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Luis M. Bernardo
Physics Division

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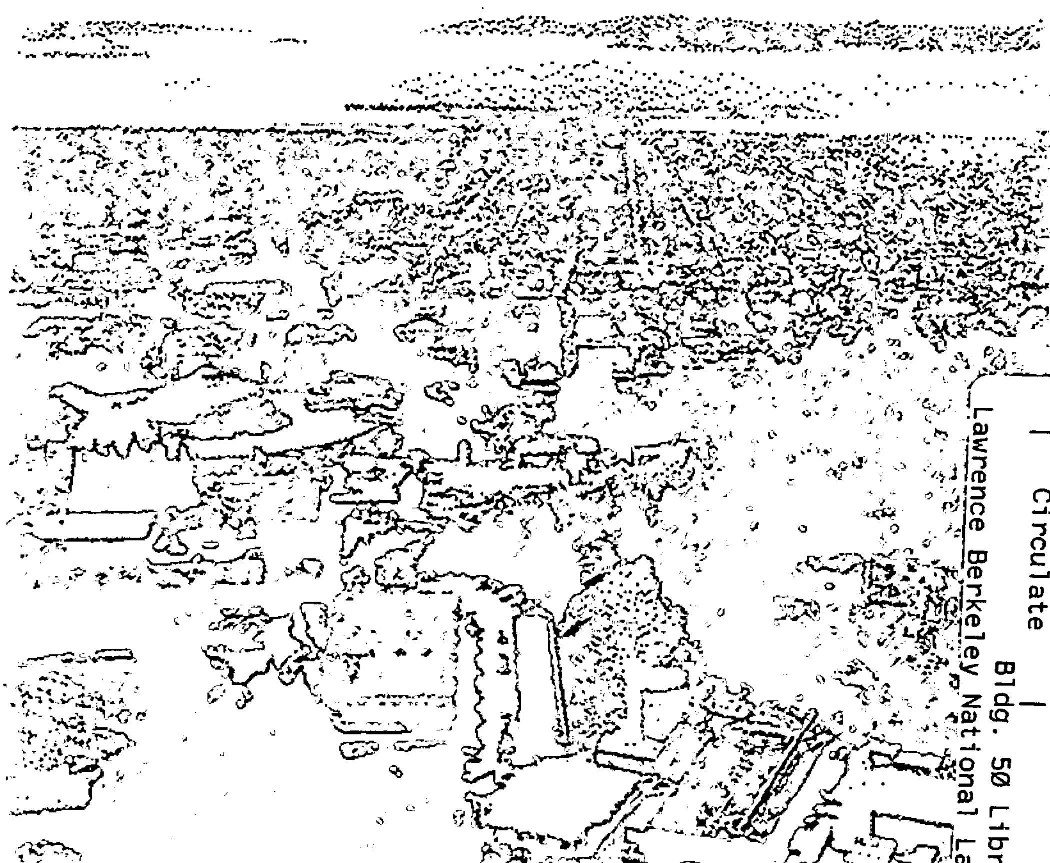
Ph.D. Thesis



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Topics in Conformal Invariance and Generalized Sigma Models

(Ph.D. Dissertation, May 1997)

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Abstract

This thesis consists of two different parts, having in common the fact that in both, conformal invariance plays a central role. In the first part, we derive conditions for conformal invariance, in the large N limit, and for the existence of an infinite number of commuting classical conserved quantities, in the Generalized Thirring Model. Our treatment uses the bosonized version of the model. Two different approaches are used to derive conditions for conformal invariance: the background field method and the Hamiltonian method based on an operator algebra, and the agreement between them is established. We construct two infinite sets of non-local conserved charges, by specifying either periodic or open boundary conditions, and we find the Poisson Bracket algebra satisfied by them. A free field representation of the algebra satisfied by the relevant dynamical variables of the model is also presented, and the structure of the stress tensor in terms of free fields (and free currents) is studied in detail. In the second part, we propose a new approach for deriving the string field equations from a general sigma model on the world sheet. This approach leads to an equation which combines some of the attractive features of both the renormalization group method and the covariant beta function treatment of the massless excitations. It has the advantage of being covariant under a very general set of both local and non-local transformations in the field space. We apply it to the tachyon, massless and first massive level, and show that the resulting field equations reproduce the correct spectrum of a left-right symmetric closed bosonic string.

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Preface

This thesis reports on work I did in collaboration with my advisor Professor Korkut Bardakci (and also with Nir Sochen, a postdoctoral fellow at the Berkeley Laboratory at the time, in the part concerning the integrability of the Generalized Thirring Model), and as such, it is also his work, and his views are present in almost every page.

The thesis consists of two very distinct parts, as reflected in its layout, and it is possible to read them independently. In common, there is the fact that in both parts conformal invariance plays a central role. In the first part we search for a conformal invariant theory, which can serve as the basis for string compactification, while in the second part, we go further and try to formulate string field theory in a background independent way.

I. Conformal Invariance and Integrability of the Generalized Thirring Model

1. Introduction

The search of new conformal field theories and integrable models in two dimensions has been an active field of research for the last fifteen years. This is due to various reasons: (1) from a mathematical viewpoint, the study of these subjects, besides being fascinating in itself, has revealed deep connection with other branches of mathematics; (2) conformal theories play an important role in string compactifications and correspond to ground states of string theory [1]; (3) statistical mechanics systems in two dimensions are conformally invariant on large distance scales at a critical point [2,3]; (4) 2D integrable models share many features with 4D gauge theories, and being exactly solvable models, can give deep insights on the physics of these theories, including non-perturbative phenomena, without relying on any approximation; and (5) matrix models formulations of string theory led to the appearance of hierarchies of classical integrable equations in the description of non-perturbative string amplitudes [4,5].

Here, we are going to study a fermionic model that we call the Generalized Thirring Model (GTM). Both the Conformal Invariance aspect and the Integrability aspect of the model will be studied.

This model is a natural generalization on the non-abelian Thirring model studied by Dashen and Frishman [6] in the early 70's, which was in turn a generalization of the original model studied by Thirring [7] in 1958. While the original Thirring model is equivalent to a free field theory, and a trivial example of a conformal theory, the non-abelian model, with a four fermion interaction invariant under $SU(n)$, has conformal invariance at quantized values of the coupling constant. Starting in the late 80's there was a renewed interest [8,9,10,11] in the conformal properties of the non-abelian Thirring model and of its relation to string dynamics. This was motivated by developments in string theory and by suggestions that abelian fermionic Thirring models were appropriate candidates to describe all toroidal string compactifications [12], while the non-abelian model was shown to led, through bosonization, to models with chiral bosons of importance for string phenomenology [13]. On the same lines, the original motivation for the work presented here, was the suggestion

[14] that the world sheet of the string theory resulting from QCD, may be described by a generalized Thirring model.

In parallel with the search of new conformal models has been the search of new integrable models. Although the quest for such models is older than string theory, in the last decade it received great impetus from its relation with the latter. The initial interest[1] of quantum field theorists in integrable two-dimensional theories arose from the fact that certain models share many features with 4D gauge theories [16,17], while being simpler at the same time. This interest led to the discovery by Lüscher and Pohlmeyer [18] of an infinite number of conserved charges in non-linear sigma models. In principal chiral models these charges were shown to satisfy a Kac-Moody algebra [19]. The non-abelian Thirring model was studied by de Vega, Eichenherr and Maillet [20] in the early 80's. They showed that the model was integrable with non-local charges that obeyed a quadratic Poisson Bracket algebra. Here, we want to study the integrability of the bosonized version of the GTM, where by integrability we mean the existence of an infinite number of conserved and commuting dynamical variables. Whether these variables are sufficient in number to make the model truly solvable will not be discussed, although we will mention some similar issues encountered in the study of the non-linear sigma model on a half line [21].

No attempt will be made to relate the conditions coming from conformal invariance with the ones coming from integrability, although that would be interesting and desirable, as was done in some other models (Toda field theory [22], $T = T_c$ Ising model [23]) [2]. The fact that the conditions for integrability were obtained only at the classical, not quantum, level, is not really an issue here. In fact, the relation between string theory and quantum integrable systems is not as clear as with classical integrable systems [26]. However, since the Yang-Baxter equation seems to play such an important role in establishing such relationship, and we never make use of it, it is possible that such relationship may be out of question at this stage.

In the rest of this section we introduce the Generalized Thirring Model (GTM), including the Poisson Bracket structure satisfied by the conserved currents and an extra set of variables introduced to form a complete set. In section 2 we derive the conditions for conformal invariance using two different approaches. In the first approach we use the background field method and use as criterion for conformal invariance the existence of a vanishing beta function. The second approach is based

[1] For a brief history of the study of integrable two-dimensional models see [15].

[2] See [24] for a pedagogical discussion and [25] for more examples.

in operator methods and uses as criterion for conformal invariance the existence of a chirally conserved stress tensor. In both cases the results are valid to first order in $1/N$, where N is the number of fermion flavors. We finish the section by finding some solutions of the conditions derived, and with an appendix on the background field approach. In section 3 we derive the conditions for integrability, where by that we mean the existence of an infinite number of conserved and commuting variables, and find some simple solutions of those conditions. We also construct the conserved "charges" with periodic and open boundary conditions and compute the respective Poisson Bracket's. In section 4 we present a (non-local) representation of the Poisson Bracket algebra of the GTM, introduced below, in terms of free fields. We conclude with a discussion of the results.

1.1. The Generalized Thirring Model

The Generalized Thirring Model (GTM) is a model of several massless fermions interacting through the most general Lorentz invariant four fermion couplings, including parity violating interactions [27,28]:

$$S_o = \int d^2x\, (\bar{\Psi} i\gamma^\mu \partial_\mu \Psi - \tilde{G}^{-1}_{ab}\; \bar{\Psi}_R t_a \Psi_R\; \bar{\Psi}_L t_b \Psi_L), \tag{1.1}$$

where R and L refer to the right and left chiral components of Ψ, and t_a are the trace orthogonal generators of $SU(n)$ in the adjoint representation[3]:

$$[t_a, t_b] = i f_{abc} t_c.$$

The coupling constant $(\tilde{G}^{-1})^{ab}$ is an invertible not necessarily symmetric matrix, resulting in parity violation, and Ψ is a Dirac fermion in the fundamental representation of $SU(n)$, (flavor group), and in the fundamental of $U(N)$, (color group). The flavor group is in general broken by the coupling constant matrix $\tilde{G}$, but the color group is an exact symmetry of the model: The color indices are contracted in the fermion biliniares in (1.1) to form singlets. The non-abelian Thirring Model of Dashen and Frishman [6], is recovered with $N = 1$ and $g_v \tilde{G} = 4I$, where I is the identity matrix and g_v is the coupling constant as defined in [6]. In what follows, the large N limit will be helpful in making the model tractable.

[3] The metric in group space is just δ_{ab} and so there is no distinction between upper and lower indices.

In one version of bosonization [27,29,10], this gives

$$S_o = W(g) + W(h^{-1}) - \frac{\tilde{N}}{4\pi}\int d^2x\, G_{ab}(ig^{-1}\partial_+ g)_a(ih^{-1}\partial_- h)_b, \qquad (1.2)$$

where X_a stands for $Tr(t_a X)$, g and h are group elements expressed in the adjoint representation, and

$$G_{ab} = \delta_{ab} - \frac{\pi}{\tilde{N}}\tilde{G}_{ab}, \qquad (1.3)$$

where $\tilde{N}$ is the number of fermion flavors (N) shifted by the Casimir of the group and W is the WZW action

$$W(g) = \frac{\tilde{N}}{8\pi}\left(\int d^2x\, \mathrm{Tr}(\partial_\mu g^{-1}\partial^\mu g) + \frac{2}{3}\int \mathrm{Tr}\left((g^{-1}dg)^3\right)\right). \qquad (1.4)$$

Here we have assumed that $(G-I)$ is an invertible matrix. From now on N stands for $\tilde{N}$. In the absence of sources the equations of motion are equivalent to conservation of two currents:

$$\partial_+ J_- = \partial_- J_+ = 0, \qquad (1.5)$$

where

$$\begin{aligned} J_+ &= i\frac{N}{4\pi}\left(-\partial_+ h h^{-1} + h t_a h^{-1} G^T_{ab}(g^{-1}\partial_+ g)_b\right), \\ J_- &= i\frac{N}{4\pi}\left(-\partial_- g g^{-1} + g t_a g^{-1} G_{ab}(h^{-1}\partial_- h)_b\right). \end{aligned} \qquad (1.6)$$

These currents are conserved by virtue of invariance of the action under

$$g \to u_-(x_-)g, \qquad h \to u_+(x_+)h,$$

where $x_\pm = \frac{1}{2}(x_0 \mp x_1)$ and $u_\pm(x_\pm)$ are arbitrary functions.

Treating x_+ as time and x_- as space, the Poisson Bracket between two J_-'s are easily evaluated [27]:

$$\{J_{-a}(x), J_{-b}(y)\} = -\frac{N}{4\pi}\delta_{ab}\delta'(x-y) + f_{abc}\delta(x-y)J_{-c}(x). \qquad (1.7)$$

where x and y are the space coordinates. These conserved currents satisfy an affine Lie algebra. However, they are not in sufficient number to span the space where the fields g and h are defined and must be supplemented by additional variables in order to be able to quantize the model. The extra set of variables was chosen [27] to be

$$M_{-a} \equiv (H_-^{\frac{1}{2}})_{ab}(ih^{-1}\partial_- h)_b,$$

where $H_- = 1 - G^T G$. The new Poisson Bracket's are

$$\{J_{-a}(x), M_{-b}(y)\} = 0, \tag{1.8}$$

and [27,28]

$$\begin{aligned}\{M_{-a}(x), M_{-b}(y)\} = &-\frac{4\pi}{N}\delta_{ab}\delta'(x-y) + \frac{4\pi}{N}F_{-abc}M_{-c}(x)\delta(x-y) \\ &+ \frac{2\pi}{N}\epsilon(x-y)A_{-ack}A_{-bdl}\left(Pe^{\int_x^y A_{+e}M_{-e}(x')dx'}\right)_{kl}M_{-c}(x)M_{-d}(y),\end{aligned} \tag{1.9}$$

where

$$F_{-abc} = H_{-ar}^{-\frac{1}{2}}H_{-bs}^{-\frac{1}{2}}H_{-ct}^{-\frac{1}{2}}f_{rst} - (H_-^{-\frac{1}{2}}G^T)_{ar}(H_-^{-\frac{1}{2}}G^T)_{bs}(H_-^{-\frac{1}{2}}G^T)_{ct}f_{rst}, \tag{1.10}$$

and

$$\begin{aligned}A_{-abc} &= -H_{-ar}^{-\frac{1}{2}}H_{-bs}^{-\frac{1}{2}}(H_+^{-\frac{1}{2}}G)_{ct}f_{rst} + (H_-^{-\frac{1}{2}}G^T)_{ar}(H_-^{-\frac{1}{2}}G^T)_{bs}H_{+ct}^{-\frac{1}{2}}f_{rst}, \\ A_{+abc} &= -H_{+ar}^{-\frac{1}{2}}H_{+bs}^{-\frac{1}{2}}(H_-^{-\frac{1}{2}}G^T)_{ct}f_{rst} + (H_+^{-\frac{1}{2}}G)_{ar}(H_+^{-\frac{1}{2}}G)_{bs}(H_{-ct}^{-\frac{1}{2}}f_{rst},\end{aligned} \tag{1.11}$$

where $H_+ = 1 - GG^T$ and the matrix A_{+e} has elements

$$(A_{+e})_{ab} = A_{+abe}.$$

The algebra (1.9) satisfied by the M_-'s is non-linear and also non-local. In section 4 we present a non-local representation of this algebra in terms of free fields. One reason for finding such a representation was to see whether the above model could be mapped into a well-known conformal theory, perhaps admitting an affine Sugawara construction (for a review of the affine Sugawara construction see [30] and references therein). As we will show, that doesn't seem to be the case, which suggests the possibility of a new conformal structure.

2. Conformal Invariance

The non-abelian Thirring model, with the four fermion interaction invariant under some Lie group, was shown [6] to be conformal invariant at quantized values of the coupling constant and to possess a stress tensor that at the conformal points is given by the affine Sugawara construction. The aim of this section is to find similar results for the classical scale invariant GTM, but we will delay the problem of whether the stress tensor admits an affine Sugawara construction until section 4.

2.1. Background Field Method and Conformal Invariance

In this section we use the background field method to investigate conformal invariance of the GTM. The criterion for conformal invariance is the vanishing of the beta function, which will be evaluated to first order in $1/N$. Two implement the background field method, we add a term to the action which represents the coupling of two external sources $K_\pm$ to two suitable currents:

$$\Delta S = \frac{N}{4\pi}\int d^2x \ \mathrm{Tr}\left(K_+(ih^{-1}\partial_- h)\right) + \frac{N}{4\pi}\int d^2x \ \mathrm{Tr}\left(K_-(ig^{-1}\partial_+ g)\right). \tag{2.1}$$

The next step is to define classical fields by solving the equations of motion in the presence of sources. A special solution we are going to use is

$$\begin{aligned} K_{-a} &= \left(-(ig^{-1}\partial_- g)_a + G_{ab}(ih^{-1}\partial_- h)_b\right)_{clas.}, \\ K_{+a} &= \left(-(ih^{-1}\partial_+ h)_a + G^T_{ab}(ig^{-1}\partial_+ g)_b\right)_{clas.}. \end{aligned} \tag{2.2}$$

These sources $K_\pm$ can be substituted back in S to give the classical action $S^{(0)}$. This defines the classical (background) fields $g_{clas.}$ and $h_{clas.}$ around which we expand the full quantum fields g and h. In the appendix, we use background field perturbation theory to derive, to one loop order, the conditions that the coupling constants G_{ab} must satisfy to have conformal invariance. This is done by first expanding the action S around $S^{(0)}$, and by calculating the one loop divergent contribution to the action. This calculation is fairly standard [14], and for the sake of completeness, it is sketched in the appendix at the end of this section. The result to one loop order is

$$S[\phi] = S^{(0)}[\phi_{clas.}] + S^{(2)}[\phi_{clas.}], \tag{2.3}$$

where $S^{(2)}$ is logarithmically divergent. Here ϕ (which stands for both ϕ and $\overline{\phi}$) is the field used to parameterize g (and h is parameterized by $\overline{\phi}$), and $\phi_{clas.}$ is defined by $g_{clas.} = g(\phi_{clas.})$. The divergent piece can be written as (from now on ϕ stands for $\phi_{clas.}$)

$$S^{(2)}[\phi] \cong \int \frac{d^2p}{p^2 - m^2}\int d^2x \ O(x), \tag{2.4}$$

where

$$\begin{aligned} O(x) = {} & Y^{(11)}_{ab} E_{a\alpha}E_{b\beta}\partial_+\phi^\alpha\partial_-\phi^\beta + Y^{(22)}_{ab}\overline{E}_{a\alpha}\overline{E}_{b\beta}\partial_+\overline{\phi}^\alpha\partial_-\overline{\phi}^\beta \\ & + Y^{(21)}_{ab}\overline{E}_{a\alpha}E_{b\beta}\partial_+\overline{\phi}^\alpha\partial_-\phi^\beta + Y^{(12)}_{ab}E_{a\alpha}\overline{E}_{b\beta}\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta, \end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
Y^{(11)}_{ab} &= -\mathrm{Tr}[GH_-^{-1}G^T f_a H_+^{-1} f_b] \quad Y^{(12)}_{ab} = \mathrm{Tr}[H_-^{-1}G^T f_a H_+^{-1} G f_b] \\
Y^{(21)}_{ab} &= \mathrm{Tr}[GH_-^{-1} f_a G^T H_+^{-1} f_b] \quad Y^{(22)}_{ab} = -\mathrm{Tr}[H_-^{-1} f_a G^T H_+^{-1} G f_b],
\end{aligned}
\tag{2.6}
$$

with

$$
H_- = 1 - G^T G, \qquad H_+ = 1 - GG^T, \tag{2.7}
$$

and the matrices f_a are defined by $(f_a)_{bc} = f_{abc}$, where f_{abc} are the structure constants of the group. The $E_{a\alpha}$'s are the vielbeins defined by

$$
E_{a\alpha}(\phi)\partial_+\phi^\alpha \equiv \mathrm{Tr}(t_a i g^{-1}\partial_\alpha g)\partial_+\phi^\alpha = \mathrm{Tr}(t_a i g^{-1}\partial_+ g), \tag{2.8}
$$

with similar definitions for $\overline{E}_{a\alpha}$'s in terms of h's. Now compare this divergent piece with the original Lagrangian, expressed in terms of classical fields

$$
\begin{aligned}
S^{(0)} =& W(g) + W(h^{-1}) + \frac{N}{4\pi}\int d^2x\; G_{ab}E_{a\alpha}\overline{E}_{b\beta}\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta \\
&- \frac{N}{4\pi}\int d^2x\; E_{a\alpha}E_{a\beta}\partial_+\phi^\alpha\partial_-\phi^\beta - \frac{N}{4\pi}\int d^2x\; \overline{E}_{a\alpha}\overline{E}_{a\beta}\partial_+\overline{\phi}^\alpha\partial_-\overline{\phi}^\beta .
\end{aligned}
\tag{2.9}
$$

Of the four distinct divergent terms that appear in (2.5), three correspond to wave function renormalizations and can be eliminated by field redefinitions. Conformal invariance is then imposed by requiring that the remaining divergence (the beta function) vanish. The field redefinitions that eliminate the spurious divergences are given by

$$
\begin{aligned}
(ig^{-1}\partial_+ g)_a &\longrightarrow (ig^{-1}\partial_+ g)_a + \lambda^{(11)}_{ab}(ig^{-1}\partial_+ g)_b + \lambda^{(12)}_{ab}(ih^{-1}\partial_+ h)_b, \\
(ih^{-1}\partial_- h)_a &\longrightarrow (ih^{-1}\partial_- h)_a + \lambda^{(21)}_{ab}(ih^{-1}\partial_- h)_b + \lambda^{(22)}_{ab}(ig^{-1}\partial_- g)_b,
\end{aligned}
\tag{2.10}
$$

where the λ's are first order in $1/N$. This corresponds, in the Polyakov-Wiegmann bosonization, to making the identification

$$
\begin{aligned}
A_{+a} &= (\delta_{ab} + \lambda^{(11)}_{ab})(ig^{-1}\partial_+ g)_b + \lambda^{(12)}_{ab}(ih^{-1}\partial_+ h)_b, \\
A_{-a} &= (\delta_{ab} + \lambda^{(22)}_{ab})(ih^{-1}\partial_- h)_b + \lambda^{(21)}_{ab}(ig^{-1}\partial_- g)_b,
\end{aligned}
\tag{2.11}
$$

instead of

$$
A_{+a} = (ig^{-1}\partial_+ g)_a, \quad A_{-a} = (ih^{-1}\partial_- h)_a. \tag{2.12}
$$

The same result can be obtained by introducing additional sources $L_{\pm}$,

$$\Delta S = \frac{N}{4\pi}\int d^2x \ \mathrm{Tr}\left(L_-(ih^{-1}\partial_+ h)\right) + \frac{N}{4\pi}\int d^2x \ \mathrm{Tr}\left(L_+(ig^{-1}\partial_- g)\right), \tag{2.13}$$

which are zero to lowest order, and by transforming $K_{\pm}$, $L_{\pm}$ linearly among themselves (source renormalization).

