UC Riverside
UC Riverside Electronic Theses and Dissertations

Title
Generalized Demazure Modules with Level Two Filtrations

Permalink
https://escholarship.org/uc/item/5026695r

Author
Davis, Justin Sean 'Tore

Publication Date
2020

Peer reviewed|Thesis/dissertation

eScholarship.org Powered by the California Digital Library
University of California

UNIVERSITY OF CALIFORNIA
RIVERSIDE

Generalized Demazure Modules with Level Two Filtrations

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Justin Sean Tore Davis

June 2020

Dissertation Committee:

Dr. Vyjayanthi Chari, Chairperson
Dr. Jacob Greenstein
Dr. Wee Liang Gan

Copyright by
Justin Sean Tore Davis
2020

The Dissertation of Justin Sean Tore Davis is approved:

Committee Chairperson

University of California, Riverside

Acknowledgments

I am extremely grateful to my advisor Dr. Vyjayanthi Chari, without whose guidance I would not be here. I would also like to thank all of my family and friends who have stuck by me through this whole journey, especially my wife for her long-lasting support, and all that she does raising our wonderful daughter and more.

To my daughter Virginia.

ABSTRACT OF THE DISSERTATION

Generalized Demazure Modules with Level Two Filtrations

by

Justin Sean Tore Davis

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, June 2020
Dr. Vyjayanthi Chari, Chairperson

We study generalized Demazure modules over the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$; or equivalently over the maximal standard parabolic subalgebra in the affine Lie algebra $\widehat{\mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra of Type B or C. More specifically, for a certain family of pairs of weights, we study generalized Demazure modules as submodules in a tensor product of level one Demazure modules. We give a presentation of these modules, and in the process give an algorithm for computing their graded characters in terms of level two Demazure modules.

Contents

Chapter 1

Introduction

Since their introduction in the seventies, classical Demazure modules have been studied extensively. One of the more important results is the Demazure character formula and the notion of Demazure operators, which generalizes the Weyl character formula.

More recently, so called generalized Demazure modules have been appearing in the literature. Although, the study of generalized Demazure modules is not very well-developed and in particular, no presentation or character formula is known in general for these modules. They show up as graded limits of tensor products of Kirillov-Reshetikhin modules and as graded limits of minimal affinizations as seen in [32]-[36], and as graded limits of prime modules in [4] and [7]. There is also a connection made with fusion products in [37]. These modules also show up in a geometric setting in [24] and more recently in [17], as well as other connected papers. There is rich combinatorics and geometry to uncover in studying these generalized Demazure modules.

In this paper we work entirely in the context of the current algebra $\mathfrak{g}[t]$ where $\mathfrak{g}$

is of type B_n or C_n and give a presentation, which in turn gives an algorithm to compute the graded character for a family of generalized Demazure modules. This work is inspired by the work first in [4] for type A_n, and in [7] for type D_n. Although the connection is not made in this paper to the work of Hernandez and Leclerc on monoidal categorification of cluster algebras introduced in [20, 21], we hope to make these connections in the future. These connections are explained in great detail in [3, 4] and in [7]. More recently in [1], the graded characters of HL-modules in type A_n are given explicitly in terms of Macdonald polynomials, and one hopes to make these connections in other types as well. There are difficulties which arise when trying to expand the family of generalized Demazure modules that we study; specifically those which involve a simple root with multiplicity two. However, we hope to develop techniques in the future to further understand the structure of these generalized Demazure modules.

In the conclusion, we explain how this is all related to a conjecture which makes a connection to the family of modules defined in [11] indexed by $|R^+|$-tuples of partitions.

Chapter 2

Generalized Demazure Modules

In this chapter we set up the basic notation and define the Demazure and generalized Demazure modules associated with an integrable highest weight representation of an untwisted affine Lie algebra.

For this manuscript, $\mathfrak{g}$ will be a simple finite–dimensional complex Lie algebra of type B_n or C_n and $\mathfrak{h}$ will be a fixed Cartan subalgebra of $\mathfrak{g}$. Let R be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$ and hence also a symmetric non–degenerate form $(\ ,\ )$ on $\mathfrak{h}^*$. We shall assume that this form on $\mathfrak{h}^*$ is normalized so that the square length of a long root is two. For $\alpha \in R$, let $t_\alpha \in \mathfrak{h}$ be the element that maps to α and set,

$$d_\alpha = \frac{2}{(\alpha,\alpha)}, \qquad h_\alpha = d_\alpha t_\alpha.$$

Fix a set of simple roots $\{\alpha_i : 1 \le i \le n\}$ where α_n is the spin node and fix a set of fundamental weights $\{\omega_i : 1 \le i \le n\}$ satisfying $\omega_i(h_{\alpha_j}) := (\omega_i, d_j\alpha_j) = \delta_{i,j}$. It will be convenient to set $\omega_0 = 0 = \omega_{n+1}$. As usual we let Q and P be the integer span of the simple roots and fundamental weights respectively; the sets Q^+ and P^+ are defined in the obvious way and we set $R^+ = R \cap Q^+$. For $\alpha = \sum_{j=1}^n m_j\alpha_j \in R^+$ let $\operatorname{ht}\alpha = \sum_{j=1}^n m_j$. For $1 \le i \le n$, we write $x_i^\pm, h_i, d_i$ for $x_{\alpha_i}^\pm, h_{\alpha_i}, d_{\alpha_i}$. Set

$$\alpha_{i,j} = \alpha_i + \cdots + \alpha_j, \quad 1 \le i \le j \le n,$$

$$h_{i,j} = h_{\alpha_{i,j}}, \quad d_{i,j} = d_{\alpha_{i,j}}$$

We shall adopt the convention that $\alpha_{i,j} = h_{i,j} = 0$ if $i > j$. Define a partial order on P by $\lambda \le \mu$ iff $\mu - \lambda \in Q^+$. Finally, let $\{x_\alpha^\pm, h_i : \alpha \in R^+, 1 \le i \le n\}$ be a Chevalley basis of $\mathfrak{g}$ and set $x_{\alpha_i}^\pm = x_i^\pm$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the corresponding triangular decomposition.

2.1 Affine Lie Algebras

Let $\widehat{\mathfrak{g}}$ be the untwisted affine Lie algebra associated to $\mathfrak{g}$ and recall that it has the following explicit realization. Given an indeterminate t, let $\mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials and set

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Define the commutator on $\widehat{\mathfrak{g}}$ by requiring c to be central, and

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r+s,0}\kappa(x, y)c, \quad [d, x \otimes t^r] = r(x \otimes t^r), \quad x, y \in \mathfrak{g}, \quad r, s \in \mathbb{Z}.$$

Then

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is a Cartan subalgebra of $\widehat{\mathfrak{g}}$ and we extend an element $\lambda \in \mathfrak{h}^*$ to an element of $\widehat{\mathfrak{h}}^*$ by setting $\lambda(c) = 0 = \lambda(d)$. Define elements $\Lambda_0, \delta, \alpha_0 \in \widehat{\mathfrak{h}}^*$ by

$$\Lambda_0(\mathfrak{h} \oplus \mathbb{C}d) = 0, \quad \Lambda_0(c) = 1, \quad \delta(\mathfrak{h} \oplus \mathbb{C}c) = 0, \quad \delta(d) = 1, \quad \alpha_0 = -\theta + \delta.$$

The set $\{\alpha_i : 0 \le i \le n\}$ is a set of simple roots for the pair $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$ and the corresponding set of affine fundamental weights are denoted by $\{\Lambda_i : 0 \le i \le n\}$. Let $\widehat{W}$ be the affine Weyl group and note that it contains an isomorphic copy of W. Define the affine root lattice $\widehat{Q}$ and affine weight lattice $\widehat{P}$ and the subsets $\widehat{Q}^+$, $\widehat{P}^+$ and the set of affine positive roots in the natural way. Let

$$\widehat{\mathfrak{b}} = \widehat{\mathfrak{h}} \oplus (\mathfrak{n}^+ \otimes 1) \oplus (\mathfrak{g} \otimes t\mathbb{C}[t]), \quad \widehat{\mathfrak{p}} = (\mathfrak{n}^- \otimes 1) \oplus \widehat{\mathfrak{b}}$$

be the Borel subalgebra and the standard maximal parabolic subalgebra of $\widehat{\mathfrak{g}}$ with respect to $\widehat{R}^+$. Let $\widehat{\mathfrak{n}}^+$ be the commutator subalgebra of $\widehat{\mathfrak{b}}$ and note that the commutator subalgebra of $\widehat{\mathfrak{p}}$ is just the current algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ associated to $\mathfrak{g}$. The action of d induces a canonical $\mathbb{Z}_+$- grading on $\mathfrak{g}[t]$ and on its universal enveloping algebra with the grade of an element $x \otimes t^r \in \mathfrak{g}[t]$ being r.

2.2 Generalized Demazure Modules

Given $\Lambda \in \widehat{P}^+$ let $\widehat{V}(\Lambda)$ be the integrable irreducible $\widehat{\mathfrak{g}}$-module generated by an element $\widehat{v}_\Lambda$ with relations

$$\widehat{\mathfrak{n}}^+ \widehat{v}_\Lambda = 0, \quad h\widehat{v}_\Lambda = \Lambda(h)\widehat{v}_\Lambda, \quad h \in \widehat{\mathfrak{h}},$$

$$(x_i^- \otimes 1)^{\Lambda(h_i)+1}\widehat{v}_\Lambda = 0, \quad 1 \le i \le n, \quad (x_\theta^+ \otimes t^{-1})^{\Lambda(h_0)+1}\widehat{v}_\Lambda = 0.$$

We have

$$\widehat{V}(\Lambda) = \bigoplus_{\Lambda' \in \widehat{P}} \widehat{V}(\Lambda)_{\Lambda'}, \quad \widehat{V}(\Lambda)_{\Lambda'} = \{v \in \widehat{V}(\Lambda) : hv = \Lambda'(h)v, \quad h \in \widehat{\mathfrak{h}}\}.$$

It is known that

$$\dim \widehat{V}(\Lambda)_{w\Lambda} = 1, \quad \text{for all } w \in \widehat{W},$$

and we let $\widehat{V}_w(\Lambda)$ be the $\widehat{\mathfrak{b}}$-module generated by $\widehat{V}(\Lambda)_{w\Lambda}$. The modules $\widehat{V}_w(\Lambda)$ are finite-dimensional and are called the Demazure modules associated to Λ. Moreover $\widehat{v}_\Lambda \in \widehat{V}_w(\Lambda)$ for all $w \in \widehat{W}$. The following is now immediate.

Lemma 1. *Suppose that $w, w' \in \widehat{W}$ and $\Lambda \in \widehat{P}^+$. Then*

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{b}}}(\widehat{V}_w(\Lambda), \widehat{V}_{w'}(\Lambda)) \leq 1,$$

and any non-zero element of this space is injective. □

Given $\Lambda, \Lambda' \in \widehat{P}^+$ and $w, w' \in \widehat{W}$ we set

$$\widehat{V}_{w,w'}(\Lambda, \Lambda') = \mathbf{U}(\widehat{\mathfrak{b}})(\widehat{V}(\Lambda)_{w\Lambda} \otimes \widehat{V}(\Lambda')_{w'\Lambda'}).$$

These (along with an obvious extension to an arbitrary number of dominant integral weights and Weyl group elements) are called the generalized Demazure modules. Furthermore, since

$$\dim \operatorname{Hom}_{\widehat{\mathfrak{g}}}(\widehat{V}(\Lambda) \otimes \widehat{V}(\Lambda'),\ \widehat{V}(\Lambda + \Lambda')) = 1$$

it is easy to see that

$$\widehat{V}(\Lambda)_{w\Lambda} \otimes \widehat{V}(\Lambda')_{w\Lambda'} \cong \widehat{V}(\Lambda + \Lambda')_{w(\Lambda+\Lambda')} \ \text{ and so } \ \widehat{V}_{w,w}(\Lambda, \Lambda') \cong \widehat{V}_w(\Lambda + \Lambda').$$

In this manuscript we shall be interested in certain special families of Demazure and generalized Demazure modules; namely those associated to pairs (w, Λ) such that the restriction of $w\Lambda$ to $\mathfrak{h}$ is in $(-P^+)$. This restriction guarantees that $\widehat{V}_w(\Lambda)$ is stable under $\widehat{\mathfrak{p}}^+$. Equivalently, we can (and do) regard these modules as $\mathbb{Z}$-graded modules for $\mathfrak{g}[t]$ and call them the stable Demazure modules. We recall a result proved independently in [23, 26].

