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Symplectic geometric methods in microlocal sheaf theory

by

Christopher Lp Kuo

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Committee in charge:

Professor Vivek Shende, Chair
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Symplectic geometric methods in microlocal sheaf theory

Abstract

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by

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Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Vivek Shende, Chair

The main goal of this dissertation is to import symplectic geometric methods into microlocal sheaf theory based on the foundational sheaf quantization construction by Guillermou, Kashiwara, and Schapira. This construction provides the notion of isotopies of sheaves and a sheaf-theoretic analogue of the notion of continuation maps in Lagrangian Floer theory.

Based on previous work by Ganatra, Pardon, and Shende, we make further examination on the category of unbounded sheaves microsupported in some singular isotropic Λ in the cosphere bundle. We show that various categorical constructions concerning this category can be described in symplectic geometric terms by using isotopies of sheaves.

The main construction is a sheaf-theoretic analogue of the wrapped Fukaya category, by localizing a category of sheaves microsupported away from some given Λ along continuation maps. When Λ is a subanalytic singular isotropic, we also construct a comparison map to the category of compact objects in the category mentioned above, and show that it is an equivalence. The last statement can be seen as a sheaf-theoretical incarnation of the sheaf-Fukaya comparison theorem of Ganatra-Pardon-Shende.

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Acknowledgements

Chapter 1

Introduction

The microlocal sheaf theory of Kashiwara and Schapira [32] relates sheaves – which are topological structures – on manifolds with the symplectic geometry of their cotangent bundles. The basic construction is the *microsupport*: to a sheaf F on a C^1 manifold M, one associates a closed conic subset $\mathrm{SS}(F)$ in the cotangent bundle T^*M. The intersection of this set with the zero section recovers the support $\mathrm{supp}(F)$; the projectivization $\mathrm{SS}^\infty(F) := (\mathrm{SS}(F) \setminus 0_M)/\mathbb{R}_{>0}$ indicates the codirections along which the sheaf changes. A key indicator of the symplectic nature of the theory is the involutivity theorem [32, Thm. 6.5.4], which asserts that $\mathrm{SS}(F)$ is always a singular coisotropic subset with respect to the canonical symplectic structure of T^*M. Under an appropriate tameness assumption, $\mathrm{SS}(F)$ is a singular Lagrangian if and only if F is constructible, i.e., there exists a stratification $\{X_s\}$ of M such that $F|_{X_s}$ is a local system for all X_s.

Deeper relationships between microlocal sheaf theory and symplectic geometry began to emerge in the mid 2000s. Nadler and Zaslow related constructible sheaves to 'infinitesimally wrapped' Floer theory [44, 40]. Meanwhile, Tamarkin introduced purely sheaf-theoretical methods into symplectic topology in his study of non-displaceability, a problem previously studied largely by Floer theoretic methods [53]. The subsequent Guillermou-Kashiwara-Schapira sheaf quantization of contact isotopies [26] — i.e. the highly nonobvious statement that contact isotopies of S^*M act on sheaves on M — led to a host of further incursions by sheaf theorists into symplectic topology [23, 24, 25, 50, 49, 10, 11, 48, 55, 8, 9, 29, 4, 5, 6, 34, 35] and vice versa [56, 57, 46].

Meanwhile on the Floer theoretic side, Abouzaid and Seidel formulated a way to incorporate contact dynamics into Fukaya categories for noncompact symplectic manifolds [2]. Their construction is roughly to localize a partially-defined infinitesimally wrapped Fukaya category along 'continuation morphisms' associated to positive isotopies. The resulting notion of wrapped Fukaya category (and its later 'partially wrapped' generalizations [51, 20, 19]) provides the correct mirrors to coherent sheaf categories on certain singular spaces. While such wrapped categories are nontrivial to compute directly (the simplest case was [1]), Nadler conjectured that they matched categories of compact objects inside categories of unbounded sheaves with prescribed microsupport [41]. This conjecture was later established by the work of Ganatra, Pardon, and Shende [21]; as a result, sheaf-theoretic methods (e.g. [41, 42, 14, 33, 18, 17]) can be used to establish homological mirror symmetry

in these settings.

While Nadler termed his category the 'wrapped sheaves', the name did not entirely reflect its construction: there is no wrapping in their definition. In this paper we study the category which this name manifestly suits — the localization of the category of sheaves along the continuation morphisms of [26] — and give an entirely sheaf-theoretic proof that it is equivalent to Nadler's category.

Remark 1.0.1. Let us compare and contrast this article with the work [21]. In that article, the authors construct an equivalence between the partially wrapped Fukaya category of a cotangent bundle, stopped along some subanalytic isotropic Λ, with the category of compact objects in the category of unbounded sheaves microsupported in Λ:

$$\operatorname{Perf} \mathcal{W}(T^*M, \Lambda)^{op} \cong \operatorname{Sh}_\Lambda(M)^c$$

Their approach was to introduce an abstract axiomatic characterization ('microlocal Morse theatre') and verify that both sides satisfy it.

Here is another approach which is illustrated in the following diagram:

$$
\begin{array}{ccccc}
\operatorname{Fuk}_\epsilon(T^*M)^{op} \supseteq \operatorname{Fuk}_\epsilon(T^*M, \Lambda)^{op} & \dashrightarrow^{l_{\operatorname{Fuk}}} & \mathcal{W}(T^*M, \Lambda)^{op} & & \\
\| [44,\,40] \qquad\qquad \| [44,\,40] & & \uparrow \quad\quad [21] & & \\
\operatorname{Sh}_{\operatorname{constr}}(M) \supseteq \operatorname{Sh}_{\operatorname{constr},S^*M\backslash\Lambda}(M) & \xrightarrow{l_{\operatorname{Sh}}} & \mathfrak{wsh}_\Lambda(M) & \xrightarrow{\mathfrak{W}_\Lambda^+} & \operatorname{Sh}_\Lambda(M)^c
\end{array}
$$

Assume given an equivalence of infinitesimally wrapped categories such as asserted in [44, 40], localize both sides along the continuation morphisms, and then show purely on the sheaf-theoretic side that the resulting localized category of sheaves is in fact equivalent to $\operatorname{Sh}_\Lambda(M)^c$. See Figure 3.2 for an illustration. The present article establishes the last step, indicated by $\mathfrak{W}_\Lambda^+$, in this argument. While this route of proof would be logically independent of [20, 19, 21], the proof of the main theorem of the present article follows a strategy adapted from [21].

Note however that, in the literature, the category $\mathcal{W}(T^*M, \Lambda)$ is not constructed as a localization of an infinitesimally wrapped category. In fact, constructions of $\mathcal{W}(T^*M, \Lambda)$ as in [2, 51, 20], always avoid infinitesimal wrappings. For example, the authors in [20] begin with the sub-semi-category $\operatorname{Fuk}^{\operatorname{pre}}(T^*M, \Lambda)$ with morphisms defined only between transversal Lagrangians, and roughly speaking localize along continuation maps so they only ever have to work with large wrappings.

We turn to a more precise discussion of our results. Classically, Kashiwara and Schapira study the totality of sheaves on manifolds $\operatorname{Sh}(M)$ and use the notion of microsupport to measure various properties regarding sheaves and morphisms between them [32]. Based on the modern higher categorical setting, recent study [21, 43] suggests that the subcategory $\operatorname{Sh}_\Lambda(M)$ of sheaves with some fixed singular isotropic microsupport condition Λ is also an important invariant and enjoys many decent properties. For example, it is shown in [21] that such categories are compactly generated and there is a procedure to find its compact
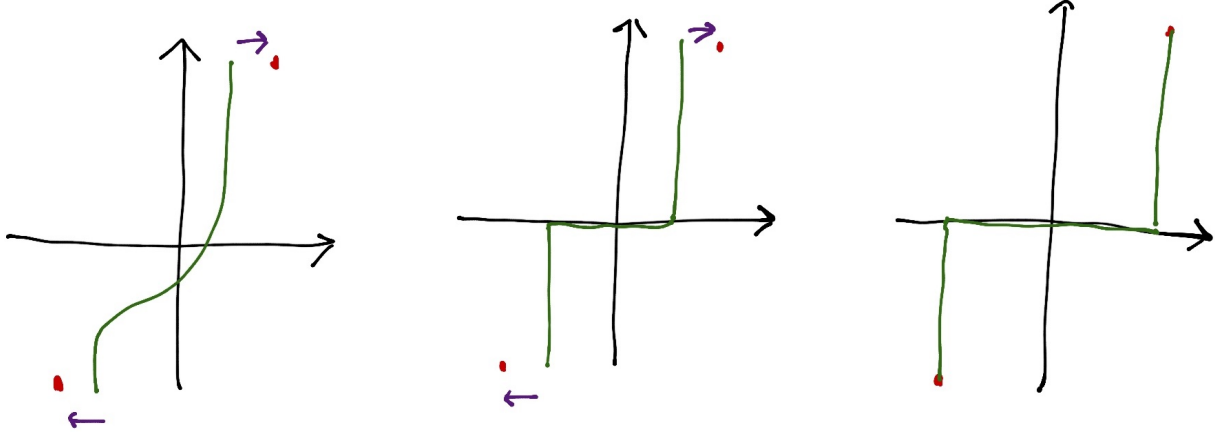
Figure 1.1: An alternative to [21], illustrated for $M = \mathbb{R}^1$. The purple arrow indicates the positive Reeb direction and the red dots are the stop Λ. The green objects, from left to right, are a smooth curve conic at infinity in $T^*\mathbb{R}^1$, the microsupport of a constant sheaf on an open interval, and the microsupport of the same sheaf after being pushed to the stop.

objects. Following this line of research, we show that categories of the form $\mathrm{Sh}_\Lambda(M)$ enjoy some compatibility property with the standard symmetric monoidal structure $\otimes$ of $\mathrm{Pr}^{\mathrm{L}}_{st}$, the category of presentable categories with morphisms being left adjoints.

Proposition 1.0.2. *Let M, N be real analytic manifolds and $\Lambda \subseteq S^*M$, $\Sigma \subseteq S^*N$ be subanalytic singular isotropics. Denote by $\Lambda \times \Sigma$ the product singular isotropic in $S^*(M \times N)$, then there is a equivalence*

$$\mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_\Sigma(N) = \mathrm{Sh}_{\Lambda \times \Sigma}(M \times N)$$
$$(F, G) \mapsto F \boxtimes G.$$

See Proposition 3.4 for details. Based on this statement, we give a geometric description for the categorical dual $\mathrm{Sh}_\Lambda(M)^\vee$ with respect to $(\mathrm{Pr}^{\mathrm{L}}_{st}, \otimes)$ as the category $\mathrm{Sh}_{-\Lambda}(M)$ with reversed microsupport. A consequence of this identification is a classification of all colimiting-preserving functors between categories of the form $\mathrm{Sh}_\Lambda(M)$:

Theorem 1.0.3. *Let M and N be real analytic manifolds and $\Lambda \subseteq S^*M$, $\Sigma \subseteq S^*N$ be closed subanalytic singular isotropics. Then, the identification $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$ induces an equivalence*

$$\mathrm{Sh}_{-\Lambda \times \Sigma}(M \times N) = \mathrm{Fun}^L(\mathrm{Sh}_\Lambda(M), \mathrm{Sh}_\Sigma(N))$$

which is given by $K \mapsto (H \mapsto K \circ H)$ for $H \in \mathrm{Sh}_\Sigma(N)$.

See Section 3.6 for details.

Now we bring in symplectic geometry and discuss the continuation maps in the sheaf-theoretical setting. Recall that for a closed conic subset $X \subseteq T^*M$, the inclusion $\iota_* : \mathrm{Sh}_X(M) \hookrightarrow \mathrm{Sh}(M)$ of sheaves microsupported in X to all sheaves has a left adjoint ι^* and

right adjoint $\iota^!$. Now consider $F \in \mathrm{Sh}(M \times \mathbb{R})$ as a one-parameter family of sheaves on M and set $F_a \coloneqq F|_{M \times \{a\}}$. An explicit description of ι^* given in [26, Proposition 4.8] shows when F satisfies the condition $\mathrm{SS}(F) \subseteq T^*M \times T^*_{\leq}\mathbb{R}$, there is a *continuation map* $F_a \to F_b$ for $a \leq b$. (See [53, 2.2.2] and [27, (77)] for the dual construction.) Now pick a contact form α on S^*M coorienting the contact structure induced from the symplectic structure on T^*M. We say a C^∞ map $\Phi : S^*M \times \mathbb{R} \to S^*M$ is an isotopy if the induced map $\phi_t \coloneqq \Phi(-, t)$ is a contactomorphism for all $t \in \mathbb{R}$ and $\phi_0 = \mathrm{id}_{S^*M}$. If Φ is a positive isotopy $(\alpha(\partial_t \phi_t) \geq 0)$ then the corresponding GKS sheaf kernel $K(\Phi)$ and hence its convolution $K(\Phi) \circ F$ with $F \in \mathrm{Sh}(M)$ will satisfy this condition and hence admit continuation maps.

Now fixed an open set $\Omega \subseteq S^*M$. Homotopy classes of compactly supported isotopies with fixed ends can be organized to an ∞-category $W(\Omega)$ whose morphisms are given by concatenating with positive isotopies. We refer this category as the category of *positive wrappings*. The discussion on continuation maps will imply that there is a *wrapping kernel* functor $w : W(\Omega) \to \mathrm{Sh}(M \times M)$ which sends isotopies to the end point of the GKS sheaf kernels and positive isotopies to continuation maps. One can use this functor to the define the *infinite wrapping* functors $\mathfrak{W}^{\pm}(\Omega) : \mathrm{Sh}(M) \to \mathrm{Sh}_{S^*M \setminus \Omega}(M)$ by sending F to the colimit $\mathrm{colim}_{\Phi}(w(\Phi) \circ F)$ or limit $\lim_{\Phi}(w(\Phi) \circ F)$ over $\Phi \in W(\Omega)$. Geometrically, we push F with increasingly positive (resp. negative) isotopies and take colimit (resp. limit) over them. These functors give a geometric description for the adjoints of the inclusion $\iota_* : \mathrm{Sh}_{S^*M \setminus \Omega}(M) \hookrightarrow \mathrm{Sh}(M)$.

Theorem 1.0.4. *Let $\iota_* : \mathrm{Sh}_{S^*M \setminus \Omega}(M) \hookrightarrow \mathrm{Sh}(M)$ denote the tautological inclusion. Then the functor $\mathfrak{W}^+(\Omega)$ (resp. $\mathfrak{W}^-(\Omega)$) is the left (resp. right) adjoint of ι_*.*

See subsection 4.2 for the proof.

One main application of the notion of isotopies of sheaves is the study of Verdier duality. Assume the manifold M is compact so all object $F \in \mathrm{Sh}_\Lambda(M)$ with perfect stalks are compact. Denote by $\mathrm{Sh}_\Lambda(M)^b$ the collection of all such sheaves so there is an inclusion $\mathrm{Sh}_\Lambda(M)^b \subseteq \mathrm{Sh}_\Lambda(M)^c$. One question is whether the Verdier duality

$$\mathrm{Sh}_\Lambda(M)^{b,op} = \mathrm{Sh}_\Lambda(M)^b$$
$$F \mapsto D_M(F) \coloneqq \mathcal{H}om(F, \omega)$$

extends to $\mathrm{Sh}_\Lambda(M)^c$. We note such an extension would produce and can be recovered from an equivalence $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$, which is in general different from the one consider for Theorem 1.0.3. We show, in Theorem 5.1.2, using the technique a perturbation trick developed in Proposition 4.2.8 that the existence of such an extension is equivalent to the invertibility of the endofunctor

$$S^+_\Lambda : \mathrm{Sh}_\Lambda(M) \to \mathrm{Sh}_\Lambda(M)$$
$$F \mapsto \mathfrak{W}^+_\Lambda(F^w)$$

where F^w is a pushoff by a small Reeb flow displacing F from Λ. We also verify, in Theorem 5.2.4, that this conditions holds for a large class of singular isotropic, and show, in Proposition 5.3.2, that the functor S^+_Λ has intrinsic categorical meaning and is in particular independent of the choice of the small Reeb flow.

The main construction of this paper is the category of *wrapped sheaves* $\mathfrak{w}\mathrm{sh}_\Lambda(M)$ where M is a real analytic manifold and Λ is a closed subset in S^*M. It is a stable category defined by first collecting sheaves which have subanalytic singular isotropic microsupport away from Λ and those which are compactly isotopic to them in $S^*M \setminus \Lambda$, and then inverting continuation maps which come from positive isotopies satisfying similar conditions. One effect of this localization is that objects which can be connected through an isotopy on $S^*M \setminus \Lambda$ will be identified. We show that Hom's in $\mathfrak{w}\mathrm{sh}_\Lambda(M)$ can be computed as colimits of Hom's between ordinary sheaves over $W(S^*M \setminus \Lambda)$. Finally, when Λ is a subanalytic singular isotropic, by using the infinite wrapping functor $\mathfrak{W}^+(S^*M \setminus \Lambda)$, we define a canonical *comparison* functor $\mathfrak{W}_\Lambda^+(M) : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to \mathrm{Sh}_\Lambda(M)^c$. The main theorem of this paper is that $\mathfrak{W}_\Lambda^+(M)$ is an equivalence.

Theorem 1.0.5. *Let $\Lambda \subseteq S^*M$ be a subanalytic singular isotropic. The comparison functor $\mathfrak{W}_\Lambda^+(M) : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to \mathrm{Sh}_\Lambda(M)^c$ is an equivalence.*

See subsection 7.3 for the proof.

Remark 1.0.6. Note that, unlike the analogous isomorphism in [21], our isomorphism is induced by an explicit functor.

Since all the above constructions are functorial on the inclusion of open sets of M, we obtain a precosheaf $\mathfrak{w}\mathrm{sh}_\Lambda$ and we refer its objects as the *wrapped sheaves*. The corollary of the above theorem is that this precosheaf is a cosheaf.

Corollary 1.0.7. *Let $\Lambda \subseteq S^*M$ be a subanalytic singular isotropic. The comparison morphism $\mathfrak{W}_\Lambda^+ : \mathfrak{w}\mathrm{sh}_\Lambda \to \mathrm{Sh}_\Lambda^c$ between precosheaves is an isomorphism. In particular, the precosheaf $\mathfrak{w}\mathrm{sh}_\Lambda$ is a cosheaf.*

The proof of Theorem 1.0.5 follows the same strategy as [21]. In short, subanalytic geometry implies that, for a subanalytic singular isotropic Λ, there exists a C^1 Whitney triangulation $\mathcal{S}$ such that Λ is contained in $N_\infty^*\mathcal{S} := \cup_{s \in \mathcal{S}} N_\infty^* X_s$. For this special case, the two categories are natural identified as $\mathrm{Perf}\,\mathcal{S}$, the category of perfect $\mathcal{S}$-modules, and hence admit a preferred set of generators which are matched under $\mathfrak{W}_{N_\infty^*\mathcal{S}}^+(M)$. We then apply the nearby cycle technology developed in [43] to conclude that $\mathfrak{W}_{N_\infty^*\mathcal{S}}^+(M)$ induces an equivalence on the Hom's for these generators and hence finished the proof for this case. To conclude the theorem for the general case, we note that the construction on both sides are contravariant on Λ. Thus we study the fiber of the canonical maps $\mathfrak{w}\mathrm{sh}_{N_\infty^*\mathcal{S}}(M) \to \mathfrak{w}\mathrm{sh}_\Lambda(M)$ and $\mathrm{Sh}_{N_\infty^*\mathcal{S}}(M)^c \to \mathrm{Sh}_\Lambda(M)^c$, and show that they are generated by a sheaf-theoretical version of the linking disks and microstalks at the smooth points of $N_\infty^*\mathcal{S} \setminus \Lambda$. Finally, we show that $\mathfrak{W}_\Lambda^+(M)$ matches those objects and thus conclude the general case.

Convention

We follows the higher categorical convention developed in [36, 37]. For example, a category unless emphasized means an ∞-category and limits and colimits are taken in this sense. We also switch between the contact and homogeneous symplectic notations. For example, we will use the same notation to denote a contact isotopy φ_t on $\subseteq S^*M$ and its on $\dot{T}^*M$ when the context is clear.

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Chapter 2

Preliminary

The goal of this chapter is to recall various notions and facts which we will be using later. None of the contents here is original and we only collect them for the convenience of the readers. Readers who are familiar with basic symplectic geometry and higher categorical theory can safely skip this section.

2.1 Homogenous symplectic geometry and contact geometry

We recall some facts from homogenous symplectic geometry and contact geometry, and explain how they are interchangeable with each other. We assume that the contact manifolds in this sections are co-orientable. Let $(X, d\alpha)$ be a Liouville manifold and let Z denote its Liouville vector field. We define a homogeneous symplectic manifold to be a Liouville manifold such that the Liouville flow induces a proper and free $\mathbb{R}_{>0}$-action. In this case, the quotient $X/\mathbb{R}_{>0}$ is a manifold.

Definition 2.1.1. A subset $Y \subseteq (X, d\alpha)$ is *conic* if it is preserved under the $\mathbb{R}_{>0}$-action.

Proposition 2.1.2. *A coisotropic submanifold Y is conic if and only if $\alpha|_{TY^{d\alpha}} = 0$ where we use $TY^{d\alpha}$ to denote the symplectic orthogonal complement of TY.*

Proof. Y is coisotropic iff $TY^{d\alpha} \subseteq TY$. Y is conic if and only if $Z(y) \in T_y Y$ for $y \in Y$ which implies $\alpha_y(w) = d\alpha_y(Z(y), w) = 0$ for all $w \in TY^{d\alpha}$. Note this direction always holds. On the other hand, the same equation implies that $Z(y) \in TY$ if $\alpha|_{TY^{d\alpha}} = 0$. Since Y is coisotropic, $Z(y)$ is in particular in $T_y Y$. $\qquad\square$

Corollary 2.1.3. *A Lagrangian submanifold $L \subseteq (X, d\alpha)$ is conic if and only if $\alpha|_L = 0$.*

Example 2.1.4. The Liouville vector field Z of the cotangent bundle T^*M can be written locally by $Z = \sum \xi_i \partial_{\xi_i}$ where the ξ_i's are the dual coordinates of local coordinates x_i of M. The Liouville flow is given by $\Phi_s^Z(x, \xi) = (x, e^s \xi)$ and the $\mathbb{R}_{>0}$-action is simply the multiplication, $r \cdot (x, \xi) = (x, r\xi)$ for $r \in \mathbb{R}_{>0}$.

Proposition 2.1.5. *The one form α descends to a contact form $\overline{\alpha}$ on $X/\mathbb{R}_{>0}$.*

Proof. Let X denote the Liouville vector field associated to α (which is non-vanishing since $\mathbb{R}_{>0}$ acts freely). By definition, $\alpha(Z) = \omega(Z, Z) = 0$ so it defines a section on $(TX/\langle Z \rangle)^* = T^*(X/\mathbb{R}_{>0})$. This is a contact form since on X, $\iota_Z \omega \wedge d(\iota_Z \omega)^{n-1} = \iota_Z \omega \wedge (\mathcal{L}_Z \omega)^{n-1} = \iota_Z \omega \wedge \omega^{n-1} = \frac{1}{n} \iota_Z \omega^{n-1}$ and $T(X/\mathbb{R}_{>0})$ can be identified as vectors transversal to Z. $\square$

Example 2.1.6. The example we will study in this paper is the cotangent bundle away from the zero section $\dot{T}^*M$ for some smooth manifold M. Pick a metric g and restrict the projection $p : \dot{T}^*M \to S^*M$ to $\{(x, \xi) | g_x(\xi, \xi) = 1\}$ and denote it as p_g. The map p_g is a diffeomorphism because its domain is transversal to the $\mathbb{R}_{>0}$-action and p_g is clearly one-to-one. Its inverse $s : S^*M \to \dot{T}^*M$ provides S^*M a global contact form $s^*\alpha_{can}$.

Note any such section gives the same contact structure but there might not be any contactomorphism sending one contact form to another. A more intrinsic description of the contact structure is $\eta_{[x,\xi]} = \ker \xi$.

Lemma 2.1.7. *A homogeneous symplectomorphism $\psi : (X, d\alpha) \to (Q, d\beta)$ preserves the Liouville form, i.e., $\psi^*\beta = \alpha$.*

Proof. let ψ be homogeneous and $\psi^*d\beta = d\alpha$. We denote the Liouville vector fields by Z and Y and the corresponding flow by ϕ_t^Z and ϕ_t^Y, $t \in \mathbb{R}$. Since ψ is a homogeneous symplectomorphism, we have $\psi(\phi_t^Z(x)) = \phi_t^Y(\psi(x))$ for all $x \in X$. Differentiate the equation and evaluate at 0, we obtain that $d\psi_x(Z(x)) = Y(\psi(x))$, i.e., $Y = \psi_* Z$. So for any differential p-form ν on Q,

$$\begin{aligned}
(\psi^*(\iota_Y \nu))(v_1, \cdots, v_{p-1}) &= (\iota_Y \nu)(\psi_* v_1, \cdots, \psi_* v_{p-1}) \\
&= \nu(Y, \psi_* v_1, \cdots, \psi_* v_{p-1}) \\
&= (\psi^*\nu)(Z, v_1, \cdots, v_{p-1}) \\
&= (\iota_Z(\psi^*\nu))(v_1, \cdots, v_{p-1}).
\end{aligned}$$

That is, $\psi^* \circ \iota_Y = \iota_Z \circ \psi^*$. In particular, $\psi^*\alpha = \psi^*\iota_Z d\alpha = \iota_Y \psi^* d\alpha = \beta$. $\square$

Proposition 2.1.8. *A co-orientation preserving contactomorphism $\varphi : (N, \xi) \to (P, \eta)$ gives rise to a unique homogeneous symplectomorphism $\tilde{\varphi} : SN \to SP$ between their symplectizations. On the other hand, a homogeneous symplectomorphism $\psi : (X, d\alpha) \to (Q, d\beta)$ induces a contactomorphism on the contact quotient in Proposition 2.1.5. These two constructions are inverse to each other if X and Q come from symplectization.*

Proof. Assume (N, ξ) and (P, η) are co-oriented by α and β. The equation $d\phi_x(\xi_x) = \eta_{\varphi(x)}$ implies that $\varphi^*\beta = h\alpha$ for some $h > 0$. More precisely, let R be the Reeb vector field of α, then $h = \beta(\varphi_* R)$. Define $\tilde{\varphi} : (N \times \mathbb{R}_{>0}, d(t\alpha)) \to (P \times \mathbb{R}_{>0}, d(s\beta))$ by $\tilde{\varphi}(x, t) = (\varphi(x), (h(x))^{-1}t)$. Then $\tilde{\varphi}^* s\beta = t\frac{1}{h}\varphi^*\beta = t\alpha$ so $\tilde{\varphi}$ is a homogeneous symplectomorphism. Now assume there is another lifting $\tilde{\varphi}'$. Since they both descend to φ, there is $g > 0$ such that $\tilde{\varphi}'(x, t) = g(x)\tilde{\varphi}(x, t)$. But then $t\alpha = (\tilde{\varphi}')^* t\beta = g\varphi^* t\beta = gt\alpha$ so $g \equiv 1$. Since ψ preserves the Liouville form, it is clear that ψ descends to a contactomorphism on the quotient. And we also see the that two constructions are inverse to each other when the homogeneous symplectic manifolds are given by symplectization. $\square$

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Example 2.1.9. We consider the case when $X = Q = \dot{T}^*M$ is the cotangent bundle away from the zero section for some manifold M. One can identify it as the symplectization of S^*M by picking a metric g. Let $\varphi : S^*M \to S^*M$ be a co-orientation preserving contactomorphism and we would like to lift it to a homogeneous symplectomorphism $\hat{\varphi} : \dot{T}^*M \to \dot{T}^*M$.

We describe here how the identification intertwines with the construction in the proposition. Denote s the section of $p : \dot{T}^*M \to S^*M$ which is given by the unit covectors. We claim that there is a (unique) section $t : S^*M \to \dot{T}^*M$ so that $\varphi^*(t^*\alpha_{can}) = s^*\alpha_{can}$. (Note we cannot just require $t = s \circ \varphi^{-1}$ since this would implies $\mathrm{id}_{S^*M} = \varphi^{-1}$.) If such t exists, then $t^*\alpha_{can} = (\varphi^{-1})^*s^*\alpha_{can} = hs^*\alpha_{can}$ for some $h \in C^\infty(S^*M; \mathbb{R}_{>0})$ given by φ. So we simply define $t : S^*M \to \dot{T}^*M$ by $t = h \cdot s$ where '$\cdot$' is the $\mathbb{R}_{>0}$ action. Then we can define $\hat{\varphi} : \dot{T}^*M \to \dot{T}^*M$ by $\hat{\varphi} = \sqrt{g} \cdot (t \circ \phi \circ p)$. One can compute that

$$\hat{\varphi}^*\alpha_{can} = \sqrt{g} \cdot (p^* \circ \varphi^* \circ t^*\alpha_{can}) = \sqrt{g} \cdot (p^* \circ s^*\alpha_{can})$$
$$= \sqrt{g} \cdot (s \circ p)^*\alpha_{can} = \sqrt{g} \cdot \frac{1}{\sqrt{g}}\alpha_{can} = \alpha_{can}$$

is a symplectomorphism. Note that we use $s \circ p(x, \xi) = \left(1/\sqrt{g_x(\xi, \xi)}\right)(x, \xi)$ for the second to last equality.

Now consider a family of isotopy $\varphi_t : S^*M \to S^*M$ such that $\varphi_0 = \mathrm{id}_{S^*M}$. The requirement $(\varphi_t^{-1})^*s^*\alpha_{can} = h_t s^*\alpha_{can}$ ensures $h_t > 0$ since $h_0 \equiv 1$. We can then lift φ_t to a family of homogeneous symplectomorphism $\hat{\varphi}_t : \dot{T}^*M \to \dot{T}^*M$ by the above process. Since this process can be reserved, we see that there is a one-to-one correspondence between contact isotropies on S^*M and homogeneous isotropies on $\dot{T}^*M$. Note that the family version of isotopies works similarly.

2.2 Stable categories

We will work in the higher categorical setting developed in [36] and [37]. The main purpose of the rest of the chapter is to fix notations. A thorough beginner guide as well as the set-up needed to work over a field k of characteristic 0 can be found in [16, Chapter I.1]. To work over more general coefficients, one requires further the theory of rigid categories from [28]. Since the relative case enjoys the same formal properties which are needed for the purpose of as the absolute case, we fix once and for all a rigid symmetric monoidal category $(\mathcal{V}_0, \otimes, 1_{\mathcal{V}})$ and its Ind-completion $\mathcal{V} := \mathrm{Ind}(\mathcal{V}_0)$. Unless specified, we will assume without mentioning that all the categories we consider will be tensored over and thus enriched in $\mathcal{V}$ and functors between those are $\mathcal{V}$-enriched as well.

The main advantages for working in the higher categorical setting is that there is an abundance of limits and colimits (in an appropriate sense). As a result, many constructions can be performed formally as universal constructions which greatly simplifies the situation. Because of the higher categorical nature of this paper, we will refer an ∞-category $\mathcal{C}$ simply as a category and when we need to emphasis that it is in particular an ordinary category, we will refer it as a 1-category.

Recall that a *presentable category* is a category with certain cardinality assumptions. Roughly speaking, such a category is large enough to contain (small) colimits but is controlled

by a small category. A main consequence of these assumptions is that the adjoint functor theorem holds. See [36, Corollary 5.5.2.9]. In addition, up to set-theoretic issues, the totality of such categories forms a (very large) category itself which has nice properties concerning limits and colimits. We will not consider the whole collection of such categories but a small portion of it which satisfies stronger finiteness conditions which we now recall.

Definition 2.2.1. Let $\mathcal{C}$ be a category. An object $c \in \mathcal{C}$ is *compact* if $\mathrm{Hom}(c, -)$ preserve (small) filtered colimit. That is for any (small) filtered index category I and any functor $X : I \to \mathcal{C}$, the canonical morphism

$$\varinjlim_{I} \mathrm{Hom}(c, X_i) \to \mathrm{Hom}(c, \varinjlim_{I} X_i)$$

is an isomorphism. Here, we use the notation $\varinjlim_{I}$ instead of $\operatorname*{colim}_{I}$ to emphasis the index category I is filtered.

Definition 2.2.2. A category $\mathcal{C}$ is *compactly generated* if there exists a small subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ consisting of compact objects of $\mathcal{C}$ such that $\mathcal{C}$ is generated by $\mathcal{C}_0$ under filtered colimits. That is, $\mathrm{Ind}(\mathcal{C}_0) \cong \mathcal{C}$ where Ind denotes the Ind-completion.

Definition 2.2.3. Let Cat denote the (very large) category of categories. We use $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ to denote the (non-full) subcategory of Cat whose objects are compactly generated categories and morphisms are functors which preserve small colimits and compact objects. We also use cat to denote the subcategory of Cat consisting of idempotent complete small categories which admit finite colimits whose morphisms are given by functors which preserve finite colimits.

Proposition 2.2.4 ([36, Proposition 5.5.7.8]). *The funtor* $\mathrm{Ind} : \mathrm{cat} \to \mathrm{Pr}^{\mathrm{L}}_{\omega}$ *taking* $\mathcal{C}_0$ *to its ind-completion* $\mathrm{Ind}(\mathcal{C}_0)$ *is an equivalence. Its inverse is given by the functor* $\theta : \mathrm{Pr}^{\mathrm{L}}_{\omega} \to \mathrm{cat}$ *sending* $\mathcal{C}$ *to* $\mathcal{C}^c$, *the subcategory of* $\mathcal{C}$ *consisting compact objects.*

Proposition 2.2.5 ([36, Proposition 5.5.7.11]). *The category* $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ *and hence* cat *admits small colimits, which can be computed in* Cat *as limits by passing to right adjoints. Here we use the fact that a morphism* $F : \mathcal{C} \to \mathcal{D}$ *in* $\mathrm{Pr}^{\mathrm{L}}_{\omega}$ *is a left adjoint since it preserves colimits.*

Classically, one use the theory of triangulated categories to encode homological information. They are 1-categories with structures and can be used to remember a small portion of homotopies. However, limits and colimits are scarce in this setting. For example, the kernel of the (unique) non-zero morphism $e : \mathbb{Z}/2 \to \mathbb{Z}/2[1]$ in $D(\mathbb{Z})$ does not exist. See, for example, [54, 2.2.1]. Hence, we use the theory of stable categories [37, Chapter 1] in this paper instead.

Definition 2.2.6. A category $\mathcal{C}$ is *pointed* if there exists a zero object 0, i.e., an object which is both initial and final.

A sequence $X \to Y \to Z$ in a pointed category $\mathcal{C}$ is a *fiber* (resp. *cofiber*) *sequence* if the diagram

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
0 & \longrightarrow & Z
\end{array}$$

is a pullback/pushforward. In this case, we say X (resp. Z) is the *fiber* (resp. *cofiber*) of the corresponding morphism $Y \to Z$ (resp. $X \to Y$).

Definition 2.2.7. A pointed category $\mathcal{C}$ is *stable* if fibers and cofibers exist and a diagram as above is a fiber sequence if and only if it is a cofiber sequence. We say a functor $F : \mathcal{C} \to \mathcal{D}$ between stable categories is *exact* if F preserves finite limits and finite colimits. Note in a stable category $\mathcal{C}$ finite limits are the same as finite colimits so preserving one kind means preserving the other.

Remark 2.2.8. We recall that in the stable case. An object X is compact if and only if $\mathrm{Hom}(X, -)$ preserves coproducts.

Example 2.2.9. A stable category $\mathcal{C}$ admits a "*shifting by 1*" automorphism $[1] : \mathcal{C} \to \mathcal{C}$ which can be defined by $X \mapsto \mathrm{cof}(X \to 0)$. Its inverse is the shifting by -1 automorphism $[-1] : \mathcal{C} \to \mathcal{C}$ which can defined by $X \mapsto \mathrm{fib}(0 \to X)$. In the case when $\mathcal{C} = \mathcal{V}$, these shiftings are given by shifting the degree of the chain complexes.

Example 2.2.10. Let $\mathcal{C}$ be a stable category. For $X, Y \in \mathcal{C}$, the direct sum $X \oplus Y$ in $\mathcal{C}$ can be computed as $\mathrm{cof}(Y[-1] \xrightarrow{0} X) = \mathrm{fib}(Y \xrightarrow{0} X[1])$.

We will use the following lemma:

Lemma 2.2.11. *Let $\mathcal{C}$ be a stable category and X, Y, Z, X', Y', and $Z' \in \mathcal{C}$. Assume we have the following commutative diagram*

$$\begin{array}{ccccc}
X & \longrightarrow & Y & \longrightarrow & Z \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} \\
X' & \longrightarrow & Y' & \longrightarrow & Z'
\end{array}$$

such that each row is a fiber sequence. Let $X'' = \mathrm{cof}(\alpha)$, $Y'' = \mathrm{cof}(\beta)$ and $Z'' = \mathrm{cof}(\gamma)$ be the corresponding cofibers of the vertical maps. Then there exist a canonical fiber sequence $X'' \to Y'' \to Z''$.

In particular, for $f_1 : X_1 \to X_2$ and $f_2 : X_2 \to X_3$, we have a fiber sequence $\mathrm{cof}(f_1) \to \mathrm{cof}(f_2 \circ f_1) \to \mathrm{cof}(f_2)$. This special case is usually referred as the octahedral axiom in the setting of triangulated categories.

Proof. The goal is to show that Z'' is the cofiber of $(X'' \to Y'')$. Recall that cofibers are computed as a colimit of the diagram $I = [\cdot \leftarrow \cdot \to \cdot]$. For example, the object Z is computed as the colimit given by the diagram $0 \leftarrow X \to Y$. Now we consider the following diagram.

$$
\begin{array}{ccccc}
0 & \longleftarrow & Y & \longrightarrow & Y' \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longleftarrow & X & \longrightarrow & X' \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longleftarrow & 0 & \longrightarrow & 0
\end{array}
$$

Taking colimit for the vertical arrows gives $0 \leftarrow Z \to Z'$ and taking the colimit again gives Z''. Similarly, taking first the horizontal arrows and then the vertical arrows gives $\mathrm{cof}(X'' \to Y'')$. But colimits commute with each other and thus $Z'' = \mathrm{cof}(X'' \to Y'')$.

Now for the special case, we apply the above result to the commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\ \ f_1\ \ } & X_2 \\
\| & & \downarrow{\scriptstyle f_2} \\
X_1 & \xrightarrow{\ f_2 \circ f_1\ } & X_3
\end{array}
$$

$\square$

Example 2.2.12. Consider two short exact sequences in a Grothendieck abelian 1-category $\mathcal{A}$ and compatible maps between them as the following.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0
\end{array}
$$

Recall that $\mathcal{A}$ naturally embeds into a stable category $D(A)$ which is usually referred as the Derived category of $\mathcal{A}$. See for example [37, Section 1.3]. An application of the above lemma implies a special case of the snake lemma in classical homological algebra.

We denote $\mathrm{Pr}^{\mathrm{L}}_{\omega,st}$ and st the subcategory of $\mathrm{Pr}^{\mathrm{L}}_\omega$ and cat which consists of stable categories. We recall that the property of being stable is compatible with the finiteness condition we discussed earlier. In particular, the Ind-completion $\mathrm{Ind}(\mathcal{C}_0)$ of a (small) stable category $\mathcal{C}_0$ is stable . Similarly, the subcategory of compact objects $\mathcal{C}^c$ of a compactly generated stable category $\mathcal{C}$ is stable.

Proposition 2.2.13. *The equivalence* $\mathrm{Ind} : \mathrm{cat} \leftrightarrows \mathrm{Pr}_\omega^{\mathrm{L}} : \theta$ *restricts to the (very large) subcategories consisting of stable categories*

$$\mathrm{Ind} : \mathrm{st} \leftrightarrows \mathrm{Pr}_{\omega,st}^{\mathrm{L}} : \theta.$$

2.3   Tensors products of stable categories

We mention that there is a symmetric monoidal structure $\otimes$ on $\mathrm{Pr}_{st}^{\mathrm{L}}$ [37, 28]. The following lemma implies that it restricts to a symmetric monoidal structure on $\mathrm{Pr}_{\omega,st}^{\mathrm{L}}$. By Proposition 2.2.13, we thus obtain a symmetric monoidal structure $\otimes$ on st by sending $(\mathcal{C}_0, \mathcal{D}_0)$ to $(\mathrm{Ind}(\mathcal{C}_0) \otimes \mathrm{Ind}(\mathcal{D}_0))^c$, and it has particular nice descriptions.

Lemma 2.3.1 ([16, Proposition 7.4.2])**.**
  Assume $\mathcal{C}$ and $\mathcal{D}$ are compactly generated stable category over $\mathcal{V}$.

  1. *The tensor product $\mathcal{C} \otimes \mathcal{D}$ is compactly generated by objects of the form $c_0 \otimes d_0$ with $c_0 \in \mathcal{C}^c$ and $d_0 \in \mathcal{D}^c$.*

  2. *For c_0, d_0 as above, and $c \in \mathcal{C}$, $d \in \mathcal{D}$, we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{C}}(c_0, c) \otimes \mathrm{Hom}_{\mathcal{D}}(d_0, d) = \mathrm{Hom}_{\mathcal{C} \otimes \mathcal{D}}(c_0 \otimes d_0, c \otimes d).$$

We will need this lemma concerning the fully-faithfulness of the tensors of functors.

Lemma 2.3.2. *If the functors $f_i : \mathcal{C}_i \to \mathcal{D}_i$ for $i = 1, 2$ in $\mathrm{Pr}_{st}^{\mathrm{L}}$ are fully-faithful, then their tensor product $f_1 \otimes f_2 : \mathcal{C}_1 \otimes \mathcal{C}_2 \to \mathcal{D}_1 \otimes \mathcal{D}_2$ is fully-faithful if one of the following condition is satisfied:*

  1. *The functor f admits a left adjoint.*

  2. *The right adjoint of f is colimit-preserving.*

Proof. We prove (1) and leave (2) to the reader. We first note that since $f_1 \otimes f_2 = (\mathrm{id}_{\mathcal{D}_1} \otimes f_2) \circ (f_1 \otimes \mathrm{id}_{\mathcal{C}_2})$. It is sufficient to prove the case when $f_2 = \mathrm{id}_{\mathcal{C}_2}$. Denote by $f_1^L : \mathcal{D}_1 \to \mathcal{C}_1$ the left adjoint of f_1. We note that since for any $Y \in \mathcal{C}_1$,

$$\mathrm{Hom}(f_1^L f_1 X, Y) = \mathrm{Hom}(f_1 X, f_1 Y) = \mathrm{Hom}(X, Y),$$

the left adjoint f_1^L is surjective. Now we notice that being surjective and being a left adjoint are both preserved under $(-) \otimes \mathrm{id}_{\mathcal{C}_2}$. Thus the right adjoint $f_1 \otimes \mathrm{id}_{\mathcal{C}_2}$ is fully-faithful since it has a surjective left adjoint by a similar argument as above. $\qquad\square$

Finally, let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal $(\infty\text{-})$category. We recall the notion of dualizability which we will use later.