Under these field redefinitions the first order correction to $S^{(0)}[\phi]$ is

$$\begin{aligned}\Delta S^{(0)} = -\frac{N}{4\pi}\int d^2x \ \Big(& ((\lambda^{(21)} + (\lambda^{12})^T)_{ab}\overline{E}_{a\alpha}E_{b\beta}\partial_+\overline{\phi}^\alpha\partial_-\phi^\beta \\ & -((\lambda^{(11)})^T G + G\lambda^{(22)})_{ab}E_{a\alpha}\overline{E}_{b\beta}\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta \\ & +((\lambda^{(11)})^T - G\lambda^{(21)})_{ab}E_{a\alpha}E_{b\beta}\partial_+\phi^\alpha\partial_-\phi^\beta \\ & +(\lambda^{(22)} - (\lambda^{(12)})^T G)_{ab}\,\overline{E}_{a\alpha}\overline{E}_{b\beta}\partial_+\overline{\phi}^\alpha\partial_-\overline{\phi}^\beta\Big).\end{aligned} \tag{2.14}$$

We now try to eliminate the divergent terms, and this leads to the matrix equations

$$\begin{aligned}-Y^{(11)} + \frac{N}{4\pi}((\lambda^{(11)})^T - G\lambda^{(21)}) = 0, \\ -Y^{(12)} - \frac{N}{4\pi}((\lambda^{(11)})^T G + G\lambda^{(22)}) = 0, \\ -Y^{(22)} + \frac{N}{4\pi}(\lambda^{(22)} - (\lambda^{(12)})^T G) = 0, \\ -Y^{(21)} + \frac{N}{4\pi}(\lambda^{(21)} + (\lambda^{(12)})^T) = 0.\end{aligned} \tag{2.15}$$

At first, one might think that these equations can be solved for the unknown λ's. If this were true, all the infinities would be absorbed by field redefinitions and conformal invariance would be automatic! In fact, the equations are linearly dependent, and for a solution to exist, the Y's must satisfy the following condition:

$$Y^{(12)} + Y^{(11)}G + GY^{(22)} + GY^{(21)}G = 0. \tag{2.16}$$

This condition is therefore equivalent to the vanishing of the beta function. Written out explicitly, this leads to the following equation between the coupling constants:

$$\begin{aligned}&\mathrm{Tr}[H_-^{-1}G^T f_a H_+^{-1} G f_b] + G_{aa'}G_{b'b}Tr[GH_-^{-1}f_{a'}G^T H_+^{-1} f_{b'}] \\ &-G_{aa'}Tr[H_-^{-1}f_{a'}G^T H_+^{-1}Gf_b] - G_{b'b}Tr[GH_-^{-1}G^T f_a H_+^{-1} f_{b'}] = 0,\end{aligned} \tag{2.17}$$

Eq. (2.17) is therefore the condition that determines the conformal points in the coupling constant space. For $G = G^T$, it agrees with the result obtained in [31], where the Q defined there is related to our G by

$$Q = 2I - G. \tag{2.18}$$

We end this section by noticing that this equation is invariant under $G \to G^{-1}$ and under $G \to O_1^T G O_2$, where O_1 and O_2 are orthogonal transformations generated by rotations in group space. The first one is the standard duality transformation (for a review see [32]), already noticed in a classical context in [33]. The second set of transformations are generated by independent group rotations of left and right fermions:

$$\Psi_R \to U_R \Psi_R, \quad \Psi_L \to U_L \Psi_L. \tag{2.19}$$

2.2. The OPE and Conformal Invariance

In this section, we shall reexamine the conformal invariance of the theory from the operator point of view, and show that again the same result as in the last section (eq. (2.17)) is obtained. Our criterion for conformal invariance is the existence of a chirally conserved stress tensor: it is well known that this is equivalent to the absence of the trace anomaly in the stress tensor [34]. Our approach will be to solve the equations of motion for the quantized fields as a power series in $1/N$, and then use this result to construct the stress tensor explicitly. We will then see that there is an anomalous term which violates chiral conservation. Conformal invariance is restored by demanding that this term vanish, and the resulting condition on the coupling constants agrees with the result derived in the last section using the background field method. Before discussing the quantum mechanical complications, we will first briefly review the classical situation. The two chiral components of the classical stress tensor, defined by

$$\begin{aligned} T_-(t,x) &= \frac{\pi}{\alpha^2} M_{-a}(t,x) M_{-a}(t,x), \\ T_+(t,x) &= \frac{\pi}{\alpha^2} M_{+a}(t,x) M_{+a}(t,x), \end{aligned} \tag{2.20}$$

where $\alpha = (4\pi/N)^{1/2}$, $t \equiv x_+, x \equiv x_-$, and

$$\begin{aligned} M_{-a} &= (H_-^{\frac{1}{2}})_{ab} (i h^{-1} \partial_- h)_b, \\ M_{+a} &= (H_+^{\frac{1}{2}} (G^T)^{-1})_{ab} (i h^{-1} \partial_+ h)_b, \end{aligned} \tag{2.21}$$

satisfy the conservation laws

$$\begin{aligned} \partial_+ T_-(t,x) &= 0, \longrightarrow T_-(t,x) = T_-(x), \\ \partial_- T_+(t,x) &= 0, \longrightarrow T_+(t,x) = T_+(t), \end{aligned} \tag{2.22}$$

and also satisfy the classical (without central charge) Virasoro algebra [27]:

$$T_-(x)T_-(y) \cong \frac{1}{(x-y)^2}(T_-(x) + T_-(y)). \tag{2.23}$$

Now, in the quantum version of the stress tensor we replace the classical expression by (we will work with the $M_{-a}(t,x)$'s, but the same applies to the $M_{+a}(t,x)$'s),

$$T_-(t,x) = \frac{\pi}{\alpha^2} \lim_{y,u\to x,t} (C_{ab} M_{-a}(t,x) M_{-b}(u,y) - \text{sing.terms}), \tag{2.24}$$

where C_{ab} is a constant matrix which starts with classical value δ_{ab}, and has higher order corrections given by

$$C_{ab} = \delta_{ab} + \sum_{n=2}^{\infty} \alpha^n C_{ab}^{(n)}, \tag{2.25}$$

due to renormalization. We will determine it by requiring that the stress tensor T_- satisfy the Virasoro algebra. To do this, and to find the singular terms to be subtracted, we need the OPE's (operator product expansion) between two M_-'s. So we will expand

$$M_{-a}(t,x) = \sum_{n=0}^{\infty} \alpha^n M_{-a}^{(n)}(t,x), \tag{2.26}$$

and carry calculations up to second order. The strategy for computing OPE's is the following. We first define $M_-^{(n)}$'s at a fixed t, say $t=0$: $M_{-a}^{(n)}(x) \equiv M_{-a}^{(n)}(t=0,x)$. The OPE's depend only on x and they can be deduced from the Poisson Bracket's at fixed t [27]. The OPE's between the M_-'s follow from the Poisson Bracket's (eq. (1.9)) and are given by [27,28]:

$$\begin{aligned} M_{-a}^{(0)}(x) M_{-b}^{(0)}(y) &\cong -\frac{1}{2\pi(x-y)^2}\delta_{ab}, \\ \sum_{n=0}^{1} M_{-a}^{(n)}(x) M_{-b}^{(1-n)}(y) &\cong -\frac{1}{4\pi(x-y)} F_{-abc}\left(M_{-c}^{(0)}(x) + M_{-c}^{(0)}(y)\right), \\ \sum_{n=0}^{2} M_{-a}^{(n)}(x) M_{-b}^{(2-n)}(y) &\cong -\frac{1}{4\pi(x-y)} F_{-abc}\left(M_{-c}^{(1)}(x) + M_{-c}^{(1)}(y)\right) \\ &\quad + \frac{1}{2\pi} E_{-ab,cd} \log(x-y) M_{-c}^{(0)}(x) M_{-d}^{(0)}(y), \end{aligned} \tag{2.27}$$

where

$$E_{-ab,cd} = A_{-cae}A_{-bde}. \tag{2.28}$$

and the constants A_{-abc} and F_{-abc} were defined before (eq. (1.10)). The $M_{+a}^{(n)}$'s obey similar OPE's, obtained from the above ones under $G \to G^T$, with the new constants A_{+abc} and F_{+abc}, and $E_{+ab,cd} = A_{+cae}A_{+bde}$. If we define $G_+ = G^T$ and $G_- = G$ then these constants can be written compactly as

$$\begin{aligned}
F_{\pm abc} &= H^{-\frac{1}{2}}_{\pm ar}H^{-\frac{1}{2}}_{\pm bs}H^{-\frac{1}{2}}_{\pm ct}f_{rst} - (H_{\pm}^{-\frac{1}{2}}G_{\mp})_{ar}(H_{\pm}^{-\frac{1}{2}}G_{\mp})_{bs}(H_{\pm}^{-\frac{1}{2}}G_{\mp})_{ct}f_{rst}, \\
A_{\pm abc} &= -H^{-\frac{1}{2}}_{\pm ar}H^{-\frac{1}{2}}_{\pm bs}(H_{\mp}^{-\frac{1}{2}}G_{\pm})_{ct}f_{rst} + (H_{\pm}^{-\frac{1}{2}}G_{\mp})_{ar}(H_{\pm}^{-\frac{1}{2}}G_{\mp})_{bs}H^{-\frac{1}{2}}_{\mp ct}f_{rst},
\end{aligned} \tag{2.29}$$

and

$$E_{\pm ab,cd} = A_{\pm cae}A_{\pm bde}.$$

Note that $A_{\pm abc}$ is antisymmetric in its first two indices.

To extend these OPE's to $t \neq 0$, we solve the equations of motion up to second order, and express the M_-'s and M_+'s at arbitrary t in terms of the same variables at $t = 0$. Since the OPE's at $t = 0$ are already known, they are then easily extended to $t \neq 0$. From

$$g^{-1}(\partial_+ J_-)g = 0 \tag{2.30}$$

we have

$$-\partial_-(g^{-1}\partial_+ g) + [(g^{-1}\partial_+ g), t_a]G_{ab}(h^{-1}\partial_- h)_b + t_a G_{ab}\partial_+(h^{-1}\partial_- h)_b = 0. \tag{2.31}$$

Now solve for $(g^{-1}\partial_+ g)$ in terms of J_+,

$$(g^{-1}\partial_+ g)_a = (G^T)^{-1}_{ab}(h^{-1}(\partial_+ - \frac{4i\pi}{N}J_+)h)_b. \tag{2.32}$$

The model is invariant under the gauge transformations $h \to u_+(x_+)h$; using this gauge invariance, we can set $J_+ = 0$ ($\partial_- J_+ = 0$, so J_+ depends only on $x_+ = t$). This gives us a special solution to the equations of motion; the general solution is obtained by applying the inverse transformation. It is interesting to notice that the equations of motion can then be written as flatness conditions for two vector fields V and W:

$$\begin{aligned}
\partial_+ V_- - \partial_- V_+ - i[V_+, V_-] = 0, \\
\partial_+ W_- - \partial_- W_+ - i[W_+, W_-] = 0,
\end{aligned} \tag{2.33}$$

where,

$$
\begin{aligned}
V_{\pm a} =&(ih^{-1}\partial_{\pm}h)_a, \\
W_{+a} = (G^T)^{-1}_{ab}V_{+b}, &\qquad W_{-a} = G_{ab}V_{-,b}.
\end{aligned}
\tag{2.34}
$$

These can be cast in a more useful form in terms of

$$
M_{-a} = (H_-^{\frac{1}{2}})_{ab}V_{-b}, \qquad M_{+a} = (H_+^{\frac{1}{2}})_{ab}W_{+b},
\tag{2.35}
$$

defined before. The equations of motion are then

$$
\begin{aligned}
&\partial_- M_{+a} + A_{+abc}M_{+b}M_{-c} = 0, \\
&\partial_+ M_{-a} + A_{-abc}M_{-b}M_{+c} = 0,
\end{aligned}
\tag{2.36}
$$

The conservation laws of the (classical) stress tensors follow at once from these equations due to the antisymmetry of $A_{\pm abc}$ in the first two indices.

The next step is to solve the equations of motion iteratively, using the expansion in α (eq. (2.26)), and a similar expansion for M_+. The zeroth and first order solutions are

$$
\begin{aligned}
&M^{(0)}_{-a}(t,x) = M^{(0)}_{-a}(x), \\
&M^{(0)}_{+a}(t,x) = M^{(0)}_{+a}(t), \\
&M^{(1)}_{-a}(t,x) = M^{(1)}_{-a}(x) - A_{-abc}M^{(0)}_{-b}(x)\int^t dt' M^{(0)}_{+c}(t'), \\
&M^{(1)}_{+a}(t,x) = M^{(1)}_{+a}(t) - A_{+abc}M^{(0)}_{+b}(t)\int^x dx' M^{(0)}_{-c}(x'),
\end{aligned}
\tag{2.37}
$$

and to second order

$$
\begin{aligned}
M^{(2)}_{-a}(t,x) = M^{(2)}_{-a}(x) &- A_{-abc}\int^t dt' M^{(0)}_{-b}(x)M^{(1)}_{+c}(t') \\
&+ A_{-abc}A_{+cde}\int^x dx'\int^t dt' M^{(0)}_{-b}(x)M^{(0)}_{-e}(x')M^{(0)}_{+d}(t') \\
&- A_{-abc}\int^t dt' M^{(1)}_{-b}(x)M^{(0)}_{+c}(t') \\
&+ A_{-abc}A_{-bde}\int^t dt'\int^{t'} dt'' M^{(0)}_{-d}(x)M^{(0)}_{+c}(t')M^{(0)}_{+e}(t'').
\end{aligned}
\tag{2.38}
$$

We will not need $M^{(2)}_{+a}(t,x)$. Therefore, $M_{-a}(t,x)$ can be expressed in terms of $M^{(n)}_{-a}(x)$'s and $M^{(n)}_{+a}(t)$'s, functions of only one variable. If we substitute the above in the definition of T_- to second order, it is easy to see that classically all of the t dependent terms cancel, as they must, because we know from the equations of motion that this is true to all orders. However this does not happen in the quantum case, where $M_{-a}^{(n)}(x)$, $n=0,1,2$, become operators that satisfy the OPE's given earlier (eq.(2.27)). First of all, as it stands, the above expression for $M_{-a}(t,x)$ is not well defined, because we haven't defined yet the product of two or more M_-'s at the same point. These products should be understood as nonsingular "normal ordered" products. For instance, $M_{-a}^{(0)}(x)M_{-b}^{(0)}(y)$ should be understood as

$$:M_{-a}^{(0)}(x)M_{-b}^{(0)}(y): \equiv M_{-a}^{(0)}(x)M_{-b}^{(0)}(y) + \frac{\delta_{ab}}{2\pi(x-y)^2}, \tag{2.39}$$

and the same applies for the $M_{+a}^{(n)}$'s. The product of a $M_{-a}^{(n)}$ and a $M_{+a}^{(n)}$ gives no problem since they are functions of different variables and commute with each other. This guarantees that $\lim_{y\to x} :M_{-a}^{(0)}(x)M_{-b}^{(0)}(y):$, and consequently the above expression for $M_{-a}^{(2)}(t,x)$ is well defined.

Next, we examine (2.24), to see what subtractions are needed to make the stress tensor well-defined, and whether it is t independent, as the conservation law (eq. (2.22)) demands. It turns out that to the order we are considering (second order in α), T_- can be made finite by making suitable subtractions, and that all of the terms in T_-, with the possible exception of one term, are t independent. The critical term in question, up to a multiplicative factor of π/α^2, is

$$T_{-\text{crit.}} = A_{-abc}A_{+cde}\int^x dx' \int^t dt' :M_{-b}^{(0)}(x)M_{-e}^{(0)}(x'): M_{-a}^{(0)}(y)M_{+d}^{(0)}(t') + (x\leftrightarrow y).$$

This term is finite as $y\to x$ and needs no subtraction. However, it is t dependent, and therefore, if it does not vanish, it violates the conservation law (eq. (2.22)). It does not automatically vanish because, while A_{-abc} is antisymmetric in a and b, $:M_{-b}^{(0)}(x)M_{-e}^{(0)}(x'):M_{-a}^{(0)}(y) + (x\leftrightarrow y)$ is not symmetric due to the normal ordering of the two M_-'s. However, the completely normal ordered product

$$:M_{-b}^{(0)}(x)M_{-e}^{(0)}(x')M_{-a}^{(0)}(y): + (x\leftrightarrow y),$$

is symmetric in a and b and vanishes when multiplied by A_{-abc}. We now make use of the identity

$$\begin{aligned}:M_{-b}^{(0)}(x)M_{-e}^{(0)}(x'):M_{-a}^{(0)}(y) = {}&:M_{-b}^{(0)}(x)M_{-e}^{(0)}(x')M_{-a}^{(0)}(y):\\ &-\frac{\delta_{ab}}{2\pi(x-y)^2}M_{-e}^{(0)}(x') - \frac{\delta_{ae}}{2\pi(x'-y)^2}M_{-b}^{(0)}(x),\end{aligned} \tag{2.40}$$

to find

$$
\begin{aligned}
\lim_{y\to x} T_{-\text{crit.}} = &- \lim_{y\to x} A_{-abc}A_{+cde} \int^t dt' M^{(0)}_{+d}(t') \times \\
&\left(\int^x dx' \frac{\delta_{ae}}{2\pi(x'-y)^2} M^{(0)}_{-b}(x) + \int^y dx' \frac{\delta_{ae}}{2\pi(x'-x)^2} M^{(0)}_{-b}(y) \right) \\
= &- \lim_{y\to x} A_{-abc}A_{+cde} \left(-\frac{1}{2\pi} \frac{M^{(0)}_{-b}(x) - M^{(0)}_{-b}(y)}{x-y} \right) \int^t dt' M^{(0)}_{+d}(t') \\
= &+ \frac{1}{2\pi} A_{-bac}A_{+dca} M'^{(0)}_{-b}(x) \int^t dt' M^{(0)}_{+d}(t').
\end{aligned}
\tag{2.41}
$$

To eliminate this anomaly and restore conformal invariance, we have to set its coefficient equal to zero,

$$
A_{-acd}A_{+bdc} = 0, \tag{2.42}
$$

recovering the same condition as before (eq. (2.17)). We note that conformal invariance imposes no restrictions on $C^{(2)}_{ab}$. These constants can be determined by requiring that the stress tensor satisfy the Virasoro algebra to second order. We therefore need the OPE of the product of two stress tensors; this is given by [27,28]:

$$
\begin{aligned}
T_-(x)T_-(y) \cong &\frac{c}{2(x-y)^4} - \frac{\pi}{(x-y)^2} \left(M^{(0)}_{-a}(x)M^{(0)}_{-a}(x) + M^{(0)}_{-a}(y)M^{(0)}_{-a}(y) \right) \\
&- \frac{\pi\alpha^2}{(x-y)^2} \left(M^{(1)}_{-a}(x)M^{(1)}_{-a}(x)M^{(1)}_{-a}(y)M^{(1)}_{-a}(y) \right) \\
&- \frac{\alpha^2}{4(x-y)^2} (F_{-adb}F_{-adc} + 2E_{-aa,bc} + 4\pi C^{(2)}_{bc}) \times \\
&\left(M^{(0)}_{-b}(x)M^{(0)}_{-c}(x) + M^{(0)}_{-b}(y)M^{(0)}_{-c}(y) \right),
\end{aligned}
\tag{2.43}
$$

where c is the central charge.

Since the Virasoro algebra reads

$$
T_-(x)T_-(y) \cong -\frac{1}{(x-y)^2}(T_-(x) + T_-(y)) + \frac{c}{2(x-y)^4}, \tag{2.44}
$$

we must have

$$
F_{-adb}F_{-adc} + 2E_{-aa,bc} + 4\pi C^{(2)}_{bc} = 0, \tag{2.45}
$$

which determines $C^{(2)}_{ab}$, and the central charge c is given by

$$
c = D - \frac{\alpha^2}{4\pi}(3E_{-aa,bb} + F_{-abc}F_{-abc}), \tag{2.46}
$$

where D is the dimension of the flavor algebra. This is the central charge of the algebra generated by T_-. The central charge of the algebra generated by the other chiral component, T_+, (see eq. (2.20)) can be gotten from (2.46), by replacing E_- by E_+ and F_- by F_+.

As a check on our formalism, we notice that, at $G_{ab} = 0$ in (1.2), the action is a sum of two decoupled WZW models and therefore it is obviously conformal. $G = 0$ indeed satisfies the condition for conformal invariance given by eq. (2.17) and so the equation passes this test. There is a further check on the central charge. The stress tensor of the WZW model is given by the Sugawara construction in terms of the currents, with the standard formula for the central charge:

$$c = \frac{2kD}{2k + c_\psi^g}, \tag{2.47}$$

where k is the level number of the affine algebra, related to our N by $2k = N$ and

$$c_\psi^g \delta_{ab} = \sum_{c,d=1}^{D} f_{acd} f_{bcd}.$$

This formula is exact. We have to compare it with (2.46) in the limit of large N (or k), with G set equal to zero. In this limit $E_{-ab,cd} = 0$, $F_{-abc} = f_{abc}$ and so from (2.46),

$$c = D\left(1 - \frac{c_\psi^g}{N}\right),$$

which agrees with the standard formula (2.47) to first order in $1/N$. This particular solution ($G = 0$) has some relation to the Dashen-Frishman conformal point [6]. It is natural to suspect such a relation, since both $G = 0$ and the Dashen-Frishman solution are $SU(n)$ symmetric. In the next section we will show how such relation can be made reasonable, but here we just note that the stress tensor of the Dashen-Frishman solution is given by the Sugawara construction and the central charge is therefore given by (2.47). But, as we pointed out above, the $G = 0$ solution, being the sum of two WZW models, has also a Sugawara stress tensor and the standard formula for the central charge. Therefore, at the level of stress tensors, there is agreement.

2.3. The Conformal Points in Coupling Space

Having derived general conditions (eq.(2.42)) that the coupling constant G_{ab}, and hence $\tilde{G}_{ab}$, must satisfy to have a conformal invariant GTM, the next step is to check if there are really solutions to those conditions.

Before we do that though, it would be useful to know if the one loop results we obtained are exact, or if there are corrections coming from higher loop contributions. In principle, the OPE method used in the previous section is easily extendable to higher orders. To get the next order correction to T_- we just need to expand $M_{-a}(t,x)$ and $M_{+a}(t,x)$ to 4th and 3rd order respectively (there are no contributions from $M^{(3)}_{-a}(t,x)$ to the stress tensor T_-) and proceed as before. However, the lengthiness of the calculations involved have prevented us from doing that.

We will restrict ourselves to diagonal matrices G_{ab} and to the $SU(2)$ and $SU(3)$ flavor groups. For $SU(2)$, a non diagonal G_{ab} can always be written as a diagonal one (in a different basis) due to the uniqueness of the structure constants, so a diagonal G_{ab} is not a restriction in that case. For $SU(3)$, more general matrices seem to necessarily lead to overdetermined systems with no non trivial solutions.

<u>$SU(2)$</u>:

$$G = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \tag{2.48}$$

From (2.42) we get then three distinct equations:

$$\begin{aligned} \frac{2(ab-c)(-b+ac)}{(-1+a^2)(-1+b^2)(-1+c^2)} &= 0, \\ \frac{2(ab-c)(-a+bc)}{(-1+a^2)(-1+b^2)(-1+c^2)} &= 0, \\ \frac{2(-b+ac)(-a+bc)}{(-1+a^2)(-1+b^2)(-1+c^2)} &= 0, \end{aligned} \tag{2.49}$$

with solutions $a = b = 0$ and $c \neq 1$ and permutations of these. Note that this solution satisfies the requirement that $(G - I)$ is invertible.

<u>$SU(3)$</u>: In this case G is a 8×8 matrix. We found that there is a solution of (2.42) with $G_{33} \neq 1$ and $G_{88} \neq 1$ with all the other diagonal elements zero.

It is also interesting to compare our results with the results obtained by Dashen and Frishman [6], even though the conformal points they found correspond to some trivial limits of the model we consider here. In their notation, the conformal points occur for $g_v = 0$ and $g_v = 4\pi/(n+1)$. The $g_v = 0$ point corresponds to $\tilde{G} = \infty$ (and also $G = \infty$). Taken as a limiting case, this point is in fact a solution of (2.42) as can be seen from (2.49), with $a = b = c$. The other point, corresponds to the $G = 0$ case. Then from (1.3) and $g_v G = 4I$ we get

$$g_v = \frac{4\pi}{\tilde{N}} = \frac{4\pi}{N+n},$$

where n is the Casimir of $SU(n)$. Since the Dashen-Frishman model corresponds to the $N = 1$ case, we seem to recover the same conformal point, and that on the other hand seems to suggest that maybe the one loop result is exact. However, since our one loop result was obtained in the large N limit no conclusion is possible.

Appendix A.

In this appendix, we fill the gaps in the evaluation of $S^{(2)}[\phi_{clas.}]$ done in §2.1. As explained there, we want to expand the action $S[\phi]$ around $S^{(0)}[\phi_{clas.}]$, the classical action. To do this, parameterize the fields g and h by:

$$g = g(\phi), \qquad h = h(\overline{\phi}); \tag{A.1}$$

where $\phi(x)$ stands for $\phi^\alpha(x)$. The ϕ^α's are the coordinates in the group manifold where g takes values, and $x \equiv x^\mu$ are coordinates in Minkowski 2-space. The classical fields $\phi^\alpha_{clas.}$ are defined by $g_{clas.} = g(\phi_{clas.})$, and similarly for $\overline{\phi}^\alpha$. From now on, unless otherwise stated, ϕ stands either for ϕ and $\overline{\phi}$.