Theorem 2. *Let $\Lambda, \Lambda' \in \widehat{P}^+$ and $w, w' \in \widehat{W}$. If $\sigma \in \widehat{W}$ is such that $\sigma(w\Lambda + w'\Lambda') \in \widehat{P}^+$, then there exists a projection of $\widehat{\mathfrak{g}}$-modules*

$$\widehat{V}(\Lambda) \otimes \widehat{V}(\Lambda') \longrightarrow \widehat{V}(\sigma(w\Lambda + w'\Lambda')) \to 0,$$

which induces an isomorphism of the one-dimensional spaces

$$\widehat{V}(\Lambda)_{w\Lambda} \otimes \widehat{V}(\Lambda')_{w'\Lambda'} \cong \widehat{V}(\sigma(w\Lambda + w'\Lambda'))_{w\Lambda + w'\Lambda'}.$$

□

The following corollary is immediate from the preceding theorem and is needed later in this section.

Corollary 3. *Assume that $\widehat{V}_w(\Lambda)$ and $\widehat{V}_{w'}(\Lambda')$ are stable Demazure modules. Then $\widehat{V}_{\sigma^{-1}}(\sigma(w\Lambda + w'\Lambda'))$ is also a stable Demazure module and there exists a surjective map of $\widehat{\mathfrak{p}}$-modules*

$$\widehat{V}_{w,w'}(\Lambda, \Lambda') \to \widehat{V}_{\sigma^{-1}}(\sigma(w\Lambda + w'\Lambda')) \to 0.$$

□

2.2.1 Over the Current Algebra

In the language of $\mathfrak{g}[t]$-modules developed above, a generalized Demazure module corresponds to considering a tensor product $\tau_s^* D(\ell, \lambda) \otimes \tau_{s'}^* D(\ell', \lambda')$ and taking the $\mathfrak{g}[t]$-module through $w_\lambda \otimes w_{\lambda'}$. Presentations of generalized Demazure modules have not been much studied although there is some work on the subject. They play an important role in the study of classical limits of representations of quantum affine algebras and first arose in this context in [34, 35] (see [37] for other examples). In this manuscript we shall give a presentation of a particular family of generalized Demazure modules, namely those of the form

$$D(\lambda, \mu) := \mathbf{U}(\mathfrak{g}[t])(w_\lambda \otimes w_\mu) \subset D(1, \lambda) \otimes D(1, \mu),$$

with some restrictions on the pair $(\lambda, \mu) \in P^+ \times P^+$. We note the following reformulation of Corollary 3.

Lemma 4. *There exists a (unique up to scalars) map* $\eta_{\lambda,\mu} : D(\lambda, \mu) \to D(2, \lambda + \mu) \to 0$, *of* $\mathfrak{g}[t]$*-modules extending the assignment* $w_\lambda \otimes w_\mu \to w_{2,\lambda+\mu}$. □

2.2.2 Stable Demazure Modules

Let $w_\circ$ be the longest element in W. Suppose that (w, Λ) and (w', Λ') are such that $\Lambda(c) = \Lambda'(c) = \ell$ and $\lambda = w_\circ w \Lambda|_{\mathfrak{h}} = w_\circ w' \Lambda'|_{\mathfrak{h}} \in P^+$. Then it is known [16, 32] that $V_w(\Lambda)$ is isomorphic to $V_{w'}(\Lambda')$ as $\mathfrak{g}[t]$-modules up to an overall shift of the action of d. In particular we can regard the stable Demazure modules as being indexed by a pair $(\ell, \lambda) \in \mathbb{N} \times P^+$.

In what follows we prefer to regard the stable (generalized) Demazure modules as $\mathbb{Z}$-graded modules for the current algebra of $\mathfrak{g}[t]$, where the grading is given by the action of d. The maps between graded modules will be of degree zero. Given $s \in \mathbb{Z}$ and a graded $\mathfrak{g}[t]$-module we denote by $\tau_s^* V$ the graded $\mathfrak{g}[t]$–module obtained by shifting the grades by s.

We recall a presentation of the stable Demazure modules given in [11]. Let $D(\ell, \lambda)$ be the $\mathfrak{g}[t]$-module generated by an element $w_{\ell,\lambda}$ with the following defining relations: for $1 \le i \le n$, $\alpha \in R^+$, $h \in \mathfrak{h}$, we have

$$\mathfrak{n}^+[t] w_{\ell,\lambda} = 0, \quad (h \otimes t^r) w_{\ell,\lambda} = \delta_{r,0} \lambda(h) w_{\ell,\lambda}, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1} w_{\ell,\lambda} = 0, \tag{2.2.1}$$

$$(x_\alpha^- \otimes t^{s_\alpha}) w_{\ell,\lambda} = 0, \quad (x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha + 1} w_{\ell,\lambda} = 0, \quad m_\alpha < \ell d_\alpha \tag{2.2.2}$$

where we write $\lambda(h_\alpha) = \ell d_\alpha(s_\alpha - 1) + m_\alpha$ with $s_\alpha \in \mathbb{N}$ and $0 < m_\alpha \leq \ell d_\alpha$. The $\mathbb{Z}$-grading on $D(\ell, \lambda)$ is given by requiring w_λ to have grade zero.

In the case $\ell = 1, 2$ the relations can be further streamlined. The following was proved when $\ell = 1$ in [10] and when $\ell = 2$ in [4].

Lemma 5. *If $\ell d_\alpha = 1$, the relations in (2.2.2) are a consequence of the relations in (2.2.1). If $\ell d_\alpha = 2$ then the second relation in (2.2.2) is a consequence of the others.* $\square$

Chapter 3

The Type B_n Case

3.1 Type B_n

In this section we assume that $\mathfrak{g}$ is a simple Lie algebra of type B_n. Recall that α_i, $1 \leq i \leq n$ denote the simple roots. Set

$$\alpha_{i,j} = \alpha_1 + \cdots + \alpha_j, \quad 1 \leq i \leq j \leq n$$

$$\beta_{i,j} = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_n), \quad 1 \leq i < j \leq n,$$

and note that

$$R^+ = \{\alpha_{i,j} : 1 \leq i \leq j \leq n\} \sqcup \{\beta_{i,j} : 1 \leq i < j \leq n\}.$$

Furthermore, for $\alpha \in R^+$ we have that $d_\alpha = 2$ if and only if $\alpha = \alpha_{i,n}$, $1 \leq i \leq n$, otherwise $d_\alpha = 1$. For computations we often use

$$\lambda(h_{i,j}) = \lambda(h_i) + \cdots + \lambda(h_j) \quad 1 \leq i \leq j < n$$

$$\lambda(h_{\beta_{i,j}}) = \lambda(h_i) + \cdots + \lambda(h_{j-1}) + 2(\lambda_j + \cdots + \lambda(h_{n-1})) + \lambda(h_n)$$

$$\lambda(h_{i,n}) = 2\lambda(h_i) + \cdots + 2\lambda(h_{i-1}) + \lambda(h_n) \quad 1 \le i < n$$

3.1.1 Interlacing Weights

Set

$$P^+(1) = \{\lambda \in P^+ : \lambda(h_i) \le d_i, \quad 1 \le i \le n\}.$$

Any $\lambda \in P^+(1)$ can be written uniquely (up to order) as a sum $\lambda = \lambda_1 + \lambda_2$ where $\lambda_s \in P^+(1)$, $s = 1, 2$ and satisfy the following: for $1 \le i \le j \le n$

$$\lambda_r(h_i) = 1 = \lambda_r(h_j) \implies \lambda_p(h_s) = 1 \text{ for some } \; i < s < j, \quad \{r, p\} = \{1, 2\},$$

and

$$\lambda(h_n) > 0 \implies \lambda_p(h_n) = 0 \;\text{ for some }\; p \in \{1, 2\}.$$

We call $(\lambda_1, \lambda_2) \in P^+ \times P^+$ an *interlacing* pair if $\lambda_1 + \lambda_2 \in P^+(1)$ and the preceding two conditions hold.

Examples. The pairs $(\omega_i, 0)$ for $0 \le i \le n$, $(2\omega_n, 0)$ and the elements of the set $\{(\omega_i, \omega_j) : 0 \le i \ne j \le n\}$ are interlacing. The pair $(\omega_3 + 2\omega_6, \omega_2 + \omega_5)$ is interlacing in B_6 but not in B_7.

Remark 6. *With our conventions, we always have* $\lambda_2(h_n) = 0$ *which implies that* $\lambda_2(h_{i,n}) \equiv 0 \mod 2$ *for all* $1 \le i \le n$

Throughout the rest of this section we shall assume without further mention that (λ_1, λ_2) *is an interlacing pair and* $\lambda = \lambda_1 + \lambda_2$. *We shall also assume without loss of generality that there exists* p *maximal with* $\lambda(h_{p+1}) \ne 0$ *with* $p \le n-1$ *and* $\lambda_1(h_{p+1}) = \lambda(h_{p+1})$.

Next we have a technical result on the properties of interlacing pairs that will be useful later.

Lemma 7. *For all* $1 \le i \le j \le n$ *and* (λ_1, λ_2) *interlacing, we have*

$$|(\lambda_1 - \lambda_2)(h_{i,j})| \le d_{i,j} \tag{3.1.1}$$

and hence $|(\lambda_1 - \lambda_2)(h_\alpha)| \le 2 \text{ for all } \alpha \in R^+$. *Further if* $\lambda(h_n) = 1$ *then*

$$|(\lambda_1 - \lambda_2)(h_\alpha)| \le 1 \text{ for all } \alpha \in R^+.$$

Proof. We prove the inequality (3.1.1) by induction on $j - i$; Since $\lambda \in P^+(1)$, induction begins at $i = j$. For $j > i$ the inductive step is clear if $\lambda(h_i) = 0$ or $\lambda(h_j) = 0$. If $\lambda(h_i) = 1 = \lambda(h_j)$ then either $\lambda(h_r) = 0$ for all $i < r < j$ or there exists $i < s < j$ minimal such that $\lambda_2(h_s) = 1$. In the first case we have proved that $(\lambda_1 - \lambda_2)(h_{i,j}) = 0$ and in the second case we have proved that $(\lambda_1 - \lambda_2)(h_{i,j}) = (\lambda_1 - \lambda_2)(h_{s+1,j})$ and the inductive step follows. For $\alpha_{i,n}$, we induct on $n - i$. Again since $\lambda \in P^+(1)$ we have $\lambda(h_n) \le 2$ and induction begins at $i = n$. For $i < n$ if $\lambda(h_i) = 0$ then the inductive step is obvious. If $\lambda(h_i) = 1$ then either $\lambda(h_r) = 0$ for all $i < r < n$ or there exists $i < s < n$ minimal such that $\lambda(h_s) = 1$. In the first case, we have $|(\lambda_1 - \lambda_2)(h_{i,n})| \le 1$ since $\lambda(h_n) \le 2$ and (λ_1, λ_2) are interlacing. In the second case we have $(\lambda_1 - \lambda_2)(h_{i,n}) = (\lambda_1 - \lambda_2)(h_{s+1,n})$ and the inductive step follows.