Definition 2.3.3. An object X in $\mathcal{C}$ is dualizable if there exists $Y \in \mathcal{C}$ and a unit and a counit $\eta : 1_{\mathcal{C}} \to Y \otimes X$ abd $\epsilon : X \otimes Y \to 1_{\mathcal{C}}$ such that the pair (η, ϵ) satisfies the standard triangle equality that the following compositions are identities

$$X \xrightarrow{\mathrm{id}_X \otimes \eta} X \otimes Y \otimes X \xrightarrow{\epsilon \otimes \mathrm{id}_X} X,$$

$$Y \xrightarrow{\eta \otimes \mathrm{id}_Y} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}_Y \otimes \epsilon} Y.$$

We note that this condition implies that the triple (Y, η, ϵ) is unique in the ∞-categorical sense.

The relevant proposition concerning dualizability which we need is the following: Let $\mathcal{A} \in \mathrm{Pr}^{\mathrm{L}}_{\omega, st}$ be compactly generated. Denote by $\mathcal{A}_0$ its compact objects and by $\mathcal{A}^{\vee} := \mathrm{Ind}(\mathcal{A}_0^{op})$ the Ind-completion of its opposite category. We first mention that the proof of the Proposition 2.3.4 below implies that $\mathcal{A}^{\vee} \otimes \mathcal{A} = \mathrm{Fun}^{ex}(\mathcal{A}_0^{op} \otimes \mathcal{A}_0, \mathcal{V}) = \mathrm{Fun}^{L}(\mathcal{A}^{\vee} \otimes \mathcal{A}, \mathcal{V})$. Here the superscript 'ex' means exact functors and the 'L' means colimit-preserving functors. As a result, the Hom-pairing $\mathrm{Hom}_{\mathcal{A}_0} : \mathcal{A}_0^{op} \otimes \mathcal{A}_0 \to \mathcal{V}$ induces a functor

$$\epsilon_{\mathcal{A}} : \mathcal{A}^{\vee} \otimes \mathcal{A} \to \mathcal{V}$$

by extending $\mathrm{Hom}_{\mathcal{A}_0}$ to the Ind-completion. On the other hand, as a functor from $\mathcal{A}_0^{op} \otimes \mathcal{A}_0$ to $\mathcal{V}$, it also defines an object in $\mathcal{A} \otimes \mathcal{A}^{\vee}$ by the above identification, which is equivalent to a functor

$$\eta_{\mathcal{A}} : \mathcal{V} \to \mathcal{A} \otimes \mathcal{A}^{\vee}.$$

Proposition 2.3.4 ([28, Proposition 4.10])**.** *If $\mathcal{A} \in \mathrm{Pr}^{\mathrm{L}}_{st}$ is compactly generated, then it is dualizable with respect to the tensor product $\otimes$ on $\mathrm{Pr}^{\mathrm{L}}_{st}$, and the triple $(\mathcal{A}^{\vee}, \eta_{\mathcal{A}}, \epsilon_{\mathcal{A}})$ exhibits $\mathcal{A}^{\vee} := \mathrm{Ind}(\mathcal{A}^{c,op})$ as a dual of it.*

Remark 2.3.5. We note that when $\mathcal{A}_0$ contains only one object, $\mathrm{Fun}^{ex}(\mathcal{A}_0^{op} \otimes \mathcal{A}_0, \mathcal{V})$ recover the notion of bi-modules. As a result, the tautological bimodule $\mathrm{Hom}_{\mathcal{A}_0}$ is usually referred as the diagonal bimodule and is denoted by $\mathrm{Id}_{\mathcal{A}_0}$.

2.4 Quotients of small stable categories

We discuss quotients of (small) stable categories. Let $\mathcal{C}_0 \in st$ be a small idempotent complete stable category and K be a collection of objects in $\mathcal{C}_0$. We would like to construct an associated localization $\mathcal{C}_0 \to \mathcal{C}_0/K$ so that a morphism $f : X \to Y$ with $\mathrm{cof}(f) \in K$ becomes an isomorphism in $\mathcal{C}_0/K$. This localization can be defined as a quotient in the following way. First, we take the stable subcategory $\langle K \rangle$ generated by K, and then take its idempotent completion which we will denote it by $\mathcal{N} = \mathcal{N}(K)$. Abstract argument implies $\mathcal{N}$ is still stable and is embedded in $\mathcal{C}_0$ as the subcategory of retracts of objects in $\langle K \rangle$ because $\mathcal{C}_0$ is idempotent complete. Let $\iota_* : \mathcal{N} \hookrightarrow \mathcal{C}_0$ denote the inclusion and we recall that colimits exist in st by Proposition 2.2.5.

Definition 2.4.1. We define the quotient $\mathcal{C}_0/K$ of $\mathcal{C}_0$ by K as the cofiber $\mathrm{cof}(\iota_*)$ taken in st and use $j^* : \mathcal{C}_0 \twoheadrightarrow \mathcal{C}_0/K$ to denote the projection.

We have the following description of morphisms in $\mathcal{C}_0/K$.

Proposition 2.4.2. *Let X, Y be objects in $\mathcal{C}_0$. Then the Hom in $\mathcal{C}_0/K$ can be computed as a colimit,*

$$\operatorname{Hom}_{\mathcal{C}_0/K}(j^*X, j^*Y) = \operatorname*{colim}_{Y \to Y'} \operatorname{Hom}_{\mathcal{C}_0}(X, Y')$$

where the colimit runs through the morphism $Y \to Y'$ whose cofiber is in $\mathcal{N}$. Alternatively, we can compute the Hom-spaces by varying the first component, i.e.,

$$\operatorname{Hom}_{\mathcal{C}_0/K}(j^*X, j^*Y) = \operatorname*{colim}_{X' \to X} \operatorname{Hom}_{\mathcal{C}_0}(X', Y)$$

with $\operatorname{cof}(X' \to X) \in \mathcal{N}$.

To prove the proposition, we first look more closely into the construction. Begin with the inclusion $\mathcal{N} \overset{\iota_*}{\hookrightarrow} \mathcal{C}_0$, we translate to the category $\operatorname{Pr}^{\mathrm{L}}_\omega$ by taking Ind by Proposition 2.2.13 and obtain

$$\operatorname{Ind}(\mathcal{N}) \xrightarrow{\operatorname{Ind}(\iota_*)} \operatorname{Ind}(\mathcal{C}_0).$$

Because $\operatorname{Ind}(\iota_*)$ preserves small colimits, it admits a right adjoint $\operatorname{Ind}(\iota_*)^R$.

Lemma 2.4.3. *For $X \in \mathcal{C}_0 \hookrightarrow \operatorname{Ind}(\mathcal{C}_0)$, the right adjoint of $\operatorname{Ind}(\iota_*)$ can be given by the formula*

$$\operatorname{Ind}(\iota_*)^R(X) = \text{``}\operatorname*{colim}_{\alpha : Z \to X, \ Z \in \mathcal{N}}\text{''} Z.$$

Here we use the quotation "colim" to emphasis the colimit is taken formally in $\operatorname{Ind}(\mathcal{N})$.

Proof. By definition, the formal colimit "$\operatorname*{colim}_{\alpha : Z \to X, \ Z \in \mathcal{N}}$"$Z$ is an object of $\operatorname{Ind}(\mathcal{N})$. Because an object of $\operatorname{Ind}(\mathcal{N})$ is of the form "$\varinjlim$"W over some filtered colimit by some objects in $\mathcal{N}$, it is sufficient to show, for all $W \in \mathcal{N}$,

$$\operatorname{Hom}_{\mathcal{C}_0}(\iota W, X) = \operatorname{Hom}_{\operatorname{Ind}(\mathcal{N})}(W, \text{``}\operatorname*{colim}_{\alpha : Z \to X, \ Z \in \mathcal{N}}\text{''} Z).$$

Since $\operatorname{Ind}(\iota)$ preserves compact objects, we compute

$$\operatorname{Hom}_{\operatorname{Ind}(\mathcal{N})}(W, \text{``}\operatorname*{colim}_{\alpha : Z \to X, \ Z \in \mathcal{N}}\text{''} Z) = \operatorname*{colim}_{\alpha : Z \to X, \ Z \in \mathcal{N}} \operatorname{Hom}_{\mathcal{N}}(W, Z) = \operatorname{Hom}_{\mathcal{C}_0}(W, X).$$

$\square$

Proof of the proposition 2.4.2. By passing to right adjoints using Proposition 2.2.13, the cofiber $\operatorname{cof}(\operatorname{Ind}(\iota_*))$ can be computed as,

$$j_* : \operatorname{fib}(\operatorname{Ind}(\iota_*)^R) \subseteq \operatorname{Ind}(\mathcal{C}_0),$$

the subcategory consisting of objects Y such that $\operatorname{Hom}(X, Y) = 0$ for all $X \in \mathcal{N}$. Because we are in the stable setting, this gives us a fiber sequence

$$\operatorname{Ind}(\iota_*)\operatorname{Ind}(\iota_*)^R \to \operatorname{id} \to j_* j^*.$$

Thus, for $X, Y \in \mathcal{C}_0$, one computes

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{C}_0/K}(j^*X, j^*Y) &= \operatorname{Hom}_{\mathcal{C}_0}(X, j_* j^* Y) \\
&= \operatorname{Hom}_{\mathcal{C}_0}\left(X, \operatorname{cof}(\operatorname{Ind}(\iota_*)\operatorname{Ind}(\iota_*)^R Y \to Y)\right) \\
&= \operatorname{Hom}_{\mathcal{C}_0}\left(X, \operatorname{cof}(\text{``} \operatorname*{colim}_{\alpha: Z \to Y,\ Z \in \mathcal{N}} \text{''} Z \to Y)\right) \\
&= \operatorname*{colim}_{\alpha: Z \to Y,\ Z \in \mathcal{N}} \operatorname{Hom}_{\mathcal{C}_0}(X, \operatorname{cof}(Z \to Y)) \\
&= \operatorname*{colim}_{Y \xrightarrow{\beta} Y',\ \operatorname{cof}(\beta) \in \mathcal{N}} \operatorname{Hom}_{\mathcal{C}_0}(X, Y').
\end{aligned}
$$

Here, we notice the last equation is simply a change the expression for the same colimit.

To obtain the similar formula which we varies the first component, we notice that there is equivalence $(\mathcal{C}_0/K)^{op} = \mathcal{C}_0^{op}/\mathcal{N}^{op}$ because they satisfies the same universal property. We thus compute

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{C}_0/K}(j^*X, j^*Y) &= \operatorname{Hom}_{(\mathcal{C}_0/K)^{op}}(j^*Y, j^*X) \\
&= \operatorname{Hom}_{\mathcal{C}_0^{op}/\mathcal{N}^{op}}(j^*Y, j^*X) \\
&= \operatorname*{colim}_{X \xleftarrow{\gamma} X',\ \operatorname{cof}(\gamma) \in \mathcal{N}^{op}} \operatorname{Hom}_{\mathcal{C}_0^{op}}(Y, X') \\
&= \operatorname*{colim}_{X' \xrightarrow{\gamma} X,\ \operatorname{cof}(\gamma) \in \mathcal{N}} \operatorname{Hom}_{\mathcal{C}_0}(X', Y).
\end{aligned}
$$

$\square$

We will use the following "snake lemma" for categories.

Lemma 2.4.4. *Consider the following diagram in* $\operatorname{Pr}^{L}_{\omega, st}$:

$$
\begin{array}{ccccc}
\mathcal{C}_0 & \xhookrightarrow{\ i\ } & \mathcal{C} & \xtwoheadrightarrow{\ p\ } & \mathcal{C}_1 \\
\downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \bar{F}} \\
\mathcal{D}_0 & \xhookrightarrow{\ j\ } & \mathcal{D} & \xtwoheadrightarrow{\ q\ } & \mathcal{D}_1
\end{array}
$$

where p and q are the quotient functor of the inclusion i and j, F_0 is the restriction of F which factors through $\mathcal{D}_0$ and $\bar{F}$ is the induced functor between the quotients. Let $\iota : \operatorname{fib}(\bar{F}) \hookrightarrow \mathcal{C}_1$ denote the fiber of $\bar{F}$, $\pi : \mathcal{D}_0 \twoheadrightarrow \operatorname{cof}(F_0)$ the cofiber of F_0 and $\partial : \operatorname{fib}(\bar{(F)}) \to \operatorname{cof}(F_0)$ the functor given by the composition $\partial = \pi \circ j^R \circ F \circ p^R \circ \iota$. If F is an equivalence, then ∂ is an equivalence.

Proof. For simplicity, we assume $\mathcal{C} = \mathcal{D}$ and F is the identity so the diagram becomes,

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We will prove that the functor $\theta := \iota^R \circ p \circ j \circ \pi^R$ is the inverse by showing that $\theta \circ \partial = \mathrm{id}_{\mathrm{fib}(\bar{F})}$. The equation $\partial \circ \theta = \mathrm{id}_{\mathrm{cof}(F_0)}$ can be proved similarly. First write out $\theta\partial$ as $\iota^R p j \pi^R \pi j^R p^R \iota$. Recall that in the stable setting, the sequence $\mathcal{C}_0 \hookrightarrow \mathcal{C} \twoheadrightarrow \mathcal{C}_1$ comes with a fiber sequence of functors

$$ii^R \to \mathrm{id}_{\mathcal{C}} \to p^R p.$$

Apply this fact to $\mathcal{C}_0 \hookrightarrow \mathcal{D}_0 \twoheadrightarrow \mathrm{cof}(F_0)$, we see there is a fiber sequence

$$F_0 F_0^R \to \mathrm{id} \to \pi^R \pi.$$

Apply $j \circ (-) \circ j^R$ and the fiber sequence becomes $ii^R \to jj^R \to j\pi^R\pi j^R$. Now apply $p \circ (-) \circ p^R$ and the we see that $pjj^R p = pj\pi^R\pi j^R p^R$ since $p \circ i = 0$. Thus, we can simplify $\theta\partial$ to $\iota^R pjj^R p^R \iota$. Similar argument allows us to further simplify $\theta\partial$ to $\iota^R pp^R \iota = \iota^R \mathrm{id}_{\mathcal{C}} \iota = \mathrm{id}_{\mathrm{fib}(\bar{F})}$. $\square$

Chapter 3

Microlocal sheaf theory

We will consider microlocal sheaf theory in the higher categorical setting over a fixed co-efficient $\mathcal{V}$ as mentioned in the last section. We begin this chapter with recalling classical definitions and results from the foundational work in [32]. Then we recall previous results concerning the category $\mathrm{Sh}_\Lambda(M)$ of sheaves microsupported in a fixed (subanalytic) Lagrangians from [21]. One key property which $\mathrm{Sh}_\Lambda(M)$ satisfies is that it is compactly generated and so an object in $\mathrm{Pr}^{\mathrm{L}}_{\omega,st}$. Based on this result, we prove a compatibility statement for $\mathrm{Sh}_\Lambda(M)$ regarding the symmetric monoidal product $\otimes$ on $\mathrm{Pr}^{\mathrm{L}}_{st}$ in Section 3.4. We also identify its categorical dual $\mathrm{Sh}_\Lambda(M)^\vee$ geometrically as $\mathrm{Sh}_{-\Lambda}(M)$ in Section 3.6. The last statement provides a classification of colimit-preserving functors between such categories in Theorem 1.0.3.

3.1   General sheaf theory

For a topological space X, the category of $\mathcal{V}$-valued presheaves is the category $\mathrm{PSh}(X) :=$ $\mathrm{Fun}(Op_X^{op}, \mathcal{V})$ of contravariant functor from the 1-category of open sets in X to $\mathcal{V}$. The category of sheaves on X, $\mathrm{Sh}(X)$, is the reflexive subcategory of $\mathrm{PSh}(X)$ consisting of those presheaves F which turn colimits in Op_X to limits. In more concrete terms, F is a sheaf if for any open cover $\mathcal{U}$ of an open set $U \subseteq X$, the canonical map $F(U) \xrightarrow{\sim} \lim_{U_I \in C(\mathcal{U})} F(U_I)$ is an isomorphism, i.e., sections over U can be computed as the totalization of the sections of the corresponding Čech nerve $C(\mathcal{U})$. Recall that the term 'reflexive' means that the inclusion $\mathrm{Sh}(X) \hookrightarrow \mathrm{PSh}(X)$ admits a left adjoint, which is usually referred as sheafification $(-)^\dagger$. Thus, the inclusion is limit-preserving. Since $\mathrm{PSh}(X)$ inherits limits and colimits from $\mathcal{V}$, so does $\mathrm{Sh}(X)$ where colimits in $\mathrm{Sh}(X)$ can be computed as the sheafification of the colimits in $\mathrm{PSh}(X)$

We also recall the six-functor formalism. First, there is a symmetric monoidal structure $(\mathrm{Sh}(X), \otimes)$ on $\mathrm{Sh}(X)$ which is induced from $\mathcal{V}$. The unit of this tensor product is the sheaf 1_X which is the sheafification of the presheaf $(U \mapsto 1_\mathcal{V})$ whose restrictions are given by the identity id_1. For a fixed sheaf F, the functor $(-) \otimes F$ which is given by tensoring with F has a right adjoint $\mathcal{H}om(F, -)$. This provides $\mathrm{Sh}(X)$ with an internal Hom. The global section of this sheaf $\mathcal{H}om$ is the $\mathcal{V}$-valued Hom, i.e., $\Gamma(X; \mathcal{H}om(G, F)) = \mathrm{Hom}(G, F) \in \mathcal{V}$ for any

$F, G \in \mathrm{Sh}(X)$.

Let $f : X \to Y$ be a continuous map. There is a pushforward functor $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ induced by pulling back open sets $f^{-1} : Op_Y \to Op_X$, $V \mapsto f^{-1}(V)$. This functor admits a left adjoint $f^* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ and the adjunction (f^*, f_*) is usually referred as the *star pullback/pushforward*. When X and Y are both locally compact Hausdorff spaces, there is another pair of adjunction $(f_!, f^!)$ such that $f_! : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f^! : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$. This adjunction is usually referred as the *shriek pullback/pushforward*. When f is proper, $f_!$ coincide with f_*.

Remark 3.1.1. Since a large portion of this paper is a sheaf-theoretic parallel of [21], we mention that their setting corresponds to the choice $\mathcal{V} = \mathbb{Z}$-Mod, the presentable stable category of modules over $\mathbb{Z}$. It can be modified as the dg category of (possibly unbounded) chain complexes of abelian groups with quasi-isomorphisms inverted. See for example [12]. This category is compactly generated and one usually denote the compact objects $(\mathbb{Z}\text{-Mod})^c$ by $\mathrm{Perf}\,\mathbb{Z}$. When representing $\mathbb{Z}$-Mod by chain complexes, $\mathrm{Perf}\,\mathbb{Z}$ consists of objects which are quasi-isomorphic to bounded chain complexes whose cohomology groups are finite rank.

Example 3.1.2. When $i : Z \hookrightarrow X$ is a locally closed subset of X, one usually use $F|_Z$ to denote i^*F for $F \in \mathrm{Sh}(X)$ and call it the restriction of F on Z.

Example 3.1.3. Consider a closed set $i : Z \hookrightarrow X$ and an open set $j : U \hookrightarrow X$. In these cases, we have $i_* = i_!$ and $j^* = j^!$. In addition, the functors i_*, j_*, $j_!$ are fully faithful with the corresponding adjoints being a left inverse.

Example 3.1.4. Let $x \in X$ be a point. For $F \in \mathrm{Sh}(X)$, we call the object $F_x \coloneqq F|_{\{x\}} \in \mathrm{Sh}(\{x\}) = \mathcal{V}$ the *stalk* of F at x. An key property of the stalks is that a morphism $G \to F$ is an isomorphism if and only if it induces isomorphism $G_x \to F_x$ on the stalk at x for all $x \in X$. We also use the convention that F has perfect stalk at x if $F_x \in \mathcal{V}_0$ or, equivalently, F_x is perfect. We say that F has perfect stalks if F_x is perfect for all $x \in X$.

Example 3.1.5. Let $a_X : X \to \{*\}$ be the projection to a point. The object $a_X^! 1$ is usually denoted as ω_X and is referred as the dualizing complex/sheaf. When X is a C^0-manifold, ω_X is a locally constant sheaf whose stalk is given by $1_\mathcal{V}[\dim X]$.

Now fix a topological space X. We see that taking integer coefficient sheaves itself forms a presheaf Sh in Cat: For an open set $U \subseteq X$, we assign the category $\mathrm{Sh}(U)$. For an inclusion of open sets $i_{U,V} : U \hookrightarrow V$, we assign the pullback functor $i_{U,V}^* : \mathrm{Sh}(V) \to \mathrm{Sh}(U)$ which is the right adjoint of $i_{U,V\,!} : \mathrm{Sh}(U) \to \mathrm{Sh}(V)$.

Proposition 3.1.6. *The presheaf* $\mathrm{Sh} : \mathrm{Op}_X \to \mathrm{Cat}$ *is a sheaf.*

Proof. Let U be an open set of X and $\mathcal{U}$ an open cover of U. The functor

$$\lim_{U_I \in C(\mathcal{U})} i_{U_I, U}^* : \mathrm{Sh}(U) \to \lim_{U_I \in C(\mathcal{U})} \mathrm{Sh}(U_I)$$

is an equivalence and its inverse $\lim_{U_I \in C(\mathcal{U})} \mathrm{Sh}(U_I) \to \mathrm{Sh}(U)$ can be described by

$$(F_{U_I})_{U_I \in C(\mathcal{U})} \mapsto \operatorname*{colim}_{U_I \in C(\mathcal{U})} (i_{U_I, U})_! F_{U_I}.$$

$\square$

We recall some standard properties of the six-functor formalism. First, the categories of sheaves has a structure of *base change*. (See [30] for the exact statement in the higher categorical setting.)

Theorem 3.1.7. *Consider a pullback diagram of locally compact Hausdorff spaces,*

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f'\ } & Y' \\
{\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

There is an equivalence $g^* f_! = f'_! g'^*$.

The push/pull functors satisfy some compatibility properties with $\otimes$ and $\mathcal{H}om$. We list a few which we will use:

Proposition 3.1.8. *Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. Then:*

1. $f^*(F \otimes G) = f^* F \otimes f^* G$, for F, $G \in \mathrm{Sh}(Y)$,

2. $(f_! G) \otimes F = f_!(G \otimes f^* F)$, for F, $G \in \mathrm{Sh}(X)$,

3. $\mathcal{H}om(f_! G, F) = f_* \mathcal{H}om(G, f^! F)$, for $F \in \mathrm{Sh}(X)$, $G \in \mathrm{Sh}(Y)$,

4. $f^! \mathcal{H}om(G, F) = \mathcal{H}om(f^* G, f^! F)$, for F, $G \in \mathrm{Sh}(Y)$.

We recall the excision fiber sequences. Let X be a locally compact Hausdorff space, $i : Z \hookrightarrow X$ be a close set, and $j : U = X \setminus Z \hookrightarrow X$ be its open complement, then $j^* i_* = 0$ by base change and there are fiber sequences

$$
j_! j^! F \to F \to i_* i^* F
$$

$$
i_! i^! F \to F \to j_* j^* F
$$

where the arrows are the units/counits of the shriek/star adjunction pairs. Such a triple $(\mathrm{Sh}(X), \mathrm{Sh}(Z), \mathrm{Sh}(U))$ is usually referred as a *recollement* in homological algebra. See for example [37, Section A.8]. When Z is locally closed, one denotes $F_Z = i_! i^* F$ and $\Gamma_Z(F) = i_* i^! F$ for $F \in \mathrm{Sh}(X)$. Thus, one can write the above fiber sequences as

$$
F_U \to F \to F_Z, \ \ \Gamma_Z(F) \to F \to \Gamma_U(F).
$$

Let $a_X : X \to \{*\}$ denote the projection to a point. We use A_X to denote the pullback $a_X^* A$ for $A \in \mathrm{Sh}(\{*\}) = \mathcal{V}$. When X is a manifold, $\mathcal{V} = \mathbb{Z}\text{-Mod}$, and A is an abelian group regarded as a chain complex concentrated as 0, a standard representative of A_X is the A-coefficient singular cochains $(U \mapsto C^*(U; A))$. When $Z \subseteq X$ is a locally closed subset of X, we abuse the notation and write A_Z for both the sheaf in $\mathrm{Sh}(Z)$ or its shriek pushforward. When $A = 1_\mathcal{V}$, we abuse the notation and write it simply as 1_Z.

Definition 3.1.9. We call A_Z the *constant sheaf* on Z with stalk A. In general, we say a sheaf F is a *locally constant sheaf* or a *local system* if there exists an open cover $\mathcal{U}$ such that $F|_U$ is constant for $U \in \mathcal{U}$ and we use $\mathrm{Loc}(X)$ to denote the subcategory spanned by such sheaves.

Example 3.1.10. Let $i : Z \hookrightarrow X$ be a local closed subset and $F \in \mathrm{Sh}(X)$. By the above (1) and (2) of Proposition 3.1.8, $F \otimes 1_Z = F \otimes i_! i^* 1_X = i_!(i^* F \otimes i^* 1_X) = i_! i^*(F \otimes 1_X) = F_Z$. A similar statement holds for open inclusions and the fiber sequence $F_U \to F \to F_Z$ can be obtained from tensoring F with the canonical one

$$1_U \to 1_X \to 1_Z.$$

We also consider set-theoretic invariants associated to sheaves.

Definition 3.1.11. Let X be a topological space and $F \in \mathrm{Sh}(X)$. The *support* of a sheaf F is defined to be the closed subset

$$\mathrm{supp}(F) = \overline{\{x \in X | F_x \neq 0\}}.$$

Example 3.1.12. Let $i : Z \hookrightarrow X$ be a closed subset. The pushforward i_* identifies $\mathrm{Sh}(Z)$ as the subcategory of $\mathrm{Sh}(X)$ consisting of sheaves F whose support $\mathrm{supp}(F)$ is contained in Z.

Before leaving this section, we recall a fundamental lemma for microlocal sheaf theory, which holds for general Hausdorff spaces.

Lemma 3.1.13 ([32, Proposition 2.7.2] , [45, Theorem 4.1])**.** *Let X be a Hausdorff space, $F \in \mathrm{Sh}(X)$. Let $\{U_s\}_{s \in \mathbb{R}}$ be a family of open subsets of X. We assume*

(a) for all $t \in \mathbb{R}, U_t = \bigcup_{s<t} U_s$,

(b) for all pairs (s,t) with $s \leq t$, the set $\overline{U_t \setminus U_s} \cap \mathrm{supp}(F)$ is compact,

(c) setting $Z_s = \cap_{t>s} \overline{U_t \setminus U_s}$, we have for all pairs (s,t) with $s \geq t$, and all $x \in Z_s \setminus U_t$,

$$(\Gamma_{X \setminus U_t} F)_x = 0.$$

Then we have the isomorphism in $\mathrm{Sh}(X)$, for all $t \in \mathbb{R}$,

$$\Gamma(\bigcup_s U_s; F) \xrightarrow{\sim} \Gamma(U_t, F).$$

3.2   Microlocal sheaf theory

Now let M be a C^α-manifold where $\alpha \in \mathbb{Z}_{>0} \cup \{\infty, \omega\}$. The term 'microlocal' usually refers to 'local' in the cotangent bundle T^*M. In [32, Section 5.1], Kashiwara and Schapira define the notion of microsupport, which is a set in T^*M enhancing the support. One description of it is the following: Let F be a sheaf and ϕ a C^1 function on M, let $m \in \phi^{-1}(t)$. We denote by $i_{\phi,t} : \{x | \phi(x) \geq t\} \hookrightarrow M$ the closed inclusion. We say m is a cohomological F-critical point of ϕ if $(i_{\phi,t}^! F)_m \neq 0$.

Definition 3.2.1. The *microsupport* of a sheaf F is defined to be the closure of the locus of differentials of the C^1 functions at their cohomological F-critical points. That is,

$$\mathrm{SS}(F) = \overline{\bigcup_{\phi \in C^1(M)} \{(x, \xi) | \exists t \in \mathbb{R}, (i^!_{\phi,t} F)_x \neq 0, \xi = d\phi_x\}}.$$

Although the microsupport is defined as a C^1-invariant, it is sufficient to check a smaller class of functions.

Proposition 3.2.2 ([32, Proposition 5.1.1]). *The microsupport of a sheaf F is the same as the closure of the locus of differentials of the C^α functions at their cohomological F-critical points. That is,*

$$\mathrm{SS}(F) = \overline{\bigcup_{\phi \in C^\alpha(M)} \{(x, \xi) | \exists t \in \mathbb{R}, (i^!_{\phi,t} F)_x \neq 0, \xi = d\phi_x\}}.$$

It is straightforward to see that the microsupport $\mathrm{SS}(F)$ of a sheaf $F \in \mathrm{Sh}(M)$ is conic and closed, and its intersection with the zero section $\mathrm{SS}(F) \cap 0_M = \mathrm{supp}(F)$ recovers the support. The involutivity theorem [32, Theorem 6.5.4] of Kashiwara and Schapira states that $\mathrm{SS}(F)$ is always a singular coisotropic. Since $\mathrm{SS}(F)$ is conic, it can be recovered from $\mathrm{supp}(F)$ and its projectivization $\mathrm{SS}^\infty(F) \coloneqq (\mathrm{SS}(F) \setminus 0_M)/\mathbb{R}_{>0}$.

Definition 3.2.3. For a closed conic subset $X \subseteq T^*M$, we use $\mathrm{Sh}_X(M)$ to denote the subcategory of sheaves consisting of those F such that $\mathrm{SS}(F) \subseteq X$. Similarly, for a closed subset $X \subseteq S^*M$, we use $\mathrm{Sh}_X(M)$ to denote the subcategory of sheaves consisting of those F such that $\mathrm{SS}^\infty(F) \subseteq X$. Note for a closed $X \subseteq S^*M$, $\mathrm{Sh}_X(M) = \mathrm{Sh}_{(\mathbb{R}_{>0}X \cup 0_M)}(M)$.

Example 3.2.4. Let M be a manifold. Being a local system is a microlocal condition. More precisely, $\mathrm{Loc}(M) = \mathrm{Sh}_{0_M}(M)$.

Example 3.2.5 ([32, Proposition 5.3.1]). More generally, let $M = \mathbb{R}^n$, γ be a closed convex cone with vertex at 0, and denote by $\gamma^\circ \coloneqq \{\xi \in (\mathbb{R}^n)^* | \xi(v) \geq 0, v \in \gamma\}$ its dual cone. One has $\mathrm{SS}(1_\gamma) \cap T^*_0\mathbb{R}^n = \gamma^\circ$. As a corollary, if $M' \subseteq M$ is a closed submanifold, then $\mathrm{SS}(1_{M'}) = N^*M'$ is the normal bundle of M'.

One might want to assign an invariant similar to the stalks for points in $(x, \xi) \in S^*M$. In general, the object $(i^!_{\phi,t} F)_x$ depends on ϕ and is not an invariant associated to the point (x, ξ). However, the situation is better when cartain transversality condition is satisfied.

Definition 3.2.6. Fix a singular isotropic $\Lambda \subseteq S^*M$, i.e, Λ is stratified by isotropic submanifolds. Let f be a function defined on some open set U of M. We say that a point $x \in U$ is a Λ-critical point of f if the graph of its differential Γ_{df} intersect $\mathbb{R}_{>0}\Lambda \cup 0_M$ at (x, df_x). A Λ-critical point x is Morse if (x, df_x) is a smooth point of $\mathbb{R}_{>0}\Lambda \cup 0_M$ and the intersection $\Gamma_{df} \cap \Lambda$ is transverse at (x, df_x). A function f is Λ-Morse if all its Λ-critical point is Morse.

Proposition 3.2.7 ([32, Proposition 7.5.3]). *Let Λ be a singular isotropic. Assume ϕ is Λ-Morse at a smooth point $(x, \xi) \in \Lambda$. For $F \in \mathrm{Sh}(M)$ such that $\mathrm{SS}^\infty(F) \subseteq \Lambda$ in a neighborhood of (x, ξ), the object $(i^!_{\phi,t} F)_m \in \mathcal{V}$ is, up to a shift, independent of ϕ.*

Definition 3.2.8. Let $\Lambda \subseteq S^*M$ be a singular isotropic and $(x, \xi) \in \Lambda$ a smooth point. For $F \in \mathrm{Sh}_\Lambda(M)$, we call functors $\mu_{(x,\xi)} : \mathrm{Sh}_\Lambda(M) \to \mathcal{V}$ of the form

$$\mu_{(x,\xi)}F := (i^!_{\phi,t}F)_x$$

microstalk functors where ϕ is any function which satisfies the assumption in the last Proposition. Since this functor is well-defined up to a shift, which will not play a significant role in the paper, we will abuse notation and call $\mu_{(x,\xi)}F$ the *microstalk* of F at (x, ξ).

Let $X \subseteq T^*M$ be conic and closed, and let $\Lambda \subseteq (T^*M \setminus X)$ be a closed conic subanalytic isotropic. Assume $\mathrm{SS}(F) \subseteq X \cup \Lambda$. To determine whether $(x, \xi) \in \Lambda$ is in the microsupport of F, by definition one has to check if $(i^!_{\phi,t}F)_x$ vanishes over all functions ϕ such that $x \in \phi^{-1}(t)$ and $d\phi_x = \xi$. However, since Λ is isotropic, it is sufficient to check the Morse ones.

Proposition 3.2.9 ([21, Proposition 4.9]). *Let X and Λ be as above. Then $\mathrm{Sh}_X(M) \subseteq \mathrm{Sh}_{X \cup \Lambda}(M)$ is the fiber of all microstalk functors $\mu_{(x,\xi)}$ for smooth Lagrangian points $(x, \xi) \in \Lambda$.*

In practice, it is hard to compute the microsupport of a sheaf directly and it is usually sufficient to deduce desired conclusions by having an upper bound. Here we collect some standard results for microsupport estimation. Let $f : M \to N$ be a map between manifolds, we use the following notations

$$T^*M \xleftarrow{\;df^*\;} M \times_N T^*N \xrightarrow{\;f_\pi\;} T^*N$$

with π_M, $\square$, π_N and $M \xrightarrow{f} N$

where the square on the right is the pullback of the cotangent bundle T^*N of N along f and df^* is given fiberwisely by the adjoint of the differential $df_x : T_xM \to T_{f(x)}N$. Let $T^*_M N$ denote the set

$$\{(x, \alpha) \in M \times_N T^*N \mid df^*_x \alpha = \alpha \circ df_x = 0\}.$$

Note in case $M \subseteq N$ is a submanifold, $T^*_M N$ is the conormal bundle of M in N.

Definition 3.2.10 ([32, Definition 5.4.12]). Let A be a closed conic subset of T^*N. We say f is *noncharacteristic* for A if

$$f^{-1}_\pi(A) \cap T^*_M N \subseteq M \times_N 0_N.$$

For a sheaf $F \in \mathrm{Sh}(M)$, we say f is *noncharacteristic* for F if it is the case for $\mathrm{SS}(F)$.

Proposition 3.2.11. *We have the following results:*

1. *([32, Proposition 5.1.3]) If $F \to G \to H$ is a fiber sequence in $\mathrm{Sh}(M)$, then*

$$(\mathrm{SS}(F) \setminus \mathrm{SS}(H)) \cup (\mathrm{SS}(H) \setminus \mathrm{SS}(F)) \subseteq \mathrm{SS}(G) \subset \mathrm{SS}(F) \cup \mathrm{SS}(H).$$

This is usually referred as the microlocal triangular inequalities.

2. ([32, Proposition 5.4.1]) For $F \in \mathrm{Sh}(M)$, $G \in \mathrm{Sh}(N)$, $\mathrm{SS}(F \boxtimes G) \subseteq \mathrm{SS}(F) \times \mathrm{SS}(G)$.

3. ([32, Proposition 5.4.4]) For $f : M \to N$ and $F \in \mathrm{Sh}(M)$, if f is proper on $\mathrm{supp}(F)$, then

$$\mathrm{SS}(f_* F) \subseteq f_\pi \left((df^*)^{-1} \, \mathrm{SS}(F) \right).$$

4. ([32, Proposition 5.4.5 and Proposition 5.4.13]) For $f : M \to N$ and $F \in \mathrm{Sh}(N)$, if f is noncharacteristic for F, then

$$\mathrm{SS}(f^* F) \subseteq df^*(f_\pi^{-1}(\mathrm{SS}(F)))$$

and the natural map $f^* F \otimes f^! 1_Y \to f^! F$ is an isomorphism. If f is furthermore smooth, the estimation is an equality.

5. ([32, Proposition 5.4.8]) Let $Z \subseteq M$ be closed. If $\mathrm{SS}(F) \cap N_{out}^*(Z) \subseteq 0_M$, then

$$\mathrm{SS}(F_Z) \subseteq N_{in}^*(Z) + \mathrm{SS}(F).$$

Similarly, let $U \subseteq M$ be open. If $\mathrm{SS}(F) \cap N_{in}^*(U) \subseteq 0_M$, then

$$\mathrm{SS}(F_U) \subseteq N_{out}^*(U) + \mathrm{SS}(F).$$

6. ([32, Proposition 5.4.14]) For F and $G \in \mathrm{Sh}(M)$. If $\mathrm{SS}(F) \cap -\mathrm{SS}(G) \subseteq 0_M$, then

$$\mathrm{SS}(F \otimes G) \subseteq \mathrm{SS}(F) + \mathrm{SS}(G).$$

7. ([32, Proposition 5.4.14 and Exercise V.13]) For F and $G \in \mathrm{Sh}(M)$. If $\mathrm{SS}(F) \cap \mathrm{SS}(G) \subseteq 0_M$, then

$$\mathcal{H}om(G, F)) \subseteq \mathrm{SS}(F) - \mathrm{SS}(G).$$

If moreover G is cohomological constructible, then the natural map

$$\mathcal{H}om(G, 1_M) \otimes F \to \mathcal{H}om(G, F)$$

is an isomorphism. If furthermore $F = \omega_M$, i.e., when $\mathcal{H}om(G, F) =: D_M(G)$ is the Verdier dual, then $\mathrm{SS}(D_X(G)) = -\mathrm{SS}(G)$.

Sometimes, when the noncharacteristic condition is absent, there is still a less refined upper bounds for pullbacks.

Definition 3.2.12 ([32, Definition 6.2.3]). We define two constructions of closed conic subsets of cotangent bundles:

1. Given closed conic subsets $A, B \subseteq T^*M$, we define $A \hat{+} B$ to be the closed subset consisting of points $(x, \xi) \in T^*M$ such that, in some local coordinate, there exist sequence $\{(x_n, \xi_n)\}$ in A and $\{(y_n, \eta_n)\}$ in B such that $x_n, y_n \to x$, $\xi_n + \eta_n \to \xi$, and $|x_n - y_n||\xi_n| \to 0$.

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2. Let $i : X \hookrightarrow M$ be a closed submanifold and choose a local coordinate (x, y, ξ, η) of T^*M such that X is given by $\{y = 0\}$. Given a closed conic subset $A \subseteq T^*M$, we define $i^\#(A)$ to be the closed subset of T^*X consisting of points (x, ξ) such that there exists $\{(x_n, y_n, \xi_n, \eta_n)\}$ in A such that $y_n \to 0$, $x_n \to x$, $\xi_n \to \xi$, and $|y_n||\eta_n| \to 0$.

In general for $f : M \to N$ and closed conic subset $A \subseteq T^*N$, $f^\#(A)$ can be defined as a special case of a more general construction which also includes $A \hat{+} B$ as a special case. Moreover, the definition can be made free of choice of local coordinates using the technique of *deformation to the normal cone* [32, Section 4.1, Section 6.2]. Also, notice that the above construction $A \hat{+} B$ and $i^\#(A)$ contain $A + B$ and $di^*(i_\pi^{-1}(\mathrm{SS}(F)))$ as closed subsets. It can be shown that they are equal when the noncharacteristic condition is satisfied.

Proposition 3.2.13. *We have the following results:*

1. *([32, Theorem 6.3.1]) Let $j : U \hookrightarrow M$ be open and $F \in \mathrm{Sh}(U)$, then*

$$\mathrm{SS}(j_*F) \subseteq \mathrm{SS}(F) \hat{+} N_{in}^*U.$$

2. *([32, Corollary 6.4.4]) Let $i : Z \to M$ be a closed submanifold and $F \in \mathrm{Sh}(M)$, then*

$$\mathrm{SS}(i^*F) \subseteq i^\# \mathrm{SS}(F).$$

3. *([32, Corollary 6.4.5.]) Let $F, G \in \mathrm{Sh}(M)$, then*

$$\mathrm{SS}(F \otimes G) \subseteq \mathrm{SS}(F) \hat{+} \mathrm{SS}(G), \ \mathrm{SS}\left(\mathcal{H}om(G, F)\right) \subseteq \mathrm{SS}(F) \hat{+} \left(-\mathrm{SS}(G)\right).$$

The notion of $\hat{+}$ can also be used to measure compatibility between !-pullback and $\otimes$.

Proposition 3.2.14 ([32, Exercise VI.4]). *Let $f : M \to N$ be a morphism of manifolds and let $F, G \in \mathrm{Sh}(N)$. If f is non-characteristic for $\mathrm{SS}(F) \hat{+} \mathrm{SS}(G)$, then*

$$f^*F \otimes f^!G = f^!(F \otimes G).$$

As a Corollary, we mention a computational tool:

Lemma 3.2.15. *Let M and N be manifolds. Then*

1. *$p_N^! 1_N = p_M^* \omega_M$,*

2. *$\omega_{M \times N} = \omega_M \boxtimes \omega_N$.*

As a Corollary, ω_M is invertible whose inverse can be given as $\Delta^! 1_{M \times M}$.

Proof. Consider the pullback diagram:

$$
\begin{array}{ccc}
M \times N & \xrightarrow{\ p_N\ } & N \\
\downarrow{\scriptstyle p_M} & & \downarrow{\scriptstyle a_N} \\
M & \xrightarrow{\ a_M\ } & \{*\}
\end{array}
$$

For (1), base change $a_M^* a_{N!} = p_{N!} p_M^*$ implies that there exists a canonical map $p_M^* a_N^! \to p_N^! a_M^*$. This map is in general not an isomorphism but in our case, we may assume M and N are Euclidean spaces by checking the map locally. Then the isomorphism follows from the isomorphism $1_{\mathcal{V}} = \Gamma_c(\mathbb{R}^k; \mathcal{V})[k]$ and $\omega_{\mathbb{R}^k} = 1_{\mathbb{R}^k}[k]$. For (2), we can use (1) and (4) of Proposition 3.2.11 and compute that

$$\omega_M \boxtimes \omega_N = p_M^* \omega_M \otimes p_N^* \omega_N = p_M^* \omega_M \otimes p_M^! 1_M = p_M^! \omega_M = \omega_{M \times N}.$$

To obtain the Corollary, we apply Proposition 3.2.14 and compute that

$$\Delta^!(1_{M \times M}) \otimes \omega_M = \Delta^!(1_{M \times M}) \otimes \Delta^*(p_1^* \omega_M) = \Delta^!(p_1^* \omega_M) = \Delta^! p_2^!(1_M) = 1_M.$$

$\square$

Finally, we mention the compatibility between microsupport and limits/colimits.