Using the vielbeins $E_{a\alpha}(\phi)$ and $\overline{E}_{a\alpha}(\overline{\phi})$ defined in §2.1, the source terms can be written as

$$\begin{aligned}\mathrm{Tr}\left(K_-(ig^{-1}\partial_+ g)\right) &= \mathrm{Tr}(K_- t_a)E_{a\alpha}\partial_+\phi^\alpha \equiv K_{-a}E_{a\alpha}\partial_+\phi^\alpha,\\ \mathrm{Tr}\left(K_+(ih^{-1}\partial_- h)\right) &= \mathrm{Tr}(K_+ t_a)\overline{E}_{a\alpha}\partial_-\overline{\phi}^\alpha \equiv K_{+a}\overline{E}_{a\alpha}\partial_-\overline{\phi}^\alpha,\end{aligned} \tag{A.2}$$

and the action becomes

$$\begin{aligned}S = W(g) + W(h^{-1}) &- \frac{N}{4\pi}\int d^2x\ G_{ab}E_{a\alpha}\overline{E}_{b\beta}\partial_+\phi^\alpha\partial_-\overline{\phi}^\beta\\ &+ \frac{N}{2\pi}\int d^2x\ K_{+a}\overline{E}_{a\alpha}\partial_-\overline{\phi}^\alpha + \frac{N}{2\pi}\int d^2x\ K_{-a}E_{a\alpha}\partial_+\phi^\alpha.\end{aligned} \tag{A.3}$$

Now we expand this action $S[\phi]$ around the classical action $S^{(0)} = S[\phi_{clas.}]$ treating $K_\pm$ as classical sources, which can then be written in terms of $\phi_{clas.}$. To expand the action, let

$$\phi(x) \longrightarrow \phi(x,s),$$

so that $\phi(x,s=0)=\phi_{clas.}(x)$ and $\phi(x,s=1)=\phi(x)$ and define

$$\xi^\alpha = \frac{d}{ds}\phi^\alpha(x,s)\bigg|_{s=0}. \tag{A.4}$$

The $\xi^\alpha(x)$'s span the tangent space at $\phi_{clas.}(x)$ and satisfy the geodesic equation

$$\frac{D}{Ds}\xi^\alpha = \frac{d}{ds}\xi^\alpha + \Gamma^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma = 0, \tag{A.5}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}E^\alpha_a\left(\frac{\partial}{\partial\phi^\beta}E_{a\gamma} + \frac{\partial}{\partial\phi^\gamma}E_{a\beta}\right), \tag{A.6}$$

is the Christoffel symbol. In general the $\xi^\alpha(x)$'s don't form an orthonormal basis but we can define new vectors

$$\zeta^a = E^a_\alpha\xi^\alpha, \tag{A.7}$$

that span the tangent space at $\phi_{clas.}(x)$ and form an orthonormal basis. The inverse relation is given by

$$\xi^\alpha = E^\alpha_a\zeta^a, \tag{A.8}$$

where $E^\alpha_a(\phi)$ is the inverse vielbein defined by

$$E^a_\alpha E^\alpha_b = \delta_{ab}. \tag{A.9}$$

Note that in the $\{\zeta^a\}$ basis the metric is δ_{ab} and so there is no difference between upper and lower indices, while in the $\{\xi^\alpha\}$ basis the metric is $g_{\alpha\beta}=E^a_\alpha E_{a\beta}$ and so an upper index is different from a lower index. The action can then be expanded as

$$S[\phi(x,s=1)] = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{d}{ds}\right)^n S[\phi]\bigg|_{s=0} \equiv \sum_{n=0}^{\infty} S^{(n)}[\phi_{clas.},\zeta], \tag{A.10}$$

and keeping terms to second order in ζ's we have (from now on $\phi(x)$ stands for $\phi_{clas.}(x)$):

$$\begin{aligned}
S^{(0)}[\phi,\zeta] &= S[\phi],\\
S^{(1)}[\phi,\zeta] &= 0 \qquad \text{(if equations of motion hold)},\\
S^{(2)}[\phi,\zeta] &= \frac{N}{8\pi}\int d^2x\, \Big(\zeta^a(-\delta_{ab}\Box + A^\mu_{ab}\partial_\mu + D_{ab})\zeta^b\\
&\quad + \overline{\zeta}^a(-\delta_{ab}\Box + \overline{A}^\mu_{ab}\partial_\mu + \overline{D}_{ab})\overline{\zeta}^b\\
&\quad + \zeta^a(G_{ab}\Box + B^\mu_{ab}\partial_\mu + C_{ab})\overline{\zeta}^b\\
&\quad + \overline{\zeta}^a(G^T_{ab}\Box + \overline{B}^\mu_{ab}\partial_\mu + C^T_{ab})\zeta^b\Big),
\end{aligned} \tag{A.11}$$

where

$$\begin{aligned}
A^\mu &= 2f_a E^a_\alpha \partial^\mu \phi^\alpha, \\
\overline{A}^\mu &= 2f_a \overline{E}^a_\alpha \partial^\mu \overline{\phi}^\alpha, \\
B^\mu &= -2f_a G E^a_\alpha (\eta^{\mu\nu} - \varepsilon^{\mu\nu}) \partial_\nu \phi^\alpha, \\
\overline{B}^\mu &= -2f_a G^T \overline{E}^a_\alpha (\eta^{\mu\nu} + \varepsilon^{\mu\nu}) \partial_\nu \overline{\phi}^\alpha, \\
C &= f_a G f_b E^a_\alpha \overline{E}^b_\beta (\eta^{\mu\nu} + \varepsilon^{\mu\nu}) \partial_\mu \phi^\alpha \partial_\nu \overline{\phi}^\beta, \\
D &= -f_a f_b E^a_\alpha E^b_\beta \partial_\mu \phi^\alpha \partial^\mu \phi^\beta, \\
\overline{D} &= -f_a f_b \overline{E}^a_\alpha \overline{E}^b_\beta \partial_\mu \overline{\phi}^\alpha \partial^\mu \overline{\phi}^\beta,
\end{aligned} \tag{A.12}$$

and the matrix f_c is defined by $(f_c)_{ab} = f_{cab}$. To compute the divergent counter term, we write $S^{(2)}$ in the form

$$S^{(2)} = \frac{N}{8\pi} \int d^2x \; Z^T (R\Box + P^\mu \partial_\mu + Q) Z, \tag{A.13}$$

where

$$Z \equiv \begin{bmatrix} \zeta \\ \overline{\zeta} \end{bmatrix} \quad \text{and} \quad \zeta = \begin{bmatrix} \zeta^1 \\ \vdots \\ \zeta^n \end{bmatrix}, \overline{\zeta} = \begin{bmatrix} \overline{\zeta}^1 \\ \vdots \\ \overline{\zeta}^n \end{bmatrix}, \tag{A.14}$$

and the matrices R, P^μ and Q are

$$R = \begin{bmatrix} -I & G \\ G^T & -I \end{bmatrix}, \quad P^\mu = \begin{bmatrix} A^\mu & B^\mu \\ \overline{B}^\mu & \overline{A}^\mu \end{bmatrix}, \quad Q = \begin{bmatrix} D & C \\ C^T & \overline{D} \end{bmatrix}. \tag{A.15}$$

After integrating over Z, we get

$$\begin{aligned}
S^{(2)} &\cong -\frac{1}{2} \mathrm{Tr} \log(R\Box + P^\mu \partial_\mu + Q) \\
&\cong -\frac{1}{2} \mathrm{Tr} \left(R^{-1} \frac{1}{\Box} Q - \frac{1}{2} R^{-1} \frac{1}{\Box} P^\mu \partial_\mu R^{-1} \frac{1}{\Box} P^\nu \partial_\nu \right) \\
&\cong \int \frac{d^2p}{p^2 - m^2} \int d^2x \; O(x),
\end{aligned} \tag{A.16}$$

where $O(x)$ was defined in §2.1.

3. Integrability

The purpose of this section is investigate the conditions under which the GTM becomes integrable. By integrability, we mean the existence of an infinite number of conserved and commuting dynamical variables. Here, we will not address the question whether these are sufficient in number to make the model truly solvable. The bosonized form of the Thirring Model will be our starting point, and our treatment from that point on will be purely classical. It should, however, be noted that this is better than treating the original fermionic model classically, since bosonization does capture some of the quantum nature of the model. From the bosonized Lagrangian, we wish to extract a Lax pair depending on a spectral parameter. For this purpose, starting from the equations of motion written as flatness conditions for two vector fields (2.33), we demand the existence of another flat vector field that interpolates between these two. This results in equations involving the coupling constants which we call the first integrability condition. Recall that to obtain these flatness conditions, we used the invariance of the model under the transformation $h \to u_+(x_+)h$ to set J_+ equal to zero. Then, what we "prove" by finding an interpolating flat vector field, is the integrability of this special solution (with $J_+ = 0$), but the general solution is also integrable by virtue of the same transformation (the situation is similar to what happens in the WZW model [35]). In general, these equations are overdetermined and they do not have solutions depending on a continuous parameter. In §3.3, we discuss four exceptional cases when such a solution exists. For each of these examples, there exists a Lax pair depending on a spectral parameter.

Given the Lax pair, in a standard fashion, one can construct conserved quantities in terms of a path ordered product. It is, however, necessary to specify boundary conditions. The simplest boundary condition is the periodic one, with the space coordinate compactified into a circle. Taking the trace of the path ordered product and expanding in powers of the spectral parameter yields an infinite number of conserved "charges". In §3.4, we compute the Poisson Bracket's of these charges and derive the conditions so that it vanishes. This is then our second integrability condition. In the four examples discussed earlier, the second integrability condition is automatically satisfied, although we are unable to prove in general that the second condition follows from the first. Another natural boundary condition is the open one: The space interval is from $-\infty$ to $+\infty$ and the fields vanish at $\pm\infty$. In this case, additional integrals of motion can be constructed by considering the matrix elements of the path ordered product, instead of just the trace. In §3.5, we compute the Poisson Bracket's of these additional integrals of motion and find that they satisfy a non-linear algebra.

3.1. Lax Pair Formulation

We start by recalling here the equations of motion obtained before (eq. (2.33)):

$$\begin{aligned} \partial_+ V_- - \partial_- V_+ - i[V_+, V_-] &= 0, \\ \partial_+ W_- - \partial_- W_+ - i[W_+, W_-] &= 0. \end{aligned} \tag{3.1}$$

These flatness conditions for the vector fields V and W are similar to the zero curvature condition of integrable systems. What is missing is a spectral parameter dependence that will provide by power expansion an infinite number of conserved currents. We will now derive conditions for a zero curvature with a spectral parameter dependence (a Lax pair) to exist. This will be the first integrability condition. The idea is to find a one parameter family of matrices that interpolate between the equations of motion. To this end we define an interpolating connection as follows:

$$B_{\pm a}(x;\lambda) = N_{\pm ab}(\lambda) M_{\pm b}(x), \tag{3.2}$$

where $N_{\pm ab}$ are constants, to be determined as functions of the spectral parameter λ. The vector field B must satisfy the zero curvature condition

$$\partial_+ B_- - \partial_- B_+ - i[B_+, B_-] = 0, \tag{3.3}$$

with boundary conditions

$$\begin{aligned} B_{\pm a}(x;\lambda = \lambda_0) &= V_{\pm a}(x), \\ B_{\pm a}(x;\lambda = \lambda_1) &= W_{\pm a}(x). \end{aligned} \tag{3.4}$$

The zero curvature reads

$$N_{-ab}\partial_+ M_{-b} - N_{+ab}\partial_- M_{+b} + f_{abc} N_{+bp} N_{-cq} M_{+p} M_{-q} = 0. \tag{3.5}$$

and using equations (2.36) and equating the coefficients of $M_{+p}M_{-q}$ to zero we get the first integrability condition

$$N_{+ab} A_{+bpq} - N_{-ab} A_{-bqp} + f_{abc} N_{+bp} N_{-cq} = 0. \tag{3.6}$$

This condition gives us a Lax pair and an infinite number of resulting conservation laws (see §3.4). One must also show that these conserved quantities are mutually commuting; that is, their Poisson Bracket's vanish. This will be the second integrability condition, derived in §3.4.

Equations (3.6) are in general an overdetermined algebraic system with $(\dim G)^3$ equations for $2(\dim G)^2$ variables $N_{\pm ab}$. In fact, since these equations are nonlinear, this counting is misleading. What we have actually is a system of polynomial equations for the variables $N_{\pm ab}$. The locus of the polynomials defines an algebraic variety $\mathcal{M}$ and the condition of integrability is not that $\mathcal{M} \neq \{0\}$ (this is guaranteed since we always have two solutions when B is equal V or W) but that $\dim \mathcal{M} \geq 1$ in order to have a spectral parameter.

Although the system is overdetermined and there are no parametric solutions for a generic coupling constants G_{ab}, there are special interactions for which the model is integrable. This is the subject of the next section.

3.2. Solutions of the First Integrability Condition

In this section we will construct some solutions to the integrability condition, proving thus, the existence in those models, of an infinite number of conserved currents. In all examples we will make use of the diagonal ansatz. We assume that G and $N_\pm$ are diagonal matrices:

$$G_{ab} = g_a \delta_{ab},$$
$$N_{\pm ab} = n_a^{\pm} \delta_{ab},$$

Then $A_{\pm abc} = A_{abc}$,

$$A_{abc} = \frac{g_a g_b - g_c}{(1-g_a^2)^{\frac{1}{2}}(1-g_b^2)^{\frac{1}{2}}(1-g_c^2)^{\frac{1}{2}}} f_{abc}, \tag{3.7}$$

and the integrability condition (3.6) reduces to

$$n_a^+ A_{abc} - n_a^- A_{acb} + f_{abc} n_b^+ n_c^- = 0,$$

or, using (3.7),

$$(n_a^+ \frac{g_a g_b - g_c}{\sqrt{(1-g_a^2)(1-g_b^2)(1-g_c^2)}} + n_a^- \frac{g_a g_c - g_b}{\sqrt{(1-g_a^2)(1-g_b^2)(1-g_c^2)}} + n_b^+ n_c^-) f_{abc} = 0. \tag{3.8}$$

Note that there is no implied sum here.

Example 1: $G = g\mathbf{1}$. (The Symmetric Case).

In this case, $N_\pm = n^\pm \mathbf{1}$, and we have one equation for two variables,

$$n^+ \frac{g^2 - g}{(1-g^2)^{\frac{3}{2}}} + n^- \frac{g^2 - g}{(1-g^2)^{\frac{3}{2}}} + n^+ n^- = 0, \tag{3.9}$$

with a one parameter family of solutions.

Example 2: $G = g_1\mathbf{1}\otimes g_2\mathbf{1}$.

This just gives two copies of the above equation (3.9).

Example 3: $SU(2)$ with $U(1)$ symmetry. $G_{ab} = g_a\delta_{ab}$ with $g_1 = g_2 \neq g_3$.

This suggests the ansatz, $N_{\pm ab} = n_a^{\pm}\delta_{ab}$ with $n_1^{\pm} = n_2^{\pm} \neq n_3^{\pm}$. Then there are three equations for four variables,

$$\begin{aligned} n_1^+(g_1^2 - g_3) + n_1^-(g_1g_3 - g_1) + (1-g_1^2)(1-g_3^2)^{\frac{1}{2}}n_1^+n_3^- &= 0, \\ n_3^+(g_3g_1 - g_1) + n_3^-(g_3g_1 - g_1) + (1-g_1^2)(1-g_3^2)^{\frac{1}{2}}n_1^+n_1^- &= 0, \\ n_1^+(g_1g_3 - g_1) + n_1^-(g_1^2 - g_3) + (1-g_1^2)(1-g_3^2)^{\frac{1}{2}}n_3^+n_1^- &= 0. \end{aligned} \tag{3.10}$$

The one parameter family of solutions is given by,

$$\begin{aligned} n_1^+ &= \left(\frac{g_1}{\lambda(1+g_3)(1-g_1^2)^2}\left(2(g_3-g_1^2) + \frac{1}{\lambda}g_1(1-g_3) + g_1(1-g_3)\lambda\right)\right)^{\frac{1}{2}}, \\ n_1^- &= \left(\frac{g_1\lambda}{(1+g_3)(1-g_1^2)^2}\left(2(g_3-g_1^2) + \frac{1}{\lambda}g_1(1-g_3) + g_1(1-g_3)\lambda\right)\right)^{\frac{1}{2}}, \\ n_3^+ &= \frac{1}{(1-g_3^2)^{\frac{1}{2}}(1-g_1^2)}\left((g_3-g_1^2) + \frac{1}{\lambda}g_1(1-g_3)\right), \\ n_3^- &= \frac{1}{(1-g_3^2)^{\frac{1}{2}}(1-g_1^2)}\left((g_3-g_1^2) + \lambda g_1(1-g_3)\right), \end{aligned} \tag{3.11}$$

where $\lambda = \frac{n_1^-}{n_1^+}$ is the free parameter with range $\frac{1}{g_1} \leq \lambda \leq g_1$. Note that the case where we also have $g_2 \neq g_1$ gives six equations for six variables.

Example 4: Symmetric spaces.

Let F be a simple group with a subgroup H. Then the Lie algebra $\mathcal{F}$ can be decomposed into the Lie algebra $\mathcal{H}$ and its orthogonal complement $\mathcal{K}$, which generates the coset F/H. This coset space is a symmetric space if $[\mathcal{K},\mathcal{K}] \subset \mathcal{H}$. In what follows we will label the generators of H with greek indices (i.e. t_α), and the generators of F/H with latin indices (i.e. t_a), and when we don't want to specify between them, we'll use dotted latin indices (i.e. $t_{\dot{a}}$).

To have a coset (symmetric) space in the present model we choose

$$G_{\dot{a}\dot{b}} = g_{\dot{a}}\delta_{\dot{a}\dot{b}},$$

with $g_a = g$ and $g_\alpha = 1$. This assures that the currents in $\mathcal{H}$ are set to zero, resulting in a coset model. We cannot use (3.6) because $A_{\pm abc}$ is not defined in this case ($H_\pm$ is not invertible), but instead, we have to use (3.1) and (3.3):

$$\begin{aligned}
\partial_+ V_{-\dot a} - \partial_- V_{+\dot a} + f_{\dot a\dot b\dot c} V_{+\dot b} V_{-\dot c} = 0,\\
\partial_+ W_{-\dot a} - \partial_- W_{+\dot a} + f_{\dot a\dot b\dot c} W_{+\dot b} W_{-\dot c} = 0,\\
\partial_+ B_{-\dot a} - \partial_- B_{+\dot a} + f_{\dot a\dot b\dot c} B_{+\dot b} B_{-\dot c} = 0.
\end{aligned} \tag{3.12}$$

The choice of G leads to the ansatz:

$$B_{\pm\dot a}(x;\lambda) = r^\pm_{\dot a}(\lambda) V_{\pm\dot a}(x),$$

with $r^\pm_a(\lambda) = r^\pm(\lambda)$ and $r^\pm_\alpha(\lambda) = 1$. Then, we can rewrite (3.12) as

$$\begin{aligned}
(1 - g_{\dot a}^2)\partial_+ V_{-\dot a} + f_{\dot a\dot b\dot c}(1 - \frac{g_{\dot a} g_{\dot c}}{g_{\dot b}}) V_{+\dot b} V_{-\dot c} = 0,\\
(1 - \frac{1}{g_{\dot a}^2})\partial_- V_{+\dot a} + f_{\dot a\dot b\dot c}(1 - \frac{g_{\dot c}}{g_{\dot a} g_{\dot b}}) V_{+\dot b} V_{-\dot c} = 0,\\
r^-_{\dot a}\partial_+ V_{-\dot a} - r^+_{\dot a}\partial_- V_{+\dot a} + f_{\dot a\dot b\dot c} r^+_{\dot b} r^-_{\dot c} V_{+\dot b} V_{-\dot c} = 0.
\end{aligned} \tag{3.13}$$

Note that $\dot a$ is fixed (no implied sum). Now if $\dot a$ is in $\mathcal{H}$, then the first two equations in (3.12) are the same, and from the third one we get the condition $r^+(\lambda) r^-(\lambda) = 1$. The fact that the first two equations become the same is a sign of gauge invariance. To fix a gauge we choose $V_{-\alpha} = 0$. Then in this gauge, if $\dot a$ is in $\mathcal{K}$, the first two equations in (3.13) solve the third one without further conditions. Hence, for symmetric spaces, we get one equation for two variables

$$r^+(\lambda) r^-(\lambda) = 1. \tag{3.14}$$

3.3. The Poisson Bracket - Periodic Boundary Condition

Having a Lax pair at hand, we can construct conserved quantities in the time variable x_+, if we also impose periodicity in the space coordinate x_-. We first define the following quantity:

$$\mathcal{U}(x_+, x_-;\lambda) = P e^{-i\int_{x_-}^{x_-} dx'_-\, B_-(x_+, x'_-;\lambda)}, \tag{3.15}$$

and take $B_\pm$ periodic in x_- with period 2π. The integral goes between x_- and $x_- + 2\pi \sim x_-$. This quantity satisfies the equation

$$\partial_- \mathcal{U} - i[B_-, \mathcal{U}] = 0, \tag{3.16}$$

which can be taken as the definition [4] of path ordering in (3.15). More importantly, the trace of this matrix, $U(\lambda) = \text{Tr}\,\mathcal{U}(x_+, x_-; \lambda)$, is conserved:

$$\begin{aligned}
\partial_+ U &= i \int_{x_-}^{x_-} dx'_- \,\text{Tr}(\partial_+ B_- \mathcal{U}) \\
(\text{Bianchi id.}) &= i \int_{x_-}^{x_-} dx'_- \,\text{Tr}\big((\partial_- B_+ + i[B_+, B_-])\mathcal{U}\big) \\
(\text{integration by parts}) &= -i \int_{x_-}^{x_-} dx'_- \,\text{Tr}\big(B_+(\partial_- \mathcal{U} - i[B_-, \mathcal{U}])\big) \\
(\text{by (3.16)}) &= 0.
\end{aligned} \tag{3.17}$$

We see that U is a conserved quantity and upon expanding it in powers of λ we get infinite number of conserved currents. Since U does not depend on x_+, we drop its dependence from U as well as the subindex $-$ from x_-. The next step is to find the algebra of the conserved currents. In particular if they all commute then we have infinite number of conserved quantities in involution, which is the trademark of integrability. We want to calculate the Poisson Bracket's of $U(\lambda)$ with $U(\mu)$:

$$\begin{aligned}
\{U(\lambda), U(\mu)\} &= \int dx dy \, \frac{\delta U(\lambda)}{\delta B_{-a}(x;\lambda)} \frac{\delta U(\mu)}{\delta B_{-b}(y;\mu)} \{B_{-a}(x;\lambda), B_{-b}(y;\mu)\} \\
&= -\int dx dy \, U_a(x;\lambda) U_b(y;\mu) \{B_{-a}(x;\lambda), B_{-b}(y;\mu)\} \\
&= -\int dx dy \, U_a(x;\lambda) U_b(y;\mu) N_{-ac}(\lambda) N_{-bd}(\mu) \{M_{-c}(x), M_{-d}(y)\}.
\end{aligned}$$

The Poisson Bracket's for the M_-'s where given in (1.9) and include a non-local term. To get a local algebra for the Poisson Bracket's for our conserved fields we need the coefficient of $\epsilon(x-y)$ to be a derivative such that by integration by parts the derivative acts on ϵ to give a δ function. For a generic coupling this cannot happen, but exactly for the models that satisfy the integrability condition this is

[4] This is the same as defining the path ordered exponential with the "upper" limit fixed, i.e., $Pe^{\int_a^b dx\, A(x)} = 1 + \int_a^b dx\, A(x) + \int_a^b dx \int_x^b dx'\, A(x)A(x') + \cdots$. Note that this is the opposite of the usual definition where the "lower" limit is fixed.

true!

$$
\begin{aligned}
&\{U(\lambda),U(\mu)\}_{non-local} = \\
&= \frac{2\pi}{N}\int dxdy\, U_a(x;\lambda)U_b(y;\mu)N_{-ac}(\lambda)N_{-bd}(\mu)\epsilon(x-y)\times \\
&\qquad A_{-ckr}A_{-dls}\big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}(x')}\big)_{rs}M_{-k}(x)M_{-l}(y) \\
&= \frac{2\pi}{N}\int dx\, U_a(x;\lambda)N_{-ac}(\lambda)A_{+ckr}M_{-k}(x)\times \\
&\qquad \int dy\, \epsilon(x-y)U_b(y;\mu)(N_{-bd}(\mu)A_{-dls})\big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}(x')}\big)_{rs}M_{-l}(y) \\
&= \frac{2\pi}{N}\int dx\, U_a(x;\lambda)N_{-ac}(\lambda)A_{-ckr}M_{-k}(x)\times \\
&\qquad \int dy\, \epsilon(x-y)U_b(y;\mu)\left(N_{+bd}(\mu)A_{+dsl}+f_{btu}N_{+ts}(\mu)N_{-ul}(\mu)\right)\times \\
&\qquad \big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}(x')}\big)_{rs}M_{-l}(y),
\end{aligned}
$$

where the last step follows from the integrability condition, (3.6). Using the following identities,

$$
\begin{aligned}
&\partial_y U_t(y;\mu) = -f_{btu}B_{-u}(y;\mu)U_b(y;\mu) = -f_{btu}N_{-ul}(\mu)M_{-l}(y)U_b(y;\mu), \\
&\partial_y\big(Pe^{\int_x^y dx'\, A_{+k}M_{-k}(x')}\big)_{rd} = -A_{+dsl}M_{-l}(y)\big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}(x')}\big)_{rs},
\end{aligned}
\tag{3.18}
$$

we can write now the integrand of the y integral as a total derivative

$$
\begin{aligned}
&U_b(y;\mu)\left(N_{+bd}(\mu)A_{+dsl}+f_{btu}N_{+ts}(\mu)N_{-ul}(\mu)\right)\times \\
&\quad \big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}(x')}\big)_{rs}M_{-l}(y) = -\partial_y\Big(U_b(y;\mu)N_{+bd}(\mu)\big(Pe^{\int_x^y dx'\, A_{+p}M_{-p}}\big)_{rd}\Big).
\end{aligned}
$$

The contribution of the non-local piece is therefore

$$
\begin{aligned}
&\{U(\lambda),U(\mu)\}_{non-local} = \frac{4\pi}{N}\int dx\, U_a(x;\lambda)U_b(x;\mu)N_{-ac}(\lambda)N_{-bd}(\mu)A_{-cld}M_{-l}(x) \\
&= \frac{2\pi}{N}\int dx\, U_a(x;\lambda)U_b(x;\mu)\times \\
&\quad \left(N_{-ac}(\lambda)N_{+bd}(\mu)A_{-cld}-N_{+ac}(\lambda)N_{-bd}(\mu)A_{-dlcl}\right)M_{-l}(x),
\end{aligned}
$$

where in the second step we (anti)symmetrized the right hand side so that the change in sign under $\lambda\leftrightarrow\mu$ is manifest. The contributions from the local terms are easily evaluated:

$$
\{U(\lambda),U(\mu)\}_{\delta} = -\frac{4\pi}{N}\int dx\, U_a(x;\lambda)U_b(x;\mu)N_{-ac}(\lambda)N_{+bd}(\mu)F_{-cdl}M_{-l}(x),
$$

and

$$\{U(\lambda), U(\mu)\}_{\delta'} = \frac{2\pi}{N} \int dx\, U_a(x;\lambda) U_b(x;\mu) \times$$
$$\left(f_{ars} N_{-sl}(\lambda) N_{-rc}(\lambda) N_{-bc}(\mu) - f_{brs} N_{-sl}(\mu) N_{-ac}(\lambda) N_{-rc}(\mu)\right) M_{-l}(x).$$

Putting everything together we get

$$\{U(\lambda), U(\mu)\} = \int dx\, U_a(x;\lambda) U_b(x;\mu) J_{abl}(\lambda,\mu) M_{-l}(x), \tag{3.19}$$

where

$$\begin{aligned} J_{abl}(\lambda,\mu) = {} & N_{-ac}(\lambda) N_{+bd}(\mu) A_{-cld} - N_{+ac}(\lambda) N_{-bd}(\mu) A_{-dlc} \\ & - 2N_{-ac}(\lambda) N_{-bd}(\mu) F_{-cdl} \\ & + f_{ars} N_{-sl}(\lambda) N_{-rc}(\lambda) N_{-bc}(\mu) - f_{brs} N_{-sl}(\mu) N_{-ac}(\lambda) N_{-rc}(\mu). \end{aligned} \tag{3.20}$$

The right hand side of equation (3.19) is zero, if the integrand is a total derivative. This motivates us to try the ansatz

$$\partial_x \left(U_a(x;\lambda) U_b(x;\mu) C_{ab}(\lambda,\mu)\right) = U_a(x;\lambda) U_b(x;\mu) J_{abl}(\lambda,\mu) M_{-l}(x). \tag{3.21}$$

The left hand side of this equation can be computed using the identities of (3.18). Note that there is no relation between this C_{ab} and the one defined in section 2. This results in the second integrability condition

$$J_{abl}(\lambda,\mu) = f_{acd} N_{-dl}(\lambda) C_{cb}(\lambda,\mu) + f_{bcd} N_{-dl}(\mu) C_{ac}(\lambda,\mu). \tag{3.22}$$

For models for which a C_{ab} that satisfies this equation can be found, we have then $\{U(\lambda), U(\mu)\} = 0$, and the conserved quantities are in involution. In the diagonal ansatz, C has the form

$$C_{ab}(\lambda,\mu) = C_a(\lambda,\mu)\delta_{ab}.$$

We will now check the models that satisfy the first integrability condition against this second integrability condition. In all examples we have, the result is the same: The second integrability condition is automatically satisfied. However, we do not know whether in general the second condition follows from the first one.