Since $\beta_{i,j} = \alpha_{i,n} + \alpha_{j,n}$ for $1 \le i < j \le n$ it follows that

$$|(\lambda_1 - \lambda_2)(h_{\beta_{i,j}})| \le 2 \text{ for all } 1 \le i < j \le n$$

since

$$|(\lambda_1 - \lambda_2)(h_{\beta_{i,j}})| = \left|\tfrac{1}{2}(\lambda_1 - \lambda_2)(h_{i,n}) + \tfrac{1}{2}(\lambda_1 - \lambda_2)(h_{j,n})\right| \le 2$$

Moreover, if $\lambda(h_n) = 1$ our assumptions imply that $\lambda_1(h_n) = 1$ and $\lambda_2(h_n) = 0$ and so using (3.1.1) we have

$$-1 \leq (\lambda_1 - \lambda_2)(h_{k,n-1}) \leq 0$$

$$\Longrightarrow \ (\lambda_1 - \lambda_2)(h_{k,n}) = 2(\lambda_1 - \lambda_2)(h_{k,n-1}) + 1 = \pm 1, \ 1 \leq k \leq n-1$$

Hence we also have

$$|(\lambda_1 - \lambda_2)(h_{\beta_{i,j}})| = \tfrac{1}{2}|(\lambda_1 - \lambda_2)(h_{i,n}) + (\lambda_1 - \lambda_2)(h_{j,n})| \leq 1$$

and the proof of the lemma is complete. $\square$

3.1.2 The Modules $V(\lambda_1 + \nu, \lambda_2 + \nu)$

Let $\lambda \in P^+(1)$ with (λ_1, λ_2) interlacing, and let $\nu \in P^+$ such that $\nu(h_n) = 0$. Define $V(\lambda_1+\nu, \lambda_2+\nu)$ to be the $\mathfrak{g}[t]$-module generated by an element $w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfying the following relations:

$$\mathfrak{n}^+[t]w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad (h \otimes t^r)w_{\lambda_1+\nu,\lambda_2+\nu} = \delta_{r,0}(\lambda + 2\nu)(h)w_{\lambda_1+\nu,\lambda_2+\nu}, \tag{3.1.2}$$

$$(x_i^- \otimes 1)^{(\lambda+2\nu)(h_i)+1}w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \tag{3.1.3}$$

$$(x_\alpha^- \otimes t^{\max\{r_{1,\alpha}, r_{2,\alpha}\}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad \alpha \in R^+ \tag{3.1.4}$$

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha+1}w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad \alpha \in R^+, \ d_\alpha = 2, \ m_\alpha < 4 \tag{3.1.5}$$

where $1 \leq i \leq n$ and $h \in \mathfrak{h}$, and $r_{j,\alpha} = \left\lceil \frac{(\lambda_j+\nu)(h_\alpha)}{d_\alpha} \right\rceil$, $j \in \{1, 2\}$. Recall that s_α and m_α are the unique positive integers such that $(\lambda+2\nu)(h_\alpha) = 2d_\alpha(s_\alpha-1)+m_\alpha$, with $0 < m_\alpha \leq 2d_\alpha$.

Define a grading on $V(\lambda_1+\nu,\lambda_2+\nu)$ by requiring the grade of $w_{\lambda_1+\nu,\lambda_2+\nu}$ to be zero. Clearly $V(\lambda_1+\nu,\lambda_2+\nu)$ is a graded quotient of $W_{\mathrm{loc}}(\lambda+2\nu)$ and $V(\lambda_1+\nu,\lambda_2+\nu)\cong V(\lambda_2+\nu,\lambda_1+\nu)$. Lemma 5 shows that if $\lambda_2+\nu=0$ then $V(\lambda_1+\nu,0)\cong D(1,\lambda_1+\nu)$. Furthermore, it is easy to check that Lemma 7 implies that $|r_{1,\alpha}-r_{2,\alpha}|\leq 1$ for all $\alpha\in R^+$ with $d_\alpha=2$, and $|r_{1,\alpha}-r_{2,\alpha}|\leq 2$ in general for all $\alpha\in R^+$.

The following is a consequence of equations (2.2.1)-(2.2.2) and Lemma 4.

Lemma 8. *The assignments $w_{\lambda_1+\nu,\lambda_2+\nu}\mapsto w_{2,\lambda+2\nu}$ and $w_{\lambda_1+\nu,\lambda_2+\nu}\mapsto w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu}$ define surjective maps of graded $\mathfrak{g}[t]$-modules*

$$\psi_{\lambda_1+\nu,\lambda_2+\nu}: V(\lambda_1+\nu,\lambda_2+\nu)\to D(2,\lambda+2\nu),\ \ \varphi_{\lambda_1+\nu,\lambda_2+\nu}: V(\lambda_1+\nu,\lambda_2+\nu)\to D(\lambda_1+\nu,\lambda_2+\nu),$$

and $\psi_{\lambda_1+\nu,\lambda_2+\nu}=\eta_{\lambda_1+\nu,\lambda_2+\nu}\circ\varphi_{\lambda_1+\nu,\lambda_2+\nu}$. □

Proof. The map $\psi_{\lambda_1+\nu,\lambda_2+\nu}$ is obvious. For $\varphi_{\lambda_1+\nu,\lambda_2+\nu}$, note that $(x_\alpha^-\otimes t^k)w_{\lambda_j+\nu}=0$ for $k\geq r_{j,\alpha},\ \alpha\in R^+$ and each $j\in\{1,2\}$. Then relation (3.1.4) follows i.e.

$$(x_\alpha^-\otimes t^{\max\{r_{1,\alpha},r_{2,\alpha}\}})(w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu})=0,\ \alpha\in R^+.$$

To show that relation (3.1.5) holds on $w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu}$, we note that

$$(x_\alpha^-\otimes t^{s_\alpha-1})^{m_\alpha+1}(w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu})=\sum_{k=0}^{m_\alpha+1}(x_\alpha^-\otimes t^{s_\alpha-1})^k w_{\lambda_1+\nu}\otimes(x_\alpha^-\otimes t^{s_\alpha-1})^{m_\alpha+1-k}w_{\lambda_2+\nu}$$

We will show that each term in the sum above is actually 0.

If $(\lambda_1+\nu)(h_\alpha)=0=(\lambda_2+\nu)(h_\alpha)$ then there is nothing to check. So suppose that $(\lambda_1+\nu)(h_\alpha)\neq 0$ or $(\lambda_2+\nu)(h_\alpha)\neq 0$. By the convention on (λ_1,λ_2) we have that $\lambda_1(h_\alpha)\geq\lambda_2(h_\alpha)$. This together with the fact that $\nu\in P^+$ is such that $\nu(h_n)=0$ implies that $\lambda_2(h_{i,n})$ and $\nu(h_{i,n})$ are both even for all i. So if $r_{1,\alpha}=r_{2,\alpha}$, i.e.,

$$\left\lceil\frac{(\lambda_1+\nu)(h_\alpha)}{2}\right\rceil=\left\lceil\frac{(\lambda_2+\nu)(h_\alpha)}{2}\right\rceil=\frac{(\lambda_2+\nu)(h_\alpha)}{2}.$$

This implies that $\lambda_1(h_{i,n}) = \lambda_2(h_{i,n})$, and $\lambda(h_{i,n})$ is even as well. So when we write $(\lambda + 2\nu)(h_\alpha) = 4(s_\alpha - 1) + m_\alpha$, we can see that $m_\alpha = 4$, so there is nothing to check.

Now suppose that $r_{1,\alpha} = r_{2,\alpha} + 1$. Then $r_{2,\alpha} = s_\alpha - 1$, then

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^{m_\alpha+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = (x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1} w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}$$

and (2.2.2) gives $(x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1} w_{\lambda_1+\nu} = 0$ in $D(1, \lambda_1 + \nu)$. So the relation holds.

$\square$

3.1.3 Statement of Main Theorem

The following is the main result of this section.

Theorem 9. *Let $(\lambda_1, \lambda_2) \in P^+ \times P^+$ be an interlacing pair. For all $\nu \in P^+$ with $\nu(h_n) = 0$, the map*

$$\varphi_{\lambda_1+\nu, \lambda_2+\nu} : V(\lambda_1 + \nu, \lambda_2 + \nu) \to D(\lambda_1 + \nu, \lambda_2 + \nu),$$

is an isomorphism. Moreover, $D(\lambda_1 + \nu, \lambda_2 + \nu)$ admits a flag and the quotients are isomorphic to $\tau_s^* D(2, \mu)$ for some $s \in \mathbb{Z}_+$, $\mu \in P^+$.

Remark 10. *This means we can write the graded character of $D(\lambda_1 + \nu, \lambda_2 + \nu)$ in terms of level two Demazure modules, and we compute an example in 4.1.10.*

Remark 11. *Analogous results were proved for A_n in [4] when $\nu = 0$ and D_n in [7]. In fact in [4], for Type A_n it was proved that*

$$V(\lambda_1, \lambda_2) \cong D(\lambda_1, \lambda_2) \cong D(2, \lambda_1 + \lambda_2).$$

The methods here work for A_n as well and prove the more general case when $\nu \neq 0$.

3.1.4 Condition For Demazure Isomorphism

The theorem is proved in steps, the first of which is the following proposition which gives a condition for the generalized Demazure module to be isomorphic to a Demazure module.

Proposition 12. *If* $\lambda(h_n) = 1$ *or if* $\lambda = \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$ *for* $1 \le i \le n$*, then for all* $\nu \in P^+$ *with* $\nu(h_n) = 0$*, we have*

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \cong D(2, 2\nu + \lambda) \cong D(\lambda_1 + \nu, \lambda_2 + \nu).$$

3.1.5 The Root β_λ

Suppose that $\lambda(h_n) \in \{0, 2\}$ and $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$ and let $p \le n-1$ be maximal such that $\lambda_1(h_{p+1}) \neq 0 \neq \lambda(h_{p+1})$. Let $1 \le p' \le p \le n-1$ be maximal so that $(\lambda_1 - \lambda_2)(h_{p',p}) = 0$. Observe that if $p' < p$ then the definition of interlacing pairs forces $\lambda_2(h_p) = 1 = \lambda_1(h_{p'})$. Set

$$\beta_\lambda = \beta_{p',p+1} = \alpha_{p'} + \cdots + \alpha_p + 2(\alpha_{p+1} + \cdots + \alpha_n)$$

and note that $\lambda_1(h_{\beta_\lambda}) = 3 - \delta_{p',p}$ and $\lambda_2(h_{\beta_\lambda}) = 1 - \delta_{p'p}$. It is helpful to note that the assumption $\lambda(h_n) \in \{0, 2\}$ implies that $p = n-1$ iff $\lambda(h_n) = 2$.

Lemma 13. *Suppose that* $\lambda(h_n) \in \{0, 2\}$ *and* $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$*. Then* $\lambda_1 - \beta_\lambda \in P^+$ *and there exists* $\nu_0 \in P^+$ *with* $\nu_0(h_n) = 0$ *such that* $(\lambda_1 - \beta_\lambda - \nu_0, \lambda_2 - \nu_0)$ *is an interlacing pair.*

Proof. It is simple to see that with our assumptions

$$\lambda_1 - \beta_\lambda = \lambda_1 - \omega_{p+1} - \delta_{p,n-1}\omega_n - \omega_{p'} + \omega_{p'-1} + \omega_p \in P^+.$$

Taking $\nu_0 = \lambda_2(h_{p'-1})\omega_{p'-1} + \lambda_2(h_p)\omega_p$ it is easy to see that $\nu_0(h_n) = 0$ and $(\lambda_1 - \beta_\lambda - \nu_0, \lambda_2 - \nu_0)$ is an interlacing pair.

$\square$

3.1.6 Upper Bound for $\dim V(\lambda_1 + \nu, \lambda_2 + \nu)$

The second reduction is the following proposition which in particular, gives an upper bound for the dimension of $V(\lambda_1+\nu, \lambda_2+\nu)$. It is useful to note that $\max\{r_{1,\beta_\lambda}, r_{2,\beta_\lambda}\} = (\lambda_1 + \nu)(h_{\beta_\lambda})$.