Proposition 3.2.16 ([32, Exercise V.7], [31, 2.7]). *Let I be a (small) set and $\{F_i\}_{i \in I}$ be a family of sheaves on M index by I. Then there are microsupport estimations,*

$$\mathrm{SS}(\bigoplus_i F_i) \subseteq \overline{\cup_i \mathrm{SS}(F_i)}, \ \mathrm{SS}(\prod_i F_i) \subseteq \overline{\cup_i \mathrm{SS}(F_i)}.$$

Let $X \subseteq T^*M$ be a conic closed subset, the above microsupport estimation combined with the adjoint functor theorem [36, Corollary 5.5.2.9] implies that the inclusion

$$\iota_{X*} : \mathrm{Sh}_X(M) \subseteq \mathrm{Sh}(M)$$

admits both a left adjoint ι_X^* and a right adjoint $\iota_X^!$. General categorical theory then implies that the category $\mathrm{Sh}_X(M)$ is its self presentable. We will see in the next two sections that, under some mild regularity condition, the category $\mathrm{Sh}_\Lambda(M)$ is in fact compactly generated where $\Lambda \subseteq S^*M$ is a singular isotropic.

3.3 Constructible sheaves

The theory of constructible sheaves is based on the results of stratified spaces. Standard references for stratified spaces are [22] and [39].

A *stratification* $\mathcal{S}$ of X is a decomposition of X into to a disjoint union of locally closed subset $\{X_s\}_{s \in \mathcal{S}}$. A set $Y \subseteq X$ is said to be $\mathcal{S}$-*constructible* if it is a union of strata in $\mathcal{S}$. We assume, without further mention, that a stratification should be locally finite and satisfies the *frontier condition* that $\overline{X_s} \setminus X_s$ is a disjoint union of strata in $\mathcal{S}$. In this case, there is an ordering which is defined by $s \leq t$ if and only if $X_t \subseteq \overline{X_s}$. We always implicitly chose this ordering when regarding $\mathcal{S}$ as a poset. For example, we will consider its linearization $\mathcal{S}\text{-Mod} := \mathrm{PSh}(\mathcal{S}^{op}, \mathcal{V})$.

Definition 3.3.1. For a given stratification $\mathcal{S}$, a sheaf F is said to be $\mathcal{S}$-constructible if $F|_{X_s}$ is a local system for all $s \in \mathcal{S}$. We denote the subcategory of $\mathrm{Sh}(X)$ consisting of such sheaves by $\mathrm{Sh}_\mathcal{S}(X)$. A sheaf F is said to be constructible if F is $\mathcal{S}$-constructible for some stratification $\mathcal{S}$.

For $s \in \mathcal{S}$, we denote by $\text{star}(s)$ the smallest $\mathcal{S}$-constructible open set containing X_s. Alternatively, $\text{star}(s) = \coprod_{t \leq s} X_t$. The poset $\mathcal{S}$ can be then identified with the subposet $\{\text{star}(s) | s \in \mathcal{S}\}$ of Op_X. Hence, there is a functor $\mathcal{S}\text{-Mod} \hookrightarrow \text{Sh}_{\mathcal{S}}(X)$ induced by the restricting along the inclusion of poset $\mathcal{S} \hookrightarrow Op_X$. A stratification is called a *triangulation* if $X = |K|$ is a realization of some simplicial complex K and $\mathcal{S} \coloneqq \{|\sigma| | \sigma \in K\}$ is given by the simplexes of K. We note that when $\mathcal{S}$ is a triangulation, the functor $\mathcal{S}\text{-Mod} \hookrightarrow \text{Sh}_{\mathcal{S}}(X)$ is an equivalence. This follows from the fact that each X_s is contractible and the following criterion:

Lemma 3.3.2 ([21, Lemma 4.2]). *Let Π be a poset with a map to Op_M, and let $\mathcal{V}[\Pi]$ denote its stabilization. The following are equivalent*

- $\Gamma(U; \mathcal{V}) = 1_{\mathcal{V}}$ *for $U \in \Pi$ and $\Gamma(U; \mathcal{V}) \to \Gamma(U \setminus V; \mathcal{V})$ whenever $U \not\subseteq V$.*

- *The composition $\mathcal{V}[\Pi] \to \mathcal{V}[Op_M] \to \text{Sh}(M)$ is fully faithful where the second map is given by !-pushforward.*

We note that the stratification given by the product of triangulations also satisfy this condition. Thus the following slight generalization of [21, Lemma 4.7] holds:

Proposition 3.3.3. *Let $\mathcal{S}$ be triangulation of M. Then $\text{Sh}_{\mathcal{S}}(M) = \mathcal{S}\text{-Mod}$. If $\mathcal{T}$ is a triangulation of N. Then $\text{Sh}_{\mathcal{S} \times \mathcal{T}}(M \times N) = (\mathcal{S} \times \mathcal{T})\text{-Mod}$. Here we denote by $\mathcal{S} \times \mathcal{T}$ the product stratification.*

Now let M be a C^{α} manifold for $\alpha \in \mathbb{N}_{>0} \cup \{\infty, \omega\}$. We consider regularity conditions for stratifications.

Definition 3.3.4. Let M and N be locally closed C^1 manifolds of $\mathbb{R}^n$, with $N \subset \overline{M} \setminus M$. Consider sequences $x_n \in M$ and $y_n \in N$ such that $x_n, y_n \to y \in N$ and $\{T_{x_n}M\}$ converges to τ (in the corresponding Grassmannian). Assume also $\{\mathbb{R}(x_n - y_n)\}$ converges to l. We say the pair (N, M) satisfies the *Whitney condition* if any such sequence satiesfies $\tau \supseteq l$. In general, we say a pair (N, M) of C^1 submanifold manifolds of a C^{α} manifold X satisfies the Whitney condition if the above condition is satisfies on local charts.

Remark 3.3.5 ([39, Chapter 4]). The Whitney condition can also be formulated without working on coordinates. Recall the normal bundle of the diagonal $\Delta_M \hookrightarrow M \times M$ can be identified as the tangent bundle TM of M. The real blow-up $Bl_{\Delta_M}(M \times M)$ can be seen as a disjoint union $\mathbb{P}(TM) \amalg (M \times M \setminus \Delta_M)$ of the projective tangent bundle and the off-diagonal. Then we say (N_1, N_2) satisfies the Whitney condition if for any sequence $(y_n, x_n) \in N_1 \times N_2 \subseteq Bl_{\Delta_M}(M \times M) \setminus \Delta_M$ such that $T_{x_n}N_2 \to \tau$ and $(y_n, x_n) \to l \in \mathbb{P}(TM)$, we have $l \subseteq \tau$.

We say a stratification $\mathcal{S}$ is C^k if each X_s is C^k locally closed manifold. A C^k stratification $\mathcal{S}$ is a Whitney stratification if (X_t, X_s) satisfies the Whitney condition for $s \leq t$. Let $N^*\mathcal{S}$ denote the union $\cup_{s \in \mathcal{S}} N^*X_s$ of the conormals of the strata. The set $N^*\mathcal{S}$ is a singular conic Lagrangian in T^*M and the Whitney condition implies a weaker property that $N^*\mathcal{S} \subseteq T^*X$ is closed. We also use $N^*_{\infty}\mathcal{S}$ to denote the corresponding singular isotropic at the infinity. The main advantage of considering Whitney stratifications are the following proposition.

Proposition 3.3.6 ([32, Prop. 8.4.1], [21, Proposition 4.8])**.** *For a Whitney stratification* $\mathcal{S}$ *of a* C^1 *manifold* M, *we have* $\mathrm{Sh}_{\mathcal{S}}(M) = \mathrm{Sh}_{N^*_\infty \mathcal{S}}(M)$ *(i.e. having microsupport contained in* $N^*\mathcal{S}$ *is equivalent to being* $\mathcal{S}$*-constructible).*

Example 3.3.7. We note that the Whitney assumption is crucial for this proposition. Consider the C^∞ map

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$
$$(x, y, z) \mapsto (z^2 x, z^2 y, z).$$

Set $V := \{x^2 + y^2 = z^4\}$, which we note is the image of $C := \{x^2 + y^2 = 1\}$ under f. Take the stratification $\mathcal{S} = \{X_1, X_2, X_3\}$ of $\mathbb{R}^3$ where $X_1 := \{x = z^2, y = 0\}$, $X_2 := V \setminus X_1$, and $X_3 := \mathbb{R}^3 \setminus X_2$.
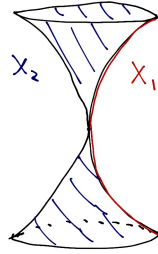

Figure 3.1: A stratification of $\mathbb{R}^3$ which provides a counterexample of the above proposition when the Whitney condition is not present. The red curve is the single 1-dimensional stratum X_1, the black locally closed surface is the single 2-dimensional stratum X_2, and their complement is the single 3-dimensional open stratum X_3.

We note that $\mathcal{S}$ does not satisfy Whitney condition (b). Now consider the sheaf $F := f_*(1_C)$. Since f is proper on C, its microsupport $\mathrm{SS}(F)$ is bounded by $f_\pi((df^*)^{-1} \mathrm{SS}(1_C))$. Because $\mathrm{SS}(1_C) = N^*C$, the slice of $(df^*)^{-1} \mathrm{SS}(1_C)$ with $\{z = 0\}$ is given by $(\mathbb{R}^*)^2 \times \{0\}$ at $(x, y, 0)$. Thus $\mathrm{SS}(F)$ is contained in $N^*X_1 \subseteq N^*\mathcal{S}$. However, by base change $F|_{X_1} = (f|_C)_*(1_{S^1 \times \{0\} \cup \{(1,0)\} \times \mathbb{R}^1})$ is not a constant sheaf.
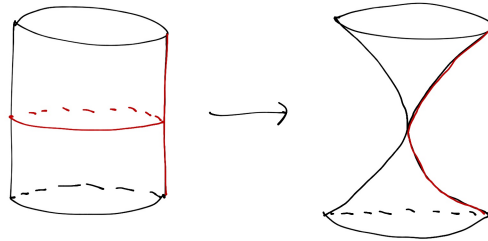

Figure 3.2: The picture exhibiting the fact that F above is not locally constant on X_1. Stalks along from 0 are given simply by $1_V = \Gamma(\{*\}; \mathcal{V})$ but the stalk at 0 is $\Gamma(S^1; \mathcal{V})$.

The above proposition is a corollary of the existence of *inward cornerings* defined by applying the following lemma which is also proved in the same paper and we will use them for the main theorem of this paper as well.

Lemma 3.3.8 ([21, Proposition 2.3]). *Fix any $1 \leq p \leq \infty$, and let $\mathcal{S}$ be a C^p Whitney stratification of M. Fix a relatively compact $\mathcal{S}$-constructible set Y. Let $\mathcal{S}_Y := \{s | X_s \subseteq Y\}$ denote the collection of strata consisting of Y and set $N^*\mathcal{S}_Y := \cup_{s \in \mathcal{S}_Y} N^* X_s$, which is closed in T^*X by the Whitney condition. Then there exists a decreasing family Y^ϵ of open neighborhoods of Y such that as $\epsilon \to 0$,*

1. *$N^* Y^\epsilon$ becomes contained in arbitrary small conic neighborhood of $N^* Y$,*

2. *$N^* Y^\epsilon \cap N^* \mathcal{S} = \varnothing$ for $\epsilon > 0$.*

Definition 3.3.9. For a relative compact $\mathcal{S}$-constructible open set U, an *inward cornering* of U is an open set of the form
$$U^{-\epsilon} := U \setminus \overline{(\partial U)^\epsilon}.$$
When $\epsilon > 0$ is small, the inward cornering $U^{-\epsilon}$ is a codimension 0 open submanifold whose closure $\overline{U^{-\epsilon}}$ is a compact manifold with corners. The family $U^{-\epsilon}$ depends smoothly on ϵ. Its outward conormal $N^*_{\infty, out} U^{-\epsilon}$ remains disjoint from $N^*_\infty \mathcal{S}$ as ϵ changes, and converges to $N^*_\infty \mathcal{S}$ uniformly as $\epsilon \to 0$.

Combining with the comment on triangulations, we obtain a simple description of sheaves microsupported in $N^*_\infty \mathcal{S}$ for some C^1 Whitney triangulation $\mathcal{S}$.

Proposition 3.3.10 ([21, Proposition 4.19]). *Let $\mathcal{S}$ be a C^1 Whitney triangulation. Then, there is an equivalence*
$$\mathrm{Sh}_{N^*_\infty \mathcal{S}}(M) = \mathcal{S}\text{-Mod}$$
$$1_{X_s} \leftrightarrow 1_s$$
where 1_s is the indicator which is defined by
$$1_s(t) = \begin{cases} 1, & t \leq s. \\ 0, & otherwise. \end{cases}$$

In particular, the category $\mathrm{Sh}_{N^*_\infty \mathcal{S}}(M)$ is compactly generated whose compact objects $\mathrm{Sh}_{N^*_\infty \mathcal{S}}(M)^c$ are given by sheaves with compact support and perfect stalks.

Before leaving this section, we mention that categories consisting of sheaves which are constructible with respect to a fixed stratification satisfy compatibility with the symmetric monoidal structure $\otimes$ on $\mathrm{Pr}^{\mathrm{L}}_{st}$. Let $\mathcal{T}$ be a Whitney stratification of another manifold N. The poset structure of $\mathcal{S} \times \mathcal{T}$ is given by the product poset structure of $\mathcal{S}$ and $\mathcal{T}$, and one can check directly that $(\mathcal{S}\text{-Mod}) \otimes (\mathcal{T}\text{-Mod}) = (\mathcal{S} \times \mathcal{T})\text{-Mod}$. Thus the above proposition implies that the desired results for wrapped sheaves:

Proposition 3.3.11. *Let $\mathcal{S}$ be a Whitney triangulation of M and $\mathcal{T}$ a Whitney triangulation of N. There is an equivalence*
$$\boxtimes : \mathrm{Sh}_{N^*_\infty \mathcal{S}}(M) \otimes \mathrm{Sh}_{N^*_\infty \mathcal{T}}(N) \xrightarrow{\sim} \mathrm{Sh}_{N^*_\infty (\mathcal{S} \times \mathcal{T})}(M \times N)$$
sending $1_{\mathrm{star}(s)} \otimes 1_{\mathrm{star}(t)}$ to $1_{\mathrm{star}(s) \times \mathrm{star}(t)}$.

3.4 Isotropic microsupport

We say a subset $\Lambda \subseteq S^*M$ is isotropic if it can be stratified by isotropic submanifolds. A standard class of isotropic subsets are given by the conormal $N^*_\infty \mathcal{S}$ of a stratification $\mathcal{S}$ which we study in the last section. Assume M is real analytic and we recall that a general isotropy which satisfies a decent regularity condition are bounded by isotropics of this form.

Definition 3.4.1. A subset Z of M is said to be subanalytic at x if there exists open set $U \ni x$, compact manifolds Y^i_j ($i = 1, 2, 1 \le j \le N$) and morphisms $f^i_j : Y^i_j \to M$ such that

$$Z \cap U = U \cap \bigcup_{j=1}^N (f^1_j(Y^1_j) \setminus f^2_j(Y^2_j)).$$

We say Z is subanalytic if Z is subanalytic at x for all $x \in M$.

Lemma 3.4.2 ([32, Corollary 8.3.22]). *Let Λ be a closed subanalytic isotropic subset of S^*M. Then there exists a C^ω Whitney stratification $\mathcal{S}$ such that $\Lambda \subseteq N^*\mathcal{S}$.*

Combining with the above proposition, we obtain a microlocal criterion for a sheaf F with subanalytic microsupport being constructible:

Proposition 3.4.3 ([32, Theorem 8.4.2]). *Let $F \in \mathrm{Sh}(M)$ and assume $\mathrm{SS}^\infty(F)$ is subanalytic. Then F is constructible if and only if $\mathrm{SS}^\infty(F)$ is a singular isotropic.*

Another feature of subanalytic geometry is that relatively compact subanalytic sets form an o-minimal structure. Thus, one can apply the result of [13] to refine a C^p Whitney stratification to a Whitney triangulation, for $1 \le p < \infty$.

Lemma 3.4.4. *Let Λ be a subanalytic singular isotropic in S^*M. Then there exists a C^1 Whitney triangulation $\mathcal{S}$ such that $\Lambda \subseteq N^*\mathcal{S}$.*

Combining the above two results, we conclude:

Theorem 3.4.5. *Let $F \in \mathrm{Sh}(M)$ and assume $\mathrm{SS}^\infty(F)$ is a subanalytic singular isotropic. Then F is $\mathcal{S}$-constructible for some C^1 Whitney triangulation $\mathcal{S}$.*

Collectively, sheaves with the same subanalytic isotropic microsupport form a category with nice finiteness properties. Let Λ be a subanalytic singular isotropic in S^*M.

Proposition 3.4.6. *Let Λ be a subanalytic singular isotropic in S^*M. The category $\mathrm{Sh}_\Lambda(M)$ is compactly generated.*

Proof. Fix a C^1 Whitney triangulation $\mathcal{S}$ such that $\Lambda \subseteq N^*_\infty \mathcal{S}$ by Lemma 3.4.4. Recall that $\mathcal{S}\text{-Mod} = \mathrm{Ind}(\mathrm{Perf}\,\mathcal{S})$ is compactly generated. For $F \in \mathrm{Sh}_\Lambda(M)$, there exists $F_i \in \mathrm{Sh}_{N^*_\infty \mathcal{S}}(M)$ such that $\iota_{\Lambda *} F = \varinjlim F_i$. Thus

$$F = \iota_\Lambda^* \iota_{\Lambda *} F = \iota_\Lambda^* \iota_{\Lambda *} \varinjlim F_i = \varinjlim \iota_\Lambda^* F_i.$$

Now note that $\iota_\Lambda{}^* F_i$ is compact in $\mathrm{Sh}_\Lambda(M)$ since $\iota_\Lambda^* \dashv \iota_{\Lambda *} \dashv \iota_\Lambda^!$ and the left adjoint of left joint preserves compact objects. $\qquad\square$

Corollary 3.4.7. *When M is compact, if $F \in \mathrm{Sh}_\Lambda(M)$ has perfect stalks, then F is in* $\mathrm{Sh}_\Lambda(M)^c$.

Proof. Apply Lemma 3.4.4 to include $\mathrm{Sh}_\Lambda(M) \subseteq \mathrm{Sh}_\mathcal{S}(M)$ as in the last proposition and conclude the statement by Proposition 3.3.10. $\qquad\square$

Now recall from Proposition 3.1.6, the presheaf Sh in Cat of ($\mathcal{V}$-valued) sheaves is itself a sheaf. Since a set can be recovered from its intersections with an open cover, the same argument shows that the assignment $U \mapsto \mathrm{Sh}_\Lambda(U)$ forms a sheaf Sh_Λ in Cat as well. In fact, since for an open inclusion $j : U \hookrightarrow M$, $j^* = j^!$ is a right adjoint, Sh_Λ forms a sheaf in $\mathrm{Pr}^{\mathrm{R}}_{st} = (\mathrm{Pr}^{\mathrm{L}}_{st})^{op}$. By Proposition 2.2.5 and Proposition 2.2.13, passing to left adjoints turns Sh_Λ to a cosheaf in $\mathrm{Pr}^{\mathrm{L}}_{\omega,st}$, and taking compact objects further turns it to a cosheaf in st.

Proposition 3.4.8. *The precosheaf $\mathrm{Sh}^c_\Lambda : \mathrm{Op}_M \to \mathrm{st}$ is a cosheaf.*

For an inclusion of subanalytic singular isotropics $\Lambda \subseteq \Lambda'$, by picking a C^1 Whitney triangulation $\mathcal{S}$ such that $\Lambda' \subseteq N^*_\infty \mathcal{S}$, a similar consideration as above shows that the inclusion $\mathrm{Sh}_\Lambda(M) \hookrightarrow \mathrm{Sh}_{\Lambda'}(M)$ has both a left and a right adjoint. Thus,

Proposition 3.4.9. *Passing to left adjoint, the inclusion $\mathrm{Sh}_\Lambda(M) \hookrightarrow \mathrm{Sh}_{\Lambda'}(M)$ induces a canonical functor $\mathrm{Sh}_{\Lambda'}(M)^c \twoheadrightarrow \mathrm{Sh}_\Lambda(M)^c$ between compact objects.*

Let Λ be a singular isotropic, $(x, \xi) \in \Lambda$, and consider the restriction of the microstalk functor to the compactly generated categories $\mu_{(x,\xi)} : \mathrm{Sh}_\Lambda(M) \to \mathcal{V}$. By applying its left adjoint to the generator $1 \in \mathcal{V}$, we see that it is tautologically corepresented by the compact object $\mu^L_{(x,\xi)}(1) \in \mathrm{Sh}_\Lambda(M)^c$. Furthermore, when there is an inclusion $\Lambda \subseteq \Lambda'$ and $(x, \xi) \in \Lambda'$, the corepresentative $\mu^L_{(x,\xi)}(1) \in \mathrm{Sh}_{\Lambda'}(M)^c$ is sent under $\mathrm{Sh}_{\Lambda'}(M)^c \twoheadrightarrow \mathrm{Sh}_\Lambda(M)^c$ to a similar corepresentative in $\mathrm{Sh}_\Lambda(M)^c$ and, they are tautologically sent to the zero object when (x, ξ) is a smooth point in $\Lambda' \setminus \Lambda$. By Proposition 3.2.9, the converse is also true:

Proposition 3.4.10 (Theorem 4.13 of [21]). *Let $\Lambda \subseteq \Lambda'$ be subanalytic isotropics and let $\mathcal{D}^\mu_{\Lambda',\Lambda}(T^*M)$ denote the fiber of the canonical functor $\mathrm{Sh}_{\Lambda'}(M)^c \twoheadrightarrow \mathrm{Sh}_\Lambda(M)^c$. Then $\mathcal{D}^\mu_{\Lambda',\Lambda}(T^*M)$ is generated by the corepresentatives of the microstalk functors $\mu_{(x,\xi)}$ for smooth Legendrian points $(x, \xi) \in \Lambda' \setminus \Lambda$.*

Before we leave this chapter, we prove Theorem 1.0.2 by generalizing of Proposition 3.3.11. Here for closed subsets $X \subseteq S^*M$ and $Y \subseteq S^*N$, we use $X \times Y$ to denote the product in $S^*(M \times N)$ which is defined by the infinite part of the product of the cones over them, i.e., the set

$$\left((0_M \cup \mathbb{R}_{>0}X) \times (0_N \cup \mathbb{R}_{>0}Y)\right)/\mathbb{R}_{>0} \subseteq S^*(M \times N).$$

Proof of Theorem 1.0.2. To deduce the general case from the triangulation case, pick a Whitney triangulation $\mathcal{T}$ of N such that $\Sigma \subseteq N^*\mathcal{T}$ and consider the following diagram:

$$
\begin{array}{ccc}
\mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_\Sigma(N) & \xrightarrow{\;\boxtimes\;} & \mathrm{Sh}_{\Lambda \times \Sigma}(M \times N) \\
\downarrow & & \uparrow \\
\mathrm{Sh}_{N^*\mathcal{S}}(M) \otimes \mathrm{Sh}_{N^*\mathcal{T}}(N) & \overset{=\!=\!=}{\underset{\boxtimes}{}} & \mathrm{Sh}_{N^*\mathcal{S} \times N^*\mathcal{T}}(M \times N)
\end{array}
$$

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The fully-faithfulness of the vertical functor on the left is implied by Lemma 2.3.2. Since the diagram commutes, the horizontal map on the upper row is also fully-faithful. Pass to the left adjoints and restrict to compact objects, the equivalence for the general case will be implied by Proposition 3.4.10 and the proposition cited bolow, whose counterpart in the Fukaya setting is discussed in a more general situation in [19, Section 6]. □

Proposition 3.4.11. *Let* $(x, \xi) \in N^*\mathcal{S}$ *and* $(y, \eta) \in N^*\mathcal{T}$. *We denote by* $D_{(x,\xi)}$ *and* $D_{(y,\eta)}$ *corepresentatives of the microstalk functors at* (x, ξ) *and* (y, η). *Then* $D_{(x,\xi)} \boxtimes D_{(y,\eta)}$ *corepresents the microstalk at* (x, y, ξ, η).

Proof. By Proposition 3.4.10, it's sufficient to show that for $F \in \mathrm{Sh}(M)$ and $G \in \mathrm{Sh}(N)$, there is an equivalence

$$\mu_{(x,\xi)}(F) \boxtimes \mu_{(y,\eta)}(G) = \mu_{(x,y,\xi,\eta)}(F \boxtimes G)$$

since corepresentative are unique. This is the Thom-Sebastiani theorem whose proof in the relevant setting can be found in for example [38, Sebastiani-Thom Isomorphism] or [47, Theorem 1.2.2]. □

Remark 3.4.12. We remark that the theorem is stated as compatibility between vanishing cycles with exterior products $\boxtimes$ in the setting of complex manifold. The proof, however, holds in our case since vanishing cycles $\phi_f(F)$ are traded with $\Gamma_{\{\mathrm{Re}\, f \geq 0\}}(F)|_{f^{-1}(0)}$ at the beginning of the proof in for example [38]. Furthermore the various computations performed there, for example,

$$f^*(D_Y(F)) \cong D_X(f^! F),$$

for a real analytic map $f : X \to Y$, require only $\mathbb{R}$-constructibility.

3.5 Convolution

One can combine the six-functors to build more general functor between sheaves on topological spaces. Let X_i, $i = 1, 2, 3$, be locally compact Hausdorff topological spaces, and write $X_{ij} = X_i \times X_j$, for $i < j$, $X_{123} = X_1 \times X_2 \to X_3$, and $\pi_{ij} : X_{123} \to X_{ij}$ for the corresponding projections. For $F \in \mathrm{Sh}(X_{12})$, $G \in \mathrm{Sh}(X_{23})$, the *convolution* is defined to be

$$G \circ_{M_2} F := \pi_{13!}(\pi_{23}^* G \otimes \pi_{12}^* F) \in \mathrm{Sh}(X_{13}).$$

When there is no confusion what X_2 is, we will usually surpass the notation and simply write it as $G \circ F$. This is usually the case when $X_1 = \{*\}$, $X_2 = X$, and $X_3 = Y$ and we think of X as the source and Y as the target, $G \in \mathrm{Sh}(X \times Y)$ as a functor sending $F \in \mathrm{Sh}(X)$ to $G \circ (F) \in \mathrm{Sh}(Y)$. Note that from its expression, this functor is colimit-preserving.

Lemma 3.5.1 ([32, Proposition 3.6.2]). *For fixed* $G \in \mathrm{Sh}(X_{23})$, *the functor* $G \circ (-) : \mathrm{Sh}(X_{12}) \to \mathrm{Sh}(X_{13})$ *induced by convoluting with* G *has a right adjoint which we denote it as* $\mathcal{H}om^\circ(G, -) : \mathrm{Sh}(X_{13}) \to \mathrm{Sh}(X_{12})$ *and is given by*

$$H \mapsto \pi_{12*} \mathcal{H}om(\pi_{23}^* G, \pi_{13}^! H). \tag{3.1}$$

Example 3.5.2. We note that convolution recovers $*$-pullback and $!$-pushforward. For example, let $f : X \to Y$ be a continuous map and denote by $i : \Gamma_f \subseteq X \times Y$ its graph. Take $X_1 = \{*\}$, $X_2 = X$, and $X_3 = Y$, then for $F \in \mathrm{Sh}(X)$,

$$1_{\Gamma_f} \circ F = \pi_{Y!}(1_{\Gamma_f} \pi_X^* F) = \pi_{Y!} i_! i^* \pi_X^* F = f_! F.$$

We note that base change implies that convolution satisfies associativity.

Proposition 3.5.3. *Let $F_i \in \mathrm{Sh}(X_{ii+1})$ for $i = 1, 2, 3$. Then*

$$F_3 \circ_{X_3} (F_2 \circ_{X_2} F_1) = (F_3 \circ_{X_3} F_2) \circ_{X_2} F_1.$$

In particular, if G_1, $G_2 \in \mathrm{Sh}(X \times X)$, then there is an identification of functors

$$G_2 \circ (G_1 \circ (-)) = (G_2 \circ_X G_1) \circ (-).$$

We will use a relative version of convolution. Let B be a locally compact Hausdorff space viewed as a parameter space. Regard $F \in \mathrm{Sh}(X_{12} \times B)$, $G \in \mathrm{Sh}(X_{23} \times B)$ as B-family sheaves, one can similarly define the relative convolution $G \circ|_B F \in \mathrm{Sh}(X_{13} \times B)$ by replacing π_{ij} with

$$\pi_{ij,B} : X_{123} \times B \to X_{ij} \times B.$$

In the case of manifolds, convolution satisfies certain compatibility with microsupport. For $A \subseteq T^* M_{12}$ and $B \subseteq T^* M_{23}$, we set

$$B \circ A = \{(x, \xi, z, \zeta) \in T^* M_{13} | \exists (y, \eta), (x, \xi, y, \eta) \in A, (y, -\eta, z, \zeta) \in B\}.$$

Note if A and B are Lagrangian correspondences satisfying appropriate transversality condition, the set $B \circ A$ is the composite Lagrangian correspondence twisted by a minus sign on the second component. Write $q_{ij} : T^* M_{123} \to T^* M_{ij}$ to be the projection on the level of cotangent bundles and $q_{2^a 3}$ the composition of q_{23} with the antipodal map on $T^* M_2$. Then $B \circ A = q_{13}(q_{2^a 3}^{-1} B \cap q_{12}^{-1} A)$ and (4), (6) and (3) of Proposition 3.2.11 implies the following corollary.

Corollary 3.5.4 ([26, (1.12)]). *Assume*

1. *p_{13} is proper on $M_1 \times \mathrm{supp}(G) \cap \mathrm{supp}(F) \times M_3$;*

2. *$q_{2^a 3}^{-1} \mathrm{SS}(G) \cap q_{12}^{-1} \mathrm{SS}(F) \cap 0_{M_1} \times T^* M_2 \times 0_{M_3} \subseteq 0_{M_{123}}$.*

$$\mathrm{SS}(G \circ F) \subseteq \mathrm{SS}(G) \circ \mathrm{SS}(F).$$

A similar microsupport estimation holds for the B-family case. One noticeable difference for the microsupport estimation is that instead of $T^* M_{ij}$ and $T^* M_{123}$ one has to consider $T^* M_{ij} \times T^* B$ and $T^* M_{123} \times (T^* B \times_B T^* B)$ instead. Here $\times_B$ is taken over the diagonal $B \hookrightarrow B \times B$. Also the projection $r_{ij} : T^* M_{123} \times (T^* B \times_B T^* B) \to T^* M_{ij} \times T^* B$ for the B-component is now given by the first projection (with a minus sign) for $ij = 12$, the addition for $ij = 13$, and the second projection $ij = 23$. Otherwise the microsupport estimation is similar to the ordinary case.

Now consider a pair of manifold M and N, fix a conic closed subset $Y \subseteq T^* N$, and $K \in \mathrm{Sh}_{T^* M \times Y}(M \times N)$. For $F \in \mathrm{Sh}(M)$, assumption (i) from the previous Corollary 3.5.4 is never satisfied for $K \circ F$. Nevertheless, we show that $\mathrm{SS}(K \circ F) \subseteq Y$.

Lemma 3.5.5. *Let $K \in \mathrm{Sh}(M \times N)$ and Y be a conic closed subset of T^*N. If the micro-support $\mathrm{SS}(K)$ is contained in $T^*M \times Y$, then $\mathrm{SS}(p_{2*}H)$ and $\mathrm{SS}(p_{2!}H)$ are both contained in Y.*

Proof. Let p_1, p_2 denote the projection from $M \times N$ to M and N. We would like to apply (2) of Proposition 3.2.11 so we need to obtain properness with respect to p_2. Pick an increasing sequence of relative compact open set $\{U_i\}_{i \in \mathbb{N}}$ of M such that $M = \cup_{i \in \mathbb{N}} U_i$ and notice that the canonical map $\mathrm{colim}_{i \in \mathbb{N}} H_{M \times U_i} \to H$ is an isomorphism. Thus by (2), (4), and (7) of Proposition 3.2.11, we can compute that

$$\mathrm{SS}(p_{2!}H) = \mathrm{SS}(\mathrm{colim}_{i \in \mathbb{N}} p_{2!}H_{U_i \times N}) \subseteq \overline{\cup_{i \in \mathbb{N}} \mathrm{SS}(p_{2!}H_{U_i \times N})}$$

$$\subseteq \overline{\cup_{i \in \mathbb{N}} p_{2\pi}(\mathrm{SS}(H_{U_i \times N}) \cap 0_M \times T^*N)}$$

$$\subseteq \overline{\cup_{i \in \mathbb{N}} p_{2\pi}(T^*M \times Y \cap 0_M \times T^*N)}$$

$$\subseteq \overline{\cup_{i \in \mathbb{N}} p_{2\pi}(0_M \times Y)} \subseteq \overline{\cup_{i \in \mathbb{N}} Y} = Y.$$

To prove the case for p_{2*} we further requires that $U_i \subseteq \overline{U_i} \subseteq U_{i+1}$ and apply the same computation to the limit $F = \lim_{i \in \mathbb{N}} \Gamma_{\overline{U_i} \times N}(H)$. $\square$

Proposition 3.5.6. *Let $K \in \mathrm{Sh}_{T^*M \times Y}(M \times N)$. Then the assignment $F \mapsto K$ defines a functor*

$$K \circ (-) : \mathrm{Sh}(M) \to \mathrm{Sh}_Y(N).$$

Proof. We recall that, for $F \in \mathrm{Sh}(M)$, $K \circ F := \pi_{N!}(K \otimes \pi_M^* F)$. By (iii) of Proposition 3.2.13, there is microsupport estimation

$$\mathrm{SS}(K \otimes \pi_M^* F) \subseteq \mathrm{SS}(K) \hat{+} (\mathrm{SS}(F) \times 0_M) \subseteq (T^*M \times Y) \hat{+} (T^*M \times 0_N).$$

Now the description of $\hat{+}$ in Definition 3.2.12 implies that if (x, ξ, y, η) is a point on the right hand side, then it comes from a limiting point of a sum from $(x_n, \xi_n, y_n, \eta_n) \in T^*M \times Y$ and $(x'_n, \xi'_n, y'_n, 0) \in T^*M \times 0_N$. Thus $(y, \eta) \in Y$, $\mathrm{SS}(K \otimes \pi_M^* F) \subseteq T^*M \times Y$, and we can apply the last lemma to conclude the proof. $\square$

We will see that in the next section that the above integral transform classifies all colimiting preserving functors between categories of the form $\mathrm{Sh}_\Lambda(M)$ where Λ is a singular closed isotropic. Before we leave this section, we notice that for a conic closed subset $X \subseteq T^*M$, we can take any $K \in \mathrm{Sh}(M \times M)$ and obtain a universal integral kernel $\iota^*_{-X \times X} K \in \mathrm{Sh}_{-X \times X}(M \times M)$. By the above proposition, $\iota^*_{-X \times X}(K)$ defines a functor $\iota^*_{-X \times X}(K) \circ (-) : \mathrm{Sh}(M) \to \mathrm{Sh}_X(M)$. On the other hand, we can consider similarly functors $\mathrm{Sh}(M) \to \mathrm{Sh}_X(M)$ which are defined by $F \mapsto \iota^*_{-X \times X}(K) \circ \iota^*_X(F)$ or $F \mapsto \iota^*_X(K \circ F)$. The claim is that they are all the same.

Lemma 3.5.7. *The following functors $\mathrm{Sh}(M) \to \mathrm{Sh}_X(M)$ are equivalent to each other:*

1. $F \mapsto \iota^*_X(K \circ \iota^*_X(F))$,

2. $F \mapsto \iota^*_{-X \times X}(K) \circ F$,

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3. $F \mapsto \iota^*_{-X \times X}(K) \circ \iota^*_X(F)$.

In particular, $\iota^*_X(F) = \iota^*_{-X \times X}(1_\Delta) \circ F$.

Proof. We note all three of the expressions on the right hand side are in $\mathrm{Sh}_X(M)$ by the previous Proposition 3.5.6, and we can see directly that, by Lemma 3.5.1 and the right adjoint version of Proposition 3.5.6 that

$$\mathrm{Hom}(\iota^*_{-X \times X}(K) \circ G, F) = \mathrm{Hom}\left(G, \mathcal{H}\mathrm{om}^\circ(\iota^*_{-X \times X}(K), F)\right)$$
$$= \mathrm{Hom}\left(\iota^*_X(G), \mathcal{H}\mathrm{om}^\circ(\iota^*_{-X \times X}(K), F)\right) = \mathrm{Hom}(\iota^*_{-X \times X}(K) \circ \iota^*_X(G), F)$$

for $F \in \mathrm{Sh}_X(M)$, $G \in \mathrm{Sh}(M)$. Thus (ii) and (iii) are the same.

Now we show that $\mathrm{Hom}(\iota^*_{-X \times X}(K) \circ G, F) = \mathrm{Hom}(\iota^*_X(K \circ \iota^*_X G), F)$ for $F \in \mathrm{Sh}_X(M)$, $G \in \mathrm{Sh}(M)$. We've seen that the left hand side is the same as $\mathrm{Hom}(G, \mathcal{H}\mathrm{om}^\circ(\iota^*_{-X \times X}(K), F))$ and the target is in $\mathrm{Sh}_X(M)$. A similarly computation will imply that the right hand side is the same as $\mathrm{Hom}(G, \iota^!_X \mathcal{H}\mathrm{om}^\circ(K, F))$ and the target is again in $\mathrm{Sh}_X(M)$. This means that we can evaluate at $G \in \mathrm{Sh}_X(M)$ and prove the quality only for this case. Assume such a case, so tautologically $G = \iota^*_X G$, and we compute that

$$\mathrm{Hom}(\iota^*_{-X \times X}(K) \circ G, F) = \mathrm{Hom}\left(\iota^*_{-X \times X}(K), \mathcal{H}\mathrm{om}(p^*_1 G, p^!_2 F)\right)$$
$$= \mathrm{Hom}\left(K, \mathcal{H}\mathrm{om}(p^*_1 G, p^!_2 F)\right) = \mathrm{Hom}(K \circ G, F) = \mathrm{Hom}(\iota^*_X(K \circ \iota^*_X(G)), F).$$

Note that, for the third equality, we use (4) and (7) to conclude that $\mathcal{H}\mathrm{om}(p^*_1 G, p^!_2 F) \in \mathrm{Sh}_{-X \times X}(M \times M)$. $\square$

3.6 Dualizability

Classifying colimiting-preserving functors shares a close relation with the notion of duality in Definition 2.3.3. In the algebraic geometric setting, this is usually referred as Fourier-Mukai [7]. One strategy to prove such a theorem is, the evaluation and coevaluation should be given by some sort of diagonals geometrically [15, Section 9]. The equivalence between such geometric diagonals and the categorical diagonals discussed in Proposition 2.3.4, which is implied by the uniqueness of duals, will provide such a classification.

In our case, we denote by $\Delta : M \hookrightarrow M \times M$ the inclusion of the diagonal and by $p : M \to \{*\}$ the projection to a point. By Proposition 1.0.2, there is an identification $\mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_{-\Lambda}(M) = \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M)$. Under this identification, we propose a duality data (η, ϵ) between $\mathrm{Sh}_\Lambda(M)$ and $\mathrm{Sh}_{-\Lambda}(M)$ in Pr^L_{st} which is given by

$$\epsilon = p_! \Delta^* : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \to \mathcal{V}$$
$$\eta = \iota^*_{\Lambda \times -\Lambda} \Delta_* p^* : \mathcal{V} \to \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M).$$

Recall that we use $\iota^*_{\Lambda \times -\Lambda} : \mathrm{Sh}(M \times M) \to \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M)$ to denote the left adjoint of the inclusion $\mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M) \hookrightarrow \mathrm{Sh}(M \times M)$. Note also that since $\mathcal{V}$ is compactly generated by $1_\mathcal{V}$, the colimit-preserving functor η is determined by its value on $1_\mathcal{V}$ so we will abuse the notation and identify it with η. In order to check the triangle equalities, we first identify $\mathrm{id} \otimes \epsilon$.

Lemma 3.6.1. *Under the identification*

$$\mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) = \mathrm{Sh}_{\Lambda \times -\Lambda \times \Lambda}(M \times M \times M),$$

the functor

$$\mathrm{id} \otimes \epsilon : \mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \to \mathrm{Sh}_\Lambda(M) \otimes \mathcal{V} = \mathrm{Sh}_\Lambda(M)$$

is identified as the functor

$$p_{1!}(\mathrm{id} \times \Delta)^* : \mathrm{Sh}_{\Lambda \times -\Lambda \times \Lambda}(M \times M \times M) \to \mathrm{Sh}_\Lambda(M).$$

Proof. Since both of the functors are colimit-preserving and the categories are compactly generated, it is sufficient to check that $p_{1!}(\mathrm{id} \times \Delta)^* \circ \boxtimes = \mathrm{id} \otimes (p_! \Delta^*)$ on pairs (F, G) for $F \in \mathrm{Sh}_\Lambda(M)^c$ and $G \in \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M)^c$ by Proposition 2.3.1.