In example 1, the symmetric model, we take $C_a(\lambda,\mu) = C(\lambda,\mu)$, and then we have one equation for one unknown, $C(\lambda,\mu)$. The condition (3.22) is satisfied, and the conserved quantities are commutative. The generalization to example 2 is trivial. In case of example 3, $SU(2)$ with $C_{ab}(\lambda,\mu) = C_a(\lambda,\mu)\delta_{ab}$ and with $C_1(\lambda,\mu) = C_2(\lambda,\mu) \neq C_3(\lambda,\mu)$, initially we have three equations for two unknowns, but using the fact that $C_a(\lambda,\mu) = -C_a(\mu,\lambda)$ we see that there are only two equations. So, in this case also, equation (3.22) can be solved, and the model is integrable. We have not checked example 4 in detail, but we suspect that the result is the same.

3.4. The Poisson Bracket - Open Boundary Condition

In this section we will show that the structure of the model is richer with open boundary conditions. We consider the open interval from $-\infty$ to $+\infty$, with fields vanishing at $\pm\infty$. With these conditions, it is possible to construct more integrals of motion than with the periodic one. These integrals do not commute in general and it is the aim of this section to calculate the algebra they generate. A subset of these integrals are the integrals we saw in the periodic case, and they commute, insuring, thus, the integrability of the model.

The calculation of the Poisson Bracket's is the same as the periodic case, the only difference will come from the boundary. Defining

$$\mathcal{U}_\lambda(x,y) = Pe^{-i\int_x^y dx'\, B_-(x';\lambda)}, \tag{3.23}$$

then, any matrix element $U_{ab}(\lambda) = \mathcal{U}_{\lambda ab}(-\infty,\infty)$, and not just the trace, is conserved, provided that $B_+(\pm\infty,\lambda) = 0$. This follows from manipulations similar to (3.17), noticing that the boundary terms coming from $\pm\infty$ can be dropped. Of course, B_- also must vanish sufficiently fast at $\pm\infty$ for the integral to exist.

Define now

$$U_{abc}(x;\lambda) = \left(\mathcal{U}_\lambda(-\infty,x)t_c\mathcal{U}_\lambda(x,\infty)\right)_{ab}, \tag{3.24}$$

then

$$\begin{aligned}\{U_{aa'}(\lambda),U_{bb'}(\mu)\} &= \int dxdy\,\frac{\delta U_{aa'}(\lambda)}{\delta B_{-c}(x;\lambda)}\frac{\delta U_{bb'}(\mu)}{\delta B_{-d}(y;\mu)}\{B_{-c}(x;\lambda),B_{-d}(y;\mu)\}\\ &= -\int dxdy\, U_{aa'c}(x;\lambda)U_{bb'd}(y;\mu)N_{-ck}(\lambda)N_{-dl}(\mu)\{M_{-k}(x),M_{-l}(y)\}.\end{aligned} \tag{3.25}$$

Being careful to take into account the boundary contribution from the non-local term in the Poisson Bracket's of the M_-'s, we get

$$\begin{aligned}\{U_{aa'}(\lambda),U_{bb'}(\mu)\} = \frac{2\pi}{N}&\int dx\, U_{aa'c}(x;\lambda)U_{bb'd}(x;\mu)J_{cde}(\lambda,\mu)M_{-e}(x)\\ &+\frac{2\pi}{N}\left(U(\lambda)t_c\right)_{aa'}\left(t_dU(\mu)\right)_{bb'}U_{lk}(0)N_{-ck}(\lambda)N_{-dl}(\mu)\\ &-\frac{2\pi}{N}\left(t_cU(\lambda)\right)_{aa'}\left(U(\mu)t_d\right)_{bb'}U_{kl}(0)N_{-ck}(\lambda)N_{-dl}(\mu),\end{aligned} \tag{3.26}$$

where J_{abc} is defined in (3.20), and $U_{ab}(0) = U_{ab}(\lambda = \lambda_0)$. If J_{abc} also satisfies the second integrability condition (3.22) then the integrand can be written as a derivative. The only contributions come from the boundary and the final result is

$$\begin{aligned}\{U_{aa'}(\lambda), U_{bb'}(\mu)\} = \frac{2\pi}{N}\Big(&\big(U(\lambda)t_c\big)_{aa'}\big(U(\mu)t_d\big)_{bb'}C_{cd}(\lambda,\mu) \\ &-\big(t_cU(\lambda)\big)_{aa'}\big(t_dU(\mu)\big)_{bb'}C_{cd}(\lambda,\mu) \\ &+\big(U(\lambda)t_c\big)_{aa'}\big(t_dU(\mu)\big)_{bb'}U_{lk}(0)N_{-ck}(\lambda)N_{-dl}(\mu) \\ &-\big(t_cU(\lambda)\big)_{aa'}\big(U(\mu)t_d\big)_{bb'}U_{kl}(0)N_{-ck}(\lambda)N_{-dl}(\mu)\Big).\end{aligned} \tag{3.27}$$

This algebra, in the symmetric case, can be compared with the algebra obtained without bosonizing the fermions obtained in [20]. Both the appearance of cubic terms and the more complicated spectral dependence of the "classical" r-matrix $C_{ab}(\lambda, \mu)$ are quantum effects due to the process of bosonization.

Equation (3.27) exhibits a closed algebra for the matrix elements $U_{ab}(\lambda)$; N_-'s and C's are the corresponding "structure constants" that depend on the particular model under consideration. The algebra is clearly non-linear, with quadratic and cubic terms in U appearing on the right hand side. We have not succeeded in identifying it with any well-known algebra, although it may possibly bear some relation to the W algebras [36]. It is also the generalization of the algebra Lüscher and Pohlmeyer [18] found for the special case of the fundamental representation of $SU(2)$. Finally we would like to comment on the relation of this algebra to the affine algebra of currents found in the principal chiral model [15]. In the symmetric case (Example 1 of §3.3), the Thirring model bears a great resemblance to the principal chiral model; for example, the equations of motion (3.1) in this case reduce to

$$\begin{aligned} g\partial_+ V_- + \partial_- V_+ &= 0, \\ \partial_+ V_- - \partial_- V_+ - i[V_+, V_-] &= 0. \end{aligned} \tag{3.28}$$

For $g = 1$, these are identical to the equations of motion of the principal chiral model. Not surprisingly, the Lax pair and the conserved quantities are in one to one correspondence, with obvious modifications in case $g \neq 1$. However, the algebra (3.27) is quite different from the affine algebra found in [15]. This is because in [15], the Poisson Bracket's of the conserved quantities were derived from the transformations they generate on the field variables, whereas we have used the standard Poisson structure given by the Lagrangian. The two Poisson structures differ in this case, as well as in the principal chiral model [37].

4. Free Field Realization

In this section, we will express the fields $M^{(n)}_{-a}(x)$ in terms of free fields $\phi_a(x)$'s so that the Poisson Bracket's in the classical case (eq. (1.9)), or the OPE in the quantum case (eq. (2.27)), between two $M_{-a}(x)$'s is still satisfied. (These ϕ's are not to be confused with the ϕ's introduced in section 2). As in section 2, the calculations will be carried only to second order in α. Our motivation for doing this is twofold: first of all, one may ask whether the relatively complicated appearance of the OPE's given by eq. (2.27) is due to our choice of fields; with a different choice of fields, a simpler algebra might emerge. Indeed, we show that one can express everything in terms of free fields; however, the simplification achieved in this way is somewhat illusory, since the expressions connecting M_-'s to free fields are non-local and complicated. Next, we reexpress the stress tensor in terms of free fields, hoping for a simple result. Indeed, the stress tensor turns out to be local and quadratic in free fields; on the other hand, an unusual term involving the second derivative of the fields makes its appearance. This term is responsible for the deviation of the central charge from the free field value and cannot be eliminated.

We start with the classical M_- fields and try to express them in terms of $\phi_a(x)$'s that satisfy the free field Poisson Bracket relations,

$$\{\phi_a(x), \phi_b(y)\} = -\log(x-y)\delta_{ab}. \tag{4.1}$$

The solution to zeroth order ($M^{(0)}_{-a}(x)$) is obvious, and the next two orders are easily constructed by guesswork. The result is,

$$\begin{aligned} M^{(0)}_{-a}(x) &= \phi'_a(x), \\ M^{(1)}_{a}(x) &= \frac{1}{3}F_{-abc}\phi'_b(x)\phi_c(x), \\ M^{(2)}_{a}(x) &= -\frac{1}{36}(F_{-ace}F_{-bde} + F_{-ade}F_{-bce})\phi'_b(x)\phi_c(x)\phi_d(x) \\ &\quad + \frac{1}{36}(F_{-abe}F_{-cde} + 9E_{-ac,bd}\phi'_b(x)\int^x dy\, \phi_c(y)\phi'_d(y). \end{aligned} \tag{4.2}$$

This can easily be extended to operators. Define now quantum free fields by OPE's

$$\phi_a(x)\phi_b(y) \cong -\frac{1}{2\pi}\log(x-y)\delta_{ab}. \tag{4.3}$$

To avoid singular expressions we work with normal ordered fields, for example,

$$\begin{aligned} \phi_a(x)\phi_b(y)\phi_c(z) = {:}\phi_a(x)\phi_b(y)\phi_c(z){:} &- \frac{1}{2\pi}\phi_c(z)\log(x-y)\delta_{ab} \\ - \frac{1}{2\pi}\phi_b(y)\log(x-z)\delta_{ac} &- \frac{1}{2\pi}\phi_a(x)\log(y-z)\delta_{bc}. \end{aligned} \tag{4.4}$$

In order to satisfy the OPE algebra given before (eq. (2.27)), we simply take over the classical expression, replacing products of fields by normal ordered products. It turns out that this almost works; however, additional terms are necessary to make it work. The final result is

$$
\begin{aligned}
M_{-a}^{(0)}(x) &= \phi_a'(x), \\
M_{-a}^{(1)}(x) &= -\frac{1}{3}F_{-abc}{:}\phi_b'(x)\phi_c(x){:}, \\
M_{-a}^{(2)}(x) &= -\frac{1}{36\pi}(2F_{-acd}F_{-bcd} + 9E_{-ab,cc})\phi_b'(x) \\
&\quad + \frac{1}{36\pi}(F_{-acd}F_{-bcd} + 9E_{-ab,cc})\int^x \frac{dy}{y-x}(\phi_b'(y) - \phi_b'(x)) \\
&\quad - \frac{1}{36}(F_{-ace}F_{-bde} + F_{-ade}F_{-bce}){:}\phi_b'(x)\phi_c(x)\phi_d(x){:} \\
&\quad + \frac{1}{36}(F_{-abe}F_{-cde} + 4E_{-ac,bd}){:}\phi_b'(x)\int^x dy\,\phi_c(y)\phi_d'(y){:}.
\end{aligned}
\tag{4.5}
$$

Using these expressions we can construct the stress tensor, which to second order, is quadratic in the free fields and is given by

$$
\begin{aligned}
\frac{\alpha^2}{\pi}T_-(x) &= {:}\phi_a'(x)\phi_a'(x){:} \\
&\quad - \frac{\alpha^2}{24\pi}(F_{-acd}F_{-bcd} + 3E_{-bc,ac})({:}\phi_a'(x)\phi_b'(x){:} + {:}\phi_a(x)\phi_b''(x){:}).
\end{aligned}
\tag{4.6}
$$

It can also be directly checked that, at least to second order, this construction in terms of free fields yields the Virasoro algebra with the correct central charge.

We could also have expressed the M_-'s in terms of currents that satisfy an affine Lie algebra; in fact, with minor modifications, (4.2) and (4.5) still hold if the $\phi_a'(x)$'s are replaced by currents. Again, the stress tensor is quadratic in the currents, as in (4.6), which looks promising for an affine Sugawara construction. But again there appears the analogue of the last term in (4.6), which, expressed in terms of the currents $J_a(x)$, looks like

$$
J_a'(x)\int^x dy\, J_a(y)
$$

and clearly does not belong in the affine Sugawara construction.

5. Discussion

We have studied the conformal invariance and integrability of the GTM, and the structure of its Poisson Bracket algebra.

The conditions that the coupling constants have to satisfy to have conformal invariance are given by (2.42) (or (2.17)). This result, valid in the large N limit, was obtained by using two different approaches, the background field method and the operator method, and agrees with the result obtained in [31] by a still different method. Among the problems that are still left open is the contribution of the higher order corrections in $1/N$ to both the condition for conformal invariance (eq. (2.17)), and to the operator algebra (eq. (2.27)). These related problems are in principle easy to answer, but the lengthiness of the calculations involved have prevented progress in that direction.

We also derived very general conditions (eqs. (3.6) and (3.22) and (3.20)) that the coupling constants should satisfy to have integrability. By that we meant the existence of an infinite number of conserved and commuting dynamical variables, but we never discussed whether this was enough to make the model really solvable. At this point it is worth mentioning some similar issues discussed in [21], where Poisson Bracket's between non-local charges, resembling the ones discussed in section 3, were also evaluated. There, it is argued that the existence of infinitely many conserved charges in involution, don't necessarily led to integrability, and it is suggested that classical non-local charges are not crucial for integrability. This is still an open question.

We have also tried to shed some light on the structure of the operator algebra mentioned above by expressing it in terms of free fields and free currents. We have found some simplification in the expression for the stress tensor, but still the result could not be reproduced by any well-known construction. It appears very likely that we have a completely new conformal model

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II. String Field Equations from Generalized Sigma Model

1. Introduction

A satisfactory formulation of string field theory continues to be one of the important open problems of string theory. There have been two main lines of approach to this problem in the past which have enjoyed varying degrees of success. The first approach (see [1–6]) starts with the BRST formalism, developed first in the context of free strings, and generalizes it to interacting strings. This approach was first successfully applied to the open string theory, and, making use of the extension of the BRST method due to Batalin and Vilkovisky [7], it was later generalized to include closed strings [5]. The great advantage of this approach is that, compared to the alternatives, it is the most developed one from the technical stand point and its correctness is beyond doubt. However, so far it has not led to any substantial advances in our understanding of string theory. This is no doubt due in part to the complexity of this method, but also, it is due to the fact that initially a fixed background has to be specified. Although there are proofs of background independence [5], to our knowledge, there is no manifestly background independent formulation. A great advantage of such a formulation would be its role in unmasking various possible hidden symmetries of string theory. From the very beginning of string theory, symmetries like invariance under the local transformations of the target space coordinates, connected with the existence of the graviton, were difficult to understand in the usual formulation that uses a flat background metric. A manifestly background invariant formulation of string theory should improve our understanding of these symmetries, and also possibly shed light on the recently discovered symmetries such as duality.

The second main line of approach (see [8], [9,10] and [11–17]) to understanding string dynamics starts with a two dimensional sigma model defined on the world sheet. In the earlier versions, the field content was restricted to massless excitations of the closed bosonic string, namely the graviton, the antisymmetric tensor and the dilaton [8]. The massive modes were neglected in order to have a renormalizable and classically scale invariant theory. The field equations are then derived by imposing quantum scale invariance, which amounts to demanding that the beta function vanish. In practice, one usually works with the beta function computed in the one loop approximation, using the background field method which preserves manifest covariance under local field redefinitions. This approach has many advantages over the first one: The background independence and the symmetries are made manifest, and the connection between conformal invariance and renormalization is made clear. There are also serious drawbacks: The massive modes are neglected and an off-shell formulation that goes beyond the equations of motion is missing. There is also the question about what happens beyond one loop; in many important cases the higher loop contributions lead to field redefinitions without changing the content of the equations of motion, although it is not clear how general this result is [17].

An important variation of this approach, which seems to overcome many of the drawbacks mentioned above, is originally due to Banks and Martinec [9]. The basic idea is to apply the renormalization group equations of Wilson and Polchinski to the two dimensional sigma model. The starting point is the most general sigma model, which includes all the massive levels of the string and is non-renormalizable in the conventional sense. A cutoff is introduced to make the model well-defined, and the equations of motion satisfied by the string fields are obtained by requiring the resulting partition function to be scale invariant. Although the classical action is scale non-invariant, and the cutoff introduces further scale breaking, the cancellation between these two effects makes the final scale invariance possible. Hughes, Liu and Polchinski [10] refined and extended this method, and they showed that the closed bosonic string scattering amplitudes in the classical (tree) limit can be derived from these equations. This approach has many nice features: It treats the whole string all at once and not just the massless levels, and it is also apparently exact and not limited to the one loop approximation. Finally, the emergence of the string amplitudes as a solution provides a stringent check on the resulting equations. Nevertheless, this approach also has some unsatisfactory features. As already noticed in [10], the equations do not seem powerful enough to eliminate all the unwanted states of the string spectrum; some additional gauge invariance needed to eliminate them is apparently missing. Another drawback is that the coordinate system in the field space is fixed right from the beginning, and as a result, covariance under field transformations, which was such an attractive feature of [8], is lost. We suspect that these problems are connected, and we offer some evidence in support of it.

Here, we propose a new approach which combines some of the advantageous features of both the renormalization group method and the covariant beta function treatment of the massless excitations. The rest of the thesis is organized as follows: In section 2 we derive our version of the renormalization group equations written in covariant form. We start with a general sigma model that is supposed to represent all the levels of the closed bosonic string, with flat metric on the worldsheet. The functional integral is written in the presence of a general background in a form completely covariant under local and non-local coordinate (field) transformations, subject only to the condition that the determinant of the transformation is unity. The field equations are then obtained by requiring invariance of the effective action under the conformal (Virasoro) group. In section 3, we apply the formalism just developed to the lowest levels of a closed, unoriented bosonic string. In §3.1 we study the tachyon and the massless level, while in §3.2 and §3.3 we study the first massive level in the non-covariant and covariant cases, respectively. In our formalism, in the tachyon and massless levels we only impose local coordinate invariance. Then, this corresponds to the familiar renormalizable sigma model [8], and our results should agree with the standard ones. This is indeed the case. When studying the first massive level, we need to consider also non-local transformations. In §3.2 we restrict ourselves to a non-covariant approach, and we find that the level structure we obtain doesn't agree with the known structure of the first massive level of the string. Like in [10], we find that there are too many states, and there is not enough gauge invariance to eliminate the spurious states. Only when we consider a covariant approach, in §3.3, do we get an agreement. However we are only able to obtain that, when the model is left-right symmetric. This difficulty, which was an unfortunate feature of our approach, was resolved by Bardakci while this thesis was being written. We conclude with a discussion of the results.

In our opinion, the main contribution of the work presented here, is that, at least in the context of a natural expansion, the field equations of motion that follow from the general sigma model can be made covariant under not only local, but also non-local transformations in the field space. Furthermore, this covariance is crucial in eliminating spurious states of the first massive level.

2. Covariant Renormalization Group Equations

In this section, we derive a set of renormalization group equations for a general sigma model in a classical background. These equations are derived by imposing conformal invariance on the sigma model in the presence of a background field; they are covariant generalizations of the string equations of motion derived in [10]. Throughout, we also work with flat worldsheet. We found the renormalization group approach of [10] advantageous for the following reason: When the generalized sigma model action S contains all the levels of the string and not just the massless ones, one is dealing with a conventionally non-renormalizable theory. In their approach, the conventionally non-renormalizable interactions coming from massive states, as well as the superrenormalizable interaction resulting from the tachyon, are treated on equal footing with the renormalizable interactions of the massless states. However, there are some problems with this approach. One of them is lack of covariance under the transformation of the target space coordinates. For example, the equations derived in [10] had a flat background; as a result, they were not explicitly covariant even under the usual coordinate transformations (local coordinate invariance) associated with gravity. Also, as pointed out by them, the equations do not seem strong enough to eliminate the states that are absent from the string spectrum. We will overcome both of these problems, at least for the first massive level, by combining the renormalization group approach with the traditional background field approach (see, for example [18]). Our approach will ensure covariance under not only local but also arbitrary non-local coordinate transformations, and by both considering a non-renormalizable action and also non-local coordinate transformations which mix up levels with different masses, the traditional treatment [8] will be extended to include massive levels of the string.

Our starting point is the partition function

$$Z[X_o, \Lambda] = \int [DX]\, e^{S'[X,\Lambda]}. \tag{2.1}$$

We will specify X_o and S' in terms of X and the action S shortly. The action S, which is a functional of the string coordinate[1] $X^{\mu\sigma}$ and a function of the cutoff parameter Λ, can be written as

$$\begin{aligned} S[X,\Lambda] &= \int d^2\sigma\, \mathcal{L}(X,\Lambda) \\ &= X^{\mu\sigma}\Delta_{\mu\sigma,\nu\sigma'}(\Lambda)X^{\nu\sigma'} + S_{\rm int}[X]. \end{aligned} \tag{2.2}$$

The cutoff appears only in the quadratic part of the action through the regularized free inverse propagator Δ; $S_{\rm int}$ is independent of the cutoff. When not essential, we will suppress the dependence on the cutoff; later, the cutoff dependence will be specified more precisely.

The primary goal of this work is to formulate the string field equations in a form covariant under arbitrary functional transformations of the background field X_o:

$$X_o^{\mu\sigma} \to F^{\mu\sigma}(X_o). \tag{2.3}$$

[1] $X^{\mu\sigma}$ is the same as $X^{\mu}(\sigma)$. Here and in the rest of this thesis σ always stands for worldsheet coordinates. All other greek indices refer to spacetime coordinates.