Proposition 14. *Suppose that $\lambda(h_n) \in \{0, 2\}$ and $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$. There exists a right exact sequence of $\mathfrak{g}[t]$-modules*

$$\tau^*_{(\lambda_1+\nu)(h_{\beta_\lambda})-1} V(\lambda_1 + \nu - \beta_\lambda, \lambda_2 + \nu) \to V(\lambda_1 + \nu, \lambda_2 + \nu) \to D(2, \lambda + 2\nu) \to 0, \quad (3.1.6)$$

with $w_{\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu} \to (x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1}) w_{\lambda_1+\nu,\lambda_2+\nu}$.

3.1.7 Level One Inclusion

Assuming Proposition 12 and Proposition 14 we complete the proof of Theorem 9. The proof is by an induction with respect to the partial order on P^+. The minimal elements with respect to this order are $0, \omega_n$ and Proposition 12 shows that induction begins. Moreover the proposition also establishes the theorem if $\lambda = \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$ or $\lambda(h_n) = 1$. Hence we have only to prove the inductive step when $\lambda(h_n) \in \{0, 2\}$ and $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$. We shall need the following result to complete the proof of the inductive step. With these assumptions on (λ_1, λ_2) and ν we have that $(\lambda_k+\nu)(h_\alpha)/d_\alpha \in \mathbb{Z}^+$ for all $\alpha \in R^+$ and $k \in \{1, 2\}$, so we omit the ceiling function.

Lemma 15. *If $\lambda(h_n) \in \{0,2\}$ and $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$, there exists an inclusion of $\mathfrak{g}[t]$-modules*

$$\tau^*_{(\lambda_1+\nu)(h_{\beta_\lambda})-1} D(1, \lambda_1 + \nu - \beta_\lambda) \hookrightarrow D(1, \lambda_1 + \nu), \tag{3.1.7}$$

which sends to $w_{\lambda_1+\nu-\beta_\lambda} \to (x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1}) w_{\lambda_1+\nu}$.

Proof. By Lemma 5, and since $(\lambda_1 - \beta_\lambda + \nu)(h_n) = 0$ by [33] we have $W_{\mathrm{loc}}(\lambda_1 - \beta_\lambda + \nu) \cong D(1, \lambda_1 - \beta_\lambda + \nu)$ it suffices to prove that $w := (x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1}) w_{\lambda_1+\nu}$ satisfies the relations in (2.2.1). If $\beta_\lambda - \alpha_i \notin R^+$ then it is clear that $(x^+_i \otimes t^r)w = 0$ for all $r \in \mathbb{Z}_+$. Otherwise we must have $i = p+1$, or $i = p'$ if $p' < p$ and we have to prove that

$$(x^-_{\beta_\lambda - \alpha_i} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1+r}) w_{\lambda_1+\nu} = 0, \quad r \in \mathbb{Z}_+.$$

But this follows from (2.2.2) since in all cases $\lambda_1(h_i) = d_i$ and so

$$(\lambda_1 + \nu)(h_{\beta_\lambda}) - 1 \geq \frac{(\lambda_1 + \nu)(h_{\beta_\lambda - \alpha_i})}{d_{\beta_\lambda - \alpha_i}}.$$

The second relation in (2.2.1) is trivial if $r = 0$ and if $r > 0$ then it follows from (2.2.2) since

$$(h \otimes t^r)w = (x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})+r-1}) w_{\lambda_1+\nu} = 0.$$

The third relation is immediate since the modules are all finite-dimensional.

□

3.1.8 Proof of Main Inductive Step

Lemma 4, Proposition 14 and the inductive hypothesis establish the following inequalities:

$$\dim D(\lambda_1+\nu,\lambda_2+\nu) \le \dim V(\lambda_1+\nu,\lambda_2+\nu),$$

$$\dim V(\lambda_1+\nu,\lambda_2+\nu) \le \dim D(2,2\nu+\lambda) + \dim D(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu).$$

The inductive step follows if we prove that

$$\dim D(\lambda_1+\nu,\lambda_2+\nu) = \dim D(2,2\nu+\lambda) + \dim D(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu). \tag{3.1.8}$$

For this, we observe that Lemma 15 gives an inclusion

$$0 \to D(1,\lambda_1+\nu-\beta_\lambda)\otimes D(1,\lambda_2+\nu) \to D(1,\lambda_1+\nu)\otimes D(\lambda_2+\nu),$$

which sends

$$w_{\lambda_1+\nu-\beta_\lambda}\otimes w_{\lambda_2+\nu} \to ((x^-_{\beta_\lambda}\otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu})\otimes w_{\lambda_2+\nu}.$$

Since $(\lambda_1+\nu)(h_{\beta_\lambda})-1 \ge (\lambda_2+\nu)(h_{\beta_\lambda})$ we see that (2.2.2) gives

$$(x^-_{\beta_\lambda}\otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1)})\,(w_{\lambda_1+\nu}\otimes w_{\lambda_2+\nu}) = \Big((x^-_{\beta_\lambda}\otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu}\Big)\otimes w_{\lambda_2+\nu}.$$

It follows that we have an inclusion

$$\iota : D(\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu) \hookrightarrow D(\lambda_1+\nu,\lambda_2+\nu)$$

and it suffices to prove that the corresponding quotient is isomorphic to $D(2,2\nu+\lambda)$. By Lemma 8 we have the following surjective maps

$$V(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(2,2\nu+\lambda).$$

These maps are all unique up to scalars and Proposition 14 shows that the kernel of the composite map is generated by the element $(x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1)})w_{\lambda_1+\nu,\lambda_2+\nu}$. Hence the kernel of

$$D(\lambda_1+\nu, \lambda_2+\nu) \twoheadrightarrow D(2, 2\nu+\lambda)$$

is generated by $(x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1)})(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})$. But this means that the latter kernel is precisely the image of ι and hence the corresponding quotient is isomorphic to $D(2, 2\nu+\lambda)$ as needed.

3.1.9 The Graded Character

Given a graded $\mathfrak{g}[t]$-module $V = \oplus_{s\in\mathbb{Z}} V[s]$ we have,

$$(\mathfrak{g} \otimes t^r)V[s] \subset V[r+s], \quad V_\mu = \bigoplus_{s\in\mathbb{Z}} V_\mu[s], \quad V_\mu[s] = V_\mu \cap V[s],$$

for all $r, s \in \mathbb{Z}$ and $\mu \in P$. Let $\mathbb{Z}[v, v^{-1}][P]$ be the group ring of P with coefficients in $\mathbb{Z}[v, v^{-1}]$ where v is an indeterminate. Let $e(\mu)$, $\mu \in P$ be a basis for the group ring and set

$$\mathrm{ch}_{\mathrm{gr}}\, V = \sum_{\mu\in P} \left(\sum_{s\in\mathbb{Z}} \dim V_\mu[s] v^s \right) e(\mu).$$

Just as in [7], notice that in the course of proving Theorem 9 we also established that the right exact sequence in Proposition 14 is exact. Hence we get an equality of graded characters

$$\mathrm{ch}_{\mathrm{gr}}\, D(\lambda_1+\nu, \lambda_2+\nu) = \mathrm{ch}_{\mathrm{gr}}\, D(2, 2\nu+\lambda) + v^{(\lambda_1+\nu)(h_{\beta_\lambda})-1}\, \mathrm{ch}_{\mathrm{gr}}\, D(\lambda_1-\beta_\lambda+\nu, \lambda_2+\nu). \quad (3.1.9)$$

Using this we can derive a formula for $\mathrm{ch}_{\mathrm{gr}}\, D(\lambda_1+\nu, \lambda_2+\nu)$ in terms of Demazure modules just as in [7]. Here we compute an example.

Example 16. *Let $\mathfrak{g}$ be of type B_7. Consider $(\lambda_1, \lambda_2) = (\omega_1+\omega_4+2\omega_7, \omega_2+\omega_5)$ and $\nu = 0$. Then $\beta_\lambda = \alpha_4+\alpha_5+2\alpha_7$, and*

$$\begin{aligned}\operatorname{ch}_{\mathrm{gr}} D(\omega_1+\omega_4+2\omega_7, \omega_2+\omega_5) &= \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+\omega_4+\omega_5+2\omega_7) \\ &\quad + v^2 \operatorname{ch}_{\mathrm{gr}} D(\omega_1+\omega_3+\omega_5, \omega_2+\omega_5)\end{aligned}$$

So our next interlacing pair is $(\omega_1+\omega_3, \omega_2)$ with $\nu_0 = \omega_5$. Now $\beta_{\lambda-\beta_\lambda-2\nu_0} = \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7$ and

$$\begin{aligned}\operatorname{ch}_{\mathrm{gr}} D(\omega_1+\omega_3+\omega_5, \omega_2+\omega_5) &= \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+\omega_3+2\omega_5) \\ &\quad + v^2 \operatorname{ch}_{\mathrm{gr}} D(\omega_2+\omega_5, \omega_2+\omega_5).\end{aligned}$$

We have $D(\omega_2+\omega_5, \omega_2+\omega_5) \cong D(2, 2\omega_2+2\omega_5)$. Therefore

$$\begin{aligned}\operatorname{ch}_{\mathrm{gr}} D(\omega_1+\omega_4+2\omega_7, \omega_2+\omega_5) &= \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+\omega_4+\omega_5+2\omega_7) \\ &\quad + v^2 \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+\omega_3+2\omega_5) \\ &\quad + v^4 \operatorname{ch}_{\mathrm{gr}} D(2, 2\omega_2+2\omega_5).\end{aligned}$$

3.2 Proof of Proposition 12 and Proposition 14.

In this section we prove Proposition 12 and Proposition 14.

3.2.1 Minimality of β_λ

Recall our convention that $\alpha_{i,j} = 0$ if $i > j$. Set

$$R(\lambda_1, \lambda_2) = \{\beta_{i,j} \in R^+ : \ (\lambda_1 - \lambda_2)(h_{\beta_{i,j}}) = \pm 2\}.$$

Note that Lemma 7 shows that $R(\lambda_1, \lambda_2) = \varnothing$ if $\lambda(h_n) = 1$ or $\lambda = \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$. The next result establishes the converse.

Lemma 17. *Suppose that* $\lambda(h_n) \in \{0,2\}$ *and* $\lambda \neq \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$*. Then* $\beta_\lambda \in R(\lambda_1, \lambda_2)$ *and more generally,*

$$\beta_{i,j} \in R(\lambda_1, \lambda_2) \iff \beta_{i,j} = \alpha_{i,p'-1} + \beta_\lambda + \alpha_{j,p}, \quad \textit{and}$$

$$(\lambda_1 - \lambda_2)(h_{i,p'-1}) = 0 = (\lambda_1 - \lambda_2)(h_{j,p}) = 0.$$

Proof. Recall from Section 3.1.5 that $\beta_\lambda = \beta_{p',p+1}$ where $1 \leq p' \leq p$ is maximal with $(\lambda_1 - \lambda_2)(h_{p',p}) = 0$ and that $(\lambda_1 - \lambda_2)(h_{\beta_\lambda}) = 2$. Hence a calculation shows that

$$\beta_{i,j} \in R(\lambda_1, \lambda_2) \implies i \leq p' \;\text{ or }\; p' < j \leq p+1,$$

which shows that $\beta_{i,j} = \alpha_{i,p'-1} + \beta_\lambda + \alpha_{j,p}$.