Note that we do not need the compactness assumption for the following computation. Let $q_1 : M^3 \to M$ and $q_{23} : M^3 \to M^2$ denote the projections $q_1(x, y, z) = x$ and $q_{23}(x, y, z) = (y, z)$. We note that $q_1 \circ (\mathrm{id} \times \Delta) = p_1$ and $q_{23} \circ (\mathrm{id} \times \Delta) = \Delta \circ p_2$. Thus,

$$\begin{aligned} p_{1!}(\mathrm{id} \times \Delta)^*(F \boxtimes G) &= p_{1!}(\mathrm{id} \times \Delta)^*(q_1^* F \otimes q_{23}^* G) = p_{1!}(p_1^* F \otimes p_2^* \Delta^* G) \\ &= F \otimes (p_{1!} p_2^* \Delta^* G) = F \otimes (p^* p_! \Delta^* G) = F \otimes_\mathcal{V} (p_! \Delta^* G). \end{aligned}$$

Note that we use the compatibility properties (1) and (2) in Proposition 3.1.8 and the base change Proposition 3.1.7 for the six-functors in this computation. □

Remark 3.6.2. A similar computation will imply that $\eta \otimes \mathrm{id}$ can be identify with

$$\iota^*_{\Lambda \times -\Lambda \times \Lambda}(1_\Delta \boxtimes \bullet) : \mathrm{Sh}_\Lambda(M) \to \mathrm{Sh}_{\Lambda \times -\Lambda \times \Lambda}(M \times M \times M).$$

Now we check the triangle equality $(\mathrm{id}_{\mathrm{Sh}_\Lambda(M)} \otimes \epsilon) \circ (\eta \otimes \mathrm{id}_{\mathrm{Sh}_\Lambda(M)}) = \mathrm{id}_{\mathrm{Sh}_\Lambda(M)}$. In other words, we check that the composition of the following functors

$$\mathrm{Sh}_\Lambda(M) \xrightarrow{(\iota^*_{\Lambda \times -\Lambda} \Delta_* p^*) \otimes \mathrm{id}} \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M) \otimes \mathrm{Sh}_\Lambda(M)$$

$$\downarrow \boxtimes$$

$$\mathrm{Sh}_{\Lambda \times -\Lambda \times \Lambda}(M \times M \times M) \xrightarrow{p_{1!}(\mathrm{id} \times \Delta)^*} \mathrm{Sh}_\Lambda(M)$$

is the identity. The other triangle equality can be checked symmetrically.

Proposition 3.6.3. *The above equality holds.*

Proof. Let $F \in \mathrm{Sh}_\Lambda(M)$. The composition of the first two arrows sends $(1_\mathcal{V}, F)$ to

$$\left(\boxtimes \circ (\iota^*_{\Lambda \times -\Lambda} \Delta_* p^*) \otimes \mathrm{id}\right)(1_\mathcal{V}, F) = (\iota^*_{\Lambda \times -\Lambda} \Delta_* 1_M) \boxtimes F.$$

Apply $p_{1!}(\mathrm{id} \times \Delta)^*$ and we obtain

$$p_{1!}(\mathrm{id} \times \Delta)^* \left((\iota^*_{\Lambda \times -\Lambda} \Delta_* 1_M) \boxtimes F\right) = p_{1!} \left((\iota^*_{\Lambda \times -\Lambda} \Delta_* 1_M) \otimes p_2^* F\right).$$

To see that $p_{1!}\left((\iota_{\Lambda\times-\Lambda}^*\Delta_*1_M)\otimes p_2^*F\right) = F$, we use the Yoneda lemma to evaluate at $\mathrm{Hom}(-,H)$ for $H \in \mathrm{Sh}_\Lambda(M)$ and compute that

$$\begin{aligned}
\mathrm{Hom}(p_{1!}\left((\iota_{\Lambda\times-\Lambda}^*\Delta_*1_M)\otimes p_2^*F\right),H) &= \mathrm{Hom}\left(\iota_{\Lambda\times-\Lambda}^*\Delta_*1_M, \mathcal{H}om(p_2^*F, p_1^!H)\right)\\
&= \mathrm{Hom}\left(\Delta_*1_M, \mathcal{H}om(p_2^*F, p_1^!H)\right)\\
&= \mathrm{Hom}\left(1_M, \Delta^!\,\mathcal{H}om(p_2^*F, p_1^!H)\right)\\
&= \mathrm{Hom}\left(1_M, \mathcal{H}om(F,H)\right) = \mathrm{Hom}(F,H).
\end{aligned}$$

For the first equality, we use (3) and (6) of Proposition 3.2.11 to obtain $\mathrm{SS}(p_2^*F) = 0_M\times\mathrm{SS}(F)$ and $\mathrm{SS}(p_1^!H) = \mathrm{SS}(H)\times 0_M$, and they further imply the microsupport estimation

$$\mathrm{SS}(\mathcal{H}om(p_2^*F, p_1^!H)) \subseteq (\mathrm{SS}(H)\times 0_M) + (0_M\times -\mathrm{SS}(F))$$

since $(0_M\times\mathrm{SS}(F))\cap(\mathrm{SS}(H)\times 0_M) \subseteq 0_{M\times M}$ and $\mathcal{H}om(p_2^*F, p_1^!H) \in \mathrm{Sh}_{-\Lambda\times\Lambda}(M)$. We use (4) of Proposition 3.1.8 to obtain the second to last equality. $\qquad\square$

Because duals are unique, there is an equivalence $\mathrm{Sh}_{-\Lambda}(M) = \mathrm{Sh}_\Lambda(M)^\vee$ and we denote by $\mathrm{D}_\Lambda : \mathrm{Sh}_{-\Lambda}(M)^c \xrightarrow{\sim} \mathrm{Sh}_\Lambda(M)^{c,op}$ the induced equivalence on wrapped sheaves associated to the pair (M,Λ). Thus, there is a commutative diagram given by the counits:

$$\begin{array}{ccc}
\mathrm{Sh}_{-\Lambda\times\Lambda}(M\times M) & \xrightarrow{\;\;p_!\Delta^*\;\;} & \mathcal{V}\\
\| & & \\
\mathrm{Sh}_{-\Lambda}(M)\otimes\mathrm{Sh}_\Lambda(M) & & \\
\mathrm{Ind}(\mathrm{D}_\Lambda)\otimes\mathrm{id}\,\| & & \|\\
\mathrm{Sh}_\Lambda(M)^\vee\otimes\mathrm{Sh}_\Lambda(M) & \xrightarrow[\;\mathrm{Hom}(-,-)\;]{} & \mathcal{V}
\end{array}$$

Here we abuse the notation and use $\mathrm{Hom}(-,-)$ to denote the functor induced by its Ind-completion. In particular, for $G \in \mathrm{Sh}_{-\Lambda}(M)^c$ and $F \in \mathrm{Sh}_\Lambda(M)$, there is an identification

$$\mathrm{Hom}(\mathrm{D}_\Lambda G, F) = p_!(G\otimes F).$$

A consequence of this identification is that colimit-preserving functors are given by integral transforms, i.e., Theorem 1.0.3 discussed in the introduction. We mention the following proof is adapted from [7] where they study a similar statement in the setting of algebraic geometry.

Proof of Theorem 1.0.3. The identification is a composition

$$\mathrm{Sh}_{-\Lambda\times\Sigma}(M\times N) = \mathrm{Sh}_{-\Lambda}(M)\otimes\mathrm{Sh}_\Sigma(N) = \mathrm{Sh}_\Lambda(M)^\vee\otimes\mathrm{Sh}_\Sigma(N) = \mathrm{Fun}^L(\mathrm{Sh}_\Lambda(M), \mathrm{Sh}_\Sigma(N)).$$

The effect of the first equivalence identifies $F\boxtimes G$ with (F,G) for $F \in \mathrm{Sh}_{-\Lambda}(M)^c$, $G \in \mathrm{Sh}_\Sigma(N)$ and these objects generate the corresponding category by Lemma 2.3.1 so it is sufficient to identify them. For $F \in \mathrm{Sh}_{-\Lambda}(M)^c$ and any $G \in \mathrm{Sh}_\Sigma(N)$, the pair (F,G) is first sent to $(\mathrm{D}_\Lambda F, G)$ and then the functor $(co\mathcal{Y}(\mathrm{D}_\Lambda F))\otimes_\mathcal{V} G$ where we use $co\mathcal{Y}$ to denote the co-Yoneda

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embedding. For this computation we use $p_M : M \times N \to M$ to denote the projection to the M-component, $a_M : M \to \{*\}$ the projection to a point, and similarly to N. We evaluate the functor at H and compute again by base change, Proposition 3.1.7, and Proposition 3.1.8 that

$$
\begin{aligned}
\left((co\mathcal{Y}(\mathrm{D}_\Lambda F)) \otimes_{\mathcal{V}} G\right)(H) &= (a_N{}^* \operatorname{Hom}(\mathrm{D}_\Lambda F, H)) \otimes G \\
&= (a_N{}^* a_{M!}(F \otimes H)) \otimes G \\
&= (p_{N!} p_M{}^*(F \otimes H)) \otimes G \\
&= p_{N!} \left(p_M{}^*(F \otimes H) \otimes p_N^* G\right) \\
&= p_{N!} \left((F \boxtimes G) \otimes p_M^* H\right) = (F \boxtimes G) \circ H.
\end{aligned}
$$

Note we use the fact that $\mathrm{D}_\Lambda F$ is the object corepresenting $(G \mapsto a_{M!}(F \otimes G))$. $\qquad \square$

Chapter 4

Isotopies of sheaves

Let (X, ω, Z) be a Liouville manifold and $\alpha := \iota_Z \omega$ be the Liouville form. Consider an isotopy of Lagrangian submanifolds conic at infinity L_t, $t \in [0, 1]$. One says that the isotopy L_t is *positive* if $\alpha(\partial_t \partial_\infty L_t) \geq 0$. Standard Floer theory implies that there is a *continuation element* $c(L_t) \in \mathrm{HF}^*(L_0, L_1)$ if L_0 and L_1 intersect transversally. For any triple (K_0, K_1, K_2) of transversally intersected Lagrangians, there exists also a multiplication map $\mu : \mathrm{HF}^*(K_0, K_1) \otimes \mathrm{HF}^*(K_1, K_2) \to \mathrm{HF}^*(K_0, K_2)$. Thus, for suitable K's, multiplying $c(L_t)$ induces a map $\mathrm{HF}^*(L_1, K) \to \mathrm{HF}^*(L_0, K)$ which is usually referred as the *continuation map* and is one key ingredient for defining the wrapped Floer category. See for example [20, Section 3.3] for details.

4.1 Continuation maps

We recall here the sheaf-theoretical continuation maps studied in [26]. A dual construction can be found in [53] and [27]. In the sheaf-theoretical setting, objects correspond to the continuation elements are simply morphisms/maps between sheaves. Thus, we simply use the term *continuation maps* to refer both the morphisms between sheaves and the induced maps on the Hom's. We denote by (t, τ) the coordinate of $T^*\mathbb{R}$ and by $T^*_{\leq}\mathbb{R} = \{\tau \leq 0\}$ the set of non-positive covectors.

Lemma 4.1.1. *Let M be a manifold, $[-\infty, \infty]$ be the compactification of $\mathbb{R}$ at the two infinities, $p : M \times \mathbb{R} \to M$ be the projection, $j : M \times \mathbb{R} \hookrightarrow M \times [-\infty, \infty]$ be the open interior, and $i_\pm : M \times \{\pm\infty\} \hookrightarrow M \times [-\infty, \infty]$ be the closed inclusion at the positive/negative infinity. Then for a sheaf $F \in \mathrm{Sh}_{T^*M \times T^*_{\leq}\mathbb{R}}(M \times \mathbb{R})$, there are isomorphisms $p_* F = i_-^* j_* F$ and $p_! F[1] = i_+^* j_* F$ identifying the two pushforwards as nearby cycles at the infinities.*

Proof. We first prove the case when $\mathrm{supp}(F) \subseteq M \times [-C, C]$ for some $C \in \mathbb{R}_{>0}$. In this case, $i_-^* j_* F = i_+^* j_* F = 0$ and $p_* F = p_! F$ since p is proper on $\mathrm{supp}(F)$. Let $x \in M$ be a point. Base change implies that $(p_* F)_x = \Gamma(\{x\} \times \mathbb{R}; F|_{\{x\} \times \mathbb{R}})$. Apply the microsupport estimation $\mathrm{SS}(f^* F) \subseteq f^\#(\mathrm{SS}(F))$ from (2) of Proposition 3.2.13 to the inclusion of the slice at x, and we obtain $\mathrm{SS}(F|_{\{x\} \times \mathbb{R}}) \subseteq T^*_{\leq}\mathbb{R}$ so we reduce to the case $M = \{*\}$. In this case, consider the family of open sets $\{(-\infty, t)\}_{t \in \mathbb{R}}$. The noncharacteristic deformation lemma, Lemma 3.1.13,

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implies that $\Gamma(\mathbb{R}; F) \xrightarrow{\sim} \Gamma((-\infty, t); F)$ for all $t \in \mathbb{R}$. Since $\mathrm{supp}(F)$ is compact, the latter vanishes for $t << 0$.

Now for the general case, we first notice there are canonical morphisms $p_* F \to i_-^* j_* F$ and $i_+^* j_* F \to p_! F[1]$ functorial on F: Let $j_- : M \times \mathbb{R} \hookrightarrow M \times [-\infty, \infty)$ denote the open embedding compactifying the negative end. For any $G \in \mathrm{Sh}(M \times [-\infty, \infty))$, there is a fiber sequence

$$j_{-!} j_-^* G \to G \to i_{-!} i_-^* G.$$

Set $G = j_{-*} F$ and recall that $j_-^* j_{-*} = \mathrm{id}$, we obtain the fiber sequence

$$j_{-!} F \to j_{-*} F \to i_{-!} i_-^* j_{-*} F.$$

Let $p_- : M \times [-\infty, \infty) \to M$ denote the projection (and similarly for p_+). The canonical morphism $p_* F \to i_-^* j_* F$ is obtained by applying p_{-*} to the above fiber sequence. The morphism $i_+^* j_* F \to p_! F[1]$ can be obtained similarly.

Recall that there is fiber sequence

$$F_{M \times (-\infty, 0]} \to F \to F_{M \times (0, \infty)}.$$

(5) of Proposition 3.2.11 implies that both $F_{M \times (-\infty, 0]}$ and $F_{M \times (0, \infty)}$ are contained in $T^* M \times T_{\leq}^* \mathbb{R}$. So it is sufficient to prove the cases when $\mathrm{supp}(F) \subseteq M \times (-\infty, C]$ and when $\mathrm{supp}(F) \subseteq \bar{M} \times [-C, \infty)$ for some $C \in \mathbb{R}_{>0}$.

We first prove the cases in which the objects vanish: Assume $\mathrm{supp}(F) \subseteq [-C, \infty)$. We claim $p_* F = 0 = i_-^* j_* F$. One computes

$$p_* F = p_* F_{[-C, \infty)} = p_* \lim_{n \to \infty} F_{[-C, n]} = \lim_{n \to \infty} p_* F_{[-C, n]} = 0$$

by the case when $\mathrm{supp}(F) \subseteq M \times [-C, C]$ for some $C \in \mathbb{R}$. Similarly, by considering the colimit

$$F_{(-\infty, C]} = \mathrm{colim}_{n \to \infty} F_{(-n, C]},$$

one conclude $p_! F = 0 = i_+^* j_* F$ when $\mathrm{supp}(F) \subseteq (-\infty, C]$.

Now assume $\mathrm{supp}(F) \subseteq (-\infty, C]$ and we claim $p_* F = i_-^* j_* F$. Consider again the fiber sequence

$$j_{-!} F \to j_{-*} F \to i_{-!} i_-^* j_{-*} F.$$

Apply p_{-*} and notice that $p_{-!} = p_{-*}$ for these sheaves because of the compact support assumption. Thus, we obtain the fiber sequence

$$p_! F \to p_* F \to i_-^* j_{-*} F.$$

Since $p_! F = 0$ by the previous case, $p_* F = i_-^* j_{-*} F$. The other isomorphism can be obtained similarly. $\square$

To define the continuation map, we need a prototype version of Theorem 1.0.4. For the rest of the section, we use I to denote an open interval and (t, τ) to denote the coordinate of its cotangent bundle $T^* I$.

Proposition 4.1.2 ([26, Proposition 4.8]). *Let $\iota_* : \mathrm{Sh}_{T^*M \times T^*_{\leq}I}(M \times I) \hookrightarrow \mathrm{Sh}(M \times I)$ denote the tautological inclusion. Then there exist left and right adjoints $\iota^* \dashv \iota_* \dashv \iota^!$ which is given by convolutions $\iota^*F = 1_{\{t'>t\}}[1] \circ F$ and $\iota^!F = \mathcal{H}om^{\circ}(1_{\{t'>t\}}[1], F)$. Here we denote by (t, t') the coordinate of I^2.*

Proof. Assume for simplicity $I = (0, 1)$. We first show that $\mathrm{SS}(1_{\{t'>t\}} \circ F) \subseteq T^*M \times T^*_{\leq}I$. Let $\pi_1, \pi_2 : M \times I \times I \to M \times I$ denote the projection $\pi_1(x, t, t') = (x, t)$ and $\pi_2(x, t, t') = (x, t')$. Then $1_{\{t'>t\}} \circ F = \pi_{2!}[(\pi_1^*F)_{M \times \{t'>t\}}]$. In order to estimate the effect of $\pi_{2!}$ on the microsupport, we need the map π_2 to be proper on the support of the sheaf. Thus, let $j : M \times I \times I \hookrightarrow M \times (-1, 1) \times I$ denote the open inclusion extending $I \hookrightarrow (-1, 1)$, $\pi'_2 : M \times (-1, 1) \times I \to M \times I$ denote the projection to the first and the third components, and factorize $1_{\{t'>t\}} \circ F$ to $\pi'_{2*}j_![(\pi_1^*F)_{M \times \{t'>t\}}]$. Before taking π'_{2*}, one observes that, by (4) and (5) of Proposition 3.2.11 and (1) of Proposition 3.2.13, none of the operations introduces non-zero covectors on the second I-component to the microsupport except when taking $(-)_{M \times \{t'>t\}}$, covectors of the form $(0, \sigma, -\sigma)$ for $\sigma \in \mathbb{R}_{>0}$ might be added to the cotangent fibers over the boundary $\{t' = t\}$. Thus

$$
\begin{aligned}
\mathrm{SS}(1_{\{t'>t\}} \circ F) &= \mathrm{SS}(\pi'_{2*}j_![(\pi_1^*F)_{M \times \{t'>t\}}]) \\
&\subseteq (\pi'_2)_\pi \left(\mathrm{SS}(j_![(\pi_1^*F)_{M \times \{t'>t\}}]) \cap T^*M \times 0_{(-1,1)} \times T^*I \right) \\
&\subseteq T^*M \times T^*_{\leq}I.
\end{aligned}
$$

For the right adjoint $\iota^!$, we note that

$$
\begin{aligned}
\lim_{r \to \infty} \left(1_{\{t-r \leq t' \leq t\}} \circ F \right) &= \lim_{r \to \infty} \pi_{2!}[(\pi_1^*F)_{M \times \{t-r \leq t' \leq t\}}] \\
&= \lim_{r \to \infty} \pi_{2*}[(\pi_1^*F)_{M \times \{t-r \leq t' \leq t\}}] \\
&= \pi_{2*} \lim_{r \to \infty} [(\pi_1^*F)_{M \times \{t-r \leq t' \leq t\}}] \\
&= \pi_{2*}[(\pi_1^*F)_{M \times \{t' \leq t\}}].
\end{aligned}
$$

Thus one can argue as the left adjoint case. (Note the last term is different from $1_{\{t' \leq t\}} \circ F$ in general since limits do not commute with convolution.)

In sum, we've shown that there are functors

$$
1_{\{t'>t\}}[1] \circ (-), \quad \lim_{r \to \infty} \left(1_{\{t-r \leq t' \leq t\}} \circ (-) \right) : \mathrm{Sh}(M \times I) \to \mathrm{Sh}_{T^*M \times T^*_{\leq}I}(M \times I).
$$

In order to show that these are indeed the desired adjoints, it is sufficient to show, for example, that the canonical morphism $1_{\Delta_I} \to 1_{\{t'>t\}}[1]$ becomes an isomorphism after convoluting with sheaves in $\mathrm{Sh}_{T^*M \times T^*_{\leq}I}(M \times I)$. Recall that convoluting with 1_{Δ_I} is the same as the identity functor.

Consider the fiber sequence

$$
1_{\{t'>t\}} \to 1_{\{t' \geq t\}} \to 1_{\Delta_I}.
$$

We have similarly $1_{\{t' \geq t\}} \circ F = \pi_{2!}(\pi_1^*F)_{M \times \{t' \geq t\}}$ and a similar microsupport estimation implies that, before applying $\pi_{2!}$, $\mathrm{SS}\left((\pi_1^*F)_{M \times \{t' \geq t\}}\right) \subseteq T^*M \times T^*_{\leq}I \times T^*I$. Thus, the last

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lemma 4.1.1 implies $\pi_{2!}[(\pi_1^* F)_{M \times \{t' \geq t\}}]$ is the nearby circle of $(\pi_1^* F)_{M \times \{t' \geq t\}}$ at ∞ along the first I-direction and it is 0. Thus $F = 1_{\Delta_I} \circ F \xrightarrow{\sim} 1_{\{t'>t\}}[1] \circ F$. A similar argument shows $\lim_{r \to \infty} (1_{\{t-r \leq t' \leq t\}} \circ F) \xrightarrow{\sim} F$ for F with the same microsupport condition.

Finally, we notice that $1_{\{t-r \leq t' \leq t\}} \circ (-)$ is the inverse of $1_{\{t+r>t'>t\}}[1] \circ (-)$. Thus the functor $1_{\{t-r \leq t' \leq t\}} \circ (-)$ is equivalent to $\mathcal{H}om^\circ(1_{\{t+r>t'>t\}}[1], -)$ and

$$\lim_{r \to \infty} (1_{\{t-r \leq t' \leq t\}} \circ F) = \lim_{r \to \infty} \mathcal{H}om^\circ(1_{\{t+r>t'>t\}}[1], F)$$
$$= \mathcal{H}om^\circ(\operatorname*{colim}_{r \to \infty} 1_{\{t+r>t'>t\}}[1], F) = \mathcal{H}om^\circ(1_{\{t'>t\}}[1], F).$$

$\square$

Now let $F \in \mathrm{Sh}_{T^*M \times T^*_\leq I}(M \times I)$ and, by the preceding lemma, $F \xrightarrow{\sim} 1_{\{t'>t\}}[1] \circ_I F$. Let $a \in I$ and let $i_a : M \hookrightarrow M \times I$ denote the slice at a. Applying i_a^* results the isomorphism $i_a^* F \xrightarrow{\sim} 1_{(-\infty,a)}[1] \circ_I F$. Recall that for $a \leq b$, there is a canonical morphism $1_{(-\infty,a)}[1] \to 1_{(-\infty,b)}[1]$ induced by the open inclusion $(-\infty, a) \hookrightarrow (-\infty, b)$.

Definition 4.1.3. For $F \in \mathrm{Sh}_{T^*M \times T^*_\leq I}(M \times I)$ and $a \leq b$ Set $F_x = i_x^* F$ for $x \in I$. We define the continuation map $c(F, a, b) : F_a \to F_b$ to be the (homotopically unique) morphism c that makes the following diagram commute:

$$
\begin{array}{ccc}
F_a & \xrightarrow{\quad c \quad} & F_b \\
\| & & \| \\
1_{(-\infty,a)}[1] \circ F & \longrightarrow & 1_{(-\infty,b)}[1] \circ F
\end{array}
$$

The continuation maps inherit various properties from $1_{(-\infty,a)}$. For example, they compose in the sense that

$$c(F, a_2, a_3) \circ c(F, a_1, a_2) = c(F, a_1, a_3)$$

since the conical map $1_{(-\infty,a_1)} \to 1_{(-\infty,a_2)} \to 1_{(-\infty,a_3)}$ compose to $1_{(-\infty,a_1)} \to 1_{(-\infty,a_3)}$. Let $p_{[a,b]} : N \times [a,b] \to N$ denote the projection. If $F|_{N \times [a,b]} = p_{[a,b]}^* G$ is a pullback from N for some $G \in \mathrm{Sh}(N)$, one can identify $F_a = G = F_b$ through the canonical map $F \to i_{a*} i_a^* F$. In this case, the continuation map $c(F, a, b)$ is equivalent to this identification $F_a = F_b$. Note because convolution $\circ$ is compatible with colimits, we have the following corollary.

Corollary 4.1.4. *For $F \in \mathrm{Sh}_{T^*N \times T^*_\leq I}(N \times I)$, the canonical map $\operatorname*{colim}_{r<t} F_r \to F_t$ is an isomorphism.*

The dual statement for limits is false in general.

Example 4.1.5. Let $N = \{*\}$ and take $F = 1_{(-\infty,0]}$. Then $F_t = 1_\mathcal{V}$ when $t \leq 0$ and 0 otherwise. Thus $F_0 = \operatorname*{colim}_{r<0} F_r$ but $F_0 \neq \lim_{s>0} F_s$.

However, we will consider the noncharacteristic situation.

Definition 4.1.6 ([43]). Let B be a manifold. For $F \in \operatorname{Sh}(M \times B)$, we say that F is *B-noncharacteristic* if the inclusion $i_b : M \times \{b\} \hookrightarrow M \times B$ is noncharacteristic for F for all $b \in B$. Equivalently, F is B-noncharacteristic if $\operatorname{SS}(F) \cap (0_M \times T^*B) \subseteq 0_{M \times B}$.

Lemma 4.1.7. *Let $F \in \operatorname{Sh}(M \times I)$ be I-noncharacteristic. Then,*

1. *The natural morphism $i_t^* F[-1] \to i_t^! F$ is an isomorphism for $t \in I$.*

2. *If $G \in \operatorname{Sh}(M \times I)$ such that $\mathcal{H}om(G, F)$ is I-noncharacteristic, then $i_t^* \mathcal{H}om(G, F) = \mathcal{H}om(i_t^* G, i_t^* F)$.*

3. *Denote by $q : M \times I \to I$ the projection. If q is proper on F, then $q_* F$ is a constant sheaf. Moreover, if $F \in \operatorname{Sh}_{T^*M \times T^*_{\leq} I}(M \times I)$, the continuation maps of F are sent to isomorphisms under q_*.*

Proof. For (i), the equivalence $i_t^* F[1] = i_t^! F$ follows directly from (4) of Proposition 3.2.11 and the observation that since $q \circ i_t = \operatorname{id}$, one has $i_t^! 1_{M \times I} = i_t^! \circ q^! 1_M[1] = 1_M[1]$.

For (ii), apply (i) and use the fact that $i_t^! \mathcal{H}om(G, F)[-1] = \mathcal{H}om(i_t^* G, i_t^! F)[-1]$.

For (iii), (3) of Proposition 3.2.11 implies that $\operatorname{SS}(q_* F) \subseteq q_\tau(\operatorname{SS}(F) \cap 0_M \times T^*I) \subseteq 0_I$ so $q_* F$ is a constant sheaf by Example 3.2.4 and the fact that $\mathbb{R}$ is contractible. For the statement of continuation maps, recall that they are given by $1_{(-\infty, s)} \circ F \to 1_{(-\infty, t)} \circ F$ for $s \leq t$. Since $q_* F = q_! F$, they are sent to $\Gamma_c((-\infty, s); q_* F) \to \Gamma_c((-\infty, t); q_* F)$ by q_* which are isomorphisms. $\qquad\square$

Corollary 4.1.8. *Let $F \in \operatorname{Sh}_{T^*N \times T^*_{\leq} I}(N \times I)$. If F is I-noncharacteristic, then the canonical map $F_t \to \varprojlim_{s > t} F_s$ is an isomorphism.*

Proof. We will use q_{ij} to denote the projection from $N \times I \times I$ to the corresponding components and p_i the projection from $N \times I$. By the above Lemma 4.1.7, $F_t = i_t^! F[1]$. Apply Proposition 4.1.2 and compute by base change and (4) of Proposition 3.1.8, we see that

$$F_t = i_t^! F[1] = i_t^! \mathcal{H}om^\circ(1_{\{s' > s\}}[1], F)[1] = i_t^! q_{12*} \mathcal{H}om(1_{N \times \{s' > s\}}, q_{13}^! F)$$
$$= p_{1*}(i_t \times \operatorname{id}_I)^! \mathcal{H}om(1_{N \times \{s' > s\}}, q_{13}^! F) = p_{1*} \mathcal{H}om(1_{N \times (t, \infty)}, F).$$

That is, for $t \leq s$, the continuation map also corresponds to the map

$$p_{1*} \mathcal{H}om(1_{N \times (t, \infty)}[1], F) \to p_{1*} \mathcal{H}om(1_{N \times (s, \infty)}[1], F)$$

and, since $*$-push commutes with limits and $\mathcal{H}om(-, F)$ turns limits to colimits, we have

$$\varprojlim_{s > t} F_s = \varprojlim_{s > t} p_{1*} \mathcal{H}om(1_{N \times (s, \infty)}[1], F) = p_{1*} \mathcal{H}om(1_{N \times (t, \infty)}[1], F) = F_t.$$

$\qquad\square$

We conclude this section with a homotopical invariant property of the continuation map in the following setting. Let I and J be open intervals and let (t, τ) and (s, σ) be the corresponding coordinates of their cotangent bundles. Let $G \in \operatorname{Sh}(M \times I \times J)$ be a sheaf such that $\operatorname{SS}(G) \subseteq \{\tau \leq 0\}$. For any $x \in I$, we use $G_{t=x} := G|_{M \times \{x\} \times J}$ to denote the

restriction and similarly for $G_{s=y}$, $y \in J$. Note by (2) of Proposition 3.2.13, the same condition $SS(G_{s=y}) \subseteq \{\tau \leq 0\}$ holds. Assume further that there exists $a \leq b$ in I such that $SS(G_{t=a})$, $SS(G_{t=b}) \subseteq T^*M \times 0_J$. By Lemma 4.2.12, this implies that there exist F_a, $F_b \in \text{Sh}(M)$ such that $G_{t=a} = p_s^*F_a$ and $G_{t=b} = p_s^*F_b$ where we use $p_s : M \times J \to M$ to denote the projection. Note that, for each $y \in J$, the restriction $G_{s=y}$ induces a continuation map $c(G, y, a, b) : F_a \to F_b$.

Proposition 4.1.9. *The morphism $c(G, y, a, b)$ is independent of $y \in J$.*

Proof. Since $SS(G) \subseteq \{\tau \leq 0\}$, a family version of Proposition 4.1.2 implies

$$G = 1_{\Delta_I \times J} \circ |_J G \xrightarrow{\sim} 1_{\{s' > s\} \times J}[1] \circ |_J G$$

is an isomorphism where $\circ|_J$ is the J-parametrized convolution. In particular, there is an isomorphism $G_{t=a} \xrightarrow{\sim} 1_{(-\infty, a) \times J}[1] \circ |_J G$ and thus a (J-parametrized) continuation map

$$c_J(G, a, b) : G_{t=a} \to G_{t=b}.$$

For $y \in J$, let $i_y : M \to M \times J$ denote the inclusion of the slice at y. By Proposition 3.1.8, there is equivalence $i_y^*(K \circ |_J G) = K|_{s=y} \circ G_{s=y}$ for $K \in \text{Sh}(I \times J)$. This implies that $c_J(G, a, b)$ restricts to $i_y^* c_J(G, a, b) = c(G_{s=y}, a, b) =: c(G, y, a, b)$. Hence, the $i_y^* \dashv i_{y_*}$ adjunction induces a commuting diagram,

$$
\begin{array}{ccc}
G_{t=a} & \xrightarrow{\ c_J(G, a, b)\ } & G_{t=b} \\
\downarrow & & \downarrow \\
i_{y_*} i_y^* G_{t=a} & \xrightarrow{\ i_{y_*} c(G, y, a, b)\ } & i_{y_*} i_y^* G_{t=b}
\end{array}
$$

which is equivalent to

$$
\begin{array}{ccc}
p_s^* F_a & \xrightarrow{\ c_J(G, a, b)\ } & p_s^* F_b \\
\downarrow & & \downarrow \\
i_{y_*} F_a & \xrightarrow{\ i_{y_*} c(G, y, a, b)\ } & i_{y_*} F_b
\end{array}
$$
.

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Since J is contractible, the horizontal arrows become isomorphism after applying p_{s*}.

$$\begin{array}{ccc}
F_a & \xrightarrow{\;\;p_{s*}c_J(G,a,b)\;\;} & F_b \\
\| & & \| \\
F_a & \xrightarrow{\;\;c(G,y,a,b)\;\;} & F_b
\end{array}\;.$$

That is, the continuation map $c(G,y,a,b)$ is equivalent to $p_{s*}c_J(G,a,b)$ for all $y \in J$.

$\square$

Remark 4.1.10. One can see from the proof that the continuation maps enjoy higher homotopical independence.

4.2 Sheaf-theoretical wrappings

We specialize to the cases of I-family sheaf which come from the *Guillermou-Kashiwara-Schapira sheaf quantization* in this section. Recall that when M is a smooth manifold, its cotangent bundle admits a canonical symplectic structure $(T^*M, d\alpha)$. The Liouville form α is compatible with the $\mathbb{R}_{>0}$-action which freely acts on $\dot{T}^*M$. Thus, there is an induced contact structure on the cosphere bundle S^*M. It can be realized as a contact hypersurface of $\dot{T}^*M$ by picking a Riemannian metric. There is a dictionary between homogeneous symplectic geometry of $\dot{T}^*M$ and contact geometry of S^*M. Thus, we will use them interchangeably when one language is more convenient. See subsection 2.1 of the Preliminary for a more detail discussion.

Definition 4.2.1. Let M, B be a manifolds and I be an open interval containing 0. We say a C^∞ map $\Phi : S^*M \times I \times B \to S^*M$ is a *B-family of contact isotopies* if for each $(t,b) \in I \times B$, the map $\varphi_{t,b} := \Phi(-,t,b)$ is a contactomorphism and $\varphi_{0,b} = \mathrm{id}_{S^*M}$ for all $b \in B$.

As remarked above, a B-family of contact isotopies Φ corresponds to a B-family of homogeneous symplectic isotopies (of degree 1), which we abuse the notation and denote it by Φ as well. For fixed $b \in B$, we let V_{Φ_b} denote the vector field generated by $\varphi_{t,b}$. Since $\varphi_{t,b}$ is homogeneous, V_{Φ_b} is a Hamiltonian vector field with $\alpha(V_{\Phi_b})$ being its Hamiltonian. The latter is the function which evaluates to $\alpha_{\varphi_{t,b}(x,\xi)}(\frac{\partial}{\partial t}\varphi_{(t,b)}(x,\xi))$ at $\varphi_{t,b}(x,\xi)$.

Proposition 4.2.2. *For each B-family of homogeneous symplectic isotopies Φ, there is a unique conic Lagrangian submanifold Λ_Φ in $\dot{T}^*(M \times M) \times T^*I \times T^*B$ which is determined by the equation $T^*_{t,b}(I \times B) \circ \Lambda_\Phi = \Lambda_{\varphi_{t,b}}$ where the later is $\{(x,-\xi,\phi_{t,b}(x,\xi)) \,|\, (x,\xi) \in \dot{T}^*M\}$, the twisted graph of $\varphi_{t,b}$. More precisely, it is given by the formula*

$$\Lambda_\Phi = \left\{ \left(x,-\xi,\varphi_{t,b}(x,\xi),t,-\alpha(V_{\Phi_b})(\varphi_{t,b}(x,\xi)),b,-\alpha_{\varphi_{t,b}(x,\xi)} \circ d(\Phi \circ i_{x,\xi,t})_b(\cdot)\right)\right\} \qquad (4.1)$$

where the parameters run through $(x,\xi) \in \dot{T}^*M$, $t \in I$, $b \in B$, and the map $i_{x,\xi,t}$ is the inclusion of B as the (x,ξ,t)-slice. We use the same notation Λ_Φ to denote its projection to $S^*(M \times M \times I \times B)$ which is a Legendrian submanifold.

The following theorem of Guillermou, Kashiwara, and Schapira is a categorification of the more classical statements of quantization which usually have operators as the quantized objects. The proof given there is the non-family case. Since the (global) existence is proved by using the uniqueness property to glue local existence and the local picture depends smoothly on the family J^n, the same proof holds for the family version with minor modification [26, Remark 3.9.].

Theorem 4.2.3 ([26, Proposition 3.2]). *Let M be a manifold. For a J^n-family contact isotopies $\Phi : S^*M \times I \times J^n \to S^*M$ where J is an open interval, there exists a unique sheaf kernel $K(\Phi) \in \mathrm{Sh}(M \times M \times I \times J^n)$ such that $\mathrm{SS}^\infty(K(\Phi)) \subseteq \Lambda_\Phi$ and $K(\Phi)|_{t=0} = 1_{\Delta_M \times J^n}$. Moreover, $\mathrm{SS}^\infty(K(\Phi)) = \Lambda_\Phi$ is simple along Λ_Φ, both projections $\mathrm{supp}(K) \to M \times I \times J^n$ are proper, and the composition is compatible with convolution in the sense that*

1. *$K(\Psi \circ \Phi) = K(\Psi) \circ |_{I \times J^n} K(\Phi)$,*

2. *$K(\Phi^{-1}) \circ |_{I \times J^n} K(\Phi) = K(\Phi) \circ |_{I \times J^n} K(\Phi^{-1}) = 1_{\Delta_M \times I \times J^n}$.*

Here Φ^{-1} is the J^n-family of isotopies given by $\Phi^{-1}(-, t, b) := \phi_{t,b}^{-1}$.

Remark 4.2.4. The equality $\mathrm{SS}^\infty(K(\Phi)) = \Lambda_\Phi$ as well as a few other properties of $K(\Phi)$ followed by the uniqueness is explained in [23].

We refer the above process of obtaining the sheaf kernel $K(\Phi)$ from a contact isotopy Φ or a J^n-family of contact isotopies Φ, for $n > 1$, as the *Guillermou-Kashiwara-Schapira sheaf quantization* or GKS sheaf quantization in short.

Example 4.2.5. The construction in [26, Example 3.10, Example 3.11] works more general: Consider a manifold M and take a Riemannian metric g. Denote by H the homogeneous Hamiltonian $H(x, \xi) := \sqrt{g_x(\xi, \xi)}$, $(x, \xi) \in \dot{T}^*M$ and Φ the corresponding positive isotopy. For small $1- << s < 0$, denote by $Z_s := \{(x, y) \in M \times M | d(x, y) \leq |s|\}$ the closed subset of the pairs of points with distance less than s where $d(x, y)$ is the metric induced on M by g. Then the slice $K(\Phi)|_s$ is given by the 1_{Z_s} and the continuation map from time-s to time-0 is given by $1_{Z_s} \to 1_\Delta$. To get the continuation map to the positive time, we note that $\mathcal{H}om(1_\Delta, p_1^*\omega_M) = \Delta_*\Delta^! p_2^! 1_M = 1_\Delta$. Since $H(x, \xi)$ is time-independent, we conclude by the uniqueness statement that, for small $0 < t << 1$, the time-t continuation map is given by $1_\Delta = \mathcal{H}om(1_\Delta, p_1^*\omega_M) \to \mathcal{H}om(1_{Z_{-t}}, p_1^*\omega_M) = K(\Phi)|_t$.

A corollary of the GKS sheaf quantization construction is that contact isotopies act on sheaves and the action is compatible with the microsupport:

Corollary 4.2.6 ([26, (4.4)]). *Let $\Phi : S^*M \times I \to S^*M$ be a contact isotopy. Then the convolution*

$$K(\Phi)|_t \circ (-) : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$$
$$F \mapsto K(\Phi)|_t \circ F$$

is an equivalence whose inverse is given by $K(\Phi^{-1})|_t \circ (-)$. For a sheaf $F \in \mathrm{Sh}(M)$, there is an equality $\dot{\mathrm{SS}}(K(\Phi) \circ F) = \Lambda_\Phi \circ \dot{\mathrm{SS}}(F)$. In particular, if we set $F_t := (K(\Phi) \circ_M F)|_{M \times \{t\}}$, then $\mathrm{SS}^\infty(F_t) = \phi_t \mathrm{SS}^\infty(F)$ for $t \in I$. Furthermore, if F has compact support, then so does F_t for all $t \in I$.

We will consider the notion of *wrapping* for sheaves. Recall that in contact geometry, a wrapping is usually referring to a one-parameter deformation of Legendrians L_t in a contact manifold Y. The wrapping is positive (resp. negative) if $\alpha(\partial_t L_t) \geq 0$ (resp. $\alpha(\partial_t L_t) \leq 0$) for some compatible contact form α. An exercise is that such a deformation L_t can always be extended to a contact isotopy Φ on Y. Since deformations of singular isotropics are not yet available at this moment, we first consider globally defined contact isotopies on S^*M, and then use them to deform sheaves through GKS sheaf quantization.

Remark 4.2.7. The term wrapping comes from the example on $\dot{T}^*(\mathbb{R}/\mathbb{Z})$ with the isotopy given by

$$\phi_t(x, \xi) = \begin{cases} (x + t, \xi), & \xi > 0, \\ (x - t, \xi), & \xi < 0. \end{cases}$$

In this paper, we will use the term *positive/negative wrapping* to mean either a positive/negative isotopy, a family of sheaves induced by such an isotopy, or the corresponding family of singular isotropics of those sheaves by taking $\mathrm{SS}^\infty(-)$. Since these two notions are dual to each other, we will mainly work with positive isotopies and simply refer them as *wrappings* when the context is clear.

We will consider the totality of all such wrappings in the next section. For now, we consider wrapping a single sheaf and develop a perturbation trick which we will use later. Let $\Phi : S^*M \times I \to S^*M$ be a contact isotopy isotopy and denote its GKS sheaf quantization by $K(\Phi)$. For $F \in \mathrm{Sh}(M)$, the convolution $K(\Phi) \circ F$ is an object in $\mathrm{Sh}(M \times I)$, which we think of it as an isotopy of F and denote it by F^Φ for simplicity. By abusing the notation, we write $F_t = i_t^*(F^\Phi) \in \mathrm{Sh}(M)$ where $i_t : M \times \{t\} \hookrightarrow M \times I$ is the slice at t so that $F_0 = F$. When Φ is positive, the expression of Λ_Φ implies that $\mathrm{SS}(K(\Phi)) \subseteq \{\tau \leq 0\}$ and there is continuation map $K(\Phi)_s \to K(\Phi)_t$ for $s \leq t$ and it induces continuation maps $F_s \to F_t$ for $s \leq t$ on F.

Proposition 4.2.8 (Perturbation trick). *Let F and $G \in \mathrm{Sh}(M)$ be sheaves such that $\mathrm{supp}(F) \cap \mathrm{supp}(G)$ is compact. Let $\Phi : S^*M \times I \to S^*M$ be a positive isotopy such that $\varphi_t(\mathrm{SS}^\infty(F)) \cap \mathrm{SS}^\infty(G) = \varnothing$ for $t > 0$. Then the continuation map $F \to F_{t_0}$ induces an isomorphism*

$$\mathrm{Hom}(G, F) \xrightarrow{\sim} \mathrm{Hom}(G, F_{t_0})$$

for $t_0 > 0$.

Remark 4.2.9. We note that when G is cohomologically constructible with perfect stalks, the object $\mathrm{Hom}(G, F_{t_0})$ from the last proposition is the same as $\Gamma(M; \mathcal{H}om(G, 1_M) \otimes F_{t_0})$ by (7) of Proposition 3.2.11. This can been seen as a sheaf-theoretic analogue of the procedure of making Lagrangians intersect transversally and will be used frequently in Chapter 5.