We shall adopt the usual language of differential geometry: Tensors will be labeled by a composite index like $\mu\sigma$, and upper and lower indices will undergo the standard transformations of contravariant and covariant tensor indices. Also, when no confusion can arise, we follow the convention of summation over repeated discrete indices $\mu\nu$, and integration over repeated continuous indices σ, σ'. Here, σ stands for the worldsheet coordinates σ_0 and σ_1; the worldsheet metric is Euclidean. In the standard background field method, it is convenient to define a new coordinate variable $X^{\mu\sigma}(s)$ as a function of an internal parameter s through the geodesic equation

$$\frac{d^2}{ds^2}X^{\mu\sigma}(s) + \Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''}(X)\frac{d}{ds}X^{\alpha\sigma'}(s)\frac{d}{ds}X^{\beta\sigma''}(s) = 0, \tag{2.4}$$

with the boundary condition that, at $s = 1$, $X^{\mu\sigma}(s = 1) \equiv X^{\mu\sigma}$, where X is the original variable that appears in (2.1). As in this case, when the parameter s is omitted, this will mean X at $s = 1$. The classical background field X_o is given by $X^{\mu\sigma}(s = 0) \equiv X_o^{\mu\sigma}$, and it is also useful to define the tangent at $s = 0$ by $(dX^{\mu\sigma}(s)/ds)_{s=0} \equiv \xi^{\mu\sigma}$. The connection Γ is yet unspecified; it is introduced in order to have covariance under (2.3). We shall see later on that quantum corrections break this group down to transformations with unit functional determinant:

$$\det\left(\frac{\delta F^{\mu\sigma}}{\delta X^{\nu\sigma'}}\right) = 1. \tag{2.5}$$

The idea of the background field method is to change variables in (2.1) from $X = X(1)$ to ξ at fixed X_o in order to exhibit the dependence on the classical field explicitly. This is conveniently done by expanding X and also the action in powers of the parameter s and setting $s = 1$ at the end. For later use, here we write down the first three terms of the expansion of X:

$$X^{\mu\sigma} = X_o^{\mu\sigma} + \xi^{\mu\sigma} - \frac{1}{2}\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''}(X_o)\xi^{\alpha\sigma'}\xi^{\beta\sigma''} + \cdots \tag{2.6}$$

In the same way, the action can be expanded:

$$\begin{aligned} S[X] &= S[X_o] + \frac{d}{ds}S[X(s)]\bigg|_{s=0} + \frac{1}{2}\frac{d^2}{ds^2}S[X(s)]\bigg|_{s=0} + \cdots \\ &= S[X_o] + \frac{\delta S[X_o]}{\delta X_o^{\alpha\sigma}}\xi^{\alpha\sigma} + \frac{1}{2}G_{\alpha\sigma,\beta\sigma'}(X_o)\xi^{\alpha\sigma}\xi^{\beta\sigma'} + S_R, \\ &\equiv S[X_o, \xi]. \end{aligned} \tag{2.7}$$

Here, S_R denotes the cubic and higher order terms in ξ in the expansion of S. The propagator G in the presence of the background field is given by

$$G_{\alpha\sigma,\beta\sigma'}(X_o) \equiv \frac{\delta^2 S[X_o]}{\delta X_o^{\alpha\sigma}\delta X_o^{\beta\sigma'}} - \frac{\delta S[X_o]}{\delta X_o^{\gamma\sigma''}}\Gamma^{\gamma\sigma''}_{\alpha\sigma,\beta\sigma'}(X_o). \tag{2.8}$$

We are now ready to define S': it is gotten from S by subtracting the term linear in ξ:

$$S'[X_o,\xi] = S[X(1)] - \frac{\delta S[X_o]}{\delta X_o^{\mu\sigma}}\xi^{\mu\sigma}. \tag{2.9}$$

It is well-known that the transition from S to S' in the background field approach is equivalent to the introduction of a background field dependent source. Changing variables of integration from X to ξ in (2.1), the partition function can be written as

$$\begin{aligned} Z[X_o] &= \int [DX]\, e^{S'} \\ &= \int [D\xi]\, e^{(S'+\mathcal{M})}, \end{aligned} \tag{2.10}$$

where we have defined

$$\det\left(\frac{\partial X}{\partial \xi}\right) \equiv \exp\left(\mathcal{M}\right). \tag{2.11}$$

From now on, we will drop the subscript on X_o; X will stand for the classical background field, and in order to avoid confusion, the original field X will be denoted by $X(1)$. $\mathcal{M}$, the log of the jacobian, can be computed from (2.6); we write down the result to quadratic order in ξ:

$$\mathcal{M} = -\left(\frac{1}{6}R_{\beta\sigma,\gamma\sigma'} + \frac{1}{2}D_{\beta\sigma}\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\gamma\sigma'}\right)\xi^{\beta\sigma}\xi^{\gamma\sigma'} + \mathcal{M}_R, \tag{2.12}$$

where $\mathcal{M}_R$ is at least cubic in ξ. The combination of the Ricci tensor and the covariant derivative that appears on the right hand side of this equation is explicitly given by

$$\begin{aligned} \frac{1}{6}R_{\beta\sigma,\gamma\sigma'} + \frac{1}{2}D_{\beta\sigma}\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\gamma\sigma'} &= \frac{1}{6}\left(\frac{\delta\Gamma^{\alpha\sigma''}_{\beta\sigma,\gamma\sigma'}}{\delta X^{\alpha\sigma''}} + \frac{\delta\Gamma^{\alpha\sigma''}_{\gamma\sigma',\alpha\sigma''}}{\delta X^{\beta\sigma}} + \frac{\delta\Gamma^{\alpha\sigma''}_{\beta\sigma,\alpha\sigma''}}{\delta X^{\gamma\sigma'}}\right) \\ &- \frac{1}{6}\left(\Gamma^{\alpha\sigma''}_{\nu\sigma''',\beta\sigma}\Gamma^{\nu\sigma'''}_{\alpha\sigma'',\gamma\sigma'} + 2\Gamma^{\alpha\sigma''}_{\alpha\sigma'',\nu\sigma'''}\Gamma^{\nu\sigma'''}_{\beta\sigma,\gamma\sigma'}\right). \end{aligned} \tag{2.13}$$

We note that:

a) $\mathcal{M}$ is of order $\hbar$; it is a quantum correction to the classical action.

b) We have dropped the term linear in ξ in (2.12); this can be taken care of by redefining S'.

c) Referring to (2.12), we see that the first term on the right, the Ricci tensor, is covariant; however, the second term, which is the covariant derivative of the contracted connection, is not. At this point, we could add a counterterm to S that would eliminate this non-covariant term. However, we decided to follow a different prescription, and impose the condition that the connection is derived from a metric. In that case, we have

$$\Gamma^{\beta\sigma'}_{\alpha\sigma,\beta\sigma'} = \frac{1}{2g}\frac{\delta g}{\delta X^{\alpha\sigma}}. \tag{2.14}$$

Here, g is the determinant of the metric. From this, one sees that this term is covariant only under coordinate transformations with unit determinant. Therefore, our prescription breaks the full diffeomorphism group down to transformations with unit determinant. We should emphasize that there are more ways to do this, and in fact, the one chosen here is not the standard one. They differ in the way the measure is defined or the cutoff breaks conformal invariance. For instance, an alternative prescription would have been to add the counterterm mentioned above, but leave the connection undefined. However, whatever the prescription, the final equations, obtained by imposing conformal invariance of the partition function Z, are the same.

The next step in our program is to expand the partition function (see (2.10)) in a perturbation series. However, in contrast to the usual perturbation series, each term in our series is invariant under the restricted (unit determinant) transformations ((2.3), (2.5)). In deriving the perturbation expansion, we follow the standard functional approach discussed in the textbooks (see for example [19]). First, we define a free partition function Z_o in the absence of interaction (except with the external field), coupled to an external source J:

$$\begin{aligned} Z_o[X,J] &= \int [D\xi]\,\exp\left(S(X) + \frac{1}{2}G_{\alpha\sigma,\beta\sigma'}(X)\xi^{\alpha\sigma}\xi^{\beta\sigma'} + iJ_{\mu\sigma}\xi^{\mu\sigma}\right) \\ &= \exp\left(S(X) - \frac{1}{2}\mathrm{Tr}\log G + \frac{1}{2}J_{\mu\sigma}G^{\mu\sigma,\nu\sigma'}(X)J_{\nu\sigma'}\right). \end{aligned} \tag{2.15}$$

Here, G with the upper indices is the inverse of G with the lower indices. The full partition function can now be written as

$$Z(X) = \exp\left(S_I(X,P)\right) Z_o(X,J)|_{J=0}, \tag{2.16}$$

where,

$$S_I(X,\xi) = S_R(X,\xi) + \mathcal{M}(X,\xi), \tag{2.17}$$

and P, which replaces ξ as the argument of S_I in (2.16), is given by

$$P^{\mu\sigma} \equiv -i\frac{\delta}{\delta J_{\mu\sigma}}. \tag{2.18}$$

In (2.16), after the functional derivatives with respect to J act on $Z_o(X, J)$, J is set equal to zero. Eq. (2.16) can be used as the starting point of a perturbation expansion in powers of S_I. It is easy to see that the invariance under the coordinate transformations (2.3), subject to the constraint (2.5), are preserved in this expansion, if at the same time, J and P are transformed by

$$J_{\mu\sigma} \to \left(\frac{\delta F^{\nu\sigma'}}{\delta X^{\mu\sigma}}\right) J_{\nu\sigma'}, \qquad P^{\mu\sigma} \to \left(\frac{\delta F^{\mu\sigma}}{\delta X^{\nu\sigma'}}\right) P^{\nu\sigma'}. \tag{2.19}$$

Since we are dealing with a non-renormalizable interaction, the series is badly divergent. To have a well defined answer, we introduce a cutoff in the quadratic term in the action (see (2.2)). This cutoff in general violates the coordinate invariance described above. We shall later see how to deal with this problem; in fact, the solution will be at the heart of the derivation of the string equations.

Among the coordinate diffeomorphisms, conformal transformations on the world sheet will play a special role. They are given by

$$\sigma_+ \to f_+(\sigma_+), \qquad \sigma_- \to f_-(\sigma_-), \tag{2.20}$$

where

$$\sigma_+ \equiv \sigma_0 + i\sigma_1, \qquad \sigma_- \equiv \sigma_0 - i\sigma_1. \tag{2.21}$$

In what follows, to save writing, we will only exhibit the formulas corresponding to the f_+ transformations; the f_- expressions can be obtained from these by an interchange of $+$ with $-$. The string field equations follow from demanding that the partition function (2.1) be invariant under the conformal transformations. The first thing to check is the invariance of the quadratic part of the action in (2.2); in the absence of the cutoff, Δ is given by[2]

$$\Delta_{\mu\sigma,\nu\sigma'}(\Lambda = 0) = -\partial_{\sigma_+}\partial_{\sigma_-}\delta^2(\sigma - \sigma')\eta_{\mu\nu}, \tag{2.22}$$

and is conformally invariant. Here, $\eta_{\mu\nu}$ is the flat Minkowski metric. We introduce the cutoff by defining

$$\Delta_{\mu\sigma,\nu\sigma'}(\Lambda) = \eta_{\mu\nu}\Delta_{\sigma,\sigma'}(\Lambda). \tag{2.23}$$

[2] Whenever ∂_{σ_+} acts on a function of only σ, not σ and σ', it will be written as just ∂_+.

It will turn out to be useful to also define the following related functions:

$$\Delta_{\sigma,\sigma'}(\Lambda) = -\partial_{\sigma_+}\partial_{\sigma_-}\,\delta^2_\Lambda(\sigma,\sigma'), \qquad \partial_{\sigma_+}\partial_{\sigma_-}\,\Delta^{\sigma,\sigma'}(\Lambda) = -\tilde{\delta}^2_\Lambda(\sigma,\sigma'),$$
$$\int d^2\!\sigma'\,\Delta_{\sigma,\sigma'}(\Lambda)\Delta^{\sigma',\sigma''}(\Lambda) = \delta^2(\sigma-\sigma''), \qquad \int d^2\!\sigma'\,\tilde{\delta}^2_\Lambda(\sigma,\sigma')\delta^2_\Lambda(\sigma',\sigma'') = \delta^2(\sigma-\sigma''). \tag{2.24}$$

The detailed structure of these functions is not important; all one needs to know is that $\delta^2_\Lambda(\sigma,\sigma')$ and $\tilde{\delta}^2_\Lambda(\sigma,\sigma')$ are smoothed out versions of the two dimensional Dirac delta function, and they are chosen so that $\Delta_{\sigma,\sigma'}(\Lambda)$, and as many derivatives of it as needed, are finite at $\sigma=\sigma'$, when the cutoff is finite. Notice that, unlike the propagator used in [10], our propagator need not vanish at $\sigma=\sigma'$. As a result, in contrast to [10], we shall encounter cutoff dependent terms in our equations. In some cases, these can be eliminated by renormalizing, for example, the slope parameter. In other cases, when such a renormalization is not possible, we will consider it as an anomaly and set its coefficient equal to zero. This will then provide additional useful information. For example, the field equation for the dilaton is derived in this fashion.

The cutoff violates conformal invariance; Δ with cutoff is no longer conformal invariant. To restore the conformal invariance, we have to supplement the transformations (2.20) by a suitable variation of the cutoff parameter(s). Specializing to infinitesimal variations, we define

$$\delta = \delta_\Lambda + \delta_v, \tag{2.25}$$

where δ_v is a "+" infinitesimal conformal transformation, which corresponds to taking the F in (2.3) and (2.19) to be[3]

$$F^{\mu\sigma}(X) \to F^{\mu\sigma}_v(X) = v(\sigma_+)\partial_+X^{\mu\sigma}, \tag{2.26}$$

with a similar expression for the "−" transformations. Here v is an arbitrary function of σ_+, parametrizing conformal transformations. The variation δ_Λ is defined so that the quadratic part of the action in (2.2) is invariant under the total variation δ, resulting in the equation

$$\partial_{\sigma_+}\left(v(\sigma_+)\Delta_{\sigma,\sigma'}(\Lambda)\right) + \partial_{\sigma'_+}\left(v(\sigma'_+)\Delta_{\sigma,\sigma'}(\Lambda)\right) - \delta_\Lambda\left(\Delta_{\sigma,\sigma'}(\Lambda)\right) = 0. \tag{2.27}$$

[3] Then $\delta_v = \int d^2\!\sigma\, v(\sigma_+)\partial_+X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}$, but in general $\delta_v = F^{\mu\sigma}_v(X)\frac{\delta}{\delta X^{\mu\sigma}}$. To be more precise, this F is not the same defined in (2.3) and (2.19), but is the $\tilde{F}$ defined by $F^{\mu\sigma}(X) = X^{\mu\sigma} + \tilde{F}^{\mu\sigma}(X)$, with the tilde dropped. Also, we define $v(\sigma_+) \equiv f_+(\sigma_+)$.

Later on, we will also need the cutoff variation of the propagator, so the variation of the function $\Delta^{\sigma,\sigma'}(\Lambda)$, which is the inverse of $\Delta_{\sigma,\sigma'}(\Lambda)$, is needed. At first, it may seem that the inverse also satisfies the same equation; and this would be true if the inverse were unique. However, there is a well-known ambiguity in going from Δ to its inverse; for example, in the absence of cutoff

$$\Delta^{\sigma,\sigma'}(\Lambda=0)=\frac{1}{4\pi}\log\left((\sigma-\sigma')^2\right)+k_+(\sigma_+)+k_-(\sigma_-). \qquad (2.28)$$

The functions k_+ and k_- are arbitrary, resulting in a non-unique inverse. The variation of the propagator under the change of the cutoff also suffers from the same ambiguity. This ambiguity can be resolved by demanding that the ultraviolet cutoff does not change the long distance behavior of the propagator[4]. Imposing this boundary condition, we have the following equation:

$$v(\sigma_+)\partial_{\sigma_+}\Delta^{\sigma,\sigma'}(\Lambda)+v(\sigma'_+)\partial_{\sigma'_+}\Delta^{\sigma,\sigma'}(\Lambda)-\delta_\Lambda\Delta^{\sigma,\sigma'}(\Lambda)=\frac{1}{4\pi}\frac{v(\sigma_+)-v(\sigma'_+)}{\sigma_+-\sigma'_+}. \qquad (2.29)$$

Comparing (2.27) to (2.29), we note that, because of the boundary conditions at large distances, an extra term appeared on the right hand side of (2.29). This term is the source of the conformal anomaly; in the string equations of [10], this anomaly is cancelled by the explicitly conformal non-invariant terms in the action. The above equation will play an important role in the calculations that follow: In applying the fundamental equation (2.38) to special cases, one needs an explicit expression for the variation of the propagator under the change of the cutoff; namely the term $\delta_\Lambda\Delta^{\sigma,\sigma'}$ in the above equation. This equation therefore provides the needed explicit expression. Another point that needs to be clarified is the dependence of $\Delta^{\sigma,\sigma'}$ on the variables σ and σ'. We would like to impose two dimensional rotation and translation invariance on the world sheet even in the presence of the cutoff. There is no problem in imposing both of these invariances for a fixed cutoff, however, when the cutoff is changed infinitesimally from this fixed value, its variation is given by (2.29) and it is clearly no longer translation and rotation invariant. This is the consequence of the translation and rotation non-invariant long distance boundary condition imposed in determining the cutoff variation.

The generators of conformal transformations, in the form they are expressed in (2.26), are not covariant under general coordinate transformations (2.3). They can be cast into a covariant form by writing

$$F^{\mu\sigma}_v(X)=v(\sigma_+)\partial_+X^{\mu\sigma}+\int d^2\sigma'\, v(\sigma'_+)f^{\mu\sigma}_{\sigma'}(X). \qquad (2.30)$$

[4] For a detailed treatment of this question, see [10].

The function f is introduced to make F_v transform as a vector in the indices $\mu\sigma$; note that the subscript σ' in $f^{\mu\sigma}_{\sigma'}$ is not a tensor index. A further constraint on f comes from demanding that δ_v satisfy the Virasoro algebra. We will specify this function later on when we discuss concrete examples.

We are now ready to write down the fundamental string field equation; it is obtained by demanding the invariance of the partition function under the conformal variation (2.25):

$$\delta Z(X,\Lambda)=0. \tag{2.31}$$

We note that this is not merely a requirement of covariance but one of invariance. In this respect, $F^{\mu\sigma}_v$ acts like a Killing vector that generates the conformal symmetry. Carrying out the operations represented by δ on the right hand side of (2.16) gives

$$\exp\left(S_I(X,P)\right)\mathcal{H}(X,P)\exp\left(\frac{1}{2}J_{\mu\sigma}G^{\mu\sigma,\nu\sigma'}J_{\nu\sigma'}\right)\Bigg|_{J=0}=0, \tag{2.32}$$

where $\mathcal{H}(X,P)$ will be defined shortly. From this equation, it is tempting to conclude that

$$\mathcal{H}(X,P)=0. \tag{2.33}$$

However, this conclusion is not correct; eq. (2.33) is too strong as it stands. This is because of the existence of an identity of the form

$$\exp(S_I(X,P))K^{\mu\sigma}(X,P)\left(\frac{\delta S_I}{\delta P^{\mu\sigma}}+G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right)\exp\left(\frac{1}{2}J_{\alpha\sigma''}G^{\alpha\sigma'',\beta\sigma'''}J_{\beta\sigma'''}\right)\Bigg|_{J=0}=0, \tag{2.34}$$

where K satisfies

$$\frac{\delta K^{\mu\sigma}}{\delta P^{\mu\sigma}}=0, \tag{2.35}$$

but is otherwise an arbitrary function of X and P. This identity, easy to verify directly, can be understood as follows. Reversing the steps leading from (2.10) to (2.16), one can get rid of the operator P and write the above identity as an integral over the variable ξ. The identity is then satisfied by virtue of the integrand being a total derivative. A total derivative corresponds to an infinitesimal change of variable in the integral in (2.10); therefore (2.34) is equivalent to the invariance of (2.10) under such a change of variable. Eq. (2.35) expresses the restriction that the Jacobian of this transformation is unity so as to leave the action unchanged, and (2.33) amounts to deducing the vanishing of the integrand from the vanishing of an integral and it is therefore too strong; the correct equation should be

$$\mathcal{H}=K^{\mu\sigma}\left(\frac{\delta S_I}{\delta P^{\mu\sigma}}+G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right), \tag{2.36}$$

and (2.32) is satisfied by virtue of (2.34).

The function $\mathcal{H}$ can be determined by carrying out the variations indicated in (2.31); the result is

$$\left(F_v^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}+P^{\nu\sigma'}\frac{\delta F_v^{\mu\sigma}}{\delta X^{\nu\sigma'}}\frac{\delta}{\delta P^{\mu\sigma}}+\delta_\Lambda\right)\left(S(X)+S_I(X,P)-\frac{1}{2}\mathrm{Tr}\log G(X)\right)$$
$$-\frac{1}{2}\left(F_v^{\alpha\sigma''}\frac{\delta G^{\mu\sigma,\nu\sigma'}}{\delta X^{\alpha\sigma''}}-\frac{\delta F_v^{\mu\sigma}}{\delta X^{\alpha\sigma''}}G^{\alpha\sigma'',\nu\sigma'}-\frac{\delta F_v^{\nu\sigma'}}{\delta X^{\alpha\sigma''}}G^{\mu\sigma,\alpha\sigma''}+\delta_\Lambda G^{\mu\sigma,\nu\sigma'}\right)$$
$$\times\left(\frac{\delta^2 S_I(X,P)}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}+\frac{\delta S_I}{\delta P^{\mu\sigma}}\frac{\delta S_I}{\delta P^{\nu\sigma'}}\right)=K^{\mu\sigma}\left(\frac{\delta S_I}{\delta P^{\mu\sigma}}+G_{\mu\sigma,\nu\sigma'}P^{\nu\sigma'}\right). \tag{2.37}$$

Eq. (2.37) is our version of renormalization group equations for the string action S. As it stands, it has two unusual features:

a) It is an equation in two variables X and P, whereas the standard renormalization group equations are in a single variable, the background field X.

b) It contains a function $K^{\mu\sigma}(X,P)$, arbitrary except for the constraint given by (2.35).

We will now show that these two seeming defects cancel each other; it is possible to convert (2.37) into an equation in a single variable X by taking advantage of the arbitrariness of the function K. To see this, imagine expanding S_I, $\mathcal{M}$ and K in a power series in the variable P. By equating different powers of P on both sides of the equation, we obtain an infinite set of equations, each in the single variable X. Let us now focus on the equation zeroth order in P. Since the right hand side of (2.37) starts with a linear term in P, this equation receives no contribution from K. This follows from the fact that S_R, $\mathcal{M}$ and therefore S_I all start at least quadratically in the expansion in powers of P. We therefore have an equation in the single variable X and free of the ambiguity coming from K:

$$E_G+E_{\mathcal{M}}=0, \tag{2.38}$$

where,

$$E_G=\left(F_v^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}+\delta_\Lambda\right)\left(S-\frac{1}{2}\mathrm{Tr}\log G\right), \tag{2.39}$$

and

$$E_{\mathcal{M}}=\frac{1}{2}\left(-F_v^{\alpha\sigma''}\frac{\delta G^{\mu\sigma,\nu\sigma'}}{\delta X^{\alpha\sigma''}}+\frac{\delta F_v^{\mu\sigma}}{\delta X^{\alpha\sigma''}}G^{\alpha\sigma'',\nu\sigma'}+\frac{\delta F_v^{\nu\sigma}}{\delta X^{\alpha\sigma''}}G^{\mu\sigma,\alpha\sigma''}-\delta_\Lambda G^{\mu\sigma,\nu\sigma'}\right)$$
$$\times\left[\frac{\delta^2\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}+\frac{\delta\mathcal{M}}{\delta P^{\mu\sigma}}\frac{\delta\mathcal{M}}{\delta P^{\nu\sigma'}}\right]_{P=0}=0. \tag{2.40}$$

This equation is the fundamental result of the section. In the next two sections, it will provide our starting point for the derivation of the string field equations.

We end this section with a couple of comments:

a) Eq. (2.38) is a a one loop result, which one can verify either by counting powers of $\hbar$ or more simply from the appearance of the "Tr log" type terms. It should therefore agree with the standard treatment [8] for a renormalizable action S. We will make this comparison in the next section.

b) This observation leads to an apparent paradox: No approximation has been made in deriving (2.38), yet it is clearly a one loop result. We have to conclude that the one loop result leads to exact string field equations.

c) There are of course an additional infinite number of equations coming from higher powers of P. These equations can then be used to determine the unknown function $K^{\mu\sigma}(X,P)$. One can then extract relations not involving K by using the constraint (2.35). These appear to come from two or more loops. We do not know whether these equations are redundant, or whether they contain additional information, which would then supplement the one loop result but not change it.

d) Eq. (2.38) contains two unknown functions: The Killing vector $F_v^{\mu\sigma}$ and implicitly, the connection $\Gamma^{\gamma\sigma''}_{\alpha\sigma,\beta\sigma'}$ (See (2.8)). They have to be expressed in terms of the fields that appear in S. This will be done in the following sections.

3. The Lowest String Levels

In this section we apply the formalism developed in the last section to the lowest levels of a closed, unoriented bosonic string: The tachyon and the massless level in §3.1 and the first massive level in §3.2 and §3.3. The fundamental equation (2.38) is too formal to be useful as it stands; one needs explicit results for the connection and the Killing vector. Since we do not have an exact expression for either of these, to make progress we resort to an expansion which we call the quasi-local expansion. This is an expansion in the number of derivatives on the world sheet and it is explained in §3.1. The local coordinate transformations associated with gravity appear at zeroth order, and each new power of the expansion parameter b brings in transformations with two more derivatives on the world sheet. The levels of the string can also be similarly organized; the nth level goes with the power b^{n-1}. In §3.1, we study the zeroth order term in the expansion. This corresponds to considering only the tachyon and the massless levels and imposing only local coordinate invariance. We are, therefore, back in familiar territory of renormalizable sigma model [8], where the connection and the generators of conformal algebra are well-known. Our reason for reexploring it is twofold: Firstly, we would like to check our formalism against standard results. This check is non-trivial since the way we treat the dilaton is different from the standard treatment [8], where the dilaton field is introduced as an independent field in the action from the beginning. In our approach, the determinant of the metric plays the role of the dilaton field, and everything works out alright. We note that this determinant cannot be gauged away, since the coordinate transformations under which the model is invariant are restricted to have unit determinant. The second question we would like to answer is what happens if we abandon covariance by, for example, setting the connection equal to zero. In this case, we recover the gravitational equations in a fixed gauge, but we loose the equation of motion for the dilaton. Therefore, covariance is important in obtaining a complete set of equations.

