \{i \in \mathbb{N} : a_i \neq 0\} \text{ where } y(t) = \Sigma a_i t^i$$

We have just seen that we can get rid of all multiple of m arising in A by applying a diffeomorphism $(x, y) \mapsto (x, y - f(x))$. We have "seived out" m from A.

Definition 3.1. *If $A \subset \mathbb{N}$ and $m \in \mathbb{N}$ then the m-seive of A, denoted $[m; A]$ is the set A with all multiples of m deleted. In set notation, $[m; A] = A \setminus m\mathbb{N}$.*

Since $c(t)$ is well-parameterized, the m-seive of A is not empty.

Let λ_1 be the smallest element of $[m; A]$. Thus, $\lambda_1 = n$ above, assuming that m does not divide n. If $gcd(m, n_1) = 1$ the process ends and the Puiseux expansion of $c(t)$ is $[m, \lambda_1]$. Otherwise, write $e_1 = gcd(m, \lambda_1)$.

We continue by setting $A_2 = [e_1; A]$ and choosing λ_2 to be the smallest element of A_2. We write $e_2 = gcd(e_1, \lambda_2)$. If $e_2 = 1$ we stop and Puiseux characteristic is $[m, \lambda_1, \lambda_2]$. Otherwise we set $A_3 = [e_2; A_2]$ and take λ_3 to be the smallest element of A_3. Iterate the process: $\lambda_i = min(A_i)$, $e_i = gcd(e_{i-1}, \lambda_i)$, and $A_{i+1} = [e_i; A_i]$. We stop with λ_g when $e_g = 1$. We have constructed an increasing list $m = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_g$ of integers which have no common divisor. Our Puiseux expansion is this list.

In the process, we have also constructed the divisor vector of $\vec{n}$, namely the e_i. It is a strictly decreasing vector $e_i < e_{i-1}$.

Example 3.3. *The Puiseux characteristic of the plane curve germ*

$$(t^8,\ t^{16} + t^{20} + a_{22} t^{22} + a_{26} t^{26} + a_{27} t^{27})$$

is $[8; 20, 22, 27]$ provided that $a_{22}, a_{27} \neq 0$. The divisor vector of this Puiseux characteristic is $\vec{e} = [8, 4, 2, 1]$. Its factor vector is $[2, 2, 2]$.

3.8. **Characterizing Puiseux characteristics.** Here are the necessary and sufficient conditions for a list of integers, labelled now $[\lambda_0, \lambda_1, \ldots, \lambda_g]$ to be the Puiseux characteristic of some curve.

1. $\lambda_0 > 1$ and the list is strictly increasing: $\lambda_i < \lambda_{i+1}$.

2. Its associated divisor vector $\vec{e} = [\lambda_0, e_1, \ldots, e_g]$ is strictly decreasing and ends with 1.

Remark 2. *A Puiseux characteristic of length 2 consists of a pair of relatively prime integers $[\lambda_0, \lambda_1]$ with $\lambda_0 < \lambda_1$. By fiat (or a logical contortion if you prefer) the only Puiseux characteristic of length 1 is the vector $\lambda = [1]$.*

3.9. **Puiseux to Semigroup.** The following iteration scheme recovers the semigroup generators $< b_0, b_1, \ldots, b_g >$ of the semigroup of a plane curve from its Puiseux characteristic $[\lambda_0; \lambda_1, \ldots, \lambda_g]$. I am endebted to Teissier, 2.2.1, formula (*) p. 122 for this scheme. C.T.C. Wall 's book , Prop. 4.3.8 on p. 86 and five or so pages on either side also covers this recursion formula. *We will see how this process works for the curve of example 3.3 at the end of this subsection.*

$$b_0 = \lambda_0$$

$$b_1 = \lambda_1$$

$$b_2 = \lambda_2 - \lambda_1 + n_1 \lambda_1$$

And inductively

$$b_i = \lambda_i - \lambda_{i-1} + n_{i-1} b_{i-1}$$

Using $n_{i-1} > 1$ one easily verifies by induction that

$$b_i > \lambda_i, i > 1.$$

and that $b_i \in < \lambda_1, \lambda_2, \ldots, \lambda_i >$.

I will not rederive the recursion formula but simply content myself with the understanding the generator b_2. So suppose that $e_1 = gcd(m, \lambda_1) > 1$ where $\lambda_0 = m$. Then we can put $c(t)$ into the form

$$x = t^m, y = t^{\lambda_1} + a t^{\lambda_2} + \ldots; a \neq 0.$$

We have that $n_1 e_1 = m$ and $\lambda_1 = \beta e_1$ for some integer β. Then

$$y^{n_1} = t^{n_1 \lambda_1} + n_1 a t^{\lambda_2} t^{(n_1 - 1)\lambda_1} + O(t^j), j > \lambda_2 + (n_1 - 1)\lambda_1$$

But $n_1 \lambda_1 = \beta n_1 e_1 = \beta m$ so that

$$x^\beta = t^{n_1 \lambda_1}$$

which shows that $ord(x^\beta - y^{n_1}) = \lambda_2 + (n_1 - 1)\lambda_1$. Hence $\lambda_2 + (n_1 - 1)\lambda_1 \in S$, where S is the semigroup of this curve. It is not difficult to show that this integer is the smallest element of S not lying in $< m, \lambda_1 > \subset S$ and hence this integer is the next generator b_2 of S after $b_1 = \lambda_1$.