Before we start the proof, we recall some constancy results from general sheaf theory. Let $f : X \to Y$ be a continuous map between locally compact Hausdorff space. Denote by $\mathrm{Sh}_f(X)$ the subcategory of $\mathrm{Sh}(X)$ consists of objects F satisfying the condition $F|_{f^{-1}(y)} \in \mathrm{Loc}(f^{-1}(y))$ for all $y \in Y$. We note that $f^*G \in \mathrm{Sh}_f(X)$ for $G \in \mathrm{Sh}(Y)$.

Proposition 4.2.10 ([32, Proposition 2.7.8]). *Assume there is an increase sequence of closed subsets $\{X_n\}$ such that $X_n \subseteq \mathrm{Int}(X_{n+1})$, $X = \cup_n X_n$, and $f_n := f|_{X_n}$ is proper with contractible fibers. Then, the adjunction $f_* : \mathrm{Sh}_f(X) \leftrightarrows \mathrm{Sh}(Y) : f^*$ is an equivalence of categories.*

Corollary 4.2.11. *Let M, B be manifolds and assume B is contractible. Let $p : M \times B \to M$ denote the projection. Then $F \in \mathrm{Sh}(M \times B)$ is of the form p^*G for some $G \in \mathrm{Sh}(M)$ if and only if $F|_{\{x\} \times B}$ is locally constant for all $x \in M$. In this case, $G = p_*F$.*

By Example 3.2.4, Corollary 4.2.11, and (2) of Proposition 3.2.13, we conclude:

Lemma 4.2.12. *Let B be a contractible manifold and $p : M \times B \to M$ be the projection. A sheaf $F \in Sh(M)$ satisfies $p^*p_*F \xrightarrow{\sim} F$ if and only if $\mathrm{SS}(F) \subseteq T^*M \times 0_B$.*

Proof of Proposition 4.2.8. We note that the microsupport estimation $\dot{\mathrm{SS}}(F^\Phi) \subseteq \Lambda_\Phi \circ \dot{\mathrm{SS}}(F)$ implies that the sheaf F^Φ is I-noncharacteristic. Thus by Corollary 4.1.8, the canonical map $F \to \lim_{t>0} F_t$ is an isomorphism. Since taking global sections $\Gamma(M; -)$ preserves limits, the continuation map $\mathrm{Hom}(G, F) \xrightarrow{\sim} \mathrm{Hom}(G, F_{t_0})$ can be obtained from the composition of

$$\mathcal{H}om(G, F_0) = \mathcal{H}om(G, \lim_{t>0} F_t) = \lim_{t>0} \mathcal{H}om(G, F_t) \to \mathcal{H}om(G, F_{t_0})$$

where the last morphism is the projection from the limit diagram. We claim that this diagram is actually a constant diagram after applying $\Gamma(M; -)$.

Consider the sheaf $\mathcal{H}om(p^*G, F^\Phi) \in \mathrm{Sh}(M \times I)$. Apply (7) of Proposition 3.2.11 and the microsupport estimation

$$\mathrm{SS}(\mathcal{H}om(p^*G, F^\Phi)) \subseteq \big(\mathrm{SS}(F^\Phi) + (-\mathrm{SS}(G) \times 0_I)\big)$$

implies that $\mathcal{H}om(p^*G, K(\Phi) \circ F) \in \mathrm{Sh}_{T^*M \times T^*_{\leq} I}(M \times I)$ is also I-noncharacteristic. A similar computation as in Corollary 4.1.8 implies that continuation maps obtained from applying $\mathcal{H}om(G, -)$ to the continuation maps of F is the same as those associated to $\mathcal{H}om(p^*G, F^\Phi)$. Denote by $q : M \times I \to I$ the projection to I. The assumption that $\mathrm{supp}(G) \cap \mathrm{supp}(F)$ is compact implies that q is proper on $\mathrm{supp}\big(\mathcal{H}om(p^*G, F^\Phi)\big)$ so we may apply the microsupport estimation to conclude, by Lemma 4.2.12 above, that $q_* \mathcal{H}om(p^*G, F^\Phi)$ is a constant sheaf. Finally, properness and I-noncharacteristic assumptions that

$$\big(q_* \mathcal{H}om(p^*G, F^\Phi)\big)_t = \mathcal{H}om(G, F_t)$$

which concludes the proof.

$\square$

4.3   The category of positive wrappings

We will define the category of positive wrappings whose morphisms will be given by concatenation. In order to define concatenation easily, we assume that the isotopies are constant near the end points, and the interval I will be a closed interval for this section. This requirement

doesn't lose much information since for any positive contact isotopy $\Phi : S^*M \times [0,1] \to S^*M$, one can always make it have constant ends through a homotopy of positive isotopies. For example, pick a non-decreasing C^∞ function ρ on $\mathbb{R}$ such that $\rho|_{(-\infty,1/3]} \equiv 0$ and $\rho|_{[2/3,\infty)} \equiv 1$, then an example of such a modification is given by $\tilde\Phi(x,\xi,t,s) = \Phi(x,\xi,(1-s)t+s\rho(t))$. By Proposition 4.1.9, they induce equivalent continuation maps and two such identifications can itself be identified by a similar consideration and so on. Thus, when we mention isotopies obtained through nature constructions such as by integrating from a time-independent vector field, we will implicitly assume such a deformation procedure.

Definition 4.3.1. Let $I = [t_0,t_1]$, $J = [s_0,s_1]$ be two closed intervals. We use $I\#J$ to denote the concatenated interval $(I \amalg J)/\{t_1 \sim s_0\}$. For isotopies $\Phi : S^*M \times I \to S^*M$, $\Psi : S^*M \times J \to S^*M$, the *concatenation* isotopies $\Psi\#\Phi : S^*M \times (I\#J) \to S^*M$ is the isotopy which is given by

$$(\Psi\#\Phi)(x,\xi,t) = \begin{cases} \Phi(x,\xi,t), & t \in I, \\ \Psi(\Phi(x,\xi,t_1),t), & t \in I'. \end{cases}$$

If $I = J$, one can also define the pointwise composition $\Psi \circ \Phi : S^*M \times I \to S^*M$ by

$$(\Psi \circ \Phi)(x,\xi,t) = \Psi(\Phi(x,\xi,t),t)).$$

Note that, up to a scaling, $\Psi \circ \Phi$ and $\Psi\#\Phi$ are homotopic to each othert and, if both Φ and Ψ are positive, they are homotopic through positive isotopies.

Definition 4.3.2. Let $\Omega \subseteq S^*M$ be an open subset. We say a contactomorphism $\varphi : S^*M \to S^*M$ is *compactly supported* on Ω if φ equals id_{S^*M} outside a compact set C in Ω. Similarly, a contact isotopy $\Phi : S^*M \times I \to S^*M$ is *compactly supported* on Ω if $\varphi_t = \mathrm{id}$ outside a fixed compact set C in Ω for all $t \in I$.

Definition 4.3.3. We define *the category $W(\Omega)$ of compactly supported positive wrappings* on Ω as follows: An object of $W(\Omega)$ is a pair $(\varphi, [\Phi])$ such that φ is a compactly supported contactomorphism and $[\Phi]$ is a homotopy class of compactly supported isotopies, defined on a closed interval I, having φ as its end point and realizing it as Hamiltonian. Note that the degenerate case $I = \{*\}$ is allowed. We will often simply write $(\varphi, [\Phi])$ by Φ without emphasizing that it is a homotopy class through the paper when it is irrelevant. A 1-morphism $\Psi : [\Phi_0] \to [\Phi_1]$ is a positive isotopy Ψ such that $[\Phi_1] = [\Psi\#\Phi_0]$. Composition of 1-morphisms is given by concatenation. For $\Psi_0, \Psi_1 : [\Phi_0] \to [\Phi_1]$, a 2-morphism is a positive family of isotopies $\Theta : S^*M \times I \times J \to S^*M$ which is constant near the end points on the J-direction such that $\Theta(-,t,s_i) = \Psi_i(-,t)$ and $\Theta(-,t_i,s) = \Phi_i(-)$, $i = 0,1$. Here t_i and s_i are the end points of I and J. An n-morphism will be a homotopy between $n-1$ morphisms with the obvious boundary conditions and similar constancy requirements.

We will later take colimit over the category $W(\Omega)$ and we show such colimit is filtered. Recall that a 1-category $\mathcal{C}$ is filtered if,

1. $\mathcal{C}$ is non-empty,

2. for any $X, Y \in \mathcal{C}$, there is $Z \in \mathcal{C}$ with morphisms $X \to Z$ and $Y \to Z$, and,

3. for any more morphism $f, g : X \to Y$, there exist $h : Y \to Z$ such that $h \circ f = h \circ g$.

This is the same as the condition that for any (ordered) n-simplex K, $n \in [-1, 1]$ and any functor $F : K \to \mathcal{C}$, there is an extension $\hat{F}$ on $K^{\triangleright}$ where $K^{\triangleright}$ is the $n+1$ simplex obtained by adding a final cone point to K. For example, we can realize a pair of morphisms $f, g : X \to Y$ as a hollowed triangle consisting of vertices X, X, Y and edges id_X, f, g without the presence of the face. A final cone point Z provides a morphism $h : Y \to Z$ for the edge between Y and Z. The existence of the three new faces and the fact that the only 2-morphism in a 1-category is the strict equality implies $h \circ f = h \circ g$.

Definition 4.3.4. A category $\mathcal{C}$ is filtered if for any simplex K and any functor $F : K \to \mathcal{C}$, there is an extension $\hat{F} : K^{\triangleright} \to \mathcal{C}$.

Example 4.3.5. Consider the case when $K = S^2$ is the 2-sphere, or more precisely, when $K = \Delta^2$ is the standard 2-simplex such that the base face has three vertices being a fixed object X, three edges being id_X, and the face being the trivial identification. This is essentially the situation that there are objects X, Y, a 1-morphism $f : X \to Y$, with a non-trivial 2-automorphism T on f. The condition of $\mathcal{C}$ being filtered means that there exist $g : Y \to Z$ such that the auto equivalence $g \circ T$ on $g \circ f$ is trivial, that is, $g \circ T = \mathrm{id}_{g \circ f}$.

The following proposition is a version of [20, Lemma 3.27].

Proposition 4.3.6. *The category $W(\Omega)$ is filtered.*

Proof. Similarly to the situation in classical algebraic topology, it is sufficient to check the case when $K = S^n$, the n-sphere for $n \in \mathbb{Z}_{\geq 0}$.

When $n = 0$, we are given two homotopy classes of contact isotopies Φ_0 and Φ_1 with the same end point φ, and the goal is to find another contact isotopy Φ and two positive contact isotopies Ψ_0 and Ψ_1 such that $[\Phi] = [\Psi_0 \# \Phi_0] = [\Psi_1 \# \Phi_1]$. We first notice that, up to a rescaling, $[\Phi_0^{-1} \# \Phi_0] = [\Phi_1^{-1} \# \Phi_1] = [\mathrm{id}_{S^*M}]$. So it is sufficient to modify Φ_0^{-1} and Φ_1^{-1} by composing some Φ' so that $\Phi' \circ \Phi_0^{-1}$ and $\Phi' \circ \Phi_1^{-1}$ are positive. Let H_0, H_1 denote their Hamiltonians. Since Φ_0^{-1} and Φ_1^{-1} are compactly supported, there exists a compact set $C \subseteq \Omega$ such that H_0 and H_1 are zero outside C. Pick a positive real number r such that $r > \max(|H_0|, |H_1|)$, relative compact open sets U, V in S^*M such that $C \subset U \subseteq \overline{U} \subseteq V \subseteq \Omega$, and a bump function ρ such that $\rho|_U \equiv 1$ and $\rho \equiv 0$ outside V. The contact isotopy Φ' generated by $r\rho$ will satisfy the requirement by the Leibniz rule.

When $n > 0$, we are given a family of morphism $\Psi_\theta : \Phi_0 \to \Phi_1$ parametrized by $\theta \in S^{n-1}$ such that $[\Phi_1] = [\Psi_\theta \# \Phi_0]$, and we have to show that, by possibly further concatenation, this family can be made to be null-homotopy through positive isotopies. By precomposing Φ_0^{-1}, we may assume there is an S^{n-1}-family of positive isotopies Ψ_θ and a fixed (not necessarily positive) isotopy Φ, such that, for each $\theta \in S^{n-1}$, there exists a homotopy $\Sigma_\theta : S^*M \times I \times [0, 1] \to S^*M$ connecting Φ to Ψ_θ. We can extend this map to a D^n-family of isotopy $\Sigma : S^*M \times I \times D^n \to S^*M$ by $\Sigma(x, \xi, t, r\theta) = \Sigma_\theta(x, \xi, t, r)$ where we write elements in D^n as $r\theta$ by $r \in [0, 1]$ and $\theta \in S^{n-1}$. Now the same compactness argument as before shows that there is a positive isotopy Φ' such that $\Phi' \circ \Sigma$ is positive. $\square$

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. For any diagram $p : \mathcal{D} \to \mathcal{E}$, the colimits $\operatorname*{colim}_{\mathcal{C}}(p \circ F)$ and $\operatorname*{colim}_{\mathcal{D}} F$ exist if either one exists. Thus, it is well-defined to write the canonical map $\operatorname*{colim}_{\mathcal{C}}(p \circ F) \to \operatorname*{colim}_{\mathcal{D}} F$.

Definition 4.3.7. A functor $F : \mathcal{C} \to \mathcal{D}$ is cofinal if, for any diagram $p : \mathcal{D} \to \mathcal{E}$, the canonical map $\operatorname*{colim}_{\mathcal{C}}(p \circ F) \to \operatorname*{colim}_{\mathcal{D}} F$ is an isomorphism.

In the 1-categorical setting, a more classical convention is that a functor is cofinal if and only if,

1. for any $d \in \mathcal{D}$, there exists $c \in \mathcal{C}$ and a morphism $d \to F(c)$,

2. for any morphism $f, g : d \to F(c)$, there exists $h : c \to c'$ such that $F(h) \circ f = F(h) \circ g$.

An equivalent way of saying it is that the fiber product $\mathcal{C} \times_{\mathcal{D}} d/\mathcal{D}$ is non-empty and connected for all $d \in \mathcal{D}$. Here, $d/\mathcal{D}$ is the over category whose objects are morphisms of the form $d \to d'$ and a morphism $h : (f : d \to d') \to (g : d \to d'')$ is given by a morphism $h : d' \to d''$ such that $h \circ f = g$, the fiber product is taken over the canonical projection $d/\mathcal{D} \to \mathcal{D}$ by $(d \to d') \mapsto d'$ and F. Recall a 1-category is said to be connected if the associated 1-groupoid (by formally inverting morphisms) is connected. The equivalence of these definitions is the Quillen's theorem A. In the ∞-categorical setting it states:

Theorem 4.3.8 (Quillen's Theorem A). *A functor $F : \mathcal{C} \to \mathcal{D}$ is cofinal if and only if the fiber product $\mathcal{C} \times_{\mathcal{D}} d/\mathcal{D}$ is contractible for any $d \in \mathcal{D}$.*

Now consider the following construction: For $n = 1, 2, \cdots$, take a family of open set $\Omega_n \subseteq S^*M$ such that $\Omega_n \subseteq \overline{\Omega_n} \subseteq \Omega_{n+1}$, $\cup_{n \in \mathbb{Z}_{\geq 0}} \Omega_n = S^*M$, and $\Omega_n \subseteq S^*M$ is relative compact. For $n > 0$, pick bump function ρ_n such that $\rho_n \leq \rho_{n+1}$, $\rho_n|_{\Omega_n} \equiv n$, and vanishes outside Ω_{n+1}. Let $\Phi_n : S^*M \times [0, n] \to S^*M$ be the isotopy generated by ρ_n. Since $\rho_1 \leq \rho_2 \leq \cdots \rho_n \leq \cdots$, there exists positive isotopy $\Psi_n : S^*M \times [n, n+1] \to S^*M$ such that the Φ_n's and Ψ_n's form a sequence $\mathrm{id} \xrightarrow{\Psi_0} \Phi_1 \xrightarrow{\Psi_1} \cdots$ in $W(\Omega)$. That is, the above data organizes to a functor $\Phi : \mathbb{Z}_{\geq 0} \to W(\Omega)$.

Lemma 4.3.9. *The functor $\Phi : \mathbb{Z}_{\geq 0} \to W(\Omega)$ is cofinal.*

Proof. By Quillen's Theorem A, we need to show that $\mathbb{Z}_{\geq 0} \times_{W(\Omega)} (\Phi/W(\Omega))$ is contractible. Let $\Phi \in W(\Omega)$ and let H denote its Hamiltonian. Since Φ is compactly supported there exist a compact set $C \subseteq S^*M$ such that H vanishes outside C. Pick n large such that $C \subseteq \Omega_n$ and $\max(H) \leq n$. Then the factorization $\Phi_n = (\Phi_n \circ \Phi^{-1}) \circ \Phi$ provides an morphism $\Phi \to \Phi_n$ since composition is homotopic to concatenation. Thus, the fiber product $\mathbb{Z}_{\geq 0} \times_{W(\Omega)} (\Phi/W(\Omega))$ is equivalent to $\{n \in \mathbb{Z}_{\geq 0} | \rho_n \geq H\}$. Since $\rho_{n+1} \geq \rho_n$ by our construction, the latter is equivalent to the poset of integers larger than $\min\{n : \rho_n \geq H\}$ and is contractible. $\square$

Now we quantize the above construction. For $(\varphi, [\Phi]) \in W(\Omega)$ where Φ is defined on $M \times [t_0, t_1]$, set $w(\Phi) := K(\Phi)|_{t=t_1}$ to be the restriction of the GKS sheaf quantization $K(\Phi_i)$ at the end point. We note that since we require the end point φ to be fixed, the sheaf

$w(\Phi) \in \mathrm{Sh}(M \times M)$ depends only on the homotopy class $[\Phi]$ by formula 4.1 and Lemma 4.2.12. For a morphism $\Psi : \Phi_0 \to \Phi_1$, since Ψ is positive, formula 4.1 again implies there is a continuation map $c(\Psi) : w(\Phi_0) \to w(\Phi_1)$. Similarly, for a pair of morphisms $\Psi_0, \Psi_1 : \Phi_0 \to \Phi_1$, if there is a homotopy Θ between Ψ_0 and Ψ_1, Proposition 4.1.9 implies that $K(\Theta)$ provides an identification between the continuation maps $c(\Psi_0), c(\Psi_1) : w(\Phi_0) \to w(\Phi_1)$.

Definition 4.3.10. Organizing the above construction, we obtain a functor $w : W(\Omega) \to \mathrm{Sh}(M \times M)$ sending an object Φ to the corresponding sheaf kernel $w(\Phi)$, a 1-morphism $\Psi : \Phi_0 \to \Phi_1$ to the continuation map $c(\Psi) : w(\Phi_1) \to w(\Phi_0)$, and higher morphisms to higher equivalences of continuation maps. We will refer this functor as the *wrapping kernel functor.*

For a sheaf $F \in \mathrm{Sh}(M)$ and a contact isotopy $\Phi : S^*M \times [0,1] \to S^*M$, convoluting with $K(\Phi)$ produces a sheaf $K(\Phi) \circ F$ on $\mathrm{Sh}(M \times [0,1])$ and we use the notation $F_t = (K(\Phi) \circ F)|_t$ for $t \in [0,1]$ as before. Base change 3.1.7 and compatibility of six-functor formalism 3.1.8 implies that there is an identification

$$(K(\Phi) \circ F)|_{t=1} = K(\Phi)|_{t=1} \circ F = w(\Phi) \circ F$$

functorial on Φ and F. When $\Psi : \Phi_0 \to \Phi_1$ is a positive family of isotopies, we use $c(\Psi, F) : w(\Phi_0) \circ F \to w(\Phi_1) \circ F$ to denote the induced continuation map. To simplify the notation, we sometimes use $F^{w(\Phi)}$ to denote $w(\Phi) \circ F$. When there is no need to specify the isotopy, we simply write it as F^w. Similarly, when Ψ is unspecified, we simply write $c : F^w \to F^{w'}$ for the continuation map. We prove a locality property which we will use later.

Proposition 4.3.11. *Let Φ_0, $\Phi_1 : S^*M \times I \to S^*M$ be contact isotopies. If $\Phi_0 = \Phi_1$ for on $\mathrm{SS}^\infty(F) \times I$, then $w(\Phi_0) \circ F = w(\Phi_1) \circ F$. Similarly, let Ψ_0, Ψ_1 be positive contact isotopies. If $\Psi_0 = \Psi_1$ on an open neighborhood Ω_0 of $\mathrm{SS}^\infty(F)$, then $c(\Psi_0, F) = c(\Psi_1, F)$.*

Proof. We abuse the notation and use Φ_i to denote the corresponding homogeneous symplectic isotopies. By convoluting with Φ_1^{-1}, it is enough to assume $\Phi_1 = \mathrm{id}_{S^*M}$ and show that $F^{w(\Phi_0)} = F$. We have $\mathrm{SS}(K(\Phi_0) \circ F) \subseteq \mathrm{SS}(K(\Phi_0)) \circ \mathrm{SS}(F) \subseteq \mathrm{SS}(F) \times 0_I$. Hence by Proposition 4.2.12, F_t is constant along constant on t.

Similarly, let $H_i \geq 0$ denote the Hamiltonian of Ψ_i, $i = 0, 1$. Let Ψ_s be the homotopy of isotopies between Ψ_0 and Ψ_1 generated by the Hamiltonian $\widehat{H}(x, \xi, t, s) = (1-s)H_0(x, \xi, t) + sH_1(x, \xi, t)$. Since $\Psi_0 = \Psi_1$ on Ω_0, $\Psi_s|_{\Omega_0}$ is constant on s. Thus Proposition 4.1.9 applies to $K(\Psi_s) \circ F$ and we conclude $c(\Psi_0, F) = c(\Psi_1, F)$. $\square$

The above construction defines a functor from $\mathrm{Sh}(M)$ to $[W(\Omega), \mathrm{Sh}(M)]$ since the expression $w(\Phi) \circ F$ is functorial on F. Further composing with the functor of taking limits and colimits defines functors $\mathfrak{W}^\pm(\Omega) : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$. Since the subcategory of $W(\Omega)$ consists of objects $(\varphi, [\Phi])$ such that there exists a positive isotopy Φ representing $[\Phi]$ is cofinal, we can informally write the formula by

$$\mathfrak{W}^+(\Omega)F = \operatorname*{colim}_{F \to F^w} F^w, \quad \mathfrak{W}^-(\Omega)G = \lim_{G^{w^-} \to G} G^{w^-}.$$

With this definition, we can generalize Proposition 4.1.2 to Theorem 1.0.4.

Proof of Theorem 1.0.4. Set $\Omega = S^*M \setminus X$ and let $F \in \mathrm{Sh}(M)$. We first show that for any $(x,\xi) \in \Omega$, $(x,\xi) \notin \mathrm{SS}^\infty(\mathfrak{W}^\pm(F))$, i.e., for any function f defined near x such that $f(x) = 0$ and $df_x \in \mathbb{R}_{>0}\xi$, the restriction map $(\mathfrak{W}^\pm(F))_x \to \Gamma_{\{f<0\}}(\mathfrak{W}^\pm(F))_x$ is an isomorphism. Since the situation is local and $df_x \neq 0$, by changing a coordinate, we may assume $f = x_1$ the first coordinate function near $x = 0$. Pick a family of open balls U_i centered at x such that $U_i \supseteq \overline{U_{i+1}} \supseteq U_{i+1}$ and $\cap_i U_i = \{x\}$. The stalk $\Gamma_{\{f<0\}}(\mathfrak{W}^\pm(F))_x$ can be computed by the colimit $\mathrm{colim}_i \Gamma(U_i \cap \{x_1 < 0\}; \mathfrak{W}^\pm(F))$.

We first prove the negative case. For each i, we take a small positive wrapping Φ_i supported in Ω such that $w(\Phi_i) \circ 1_{U_i \cap \{x_1 < 0\}} = 1_{\tilde{U}_i}$ with $0 \in \tilde{U}_i$ and $\tilde{U}_i$ shrinks to x as $i \to \infty$. For example, take $U_i \times C_i$ in Ω containing (x, ξ) where $\{C_i\}$ is a family of small balls on the fiber direction with a condition similar to the $\{U_i\}$. For each i, pick a bump function ρ_i on S^*M supported on $U_i \times C_i$ and equals 1 near (x, ξ). Take H_i to be the Hamiltonian associated to the Reeb flow with shrinking speed and modify it to $\rho_i H_i$. Finally, take Φ_i to be the isotopy associated to $\rho_i H_i$.

We compute,

$$
\begin{aligned}
\Gamma\left(U_i \cap \{x_1 < 0\}; \mathfrak{W}^-(F)\right) &= \lim_{W(\Omega)} \mathrm{Hom}\left(1_{U_i \cap \{x_1 < 0\}}, w(\Phi) \circ F\right) \\
&= \lim_{W(\Omega)} \mathrm{Hom}\left(w(\Phi_i) \circ 1_{U_i \cap \{x_1 < 0\}}, w(\Phi_i) \circ w(\Phi) \circ F\right) \\
&= \lim_{W(\Omega)} \mathrm{Hom}\left(1_{\tilde{U}_i}, w(\Phi_i \circ |_I \Phi) \circ F\right) \\
&= \Gamma\left(\tilde{U}_i; \mathfrak{W}^-(F)\right).
\end{aligned}
$$

Here we use the fact that $w(\Phi_i)\circ$ is an equivalence for the second equation. For the last equation, we use the fact that negative wrappings of the form $\Phi_i \circ \Phi$ is initial in $W(\Omega)$. Take $i \to \infty$ and we conclude $(\mathfrak{W}^-(F))_x \xrightarrow{\sim} \Gamma_{\{f<0\}}(\mathfrak{W}^-(F))_x$ and $\Gamma_{\{f\geq 0\}}(\mathfrak{W}^-(F))_x = 0$.

Now we turn to the positive case. We take the same family of U_i, Φ_i and $\tilde{U}_i$, and compute,

$$
\begin{aligned}
\Gamma\left(U_i \cap \{x_1 < 0\}; \mathfrak{W}^+(F)\right) &= \mathrm{Hom}\left(1_{U_i \cap \{x_1 < 0\}}, \mathrm{colim}_{W(\Omega)}(w(\Phi) \circ F)\right) \\
&= \mathrm{Hom}\left(w(\Phi_i) \circ 1_{U_i \cap \{x_1 < 0\}}, w(\Phi_i) \circ \left(\mathrm{colim}_{W(\Omega)} w(\Phi) \circ F\right)\right) \\
&= \mathrm{Hom}\left(1_{\tilde{U}_i \cap \{x_1 < 0\}}, \mathrm{colim}_{W(\Omega)}(w(\Phi_i \circ |_I \Phi) \circ F)\right) \\
&= \Gamma\left(\tilde{U}_i; \mathfrak{W}^+(F)\right).
\end{aligned}
$$

We use the fact that $w(\Phi_i)\circ$ is a left adjoint so it commutes with colimits for the third equation. Take $i \to \infty$ and we get $(\mathfrak{W}^+(F))_x \xrightarrow{\sim} \Gamma_{\{f<0\}}(\mathfrak{W}^+(F))_x$ and $\Gamma_{\{f\geq 0\}}(\mathfrak{W}^+(F))_x = 0$.

From the above computation, we see that $\mathfrak{W}^\pm(\Omega) : \mathrm{Sh}(M) \to \mathrm{Sh}(M)$ factorizes to $\mathrm{Sh}_X(M)$. Finally, we show that $\mathfrak{W}^+(\Omega) \dashv \iota_* \dashv \mathfrak{W}^-(\Omega)$. Take $G \in \mathrm{Sh}_X(M)$ and $F \in \mathrm{Sh}(M)$.

We compute,

$$\mathrm{Hom}\left(G, \mathfrak{W}^-(F)\right) = \mathrm{Hom}\left(G, \lim_{W(\Omega)} w(\Phi) \circ F\right)$$

$$= \lim_{W(\Omega)} \mathrm{Hom}\left(G, w(\Phi) \circ F\right)$$

$$= \lim_{W(\Omega)} \mathrm{Hom}\left(w(\Phi^{-1}) \circ G, F\right)$$

$$= \lim_{W(\Omega)} \mathrm{Hom}\left(G, F\right) = \mathrm{Hom}\left(\iota_* G, F\right).$$

The second to last equality is implied by Proposition 4.3.11 since Φ is compactly supported away from $\Lambda \supseteq \mathrm{SS}^\infty(G)$. A similar computation shows that

$$\mathrm{Hom}\left(\mathfrak{W}^+(F), G\right) = \mathrm{Hom}\left(F, \iota_* G\right).$$

$\square$

Remark 4.3.12. We mention that, aside from the prototype cases [53, 26, 27] mentioned in the introduction, special cases for such geometric descriptions can be found in, for example, [33] in the setting of toric homological mirror symmetry which are defined by using the group structure of the torus and are crucial for matching the data with the coherent side.

Notation 4.3.13. Recall that we use $\iota_X^* : \mathrm{Sh}(M) \to \mathrm{Sh}_X(M)$ to denote the left adjoint of the inclusion $\iota_{X*} : \mathrm{Sh}_X(M) \hookrightarrow \mathrm{Sh}(M)$ for a conic closed subset $X \subseteq T^*M$. By the above Theorem 1.0.4, when $\Lambda \subseteq S^*M$ is a singular isotropic, we will use the notation $\mathfrak{W}_\Lambda^+(M) = \iota_\Lambda^* : \mathrm{Sh}(M) \to \mathrm{Sh}_\Lambda(M)$ when emphasizing that ι_Λ^* is given by wrappings. When there is no ambiguity for the ambient manifold M, we simply write it as $\mathfrak{W}_\Lambda^+$. We will use a similar notation for the right adjoints.

4.4 Duality in dimension one

Recall in Section 3.6, we show that $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$ and use the uniqueness of counits to conclude Theorem 1.0.3. We show in this section that the uniqueness of units provides a formula for the equivalence on compact objects $\mathrm{Sh}_\Lambda(M)^{c,op} = \mathrm{Sh}_{-\Lambda}(M)^c$ and we use Theorem 1.0.4 to compute some examples in dimension one. Thus, we consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\iota_{\Lambda \times -\Lambda}^* \Delta_* p^*} & \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M) \\
\| & & \| \\
& & \mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_{-\Lambda}(M) \\
& & \Big\| \mathrm{id} \otimes \mathrm{Ind}(\mathrm{D}_\Lambda) \\
\mathcal{V} & \xrightarrow[\mathrm{Id}_{\mathrm{Sh}_\Lambda(M)}]{} & \mathrm{Sh}_\Lambda(M) \otimes \mathrm{Sh}_\Lambda(M)^\vee
\end{array}
$$

Here we recall that $\mathrm{Id}_{\mathrm{Sh}_\Lambda(M)}$ is the diagonal bimodule induced by the Hom pairing $(G, F) \mapsto \mathrm{Hom}(G, F)$ for F, $G \in \mathrm{Sh}_\Lambda(M)^c$. Thus, the object $\iota^*_{\Lambda \times -\Lambda} 1_\Delta$ represents it and, for $F \in \mathrm{Sh}_\Lambda(M)$ and $G \in \mathrm{Sh}_{-\Lambda}(M)^c$, we have the identification,

$$\mathrm{Hom}(F \boxtimes G, \iota^*_{\Lambda \times -\Lambda} 1_\Delta) = \mathrm{Hom}(F, \mathrm{D}_\Lambda G).$$

Proposition 4.4.1. *The equivalence* $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$ *provided by Proposition 5.1 induces an equivalence on compact objects*

$$\mathrm{Sh}_{-\Lambda}(M)^c = \mathrm{Sh}_\Lambda(M)^{c,op}$$
$$G \mapsto \mathrm{D}_\Lambda(G) = p_{1*} \, \mathcal{H}om(p_2^* G, \iota^*_{\Lambda \times -\Lambda} 1_\Delta).$$

By Theorem 1.0.4, $\iota^*_{\Lambda \times -\Lambda} 1_\Delta = \mathfrak{W}^+_{\Lambda \times -\Lambda} 1_\Delta$ and we compute some examples in dimension one using this fact.

Example 4.4.2. Let $M = \mathbb{R}^1$ and $\Lambda = 0_{\mathbb{R}^1}$, i.e., the case of local systems $\mathrm{Loc}(\mathbb{R}^1) = \mathcal{V}$ is generated by $1_{\mathbb{R}^1}$, the constant sheaf on $\mathbb{R}^1$. Since there is no microsupport condition, $\mathfrak{W}^+_{\Lambda \times -\Lambda} 1_\Delta = 1_{\mathbb{R}^2}[1]$ and $\mathrm{D}_\Lambda(1_{\mathbb{R}^1}) = 1_{\mathbb{R}^1}[1]$.

Example 4.4.3. Let $M = \mathbb{R}^1$ and $\Lambda = 0_{\mathbb{R}^1} \cup T^*_{0,\leq} \mathbb{R}^1$ where we use $T^*_{0,\leq} \mathbb{R}^1$ to denote the non-positive cotangent fiber at 0. In this case, the wrapped sheaves $\mathrm{Sh}_{-\Lambda}(M)^c$ is generated by $\{1_{\mathbb{R}^1}, 1_{(-\infty,0)}\}$ and the product microsupport condition is given by

$$\Lambda \times -\Lambda = \{(x, y, 0, 0)\} \cup \{(0, y, -\xi, 0)\} \cup \{(x, 0, 0, \eta)\} \cup \{(0, 0, -\xi, \eta)\}$$

where x, y run through $\mathbb{R}$ and ξ, η run through $[0, \infty)$.
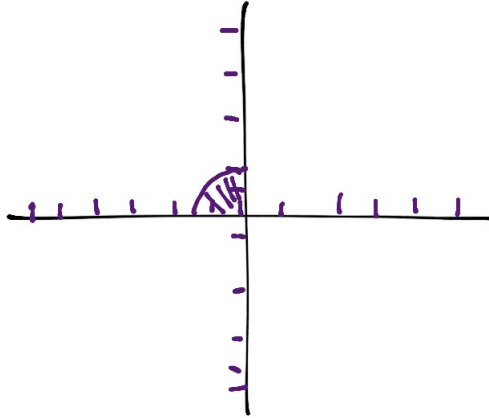

Figure 4.1: The product microsupport condition (in purple).

To compute $\mathfrak{W}^+_{\Lambda \times -\Lambda} 1_\Delta$, we note that since there is no stop toward the $(1, -1)$-direction and the diagonal is allowed to expand to an open region drawn below. In addition, when
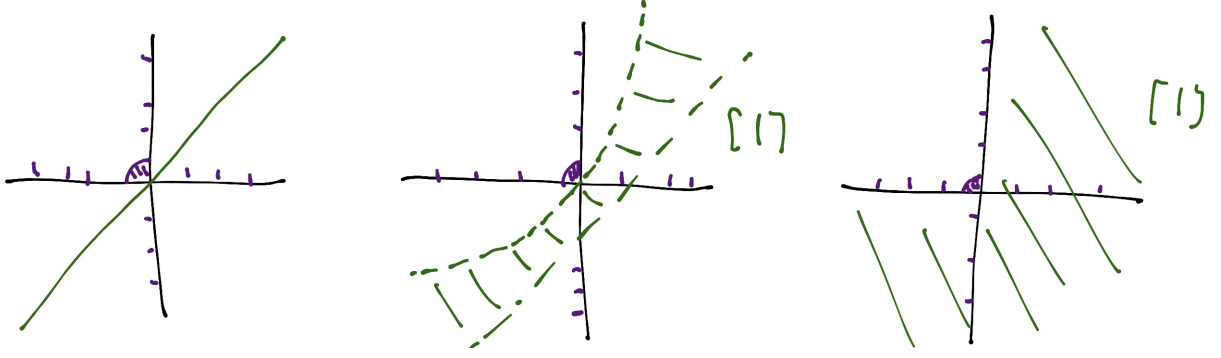
Figure 4.2: We use purple to indicate the microsupport condition and green for the support of the "expanded diagonal".

away from the origin, the upper boundary is allowed to bent toward the second quadrant. Take the colimit and we see that $\mathfrak{W}_{\Lambda \times -\Lambda}^+ 1_\Delta = 1_U[1]$ where $U = \{x > 0\} \cup \{y < 0\}$.

Recall that, for a locally close subset $i : Z \hookrightarrow X$, the functor Γ_Z is defined to be $i_* i^!$ and this implies the identification $\Gamma_Z \circ \Gamma_{Z'} = \Gamma_{Z \cap Z'}$. We also use (3) of Proposition 3.1.8 to get the identification $\mathcal{H}om(1_Z, F) = \Gamma_Z(F)$. We can thus compute that $D_\Lambda(1_{\mathbb{R}^1}) = p_{1*} 1_U[1] = 1_{(0,\infty)}[1]$, and

$$
\begin{aligned}
D_\Lambda(1_{(-\infty,0)}) &= p_{1*} \mathcal{H}om(1_{\mathbb{R}^1 \times (-\infty,0)}, 1_U[1]) = p_{1*} \Gamma_{\mathbb{R}^1 \times (-\infty,0)} \circ \Gamma_{\overline{U}}(1_{\mathbb{R}^2})[1] \\
&= p_{1*} \Gamma_{\left(\mathbb{R}^1 \times (-\infty,0) \cap \overline{U}\right)}(1_{\mathbb{R}^2})[1] = p_{1*} \Gamma_{\mathbb{R}^1 \times (-\infty,0)}(1_{\mathbb{R}^2})[1] \\
&= p_{1*} 1_{\mathbb{R}^1 \times (-\infty,0]}[1] = 1_{\mathbb{R}^1}[1].
\end{aligned}
$$

In other words, up to a shift and a sign, D_Λ swap $1_{\mathbb{R}^1}$ and $1_{(-\infty,0)}$ in this case.

Example 4.4.4. Let $M = \mathbb{R}^1$ and $\Lambda = 0_{\mathbb{R}^2} \cup T_0^* \mathbb{R}^1$. In this case, the wrapped sheaves $\mathrm{Sh}_{-\Lambda}(M)^c$ is generated by $\{1_{\mathbb{R}^1}, 1_{(-\infty,0)}, 1_{(0,\infty)}\}$. Equivalently, $\Lambda = N^* \mathcal{S}$ where $\mathcal{S}$ is the stratification $\mathcal{S} = \{\{*\}, (-\infty, 0), (0, \infty)\}$ of $\mathbb{R}^1$ and $\mathrm{Sh}_{-\Lambda}(M) = \mathrm{Sh}_\Lambda(M) = \mathrm{Sh}_\mathcal{S}(M)$. The product microsupport condition $\Lambda \times -\Lambda$ is given by $N^*(\mathcal{S} \times \mathcal{S})$.

To compute $\mathfrak{W}_{\Lambda \times -\Lambda}^+ 1_\Delta$, we note that 1_Δ can be computed as the cofiber

$$
1_\Delta = \mathrm{cof}(1_{\{x-y<0\}} \oplus 1_{\{x-y>0\}} \to 1_{\mathbb{R}^2})
$$

or alternatively

$$
1_\Delta = \mathrm{cof}(1_{\mathbb{R}^2} \to 1_{\{x-y\leq 0\}} \oplus 1_{\{x-y\geq 0\}}).
$$

Take the second expression, then we see that

$$
\mathfrak{W}_{\Lambda \times -\Lambda}^+ 1_\Delta = \mathrm{cof}(1_{\mathbb{R}^2} \to 1_{\{x\leq 0, y\geq 0\}} \oplus 1_{\{x\geq 0, y\leq 0\}}).
$$

Since $1_{\{x\leq 0, y\geq 0\}} = \Gamma_{\{x<0, y>0\}}(1_{\mathbb{R}^2})$, and there is a fiber sequence

$$
\Gamma_{\{x,y\geq 0\} \cup \{x,y\leq 0\}}(F) \to F \to \Gamma_{\{x<0, y>0\} \cup \{x>0, y<0\}}(F)
$$

for any $F \in \mathrm{Sh}(\mathbb{R}^2)$, there is an identification $\mathfrak{W}_{\Lambda \times -\Lambda}^+ 1_\Delta = i_* i^! 1_{\mathbb{R}^2}[1]$ where we use i to denote the closed inclusion $i : \{x, y \geq 0\} \cup \{x, y \leq 0\} \hookrightarrow \mathbb{R}^2$.
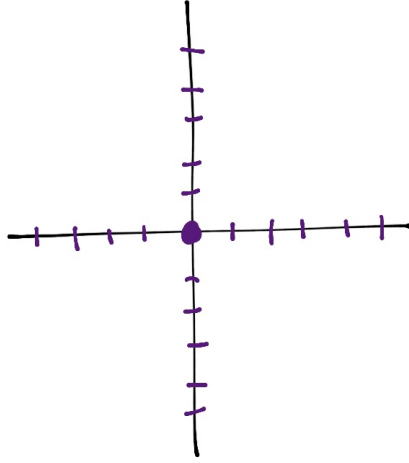
55

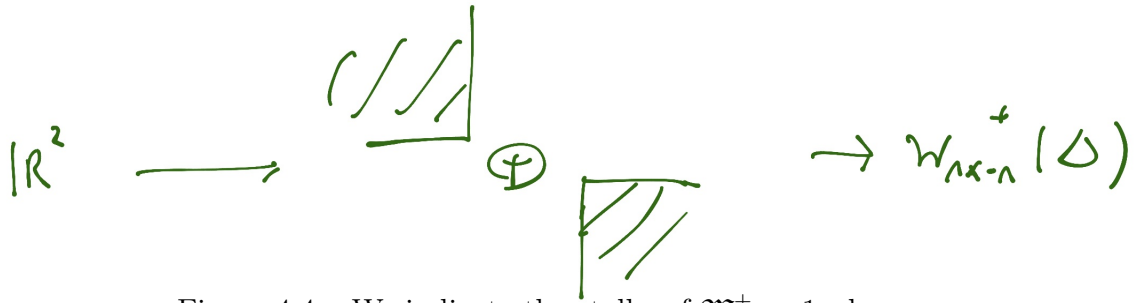Figure 4.3: The product microsupport condition (in purple).

Figure 4.4: We indicate the stalks of $\mathfrak{W}^+_{\Lambda \times - \Lambda} 1_\Delta$ by green.