Since $\beta_{i,p+1} \in R^+$ if $i < p'$ we have

$$(\lambda_1 - \lambda_2)(h_{\beta_{i,p+1}}) = (\lambda_1 - \lambda_2)(h_{i,p'-1}) + 2$$

and Lemma 7 forces $(\lambda_1 - \lambda_2)(h_{i,p'-1}) \in \{-1, 0\}$. If $j < p+1$ we have $\beta_{p',j} = \beta_{p',p+1} + \alpha_{j,p}$ and hence a similar argument shows that $(\lambda_1 - \lambda_2)(h_{j,p}) \in \{-1, 0\}$. Together with hypothesis that $(\lambda_1 - \lambda_2)(h_{\beta_{i,j}}) = \pm 2$ we get $(\lambda_1 - \lambda_2)(h_{i,p'-1}) = 0 = (\lambda_1 - \lambda_2)(h_{j,p})$ as needed. The converse is obvious. □

Remark 18. *Notice that in particular we have established (with our conventions on* (λ_1, λ_2)*) that* $\beta \in R(\lambda_1, \lambda_2)$ *iff* $(\lambda_1 - \lambda_2)(h_\beta) = 2$*.*

3.2.2 Induction Begins

Assume that $\lambda(h_n) = 1$ or that $\lambda = \omega_{i-1} + \omega_i + \delta_{i,n}\omega_n$. By Lemma 7 we have $(\lambda_1 - \lambda_2)(h_\alpha) \in \{-1, 0, 1\}$ for all $\alpha \in R^+$ with $d_\alpha = 1$, and so $R(\lambda_1, \lambda_2) = \varnothing$. The defining relations give

$$V(\lambda_1+\nu,\lambda_2+\nu)\cong D(2,\lambda+2\nu).$$

Since the maps in Lemma 4 and Lemma 8 are unique up to scalars, it now follows that the map

$$V(\lambda_1+\nu,\lambda_2+\nu)\twoheadrightarrow D(\lambda_1+\nu,\lambda_2+\nu)\twoheadrightarrow D(2,2\nu+\lambda_1+\lambda_2),$$

is an isomorphism and hence all the maps are isomorphisms. Proposition 12 is proved.

3.2.3 Cyclic Kernel

We turn to the proof of Proposition 14. We have $R(\lambda_1,\lambda_2)\neq\varnothing$ and our conventions imply that $\lambda_1(h_{\beta_\lambda})=3-\delta_{p',p}$ and $\lambda_2(h_{\beta_\lambda})=1-\delta_{p',p}$. Using the remark following Lemma 17 we have $\lambda_1(h_\beta)\geq\lambda_2(h_\beta)$ for all $\beta\in R(\lambda_1,\lambda_2)$. Hence for all $\beta\in R(\lambda_1,\lambda_2)$,

$$(\nu+\lambda_1)(h_\beta)=s_\beta+1.$$

Furthermore, by writing

$$s_\alpha=\left\lceil\frac{(\lambda+2\nu)(h_\alpha)}{2d_\alpha}\right\rceil$$

and using Lemma 7, we have $\max\{r_{1,\alpha},r_{2,\alpha}\}=s_\alpha$ for $\alpha\notin R(\lambda_1,\lambda_2)$.

An inspection of the defining relations of the modules shows that the kernel K of the canonical map

$$\psi_{\lambda_1+\nu,\lambda_2+\nu}:V(\lambda_1+\nu,\lambda_2+\nu)\to D(2,\lambda+\nu)\to 0,\quad w_{\lambda_1+\nu,\lambda_2+\nu}\to w_{2\nu+\lambda},$$

is generated by the elements

$$(x_\beta^-\otimes t^{(\nu+\lambda_1)(h_\beta)-1})w_{\lambda_1+\nu,\lambda_2+\nu},\quad \beta\in R(\lambda_1,\lambda_2).$$

The proof of Proposition 14 is now shown in two steps. The first step is to show that

$$K = \mathbf{U}(\mathfrak{g}[t])(x^-_{\beta_\lambda} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1)})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

For this, let $\beta_{i,j} \in R(\lambda_1, \lambda_2)$ and write $\beta_{i,j} = \alpha_{i,p'-1} + \beta_{p',p+1} + \alpha_{j,p}$ as in Lemma 17 and assume that $i \leq p' - 1$ or $j \leq p$ (otherwise $\beta = \beta_\lambda$ and there is nothing to prove). The defining relations

$$(x^-_{i,p'-1} \otimes t^{(\nu+\lambda_1)(h_{i,p'-1})})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x^-_{j,p} \otimes t^{(\nu+\lambda_1)(h_{j,p})})w_{\lambda_1+\nu,\lambda_2+\nu},$$

imply if $i \leq p' - 1$ and $j \leq p$

$$(x^-_{i,p'-1} \otimes t^{(\nu+\lambda_1)(h_{i,p'-1})})(x^-_{j,p} \otimes t^{(\nu+\lambda_1)(h_{j,p})})(x^-_{\beta_\lambda} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu} =$$

$$[x^-_{i,p'-1}, [x^-_{j,p}, x^-_{\beta_\lambda}]] \otimes t^{(\nu+\lambda_1)(h_{\beta_{i,j}})-1})w_{\lambda_1+\nu,\lambda_2+\nu} = (x^-_{\beta_{i,j}} \otimes t^{(\nu+\lambda_1)(h_{\beta_{i,j}})-1})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

An obvious modification works if $i \geq p'$ or $j \geq p+1$ and the first step is proved.

The next step is to show that the element $(x^-_{\beta_\lambda} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the defining relations of the element $w_{\lambda_1+\nu-\beta_\lambda,\lambda_2+\nu} \in V(\lambda_1+\nu-\beta_\lambda, \lambda_2+\nu)$. This establishes the existence of the map $V(\lambda_1 + \nu - \beta_\lambda, \lambda_2 + \nu) \to K \to 0$.

3.2.4 Quotient of a Local Weyl Module

Here we prove that $(x^-_{\beta_\lambda} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the relations in (3.1.2) with (λ_1, λ_2) replaced by $(\lambda_1 - \beta_\lambda, \lambda_2)$. Since $(x^+_i \otimes t^r)w_{\lambda_1+\nu,\lambda_2+\nu} = 0$ for $1 \leq i \leq n$ and $r \in \mathbb{Z}_+$ the first relation in (3.1.2) is immediate if $\beta_\lambda - \alpha_i \notin R^+$. Otherwise we must prove that

$$(\dagger) \quad (x^-_{\beta_\lambda-\alpha_i} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1+r})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Since $\beta_\lambda = \beta_{p',p+1}$ it follows that either $i = p+1$ and if $p' < p$ then we can also have $i = p'$ and if $p = n-2$ then, in addition we can have $i = n$. In the first two cases we have $\lambda_1(h_i) = 1$ and $\lambda_2(h_i) = 0$ (see Section 3.1.5). In particular

$$(x^-_{\beta_\lambda - \alpha_i} \otimes t^{\nu(h_{\beta_\lambda - \alpha_i})+2-\delta_{p,p'}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

is a defining relation and (†) follows by applying $(h \otimes t^r)$ since

$$(\nu + \lambda_1)(h_{\beta_\lambda}) - 1 + r = \nu(h_{\beta_\lambda}) + 2 - \delta_{p,p'} + r \geq \nu(h_{\beta_\lambda - \alpha_i}) + 2 - \delta_{p,p'}, \quad r \in \mathbb{Z}_+.$$

A similar argument establishes the case when $p = n-2$ and $i = n$.

It is trivial to see that the second relation in (2.2.1) holds and the third is then immediate since $V(\nu + \lambda_1, \nu + \lambda_2)$ is finite-dimensional.

3.2.5 Reduction of Relations

We need the following result to prove that $(x^-_{\beta_\lambda} \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the relation (3.1.4).

Lemma 19. *The relation $(x^-_\alpha \otimes t^{\max\{r_{1,\alpha}, r_{2,\alpha}\}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$ holds in $V(\lambda_1 + \nu, \lambda_2 + \nu)$ for all $\alpha \in R^+$ iff it holds for all the elements of the set*

$$\{\alpha_i\} \cup \{\alpha_{i,j} : 1 \leq i \leq j \leq n, \quad \lambda(h_i) \neq 0 \neq \lambda(h_j), \quad \lambda(h_s) = 0, \quad i < s < j\}.$$

Proof. The forward direction of the Lemma is obvious. For the reverse direction we proceed by induction with respect to the partial order on R^+ induced by the partial order $\leq$ on P. If $\alpha = \alpha_i$ for some $1 \leq i \leq n$ there is nothing to prove. For the inductive step choose

$1 \leq r \leq n$ so that $\alpha - \alpha_r \in R^+$. If $\lambda(h_r) = 0$ then the result follows from $[x_r^-, x_{\alpha-\alpha_r}^-] = x_\alpha^-$ and

$$(x_r^- \otimes t^{\frac{\nu(h_r)}{d_r}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x_{\alpha-\alpha_r}^- \otimes t^{\max\left\{\frac{\lambda_1(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}, \frac{\lambda_2(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}\right\}+\frac{\nu(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}})w_{\lambda_1+\nu,\lambda_2+\nu},$$

by the inductive hypothesis. In any case, we have $\frac{\nu(h_r)}{d_r} + \frac{\nu(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}} = \frac{\nu(h_\alpha)}{d_\alpha}$, and $\frac{\lambda_k(h_\alpha)}{d_\alpha} = \frac{\lambda_k(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}$. Therefore

$$\frac{\nu(h_r)}{d_r} + \max\left\{\frac{\lambda_1(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}, \frac{\lambda_2(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}}\right\} + \frac{\nu(h_{\alpha-\alpha_r})}{d_{\alpha-\alpha_r}} = \max\{r_{1,\alpha}, r_{2,\alpha}\},$$

and the relation holds.

Otherwise, $\lambda(h_r) > 0$ for all $1 \leq r \leq n$ with $\alpha - \alpha_r \in R^+$, in which case α is one of the following:

- $\alpha_{i,j}$ with $\lambda(h_i) = 1,\ \lambda(h_j) \neq 0$,
- $\beta_{i,j}$ with $i < j-1$ and $\lambda(h_i) = 1,\ \lambda(h_j) \neq 0$,
- $\beta_{j-1,j}$ with $\lambda(h_j) \neq 0$.

In the first two cases choose $i < s \leq j$ minimal with $\lambda(h_s) \neq 0$ and $\lambda(h_{i,s}) = \lambda(h_s) + 1$. If $\alpha = \alpha_{i,n}$ and $s = j = n$ then we are done, otherwise the result follows since Lemma 7 gives $\max\{\lambda_1(h_{i,s}), \lambda_2(h_{i,s})\} = 1$ and in any case we have $\nu(h_{i,s}) + \nu(h_{\alpha-\alpha_{i,s}})/d_{\alpha-\alpha_{i,s}} = \nu(h_\alpha)/d_\alpha$ and $d_{\alpha-\alpha_{i,s}} = d_\alpha$ so that

$$\max\left\{\frac{\lambda_1(h_{\alpha-\alpha_{i,s}})}{d_{\alpha-\alpha_{i,s}}}, \frac{\lambda_2(h_{\alpha-\alpha_{i,s}})}{d_{\alpha-\alpha_{i,s}}}\right\} = \max\left\{\frac{\lambda_1(h_\alpha)}{d_\alpha}, \frac{\lambda_2(h_\alpha)}{d_\alpha}\right\} - 1.$$

Then the inductive hypothesis gives

$$(x_{i,s}^- \otimes t^{\nu(h_{i,s})+1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x_{\alpha-\alpha_{i,s}}^- \otimes t^{\frac{\nu(h_{\alpha-\alpha_{i,s}})}{d_{\alpha-\alpha_{i,s}}}+\max\left\{\frac{\lambda_1(h_\alpha)}{d_\alpha}, \frac{\lambda_2(h_\alpha)}{d_\alpha}\right\}-1})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

This is of course assuming that $s \neq j-1$ if $\alpha = \beta_{i,j}$. If this is the case then the result holds in a similar fashion by working with the pair $(\alpha_{j-1,j}, \alpha - \alpha_{j-1,j})$, and noting that $\max\left\{\frac{\lambda_1(h_{j-1,j})}{d_{j-1,j}}, \frac{\lambda_2(h_{j-1,j})}{d_{j-1,j}}\right\} = 1$.