Remark 3. *The b_i obtained from this formula have the same divisor vector $\vec{e} = [e_0, e_1, \ldots, e_g]$ and factor vector $[n_1, n_2, \ldots, n_g]$ as that of the Puiseux vector. This follows directly from the recursion relation. Using the same recursion relation, we can invert an associated matrix and solve for the λ_i given the b_i.*

Example 3.4. *Return to our earlier curve $(t^8,\ t^{16} + t^{20} + a_{22} t^{22} + a_{26} t^{26} + a_{27} t^{27})$ whose Puiseux characteristic is $[8; 20, 22, 27]$ provided that $a_{22}, a_{27} \neq 0$. To find its semigroup compute that*

$$[e_0, e_1, e_2, e_3] = [8, 4, 2, 1]$$

while

$$[n_1, n_2, n_3] = [2, 2, 2]$$

We compute $b_0 = \lambda_0 = 8, b_1 = \lambda_1 = 20$

$$b_2 = 22 - 20 + 2 * 20 = 42$$

$$b_3 = (27 - 22) + 2 * b_2 = 5 + 84 = 89$$

so that $S =< 8, 20, 42, 89 >$

As a reality check we verify that we can get the integers $42, 89$ *as orders of polynomials pulled back to the curve. Take the case* $a_{22} = a_{27} = 1, a_{26} = 0$ *for simplicity so that* $y(t) = t^{20} + t^{22} + t^{27}$. *Then* $y^2 - x^5 = (t^{40} + 2t^{42} + 2t^{47} + \ldots.) - t^{40} = 2t^{42} + 2t^{47}$ *which has order* 42. *To get* 89 *note that* $(y^2 - x^5)^2 = 4t^{84} + 8t^{89} + \ldots$ *Now* 84 *is a multiple of* 4. *Since* $[8, 20] = 4[2, 5]$ *and since the conductor of* $< 2, 5 >$ *is* 4*, from* 16 *onwards all integers which are multiples of* 4 *occur in any semigroup having* 8 *and* 20 *as generators. It follows that we can kill the* t^{84} *term of* $(y^2 - x^5)^2$ *with some polynomial in* x*, and* y. *Indeed* $84 = 64 + 20$ *is the valuation of* $x^8 y$. *We see that*

$$(y^2 - x^5)^2 - 4x^8 y = 8t^{89} + \ldots$$

which has a valuation of 89.

3.10. **Sketch of a proof of Teissier's proposition 3.1.** Given a plane curve, form its Puiseux expansion. Run the recursion to obtain the minimum generating list for the semigroup of the curve. Use remark 3 to see that the divisor vector of the list semigroup generators is strictly decreasing. Inductively verify conditions (2) and (3).

Conversely, given a semigroup whose minimal generating list satisfies the given relation run the recursion relation backwards to obtain a Puiseux expansion $[\lambda_0, \lambda_1, \ldots, \lambda_g]$ whose semigroup is the give semigroup. The curve $x = t^{\lambda_0}, y = t^{\lambda_1} + t^{\lambda_2} + \ldots + t^{\lambda_g}$ has S as its semigroup.

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References

[1] V.I. Arnol'd, *Simple singularities of curves*, Proc. Steklov Inst. Math., no. 3 (226), 20-28, (1999).

[2] J. Castellanos, *The semigroup of a space curve singularity*, Pacific J. Math., v. 221, no. 2, 227- 251, (2005).

[3] J.C. Rosales and P.A. García-Sánchez, **Numerical Semigroups**, vol. 20 in the series *Developments in Mathematics*, Springer, (2009).

[4] E. Kunz, Über die Klassifikation numerischer Halbgruppen, Regensburger Mathematische Schriften 11, (1987).

[5] A. Araujo, O. Neto, *Moduli of Legendrian Curves*, Annales de la faculté des sciences de Toulouse Mathématiques 18.4, 797-809, `http://eudml.org/doc/10127` or `https://arxiv.org/abs/0910.5623`, (2009)

[6] R. Montgomery and M. Zhitomirskii, **Points and Curves in the Monster Tower**, Memoirs of the A.M.S. , vol. 205 (2010).

[7] N. Kaplan and C. O'Neill, *Numerical semigroups, polyhedra, and posets I : the group cone* `https://arxiv.org/abs/1912.03741v4`.

[8] E. O'Sullivan, **Understanding the Face Structure of the Kunz Cone**, Master's thesis, Cal State Univ. San Diego (2023).

[9] W. Bruns, P. Garcia-Sanchez, C. O'neill, and D. Wilburne, *Wilf's Conjecture in Fixed Multiplicity*, (2019), `https://arXiv.org/abs/1903.04342`

[10] B. Teissier, *Appendix* to the monograph **The Moduli Problem for Plane Branches**, by O. Zariski, translated from the French, Univ. Lecture Series v. 39, AMS, (2006)

[11] C.T.C. Wall, **Singular Points of Plane Curves**, London Math. Soc. Student Text 63, Cambridge Univ. Press, Cambridge, (2004).
[12] M. Zhitomirski, *Germs of integral curves in contact 3-space, plane and space curves*, Isaac Newton Institute for Mathematical Sciences. Preprint NI00043-SGT, December 2000

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