We compute that

$$D_\Lambda(1_{\mathbb{R}^1}) = p_{1*} \operatorname{cof}(1_{\mathbb{R}^2} \to 1_{\{x \leq 0, y \geq 0\}} \oplus 1_{\{x \geq 0, y \leq 0\}})$$
$$= \operatorname{cof}(1_{\mathbb{R}^1} \to 1_{(-\infty, 0]} \oplus 1_{[0, \infty)}) = 1_{\{0\}}.$$

Similarly, let $j : \mathbb{R}^1 \times (-\infty, 0) \hookrightarrow \mathbb{R}^2$ denote the open inclusion and consider the fiber product

$$
\begin{array}{ccc}
\{x \leq 0, y < 0\} & \xrightarrow{\ i'\ } & \mathbb{R}^1 \times (-\infty, 0) \\
{\scriptstyle j'} \downarrow & & \downarrow {\scriptstyle j} \\
\{x, y \geq 0\} \cup \{x, y \leq 0\} & \xrightarrow[\ i\ ]{} & \mathbb{R}^2
\end{array}
$$

and compute

$$\begin{aligned}
\mathrm{D}_\Lambda(1_{(-\infty,0)}) &= p_{1*}\,\mathcal{H}om(1_{\mathbb{R}^1\times(-\infty,0)}, i_*i^!1_{\mathbb{R}^2}[1])\\
&= p_{1*}j_*\,\mathcal{H}om(1_{\mathbb{R}^1\times(-\infty,0)}, j^!i_*i^!1_{\mathbb{R}^2}[1])\\
&= (p_1\circ j)_*\,\mathcal{H}om(1_{\mathbb{R}^1\times(-\infty,0)}, , i'_*i'^!1_{\mathbb{R}^1\times(-\infty,0)}[1])\\
&= (p_1\circ j)_*1_{\{x<0,y<0\}}[1] = 1_{(-\infty,0)}[1].
\end{aligned}$$

In sum, the functor $\mathrm{D}_\mathbb{S}$ shifts $1_{(-\infty,0)}$ and $1_{(0,\infty)}$ by $[1]$ and interchange $1_{\mathbb{R}^1}$ and $1_{\{0\}}$.

Now we turn to the S^1 case.

Example 4.4.5. We compute the coevaluation sheaf $\mathfrak{W}^+_{0_{T^2}}1_\Delta$. First recall a similar but simpler computation $\mathfrak{W}^+_{0_{S^1}}1_{\{0\}}$. It can be computed as the colimit $\underset{n}{\mathrm{colim}}(\pi_!1_{[-n,n]})[1] = \pi_!1_{\mathbb{R}^1}$ where $\pi:\mathbb{R}^1\to S^1$ is the projection.
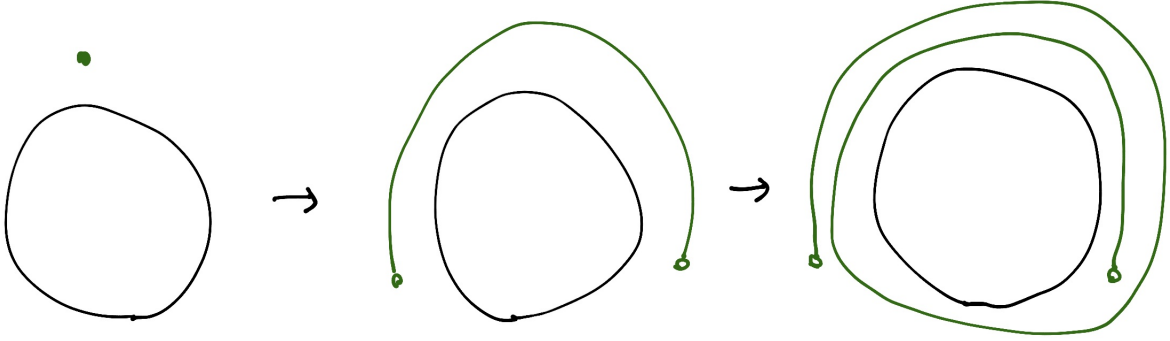

Figure 4.5: The first few sheaves of the colimit $\underset{n}{\mathrm{colim}}(\pi_!1_{[-n,n]})$ where the green indicates the support in the universal cover $\mathbb{R}^1$.

As one can see form the picture, the local system $\pi_!1_{\mathbb{R}^1}[1]$ has, up to the shift $[1]$, $1_\mathcal{V}^{\oplus\mathbb{Z}}$ as its stalks and the monodromy $m:1_\mathcal{V}^{\oplus\mathbb{Z}}\to 1_\mathcal{V}^{\oplus\mathbb{Z}}$ is given by shifting to the right by 1. The sheaf $\mathfrak{W}^+_{0_{T^2}}1_\Delta$ can be computed similarly and the resulting local system again has, up to the shift $[1]$, $1_\mathcal{V}^{\oplus\mathbb{Z}}$ as its stalks. The monodromy induced by the two standard generators of $\pi_1(T^2) = \pi_1(S^1\times S^1)$ is again m for each direction.
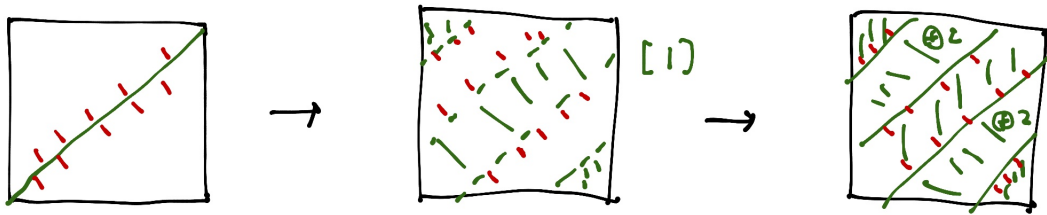

Figure 4.6: The first few sheaves of the colimit $\mathfrak{W}^+_{S^1\times S^1}1_\Delta$. We use green again to indicate the support and use red to indicate the microsupport at infinity of the sheaf at each step.

To compute $D_\Lambda(\pi_! 1_{\mathbb{R}^1})$, we first consider the fiber product

$$
\begin{array}{ccc}
S^1 \times \mathbb{R}^1 & \xrightarrow{\tilde{p}_2} & \mathbb{R}^1 \\
\downarrow{\scriptstyle (\mathrm{id},\pi)} & & \downarrow{\scriptstyle \pi} \\
S^1 \times S^1 & \xrightarrow{p_2} & S^1
\end{array}.
$$

We thus compute that

$$
\begin{aligned}
D_\Lambda(\pi_! 1_{\mathbb{R}^1}) &= p_{1*} \mathcal{H}om(p_2^* \pi_! 1_{\mathbb{R}^1}, \mathfrak{W}^+_{0_{T^2}} 1_\Delta) \\
&= p_{1*} \mathcal{H}om((\mathrm{id}\times\pi)_! 1_{S^1\times\mathbb{R}^1}, \mathfrak{W}^+_{0_{T^2}} 1_\Delta) \\
&= p_{1*}(\mathrm{id}\times\pi)_*(\mathrm{id}\times\pi)^* \mathfrak{W}^+_{0_{T^2}} 1_\Delta \\
&= (S^1 \times \mathbb{R}^1 \to S^1)_*(\mathrm{id}\times\pi)^* \mathfrak{W}^+_{0_{T^2}} 1_\Delta = \pi_! 1_{\mathbb{R}^1}[1].
\end{aligned}
$$

Here we note that the functor $(\mathrm{id}\times\pi)^*$ has the effect of forgetting the monodromy on the vertical direction and the projection $S^1 \times \mathbb{R}^1 \to S^1$ being an homotopic equivalence induces an equivalence between local systems.

Example 4.4.6. Let $M = S^1 = \mathbb{R}^1/\mathbb{Z}^1$ and $\Lambda = 0_{S^1} \cup T^*_{0,\le} S^1$ where we use $T^*_{0,\le} S^1$ to denote the non-positive cotangent fiber at 0. Then the product support condition $\Lambda \times -\Lambda$ is the projection of the $\mathbb{R}^1$ case we considered above. The coevaluation sheaf $\mathfrak{W}^+_{\Lambda\times-\Lambda} 1_\Delta$ can be computed similarly as the local system case except the $(-1,1)$-direction is stopped.

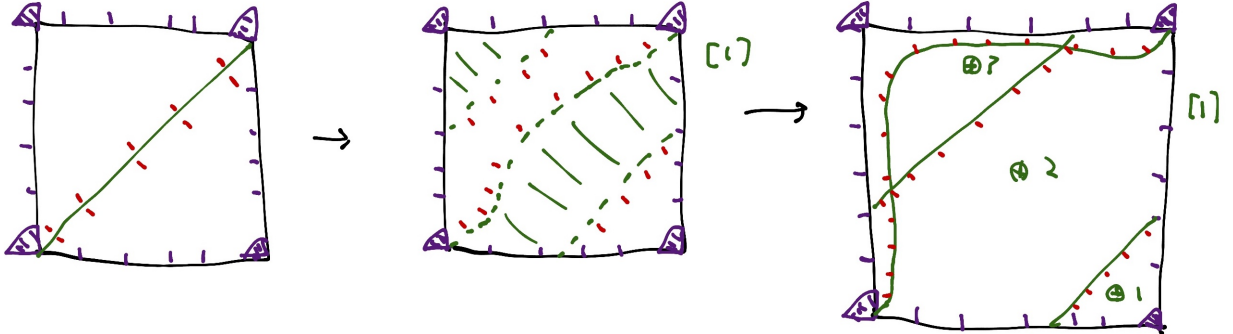Figure 4.7: The wrapping process for $\mathfrak{W}^+_{\Lambda\times-\Lambda} 1_\Delta$. Again, we use purple for the microsupport condition, green for the support, and red for the microsupport at the infinity.

Thus, the resulting local system has asymmetric slices, or more precisely, the horizontal slice is given by $\pi_! 1_{(-\infty,0)}[1]$ and the vertical slice is given by $\pi_! 1_{(0,\infty)}[1]$.

$$\pi_! 1_{(-\infty,0)}[1] \qquad\qquad \pi_! 1_{(0,\infty)}[1]$$
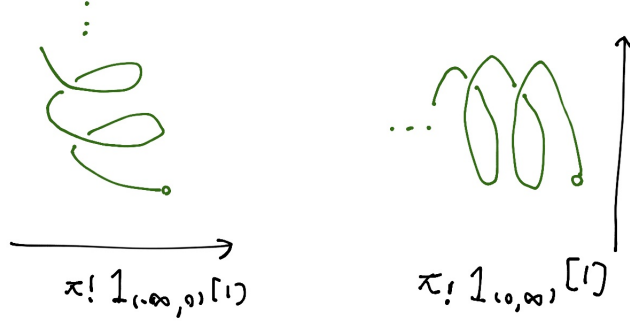
Figure 4.8: Horizontal and vertical direction of $\mathfrak{W}^+_{-\Lambda\times\Lambda}1_\Delta$.

Unlike the $\mathbb{R}^1$ case, the wrapped sheaves $\mathrm{Sh}_\Lambda(S^1)^c$ is generated by one object $\pi_!1_{(0,\infty)}$. Thus, we compute

$$
\begin{aligned}
\mathrm{D}_\Lambda(\pi_!1_{(0,\infty)}) &= p_{1*}\,\mathcal{H}om(p_2^*\pi_!1_{(0,\infty)}, \mathfrak{W}^+_{0_{T^2}}1_\Delta) \\
&= p_{1*}\,\mathcal{H}om((\mathrm{id}\times\pi)_!1_{S^1\times(0,\infty)}, \mathfrak{W}^+_{0_{T^2}}1_\Delta) \\
&= (S^1\times\mathbb{R}^1\to S^1)_*\,\mathcal{H}om(1_{S^1\times(0,\infty)}, (\mathrm{id}\times\pi)^*\mathfrak{W}^+_{0_{T^2}}1_\Delta) \\
&= \pi_!1_{(-\infty,0)}[1].
\end{aligned}
$$

As in the local system case, we pick out the horizontal slice.

Chapter 5

Verdier duality

Let $\mathrm{Sh}(M)^b \coloneqq \{F \in \mathrm{Sh}(M) | F_x \in \mathcal{V}_0, x \in M\}$ be the category of sheaves with perfect stalks, and set $\mathrm{Sh}_\Lambda(M)^b \coloneqq \mathrm{Sh}_\Lambda(M) \cap \mathrm{Sh}(M)^b$. Recall that the Verdier dual

$$D_M : \mathrm{Sh}_\Lambda(M)^{b,op} \to \mathrm{Sh}_{-\Lambda}(M)^b$$
$$F \mapsto D_M(F) \coloneqq \mathcal{H}om(F, \omega_M)$$

is an equivalence since the double dual $F \to D_M(D_M(F))$ is an isomorphism by [32, Proposition 3.4.3.].

Assume M is compact for this chapter. Then by Corollary 3.4.7, $\mathrm{Sh}_\Lambda(M)^b \subseteq \mathrm{Sh}_\Lambda(M)^c$ by Lemma 3.4.4. The goal of this chapter is to study conditions to extend the Verdier dual $D_M : \mathrm{Sh}_\Lambda(M)^{b,op} = \mathrm{Sh}_{-\Lambda}(M)^b$ to the whole $\mathrm{Sh}_\Lambda(M)^c$. We will consider a pair (ϵ, η) so that ϵ is given by

$$\epsilon = p_* \Delta^! : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \to \mathcal{V}.$$

The sheaf kernel $\eta \in \mathrm{Sh}_{\Lambda \times -\Lambda}(M \times M)$, if exists, will be given as the convolution right inverse of $\iota^*_{\Lambda \times -\Lambda}(w(\Phi)) \otimes p_1^* \omega_M^{-1}$ where Φ is a small Reeb pushoff of Λ displacing it from itself. Here we following the notation from the previous chapter and denote by $w(\Phi) \in \mathrm{Sh}(M \times M)$ the restriction $K(\Phi)|_{t_0}$ of the GKS sheaf quantization at some unspecified small t_0. We will show that assume the existence of η, this pair provides a duality $\mathrm{Sh}_\Lambda(M)^\vee = \mathrm{Sh}_{-\Lambda}(M)$, which restricts to the Verdier duality D_M on $\mathrm{Sh}_\Lambda(M)^b$. We also prove that for the case when $\Lambda = N^*\mathcal{S}$ where $\mathcal{S}$ is a Whitney triangulation of M, the sheaf kernel $\iota^*_{\Lambda \times -\Lambda}(w(\Phi))$ does admit a convolution inverse and so such η exists. Finally, we show that $\iota^*_{\Lambda \times -\Lambda}(w(\Phi))$, regarded as an endofunctor by convolution, has a purely sheaf-theoretic definition without referencing to symplectic geometry.

5.1 Verdier duality as a categorical dual

We begin by recalling a classical identification.

Proposition 5.1.1 ([32, Proposition 3.4.4]). *Let F, $G \in \mathrm{Sh}(M)$. If $G \in \mathrm{Sh}(M)^b$ is constructible, then*

$$D_M G \boxtimes F = \mathcal{H}om(p_1^* G, p_2^! F).$$

This implies that if the Verdier duality were to come from a duality on $\mathrm{Sh}_\Lambda(M)^c$ given by a pair of unit/counit (η, ϵ), then the unit ϵ must by of the form

$$\epsilon : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \to \mathcal{V}$$
$$H \mapsto p_* \Delta^! H.$$

For if $F \in \mathrm{Sh}_\Lambda(M)$, $G \in \mathrm{Sh}_{-\Lambda}(M)^b$, we have

$$p_* \Delta^! (G \boxtimes F) = p_* \Delta^! (D_M(D_M(G)) \boxtimes F)$$
$$= p_* \Delta^! \mathcal{H}om(p_1^*(D_M(G)), p_2^! F) = p_* \mathcal{H}om((D_M(G)), F) = \mathrm{Hom}(D_M(G), F).$$

Now take a small Reeb flow $\Phi : S^* M \times I \to S^* M$ so that $\varphi_t(\Lambda) \cap \Lambda = \varnothing$ for $I \setminus \{0\}$ by Lemma 6.2.8. Equivalently, Φ is the homogeneous Hamiltonian isotopy associated to $H(x, \xi) := \sqrt{g_x^*(\xi, \xi)}$ on $\dot{T}^* M$ for some Riemannian metric g on M. Write $w(\Phi) = K(\Phi)|_t$ to be some the restriction of the GKS sheaf quantization at some unspecified positive time t_0. The main theorem of this section is the following:

Theorem 5.1.2. *Assume $\iota_{\Lambda \times -\Lambda}^*(w(\Phi))$ admits a left inverse. Then the proposed (ϵ, η) in the beginning of the chapter exhibits $\mathrm{Sh}_{-\Lambda}(M)^\vee = \mathrm{Sh}_\Lambda(M)$ in Pr_{st}^L. Furthermore, the induced equivalence $\mathrm{Sh}_{-\Lambda}(M)^{c,op} = \mathrm{Sh}_\Lambda(M)^c$ restricts to the Verdier duality on $\mathrm{Sh}_\Lambda(M)^b \subseteq \mathrm{Sh}_\Lambda(M)^c$.*

The proof of this theorem follows the same logic as its counterpart in Section 3.6. That is, we first identify the tensored functor $\mathrm{id} \otimes \epsilon$ and then check the triangle equality $(\mathrm{id} \otimes \epsilon)(\eta \otimes \mathrm{id}) = \mathrm{id}$. However, the situation is more complicated here since the definition of ϵ involves !-pullback, which is only a right adjoint in general and usually doesn't play well with $\boxtimes$ product. The upshot of this section is that when restricting to sheaves with a fixed singular isotropic microsupport, we can trade the !-pullbacks with left adjoints by the perturbation trick Proposition 4.2.8 we developed earlier.

Lemma 5.1.3. *The functor $\epsilon := p_* \Delta^! : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \to \mathcal{V}$ is colimit preserving and thus a well-defined morphism in Pr_{st}^L.*

Proof. We define the product isotopy $\varphi \times \varphi$ to be the isotopy on $\dot{T}^*(M \times M)$ whose Hamiltonian is given by $(H \times H)(x, y, \xi, \eta) := \sqrt{H(x, \xi)^2 + H(y, \eta)^2}$. Recall that we use $H^{\varphi \times \varphi}$ to denote an unspecified positive pushforward by $\varphi \times \varphi$. Then the perturbation trick (Proposition 4.2.8) implies that $p_* \Delta^! H = \mathrm{Hom}(1_\Delta, H) \xrightarrow{\sim} \mathrm{Hom}(1_\Delta, H^{\varphi \times \varphi}) = p_* \Delta^! H^{\varphi \times \varphi}$. Then (4) of Proposition 3.2.11 implies that the last term is the same as $p_* \Delta^* \left((w(\varphi \times \varphi) \circ H) \otimes \omega_M^{-1}\right)$, which is colimit-preserving on H. $\square$

For the rest of the chapter, we will use q_i, q_{ij}, $i < j$, to denote projections from M^3 to the corresponding components and p_i for projections from M^2.

Lemma 5.1.4. *Under the identification*

$$\mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \otimes \mathrm{Sh}_{-\Lambda}(M) = \mathrm{Sh}_{-\Lambda \times \Lambda \times -\Lambda}(M \times M \times M),$$

the functor

$$\epsilon \otimes \mathrm{id} : \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M) \otimes \mathrm{Sh}_{-\Lambda}(M) \to \mathcal{V} \otimes \mathrm{Sh}_{-\Lambda}(M) = \mathrm{Sh}_{-\Lambda}(M)$$

is identified as the functor

$$p_{2*}(\Delta \times \mathrm{id})^! : \mathrm{Sh}_{-\Lambda \times \Lambda \times -\Lambda}(M \times M \times M) \to \mathrm{Sh}_{-\Lambda}(M).$$

Proof. Note that this lemma is a variant of Lemma 3.6.1. For this lemma, we will apply a slightly generalized version of the perturbation trick in Proposition 4.2.8. That is, in addition to applying Proposition 4.2.8 to obtain $Hom(1_\Delta, H) = \mathrm{Hom}(w(\Phi^-), H)$, for $H \in \mathrm{Sh}_{-\Lambda \times \Lambda}(M \times M)$, by wrapping back along the continuation map $w(\Phi^-) \to 1_\Delta$, we note that it induces an isomorphism $q_{3*}(\mathcal{H}om(q_{12}^* 1_\Delta, H \boxtimes F)) = q_{3*}(\mathcal{H}om(q_{12}^* w(\Phi^-), H \boxtimes F))$, for $F \in \mathrm{Sh}{-\Lambda}(M)$. We remark that the proof is essentially the same since the extra M-component is trivial.

This way, we trade $(\epsilon \otimes \mathrm{id})(H, F) := (p^* \mathrm{Hom}(1_\Delta, H)) \otimes F$ with $(p^* \mathrm{Hom}(w(\Phi^-), H)) \otimes F$. We can then do a similar computation as in Lemma 3.6.1 that

$$
\begin{aligned}
\left(p^* \mathrm{Hom}(w(\Phi^-), H)\right) \otimes F &= \left(p^* \Gamma(M \times M; \mathcal{H}om\left(w(\Phi^-), H\right) \otimes\right) F \\
&= \left(q_{3*} q_{12}^* \mathcal{H}om\left(w(\Phi^-), H\right) \otimes\right) F.
\end{aligned}
$$

Now the key reason for us to use the perturbation trick is that $w(\Phi^-)$ avoids $-\Lambda \times \Lambda$ and we can apply (7) of Proposition 3.2.11 since $w(\Phi^-)$ has perfects stalks by Example 4.2.5. That is, we trade $\mathcal{H}om\left(w(\Phi^-), H\right)$ with $\mathcal{H}om\left(w(\Phi^-), 1_{M \times M}\right) \otimes H$ and compute that

$$
\begin{aligned}
\left(q_{3*} q_{12}^* \mathcal{H}om\left(w(\Phi^-), H\right) \otimes\right) F &= q_{3*}\left(\mathcal{H}om\left(q_{12}^* w(\Phi^-), q_{12}^* H\right) \otimes q_3^* F\right) \\
&= q_{3*}\left(\mathcal{H}om\left(q_{12}^* w(\Phi^-), H \boxtimes F\right)\right) \\
&\xleftarrow{\sim} q_{3*}\left(\mathcal{H}om\left(q_{12}^* 1_\Delta, H \boxtimes F\right)\right) \\
&= p_{2*}(\Delta \times \mathrm{id})^!(H \boxtimes F)
\end{aligned}
$$

by perturbing $w(\Phi^-)$ back to 1_Δ. $\qquad \square$

Proof of Theorem 5.1.2. Again, this proof is a variant of the proof of Proposition . We have to show that with the assumption on η, $(\mathrm{id} \otimes \epsilon)(\eta \otimes \mathrm{id}) = \mathrm{id}$, or equivalently, $p_{1*}(\mathrm{id} \times \Delta)^!(\eta \boxtimes F) = F$ for $F \in \mathrm{Sh}_{-\Lambda}(M)$ by the last lemma. Again we use a slight generalized version of the Perturbation trick from Proposition 4.2.8 to trade 1_Δ with $w(\Phi^-)$ whose microsupport avoids $-\Lambda \times \Lambda \times -\Lambda$. This way, we can do a computation similar to the one for the lemma above and see that

$$
\begin{aligned}
p_{2*}(\Delta \times \mathrm{id})^!(F \boxtimes \eta) &= q_{3*}(\Delta \times \mathrm{id})_* \mathcal{H}om\left(1_{M \times M}, (\Delta \times \mathrm{id})^!(F \boxtimes \eta)\right) \\
&= q_{3*} \mathcal{H}om\left(1_{\Delta \times M}, (\Delta \times \mathrm{id})^!(F \boxtimes \eta)\right) \\
&= q_{3*} \mathcal{H}om\left(q_{12}^* w(\Phi^-), F \boxtimes \eta\right) \\
&= q_{3*}\left(\mathcal{H}om\left(q_{12}^* w(\Phi^-), 1_{M^3}\right) \otimes (F \boxtimes \eta))\right) \\
&= q_{3*}\left(q_{12}^* \mathcal{H}om\left(w(\Phi^-), 1_{M \times M}\right) \otimes q_{23}^* \eta \otimes q_1^* F)\right)
\end{aligned}
$$

Our goal now is to organize the pull/push functors associated to the projections, by (1) and (2) of Proposition 3.1.8, into the form of convolutions. This process is a special case of

the proof for Proposition 3.5.3.

$$q_{3*}\left(q_{12}^*\,\mathcal{H}om\left(w(\Phi^-),1_{M\times M}\right)\otimes q_{23}^*\eta\otimes q_1^*F)\right))$$
$$= p_{2*}q_{13*}\left([q_{12}^*\,\mathcal{H}om\left(w(\Phi^-),1_{M\times M}\right)\otimes q_{23}^*\eta]\otimes q_{13}^*p_1^*F\right)$$
$$= p_{2*}\left(q_{13*}[q_{12}^*\,\mathcal{H}om\left(w(\Phi^-),1_{M\times M}\right)\otimes q_{23}^*\eta]\otimes p_1^*F\right)$$
$$= p_{2*}(\eta\circ_M\,\mathcal{H}om\left(w(\Phi^-),1_{M\times M}\right)]\otimes p_1^*F)$$
$$= \left(\eta\circ_M\,\mathcal{H}om(w(\Phi^-),1_{M\times M})\right)\circ F.$$

Now we note that by Example 4.2.5, $w(\Phi)=\mathcal{H}om(w(\Phi^-),p_1^*\omega_M)$. We recall also that, for any $L\in\mathrm{Loc}(M)$ and $G\in\mathrm{Sh}(M)$, $L\otimes G=p_1^*L\circ G$. Thus by Lemma 3.2.15, the desired sheaf kernel η exists if $\iota_{\Lambda\times-\Lambda}^*\left(w(\Phi)\right)$ has a convolution right inverse of in view of Theorem 1.0.3 which identifies sheaf kernel with colimit-preserving endofunctors $\mathrm{Sh}_{\Lambda\times-\Lambda}(M\times M)=\mathrm{End}^L(\mathrm{Sh}_{-\Lambda}(M))$. $\qquad\square$

5.2   Wrap-once functor

Recall that we construct a sheaf kernel $\iota_{\Lambda\times-\Lambda}^*(w(\Phi))\in\mathrm{Sh}_{\Lambda\times-\Lambda}(M)$ where $\Phi:S^*M\times I\to S^*M$ is a small Reeb flow such that $\varphi_t(\Lambda)\cap\Lambda=\varnothing$ for $tinI$. To simplify the notation, we consider its counterpart acting on $\mathrm{Sh}_\Lambda(M)$, and recall that $\iota_{-\Lambda\times\Lambda}^*(\phi)\circ F=\iota_\Lambda^*(F^\phi)=\mathfrak{W}_\Lambda^+(F^\phi)$ by Lemma 3.5.7 and theorem 1.0.4 . Geometrically, we wrap F forward away from Λ and then wrap it all the way to back to Λ positively.

Definition 5.2.1. With the setting above, we use S_Λ^+ to denote the functor

$$S_\Lambda:\mathrm{Sh}_\Lambda(M)\to\mathrm{Sh}_\Lambda(M)$$
$$F\mapsto\iota_\Lambda^*(w(\Phi)\circ F)$$

and call it the *wrap-once (positively) functor*. When $\Lambda=N^*\mathcal{S}$ for some Whitney triangulation $\mathcal{S}$, we abuse the notation and write it simply as $S_\mathcal{S}^+$.

We remark that the definition of S_Λ^+ does not require of the compactness assumption of M. We will show in the next section that the functor S_Λ^+ does not depend on the choice of the small Reeb pushforward Φ.

Example 5.2.2. Consider the case when $M=\mathbb{R}^1$ and $\Lambda=0_{\mathbb{R}^1}\cup(\cup_{n\in\mathbb{Z}}T_{n,\leq}^*\mathbb{R}^1)$. The corresponding case over S^1 is a on of projection of this picture. In this case, the functor S_Λ^+ and its inverse is given by, up to a shift by [1], the constant sheaves supported in the shaded areas below:
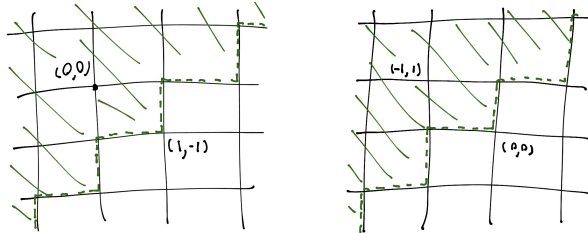
A simple computation shows that convoluting by the first sheaf kernel sends $1_{(n,\infty)}$ to $1_{(n-1,\infty)}$ for $n \in \mathbb{Z}$. Similarly, the second sheaf kernel sends $1_{(n,\infty)}$ to $1_{(n+1,\infty)}$ for $n \in \mathbb{Z}$.

Remark 5.2.3. A parallel construction in the Fukaya setting is considered by Sylvan in [52] where the invertibility of the wrap-once functor and a monodromy functor is proven under a swappability assumption.

The main theorem of this section is that $S_{\mathcal{S}}^+$ is invertible.

Theorem 5.2.4. *Assume M is compact. The functor $S_{\mathcal{S}}^+ : \mathrm{Sh}_{\mathcal{S}}(M) \to \mathrm{Sh}_{\mathcal{S}}(M)$ is an equivalence.*

We will mainly use the convolution expression of $\iota_{-\Lambda \times \Lambda}(w(\Phi))^* \circ F$ for $S_{\mathcal{S}}^+(F)$. The main strategy is to use general machinery of microlocal theory to write down an explicit formula of its inverse.

Lemma 5.2.5. *The right adjoint of the wrap-once functor $S_{\mathcal{S}}^+ : \mathrm{Sh}_{\mathcal{S}}(M) \to \mathrm{Sh}_{\mathcal{S}}(M)$, which we will denote it by $S_{\mathcal{S}}^-$, is given by a convolution*

$$S_{\mathcal{S}}^-(F) = \mathcal{H}om\left(\iota_{\mathcal{S} \times \mathcal{S}}^*(1_\Delta), p_1^* \omega\right) \circ F$$

and is, in particular, colimit preserving.

Proof. By definition, the functor $S_{\mathcal{S}}^-$ is classify by $\mathrm{Hom}(S_{\mathcal{S}}^+(G), F) = \mathrm{Hom}(G, S_{\mathcal{S}}^-(F))$ for all $F, G \in \mathrm{Sh}_\Lambda(M)$. And moving everything except $\iota_{\mathcal{S} \times \mathcal{S}}^*(w(\Phi))$ to the right hand side, we see that

$$\mathrm{Hom}(S_{\mathcal{S}}^+(G), F) = \mathrm{Hom}\left(\iota_{\mathcal{S} \times \mathcal{S}}^* w(\Phi), \mathcal{H}om(p_1^* G, p_2^! F)\right) = \mathrm{Hom}\left(w(\Phi), \mathcal{H}om(p_1^* G, p_2^! F)\right).$$

The virtual of $w(\Phi)$ over $\iota_{\mathcal{S} \times \mathcal{S}}^* w(\Phi)$ is that its microsupport avoids $0 \times N^* \mathcal{S}$ and we can apply (7) of Proposition 3.2.11 to trade $\mathcal{H}om(w(\Phi), p_2^! F)$ with $\mathcal{H}om(w(\Phi), 1_{M \times M}) \otimes p_2^! F$ and compute that

$$
\begin{aligned}
\mathrm{Hom}\left(w(\Phi), \mathcal{H}om(p_1^* G, p_2^! F)\right) &= \mathrm{Hom}\left(p_1^* G, \mathcal{H}om(w(\Phi), p_2^! F)\right) \\
&= \mathrm{Hom}\left(p_1^* G, \mathcal{H}om(w(\Phi), 1_{M \times M}) \otimes p_2^! F\right) \\
&= \mathrm{Hom}\left(p_1^* G, \mathcal{H}om(w(\Phi), p_2^! 1_M) \otimes p_2^* F\right) \\
&= \mathrm{Hom}(G, w(\Phi^-) \circ F).
\end{aligned}
$$

Here we use the fact that $\mathcal{H}om(w(\Phi), p_2^! 1_M) = w(\Phi^-)$ and the fact that $w(\Phi)$ is symmetric with respect the two components of $M \times M$ to obtain the last equality. Since $w(\Phi^-) \circ F$ is not in $\mathrm{Sh}_{\mathcal{S}}(M)$, we use the tautological identity $G = (\iota_{\mathcal{S} \times \mathcal{S}}^* 1_\Delta) \circ G$ to further change the expression to

$$
\begin{aligned}
\mathrm{Hom}(G, w(\Phi^-) \circ F) &= \mathrm{Hom}(\iota_{\mathcal{S} \times \mathcal{S}}^* 1_\Delta \circ G, w(\Phi^-) \circ F) \\
&= \mathrm{Hom}\left(p_1^* G, \mathcal{H}om(\iota_{\mathcal{S} \times \mathcal{S}}^* 1_\Delta, p_2^!(w(\Phi^-) \circ F))\right).
\end{aligned}
$$

Now the main reason to consider the case of triangulation $\mathcal{S}$ is that the object $(\iota_{\mathcal{S} \times \mathcal{S}}^* 1_\Delta)$, which is a compact object in $\mathrm{Sh}_{\mathcal{S} \times \mathcal{S}}(M \times M)$ has perfect stalks by Proposition 3.3.3. Thus for a similar reason as above, we can apply (7) of Proposition 3.2.11 again and conclude that

$$\text{Hom}\left(p_1^*G, \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_2^!(w(\Phi^-)\circ F))\right) = \text{Hom}\left(p_1^*G, \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_2^!1_M)\otimes p_2^*(w(\Phi^-)\circ F)\right)$$
$$= \text{Hom}\left(G, \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_1^*\omega_M)\circ(w(\Phi^-)\circ F)\right)$$

Thus $S_\mathcal{S}^-(F) = \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_1^*\omega_M)\circ(w(\Phi^-)\circ F)$. However, by Lemma 3.5.7,

$$\mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_1^*\omega_M)\circ(w(\Phi^-)\circ F = \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_1^*\omega_M)\circ\left(\iota_{\mathcal{S}\times\mathcal{S}}^*w(\Phi^-)\right)\circ F$$
$$= \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta, p_1^*\omega_M)\circ F$$

since $\iota_{\mathcal{S}\times\mathcal{S}}^* = \mathfrak{W}_{\mathcal{S}\times\mathcal{S}}^+$ by Theorem 1.0.4 and we can wrap $w(\Phi^-)$ tautologically forward to 1_Δ.
$\square$

Proof of Theorem. 5.2.4 The adjunction $\iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ(-) \dashv \mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right)\circ(-)$ implies that there is a canonical map

$$\mathcal{H}om\left((\iota_{\mathcal{S}\times\mathcal{S}}^*1_\Delta), p_1^*\omega_M\right)\circ\left(\iota_{\mathcal{S}\times\mathcal{S}}^*w(\Phi)\circ(F)\right) \to F,$$

and we show that this is an equivalence by a similar computation as the last lemma in a revered order. First, one notice that the second $\iota_{\mathcal{S}\times\mathcal{S}}^*w(\Phi)$ is redundant since we can apply Lemma 3.5.7 and (7) of Proposition 3.2.11 again and compute that

$$\mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M)\circ\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ F\right)$$
$$= \mathcal{H}om(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M)\circ(w(\Phi)\circ F)$$
$$= \mathcal{H}om^\circ(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), w(\Phi)\circ F) = \iota_\mathcal{S}^!(w(\Phi)\circ F).$$

Note for the last equality, we use the right adjoint version of the equality $\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta)\circ(G) = \iota_\mathcal{S}^*(G)$ for $G\in\text{Sh}(M)$. Now we note that $\iota_\mathcal{S}^!(w(\Phi)\circ F) = \mathfrak{W}_\mathcal{S}^-(w(\Phi)\circ F) = F$ by wrapping $w(\Phi)$ backward to 1_Δ tautologically.

We also have to show that the canonical map,

$$G \to \iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right)\circ G)$$

is also an equivalence.

Take $H\in\text{Sh}_\mathcal{S}(M)$ and we compute that with similar techniques as above that

$$\text{Hom}(\iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right)\circ G), H)$$
$$= \text{Hom}(w(\Phi)\circ(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right)\circ G), H)$$
$$= \text{Hom}(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right)\circ G, w(\varphi^-)\circ H)$$
$$= \text{Hom}\left(p_1^*G, \mathcal{H}om(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right), p_2^!(w(\varphi^-)\circ H))\right)$$
$$= \text{Hom}\left(p_1^*G, \mathcal{H}om(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right), p_1^*\omega_M)\otimes p_2^*(w(\varphi^-)\circ H)\right).$$

Now we note that, since $\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta)$ has perfect stalks, the canonical map for the double dual

$$\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta) \to \mathcal{H}om(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta), p_1^*\omega_M\right), p_1^*\omega_M)$$

is an isomorphism. Thus we have

$$\mathrm{Hom}(\iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta),p_1^*\omega_M\right)\circ G),H)=\mathrm{Hom}\left(p_1^*G,\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta)\otimes p_2^*(w(\varphi^-)\circ H)\right)$$

$$=\mathrm{Hom}\left(G,\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta)\circ(w(\varphi^-)\circ H)\right)=\mathrm{Hom}\left(G,\iota_{\mathcal{S}}^*(w(\varphi^-)\circ H)\right)=\mathrm{Hom}(G,H),$$

which implies $G=\iota_{\mathcal{S}\times\mathcal{S}}^*(w(\Phi))\circ(\mathcal{H}om\left(\iota_{\mathcal{S}\times\mathcal{S}}^*(1_\Delta),p_1^*\omega_M\right)\circ G.$

$\square$

5.3 Wrap-once as dual cotwist

For this section, we do not need the assumption of M being compact. It is proven in [52], that the Fukaya counterpart of the wrap-once functor S_Λ^+ is the dual cotwist of a certain spherical functor in the sense of [3]. We will consider a much simpler statement in our setting.

Recall that the st-valued sheaf Sh_Λ on M can be microlocalized to T^*M in the sense that there is a conic sheaf $\mu\mathrm{sh}_\Lambda$ on T^*M such that $\mu\mathrm{sh}_\Lambda\,|_{0_M}$ and, for each conic open subset $\Omega\subseteq T^*M$, the category $\mu\mathrm{sh}_\Lambda(\Omega)$ is a compactly generated stable category, and for each inclusion of conic opens $\Omega\subseteq\Omega'$, the restriction map $\mu\mathrm{sh}_\Lambda(\Omega')\to\mu\mathrm{sh}_\Lambda(\Omega)$ preserves both limits and colimits. A more detailed discussion of $\mu\mathrm{sh}_\Lambda$, the sheaf of microsheaves, can be found in [41, Chapter 3].

Denote by $q^*:\mathrm{Sh}_\Lambda(M)\rightleftharpoons\mu\mathrm{sh}_\Lambda(\dot{T}^*M):q_!$ the restriction associated to $\mu\mathrm{sh}_\Lambda$ from T^*M to $\dot{T}^*M$ and its left adjoint. Note that $\mu\mathrm{sh}_\Lambda(M)=\mathrm{Sh}_\Lambda(M)$ since $\mu\mathrm{sh}_\Lambda\,|_{0_M}$ and $\mu\mathrm{sh}_\Lambda$ is conic.

Definition 5.3.1. We denote by $S_\Lambda^{+,c}:\mathrm{Sh}_\Lambda(M)\to\mathrm{Sh}_\Lambda(M)$, the dual cotwist associated to q^* which by definition is defined by the fiber sequence,

$$q_!q^*\to\mathrm{id}_{\mathrm{Sh}_\Lambda(M)}\to S_\Lambda^{+,c}$$

where the map $q_!q^*\to\mathrm{id}_{\mathrm{Sh}_\Lambda(M)}$ is the counit of the $q_!\dashv q^*$ adjunction.

Proposition 5.3.2. *There is an identification $S_\Lambda^{+,c}=S_\Lambda^+:\mathrm{Sh}_\Lambda(M)\to\mathrm{Sh}_\Lambda(M)$. In particular, the definition of S_Λ^+ does not depend on the choice of the small Reeb push Φ.*

What we need for $\mu\mathrm{sh}_\Lambda$ is its local description from [41, 3.4]: For any $(x,\xi)\in\Lambda$, we may chose a small open ball $\Omega\subseteq S^*M$ containing Λ such that $\mu\mathrm{sh}_\Lambda(\Omega)$ fits in a fiber sequence,

$$K(B,\Omega)\hookrightarrow\mathrm{Sh}_\Lambda(B,\Omega)\to\mu\mathrm{sh}_\Lambda(\Omega),$$

where $B=\pi_\infty(\Omega)$, $\mathrm{Sh}_\Lambda(B,\Omega)$ consists of sheaves F on B such that $\mathrm{SS}^\infty(F)\cap\Omega\subseteq\Lambda$, and $K(B,\Omega)$ is its subcategory so that $\mathrm{SS}^\infty(F)\cap\Omega=\varnothing$.

Lemma 5.3.3. *The composition $q_!q^*F\to F\xrightarrow{c}S_\Lambda^+(F)$ is zero where $c:F\to S_\Lambda^+(F)$ is the continuation map obtained from the expression $S_\Lambda^+(F)=\mathfrak{W}_\Lambda^+(F^\varphi)$. As a result, there is a canonical map $S_\Lambda^{+,c}\to S_\Lambda^+$.*

Proof. Recall that for an adjunction $s : \mathcal{A} \leftrightarrows \mathcal{B} : l$, there is an equivalence

$$\mathrm{Hom}(lx, y) = \mathrm{Hom}(x, sy)$$
$$\alpha \mapsto s(\alpha) \circ \eta$$

where $\eta : x \to slx$ is the unit. In our case, the object x is q^*F and the unit $\eta : q^*F \xrightarrow{\sim} q^*q_! q^*F$ is an equivalence so the composition being zero is the same as the morphism q^*c being zero in $\mu\mathrm{sh}_\Lambda(\dot{T}^*M)$. Now $\mu\mathrm{sh}_\Lambda$ being a sheaf of categories means morphisms associated to these categories form sheaves, and it is enough to show that, for all $(x, \xi) \in \mathbb{R}_{>0}\Lambda$, there exists $\Omega \ni (x, \xi)$ such that q^*c restricts to zero on Ω. Since φ displace Λ from itself, shrink Ω if needed, we may assume that $\varphi(\mathrm{SS}^\infty(F)) \cap \Omega = \varnothing$. But then $c : F \to S_\Lambda^+(F)$ factors through $F \to F^\varphi \to S_\Lambda^+(F)$ with the object in the middle being 0. $\qquad\square$

In order to prove Proposition 5.3.2, we recall some implication of Sh_Λ being a sheaf: Let $\{U_i\}_{i \in I}$ be an open cover of M. Then the statement $\mathrm{Sh}(M) \xrightarrow{\sim} \lim_i \mathrm{Sh}(U_i)$ implies that, for $F \in \mathrm{Sh}(M)$, the counit

$$\operatorname*{colim}_I F_{U_i} \to F$$

is an equivalence. This description is inherited by Sh_Λ, i.e., for an open set $j : U \subset M$, the left adjoint of $j^* : \mathrm{Sh}_\Lambda(M) \to \mathrm{Sh}_{\Lambda|_U}(U)$ is given by

$$\mathrm{Sh}_{\Lambda|_U}(U) \to \mathrm{Sh}(M)$$
$$G \mapsto \mathfrak{W}_\Lambda^+(q_! G)$$

where we use $j : U \subseteq M$ to denote the inclusion, and the counit, now with the wrapping,

$$\operatorname*{colim}_I \mathfrak{W}_\Lambda^+(j_{i!} F_{U_i}) \to F$$

is an equivalence for $F \in \mathrm{Sh}_\Lambda(M)$. In fact, this follows from the simple fact that $\mathfrak{W}_\Lambda^+$ is colimit-preserving so that

$$\operatorname*{colim}_I \mathfrak{W}_\Lambda^+(j_{i!} F_{U_i}) = \mathfrak{W}_\Lambda^+(\operatorname*{colim}_I j_{i!} F_{U_i})$$
$$= \mathfrak{W}_\Lambda^+ F \xleftarrow{\sim} F.$$

We would like to utilize this last computation to prove the main proposition.