In the third case if $\lambda(h_{j-1}) = 1$ or if $\lambda(h_{j-1}) = 0$ and there exists $s > j$ minimal with $\lambda(h_s) = 1$ then the result follows as before by working with the pairs $(\alpha_{j-1,j}, \alpha - \alpha_{j-1,j})$ and $(\alpha_{j-1,s}, \alpha - \alpha_{j-1,s})$ respectively. Otherwise we have $\max\{\lambda_1(h_\alpha), \lambda_2(h_\alpha)\} = 2$ and $\max\left\{\frac{\lambda_1(h_{j,n})}{2}, \frac{\lambda_2(h_{j,n})}{2}\right\} = 1$. The result now follows from

$$(x^-_{j-1,n} \otimes t^{\nu(h_{j-1,n})+1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x^-_{j,n} \otimes t^{\frac{\nu(h_{j,n})}{2}+1})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

and also noting that $\nu(h_{j-1,n}) + \nu(h_{j,n})/2 = \nu(\beta_{j-1,j})$. □

3.2.6 Result From Garland's Relations

This is a result that will need for part of the proof in the next subsection.

Lemma 20. *Retain the notation established so far. For $\alpha \in R^+$, The following relation holds in $V(\lambda_1+\nu, \lambda_2+\nu)$*

$$(x^-_\alpha \otimes t^{\max\{r_{1,\alpha},r_{2,\alpha}\}-1})(x^-_\alpha \otimes t^{\min\{r_{1,\alpha},r_{2,\alpha}\}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Proof. The defining relation of $V(\lambda_1+\nu, \lambda_2+\nu)$ gives

$$(x^-_\alpha \otimes 1)^{2\nu(h_\alpha)+\lambda(h_\alpha)+1}w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

and hence

$$(x^+_\alpha \otimes t)^{2\nu(h_\alpha)+\lambda(h_\alpha)-1}(x^-_\alpha \otimes 1)^{2\nu(h_\alpha)+\lambda(h_\alpha)+1}w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

It was shown in [18] that

$$(x_\alpha^+ \otimes t)^{(2\nu+\lambda)(h_\alpha)-1}(x_\alpha^- \otimes 1)^{(2\nu+\lambda)(h_\alpha)+1} - \sum_{k=0}^{(2\nu+\lambda)(h_\alpha)-1} (x_\alpha^- \otimes t^k)(x_\alpha^- \otimes t^{(2\nu+\lambda)(h_\alpha)-1-k})$$

is in $\mathbf{U}(\mathfrak{g}[t])\,(\mathfrak{n}^+[t] + (\mathfrak{h}\otimes\mathbb{C}[t]))$ Hence we get

$$\sum_{k=0}^{2\nu(h_\alpha)+\lambda(h_\alpha)-1} (x_\alpha^- \otimes t^k)(x_\alpha^- \otimes t^{(2\nu+\lambda)(h_\alpha)-1-k}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Assume without loss of generality that $(\lambda_1-\lambda_2)(h_\alpha) \geq 0$ and recall that Lemma 7 and Lemma 13 give $|(\lambda_1-\lambda_2)(h_\alpha)| \leq 2$. Consider the defining relation

$$(x_\alpha^- \otimes t^k) w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad k \geq r_{1\alpha}.$$

If $k < r_{1,\alpha}$ then $(2\nu+\lambda)(h_\alpha) - 1 - k \geq r_{2,\alpha}$. The left hand side is identically zero if $r_{1,\alpha} = r_{2,\alpha}$, and if

$$1 \leq r_{1,\alpha} - r_{2,\alpha} \leq 2$$

we get the assertion of the Lemma. □

3.2.7 Existence of Map Onto the Kernel

To complete the proof of the second step, we now show that (3.1.4) also holds. Here it is useful to note that $(\lambda - \beta_\lambda - 2\nu_0)(h_n) = 0$, so by the preceding Lemma it suffices to prove that

$$(*)\quad (x_\alpha^- \otimes t^{\nu(h_\alpha)+\max\{(\lambda_1-\beta_\lambda)(h_\alpha),\lambda_2(h_\alpha)\}})(x_{\beta_\lambda}^- \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

for simple roots α_i and roots of the form $\alpha = \alpha_{i,j}$, $1 \leq i \leq j < n$ with

$$(**)\quad (\lambda - \beta_\lambda - 2\nu_0)(h_i) = 1 = (\lambda - \beta_\lambda - 2\nu_0)(h_j), \quad (\lambda - \beta_\lambda - 2\nu_0)(h_s) = 0, \quad i < s < j,$$

where $\nu_0 = \lambda_2(h_{p'-1})\omega_{p'-1} + \lambda_2(h_p)\omega_p$. For simple roots α_i with $1 \leq i < n$, we need to show that

$$(x_i^- \otimes t^{\nu(h_i)+\max\{(\lambda_1-\beta_\lambda)(h_i),\lambda_2(h_i)\}})(x_{\beta_\lambda}^- \otimes t^{\nu(h_{\beta_\lambda})+\lambda_1(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

First suppose that $\beta_\lambda(h_i) = 0$, then $\max\{(\lambda_1 - \beta_\lambda)(h_i), \lambda_2(h_i)\} = \max\{\lambda_1(h_i), \lambda_2(h_i)\}$, and $\beta_\lambda + \alpha_i \notin R^+$, so the relation holds.

Now suppose that $\beta_\lambda(h_i) = -1$. In this case, since (λ_1, λ_2) are interlacing and

$$\max\{(\lambda_1 - \beta_\lambda)(h_i), \lambda_2(h_i)\} = \lambda_1(h_i) + 1$$

Furthermore, by the definition of β_λ we have that $i = p' - 1$ or $i = p \neq p'$, and $\beta_\lambda + \alpha_i \in R^+$. Then

$$\begin{aligned}
&(x_i^- \otimes t^{\nu(h_i)+\lambda_1(h_i)+1})(x_{\beta_\lambda}^- \otimes t^{\nu(h_{\beta_\lambda})+\lambda_1(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu} \\
&= (x_{\beta_\lambda}^- \otimes t^{\nu(h_{\beta_\lambda})+\lambda_1(h_{\beta_\lambda})-1})(x_i^- \otimes t^{\nu(h_i)+\lambda_1(h_i)+1})w_{\lambda_1+\nu,\lambda_2+\nu} \\
&+ (x_{\beta_\lambda+\alpha_i}^- \otimes t^{\nu(h_{\beta_\lambda+\alpha_i})+\lambda_1(h_{\beta_\lambda+\alpha_i})})w_{\lambda_1+\nu,\lambda_2+\nu}
\end{aligned}$$

The first term is 0 since $\lambda_1(h_i) + 1 \geq \max\{\lambda_1(h_i), \lambda_2(h_i)\}$, and the second term is 0 since $\lambda_1(h_{\beta_\lambda}) = \lambda_2(h_{\beta_\lambda}) + 2$ implies that

$$\max\{\lambda_1(h_{\beta_\lambda+\alpha_i}), \lambda_2(h_{\beta_\lambda+\alpha_i})\} = \lambda_1(h_{\beta_\lambda+\alpha_i}).$$

Lastly, suppose that $\beta_\lambda(h_i) = 1$. In this case we have

$$\max\{(\lambda_1 - \beta_\lambda)(h_i), \lambda_2(h_i)\} \geq \lambda_1(h_i) - 1.$$

The definition of β_λ implies that $i = p+1$ or $i = p' \neq p$, and $\lambda_1(h_i) = 1$, with $\beta_\lambda + \alpha_i \notin R^+$. By 20 we have

$$(x_i^- \otimes t^{(\nu+\lambda_1)(h_i)-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

this together with the fact that $\beta_\lambda+\alpha_i \notin R^+$ and $\max\{\lambda_1(h_{\beta_\lambda-\alpha_i}), \lambda_2(h_{\beta_\lambda-\alpha_i})\} = \lambda_1(h_{\beta_\lambda-\alpha_i})$ implies

$$0 = (x^-_{\beta-\alpha_i} \otimes t^{\nu(h_{\beta-\alpha_i})+\lambda_1(h_{\beta-\alpha_i})})(x^-_i \otimes t^{(\nu+\lambda_1)(h_i)-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= 2(x^-_i \otimes t^{(\nu+\lambda_1)(h_i)-1})(x^-_{\beta_\lambda} \otimes t^{(\lambda_1+\nu)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu}$$

and the relation holds.

Now we show $(*)$ for $\alpha_{i,j}$ satisfying $(**)$. Note that this forces $\beta_\lambda(h_i) \leq 0$ and $\beta_\lambda(h_j) \leq 0$. We claim that $(\beta_\lambda, \alpha) \leq 0$. Otherwise we would have that $\beta_\lambda - \alpha \in R$ and since $\alpha - \beta_\lambda \notin R^+$ we must have $\beta_\lambda - \alpha \in R^+$. It is elementary to see that this is only possible if $\alpha_{i,j} = \alpha_{p',r}$ or $\alpha_{p+1,r}$ for some $r \neq p$ and $p' < r \leq n$. Since $\beta_{p',p+1}(h_{p'}) = 1 - \delta_{p',p}$, and $\beta_{p',p+1}(h_{p+1}) = 1$ we see that $(**)$ forces $p' = p$ and $\alpha = \alpha_{p,r}$. However, if $p' = p$ then we have $\lambda(h_p) = 0$ by definition and $\beta_\lambda(h_p) = 0$ which again contradicts $(**)$ and the claim is proved.

If $\alpha + \beta_\lambda \notin R^+$, i.e. $(\alpha, \beta_\lambda) = 0$ then

$$(x^-_\alpha \otimes t^{\frac{\nu(h_\alpha)}{d_\alpha}+\max\left\{\frac{(\lambda_1-\beta_\lambda)(h_\alpha)}{d_\alpha}, \frac{\lambda_2(h_\alpha)}{d_\alpha}\right\}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

is a defining relation and $(*)$ follows since $[x^-_\alpha, x^-_{\beta_\lambda}] = 0$. If $(\alpha, \beta_\lambda) < 0$ then $\alpha + \beta_\lambda \in R^+$, $d_{\alpha+\beta_\lambda} = 1$ and $[x^-_\alpha, x^-_{\beta_\lambda}] = x^-_{\alpha+\beta_\lambda}$. It suffices to show that

$$(x^-_{\alpha+\beta_\lambda} \otimes t^{\nu(h_\alpha)+\max\{(\lambda_1-\beta_\lambda)(h_\alpha),\lambda_2(h_\alpha)\}+(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

But this holds since

$$\begin{aligned}\max\{\lambda_1(h_{\alpha+\beta_\lambda}), \lambda_2(h_{\alpha+\beta_\lambda})\} &= \max\{\lambda_1(h_\alpha) + 3 - \delta_{p',p}, \lambda_2(h_\alpha) + 1 - \delta_{p',p}\} \\ &\leq \max\{\lambda_1(h_\alpha) + 3 - \delta_{p',p}, \lambda_2(h_\alpha) + 2 - \delta_{p',p}\} \\ &= \max\{\lambda_1(h_\alpha) + 1, \lambda_2(h_\alpha)\} + 2 - \delta_{p',p}.\end{aligned}$$

This completes the proof of Proposition 14.