The next step is to go to first order in the expansion, which is the subject of §3.2. The first massive level of the string enters at this order, and also the coordinate (field) transformations include non-local terms for the first time. The important question is whether in this case, a suitable metric and a Killing vector that generates the conformal algebra exist. We show how to construct both the metric and the Killing vector to this order, and we derive the resulting equations of motion for the first massive level. An important check on the method is to find out whether the level structure agrees with that of the first massive level of the string. Again, to see what difference covariance makes, we check this for the non-covariant version, when the connection is set equal to zero. As mentioned before, we find that there are too many states, and there is not enough gauge invariance to eliminate the spurious states. In §3.3, we investigate the first massive level in the covariant case. Here, the situation is the opposite; for a general left-right non-symmetric model, there are too few states. Only when the model is left-right symmetric, there is an exact match. At the end of this thesis we mention some recent work done by Bardakci that overcomes this restriction.

3.1. The Tachyon and the Massless Level

We will start by studying the two lowest levels of a closed, unoriented bosonic string: The tachyon and the massless level. Ideally, one would like to start with S as an arbitrary functional of $X^{\mu\sigma}$ and try to solve (2.38) in all its generality. However, this direct approach seems hopelessly complicated and not particularly useful. Instead, the problem is made tractable by expanding S in powers of the σ derivatives of $X^{\mu\sigma}$. We will call this the quasi-local expansion. This expansion is quite natural from the point of two dimensional field theory on the world sheet and it has been the basis of most of the work done on this subject. From the string point of view, it is an expansion in the level number, two derivatives in σ corresponding to an increase of one unit in level number. We note that as a consequence of two-dimensional rotation invariance on the world sheet, which we shall always assume, there is always an equal number of derivatives with respect to σ_+ and σ_-. It is then convenient to introduce a parameter "b" to keep track of the expansion: The field representing the nth level of the string will be multiplied by $b^{(n-1)}$. For example, the tachyon has coefficient b^{-1} and the massless level is independent of b. The first two terms in the quasi-local expansion of S are then given by

$$S = b^{-1}S^{(-1)} + S^{(0)} + \cdots = \int d^2\sigma \, \left(b^{-1}\Phi(X(\sigma)) + \tilde{g}_{\mu\nu}(X(\sigma))\partial_+ X^{\mu\sigma}\partial_- X^{\nu\sigma} + \cdots\right). \tag{3.1}$$

Here and in the sequel, we have adopted the following notation: The superscripts (-1), (0), etc., refer to terms in S proportional to the corresponding powers of b. Expressions like $\Phi(X(\sigma))$ denote local functions of the coordinate $X(\sigma)$, whereas expressions such as $F^{\mu\sigma}(X)$ denote functionals in the same coordinate. Also, we should make clear that the parameter "b" is merely a bookkeeping device and can be set equal to one at the end of the calculation.

In the same spirit, the coordinate transformations (2.3) have a quasi-local expansion:

$$F^{\mu\sigma}(X) = f^{\mu}(X(\sigma)) + bf^{\mu}{}_{\nu\lambda}(X(\sigma))\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + bf^{\mu}{}_{\nu}\partial_+\partial_- X^{\nu\sigma}(X(\sigma)) + \cdots. \tag{3.2}$$

The first term is the local coordinate transformation associated with gravity; terms with increasing powers of b contain higher derivatives of σ and become increasingly non-local. In this section, we will only be concerned with invariance under local transformations represented by the first term in (3.2). However, the general strategy, pursued in the next section, is to determine $F_v^{\mu\sigma}$, the generator of the conformal transformation (see (2.30)), and the connection Γ as a power series in b so as to achieve covariance under both local and non-local transformations. To zeroth order in b, F_v, the generator of the conformal transformations, is given by the first term in (2.30); the function $f_{\sigma'}^{\mu\sigma}$ is at least of first order in b.

To simplify the exposition, we have so far neglected the cutoff dependence in S (see (2.2)). With the cutoff restored, the second term in (3.1) should read

$$S^{(0)} = \int d^2\sigma d^2\sigma' \, \tilde{g}_{\mu\sigma,\nu\sigma'}\partial_{\sigma_+} X^{\mu\sigma}\partial_{\sigma'_-} X^{\nu\sigma'}, \tag{3.3}$$

where $\tilde{g}$ is given by

$$\tilde{g}_{\mu\sigma,\nu\sigma'} = \eta_{\mu\nu}\delta^2_\Lambda(\sigma,\sigma') + \tilde{h}_{\mu\nu}(X(\sigma))\delta^2(\sigma-\sigma'). \tag{3.4}$$

$\delta^2_\Lambda(\sigma,\sigma')$ is defined in (2.24) and $\tilde{h}$ is a cutoff independent local function of $X(\sigma)$.

We have now to determine the connection Γ and the generator of conformal transformations $F^{\mu\sigma}_v$ to zeroth order in b. We have already observed above that F_v is given by (2.26) to zeroth order, since f in (2.30) is already first order in b. As for the connection, it will be derived from a metric that transforms correctly under local transformations. The standard choice for the metric made in sigma model calculations, which we shall adopt, is the symmetric part of $\tilde{g}$ in (3.3):

$$g_{\mu\sigma,\nu\sigma'} = \frac{1}{2}\left(\tilde{g}_{\mu\sigma,\nu\sigma'} + \tilde{g}_{\nu\sigma',\mu\sigma}\right) = \eta_{\mu\nu}\delta^2_\Lambda(\sigma,\sigma') + h_{\mu\nu}(X(\sigma))\delta^2(\sigma-\sigma'), \tag{3.5}$$

where,

$$h_{\mu\nu} = \frac{1}{2}\left(\tilde{h}_{\mu\nu} + \tilde{h}_{\nu\mu}\right). \tag{3.6}$$

The connection, to zeroth order in b, is given in terms of the metric by the standard formula:

$$\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''} = \frac{1}{2}g^{\mu\sigma,\lambda\sigma'''}\left(\frac{\delta g_{\lambda\sigma''',\beta\sigma''}}{\delta X^{\alpha\sigma'}} + \frac{\delta g_{\alpha\sigma',\lambda\sigma'''}}{\delta X^{\beta\sigma''}} - \frac{\delta g_{\alpha\sigma',\beta\sigma''}}{\delta X^{\lambda\sigma'''}}\right). \tag{3.7}$$

With these preliminaries out of the way, we are ready to write down the field equation for the tachyon field. This we do by extracting terms lowest order in b, proportional to b^{-1}, from (2.38). Notice that only E_G contributes. The equation reduces to the following simple form

$$\left(\int d^2\!\sigma\, v(\sigma_+)\partial_+X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda\right)\left(S^{(-1)}_{\text{int}}[X] - \frac{1}{2}\text{Tr}\log G^{(-1)}\right) = 0. \tag{3.8}$$

The first term on the left is easy to calculate:

$$\int d^2\!\sigma\, v(\sigma_+)\partial_+X^{\mu\sigma}\frac{\delta}{\delta X^{\mu\sigma}}\left(S^{(-1)}_{\text{int}}[X]\right) = -\int d^2\!\sigma\, v'(\sigma_+)\Phi(X(\sigma)). \tag{3.9}$$

In calculating the contribution of the second term, we take advantage of the following simplifications: We are going to drop all the non-local terms that can arise in the expansion of this term. Such non-local terms are in general present since the action S that satisfies (2.37) is not necessarily one particle irreducible, and we wish to extract the one particle irreducible part that is local[5]. Another simplification follows from the fact that the result clearly is going to be a covariant Klein-Gordon equation for the tachyon field Φ in the background metric g. We can then first linearize this equation by expanding to first order in $h_{\mu\nu}$ of (3.5) around the flat background, and then covariantize the result to arrive at the full answer in an arbitrary background. This will be our strategy in the rest of this thesis; only the linear part of the field equations will be computed in the presence of a flat background, and the result will be generalized to a non-trivial background, making use of the powerful restrictions resulting from covariance.

Eq. (3.8) is linearized by setting

$$G_{\mu\sigma,\nu\sigma'}(X) = 2\Delta_{\mu\sigma,\nu\sigma'} + H_{\mu\sigma,\nu\sigma'}(X), \tag{3.10}$$

and by expanding the "Tr log" to first order in H:

$$\mathrm{Tr}\log G \cong \frac{1}{2}\Delta^{\mu\sigma,\nu\sigma'} H_{\mu\sigma,\nu\sigma'}. \tag{3.11}$$

The linear part of H, to order b^{-1}, is given by

$$H^{(-1)}_{\mu\sigma,\nu\sigma'} \cong \frac{\delta^2 S_{\mathrm{int}}[X]}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \cong \delta^2(\sigma-\sigma')\partial_\mu\partial_\nu\Phi(X(\sigma)). \tag{3.12}$$

In calculating the left hand side of (3.8), the following identity proves useful:

$$\begin{aligned}\Delta^{\mu\sigma,\nu\sigma'}\int d^2\sigma''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}}\left(\frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\right) &= \Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\\ \times\left(\int d^2\sigma''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''}\frac{\delta S}{\delta X^{\lambda\sigma''}}\right) - \frac{\delta^2 S}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}&\left(v(\sigma_+)\partial_{\sigma_+} + v(\sigma'_+)\partial_{\sigma'_+}\right)\Delta^{\mu\sigma,\nu\sigma'}.\end{aligned} \tag{3.13}$$

The second term on the right hand side can be calculated making use of (2.29), leading to the equation

$$\int d^2\sigma\, v'(\sigma_+)\left[-\Phi(X(\sigma)) + \frac{1}{16\pi}\partial^\mu\partial_\mu\Phi(X(\sigma)) + \frac{1}{4}\Delta^{(0)}(\Lambda)\partial^\mu\partial_\mu\Phi(X(\sigma))\right] = 0, \tag{3.14}$$

[5] See [20] for a version of renormalization group equations that are one particle irreducible.

where $\Delta^{(0)}(\Lambda)$ is the propagator $\Delta^{\sigma,\sigma'}(\Lambda)$, evaluated at $\sigma = \sigma'$. By translation invariance, it is independent of σ. Since Φ is only a function of $X^{\mu\sigma}$ and not of its derivatives with respect to σ, it follows that

$$\left(\frac{1}{16\pi} + \frac{1}{4}\Delta^{(0)}(\Lambda)\right) \partial^\mu \partial_\mu \Phi(X(\sigma)) - \Phi(X(\sigma)) = 0. \tag{3.15}$$

The term $\Delta^{(0)}(\Lambda)$ is cutoff dependent and it blows up as $\Lambda \to \infty$. This cutoff dependent term can be eliminated by explicitly introducing the slope parameter which we have suppressed and by renormalizing it. The same cutoff dependent term is encountered in the equations for the higher levels and it is again eliminated by the same slope renormalization. Finally, (3.15) can easily be generalized to an arbitrary background by using the metric given by (3.5) and casting it into a covariant form.

The next step is to derive the field equation for $\tilde{g}$, which includes both the metric ((3.5)), and the antisymmetric tensor

$$B_{\mu\nu} = \frac{1}{2}(\tilde{h}_{\mu\nu} - \tilde{h}_{\nu\mu}). \tag{3.16}$$

To do this, we have to extract zeroth order terms in b from (2.38). It is useful to distinguish between the two terms E_G and $E_{\mathcal{M}}$, the former coming from the variation of the $\mathrm{Tr}\log G$, and the latter coming from the variation of $\mathcal{M}$. The reason for this distinction is that the E_G is cutoff independent, whereas $E_{\mathcal{M}}$ is proportional to a cutoff dependent factor. We will argue later that these two terms must vanish separately, yielding two separate equations. The first of these will be the equation for the metric g and the antisymmetric tensor B; the second will provide the equation for the dilaton. Our strategy is again to expand around the flat background to first order in h and B, and use covariance to arrive at the full answer. We make use of (3.11) to calculate $\mathrm{Tr}\log G$, extracting the linear piece in $\tilde{h}$, and using (2.29), we find

$$\begin{aligned} E_G^{(0)} &= \left(\int d^2\!\sigma\, v(\sigma_+)\partial_+ X^{\mu\sigma} \frac{\delta}{\delta X^{\mu\sigma}} + \delta_\Lambda\right) \left(H^{(0)}_{\mu\sigma',\nu\sigma'} \Delta^{\mu\sigma',\nu\sigma'}(\Lambda)\right)\Big|_{\text{lin}} = \\ &= -\frac{1}{4\pi}\int d^2\!\sigma\, d^2\!\sigma'\, \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \\ &\quad \times \left(\frac{\delta^2 S^{(0)}_{\text{int}}[X]}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}}\Big|_{\text{lin}} + 2\delta^2(\sigma - \sigma')\Gamma^{\lambda}_{\mu\mu}(X(\sigma))\partial_+\partial_- X^{\lambda\sigma}\right). \end{aligned} \tag{3.17}$$

Here, the subscript "lin" refers to terms linear in $\tilde{h}$. The Γ that appears on the right hand side is the linearized form of the connection (3.7):

$$\Gamma^{\mu\sigma}_{\alpha\sigma',\beta\sigma''} \cong \tilde{\delta}^2_\Lambda(\sigma,\sigma')\delta^2(\sigma'-\sigma'')\Gamma^\mu_{\alpha\beta}(X(\sigma')), \tag{3.18}$$

where,

$$\Gamma^\mu_{\alpha\beta}(X(\sigma)) \cong \frac{1}{2}\eta^{\mu\nu}\Big(\partial_\alpha h_{\nu\beta}(X(\sigma)) + \partial_\beta h_{\alpha\nu}(X(\sigma)) - \partial_\nu h_{\alpha\beta}(X(\sigma))\Big). \tag{3.19}$$

The functional derivative of $S^{(0)}_{\rm int}$ can be calculated from (3.1), and repeating the steps that led to (3.15) gives the following field equation:

$$\Box\tilde{h}_{\nu\lambda} - \partial_\nu\partial_\mu\tilde{h}_{\mu\lambda} + \partial_\mu\partial_\lambda\tilde{h}_{\mu\nu} - 2\partial_\lambda\Gamma^\nu_{\mu\mu} = 0, \tag{3.20}$$

where $\Box = \partial_\mu\partial^\mu$. Here, since we use a flat metric to raise and lower indices, there is no real distinction between upper and lower indices. This will be understood whenever we have repeated upper or lower indices.

The equation above came from the conformal transformations in the variable σ_+. The other set of conformal transformations in the variable σ_- result in an additional equation:

$$\Box\tilde{h}_{\nu\lambda} - \partial_\lambda\partial_\mu\tilde{h}_{\nu\mu} + \partial_\nu\partial_\mu\tilde{h}_{\lambda\mu} - 2\partial_\nu\Gamma^\lambda_{\mu\mu} = 0. \tag{3.21}$$

It is now convenient to combine these two equations and rewrite them in terms of h and the antisymmetric tensor B. Interestingly, we find that, without any reference to (3.19), these equations fix the contracted connection Γ up to an arbitrary scalar field ϕ, which we shall identify with the dilaton field:

$$\Gamma^\lambda_{\mu\mu} = \partial_\mu h_{\mu\lambda} + \partial_\lambda\phi, \tag{3.22}$$

and we arrive at the following equations for h and B:

$$\Box h_{\nu\lambda} - \partial_\nu\partial_\mu h_{\mu\lambda} - \partial_\lambda\partial_\mu h_{\mu\nu} - 2\partial_\nu\partial_\lambda\phi = 0, \tag{3.23}$$

$$\Box B_{\nu\lambda} - \partial_\nu\partial_\mu B_{\mu\lambda} + \partial_\lambda\partial_\mu B_{\mu\nu} = 0. \tag{3.24}$$

Comparing with (3.19) determines ϕ:

$$\phi = -\frac{1}{2}h_{\mu\mu}, \tag{3.25}$$

and substituting this result back into (3.23) gives the standard equations of gravity without source in linearized form. The equation for the antisymmetric tensor $B_{\mu\nu}$ is the linearized version of the standard result of [8].

Up to this point, we have not taken into account $E_{\mathcal{M}}$. Let us first calculate the $\mathcal{M}$ dependent factor in (2.40). From equation (2.12), we find that

$$\left.\frac{\delta^2 \mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right|_{P=0} = -\frac{1}{3}\left(\frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\mu\sigma}}{\delta X^{\nu\sigma'}} + \frac{\delta\Gamma^{\lambda\sigma''}_{\lambda\sigma'',\nu\sigma'}}{\delta X^{\mu\sigma}} + \frac{\delta\Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'}}{\delta X^{\lambda\sigma''}}\right)$$
$$= -\frac{1}{3}\Delta^{(0)}(\Lambda)\delta^2(\sigma-\sigma')\Big(\partial_\nu\Gamma^{\lambda}_{\lambda\mu}(X(\sigma)) + \partial_\mu\Gamma^{\lambda}_{\lambda\nu}(X(\sigma)) + \partial_\lambda\Gamma^{\lambda}_{\mu\nu}(X(\sigma))\Big).$$

The term which is quadratic in $\mathcal{M}$ will not contribute since it is non-linear (and also non-local). To the order we are considering, the first factor on the right in (2.40) can be evaluated by setting $G^{\alpha\sigma,\beta\sigma'}$ equal to $\Delta^{\alpha\sigma,\beta\sigma'}$, with the result,

$$E^{(0)}_{\mathcal{M}} = \frac{1}{2}\left(v(\sigma_+)\partial_{\sigma_+}\Delta^{\mu\sigma,\nu\sigma'} + v(\sigma'_+)\partial_{\sigma'_+}\Delta^{\mu\sigma,\nu\sigma'} - \delta_\Lambda\Delta^{\mu\sigma,\nu\sigma'}\right)\left.\frac{\delta^2\mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\right|_{P=0}$$
$$= -\frac{1}{24\pi}\Delta^{(0)}(\Lambda)\eta^{\mu\nu}\int d^2\sigma\, v'(\sigma_+)\left(2\partial_\nu\Gamma^{\lambda}_{\lambda\mu}(X(\sigma)) + \partial_\lambda\Gamma^{\lambda}_{\mu\nu}(X(\sigma))\right). \tag{3.26}$$

We see that unlike the cutoff independent E_G, $E_{\mathcal{M}}$ is proportional to the cutoff dependent factor $\Delta^{(0)}(\Lambda)$. This term is a consequence of the prescription chosen, and being cutoff dependent, must be set equal to zero by itself. This gives us the additional equation

$$2\partial_\mu\Gamma^{\lambda}_{\lambda\mu} + \partial_\lambda\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2}\Box h_{\mu\mu} + \partial_\mu\partial_\nu h_{\mu\nu} = 0, \tag{3.27}$$

and combining this with (3.23) and (3.25), we find that

$$\Box h_{\mu\mu} = 0, \qquad \partial_\mu\partial_\nu h_{\mu\nu} = 0, \tag{3.28}$$

and

$$\Box h_{\mu\nu} - \partial_\mu\partial_\lambda h_{\lambda\nu} - \partial_\nu\partial_\lambda h_{\lambda\mu} - \partial_\mu\partial_\nu h_{\lambda\lambda} = 0. \tag{3.29}$$

Equations (3.28) and (3.29) describe the coupled graviton-dilaton system in the linear approximation. To see this, we note that these equations are invariant only under coordinate transformations of unit determinant, which, linearized, results in invariance under gauge transformations

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\kappa_\nu + \partial_\nu\kappa_\mu, \tag{3.30}$$

with the important restriction that $\partial_\mu \kappa^\mu = 0$. This restriction to unit determinant was explained in section 2 in the paragraph following (2.14). As a consequence, the trace of h, $h_{\mu\mu}$, which can be gauged away if there is invariance under unrestricted coordinate transformations, can no longer be eliminated and becomes a dynamical degree of freedom. Up to normalization, we identify it with the dilaton field ϕ. The natural candidate for the graviton field is the traceless component of h:

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{D}\eta_{\mu\nu} h_{\lambda\lambda}, \tag{3.31}$$

where D is the dimension of space. We identify $\bar{h}$ with the graviton field in the gauge where the metric has unit determinant and as a consequence, the graviton field is traceless. $\bar{h}$ also satisfies (3.29), which is the correct equation for the graviton coupled to the dilaton in this gauge. We would like to point out the difference between our treatment of the dilaton and the standard approach. In the standard treatment, in addition to $X^{\mu\sigma}$, the dilaton field ϕ is introduced in the action from the beginning, and the theory is regularized by going from 2 to $2+\epsilon$ dimensions on the world sheet. We stay with a two dimensional world sheet, regularize only the free propagator (see (2.2)), and the dilaton field is identified with the log of the determinant of the metric. This identification is only possible because the full coordinate invariance was broken down to transformations of unit determinant.

We end this section with a few observations:

a) So far, we have worked out the coupled system of the graviton, dilaton and the antisymmetric tensor only in the linear approximation. As stressed earlier, the full dependence on the graviton field follows from covariance. However, we have not calculated the higher order contributions in the dilaton field ϕ and the antisymmetric tensor field $B_{\mu\nu}$. It would be interesting to compare these to the results of [8], although such a comparison is plagued with ambiguities due to possible field redefinitions involving the dilaton field. It is also not clear that we should even consider the antisymmetric tensor: Our approach works only for the left-right symmetric string models and the antisymmetric tensor decouples in that case.

b) In the presence of the cutoff, the coordinate transformation, given by (3.2), has to be modified to preserve the invariance of the action. In the linear approximation, the modification is

$$F^{\mu\sigma}(X) \cong \int d\tilde{\sigma}'\, \tilde{\delta}^2_\Lambda(\sigma,\sigma') f^\mu(X(\sigma')) + \cdots. \tag{3.32}$$

Although they are not needed in the work reported here, the non-linear corrections to (3.2) can, in principle, be worked out.

c) There is an ambiguity in the expression for the connection given by (3.19), which is the standard result of differential geometry derived from the metric. However, since we insist on invariance under transformations with unit determinant, we are free to modify the metric by, for example

$$g_{\mu\nu} \longrightarrow g_{\mu\nu}(\det g)^k, \tag{3.33}$$

where k is an arbitrary constant. The modified metric leads to a modified connection, and to a new set of equations. These equations are not, however, physically different from (3.28) and (3.29); they correspond to field redefinitions involving the dilaton field mentioned above. This becomes clear by noticing that the dilaton field can be taken to be the log of the determinant of g; then (3.33) is a dressing of the metric by the dilaton field.

d) It is of some interest to find out what would have happened, if we had carried out a non-covariant calculation. This means setting the connection Γ equal to zero throughout, and referring to the equations (3.20) and (3.21), it amounts to choosing the gauge

$$\Gamma^{\lambda}_{\mu\mu} \cong \partial_\mu h_{\mu\lambda} - \frac{1}{2}\partial_\lambda h_{\mu\mu} = 0. \tag{3.34}$$

Therefore, the equation for the graviton comes out gauge fixed, but otherwise correct. What is missing is (3.28), the equation for the dilaton field. This is because the equation of motion for the dilaton comes entirely from $\mathcal{M}$, and with connection equal to zero, $\mathcal{M}$ is also zero.

3.2. The First Massive Level - Non-Covariant Approach

In this section, we shall investigate the first massive state, using once again the tools developed in section 2. The particular question we would like to address is whether the spectrum of states that follows from the linear (free) part of the equations of motion we are going to derive is consistent with the known spectrum of the first massive level of the string. This is clearly a necessary test any successful candidate for string field equations must pass. Of course, in addition, the nonlinear part of the equations should reproduce the interactions of the string theory. We will not address the question of interactions here, apart from observing that the stringent requirements of covariance we are going to impose probably fix the interaction uniquely.

The field equations will again follow from (2.38), given F ((2.30)) and the connection Γ to first order in b. For the sake of comparison with the non-covariant renormalization group approach of [10], we will first carry out a calculation with vanishing connection and F given by (2.26). Comparing the resulting physical states to those of the string, we will find that there are too many of them. In the next section the calculation is done covariantly: We start with Γ and F_v derived from a metric, suitably defined so as to satisfy invariance under coordinate transformations (2.3) and (2.5) to first order in b. The resulting set of states appear to be consistent with those of the left-right symmetric string model. We conclude that only the covariant approach yields equations powerful enough to produce the spectrum of at least the left-right symmetric string theory; the equations resulting from the non-covariant approach turn out to be too weak.