Proof of Proposition 5.3.2. Take an open cover $\{\Omega_i\}$ of Λ such that each $\mathbb{R}_{>0}\Omega_i$ is a small cone so that $\mu\mathrm{sh}_\Lambda(\Omega_i)$ fits in a fiber sequence

$$K(B_i, \Omega_i) \hookrightarrow \mathrm{Sh}_\Lambda(B_i, \Omega_i) \to \mu\mathrm{sh}_\Lambda(\Omega_i)$$

where $B_i = \pi_\infty(\Omega_i)$. We denote by $\dot{I}$ its index set. We notice that $\{B_i\}_{\dot{I}} \cup \{M \setminus \pi_\infty(\Lambda)\}$ forms an open cover of M, which we will simply denote it by $\{U_i\}_{i \in I}$. By Theorem 1.0.4, the left adjoint of the inclusion $K(B_i, \Omega_i) \hookrightarrow \mathrm{Sh}_\Lambda(B_i, \Omega_i)$ can be described by

$$\mathrm{Sh}_\Lambda(B_i, \Omega_i) \to K(B_i, \Omega_i)$$
$$G \mapsto \operatorname*{colim}_{w:\Omega_i}(G^w) =: \mathfrak{W}_i^+(G)$$

where $w : \Omega_i$ means positive wrappings compactly supported in Ω_i. Denote by

$$q_i^* : \mathrm{Sh}_\Lambda(B_i, \Omega_i) \leftrightarrows \mu\mathrm{sh}_\Lambda(\Omega_i) : q_{i!}$$

the adjunction over Ω_i. Fix an $F \in \mathrm{Sh}_\Lambda(M)$, for and $i \in \dot{I}$, there exists a fiber sequence

$$q_{i!}q_i^* F|_{B_i} \to F|_{B_i} \to \mathfrak{W}_i^+(F|_{B_i}).$$

Denote by $r_i : \Omega_i \hookrightarrow S^*M$ the inclusion and $r_i^* : \mu\mathrm{sh}_\Lambda(\Lambda) \leftrightarrows \mu\mathrm{sh}_\Lambda(\Omega_i) : r_{i!}$ the induced adjunction on microsheaves. The fact that $\mu\mathrm{sh}_\Lambda$ is a sheaf implies that we can glue q^*F from $r_{i!}r_i^* q^*F$ and the canonical map

$$q^*F \xleftarrow{\sim} \operatorname*{colim}_{\dot{I}} r_{i!}r_i^* q^*F$$

is an isomorphism. We recall that we have the following commutative diagram of categories induced by inclusions of opens.

$$
\begin{array}{ccc}
\mathrm{Sh}_\Lambda(M) & \xrightarrow{\;\;j_i^*\;\;} & \mathrm{Sh}_\Lambda(B_i) \\
\Big\downarrow{\scriptstyle q^*} & & \Big\downarrow{\scriptstyle q_i^*} \\
\mu\mathrm{sh}_\Lambda(\dot{T}^*M) & \xrightarrow{\;\;r_i^*\;\;} & \mu\mathrm{sh}_\Lambda(\Omega_i)
\end{array}
$$

Thus $r_{i!}r_i^* q^*F = r_{i!}q_i^* F|_{B_i}$. A similar diagram for the left adjoints implies that

$$
\begin{aligned}
q_! q^* F &= q_! \operatorname*{colim}_{\dot{I}} r_{i!}r_i^* q^*F \\
&= \operatorname*{colim}_{\dot{I}} q_! r_{i!}r_i^* q^*F \\
&= \operatorname*{colim}_{\dot{I}} \mathfrak{W}_\Lambda^+ j_{i!} q_{i!} q_i^* F|_{B_i}.
\end{aligned}
$$

Since $F = \operatorname*{colim}_{I} \mathfrak{W}_\Lambda^+ j_{i!} F|_{B_i}$, we get that

$$S_\Lambda^{+,c}(F) = \operatorname*{colim}_{I} \mathfrak{W}_\Lambda^+ j_{i!} \mathfrak{W}_i^+(F|_{B_i}).$$

Note that we use the fact that the colimit for $q_! q^*F$ can be trivially extended to be taken over I. To get rid of the $\mathfrak{W}_i^+$ term, we further assume that $\mathrm{supp}(\varphi_t) \cap S^*B_i \subseteq \Omega_i$ for all $i \in \dot{I}$. This assumption ensures that φ_t restricts to S^*B_i and $(F|_{B_i})^\varphi = F^\varphi|_{B_i}$, for $i \in \dot{I}$. Now, the key observation is that, since φ_t has pushed $\mathrm{SS}(F)$ away from Λ and Ω_i has the form of a product of small balls,

$$\mathfrak{W}_i^+(F|_{B_i}) := \mathfrak{W}(\Omega_i)^+(F|_{B_i}) = \mathfrak{W}(\Omega_i \setminus \Lambda)^+(F^\varphi|_{B_i}).$$

To see what $\mathfrak{W}_\Lambda^+ j_{i!}\mathfrak{W}(\Omega_i \setminus \Lambda)^+(F^\varphi|_{B_i}) = \mathfrak{W}_\Lambda^+((F^\varphi)_{B_i}) \in \mathrm{Sh}_\Lambda(M)$, we take $G \in \mathrm{Sh}_\Lambda(M)$ and compute that

$$
\begin{aligned}
\mathrm{Hom}(\mathfrak{W}_\Lambda^+ j_{i!}\mathfrak{W}(\Omega_i \setminus \Lambda)^+(F^\varphi|_{B_i}), G) &= \mathrm{Hom}(\mathfrak{W}(\Omega_i \setminus \Lambda)^+(F^\varphi|_{B_i}), G|_{B_i}) \\
&= \mathrm{Hom}(F^\varphi|_{B_i}, \mathfrak{W}(\Omega_i \setminus \Lambda)^- G|_{B_i}) = \mathrm{Hom}(F^\varphi|_{B_i}, G|_{B_i}) = \mathrm{Hom}\left(\mathfrak{W}_\Lambda^+((F^\varphi)_{B_i}), G\right).
\end{aligned}
$$

Finally, we patch everything back and get

$$S_\Lambda^{+,c}(F) = \operatorname*{colim}_I \mathfrak{W}_\Lambda^+ \left((F^\varphi)_{B_i}\right) = \mathfrak{W}_\Lambda^+ \operatorname*{colim}_I (F^\varphi)_{B_i} = \mathfrak{W}_\Lambda^+ F^\varphi = S_\Lambda^+(F).$$

$\square$

Chapter 6

The category of wrapped sheaves

In this section, we make our main construction of this dissertation and mimic the definition of $\mathcal{W}(T^*M, \Lambda)$, the (partially) wrapped Fukaya category associated to the pair (T^*M, Λ), and define the category of wrapped sheaves $\mathfrak{wsh}_\Lambda(M)$ using the techniques developed in the Chapter 4. We note that although this and the next chapter is logically independent from [20, 19], many of the proofs are adaptations from the wrapped Fukaya setting to the wrapped sheaves setting.

6.1 Definition

Let M be a real analytic manifold and $\Lambda \subseteq S^*M$ be a closed subset. Let $\widetilde{\mathfrak{wsh}}_\Lambda(M)$ be the small subcategory of $\mathrm{Sh}(M)$ generated under finite colimits and retracts by sheaves of the form $F^{w(\Phi)}$ where F is a sheaf with compact support such that $\mathrm{SS}^\infty(F)$ is a subanalytic singular isotropic and $\mathrm{SS}^\infty(F) \cap \Lambda = \varnothing$, and Φ is a contact isotopy compactly supported away from Λ. To encode the effect of wrappings, we take

$$\mathcal{C}_\Lambda(M) := \langle \mathrm{cof}(c(\Psi, F)) | \Psi \in \mathrm{Mor}(W(S^*M \setminus \Lambda)), F \in \widetilde{\mathfrak{wsh}}_\Lambda(M) \rangle$$

to be the subcategory of $\widetilde{\mathfrak{wsh}}_\Lambda(M)$ generated by the cofibers of the continuation maps.

Definition 6.1.1. We define the category of *wrapped sheaves* associated to (M, Λ) to be the quotient category

$$\mathfrak{wsh}_\Lambda(M) := \widetilde{\mathfrak{wsh}}_\Lambda(M)/\mathcal{C}_\Lambda(M)$$
$$:= \mathrm{cof}\left(\mathcal{C}_\Lambda(M) \hookrightarrow \widetilde{\mathfrak{wsh}}_\Lambda(M)\right)$$

where the cof is taken in st.

Remark 6.1.2. Localization identifies sheaves which are isotopic to each other: Let $F \xrightarrow{c} F^w \to \mathrm{cof}(c)$ be a fiber sequence in $\widetilde{\mathfrak{wsh}}_\Lambda(M)$ induced by a continuation map. Since a quotient map is exact and $\mathrm{cof}(c) = 0$ in $\mathfrak{wsh}_\Lambda(M)$, the fiber sequence becomes $F \xrightarrow{c} F^w \to 0$ and hence $c : F \to F^w$ is an isomorphism in $\mathfrak{wsh}_\Lambda(M)$. Now let Φ be any isotopy compactly supported away from Λ. By Proposition 4.3.6, Φ can be modified to be positive by a further

wrapping. That is, there exists $\Psi : \mathrm{id} \to \Phi'$ and $\Psi' : \Phi \to \Phi'$ in $W(S^*M \setminus \Lambda)$ and thus there are continuation maps $F \xrightarrow{c(\Psi,F)} F^\Phi$ and $F \xrightarrow{c(\Psi',F)} F^{\Phi'}$. As a result, the two objects F and F^Φ are isomorphic in $\mathfrak{w}\mathrm{sh}_\Lambda(M)$.

To simplify the notation, we will use Hom_w to denote the Hom spaces of the localized category when the context is clear. As the localization is essentially surjective, we usually implicitly assume a preimage $F \in \widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)$ for objects in $\mathfrak{w}\mathrm{sh}_\Lambda(M)$. By Proposition 2.4.2, there are identifications

$$\mathrm{Hom}_w(X,Y) = \operatorname*{colim}_{Y \xrightarrow{\alpha} Y'} \mathrm{Hom}(X,Y') = \operatorname*{colim}_{X' \xrightarrow{\beta} X} \mathrm{Hom}(X',Y)$$

where α and β run through morphisms whose cofibers $\mathrm{cof}(\alpha)$, $\mathrm{cof}(\beta)$ are in $\mathcal{C}_\Lambda(M)$. We will show that it is enough to take the colimit over $W(S^*M \setminus \Lambda)$ in our case, which is the same colimit over all continuation maps $c : F \to F^w$ by cofinality. We begin with the case of Homing out of objects in $\mathcal{C}_\Lambda(M)$ and in this case Hom_w vanishes.

Lemma 6.1.3. *Let $G \in \mathcal{C}_\Lambda(M)$ and $F \in \widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)$, we have*

$$\operatorname*{colim}_{c:F \to F^w} \mathrm{Hom}(G, F^w) = 0 = \mathrm{Hom}_w(G,F)$$

where $F \xrightarrow{c} F^w$ runs through all continuation maps.

Proof. We first consider the case when $G \in \mathcal{C}_\Lambda(M)$ is built from iterated cones and shifts of $\mathrm{cof}(c)$ for some continuation map c. For such a G, we may assume G fits into a cofiber sequence $H \xrightarrow{c} H^{w(\Phi)} \to G$ by induction. Apply $\mathrm{Hom}(-, F^{w(\Phi')})$ and we obtain the cofiber sequence

$$\mathrm{Hom}(G, F^{w(\Phi')}) \to \mathrm{Hom}(H^{w(\Phi)}, F^{w(\Phi')}) \to \mathrm{Hom}(H, F^{w(\Phi')}).$$

Then one compute

$$\begin{aligned}
\mathrm{Hom}(H^{w(\Phi)}, F^{w(\Phi')}) &= \mathrm{Hom}(w(\Phi) \circ H, w(\Phi') \circ F) \\
&= \mathrm{Hom}(H, w(\Phi^{-1}) \circ w(\Phi') \circ F) \\
&= \mathrm{Hom}(H, w(\Phi^{-1} \circ \Phi') \circ F).
\end{aligned}$$

Take $\operatorname*{colim}_{F \to F^{w'}}$ over isotopies Φ' of the form $\Phi \circ \Psi$, which is cofinal, and we obtain the fiber sequence

$$\operatorname*{colim}_{F \to F^{w'}} \mathrm{Hom}(G, F^{w'}) \to \operatorname*{colim}_{F \to F^{w'}} \mathrm{Hom}(H, F^{w'}) \xrightarrow{\sim} \operatorname*{colim}_{F \to F^{w'}} \mathrm{Hom}(H, F^{w'}).$$

This implies that $\operatorname*{colim}_{F \to F^{w'}} \mathrm{Hom}(G, F^{w'}) = 0$.

Now let G' be a retract of the same G as above. Taking $\operatorname*{colim}_{F \to F^w} \mathrm{Hom}(-, F^w)$ makes $\operatorname*{colim}_{F \to F^w} \mathrm{Hom}(G', F^w)$ a retract of $\operatorname*{colim}_{F \to F^{w'}} \mathrm{Hom}(G, F^{w'}) = 0$. Since the only retract of a zero object is a zero object, we conclude that $\operatorname*{colim}_{F \to F^w} \mathrm{Hom}(G', F^w) = 0$. $\square$

Proposition 6.1.4. *For $F, G \in \widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)$, we have*

$$\mathrm{Hom}_w(G, F) = \underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}(G, F^w)$$

where $F \xrightarrow{c} F^w$ runs through all continuation maps.

Proof. Consider any morphism $\alpha : G' \to G$ such that $\mathrm{fib}(\alpha) \in \mathcal{C}_\Lambda(M)$ which we will denote it as $G' \xrightarrow{qis.} G$ when the exact morphism α is not relevant. Now take a continuation map $c : F \to F^w$ and apply $\mathrm{Hom}(-, F^w)$ to the fiber sequence $\mathrm{fib}(\alpha) \to G' \to G$, we obtain the fiber sequence

$$\mathrm{Hom}(G, F^w) \to \mathrm{Hom}(G', F^w) \to \mathrm{Hom}(\mathrm{fib}(\alpha), F^w).$$

Now recall that Hom_w can be computed by either varying the first or the second factor. As a result, we can first take colimit over such $\alpha : G' \to G$ and we obtain the fiber sequence

$$\mathrm{Hom}(G, F^w) \to \mathrm{Hom}_w(G, F^w) \to \underset{G' \xrightarrow{qis.} G}{\mathrm{colim}} \, \mathrm{Hom}(\mathrm{fib}(\alpha), F^w).$$

Then we take colimit over $F \to F^w$ and get

$$\underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}(G, F^w) \to \underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}_w(G, F^w) \to \underset{F \to F^w}{\mathrm{colim}} \, \underset{G' \xrightarrow{qis.} G}{\mathrm{colim}} \, \mathrm{Hom}(\mathrm{fib}(\alpha), F^w).$$

Since colimits commute with each other, the above Lemma 6.1.3 implies

$$\underset{F \to F^w}{\mathrm{colim}} \, \underset{G' \xrightarrow{qis.} G}{\mathrm{colim}} \, \mathrm{Hom}(\mathrm{fib}(\alpha), G') = \underset{G' \xrightarrow{qis.} G}{\mathrm{colim}} \, \underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}(\mathrm{fib}(\alpha), G') = 0$$

or, equivalently

$$\underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}(G, F^w) \xrightarrow{\sim} \underset{F \to F^w}{\mathrm{colim}} \, \mathrm{Hom}_w(G, F^w) \xrightarrow{\sim} \mathrm{Hom}_w(G, F).$$

Here we use that fact that $F \to F^w$ is an isomorphism in $\mathfrak{w}\mathrm{sh}_\Lambda(M)$. $\qquad\square$

We note that the above construction is covariant on the open sets of M. For an open set $U \subseteq M$, we set $\Lambda|_U = \Lambda \cap S^*U$. We abuse the notation and use $\widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(U)$ to denote the category $\widetilde{\mathfrak{w}\mathrm{sh}}_{\Lambda|_U}(U)$. When there is an inclusion of open sets $U \subseteq V$, objects in $\widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(U)$ can be naturally regarded as objects in $\widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(V)$ since we require them to have compact support in U. We define $\mathcal{C}_\Lambda(U)$ similarly and note that $\mathcal{C}_\Lambda(U) \subseteq \widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(U) \cap \mathcal{C}_\Lambda(V)$. Thus there is a canonical map $\mathfrak{w}\mathrm{sh}_\Lambda(U) \to \mathfrak{w}\mathrm{sh}_\Lambda(V)$.

Definition 6.1.5. The above construction defines a covariant functor $\mathfrak{w}\mathrm{sh}_\Lambda : \mathrm{Op}_M \to \mathrm{st}$, i.e., a precosheaf with coefficient in idempotent complete small stable categories. We refer it as the precosheaf of *wrapped sheaves* associated to Λ.

Note also that this construction is contravariant on the closed set Λ. That is, if $\Lambda \subseteq \Lambda'$ is an inclusion of closed subset of S^*M, there is a canonical map $\mathfrak{w}\mathrm{sh}_{\Lambda'}(U) \to \mathfrak{w}\mathrm{sh}_\Lambda(U)$ for $U \subseteq M$ by a similar consideration. In other words, there is a morphism $\mathfrak{w}\mathrm{sh}_{\Lambda'} \to \mathfrak{w}\mathrm{sh}_\Lambda$ between precosheaves.

Remark/Conjecture 6.1.6. Consider the case when Λ is a subanalytic singular isotropic. Inspired by homological mirror symmetry, Nadler defines in [41] a conic cosheaf $\mu\mathrm{sh}^w_\Lambda :$ $\mathrm{Op}_{T^*M} \to \mathrm{st}$ through a purely categorical construction and term it as the cosheaf of *wrapped microlocal sheaves*. This cosheaf can be obtained from and determines the sheaf $\mu\mathrm{sh}_\Lambda$, which we used in Section 5.3, by passing to compact objects. One main property of $\mu\mathrm{sh}^w_\Lambda$ is that its restriction to the zero section, the 'wrapped sheaves', is the cosheaf Sh^c_Λ discussed in Proposition 3.4.8. We will reserve the term 'wrapped sheaves' for the geometrically constructed category $\mathfrak{wsh}_\Lambda$ through this paper. Corollary 1.0.7 of the main theorem asserts that these two cosheaves are the same after all. As a result, we expect to extend the construction $\mathfrak{wsh}_\Lambda$ to the cotangent bundle as well.

6.2   Generation

We find a set of generators of $\mathfrak{wsh}_\Lambda(M)$ when Λ is a singular isotropic. We first prove a special case of the Künneth formula which we will refer it as the stabilization lemma. The corresponding lemma in the Fukaya setting can be found in [19, (1.7), (1.8)]. Fix $n \in \mathbb{R}^n$. Let M be a real analytic manifold and $\Lambda \subseteq S^*M$ be a closed subset. We set

$$\Lambda^{st} = ((\mathbb{R}_{>0}\Lambda \cup 0_M) \times 0_{\mathbb{R}^n})^\infty \subseteq S^*(M \times \mathbb{R}^n).$$

Pick a small ball $B \subseteq \mathbb{R}^n$ centered at 0. For $F \in \mathrm{Sh}(M)$, by (6) of Proposition 3.2.11, there is microsupport estimation

$$\mathrm{SS}(F \boxtimes 1_B) \subseteq \mathrm{SS}(F) \times N^*_{out}(B).$$

As a result, exterior tensoring with 1_B induces a functor

$$- \boxtimes 1_B : \widetilde{\mathfrak{wsh}}_\Lambda(M) \to \widetilde{\mathfrak{wsh}}_{\Lambda^{st}}(M \times \mathbb{R}^n).$$

We claim that this functor induces a fully faithful functor on the quotient. We first recall a lemma.

Lemma 6.2.1. *Let* $\mathcal{C}$, $\mathcal{D}$ *be stable categories,* S *and* T *be sets of morphisms in* $\mathcal{C}$ *and* $\mathcal{D}$ *which are closed under composition and contain identities. Set* $\mathcal{C}_0 := \langle \mathrm{cof}(s) | s \in S \rangle$ *and* $\mathcal{D}_0 := \langle \mathrm{cof}(t) | t \in T \rangle$. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor such that for all* $X_0 \xrightarrow{s} X_1 \in S$, *there exists* $F(X_1) \xrightarrow{t} Y \in T$ *such that* $t \circ F(s) \in T$. *Then* $F|_{\mathcal{C}_0}$ *factors through* $\mathcal{D}_0$ *and* F *descends to a functor* $\bar{F} : (\mathcal{C}/\mathcal{C}_0) \to (\mathcal{D}/\mathcal{D}_0)$ *filling the commutative square with the quotient functors:*

$$
\begin{array}{ccccc}
\mathcal{C}_0 & \hookrightarrow & \mathcal{C} & \twoheadrightarrow & \mathcal{C}/\mathcal{C}_0 \\
{\scriptstyle F|_{\mathcal{C}_0}}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle \bar{F}}\downarrow \\
\mathcal{D}_0 & \hookrightarrow & \mathcal{D} & \twoheadrightarrow & \mathcal{D}/\mathcal{D}_0
\end{array}
$$
.

Proof. Let $X_0 \xrightarrow{s} X_1$ and $F(X_1) \xrightarrow{t} Y \in T$ be as above. By the Lemma 2.2.11, there exists a fiber sequence $F(\mathrm{cof}(s)) \to \mathrm{cof}(t \circ F(s)) \to \mathrm{cof}(t)$. $\square$

Proposition 6.2.2 (The stabilization lemma). *The functor* $- \boxtimes 1_B$ *defined above descends to a fully faithful functor on the quotient*

$$\mathfrak{wsh}_\Lambda(M) \hookrightarrow \mathfrak{wsh}_{\Lambda^{st}}(M \times \mathbb{R}^n)$$

which we will refer it as the stabilization functor.

Proof. Let $\Phi : \dot{T}^*M \times I \to \dot{T}^*M$ be a positive homogenous symplectic isotopy and $H = \alpha(\Phi_*\partial_t)$ be its corresponding Hamiltonian. We note that since H is not defined on the entire T^*M, it is not always possible to extend it to $\dot{T}^*(M \times \mathbb{R}^n)$ by setting the dependence on the second component to be constant. Nevertheless, H^2 is defined on T^*M since it is homogeneous of degree 1. Pick a bump function ρ on $S^*(M \times \mathbb{R}^n)$ such that $(\mathrm{supp}(H) \times \overline{T^*B})^\infty \subseteq \mathrm{Int}(\mathrm{supp}(\rho))$ so that $H^2 + \rho|\xi|^2 > 0$ when $H > 0$. Here we use the same notation ρ to denote its pullback on $\dot{T}^*M$. Then set $\tilde{H} := \sqrt{H^2 + \rho|\xi|^2}$ and we denote its corresponding homogeneous isotopy on $\dot{T}^*(M \times \mathbb{R}^n)$ by $\tilde{\Phi}$.

The above construction implies that

$$\mathrm{SS}\left((K(\Phi^{-1}) \boxtimes_I 1_{\Delta_{\mathbb{R}^n} \times I}) \circ_I K(\tilde{\Phi}) \right) \subseteq \mathrm{SS}(K(\Phi^{-1}) \boxtimes_I 1_{\Delta_{\mathbb{R}^n} \times I}) \circ_I \mathrm{SS}(K(\tilde{\Phi})) \subseteq \{\tau \leq 0\}$$

since $H(x,\xi) \leq \tilde{H}(x,\xi,t,\tau)$. Thus there is a continuation map

$$1_{\Delta_{M \times \mathbb{R}^n}} \to (w(\Phi^{-1}) \boxtimes 1_{\Delta_{\mathbb{R}^n}}) \circ w(\tilde{\Phi})$$

or equivalently

$$w(\Phi) \boxtimes 1_{\Delta_{\mathbb{R}^n}} \to w(\tilde{\Phi})$$

which precomposes with $1_{\Delta_{M \times \mathbb{R}^n}} \to w(\Phi) \boxtimes_I 1_{\Delta_{\mathbb{R}^n}}$ to $1_{\Delta_{M \times \mathbb{R}^n}} \to w(\tilde{\Phi})$. Thus, the last lemma implies that the functor

$$\mathfrak{wsh}_\Lambda(M) \to \mathfrak{wsh}_{\Lambda^{st}}(M \times \mathbb{R}^n)$$

is well-defined.

A similar argument implies that for any positive isotopy Ψ on $\dot{T}^*(M \times \mathbb{R}^n)$ with Hamiltonian $\tilde{H}$, there exists H on $\dot{T}^*M$ and ρ on $\dot{T}^*\mathbb{R}^n$ such that $\tilde{H} \leq \sqrt{H^2 + \rho^2}$. Thus wrappings coming from the product is cofinal. Since B is contractible, there is an isomorphism $\mathrm{Hom}(F \boxtimes 1_B, G \boxtimes 1_{\tilde{B}}) = \mathrm{Hom}(F, G)$ for any larger ball $\tilde{B}$ in $\mathbb{R}^n$. This implies

$$
\begin{aligned}
\mathrm{Hom}_{\mathfrak{wsh}_{\Lambda^{st}}(M \times \mathbb{R}^n)}(G \boxtimes 1_B, F \boxtimes 1_B) &= \operatorname*{colim}_{\tilde{\Phi} \in W(S^*(M \times \mathbb{R}^n) \backslash \Lambda^{st})} \mathrm{Hom}\left(G \boxtimes 1_B, (F \boxtimes 1_B)^{\tilde{\Phi}} \right) \\
&= \operatorname*{colim}_{\tilde{\Phi} \in W(S^*(M \times \mathbb{R}^n) \backslash \Lambda^{st})} \mathrm{Hom}\left(G \boxtimes 1_B, F^\Phi \boxtimes 1_{\tilde{B}} \right) \\
&= \operatorname*{colim}_{\Phi \in W(S^*M \backslash \Lambda)} \mathrm{Hom}\left(G, F^\Phi \right) \\
&= \mathrm{Hom}_{\mathfrak{wsh}_\Lambda(M)}(G, F).
\end{aligned}
$$

Thus, the stabilization functor $\mathfrak{wsh}_\Lambda(M) \hookrightarrow \mathfrak{wsh}_{\Lambda^{st}}(M \times \mathbb{R}^n)$ is fully faithfull. $\square$

Now we show that the category $\mathfrak{wsh}_\varnothing(M)$ is generated by one object. More precisely, we say an open set $B \subseteq M$ is a ball if B is relative compact, contractible and $\overline{B}$ is a closed disk. Now let B be a ball such that there exists an open chart U containing $\overline{B}$. Since all such balls are smoothly isotopic to each other inside M, lifting such isotopies implies that the object 1_B, where B is a such ball, is independent of the choice of the exact ball. In order to show that 1_B is a generator, we need a class of auxiliary objects.

Definition 6.2.3. We say that an open set $B \subseteq M$ is a stable ball if it is relative compact, contractible, and $\overline{B}$ has a smooth boundary in M.

One can check that a stable ball is a ball up to a stabilization by the famous corollary of the cobordism theorem. The following statements are Theorem 5.12 and Corollary 5.13 in [21].

Theorem 6.2.4. *A stable ball of dimension ≥ 6 with simply connected boundary is a ball.*

Corollary 6.2.5. *Let M be a stable ball. Then $B \times I^k$ is a ball provided $\dim B + k \geq 6$ and $k \geq 1$.*

Proof. This is implied by a combination of the van Kampen Theorem and the Poincáre duality for manifolds with boundary

$$H^{\dim N - k}(N, \partial N) = H_k(N).$$

$\square$

Lemma 6.2.6. *Assume M is connected. The category $\mathfrak{wsh}_\varnothing(M)$ is generated under finite colimits and retractions by 1_B for any small ball B.*

Proof. Let $F \in \mathfrak{wsh}_\varnothing(M)$ be an object. By Remark 6.1.2, we may assume F is a sheaf with compact support, subanalytic isotropic microsupport, and perfect stalks. By Proposition 3.4.5, there is a Whitney triangulation $\mathcal{T}$ such that F is $\mathcal{T}$-constructible. Since $\mathrm{Sh}_{N^*_\infty \mathcal{T}}(M)^c = \mathrm{Perf}\,\mathcal{T}$ is generated under finite colimits and retractions by $1_{\mathrm{star}(t)}$ for $t \in \mathcal{T}$, we may assume $F = 1_{\mathrm{star}(t)}$. We claimed that the object $1_{\mathrm{star}(t)}$ is isomorphic to 1_B for some small ball $B \subseteq \mathrm{star}(t)$. Note the open set $\mathrm{star}(t)$ is relatively compact and contractible, however, $\overline{\mathrm{star}(r)}$ might not be a manifold with boundary and modification needs to be made.

Apply the inward cornering construction in Definition 3.3.9 to $U = \mathrm{star}(t)$, we obtain a family of $\mathrm{star}(t)^{-\epsilon}$ depending smoothly on ϵ. When ϵ is small, the object $1_{\mathrm{star}(t)^{-\epsilon}}$ in $\mathfrak{wsh}_\Lambda(M)$ is independent of ϵ so we abuse the notation and simply denote it by $1_{\mathrm{star}(t)^-}$. As there is no stop restriction, the canonical map $1_{\mathrm{star}(t)^-} \to 1_{\mathrm{star}(t)}$ becomes an isomorphism in $\mathfrak{wsh}_\varnothing(M)$ through the positive wrapping obtained by taking $\epsilon \to 0$. The closure $\overline{\mathrm{star}(t)^-}$ is a manifold with corners when ϵ is small, i.e., the boundary $\overline{\mathrm{star}(t)^-}$ can be modified by the boundary of the inclusion $[0,\infty)^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^n$ for some $k \geq 1$. By Example 3.2.5, $\mathrm{SS}^\infty(1_{(0,\infty)^k \times \mathbb{R}^{n-k}})$ is smooth and $1_{(0,\infty)^k \times \mathbb{R}^{n-k}}$ can be wrapped to some 1_{V_δ} where $V_\delta := \{x \in \mathbb{R}^n | d(x, [0,\infty)^k \times \mathbb{R}^{n-k}) < \delta\}$ by the Reeb flow. One can see from the local model that the boundary of V_δ is smooth for small δ. Thus, we may further replace $1_{\mathrm{star}(t)^-}$ by some 1_U such that U is relative compact, contractible and has smooth boundary, i.e., a stable ball.

Finally pick a ball $B \subseteq U$ and consider the canonical morphism $1_B \to 1_U$ induced by the inclusion. Apply the stabilization lemma for n large and we see from the last corollary that we may assume U to be a ball as well. In this case, the canonical map $1_B \to 1_U$ coincide with the continuation map obtained by the standard Reeb flow and is an isomorphism. Thus, the original map is an isomorphism in $\mathfrak{wsh}_{\varnothing}(T^*M)$ and we see that 1_B generates. $\qquad\square$

We assume for the rest of this section that Λ is a singular isotropic. To study generation for the general case, we need the following lemma to perform general position argument.

Definition 6.2.7 ([19, Definition 1.6]). Let Y^{2n-1} be a contact manifold. We say a set $\mathfrak{f}$ is mostly Legendrian if there is a decomposition $\mathfrak{f} = \mathfrak{f}^{subcrit} \cup \mathfrak{f}^{crit} \subseteq Y$ for which $\mathfrak{f}^{subcrit}$ is closed and is contained in the smooth image of a second countable manifold of dimension $< n - 1$, and $\mathfrak{f}^{crit}$ is a Legendrian submanifold.

We note that a closed singular isotropic Λ is in particular mostly Legendrian.

Lemma 6.2.8 ([19, Lemma 2.2 and Lemma 2.3]). *Let Y^{2n-1} be a contact manifold and $\mathfrak{f}$ be mostly Legendrian.*

1. *Let $\Lambda \subseteq Y$ be a compact Lagrangian. Then Λ admits cofinal wrappings $\Lambda \rightsquigarrow \Lambda^w$ with Λ^w disjoint from $\mathfrak{f}$.*

2. *Let $\Lambda_1, \Lambda_2 \subseteq Y$ be compact Legendrians disjoint from $\mathfrak{f}$. Consider the space of positive Legendrian isotopies $\Lambda_1 \rightsquigarrow \Lambda_2$. Then the subspace of isotopies which*

   2.1. *remains disjoint from $\mathfrak{f}^{subcrit}$ and*

   2.2. *intersect $\mathfrak{f}^{crit}$ only finitely many times, each time passing transversally at a single point,*

   is open and dense.

For inclusion of singular isotropics $\Lambda \subseteq \Lambda' \subseteq S^*M$, general position argument implies that the induced map $\mathfrak{wsh}_{\Lambda'}(M) \to \mathfrak{wsh}_{\Lambda}(M)$ is always essentially surjective. In order to study the fiber of this map, we consider the following objects:

Let Λ be a subanalytic singular isotropic and let $(x, \xi) \in \mathbb{R}_{>0}\Lambda$ be a smooth point. Consider a proper analytic Λ-Morse function $f : M \to \mathbb{R}$. We assume there exists $\epsilon > 0$ such that x is the only Λ-critical point over $f^{-1}([-\epsilon, \epsilon])$ with critical value 0, $df_x = \xi$ and $f^{-1}(-\infty, \epsilon)$ is relatively compact. By our assumption, both $1_{f^{-1}(-\infty, \pm\epsilon)}$ are objects of $\widetilde{\mathfrak{wsh}}_{\Lambda}(M)$.

Definition 6.2.9. A sheaf-theoretical linking disk at (x, ξ) (with respect to Λ) is an object $D_{(x, \xi)}$ of the form
$$\mathrm{cof}(1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)}) \in \mathfrak{wsh}_{\Lambda}(M)$$
where the arrow is induced by the inclusion of opens $f^{-1}(-\infty, -\epsilon) \subseteq f^{-1}(-\infty, \epsilon)$ given by a function f with the above properties. Note that by scaling f with $r \in \mathbb{R}_{>0}$, we see that the object $D_{(x, \xi)}$ depends only on (x, ξ)'s image in S^*M. Thus, we also use the same notation $D_{(x, \xi)}$ for $(x, \xi) \in S^*M$.

Remark 6.2.10. Since $df \neq 0$ over $f^{-1}([-\epsilon, \epsilon])$, the fibers $f^{-1}(t)$ for $t \in [-\epsilon, \epsilon]$ are smooth submanifolds. Thus, the canonical map $1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)}$ is also given by the continuation map of the wrapping $\{N^*_{\infty, out} f^{-1}(-\infty, t)\}_{t \in [-\epsilon, \epsilon]}$ which passes through Λ transversely exactly once at $(x, [\xi])$. Extend this wrapping to a global one Ψ. Since there is no other intersection with Λ, we can decompose Ψ to $\Psi_+ \# \Psi_0 \# \Psi_-$ so that $\Psi_\pm$ do not intersect Λ and Ψ_0 only moves points near (x, ξ). This way, we can see $D_{(x, \xi)}$ can be presented as a cofiber induced by an expanding open half-plane.

Thus, $D_{(x, \xi)}$ can also be presented as a cofiber induced by inclusions of small balls.
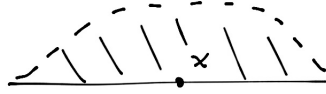
Proposition 6.2.11. *Let $\Lambda \subseteq \Lambda'$ be subanalytic singular isotropics and let $\mathcal{D}^w_{\Lambda', \Lambda}(M)$ denote the fiber of the canonical map $\mathfrak{wsh}_{\Lambda'}(M) \to \mathfrak{wsh}_\Lambda(M)$. Then $\mathcal{D}^w_{\Lambda', \Lambda}(M)$ is generated by the sheaf-theoretical linking disk $D_{(x, \xi)}$ for smooth Legendrian points $(x, \xi) \in \Lambda' \setminus \Lambda$.*

Proof. Let $F \in \mathfrak{wsh}_\Lambda(M)$. We assume that $\mathrm{SS}^\infty(F)$ is a subanalytic isotropic and pick a Whitney triangulation $\mathcal{T}$ such that $\mathrm{SS}^\infty(F) \subseteq N^*_\infty \mathcal{T}$. Fixed a particular way to construct F out of sheaves of the form $M_{\mathrm{star}(t)}$ for some $M \in \mathcal{V}_0$ by taking finite steps of cofibers and use $\{F_i\}_{\{i \in A\}}$ to denote those $M_{\mathrm{star}(t)}$'s which show up in these steps. Note that it is possible that their microsupport $\mathrm{SS}^\infty(F_i)$ intersect Λ. However, we see from the proof of Lemma 6.2.6, the microsupport $\mathrm{SS}^\infty(F_i)$ of these F_i's are smooth Legendrians in $S^* M$. Thus, we can apply Lemma 6.2.8 and assume $\mathrm{SS}^\infty(F_i) \cap \Lambda' = \varnothing$ for $i \in A$. By the microsupport triangular inequality (1) of Proposition 3.2.11, the sheaves appear in the the cofiber sequences which build F from these F_i's do not intersect Λ' as well. In particular, $\mathrm{SS}^\infty(F) \cap \Lambda' = \varnothing$. Similarly, an application of Lemma 6.2.8 implies that we can assume the existence of a cofinal wrapping sequence $F \to F^{w_1} \to F^{w_2} \to \cdots$ such that $\mathrm{SS}^\infty(F^{w_n}) \cap \Lambda' = \varnothing$. This implies that the canonical map

$$\widetilde{\mathfrak{wsh}}_{\Lambda'}(M) / \left(\mathcal{C}_\Lambda(M) \cap \widetilde{\mathfrak{wsh}}_{\Lambda'}(M) \right) \to \mathfrak{wsh}_\Lambda(M) := \widetilde{\mathfrak{wsh}}_\Lambda(M) / \mathcal{C}_\Lambda(M)$$

is an equivalence. Thus, we can apply Lemma 2.4.4 to the diagram

$$\begin{array}{ccccc}
\mathcal{C}_{\Lambda'}(M) & \xhookrightarrow{\quad i \quad} & \widetilde{\mathfrak{wsh}}_{\Lambda'}(M) & \xtwoheadrightarrow{\quad p \quad} & \mathfrak{wsh}_{\Lambda'}(M) \\
\downarrow & & \| & & \downarrow \\
\mathcal{C}_{\Lambda}(M) \cap \widetilde{\mathfrak{wsh}}_{\Lambda'}(M) & \xhookrightarrow{\quad j \quad} & \widetilde{\mathfrak{wsh}}_{\Lambda'}(M) & \xtwoheadrightarrow{\quad q \quad} & \mathfrak{wsh}_{\Lambda}(M)
\end{array}$$

which implies that $\mathcal{D}^{w}_{\Lambda',\Lambda}(M) = \left(\mathcal{C}_{\Lambda}(M) \cap \widetilde{\mathfrak{wsh}}_{\Lambda'}(M)\right)/\mathcal{C}_{\Lambda'}(M)$. Now $\mathcal{C}_{\Lambda}(M)$ is the category generated by the cofibers $\mathrm{cof}(c(\Psi, F))$ of continuation maps whose wrapping $\psi_t(\mathrm{SS}^{\infty}(F))$ avoids Λ, and $\mathcal{C}_{\Lambda}(M) \cap \widetilde{\mathfrak{wsh}}_{\Lambda'}(M)$ is generated by a similar construction except we now only requires the end points to avoid Λ. The claim is that the quotient is generated by the sheaf-theoretic linking disks $D_{(x,\xi)}$ for smooth Legendrian points $(x,\xi) \in \Lambda' \setminus \Lambda$.

Lemma 2.2.11 implies that if $H_1 \to H_2 \to H_3$ is a cofiber sequence, then $\mathrm{cof}(c(\Psi, H_1)) \to \mathrm{cof}(c(\Psi, H_2)) \to \mathrm{cof}((c(\Psi, H_3))$ is also a fiber sequence. Thus, it is enough to assume F has smooth Legendrian microsupport by the discussion at the beginning of the proof. Let Ψ be a positive isotopy such that $\psi_t(\mathrm{SS}^{\infty}(F))$ does not touch Λ and, by general position argument, we may assume $\psi_t(\mathrm{SS}^{\infty}(F))$ touches Λ' for finitely many times and transversally through one point $p \in \Lambda' \setminus \Lambda$ each time. Decomposing Ψ to $\Psi = \Psi_k \# \cdots \# \Psi_1$ so that passing happens only once during the duration of each Ψ_i. Since $c(\Psi) = c(\Psi_k) \circ \cdots \circ c(\Psi_1)$, it is sufficient to prove the case when only one such passing at (x, ξ) appears by induction with Lemma 2.2.11.