Lastly, we need to show that relation (3.1.5) holds, i.e.,

$$(x_\alpha^- \otimes t^{s'_\alpha - 1})^{m'_\alpha + 1}(x_{\beta_\lambda}^- \otimes t^{(\nu+\lambda_1)(h_{\beta_\lambda})-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

where s'_α and m'_α are the unique nonnegative integers satisfying $(\lambda - \beta_\lambda + 2\nu)(h_\alpha) = 4(s'_\alpha - 1) + m'_\alpha$, with $1 < m'_\alpha \leq 4$. We only have to show the relation when $(\lambda - \beta_\lambda + 2\nu)(h_\alpha) > 4$, and $m'_\alpha \neq 4$. First we see that

$$\beta_\lambda(h_{i,n}) = \begin{cases} 2, & i \in \{p', p+1\} \\ 0, & \text{otherwise.} \end{cases}$$

We also have that $\lambda_1(h_{p'}) = 2 = \lambda_1(h_{p+1})$, while $\lambda_2(h_{p'}) = 0 = \lambda_2(h_{p+1})$. This implies that 4 divides $(\lambda - \beta_\lambda + 2\nu)(h_{i,n})$, so $m'_\alpha \in \{0, 4\}$ and there is nothing to show.

If $\beta_\lambda(h_{i,n}) = 0$ then $m_\alpha = m'_\alpha$ and $s_\alpha = s'_\alpha$, and since $\alpha_{i,n} + \beta_\lambda \notin R^+$ for all $1 \leq i \leq n$

$$(x_\alpha^- \otimes t^{s'_\alpha - 1})^{m'_\alpha + 1} w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

is a defining relation in $V(\lambda_1 + \nu, \lambda_2 + \nu)$.

Chapter 4

The Type C_n Case

4.1 Type C_n

In this section we assume that $\mathfrak{g}$ is a simple Lie algebra of type C_n. Recall that α_i, $1 \leq i \leq n$ denote the simple roots. Set

$$\alpha_{i,j} = \alpha_1 + \cdots + \alpha_j, \quad 1 \leq i \leq j \leq n-1$$

$$\beta_{i,j} = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n, \quad 1 \leq i < j \leq n,$$

$$\beta_j = 2(\alpha_i + \cdots + \alpha_{n-1}) + \alpha_n, \ 1 \leq j \leq n$$

and note that

$$R^+ = \{\alpha_{i,j} : 1 \leq i \leq j \leq n-1\} \sqcup \{\beta_{i,j} : 1 \leq i < j \leq n\} \sqcup \{\beta_j : 1 \leq j \leq n\}.$$

Furthermore, for $\alpha \in R^+$ we have that $d_\alpha = 1$ if and only if $\alpha = \beta_j$, $1 \leq j \leq n$, otherwise $d_\alpha = 2$. For computations we often use

$$\lambda(h_{i,j}) = \lambda(h_i) + \cdots + \lambda(h_j)$$

$$\lambda(h_{\beta_{i,j}}) = \lambda(h_i) + \cdots + \lambda(h_{j-1}) + 2(\lambda(h_j) + \cdots + \lambda(h_n))$$

$$\lambda(\beta_j) = \lambda(h_j) + \cdots + \lambda(h_n)$$

for $\lambda \in P$.

4.1.1 Interlacing Weights

Let

$$P^+(1) = \{\lambda \in P^+ : \lambda(h_i) \leq 1, \quad 1 \leq i \leq n\}.$$

Any $\lambda \in P^+(1)$ can be written uniquely (up to order) as a sum $\lambda = \lambda_1 + \lambda_2$ where $\lambda_s \in P^+(1)$, $s = 1, 2$ and satisfy the following: for $1 \leq i \leq j \leq n$

$$\lambda_r(h_i) = 1 = \lambda_r(h_j) \implies \lambda_p(h_s) = 1 \text{ for some } \ i < s < j, \quad \{r, p\} = \{1, 2\},$$

We call $(\lambda_1, \lambda_2) \in P^+ \times P^+$ an *interlacing* pair if $\lambda_1 + \lambda_2 \in P^+(1)$ and the preceding condition holds.

Examples. The pairs $(\omega_i, 0)$ for $0 \leq i \leq n$ and the elements of the set $\{(\omega_i, \omega_j) : 0 \leq i \neq j \leq n\}$ are interlacing. The pair $(\omega_2 + \omega_3, \omega_4 + \omega_6)$ is not interlacing in C_6, but the pair $(\omega_2 + \omega_4, \omega_3 + \omega_6)$ is interlacing in C_6.

In this section we first establish some technical results on the properties of interlacing pairs.

Throughout this section we shall assume without further mention that (λ_1, λ_2) *is an* *interlacing pair and $\lambda = \lambda_1 + \lambda_2$, and ν is (λ_1, λ_2)-compatible. We shall also assume without loss of generality that there exists p maximal with $\lambda(h_p) \neq 0$ with $p \leq n$ and $\lambda_1(h_p) = \lambda(h_p)$.*

Lemma 21. *For all $1 \leq i \leq j \leq n-1$ and (λ_1, λ_2) interlacing, we have*

$$|(\lambda_1 - \lambda_2)(h_{i,j})| \leq 1, \tag{4.1.1}$$

and

$$|(\lambda_1 - \lambda_2)(h_\alpha)| \leq d_\alpha \text{ for all other } \alpha \in R^+.$$

Proof. We prove the inequality (4.1.1) by induction on $j - i$; Since $\lambda \in P^+(1)$, induction begins at $i = j$. For $j > i$ the inductive step is clear if $\lambda(h_i) = 0$ or $\lambda(h_j) = 0$. If $\lambda(h_i) = 1 = \lambda(h_j)$ then either $\lambda(h_r) = 0$ for all $i < r < j$ or there exists $i < s < j$ minimal such that $\lambda_2(h_s) = 1$. In the first case we have proved that $(\lambda_1 - \lambda_2)(h_{i,j}) = 0$ and in the second case we have proved that $(\lambda_1 - \lambda_2)(h_{i,j}) = (\lambda_1 - \lambda_2)(h_{s+1,j})$ and the inductive step follows. For β_j, we induct on $n - j$, and the proof is identical to the previous case. For $\beta_{i,j}$ we write $\beta_{i,j} = \alpha_{i,j-1} + \beta_j$. If $(\lambda_1 - \lambda_2)(h_{\beta_j}) = 1$, then since (λ_1, λ_2) are interlacing, we must have $-1 \leq (\lambda_1 - \lambda_2)(h_{i,j-1}) \leq 0$. Therefore,

$$|(\lambda_1 - \lambda_2)(h_{\beta_{i,j}})| = |(\lambda_1 - \lambda_2)(h_{i,j-1}) + 2(\lambda_1 - \lambda_2)(h_{\beta_j})| \leq 2,$$

and so the inequalities hold. $\square$

4.1.2 The Modules $V(\lambda_1 + \nu, \lambda_2 + \nu)$

For an interlacing pair (λ_1, λ_2) with $\lambda = \lambda_1 + \lambda_2$, If $\lambda = 0$ set $p = p' = p'' = 0$. If $\lambda = \omega_j$, set $p = j$, and $p' = p'' = 0$. If $\lambda = \omega_i + \omega_j$, set $p = j$, $p' = i$, and $p'' = 0$. If

$\lambda(h_{1,n-1}+h_n)\geq 3$, let $p>p'>p''$ be maximal with $\lambda(h_p+h_{p'}+h_{p''})=3$.

For an interlacing pair (λ_1,λ_2), we say that $\nu\in P^+$ is (λ_1,λ_2)-*compatible* if $p''<\min\nu$ and $\nu(h_{p''+1,p-1})\leq 1$, where $\min\nu=\min\{i\ :\ \nu(h_i)\neq 0\}$. If $\nu=0$ we take $\min\nu=n+1$.

For an interlacing pair (λ_1,λ_2) with $\lambda=\lambda_1+\lambda_2$ and a (λ_1,λ_2)-compatible $\nu\in P^+$, we set

$$R(\lambda_1,\lambda_2,\nu)=\{\beta_{i,j}\in R^+\ :\ (\lambda_1-\lambda_2)(h_{\beta_{i,j}})=\pm 2,\ (\lambda_1+\nu)(h_{\beta_{i,j}})\equiv(\lambda_2+\nu)(h_{\beta_{i,j}})\equiv 1 \mod 2\}.$$

Define $V(\lambda_1+\nu,\lambda_2+\nu)$ to be the $\mathfrak{g}[t]$-module generated by an element $w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfying the following relations:

$$\mathfrak{n}^+[t]w_{\lambda_1+\nu,\lambda_2+\nu}=0,\quad (h\otimes t^r)w_{\lambda_1+\nu,\lambda_2+\nu}=\delta_{r,0}(\lambda+2\nu)(h)w_{\lambda_1+\nu,\lambda_2+\nu},\tag{4.1.2}$$

$$(x_i^-\otimes 1)^{(\lambda+2\nu)(h_i)+1}w_{\lambda_1+\nu,\lambda_2+\nu}=0,\tag{4.1.3}$$

$$(x_\alpha^-\otimes t^{\max\{r_{1,\alpha},r_{2,\alpha}\}})w_{\lambda_1+\nu,\lambda_2+\nu}=0,\quad \alpha\in R^+\tag{4.1.4}$$

$$(x_\alpha^-\otimes t^{s_\alpha-1})^{m_\alpha+1}w_{\lambda_1+\nu,\lambda_2+\nu}=0,\quad \alpha\notin R(\lambda_1,\lambda_2,\nu),\ m_\alpha<4\tag{4.1.5}$$

$$(x_\beta^-\otimes t^{s_\beta})^2 w_{\lambda_1+\nu,\lambda_2+\nu}=0,\quad \beta\in R(\lambda_1,\lambda_2,\nu)\tag{4.1.6}$$

where $1\leq i\leq n$ and $h\in\mathfrak{h}$. Where $r_{j,\alpha}=\left\lceil\frac{(\lambda_j+\nu)(h_\alpha)}{d_\alpha}\right\rceil$, $j\in\{1,2\}$, and recall s_α and m_α are the unique positive integers such that $(\lambda+2\nu)(h_\alpha)=2d_\alpha(s_\alpha-1)+m_\alpha$, with $0<m_\alpha\leq 2d_\alpha$.

Define a grading on $V(\lambda_1+\nu,\lambda_2+\nu)$ by remark by requiring the grade of $w_{\lambda_1+\nu,\lambda_2+\nu}$ to be zero. Clearly $V(\lambda_1+\nu,\lambda_2+\nu)$ is a graded quotient of $W_{\rm loc}(\lambda+2\nu)$ and $V(\lambda_1+\nu,\lambda_2+$

$\nu) \cong V(\lambda_2+\nu, \lambda_1+\nu)$. Lemma 5 shows that if $\lambda_2+\nu = 0$ then $V(\lambda_1+\nu, 0) \cong D(1, \lambda_1+\nu)$.

Remark 22. *It is easy to check that Lemma 21 implies that* $|r_{1,\alpha} - r_{2,\alpha}| \leq 1$ *for all* $\alpha \in R^+$. *Moreover,* $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{2,\alpha} = r_{1,\alpha} + 1$ *only when* $\alpha = \alpha_{i,j}$ *and* $(\lambda_1+\nu)(h_{i,j}) = (\lambda_2+\nu)(h_{i,j}) - 1$ *with* $(\lambda_1+\nu)(h_{i,j})$ *odd. Otherwise* $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{1,\alpha}$.

The following is a consequence of equations (4.1.2)-(4.1.6) and Lemma 4.