The starting point is the first massive level, written out in full generality:

$$\begin{aligned} S^{(1)} = \int d^2\sigma \Big(& e^{(1)}_{\mu_1\mu_2,\nu_1\nu_2}\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2} + e^{(2)}_{\mu_1\mu_2\mu_3}\partial_+^2 X^{\mu_1}\partial_- X^{\mu_2}\partial_- X^{\mu_3} \\ & + e^{(1)}_{\mu_1\mu_2\mu_3}\partial_+\partial_- X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\mu_3} + e^{(3)}_{\mu_1\mu_2\mu_3}\partial_-^2 X^{\mu_1}\partial_+ X^{\mu_2}\partial_+ X^{\mu_3} \\ & + e^{(1)}_{\mu_1\mu_2}\partial_+\partial_- X^{\mu_1}\partial_+\partial_- X^{\mu_2} + e^{(2)}_{\mu_1\mu_2}\partial_+^2\partial_- X^{\mu_1}\partial_- X^{\mu_2} \\ & + e^{(3)}_{\mu_1\mu_2}\partial_-^2\partial_+ X^{\mu_1}\partial_+ X^{\mu_2} + e^{(4)}_{\mu_1\mu_2}\partial_+^2 X^{\mu_1}\partial_-^2 X^{\mu_2}\Big), \end{aligned} \tag{3.35}$$

where the e's in this expression are local functions of the field $X^{\mu\sigma}$. Here and in many of the equations that follow, we have also simplified writing by replacing, for example, $X^{\mu_1\sigma}$ by X^{μ_1}.

Eq. (3.35) is highly redundant because of the existence of linear gauges. These result from the possibility of adding zero to (3.35) by adding a total derivative in σ_+ or in σ_- to the integrand. Such a possibility already exists for the zero mass level; adding

$$0 = \int d^2\sigma \left(\partial_+\left(\partial_- X^{\mu\sigma}\Lambda_\mu(X(\sigma))\right) - \partial_-\left(\partial_+ X^{\mu\sigma}\Lambda_\mu(X(\sigma))\right)\right) \tag{3.36}$$

to (3.1) amounts to the well-known gauge transformation of the antisymmetric tensor B:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu. \tag{3.37}$$

For the first massive level, the situation is more complicated; there are six distinct linear gauge transformations. These are discussed in Appendix A, where it is also shown that, making use of these gauges, all but three of the fields appearing in (3.35) can be eliminated. The resulting linear gauge fixed form of $S^{(1)}$ reads

$$S^{(1)} = \int d^2\sigma \, \big(e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} + e_{\mu_1\mu_2\mu_3}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\mu_3} + e_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}\big). \tag{3.38}$$

It is also shown in Appendix A that this form of $S^{(1)}$ is in fact completely gauge fixed; in contrast to the massless level, there are no linear gauge transformations left of the form (3.37) that map it into itself. It is now easy to carry out the non-covariant calculation by substituting S given by (3.38) in (2.38), and setting $\Gamma = 0$ and F_v to the value given by (2.26). The resulting equation is

$$\left(\int d^2\sigma'' \, v(\sigma''_+)\partial_+X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\left(S^{(1)} - \frac{1}{4}\Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\right) = 0. \tag{3.39}$$

The second term of this equation can be evaluated with help of the identities (3.13) and (2.29):

$$\begin{aligned}\left(\int d^2\sigma'' \, v(\sigma''_+)\partial_+X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} &= \\ = \Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\int d^2\sigma'' v(\sigma''_+)\partial_{\sigma''_+}X^{\lambda\sigma''}\frac{\delta S^{(1)}}{\delta X^{\lambda\sigma''}} & \\ - \frac{1}{4\pi}\int d^2\sigma d^2\sigma' \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\mu\sigma'}}.&\end{aligned} \tag{3.40}$$

Setting $S^{(1)} = \int d^2\sigma \, U(X(\sigma))$ and using the above results gives

$$\int d^2\sigma \, v'(\sigma_+)U(X(\sigma)) - \frac{1}{4}\Delta^{\mu\sigma,\nu\sigma'}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\int d^2\sigma'' \, v'(\sigma''_+)U(X(\sigma'')) + \frac{1}{16\pi}\int d^2\sigma d^2\sigma' \, \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} = 0. \tag{3.41}$$

The last term in this equation can be evaluated after a tedious but straightforward calculation, with the result

$$\int d^2\sigma d^2\sigma' \, \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+} \, \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma} \delta X^{\mu\sigma'}} = \int d^2\sigma \, v'(\sigma_+)$$

$$\times \Big((\Box e_{\mu_1\mu_2,\nu_1\nu_2} - 2\partial_{\mu_1}\partial_\mu e_{\mu\mu_2,\nu_1\nu_2} + \frac{1}{3}\partial_{\mu_1}\partial_{\mu_2} e_{\mu\mu,\nu_1\nu_2})\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2}$$
$$+ (\Box e_{\mu_1\mu_2\mu_3} - 4\partial_\mu e_{\mu\mu_2,\mu_1\mu_3} - \partial_{\mu_2}\partial_\mu e_{\mu_1\mu\mu_3} + \frac{4}{3}\partial_{\mu_2} e_{\mu\mu,\mu_1\mu_3})\partial_+\partial_- X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\mu_3}$$
$$+ (\Box e_{\mu_1\mu_2} - \partial_\mu e_{\mu_1\mu\mu_2} + \frac{2}{3} e_{\mu\mu,\mu_1\mu_2})\partial_+\partial_- X^{\mu_1}\partial_+\partial_- X^{\mu_2}$$
$$+ (-2\partial_\mu e_{\mu\mu_1,\mu_2\mu_3} + \frac{1}{3}\partial_{\mu_1} e_{\mu\mu,\mu_2\mu_3})\partial_+^2 X^{\mu_1}\partial_- X^{\mu_2}\partial_- X^{\mu_3}$$
$$+ (-\partial_\mu e_{\mu_1\mu\mu_2} + \frac{2}{3} e_{\mu\mu,\mu_1\mu_2})\partial_+^2\partial_- X^{\mu_1}\partial_- X^{\mu_2} \Big). \tag{3.42}$$

One has to take into account possible gauge invariance of the integral on the right hand side of this equation. Because of the presence of the factor $v'(\sigma_+)$, the gauges are generated by adding a total derivative with respect to σ_- only, and as a result, there are only three of them, as opposed to six in the case of (3.35). In writing down (3.42), we have already eliminated all redundant terms and fixed the linear gauges completely.

Let us now evaluate the second term in (3.41). As opposed to (3.42), which is cutoff independent, here we encounter only cutoff dependent terms. These terms are proportional to $\Delta^{\sigma,\sigma'}(\Lambda)$ and its derivatives, evaluated at $\sigma = \sigma'$. By rotation invariance on the world sheet, the number of derivatives with respect to σ_+ must match those with respect to σ_-. Defining

$$\begin{aligned} \partial_{\sigma_+}\partial_{\sigma_-}\Delta^{\sigma,\sigma'}(\Lambda)|_{\sigma=\sigma'} &\equiv \Delta_2^{(0)}(\Lambda), \\ \partial_{\sigma_+}^2\partial_{\sigma_-}^2\Delta^{\sigma,\sigma'}(\Lambda)|_{\sigma=\sigma'} &\equiv \Delta_4^{(0)}(\Lambda), \end{aligned} \tag{3.43}$$

we have,

$$\Delta^{\mu\sigma,\nu\sigma'} \frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} \int d^2\sigma'' \, v'(\sigma''_+) U(X(\sigma''))\Big|_{\text{sing}} =$$
$$= \int d^2\sigma \, v'(\sigma_+) \Big(\Delta^{(0)}(\Lambda) \Box e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2}$$
$$+ \Box e_{\mu_1\mu_2\mu_3}\partial_+\partial_- X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\mu_3} + \Box e_{\mu_1\mu_2}\partial_+\partial_- X^{\mu_1}\partial_+\partial_- X^{\mu_2}$$
$$+ 2\Delta_2^{(0)}(\Lambda)(-4e_{\mu\mu_1,\mu\mu_2} + \partial_\mu e_{\mu\mu_1\mu_2} + \partial_{\mu_2} e_{\mu_1\mu\mu} - 2\partial_\mu\partial_{\mu_2} e_{\mu\mu_1})\partial_+ X^{\mu_1}\partial_- X^{\mu_2}$$
$$+ 2\Delta_4^{(0)}(\Lambda) e_{\mu\mu} \Big).$$

These singular terms can be eliminated by renormalization as follows: The same slope renormalization that got rid of the cutoff dependent term in the equation for the tachyon (see (3.15) and the discussion that follows) also eliminates the term proportional to $\Delta^{(0)}(\Lambda)$ here. The other cutoff dependent terms are due to the contraction of two X's in the same vertex, and they can be taken care of by vertex renormalization. This amounts to eliminating them by introducing local counterterms in S of the form, for example,

$$\Delta S = \text{const} \times \Delta_2^{(0)}(\Lambda) \int d^2\sigma \, e_{\mu\mu_1,\mu\mu_2} \partial_+ X^{\mu_1} \partial_- X^{\mu_2}. \tag{3.44}$$

In the operator formulation of the string theory, these divergent terms are eliminated by the operator normal ordering of the vertex.

After renormalization, one is left with the finite equations given by (3.42). They fall into two classes: Propagating equations of motion are (with the unconventionally normalized mass squared given by 16π)

$$\begin{aligned} \Box e_{\mu_1\mu_2,\nu_1\nu_2} + 16\pi e_{\mu_1\mu_2,\nu_1\nu_2} &= 0, \\ \Box e_{\mu_1\mu_2\mu_3} + 16\pi e_{\mu_1\mu_2\mu_3} &= 0, \\ \Box e_{\mu_1\mu_2} + 16\pi e_{\mu_1\mu_2} &= 0, \end{aligned} \tag{3.45}$$

plus constraints

$$\begin{aligned} \partial_\mu e_{\mu\mu_1,\nu_1\nu_2} - \frac{1}{6}\partial_{\mu_1} e_{\mu\mu,\nu_1\nu_2} &= 0, \\ \partial_\mu e_{\mu_1\mu\mu_2} - \frac{2}{3} e_{\mu\mu,\mu_1\mu_2} &= 0, \end{aligned} \tag{3.46}$$

and also the constraints that come from $v'(\sigma_-)$

$$\begin{aligned} \partial_\nu e_{\mu_1\mu_2,\nu\nu_2} - \frac{1}{6}\partial_{\nu_2} e_{\mu_1\mu_2,\nu\nu} &= 0, \\ \partial_\mu e_{\mu_1\mu_2\mu} - \frac{2}{3} e_{\mu_1\mu_2,\nu\nu} &= 0. \end{aligned} \tag{3.47}$$

Comparing with the structure of the first massive level of the string (see Appendix B), it is clear that the above constraints are too weak. For example, in string theory, everything is expressible in terms of the analogue of $e_{\mu_1\mu_2,\nu_1\nu_2}$, whereas here $e_{\mu_1\mu_2}$ and most of $e_{\mu_1\mu_2\mu_3}$ cannot be so expressed. Clearly, the latter fields are spurious and should somehow be eliminated. In the next section, we will see that the covariant approach overcomes this problem.

In this section, the field equations for the first massive level will be rederived, this time imposing covariance under coordinate transformations given by (3.2). When treating the massless levels, covariance under only the local transformations (first term in (3.2)) was imposed; we now require, in addition, covariance under transformations first order in b. We will initially simplify the problem by starting with flat Minkowski metric, with $\tilde{h}=0$, in (3.4), and with the action

$$S = X^{\mu\sigma}\Delta_{\mu\sigma,\nu\sigma'}(\Lambda)X^{\nu\sigma'} + bS^{(1)} = S^{(0)} + bS^{(1)}, \tag{3.48}$$

where $S^{(1)}$ is given by (3.38). Because the metric is flat, we have to set the first term in (3.2) equal to zero, and also take into account the introduction of the cutoff in (3.48) by modifying the transformations. The modification needed is similar to (3.32), $X^{\mu\sigma} \to X'^{\mu\sigma}$, with:

$$X'^{\mu\sigma} = X^{\mu\sigma} + b\int d^2\tilde{\sigma}'\, \tilde{\delta}^2_{\Lambda}(\sigma,\sigma')\Big(f_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + f_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\Big). \tag{3.49}$$

It is easy to check that, to first order in b, (3.48) is invariant under (3.49), if at the same time, the fields transform by

$$\begin{aligned} e_{\mu\nu\lambda} &\to e_{\mu\nu\lambda} + 2f_{\mu\nu\lambda}, \\ e_{\mu\nu} &\to e_{\mu\nu} + f_{\mu\nu} + f_{\nu\mu}. \end{aligned} \tag{3.50}$$

Since only the symmetric part of $f_{\mu\nu}$ appears, from now on we will impose the condition

$$f_{\mu\nu} = f_{\nu\mu}. \tag{3.51}$$

As we have mentioned earlier, we initially work with flat metric in order to simplify the exposition. After having derived the field equations with the flat metric as background, we will then show that everything can easily be generalized to accommodate an arbitrary metric.

The above transformations are subject to the condition of unit determinant (see (2.5)). This translates into

$$\begin{aligned} 0 &= \operatorname{Tr}\log\left(\frac{\delta X'}{\delta X}\right) \\ &\cong \int d^2\tilde{\sigma}\, d^2\tilde{\sigma}'\, \tilde{\delta}^2_{\Lambda}(\sigma,\sigma')\frac{\delta}{\delta X^{\mu\sigma}}\Big(f_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + f_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\Big). \end{aligned}$$

Using two dimensional rotational invariance, several cutoff dependent terms vanish, giving us

$$\tilde{\delta}^2_\Lambda(0)\int d^2\sigma \ (\partial_\mu f_{\mu\nu\lambda}\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + \partial_\mu f_{\mu\nu}\partial_+\partial_- X^{\nu\sigma}) = 0, \tag{3.52}$$

and

$$f_{\mu\mu} = 0, \tag{3.53}$$

where we have used the fact that, by translation invariance, $\tilde{\delta}^2_\Lambda(\sigma,\sigma)$, which will be shortened to $\tilde{\delta}^2_\Lambda(0)$, does not depend on σ.

The first condition is satisfied by setting

$$\partial_\mu f_{\mu\nu\lambda} - \partial_\lambda\partial_\mu f_{\mu\nu} + \partial_\lambda\Lambda_\nu - \partial_\nu\Lambda_\lambda = 0, \tag{3.54}$$

where Λ_μ is arbitrary. It is interesting to identify field combinations that are invariant under the transformations (3.50), subject to the constraints (3.53) and (3.54). $e_{\mu\mu}$ is clearly one such invariant; another combination which is almost invariant is given by

$$k_{\nu\lambda} = \partial_\mu e_{\mu\nu\lambda} - \partial_\lambda\partial_\mu e_{\mu\nu}. \tag{3.55}$$

Under (3.50), $k_{\nu\lambda}$ undergoes the following gauge transformation:

$$k_{\nu\lambda} \to k_{\nu\lambda} + 2(\partial_\nu\Lambda_\lambda - \partial_\lambda\Lambda_\nu), \tag{3.56}$$

and so it is the appropriate gauge invariant field strength constructed out of $k_{\nu\lambda}$ that is invariant. Later, we will see that this gauge invariance is broken for reasons that will become clear.

We can now extend the metric given by (3.5) to include the first order correction in b. The key observation is that if there were no restrictions on the f's, we could gauge away the fields $e_{\mu\nu\lambda}$ and $e_{\mu\nu}$ by a transformation of the form (3.49) by setting

$$\begin{aligned} f_{\mu\nu\lambda} &= \frac{1}{2}e_{\mu\nu\lambda}, \\ f_{\mu\nu} &= \frac{1}{2}e_{\mu\nu}. \end{aligned} \tag{3.57}$$

The metric extended to first order in b is then constructed starting with flat metric to zeroth order in b and carrying out the transformation (3.49), with the f's given by (3.57):

$$X^{\mu\sigma} \to X^{\mu\sigma} - \frac{b}{2}\int d^2\sigma'\, \tilde{\delta}^2_\Lambda(\sigma,\sigma')\Big(e_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\Big). \tag{3.58}$$

The result is

$$
\begin{aligned}
g_{\mu\sigma,\nu\sigma'} &= \eta_{\mu\nu}\delta^2_\Lambda(\sigma,\sigma') + b h^{(1)}_{\mu\sigma,\nu\sigma'},\\
h^{(1)}_{\mu\sigma,\nu\sigma'} &= -\frac{1}{2}\frac{\delta}{\delta X^{\nu\sigma'}}\left(e_{\mu\alpha\lambda}(X(\sigma))\partial_+ X^{\alpha\sigma}\partial_- X^{\lambda\sigma} + e_{\mu\alpha}(X(\sigma))\partial_+\partial_- X^{\alpha\sigma}\right)\\
&\quad + (\mu\sigma \leftrightarrow \nu\sigma').
\end{aligned}
\tag{3.59}
$$

If the constraints (3.53) and (3.54) did not exist, this would be a trivial metric, equivalent to a flat metric. In that case, there would be no need to go to the trouble of constructing it; it would have been simpler to fix gauge by eliminating the fields $e_{\mu\nu\lambda}$ and $e_{\mu\nu}$. However, the constraints on the f's make (3.59) a non-trivial metric: Because of these constraints, $h_{\mu\sigma,\nu\sigma'}$ can no longer be transformed away, and neither can the e's be completely eliminated. It is easy to check directly, using (3.50), that, even in the presence of the constraints, (3.59) transforms correctly to first order in b under (3.49).

There is a somewhat subtle issue of the invariance of this metric under the linear gauges given in Appendix A. So far, we have been working with the gauge fixed form of the action given by (3.38), and since the action was gauge invariant to start with, this gauge fixing is legitimate. We have to show that, the metric proposed above, is also the gauge fixed form of a gauge invariant expression. In fact, the functions $e_{\mu\nu\lambda}$ and $e_{\mu\nu}$ can be written in gauge invariant form; this is implicit in the gauge fixing procedure that led to (3.38). Explicitly, we can define two tensors invariant under linear gauges by

$$
\begin{aligned}
T_{\mu\nu\lambda} &= e^{(1)}_{\mu\nu\lambda} - 2e^{(2)}_{\nu\mu\lambda} - 2e^{(3)}_{\lambda\nu\mu} - \partial_\nu e^{(2)}_{\mu\lambda} - \partial_\lambda e^{(3)}_{\mu\nu} + \partial_\lambda e^{(4)}_{\nu\mu} + \partial_\nu e^{(4)}_{\mu\lambda} + \partial_\mu e^{(4)}_{\nu\lambda},\\
T_{\mu\nu} &= e^{(1)}_{\mu\nu} - \frac{1}{2}e^{(2)}_{\mu\nu} - \frac{1}{2}e^{(2)}_{\nu\mu} - \frac{1}{2}e^{(3)}_{\mu\nu} - \frac{1}{2}e^{(3)}_{\nu\mu} + \frac{1}{2}e^{(4)}_{\mu\nu} + \frac{1}{2}e^{(4)}_{\nu\mu}.
\end{aligned}
\tag{3.60}
$$

After gauge fixing, $T_{\mu\nu\lambda}$ and $T_{\mu\nu}$ reduce to $e^{(1)}_{\mu\nu\lambda}$ and $e^{(1)}_{\mu\nu}$ respectively.

From the metric given above, one can find the first order correction in b to the connection and the generators of the conformal transformations (see (2.30) and the related discussion). The standard formula of differential geometry expressing the connection in terms of metric gives

$$
\begin{aligned}
\Gamma^{(1)\lambda\sigma'''}_{\mu\sigma,\nu\sigma'} &= -\frac{1}{2}\frac{\delta^2}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\int d\tilde{\sigma}''\,\tilde{\delta}^2_\Lambda(\sigma''',\sigma'')\\
&\quad\times\left(e_{\lambda\alpha\beta}(X(\sigma''))\partial_+ X^{\alpha\sigma''}\partial_- X^{\beta\sigma''} + e_{\lambda\alpha}(X(\sigma''))\partial_+\partial_- X^{\alpha\sigma''}\right).
\end{aligned}
\tag{3.61}
$$

Now let us compute the corrected conformal generators. Write (2.30) as

$$F^{\mu\sigma}_{v}(X) = \int d^2\!\sigma'\, v(\sigma'_+) F^{\mu\sigma}_{\sigma'}(X),$$
$$F^{\mu\sigma}_{\sigma'}(X) = \delta^2(\sigma' - \sigma)\partial_+ X^{\mu\sigma} + f^{\mu\sigma}_{\sigma'}(X),$$

where $f^{\mu\sigma}_{\sigma'}$ starts at first order in b. Taking advantage of the fact that $F^{\mu\sigma}_{v}$ transforms like a vector in the indices $\mu\sigma$, the first order correction is computed exactly as in the case of the metric. Start with

$$F^{\mu\sigma}_{\sigma'}(X) = \delta^2(\sigma' - \sigma)\partial_+ X^{\mu\sigma}$$

at zeroth order, and apply the vector transformation law to it under coordinate transformation (3.58), with the result

$$\begin{aligned}\int d^2\!\sigma'\, v(\sigma'_+) f^{\mu\sigma}_{\sigma'} =& -\frac{b}{2}\int d^2\!\sigma'\, \partial_{\sigma_+}\tilde{\delta}^2_{\Lambda}(\sigma,\sigma')(v(\sigma_+) - v(\sigma'_+)) \\ &\times \left(e_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\right).\end{aligned} \tag{3.62}$$

This result can be simplified in the limit of large cutoff. As Λ becomes large, $\delta^2_{\Lambda}(\sigma,\sigma') \to \delta^2(\sigma - \sigma')$, and $\partial_{\sigma_+}\tilde{\delta}^2_{\Lambda}(\sigma,\sigma')(v(\sigma_+) - v(\sigma'_+)) \to -\tilde{\delta}^2_{\Lambda}(\sigma,\sigma')v'(\sigma'_+)$, so we can write

$$\begin{aligned}\int d^2\!\sigma'\, v(\sigma'_+) f^{\mu\sigma}_{\sigma'} \cong& \frac{b}{2}\int d^2\!\sigma'\, v'(\sigma'_+)\tilde{\delta}^2_{\Lambda}(\sigma,\sigma') \\ &\times \left(e_{\mu\nu\lambda}(X(\sigma'))\partial_+ X^{\nu\sigma'}\partial_- X^{\lambda\sigma'} + e_{\mu\nu}(X(\sigma'))\partial_+\partial_- X^{\nu\sigma'}\right).\end{aligned}$$

We have now at hand all the information needed to evaluate (2.38) to first order in b; the action is given by (3.48), the connection by (3.61), and the conformal generator by (3.62). Since the calculation is straightforward but somewhat tedious, we skip the details and instead, indicate the main steps. Part of the calculation was already carried out for the non-covariant case in section 4; all we have to do is to add the extra terms that arise from the connection and from f in (3.62). We first calculate the terms that contribute to E_G in (2.38); a simple calculation gives

$$\int d^2\!\sigma' v(\sigma'_+) f^{\mu\sigma}_{\sigma'} \frac{\delta S^{(0)}}{\delta X^{\mu\sigma}} = -b\int d^2\!\sigma\, v'(\sigma_+)\partial_+\partial_- X^{\mu\sigma}\left(e_{\mu\nu\lambda}\partial_+ X^{\nu\sigma}\partial_- X^{\lambda\sigma} + e_{\mu\nu}\partial_+\partial_- X^{\nu\sigma}\right),$$

and therefore, to first order in b,

$$\left(\int d^2\!\sigma'\, v(\sigma'_+) F^{\mu\sigma}_{\sigma'} \frac{\delta S}{\delta X^{\mu\sigma}}\right)^{(1)} \cong \int d^2\!\sigma\, v'(\sigma_+)\left(e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2}\right),$$

since the other two terms, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$, cancel. Next, expanding the Tr log as in (3.11), we compute $H_{\mu\sigma,\nu\sigma'}$ to first order in b:

$$
\begin{aligned}
H^{(1)}_{\mu\sigma,\nu\sigma'} =& \frac{\delta^2 S^{(1)}}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} - \Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'}\frac{\delta S^{(0)}}{\delta X^{\lambda\sigma''}} \\
=& \frac{\delta^2 S'}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}} + \partial_{\sigma_+}\partial_{\sigma_-}\left(\frac{\delta}{\delta X^{\nu\sigma'}}\left(e_{\mu\alpha\beta}\partial_+ X^{\alpha\sigma}\partial_- X^{\beta\sigma} + e_{\mu\alpha}\partial_+\partial_- X^{\alpha\sigma}\right)\right) \\
&+ (\mu\sigma \leftrightarrow \nu\sigma'),
\end{aligned}
\tag{3.63}
$$

where,

$$
S' = \int d^2\!\sigma\, e_{\mu_1\mu_2,\nu_1\nu_2}\partial_+ X^{\mu_1}\partial_+ X^{\mu_2}\partial_- X^{\nu_1}\partial_- X^{\nu_2}. \tag{3.64}
$$

Next, we apply δ (see (2.25)) to $H_{\mu\sigma,\nu\sigma'}$. The contribution coming from the first term on the right in (3.63),

$$
\left(\int d^2\!\sigma''\, v(\sigma'')\partial_+ X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\left(\eta^{\mu\nu}\Delta^{\sigma,\sigma'}(\Lambda)\frac{\delta^2 S'}{\delta X^{\mu\sigma}\delta X^{\nu\sigma'}}\right),
$$

has already been calculated in the last section; it is given by setting $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ in (3.42) equal to zero. The contribution of the second term in (3.63), after a somewhat lengthy computation, is given by

$$
\begin{aligned}
&\frac{1}{2}\left(\int d^2\!\sigma''\, v(\sigma''_+)\partial_+ X^{\lambda\sigma''}\frac{\delta}{\delta X^{\lambda\sigma''}} + \delta_\Lambda\right)\eta^{\mu\nu}\Delta^{\sigma,\sigma'}(\Lambda) \\
&\quad\times\left(\partial_{\sigma_+}\partial_{\sigma_-}\frac{\delta}{\delta X^{\nu\sigma'}}\left(e_{\mu\alpha\beta}\partial_+ X^{\alpha\sigma}\partial_- X^{\beta\sigma} + e_{\mu\alpha}\partial_+\partial_- X^{\alpha\sigma}\right) + (\mu\sigma \leftrightarrow \nu\sigma')\right) = \\
&= -\tilde{\delta}^2_\Lambda(0)\int d^2\!\sigma\, v'(\sigma_+)\partial_+ X^\alpha\partial_- X^\beta(\partial_\mu e_{\mu\alpha\beta} - \partial_\beta\partial_\mu e_{\mu\alpha}) \\
&\quad - \partial_+\partial_-\tilde{\delta}^2_\Lambda(0)\int d^2\!\sigma\, v'(\sigma_+)e_{\mu\mu},
\end{aligned}
\tag{3.65}
$$

with

$$
\partial_+\partial_-\tilde{\delta}^2_\Lambda(0) \equiv \left(\partial_{\sigma_+}\partial_{\sigma_-}\tilde{\delta}^2_\Lambda(\sigma,\sigma')\right)_{\sigma=\sigma'}.
$$

The main steps in the computation are the following: The critical term to be evaluated turns out to be

$$
\begin{aligned}
&\int d^2\!\sigma d^2\!\sigma'\,(\delta_\Lambda\Delta^{\sigma,\sigma'}(\Lambda))\left(\partial_{\sigma_+}\partial_{\sigma_-}\frac{\delta}{\delta X^{\mu\sigma'}}\left(e_{\mu\alpha\beta}\partial_+ X^{\alpha\sigma}\partial_- X^{\beta\sigma} + e_{\mu\alpha}\partial_+\partial_- X^{\alpha\sigma}\right)\right) = \\
&= -\int d^2\!\sigma\left(\delta_\Lambda\tilde{\delta}^2_\Lambda(\sigma,\sigma')\partial_\mu e_{\mu\alpha\beta}\partial_+ X^{\alpha\sigma}\partial_- X^{\beta\sigma} + \partial_{\sigma_+}\partial_{\sigma_-}\delta_\Lambda\tilde{\delta}^2_\Lambda(\sigma,\sigma')e_{\mu\mu}\partial_+\partial_- X^{\alpha\sigma}\right)_{\sigma'=\sigma},
\end{aligned}
\tag{3.66}
$$

and the cutoff dependent factors can be simplified using (2.29):

$$\left(\delta_\Lambda \tilde{\delta}^2_\Lambda(\sigma,\sigma')\right)_{\sigma=\sigma'} = v'(\sigma_+)\tilde{\delta}^2_\Lambda(0). \tag{3.67}$$

To obtain E_G to first order, the above correction term should be added to (3.42), with $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ set equal to zero.