Let $q \in \mathrm{SS}^{\infty}(F)$ be the point so that the path $\psi_t(q)$ pass $(x, \xi) \in \Lambda'$ and U small open ball near q in S^*M. We again decompose Ψ to $\Psi = \Psi_+ \# \Psi_0 \# \Psi_-$ such that there's no passing happing during $\Psi_{\pm}$ and Ψ_0 only moves points in U. In this case, $c(\Psi_{\pm}, F)$ are isomorphisms and we can further assume Ψ only moves points in U. Now set $F_t = (K(\Psi) \circ F)|_{M \times \{t\}}$ so $\mathrm{SS}^{\infty}(F_t) = \psi_t(\mathrm{SS}^{\infty}(F))$. We use one last general position argument to assume the front projection $\pi_{\infty} : \mathrm{SS}^{\infty}(F_t) \to \pi_{\infty}(\mathrm{SS}^{\infty}(F_t))$ is finite near $\psi_t(U)$ so $\pi(U \cap \mathrm{SS}^{\infty}(F_t)) \subseteq \pi(U)$ is a hyperplane. Thus, we reduce to the local picture defining $D_{(x,\xi)}$ discussed in the last Remark 6.2.10. $\qquad\square$

We combine the above two results to deduce a generation result for a special case. Let $\mathcal{S}$ be a Whitney triangulation. For each stratum $s \in \mathcal{S}$, we pick a small ball B_s which centered at X_s and contained in $\mathrm{star}(s)$ such that $N^*_{\infty,out}B_s \cap N^*_{\infty}\mathcal{S} = \varnothing$ and consider $1_{B_s} \in \mathfrak{wsh}_{N^*_{\infty,out}\mathcal{S}}(M)$. This is possible because of the Whitney condition. Again different choices of such small balls induce the same objects in $\mathfrak{wsh}_{N^*_{\infty}\mathcal{S}}(M)$ since they are isotopic to each other in the base by isotopies respecting the stratification and the lifting isotopies on the microsupport won't touch $N^*_{\infty}(\mathcal{S})$. (See [39])

Proposition 6.2.12. *The set $\{1_{B_s}\}_{s \in \mathcal{S}}$ generates $\mathfrak{wsh}_{N^*_{\infty}\mathcal{S}}(M)$ under finite colimits and retractions.*

Proof. Set $\mathcal{S}_{\leq k} = \{s \in \mathcal{S} | \dim X_s \leq k\}$. We claim, when $k < n - 1$, $\{1_{B_s}\}_{s \in \mathcal{S}_{\leq k}}$ plus 1_B for any small ball B whose closure $\overline{B}$ is disjoint from any stratum of dimension $\leq k$ generates

$\mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq k}}(M)$. To see this, we note that the case $k = -1$ is Lemma 6.2.6. Assume the case for $k < n - 2$ and consider the projection $\mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq k}}(M) \twoheadrightarrow \mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq k-1}}(M)$. We note that $M \setminus \cup_{s \in \mathcal{S}_{k+1}} X_s$ is path connected by standard Hausdorff dimension theory since these X_s's have codimension ≥ 2. Thus, 1_B is independent of the choice of B. By Proposition 6.2.11, the fiber of the above projection is generated under finite colimits and retracts by sheaf-theoretic linking disks $D_{(x,\xi)}$ for $(x, \xi) \in N_\infty^* \mathcal{S}_{\leq k+1} \setminus N_\infty^* \mathcal{S}_{\leq k}$. But $D_{(x,\xi)}$ can be written as the cofiber $\mathrm{cof}(1_B \to 1_{B_s})$ by the local picture mentioned in Remark 6.2.10. Finally, apply a similar argument to the projection $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M) = \mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq n-1}}(M) \to \mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq n-2}}(M)$ implies the proposition. $\qquad \square$

Recall in the proof of Lemma 6.2.6, we show that $1_{B_s} \to 1_{\mathrm{star}(s)}$ is an isomorphism in $\mathfrak{wsh}_\varnothing(M)$. The later object is, however, not an object in $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M)$. Instead, we consider the object $1_{\mathrm{star}(s)^-}$ where $\mathrm{star}(s)^-$ is a small inward cornering of $\mathrm{star}(s)$ of Definition 3.3.9. Choose B_s small so that $B_s \subseteq \mathrm{star}(s)^-$. We claim that the canonical map $1_{B_s} \to 1_{\mathrm{star}(s)^-}$ is an isomorphism. By the Yoneda embedding, it is an isomorphism if the corresponding morphism $\mathrm{Hom}_w(-, 1_{B_s}) \to \mathrm{Hom}_w(-, 1_{\mathrm{star}(s)^-})$ is an isomorphism as presheaves on $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M)$. The following statements are in directly parallel with Proposition 5.18, Lemma 5.21, and Proposition 5.24 in [21].

Lemma 6.2.13. *For a $\mathcal{S}$-constructible relatively compact open set U, we have*

$$\mathrm{Hom}_w(1_{B_s}, 1_{U^-}) = \begin{cases} 1 & \mathrm{star}(s) \subseteq U \\ 0 & otherwise \end{cases}$$

Proof. The construction of U^- tautologically provides a cofinal sequence $1_{U^{-\epsilon}}$ so

$$\mathrm{Hom}_w(1_{B_s}, 1_{U^-}) = \operatorname*{colim}_{\epsilon \to 0} \mathrm{Hom}(1_{B_s}, 1_{U^{-\epsilon}}).$$

First consider the case when $\mathrm{star}(s) \subseteq U$. Since $B_s \subseteq \mathrm{star}(s)$ has a non-zero distance from ∂U, it is contained in $U^{-\epsilon}$ for $\epsilon << 1$ and the left hand side is 1. When $\mathrm{star}(s) \cap U = \varnothing$, the Hom is clearly 0 so we assume s is a stratum on the boundary of U. In this case, one can conclude the result by refining the wrapping to a family $U^{-\epsilon, -\delta}$ where we add the centered of the ball B_s to the stratification and δ denotes the parameter which corresponds to this new stratum. See [21, Proposition 2.10]. $\qquad \square$

Proposition 6.2.14. *The canonical map $1_{B_s} \to 1_{\mathrm{star}(s)^-}$ is an isomorphism in $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M)$.*

Proof. We proceed by induction on the codimension of s. When s has codimension zero, we may replacing M by $\mathrm{star}(s)$ and it becomes Lemma 6.2.6.

Now the previous lemma and the proposition 6.2.12 implies

$$\mathrm{Hom}(1_{B_s}, 1_{\mathrm{star}(t)^-}) = \mathrm{Hom}(1_{\mathrm{star}(s)^-}, 1_{\mathrm{star}(t)^-}) = 0$$

for t of strictly smaller codimension than s. By induction, $1_{B_t} \xrightarrow{\sim} 1_{\mathrm{star}(t)^-}$ for such t's. The later generates a subcategory which contains the fiber of the projection $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(T^* M) \to \mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq \dim s}}(T^* M)$. This implies that it is enough to show the isomorphism in the category $\mathfrak{wsh}_{N_\infty^* \mathcal{S}_{\leq \dim s}}(T^* M)$. This is a special case fo the following lemma applying to the case $Y = \mathrm{star}(s)^-$, $X = s \cap \mathrm{star}(s)^-$, and $Z = t \cap \mathrm{star}(s)^-$. $\qquad \square$

Lemma 6.2.15. *Let $X^m \subseteq Y^n$ be an inclusion of stable balls, with $\partial X \subseteq \partial Y$. Assume there exists another stable ball (with corners) $Z^{m+1} \subseteq Y^n$ such that ∂Z is the union of X with a smooth submanifold of ∂Y. Then the canonical map $1_{B_\epsilon(x)} \to 1_Y$ is an isomorphism in $\mathfrak{wsh}_{N_\infty^* X}(Y)$ for any $x \in X$.*

Proof. Reduce to the case of balls by stabilization. In this case, Y is a unit ball, X is the intersection of Y with a linear subspace, and Z is the intersection of Y with a closed half-plane with the boundary being the linear subspace. The positive isotopy which expands $1_{B_\epsilon(x)}$ to 1_Y is disjoint from $N_\infty^* X$. $\qquad\square$

Corollary 6.2.16. *The set $\{1_{\mathrm{star}(s)^-}\}_{s \in \mathbb{S}}$ generates $\mathfrak{wsh}_{N_\infty^* \mathbb{S}}(M)$ under finite colimits and retractions.*

Chapter 7

The comparison morphism

Let M be a real analytic manifold and $\Lambda \subseteq S^*M$ a subanalytic singular isotropic. We define in this section, by an abuse of notation, a *comparison* functor

$$\mathfrak{W}^+_\Lambda(M) : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to \mathrm{Sh}_\Lambda(M)^c$$

and show that it is an equivalence of category. Since such functors combine to a *comparison* morphism $\mathfrak{W}^+_\Lambda : \mathfrak{w}\mathrm{sh}_\Lambda \to \mathrm{Sh}^c_\Lambda$ between precosheaves, the last statement will implies that $\mathfrak{w}\mathrm{sh}_\Lambda$ is in particular a cosheaf for this case.

7.1 Definition

Let $\Lambda \subseteq S^*M$ be a closed singular isotropic subset. Recall from Proposition 1.0.4 that the inclusion $\mathrm{Sh}_\Lambda(M) \hookrightarrow \mathrm{Sh}(M)$ has a left adjoint given by the positive infinite wrapping functor

$$\mathfrak{W}^+_\Lambda(M) : \mathrm{Sh}(M) \to \mathrm{Sh}_\Lambda(M).$$

Geometrically, it takes a sheaf F to the limiting object over increasingly positive wrappings. Since a continuation map $c : F \to F^w$ tautologically becomes an isomorphism after applying $\mathfrak{W}^+_\Lambda(M)$, the functor $\mathfrak{W}^+_\Lambda(M)$ vanishes on $\mathcal{C}_\Lambda(M)$. We abuse the notation and denote the resulting functor on the quotient category also by

$$\mathfrak{W}^+_\Lambda(M) : \mathfrak{w}\mathrm{sh}_\Lambda(M) \to \mathrm{Sh}_\Lambda(M).$$

Remark 7.1.1. We note that in general when Λ is not a singular isotropic, the category on the right hand side is much larger. For example, when $\Lambda = S^*M$ and M is non-compact, $\mathfrak{W}^+_{S^*M}(M)$ is the trivial inclusion $\{0\} \hookrightarrow \mathrm{Sh}(M)$.

We first notice that in this case the restriction of $\mathfrak{W}^+_\Lambda(M)$ on $\mathfrak{w}\mathrm{sh}_\Lambda(M)$ takes image in the subcategory consisting of compact objects.

Lemma 7.1.2. *Let Λ be a subanalytic singular isotropic. For $F \in \mathfrak{w}\mathrm{sh}_\Lambda(M)$, the sheaf $\mathfrak{W}^+_\Lambda(M)(F)$ is a compact object.*

Proof. Let $F \in \mathfrak{wsh}_\Lambda(M)$ and $\varinjlim F_i$ be a filtered colimit in $\mathrm{Sh}_\Lambda(M)$. We compute,

$$\mathrm{Hom}(\mathfrak{W}_\Lambda^+(M)(F), \varinjlim F_i) = \operatorname*{colim}_{\Phi \in W(S^*M \backslash \Lambda)} \mathrm{Hom}(w(\Phi) \circ F, \varinjlim F_i)$$

$$= \operatorname*{colim}_{\Phi \in W(S^*M \backslash \Lambda)} \mathrm{Hom}(F, w(\Phi^{-1}) \circ \varinjlim F_i)$$

$$= \operatorname*{colim}_{\Phi \in W(S^*M \backslash \Lambda)} \mathrm{Hom}\left(F, \varinjlim(w(\Phi^{-1}) \circ F_i)\right)$$

$$= \operatorname*{colim}_{\Phi \in W(S^*M \backslash \Lambda)} \mathrm{Hom}(F, \varinjlim F_i) = \mathrm{Hom}(F, \varinjlim F_i).$$

For the last equality, we use the fact that Φ is supported away from $\Lambda \supseteq \mathrm{SS}^\infty(F_i)$ so $w(\Phi^{-1}) \circ F_i = F_i$ by Lemma 4.3.11. Now pick a Whitney triangulation $\mathcal{S}$ such that F is $\mathcal{S}$-constructible and $\Lambda \subseteq N^*\mathcal{S}$. In this case, the Hom can be computed in $\mathrm{Sh}_{N_\infty^*\mathcal{S}}(M) = \mathcal{S}\text{-Mod}$. Since $\mathrm{Sh}_{N_\infty^*\mathcal{S}}(M)^c$ consists exactly objects with compact support and perfect stalks, F is compact in $\mathrm{Sh}_{N_\infty^*\mathcal{S}}(M)$. Thus $\mathrm{Hom}(F, \varinjlim F_i) = \varinjlim \mathrm{Hom}(F, F_i)$ and a backward computation as above implies that

$$\mathrm{Hom}(\mathfrak{W}_\Lambda^+(M)(F), \varinjlim F_i) = \varinjlim \mathrm{Hom}(\mathfrak{W}_\Lambda^+(M)(F), F_i)$$

so $\mathfrak{W}_\Lambda^+(M)(F) \in \mathrm{Sh}_\Lambda(M)^c$ is compact. $\qquad \square$

We note that this map is compatible with the precosheaf structure on both side.

Lemma 7.1.3. *Let $j : U \subseteq M$ be an open set. The restriction $j^* : \mathrm{Sh}_\Lambda(M) \to \mathrm{Sh}_{\Lambda|_U}(U)$ has left and right adjoints which are given by $\mathfrak{W}_\Lambda^+ \circ j_!$ and $\mathfrak{W}_\Lambda^- \circ j_*$. Hence, taking left adjoint induces a functor $\mathfrak{W}_\Lambda^+ \circ j_! : \mathrm{Sh}_{\Lambda|_U}(U)^c \to \mathrm{Sh}_\Lambda(M)^c$ between compact objects.*

Proof. We use the fact that the left adjoint of a left adjoint preserves compact objects. $\quad \square$

Note when $\Omega \subseteq \Omega'$, there is equivalence $\mathfrak{W}^+(\Omega') \circ \mathfrak{W}^+(\Omega) = \mathfrak{W}^+(\Omega) \circ \mathfrak{W}^+(\Omega') = \mathfrak{W}^+(\Omega')$. Thus, by the above lemma, there is commuting diagram for an inclusion of opens $j : U \hookrightarrow V$:

$$
\begin{array}{ccc}
\mathfrak{wsh}_\Lambda(U) & \xrightarrow{\ \mathfrak{W}_{\Lambda|_U}^+(U)\ } & \mathrm{Sh}_{\Lambda|_U}(U)^c \\
{\scriptstyle j_!}\Big\downarrow & \quad {\scriptstyle \mathfrak{W}_{\Lambda|_V}^+(V) \circ j_!}\Big\downarrow & \\
\mathfrak{wsh}_\Lambda(V) & \xrightarrow[\ \mathfrak{W}_{\Lambda|_V}^+(V)\ ]{} & \mathrm{Sh}_{\Lambda|_V}(V)^c
\end{array}
$$

Definition 7.1.4. We will refer the morphism $\mathfrak{W}_\Lambda^+ : \mathfrak{wsh}_\Lambda \to \mathrm{Sh}_\Lambda^c$ between precosheaves defined by the above diagram as the *comparison* morphism.

Similarly, when $\Lambda \subseteq \Lambda'$, recall the left adjoint of the inclusion $\mathrm{Sh}_\Lambda(M) \hookrightarrow \mathrm{Sh}_{\Lambda'}(M)$ is given by $\mathfrak{W}_\Lambda^+(M)$ and thus there is a commuting diagram:

$$\begin{CD}
\mathfrak{wsh}_{\Lambda'}(M) @>\mathfrak{W}^+_{\Lambda'}(M)>> \mathrm{Sh}_{\Lambda'}(M)^c \\
@VVV @VV\mathfrak{W}^+_{\Lambda}(M)V \\
\mathfrak{wsh}_{\Lambda}(M) @>\mathfrak{W}^+_{\Lambda'}(M)>> \mathrm{Sh}_{\Lambda}(M)^c
\end{CD}$$

One can see this is compatible with the corestrictions on both side. Thus, there is a commuting diagram in precosheaves with coefficient in st_w:

$$\begin{CD}
\mathfrak{wsh}_{\Lambda'} @>\mathfrak{W}^+_{\Lambda}>> \mathrm{Sh}^c_{\Lambda'} \\
@VVV @VVV \\
\mathfrak{wsh}_{\Lambda} @>\mathfrak{W}^+_{\Lambda'}>> \mathrm{Sh}^c_{\Lambda}
\end{CD}.$$

The main theorem of this paper, Theorem 1.0.5, is that the comparison functor

$$\mathfrak{W}^+_{\Lambda}(M) : \mathfrak{wsh}_{\Lambda}(M) \to \mathrm{Sh}_{\Lambda}(M)$$

is an equivalence. As a corollary, the comparison morphism

$$\mathfrak{W}^+_{\Lambda} : \mathfrak{wsh}_{\Lambda} \to \mathrm{Sh}^c_{\Lambda}$$

is an isomorphism so, in particular, $\mathfrak{wsh}_{\Lambda}$ is a cosheaf since Sh^c_{Λ} is.

7.2 Sufficient condition for fully faithfulness

For the rest of the section, we work with a fixed pair (M, Λ) such that $\Lambda \subseteq S^*M$ is a subanalytic singular isotropic. We would like to study the effect of $\mathfrak{W}^+_{\Lambda}$ on the Hom. Since $\mathfrak{W}^+_{\Lambda}$ is defined by a colimit, the canonical $\mathrm{Hom}_w(G, F) \to \mathrm{Hom}(\mathfrak{W}^+_{\Lambda}G, \mathfrak{W}^+_{\Lambda}F)$ can be obtained from the following few steps. By definition of colimits, there is a canonical map *colimiting continuation map* $F^w \to \mathfrak{W}^+_{\Lambda}F$ for any wrapping w. This induces, for any other wrapping w', a map between the Hom's $\mathrm{Hom}(G^{w'}, F^w) \to \mathrm{Hom}(G^{w'}, \mathfrak{W}^+_{\Lambda}F)$. Since convoluting with $w(\Phi)$ is an auto-equivalence on $\mathrm{Sh}(M)$, there is a canonical map

$$\mathrm{Hom}(G, F^w) = \mathrm{Hom}(G^{w'}, (F^w)^{w'}) \to \mathrm{Hom}(G^{w'}, \mathfrak{W}^+_{\Lambda}(F)).$$

Take limit over w' and then colimit over w, we obtain the map between Hom's

$$\mathrm{Hom}_w(G, F) = \operatorname*{colim}_w \mathrm{Hom}(G, F^w) \to \varprojlim_{w'} \mathrm{Hom}(G^{w'}, \mathfrak{W}^+_{\Lambda}(F)) = \mathrm{Hom}(\mathfrak{W}^+_{\Lambda}(G), \mathfrak{W}^+_{\Lambda}(F)).$$

In short, we have the following lemma.

Lemma 7.2.1. *Running F and G through a set of generators of $\mathfrak{w}\mathrm{sh}_\Lambda(M)$. If the limiting continuation map $F \to \mathfrak{W}_\Lambda^+ F$ becomes an isomorphism after applying $\mathrm{Hom}(G, -)$ for all such G, then the canonical map on Hom $\mathrm{Hom}_w(G, F) \to \mathrm{Hom}(\mathfrak{W}_\Lambda^+(G), \mathfrak{W}_\Lambda^+(F))$ is an isomorphism for any $F, G \in \mathfrak{w}\mathrm{sh}_\Lambda(M)$.*

Pick any cofinal functor $\Psi : \mathbb{Z}_{\geq 0} \to W(S^*M \setminus \Lambda)$ which corresponds to a sequence of wrappings $\mathrm{id} \xrightarrow{\Psi_0} \Phi_1 \xrightarrow{\Psi_1} \Phi_2 \to \cdots$. For convenience, we scale it so that Ψ_i has domain $S^*M \times [i, i+1]$. This sequence of positive family of isotopies patches to a positive isotopy $\Psi : S^*M \times [0, \infty) \to S^*M$ whose restriction on $S^*M \times [i, i+1]$ is $\Psi_i \# \Phi_i$. Note that Ψ has a non-compact support by the cofinal criterion Lemma 4.3.9. By the GKS sheaf quantization, there is a sheaf kernel $K(\Psi)$ on $M \times M \times [0, \infty)$ such that $K(\Psi)|_{M \times M \times [i, i+1]} = K(\Psi_i)$. For $F \in \widetilde{\mathfrak{w}\mathrm{sh}_\Lambda}(M)$, we write $F^\Psi = K(\Psi) \circ F$ and let F^{w_n} denote $F^{w(\Phi_n)} = F^\Psi|_{M \times \{n\}}$ the resulting sheaves under the wrapping Ψ. it is enough to study the morphism

$$\mathrm{Hom}(G, F^{w_n}) \to \mathrm{Hom}(G, \mathfrak{W}_\Lambda^+(F))$$

which is induced from the sequence of wrappings

$$F \to F^{w_1} \to \cdots \to F^{w_n} \to \cdots \to \mathfrak{W}_\Lambda^+(F).$$

Definition 7.2.2. Let X be a topological space, $j : X \times \mathbb{R} \hookrightarrow X \times (-\infty, \infty]$ and $i : X \times \{\infty\} \to X \times (-\infty, \infty]$ be the inclusions as open and closed subset. We call the composition $\psi = i^* \circ j_* : \mathrm{Sh}(X \times (-\infty, \infty)) \to \mathrm{Sh}(X)$ the *nearby cycle* functor.

Lemma 7.2.3. *The colimit $\mathfrak{W}_\Lambda^+ F$ can be computed as the nearby cycle at infinity of the sheaf F^Ψ. That is, $\mathfrak{W}_\Lambda^+ F = \psi F^\Psi$.*

Proof. Since Ψ_n are cofinal, $\mathfrak{W}_\Lambda^+(F) = \underset{n \in \mathbb{N}}{\mathrm{colim}}\, F^{w_n}$. By the construction above, for each $n > 1$, F^{w_n} is given by $1_{\{n\}} \circ F^\Psi \xrightarrow{\sim} 1_{(0,n)}[1] \circ F^\Psi$ and the continuation map between them is induced by $1_{(0,n)} \to 1_{(0,m)}$ for $m \geq n$. Since convolution commutes with colimit, $\mathfrak{W}_\Lambda^+(F) = (\underset{n \in \mathbb{N}}{\mathrm{colim}}\, 1_{(0,n)}[1]) \circ F^\Psi = 1_{(0,\infty)}[1] \circ F^\Psi = p_! F^\Psi[1]$ where $p : M \times [0, \infty) \to M$ is the projection. The latter is the same as $\psi F^\Psi = i^* j_* F^\Psi$ by Lemma 4.1.1 because Ψ is a positive isotopy and so $\mathrm{SS}(F^\Psi) \subseteq \{\tau \leq 0\}$. $\qquad\square$

To study the (co)limiting continuation map $\mathrm{Hom}(G, F) \to \mathrm{Hom}(G, \psi F^\Psi)$, we use a similar trick as in Proposition 4.2.8 and consider the object $\mathcal{H}om(p^*G, F^\Psi)$ where we use $p : M \times \mathbb{R} \to M$ to denote the projection. This time, we have to study its behavior near the infinity.

Lemma 7.2.4. *Let $F, G \in \mathrm{Sh}(M \times \mathbb{R})$ be sheaves on $M \times \mathbb{R}$ such that F and $\mathcal{H}om(G, F)$ are $\mathbb{R}$-noncharacteristic, and q is proper on $\mathrm{supp}(G)$ and $\mathrm{supp}(\psi G)$ is compact. Then $\mathrm{Hom}(i_s^* G, i_s^* F)$ is contant on $s \in \mathbb{R}$ and equals to $\Gamma(M; \psi \mathcal{H}om(G, F))$*

Proof. The statement over $\mathbb{R}$ follows from Lemma 4.1.7. Since we assume $\mathrm{supp}(\psi(G))$ is proper, we can apply base change on the lager space $M \times (-\infty, \infty]$ to obtain the statement for $\Gamma(M; \psi \mathcal{H}om(G, F))$ since $*$-push of $(a, b) \hookrightarrow [a, b]$ sends constant sheaves to constant sheaves. $\qquad\square$

Since ∞ is a boundary point, we cannot conclude the equivalence using transversality. Such situation is considered by Nadler and Shende in [43] where they developed the theory of nearby cycle to study the canonical map

$$\Gamma\left(M; \psi\,\mathcal{H}om(G, F)\right) \to \operatorname{Hom}(\psi G, \psi F)$$

which we recall now.

Definition 7.2.5 ([43, Definition 2.2]). A closed subset $X \subseteq S^*M$ is *positively displaceable from legendrians* (pdfl) if given any Legendrian submanifold L (compact in a neighborhood of X), there is a 1-parameter positive family of Legendrians L_s, $s \in (-\epsilon, \epsilon)$ (constant outside a compact set), such that L_s is disjoint from X except at $s = 0$.

Definition 7.2.6 ([43, Definition 2.7]). Fix a co-oriented contact manifold $(\mathcal{V}, \xi)$ and positive contact isotopy η_s. For any subset $Y \subseteq \mathcal{V}$ we write $Y[s] := \eta_s(Y)$. Given $Y, Y' \subseteq \mathcal{V}$ we define the *chord length spectrum* of the pair to be the set lengths of Reeb trajectories from Y to Y':

$$cls(Y \to Y') = \{s \in \mathbb{R} | Y[s] \cap Y' \neq \varnothing\}$$

we term $cls(Y) := cl(Y \to Y)$ the chord length spectrum of Y.

Definition 7.2.7 ([43, Definition 2.9]). Given a parameterized family of pairs (Y_b, Y_b') in S^*M over $b \in B$ we say it is gapped if there is some interval $(0, \epsilon)$ uniformly avoided by all $cl(Y_b \to Y_b')$. In case $Y = Y'$, we simply say Y is gapped.

Definition 7.2.8 ([43, Definition 3.17]). Given a subset $X \subseteq T^*(M \times J)$, we define its *nearby subset* by

$$\psi(X) := \overline{\Pi(X)} \cap T^*(M \times (-1, \infty])|_{M \times \{\infty\}}.$$

The main theorem for the nearby cycles in [43] is the following:

Theorem 7.2.9 ([43, Theorem 4.2]). *Let F, G be sheaves on $M \times J$. Assume*

1. *$\operatorname{SS}(F)$ and $\operatorname{SS}(G)$ are J-noncharacteristic;*

2. *$\psi(\operatorname{SS}(F))$ and $\psi(\operatorname{SS}(G))$ are pdfl;*

3. *The family of pairs in S^*M determined by $(\operatorname{SS}_\pi(F), \operatorname{SS}_\pi(G))$ is gapped for some fixed contact form on S^*M.*

Then

$$\Gamma(M; \psi\,\mathcal{H}om(F, G)) \to \operatorname{Hom}(\psi(F), \psi(G))$$

is an isomorphism.

Now we apply the theory of nearby cycle to the infinite wrapping functor.

Lemma 7.2.10. *Let $F \to F^{w_1} \to F^{w_2} \to \cdots$ be a sequence in $\widetilde{\mathfrak{w}\mathrm{sh}}_\Lambda(M)$ as in Lemma 7.2.3. If for any conic open neighborhood $\mathcal{U}$ of Λ, there exists n such that $\operatorname{SS}(F^{w_n}) \subseteq \mathcal{U}$. Then the sequence is cofinal.*

Proof. Pick a decreasing conic open neighborhood $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \mathcal{U}_3 \supseteq \cdots \supseteq \Lambda$ of Λ such that $\overline{\mathcal{U}_n} \subseteq \mathcal{U}_{n+1}$ and $\cap_n \mathcal{U}_n = \Lambda$. By taking a subsequence of the F^{w_n}'s, we may assume $\mathrm{SS}(F^{w_n}) \subseteq \mathcal{U}_n$ for $n \geq 1$. Again by taking a subsequence of both the F^{w_n}'s and $\mathcal{U}_n$'s, we may further assume $\mathcal{U}_{n+1} \cap \mathrm{SS}(F^{w_n}) = \varnothing$. By the locality property 4.3.11, the continuation map $c : F^{w_n} \to F^{w_{n+1}}$ depends only on the value of Ψ on $\mathcal{U}_n$. Thus, we may modify Ψ on $T^*M \setminus \overline{\mathcal{U}_{n-1}}$ to satisfy the condition in Lemma 4.3.9. Since taking subsequence won't change the colimit, the original sequence is cofinal. $\qquad\square$

Theorem 7.2.11. *Let $F \in \widetilde{\mathfrak{wsh}}_\Lambda(M)$. Assume there is a sequence of wrappings*

$$\Phi_0 \xrightarrow{\Psi_0} \Phi_1 \to \cdots$$

which glues to a (non-compactly supported) positive contact isotopy $\Psi : S^*M \times [0,\infty) \to S^*M$ such that, for any neighborhood $\mathcal{U}$ of Λ, $\psi_s(\mathrm{SS}^\infty(F)) \subseteq \mathcal{U}$ for $s >> 0$. Then for $G \in \widetilde{\mathfrak{wsh}}_\Lambda(M)$ the canonical map,

$$\mathrm{Hom}(G, F^{w_n}) \to \mathrm{Hom}(G, \psi F^\Psi)$$

is an isomorphism for $n >> 0$. Thus, the canonical map

$$\mathrm{Hom}_w(G, F) \to \mathrm{Hom}(\mathfrak{W}_\Lambda^+ G, \mathfrak{W}_\Lambda^+ F)$$

is an isomorphism.

Proof. By Lemma 4.3.9, the sequence F^{w_n} is cofinal and $\mathfrak{W}_\Lambda^+ F$ is computed by $\operatorname*{colim}_{n \in \mathbb{N}} F^{w_n}$. Thus, the first statement implies that $\mathfrak{W}_\Lambda^+$ induces isomorphisms on the Hom by Lemma 7.2.1.

Now note that $\mathrm{SS}^\infty(G)$ in S^*M is compact since $\mathrm{supp}(G)$ is compact and the front projection $\pi_\infty : S^*M \to M$ is proper. Since a manifold is in particular a regular topological space, there exist open sets $\mathcal{U}$ and $\mathcal{V}$ containing Λ and $\mathrm{SS}^\infty(G)$ such that $\mathcal{U} \cap \mathcal{V} = \varnothing$. By restricting to $n >> 0$, we may assume $\psi_s(\mathrm{SS}^\infty(F)) \subseteq \mathcal{U}$ is thus disjoint from $\mathrm{SS}^\infty(G)$, which implies that $\mathcal{H}om(p^*G, F^\Psi)$ is J-noncharacteristic. Lemma 7.2.4 then implies that $\mathrm{Hom}(G, F^{w_n}) = \Gamma(M; \psi \mathcal{H}om(p^*G, F^\Psi))$.

So it is sufficient to check the conditions of Theorem 7.2.9 hold for the pair p^*G and F^Ψ. The set $\mathrm{SS}(p^*G) = \mathrm{SS}(G) \times 0_J$ is tautologically J-noncharacteristic. For F^Ψ, we recall that $\dot{\mathrm{SS}}(F^\Psi) = \Lambda_\Psi \circ F = \{(\Psi(x, \xi, t) | (x, \xi) \in \dot{\mathrm{SS}}(F), t \in [0, \infty)\}$ which implies that F^Ψ is J-noncharacteristic.

By picking a shrinking neighborhood $\mathcal{V}_n$ of Λ, we see that the nearby set $\psi(\mathrm{SS}^\infty(F^\Psi))$ is contained in Λ. One can pick a Whitney triangulation $\mathcal{S}$ such that $\Lambda \subseteq N_\infty^* \mathcal{S}$. Similarly, up to an isotopy, there exists a Whitney triangulation $\mathcal{T}$ such that $\mathrm{SS}^\infty(G) \subseteq N_\infty^* \mathcal{T}$. The singular isotropics $N_\infty^* \mathcal{S}$ and $N_\infty^* \mathcal{T}$ are pdfl by Lemma 6.2.8.

Now the same argument showing $\mathcal{H}om(p^*G, F^\Psi)$ is J-noncharacteristic implies that there is an $N \in \mathbb{N}$ such that $\psi_s(\mathrm{SS}^\infty(F)) \subseteq \mathcal{V}$ for $s \geq N$. Thus, when restricting to $M \times [N, \infty)$, p^*G and F^Ψ are microlocally disjoint and the gapped condition is tautologically satisfied. $\quad\square$

Corollary 7.2.12. *If there exists a generating set S in $\widetilde{\mathfrak{wsh}}_\Lambda(M)$ such that each $F \in S$ admits a sequence F^{w_n} constructed in the above manner such that $\mathrm{SS}^\infty(F^{w_n})$ is contained in arbitrary small neighborhood of Λ for n large, then the comparison functor*

$$\mathfrak{W}_\Lambda^+ : \mathfrak{wsh}_\Lambda(M) \hookrightarrow \mathrm{Sh}_\Lambda(M)^c$$

is fully faithful.

7.3 Proof of the main theorem

We first consider the special case when $\Lambda = N_\infty^* \mathcal{S}$ for some Whitney triangulation $\mathcal{S}$.

Theorem 7.3.1. *The comparison functor $\mathfrak{W}_{N_\infty^* \mathcal{S}}^+ : \mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M) \to \mathrm{Sh}_{N_\infty^* \mathcal{S}}(M)^c$ is an equivalence.*

Proof. By Proposition 6.2.6 and Proposition 6.2.14, $\mathfrak{wsh}_\mathcal{S}(M)$ has $\{1_{\mathrm{star}(s)^-}\}$ as a set of generators. Recall that $1_{\mathrm{star}(s)^-}$ is defined to be an unspecified inward cornering $\mathrm{star}(s)^{-\epsilon}$ for small enough ϵ. As mentioned in Definition 3.3.9, the construction of $\mathrm{star}(s)^{-\epsilon}$ is made so that $N_{\infty, out}^* \mathrm{star}(s)^{-\epsilon}$ is disjoint from $N_\infty^* \mathcal{S}$ and is contained in arbitrary small neighborhood of $N_\infty^* \mathcal{S}$ as $\epsilon \to 0$. Since $\mathrm{SS}(1_{\mathrm{star}(s)^{-\epsilon}}) = N_{\infty, out}^* (\mathrm{star}(s)^{-\epsilon})$, Theorem 7.2.11 applies and $\mathfrak{W}_{N_\infty^* \mathcal{S}}^+$ is fully faithful. We see from the same generators $1_{\mathrm{star}(s)^-}$ that $\mathfrak{W}_{N_\infty^* \mathcal{S}}^+$ is essential surjective since $\{1_{\mathrm{star}(s)}\}$ form a set of generators of $\mathrm{Sh}_{N_\infty^* \mathcal{S}}(M)^c$ by Proposition 3.3.10. $\qquad\square$

To prove the general case, we have to match a special class of objects on both sides. The following lemma is a sheaf-theoretic variant of the proof [21, Theorem 5.36].

Lemma 7.3.2. *Let Λ be a subanalytic singular isotropic and $(x, \xi) \in \Lambda$ be a smooth point. For any $F \in \mathrm{Sh}(M)$ such that $\mathrm{SS}(F)$ is contained in Λ near (x, ξ), there is an equivalence*

$$\mathrm{Hom}(\mathfrak{W}_\Lambda^+ D_{(x, \xi)}, F) = \mu_{(x, \xi)} F.$$

That is, the object $\mathfrak{W}_\Lambda^+ D_{(x, \xi)}$ co-represents $\mu_{(x, \xi)}$.

Proof. Recall $D_{(x, \xi)}$ is defined to be the cofiber of the canonical map $1_{f^{-1}(-\infty, -\epsilon)} \to 1_{f^{-1}(-\infty, \epsilon)}$ where f is a proper analytic function defined near x satisfying the following conditions: There exists an $\epsilon > 0$, so that f has only one Λ-critical point x over $f^{-1}[-\epsilon, \epsilon]$ with $f(x) = 0$, $df_x = \xi$ and $f^{-1}(-\infty, \epsilon)$ is relatively compact. Proposition 3.2.7 implies the function f defines a microstalk functor $\mu_{(x, \xi)}$, by

$$\mu_{(x, \xi)}(F) := \Gamma_{\{f \geq 0\}}(F)_x.$$

As in Remark 6.2.10, the local picture of the fibers $f^{-1}(\{t\})$ for $t \in [-\epsilon, \epsilon]$ is a hyperplane near x. Let Ψ denote any global extension of the wrapping $N_{\infty, out}^* f^{-1}(-\infty, t)$ for $t \in [-\epsilon, \epsilon]$. If we modify Ψ to Ψ_0 by multiplying a bump function supported near (x, ξ) on its Hamiltonian, the resulting wrapping $(1_{f^{-1}(-\infty, \epsilon)})^{\Psi_0}$ will appear as expanding $f^{-1}(-\infty, \epsilon)$ to some large open set where the expansion happens only near x in the ξ codirection. The cofiber

cof $\left(c(\Psi_0, 1_{f^{-1}(-\infty,\epsilon)})\right)$ can be seen as the cofiber induced by some small open neighborhood U and its open subset $U \cap 1_{f^{-1}(-\infty,\epsilon)}$.
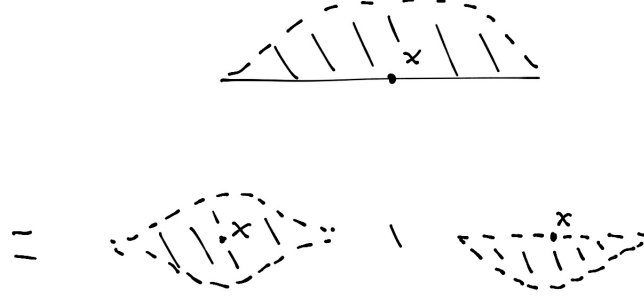
Since the graph Γ_{df} does not intersection Λ except at (x,ξ), there is an isomoprhism $\Gamma(f^{-1}(-\infty,\epsilon); F) = \Gamma(f^{-1}(-\infty,0); F)$ by the non-characteristic deformation lemma 3.1.13. Thus, $\Gamma_{\{f\geq 0\}}(F)_x$ can be computed as a colimit

$$\Gamma_{\{f\geq 0\}}(F)_x = \operatorname*{colim}_{\Psi_0} \operatorname{Hom}(\operatorname{cof}\left(1_{f^{-1}(-\infty,0)} \to (1_{f^{-1}(-\infty,0)})^{w(\Phi_0)}\right), F)$$

$$= \operatorname*{colim}_{\Psi_0} \operatorname{Hom}(\operatorname{cof}\left(1_{f^{-1}(-\infty,\epsilon)} \to (1_{f^{-1}(-\infty,0)})^{w(\Phi_0)}\right), F)$$

by picking Ψ_0 so that the corresponding U as above forms a neighborhood basis of x. Similarly, $\operatorname{Hom}((1_{f^{-1}(-\infty,0)})^w, F)$ can be replaced by $\Gamma(f^{-1}(-\infty,\epsilon); F)$ such that the maps are compatible with inclusions of the corresponding open sets. That is, we are taking colimit over a constant functor and thus

$$\Gamma_{\{f\geq 0\}}(F)_x = \operatorname*{colim}_{\Psi_0} \operatorname{Hom}(\operatorname{cof}\left(1_{f^{-1}(-\infty,\epsilon)} \to (1_{f^{-1}(-\infty,0)})^{w(\Phi_0)}\right), F)$$

$$= \operatorname*{colim}_{\Psi_0} \operatorname{Hom}(\operatorname{cof}(1_{f^{-1}(-\infty,-\epsilon)} \to 1_{f^{-1}(-\infty,\epsilon)}), F)$$

$$= \operatorname{Hom}(\operatorname{cof}(1_{f^{-1}(-\infty,-\epsilon)} \to 1_{f^{-1}(-\infty,\epsilon)}), F) = \operatorname{Hom}(D_{(x,\xi)}, F)$$

Finally, we recall that $\mathfrak{W}_\Lambda^+$ is defined by the restriction of the left adjoint of the tautological inclusion $\operatorname{Sh}_\Lambda(M) \hookrightarrow \operatorname{Sh}(M)$ on $\widetilde{\mathfrak{w}\operatorname{sh}}_\Lambda(M)$, and we conclude that $\Gamma_{\{f\geq 0\}}(F)_x = \operatorname{Hom}(\mathfrak{W}_\Lambda^+ D_{(x,\xi)}, F)$. $\qquad\square$

Remark 7.3.3. Corepresentatives of the microstalk functors $\mu_{(x,\xi)} : \operatorname{Sh}_\Lambda(M) \to \mathcal{V}$ are frequently considered since they often provide a preferred set of generators. For example, Zhou in [58] finds an explicit description of corepresentatives in the case of FLTZ skeleton first considered in [14], and uses it to match them with certain line bundles on the coherent side, which gives an explicit description to the equivalence proved in [33] through descent argument. A common recipe for finding such a description is to first find a sheaf F which is constructed locally near x, and is thus not necessarily in $\operatorname{Sh}_\Lambda(M)$, but still satisfies the identification $\operatorname{Hom}(F, -) = \mu_{(x,\xi)}$ on $\operatorname{Sh}_\Lambda(M)$. Then one constructs a one-parameter family of sheaves F_t, $t \in [0,1]$, such that $F_0 = F$, $F_1 \in \operatorname{Sh}_\Lambda(M)$, and $\operatorname{Hom}(F_t, -)$ remains constant as t varies. This lemma can be seen as an abstraction for such a construction when subanalytic structure is presented.

Proof of Theorem 1.0.5. Pick a Whitney triangulation $\mathcal{S}$ such that $\Lambda \subseteq N_\infty^* \mathcal{S}$. We use $D^w_{N_\infty^* \mathcal{S}, \Lambda}(M)$ denote to the subcategory in $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M)$ generated by the sheaf-theoretical linking discs $D_{(x, \xi)}$ at Legendrian points of $N_\infty^* \mathcal{S} \setminus \Lambda$ and, similarly, $D^\mu_{N_\infty^* \mathcal{S}, \Lambda}(M)$ the subcategory in $\mathrm{Sh}_{N_\infty^* \mathcal{S}}(M)$ generated by the corresponding microstalk representatives. By Proposition 3.4.10 and Proposition 6.2.11, they are the fiber of the projections $\mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M) \to \mathfrak{wsh}_\Lambda(M)$ and $\mathrm{Sh}_{N_\infty^* \mathcal{S}}(M)^c \to \mathrm{Sh}_\Lambda(M)^c$ respectively. Thus, there is a commuting diagram

$$
\begin{array}{ccccc}
D^w_{N_\infty^* \mathcal{S}, \Lambda}(M) & \longrightarrow & \mathfrak{wsh}_{N_\infty^* \mathcal{S}}(M) & \longrightarrow & \mathfrak{wsh}_\Lambda(M) \\
\big\uparrow & & \Big\| \, \mathfrak{W}^+_{N_\infty^* \mathcal{S}} & & \Big\downarrow \mathfrak{W}^+_\Lambda \\
D^\mu_{N_\infty^* \mathcal{S}, \Lambda}(M) & \longrightarrow & \mathrm{Sh}_{N_\infty^* \mathcal{S}}(M)^c & \longrightarrow & \mathrm{Sh}_\Lambda(M)^c
\end{array}
.
$$

The last lemma implies that the equivalence $\mathfrak{W}^+_{N_\infty^* \mathcal{S}} : \mathfrak{wsh}_{\mathcal{S}}(T^* M) \xrightarrow{\sim} \mathrm{Sh}_{\mathcal{S}}(M)^c$ restricts to

$$
\mathfrak{W}^+_{N_\infty^* \mathcal{S}} : D^w_{N_\infty^* \mathcal{S}, \Lambda}(M) \xrightarrow{\sim} D^\mu_{N_\infty^* \mathcal{S}, \Lambda}(M).
$$

Hence, Lemma 2.4.4 implies that $\mathfrak{W}^+_\Lambda$ is an equivalence as well. $\qquad\square$

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