We now consider the term $E_{\mathcal{M}}$ in (2.38); a straightforward calculation gives the result

$$\begin{aligned}
E^{(1)}_{\mathcal{M}} &= \frac{1}{2}\int d^2\sigma''\, v(\sigma''_+) \\
&\quad \times \left(\frac{\delta F^{\mu\sigma}_{\sigma''}}{\delta X^{\alpha\sigma'''}} G^{\alpha\sigma''',\nu\sigma'} + \frac{\delta F^{\nu\sigma'}_{\sigma''}}{\delta X^{\alpha\sigma'''}} G^{\mu\sigma,\alpha\sigma'''} - \delta_\Lambda G^{\mu\sigma,\nu\sigma'}\right) \frac{\delta^2 \operatorname{Tr}\log \mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\bigg|_{P=0} \\
&= \frac{1}{2}\left(\left(v(\sigma_+)\partial_{\sigma_+} + v(\sigma'_+)\partial_{\sigma'_+}\right)\Delta^{\mu\sigma,\nu\sigma'} - \delta_\Lambda \Delta^{\mu\sigma,\nu\sigma'}\right) \frac{\delta^2 \operatorname{Tr}\log \mathcal{M}}{\delta P^{\mu\sigma}\delta P^{\nu\sigma'}}\bigg|_{P=0} \\
&= \frac{1}{24\pi}\int d^2\sigma d^2\sigma'\, \frac{v(\sigma_+) - v(\sigma'_+)}{\sigma_+ - \sigma'_+}\eta^{\mu\nu}\left(\frac{\delta \Gamma^{\lambda\sigma''}_{\lambda\sigma'',\mu\sigma}}{\delta X^{\nu\sigma'}} + \frac{\delta \Gamma^{\lambda\sigma''}_{\lambda\sigma'',\nu\sigma'}}{\delta X^{\mu\sigma}} + \frac{\delta \Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'}}{\delta X^{\lambda\sigma''}}\right) \\
&= -\frac{b}{16\pi}\int d^2\sigma\, v'(\sigma_+)\Big(\partial_+\partial_-\tilde{\delta}^2_\Lambda(0)\Box e_{\lambda\lambda} \\
&\qquad + \tilde{\delta}^2_\Lambda(0)\left(\Box k_{\nu\lambda} - \partial_\nu\partial_\mu k_{\mu\lambda} + \partial_\lambda\partial_\mu k_{\mu\nu}\right)\partial_+ X^\nu \partial_- X^\lambda\Big),
\end{aligned} \tag{3.68}$$

where Γ is given by (3.61), and k is defined by (3.55).

Putting together (3.42), (3.65) and (3.68) in (2.38), we finally get the equations for the first massive level. These equations contain cutoff independent terms, which come only from (3.42), and cutoff dependent terms, which all come from (3.65) and (3.68). We note that all the cutoff dependent contributions come from terms proportional to the connection Γ, and therefore they are absent from a non-covariant calculation. We first write down the cutoff independent equations:

$$\begin{aligned}
\Box e_{\mu_1\mu_2,\nu_1\nu_2} + 16\pi e_{\mu_1\mu_2,\nu_1\nu_2} &= 0, \\
e_{\mu\mu,\nu_1\nu_2} = 0, \qquad e_{\mu_1\mu_2,\nu\nu} &= 0, \\
\partial_\mu e_{\mu\mu_1,\nu_1\nu_2} = 0, \quad \partial_\nu e_{\mu_1\mu_2,\nu\nu_1} &= 0.
\end{aligned} \tag{3.69}$$

We have one equation of motion and four constraints. In addition, we have three cutoff dependent equations. Two of them follow from the conformal transformations in σ_+:

$$\begin{aligned}
\frac{1}{16\pi}\left(\Box k_{\nu\lambda} - \partial_\nu\partial_\mu k_{\mu\lambda} + \partial_\lambda\partial_\mu k_{\mu\nu}\right) + k_{\nu\lambda} &= 0, \\
\frac{1}{16\pi}\Box e_{\lambda\lambda} + e_{\lambda\lambda} &= 0,
\end{aligned} \tag{3.70}$$

where k is defined by (3.55). The remaining equation (there is a fourth, repeated equation, for $e_{\lambda\lambda}$), results from conformal transformations in σ_- and it is conveniently written in terms of a field $\bar{k}$, defined by

$$\bar{k}_{\nu\lambda} = \partial_\mu e_{\mu\nu\lambda} - \partial_\nu \partial_\mu e_{\mu\lambda}, \tag{3.71}$$

and it reads

$$\frac{1}{16\pi}\left(\Box \bar{k}_{\nu\lambda} - \partial_\lambda \partial_\mu \bar{k}_{\nu\mu} + \partial_\nu \partial_\mu \bar{k}_{\lambda\mu}\right) + \bar{k}_{\nu\lambda} = 0. \tag{3.72}$$

The equation satisfied by k is not invariant under the gauge transformations given by (3.56). The reason for this is the following: In the computation of the determinant, the cutoff dependent factor $\tilde{\delta}^2_\Lambda(\sigma, \sigma')$ at $\sigma = \sigma'$ is σ independent and therefore it can be put in front of the integral in (3.52). The integral itself is then invariant under the gauge transformation (3.56). On the other hand, in the main step leading to (3.65), the cutoff variation of the same factor at $\sigma = \sigma'$ is σ dependent (see (3.66), (3.67), and also the discussion following (2.29)). As a consequence, an additional factor $v'(\sigma_+)$, as compared to (3.52), appears in the integral on the right hand side of (3.65), and this spoils gauge invariance under (3.56). It is, therefore, necessary to modify the condition (3.54); it should be replaced by

$$\begin{aligned} \partial_\mu f_{\mu\nu\lambda} &= 0, \\ \partial_\mu f_{\mu\nu} &= 0. \end{aligned} \tag{3.73}$$

Both k and $\bar{k}$ are invariant under the transformations satisfying these more stringent conditions.

Going back to the equations (3.69), we see that two of the constraints are too stringent,

$$e_{\mu\mu,\nu_1\nu_2} = 0, \qquad e_{\mu_1\mu_2,\nu\nu} = 0, \tag{3.74}$$

eliminating degrees of freedom from the field $e_{\mu_1\mu_2,\nu_1\nu_2}$ which are present in the string spectrum (see Appendix B). The hope is that k and $\bar{k}$ could supply the missing degrees of freedom. We shall see below that this happens in the left-right symmetric case, with parity invariance on the world sheet, which interchanges σ_+ and σ_-. In this case, e is invariant under the interchange of the μ's with ν's, and the components eliminated by (3.74) are the same as those of a symmetric second rank tensor. We have analyzed equations (3.70) and (3.72) in the left-right symmetric case, when

$$e_{\mu\nu\lambda} = e_{\mu\lambda\nu}.$$

Defining

$$l_{\nu\lambda} \equiv \partial_\mu e_{\mu\nu\lambda}, \qquad l_\nu \equiv \partial_\mu e_{\mu\nu},$$

and

$$A_{\mu\nu} \equiv 2l_{\mu\nu} - \partial_\mu l_\nu - \partial_\nu l_\mu, \qquad L_{\mu\nu} \equiv \partial_\mu l_\nu - \partial_\nu l_\mu,$$

one can easily show that equations (3.70) and (3.72) are equivalent to the equations

$$\begin{aligned} &\frac{1}{16\pi}\Box A_{\mu\nu} + A_{\mu\nu} = 0, \\ &\frac{1}{16\pi}\Box L_{\mu\nu} + L_{\mu\nu} = 0, \end{aligned} \tag{3.75}$$

plus the constraint

$$\partial_\mu \partial_\nu A_{\mu\lambda} - \partial_\mu \partial_\lambda A_{\mu\nu} = \Box L_{\lambda\nu}. \tag{3.76}$$

The number of independent degrees of freedom of the above system is the same as that of a symmetric second order tensor minus a scalar. The missing scalar is provided by $e_{\mu\mu}$, so in the final count, the fields k and $\bar{k}$ provide the missing degrees of freedom needed to establish agreement with the string theory spectrum. Unfortunately, in the general case with no left-right symmetry, there are still missing degrees of freedom, and at the present time, we have no solution to this problem. Our suspicion is that our method in its present form is applicable only in the symmetric case, and some new ideas are needed to extend it to the general case[6].

We close this section by a brief description of the promised extension of the results of this section to the case of a general gravitational background. This means replacing the flat background given by $\eta_{\mu\nu}\delta^2_\Lambda(\sigma,\sigma')$ in (3.59) by the metric $g_{\mu\sigma,\nu\sigma'}$ of (3.5). We have to show that the equations of this section can be covariantized with respect to this metric. Most of the time, the task is trivial; one has to keep the upper and lower indices of tensors match correctly and use the metric to raise and lower indices as needed. For example, in (3.38), the first term on the right is correctly written, since $\partial_+ X^\mu$ and $\partial_- X^\mu$ transform as contravariant vectors. On the other hand, $\partial_+\partial_- X^\mu$ is not a vector; it should be replaced by

$$\partial_+\partial_- X^\mu \rightarrow \partial_+\partial_- X^\mu + \partial_+ X^\alpha \partial_- X^\beta \Gamma^\mu_{\alpha\beta}(X(\sigma)),$$

6 See the Epilogue section for a brief note on new work done by Bardakci

where the connection Γ is given by (3.18). Similarly, the partial derivative with respect to X in (3.59) should be replaced by the covariant derivative using the same connection: For example,

$$\frac{\delta}{\delta X^{\mu\sigma}}(V_{\nu\sigma'}) \rightarrow \frac{\delta}{\delta X^{\mu\sigma}}(V_{\mu\sigma'}) - \Gamma^{\lambda\sigma''}_{\mu\sigma,\nu\sigma'} V_{\lambda\sigma''},$$

for a vector $V_{\nu\sigma'}$. One can easily show that everything in this section goes through with these modifications. Notice, however, that in all this we have worked only with the metric, which is symmetric, and we have dropped the antisymmetric tensor altogether. This is clearly permissible only in a left-right symmetric model. It is clear that, in order to generalize our treatment to the left-right non-symmetric string, we have to figure out how to incorporate the antisymmetric tensor in the discussion above.

4. Discussion

With this work we have proposed a new approach for deriving the string field equations from a general sigma model on the world sheet. Those equations can be made covariant under not only local, but also non-local transformations in the field space. In this approach the world sheet one loop result is exact, although it may only give incomplete information, to be supplemented by higher loop results. We applied this method to derive the equations for the tachyon, massless and first massive level. The spectrum of states that follows from the linear part of these equations of motion was shown to agree with the known spectrum of strings. This is in contrast with a non-covariant approach, where the equations are too weak to produce the right spectrum.

In this work we only analyzed the linear part of the equations. We did not address the question of string interactions, neither did we attempt to extend our results to higher string loops. It would be desirable to go beyond the expansion we have used, and establish exact covariance under non-local transformations. Also, natural generalizations such as a better treatment of left-right non-symmetric closed string (see Epilogue below), strings with boundaries (open strings) and fermionic strings are worthy of investigation.

Epilogue

While this thesis was being written, Bardakci pushed this direction of research further and solved some of the difficulties present in this work. For instance, there is no need anymore to have left-right symmetry. To achieve that, one shouldn't assume that the connection is derived from a metric, as we did here. Instead, it should be left undefined as much as possible, the only constrain being in the number of derivatives present to give the right conformal dimension. Due to the extra freedom this gives, many constraints are not present anymore because they can be absorbed in the definition of the connection. The equations and constraints one gets in the end are just enough to get the right number of degrees of freedom, without the need to impose some left-right symmetry; and the one to one correspondence between the known string spectrum and those equations and constraints is easily established. Also, since the connection doesn't come from a metric, the restriction to coordinate transformations of unit determinant is not present anymore. The Virasoro symmetries were also shown to be satisfied.

Appendix A.

In this appendix we fill up the steps that lead from (3.35) to (3.38). As it was said in the comments that follow (3.35), of the eight fields present there, all but three can be eliminated by linear gauge transformations. The six distinct linear gauge transformations that we can add to (3.35) are:

$$
\begin{aligned}
1)\quad & \partial_+(\varepsilon_{\mu,\nu_1\nu_2}\partial_+X^\mu\partial_-X^{\nu_1}\partial_-X^{\nu_2}) = \partial_{\mu_1}\varepsilon_{\mu_2,\nu_1\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} \\
& \quad + \varepsilon_{\mu,\nu_1\nu_2}(\partial_+^2X^\mu\partial_-X^{\nu_1}\partial_-X^{\nu_2} + 2\partial_+X^\mu\partial_+\partial_-X^{\nu_1}\partial_-X^{\nu_2}) \\
2)\quad & \partial_-(\bar\varepsilon_{\mu_1\mu_2,\nu}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^\nu) = \partial_{\nu_1}\bar\varepsilon_{\mu_1\mu_2,\nu_2}\partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^{\nu_1}\partial_-X^{\nu_2} \\
& \quad + \bar\varepsilon_{\mu_1\mu_2,\nu}(2\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}\partial_-X^\nu + \partial_+X^{\mu_1}\partial_+X^{\mu_2}\partial_-^2X^\nu) \\
3)\quad & \partial_+(\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_-X^{\mu_2}) = \partial_\nu\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+X^\nu\partial_+\partial_-X^{\mu_1}\partial_-X^{\mu_2} \\
& \quad + \varepsilon^{(1)}_{\mu_1\mu_2}(\partial_+^2\partial_-X^{\mu_1}\partial_-X^{\mu_2} + \partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}) \\
4)\quad & \partial_-(\bar\varepsilon^{(1)}_{\mu_1\mu_2}\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2}) = \partial_\nu\bar\varepsilon^{(1)}_{\mu_1\mu_2}\partial_-X^\nu\partial_+\partial_-X^{\mu_1}\partial_+X^{\mu_2} \\
& \quad + \bar\varepsilon^{(1)}_{\mu_1\mu_2}(\partial_-^2\partial_+X^{\mu_1}\partial_+X^{\mu_2} + \partial_+\partial_-X^{\mu_1}\partial_+\partial_-X^{\mu_2}) \\
5)\quad & \partial_+(\varepsilon^{(2)}_{\mu_1\mu_2}\partial_+X^{\mu_1}\partial_-^2X^{\mu_2}) = \partial_\nu\varepsilon^{(2)}_{\mu_1\mu_2}\partial_+X^\nu\partial_+X^{\mu_1}\partial_-^2X^{\mu_2} \\
& \quad + \varepsilon^{(2)}_{\mu_1\mu_2}(\partial_+^2X^{\mu_1}\partial_-^2X^{\mu_2} + \partial_+X^{\mu_1}\partial_+\partial_-^2X^{\mu_2}) \\
6)\quad & \partial_-(\bar\varepsilon^{(2)}_{\mu_1\mu_2}\partial_-X^{\mu_1}\partial_+^2X^{\mu_2}) = \partial_\nu\bar\varepsilon^{(2)}_{\mu_1\mu_2}\partial_-X^\nu\partial_-X^{\mu_1}\partial_+^2X^{\mu_2} \\
& \quad + \bar\varepsilon^{(2)}_{\mu_1\mu_2}(\partial_-^2X^{\mu_1}\partial_+^2X^{\mu_2} + \partial_-X^{\mu_1}\partial_+^2\partial_-X^{\mu_2}).
\end{aligned}
\tag{A.1}
$$

To eliminate $e^{(4)}_{\mu\nu}$ choose

$$e^{(4)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0. \tag{A.2}$$

To eliminate $e^{(2)}_{\mu\nu}$ choose

$$e^{(2)}_{\mu_1\mu_2} + \varepsilon^{(1)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0. \tag{A.3}$$

To eliminate $e^{(3)}_{\mu\nu}$ choose

$$e^{(3)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(1)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_2\mu_1} = 0. \tag{A.4}$$

To eliminate $e^{(2)}_{\mu\nu\lambda}$ choose

$$e^{(2)}_{\mu_1\mu_2\mu_3} + \varepsilon_{\mu_1,\mu_2\mu_3} + \frac{1}{2}(\partial_{\mu_2}\bar{\varepsilon}^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\bar{\varepsilon}^{(2)}_{\mu_2\mu_1}) = 0. \tag{A.5}$$

To eliminate $e^{(3)}_{\mu\nu\lambda}$ choose

$$e^{(3)}_{\mu_1\mu_2\mu_3} + \bar{\varepsilon}_{\mu_2\mu_3,\mu_1} + \frac{1}{2}(\partial_{\mu_2}\varepsilon^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\varepsilon^{(2)}_{\mu_2\mu_1}) = 0. \tag{A.6}$$

By choosing $\varepsilon^{(2)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1}$, $\varepsilon^{(1)}_{\mu_1\mu_2}$, $\bar{\varepsilon}^{(1)}_{\mu_1\mu_2}$, $\varepsilon_{\mu_1,\mu_2\mu_3}$, and $\bar{\varepsilon}_{\mu_2\mu_3,\mu_1}$ properly, we can eliminate everything except $e_{\mu_1\mu_2,\nu_1\nu_2}$, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$; we dropped the superscript (1) after gauge fixing. The transformations that preserve this gauge are

$$\begin{gathered}
\varepsilon^{(2)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \qquad \varepsilon^{(1)}_{\mu_1\mu_2} + \bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \qquad \bar{\varepsilon}^{(1)}_{\mu_1\mu_2} + \varepsilon^{(2)}_{\mu_2\mu_1} = 0, \\
2\varepsilon_{\mu_1,\mu_2\mu_3} + \partial_{\mu_2}\bar{\varepsilon}^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\bar{\varepsilon}^{(2)}_{\mu_2\mu_1} = 0, \\
2\bar{\varepsilon}_{\mu_2\mu_3,\mu_1} + \partial_{\mu_2}\varepsilon^{(2)}_{\mu_3\mu_1} + \partial_{\mu_3}\varepsilon^{(2)}_{\mu_2\mu_1} = 0,
\end{gathered} \tag{A.7}$$

plus, we could also add

$$\partial_+(\varepsilon_{\mu_1}\partial_+\partial_-^2 X^{\mu_1}) - \partial_-(\varepsilon_{\mu_1}\partial_-\partial_+^2 X^{\mu_1}) = \partial_{\mu_1}\varepsilon_{\mu_2}(\partial_+X^{\mu_1}\partial_+\partial_-^2 X^{\mu_2} - \partial_-X^{\mu_1}\partial_-\partial_+^2 X^{\mu_2}). \tag{A.8}$$

All of these linear transformations act trivially on (3.38); they leave $e_{\mu_1\mu_2,\nu_1\nu_2}$, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ invariant. This means that linear gauges are completely fixed in the form given by (3.38).

Appendix B.

In this appendix we show that the constraints (equations (3.46) and (3.47)) obtained in section 4 for the first massive level fields are too weak. This will be done by comparing those constraints with the analogue constraints of the first massive level of the string.

To obtain these consider the most general level 2 state $|s\rangle$ given by

$$|s\rangle = \Big(E_{\mu_1\mu_2,\nu_1\nu_2} a_1^{\dagger\mu_1} a_1^{\dagger\mu_2} b_1^{\dagger\nu_1} b_1^{\dagger\nu_2} + E_{\mu,\nu_1\nu_2} a_2^{\dagger\mu} b_1^{\dagger\nu_1} b_1^{\dagger\nu_2} + E_{\mu_1\mu_2,\nu} a_1^{\dagger\mu_1} a_1^{\dagger\mu_2} b_2^{\dagger\nu} + E_{\mu,\nu} a_2^{\dagger\mu} b_2^{\dagger\nu}\Big)|0\rangle, \tag{B.1}$$

where a_n^μ, $a_n^{\dagger\mu}$ and b_n^μ, $b_n^{\dagger\mu}$ are the closed string operators. This state satisfies the relations

$$(L_0 - 1)|s\rangle = (\bar{L}_0 - 1)|s\rangle = 0$$

and

$$L_1|s\rangle = \bar{L}_1|s\rangle = 0, \qquad L_2|s\rangle = \bar{L}_2|s\rangle = 0,$$

from which we get some conditions between the E's. The gauge freedom present in these conditions can be taken care of by adding zero norm states to $|s\rangle$. After that is done we get the constraints

$$\begin{aligned}
p^\mu E_{\mu\mu_1,\nu_1\nu_2} + \sqrt{2} p_{\mu_1} E_{,\nu_1\nu_2} &= 0, \\
p^\nu E_{\mu_1\mu_2,\nu_1\nu} + \sqrt{2} p_{\nu_1} E_{\mu_1\mu_2,} &= 0, \\
p^\mu E_{\mu\mu_1,} + \sqrt{2} p_{\mu_1} E &= 0, \\
p^\nu E_{,\nu_1\nu} + \sqrt{2} p_{\nu_1} E &= 0, \\
4\sqrt{2} E_{,\nu_1\nu_2} - E_{\mu\mu,\nu_1\nu_2} &= 0, \\
4\sqrt{2} E_{\mu_1\mu_2,} - E_{\mu_1\mu_2,\nu\nu} &= 0,
\end{aligned} \tag{B.2}$$

where the new E's are related to the old ones by

$$E_{\mu,\nu_1\nu_2} = p_\mu E_{,\nu_1\nu_2}, \qquad E_{\mu_1\mu_2,\nu} = p_\nu E_{\mu_1\mu_2,}, \qquad E_{\mu,\nu} = p_\mu p_\nu E.$$

Combining, the only constraints on $E_{\mu_1\mu_2,\nu_1\nu_2}$ are

$$\begin{aligned}
p^\mu E_{\mu\mu_1,\nu_1\nu_2} + \frac{1}{4} p_{\mu_1} E_{\mu\mu,\nu_1\nu_2} &= 0, \\
p^\nu E_{\mu_1\mu_2,\nu_1\nu} + \frac{1}{4} p_{\nu_1} E_{\mu_1\mu_2,\nu\nu} &= 0,
\end{aligned} \tag{B.3}$$

and all other E's are given in terms of $E_{\mu_1\mu_2,\nu_1\nu_2}$. If we consider that the E's play the analogue roll of the e's in section 4, then the constraints (3.46) and (3.47) are not powerfull enough, because for example, $e_{\mu_1\mu_2\mu_3}$ and $e_{\mu_1\mu_2}$ are not completely determined in terms of $e_{\mu_1\mu_2,\nu_1\nu_2}$.

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