Lemma 23. *The assignments* $w_{\lambda_1+\nu,\lambda_2+\nu} \mapsto w_{2,\lambda+2\nu}$ *and* $w_{\lambda_1+\nu,\lambda_2+\nu} \mapsto w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}$ *define surjective maps of graded* $\mathfrak{g}[t]$*-modules*

$$\psi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1+\nu, \lambda_2+\nu) \to D(2, \lambda+2\nu), \quad \varphi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1+\nu, \lambda_2+\nu) \to D(\lambda_1+\nu, \lambda_2+\nu) \to 0,$$

and $\psi_{\lambda_1+\nu,\lambda_2+\nu} = \eta_{\lambda_1+\nu,\lambda_2+\nu} \circ \varphi_{\lambda_1+\nu,\lambda_2+\nu}$. ☐

Proof. The map $\psi_{\lambda_1+\nu,\lambda_2+\nu}$ is obvious. For $\varphi_{\lambda_1+\nu,\lambda_2+\nu}$, note that $(x_\alpha^- \otimes t^k)w_{\lambda_j+\nu} = 0$ for $k \geq r_{j,\alpha}$, $\alpha \in R^+$ and each $j \in \{1, 2\}$. Then relation (4.1.4) follows i.e.

$$(x_\alpha^- \otimes t^{\max\{r_{1,\alpha}, r_{2,\alpha}\}})(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = 0, \ \alpha \in R^+.$$

To show that relation (4.1.5) holds on $w_{\lambda_1\nu} \otimes w_{\lambda_2+\nu}$, recall that s_α and m_α are the unique nonnegative integers such that $\lambda(h_\alpha) = \ell d_\alpha(s_\alpha - 1) + m_\alpha$ with and $0 < m_\alpha \leq \ell d_\alpha$. we note that

$$(x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = \sum_{k=0}^{m_\alpha+1} (x_\alpha^- \otimes t^{s_\alpha-1})^k w_{\lambda_1+\nu} \otimes (x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1-k} w_{\lambda_2+\nu}$$

We will show that each term in the sum above is actually 0.

When $d_\alpha = 2$ equation (2.2.2) gives $(x_\alpha^- \otimes t^{r_{k,\alpha}-1})^2 w_{\lambda_k+\nu} = 0$ when $(\lambda_k+\nu)(h_\alpha)$ is odd, and $(x_\alpha^- \otimes t^{r_{k,\alpha}-1})^3 w_{\lambda_k+\nu} = 0$ when $(\lambda_k+\nu)(h_\alpha)$ is even.

First suppose that $r_{1,\alpha} = r_{2,\alpha}$ with $(\lambda_1+\nu)(h_\alpha) = (\lambda_2+\nu)(h_\alpha)$, then $r_{1,\alpha} = s_\alpha$. If $(\lambda_k+\nu)(h_\alpha)$ are even then $m_\alpha = 4$ and there is nothing to show. If $(\lambda_k+\nu)(h_\alpha)$ are odd then $m_\alpha = 2$, and

$$\begin{aligned}&(x_\alpha^- \otimes t^{s_\alpha-1})^3(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})\\ &= \sum_{k=0}^{3}(x_\alpha^- \otimes t^{s_\alpha-1})^k w_{\lambda_1+\nu} \otimes (x_\alpha^- \otimes t^{s_\alpha-1})^{3-k} w_{\lambda_2+\nu}\end{aligned}$$

So we see that each term is zero. If $\lambda_1(h_\alpha) = \lambda_2(h_\alpha)+1$, then $r_{1,\alpha} = r_{2,\alpha}$ implies that $(\lambda_1+\nu)(h_\alpha)$ is even and $(\lambda_1+\nu)(h_\alpha)$ is odd. Then $m_\alpha = 3$ and

$$\begin{aligned}&(x_\alpha^- \otimes t^{s_\alpha-1})^4(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})\\ &= \sum_{k=0}^{4}(x_\alpha^- \otimes t^{s_\alpha-1})^k w_{\lambda_1+\nu} \otimes (x_\alpha^- \otimes t^{s_\alpha-1})^{4-k} w_{\lambda_2+\nu}\end{aligned}$$

So we see that each term is zero. The case when $\lambda_1(h_\alpha) = \lambda_2(h_\alpha)-1$ is symmetric.

Now suppose that $r_{1,\alpha} = r_{2,\alpha}+1$. Then since $\alpha \notin R(\lambda_1,\lambda_2,\nu)$, we have $r_{1,\alpha} = s_\alpha$ and $r_{2,\alpha} = s_\alpha - 1$. If $(\lambda_1+\nu)(h_\alpha)$ is odd, then $m_\alpha = 2$, and if it is even then $m_\alpha = 3$. In any case we have

$$(x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1}(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = (x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1} w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu} = 0$$

since $(x_\alpha^- \otimes t^{s_\alpha-1})^{m_\alpha+1} w_{\lambda_1+\nu} = 0$ by (2.2.2) in $D(1,\lambda_1+\nu)$. The case when $r_{1,\alpha} = r_{2,\alpha}-1$ is symmetric.

Lastly for $\beta \in R(\lambda_1,\lambda_2,\nu)$, we need to show that (4.1.6) holds. In this case, we have $r_{1,\alpha} = r_{2,\beta}+1 = s_\beta+1$, and that $(\lambda_1+\nu)(h_\beta)$ is odd. Then

$$(x_\beta^- \otimes t^{s_\beta})^2(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = (x_\beta^- \otimes t^{s_\beta})^2 w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}.$$

This proves the existence of the map $\varphi_{\lambda_1+\nu,\lambda_2+\nu}$. $\square$

4.1.3 Statement of Main Theorem

The following is the main result of this section.

Theorem 24. *Let $(\lambda_1, \lambda_2) \in P^+ \times P^+$ be an interlacing pair. For any (λ_1, λ_2)-compatible $\nu \in P^+$ the map*

$$\varphi_{\lambda_1+\nu,\lambda_2+\nu} : V(\lambda_1+\nu, \lambda_2+\nu) \to D(\lambda_1+\nu, \lambda_2+\nu),$$

is an isomorphism. Moreover $D(\lambda_1+\nu, \lambda_2+\nu)$ admits a flag and the quotients are isomorphic to $\tau_s^* D(2, \mu)$ for some $s \in \mathbb{Z}_+$, $\mu \in P^+$.

4.1.4 Condition for Demazure Isomorphism

Similar to the first half of the paper, the theorem is proved in several steps. We start with the following proposition which gives a condition for the generalized Demazure module to be isomorphic to a Demazure module.

Proposition 25.

4.1.5 The Root $\beta_{\lambda,\nu}$

Suppose that $p' < \min \nu$. If $\lambda(h_{1,n-1} + h_n) \geq 3$ or $\lambda(h_{1,n-1} + h_n) \leq 2$ with $p' < \min \nu < p$. Then set

$$\beta_{\lambda,\nu} = \beta_{s,p}, \quad \text{where} \quad s = \begin{cases} p'' & \min \nu \geq p \\ \\ \min \nu & p' < \min \nu < p \end{cases}$$

and note that $\lambda_1(h_{\beta_{\lambda,\nu}}) = 3 - \delta_{s,\min \nu}$ and $\lambda_2(h_{\beta_{\lambda,\nu}}) = 1 - \delta_{s,\min \nu}$.

Lemma 26. *Retain the notation above. If $s = p''$ then $\lambda_1 - \beta_{\lambda,\nu} \in P^+$ and there exists $\nu_0 \in P^+$ such that*

$$(\lambda_1 - \beta_{\lambda,\nu} - \nu_0, \lambda_2 - \nu_0)$$

is an interlacing pair and $\nu + \nu_0$ is

$$(\lambda_1 - \beta_{\lambda,\nu} - \nu_0, \lambda_2 - \nu_0) - \textit{compatible.}$$

If $s = \min \nu$, then $\lambda_1 - \beta_{\lambda,\nu} + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} \in P^+$ and there exists ν_0 such that

$$(\lambda_1 - \beta_{\lambda,\nu} + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0, \lambda_2 + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0)$$

is interlacing and $\nu + \nu_0 - (1 - \delta_{\min \nu, p-1})\omega_{\min \nu}$ is

$$(\lambda_1 - \beta_{\lambda,\nu} + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0, \lambda_2 + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0) - \textit{compatible.}$$

Proof. Suppose $s = p''$, then $\beta_{p'',p} = -\omega_{p''-1} + \omega_{p''} - \omega_{p-1} + \omega_p$. Set $\nu_0 = \lambda_2(h_{p''-1})\omega_{p''-1} + \lambda_2(h_{p-1})\omega_{p-1}$ and it is easy to check that $(\lambda_1 - \beta_{\lambda,\nu} - \nu_0, \lambda_2 - \nu_0)$ is an interlacing pair. Suppose that q, q', q'' are analogous to p, p', p'' for the pair $(\lambda_1 - \beta_{\lambda,\nu} - \nu_0, \lambda_2 - \nu_0)$. It is also not hard to see that $\min(\nu_0 + \nu) \geq p'' - 1 < q''$, and $(\nu_0 + \nu)(h_{q''+1,q-1}) \leq 1$.

If $s = \min \nu$, then $\beta_{\min \nu, p} = -\omega_{\min \nu - 1} + \omega_{\min \nu} - \omega_{p-1} + \omega_p$. Set $\nu_0 = \lambda_2(h_{\min \nu - 1})\omega_{\min \nu - 1}$ and it is easy to check that

$$(\lambda_1 - \beta_{\lambda,\nu} + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0, \lambda_2 + (1 - \delta_{\min \nu, p-1})\omega_{\min \nu} - \nu_0)$$

is interlacing. Suppose that q, q', q'' are analogous to p, p', p'' for the new interlacing pair. Then we can see that in any case $\min(\nu + \nu_0 - (1 - \delta_{\min \nu, p-1})\omega_{\min \nu}) \geq p'' < q''$, and $(\nu + \nu_0 - (1 - \delta_{\min \nu, p-1})(h_{q''+1,q-1}) \leq 1$

□

4.1.6 Upper Bound for $\dim V(\lambda_1+\nu, \lambda_2+\nu)$

The second reduction is the following proposition which will give an upper bound for the dimension of $V(\lambda_1+\nu, \lambda_2+\nu)$.

Proposition 27. *Suppose that $p' < \min\nu$. If $\lambda(h_{1,n-1}+h_n) \geq 3$ or $\lambda(h_{1,n-1}+h_n) \leq 2$ with $p' < \min\nu < p$, then there exists a right exact sequence of $\mathfrak{g}[t]$-modules*

$$\tau^*_{r_{1,\beta_{\lambda,\nu}}-1} V(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu) \to V(\lambda_1+\nu, \lambda_2+\nu) \to D(2, \lambda+2\nu) \to 0, \qquad (4.1.7)$$

with $w_{\lambda_1+\nu-\beta_{\lambda,\nu},\lambda_2+\nu} \to (x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu}$.

4.1.7 Level One Inclusion

Assuming Proposition 25 and Proposition 27 we complete the proof of Theorem 9. The proof is by an induction with respect to the partial order on P^+. The minimal elements with respect to this order are $0, \omega_1$ and Proposition 12 shows that induction begins. Moreover the proposition also establishes the theorem if $p" \min\nu \leq p'$, or $\lambda(h_{1,n-1}+h_n) \leq 2$ with $\min\nu \geq p$. Hence we have only to prove the inductive step when $\lambda(h_{1,n-1}+h_n) \geq 3$ or $\lambda(h_{1,n-1}+h_n) \leq 2$ with $p' < \min\nu < p$. We need the following result to complete the proof of the inductive step.

Lemma 28. *Suppose $p' < \min\nu$. If $\lambda(h_{1,n-1}+h_n) \geq 3$ or $\lambda(h_{1,n-1}+h_n) \leq 2$ with $p' < \min\nu < p$, there exists an inclusion of $\mathfrak{g}[t]$-modules*

$$\tau^*_{r_{1,\beta_{\lambda,\nu}}-1} D(1, \lambda_1+\nu-\beta_{\lambda,\nu}) \hookrightarrow D(1, \lambda_1+\nu), \qquad (4.1.8)$$

which sends to $w_{\lambda_1+\nu-\beta_{\lambda,\nu}} \to (x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu}$.

Proof. By Lemma 5 it suffices to prove that $w := (x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu}$ satisfies the relations in (2.2.1), and the first relation in (2.2.2) when $d_\alpha = 2$. If $\beta_{\lambda,\nu} - \alpha_i \notin R^+$ then it is clear that $(x_i^+ \otimes t^r)w = 0$ for all $r \in \mathbb{Z}_+$. Otherwise we must have $i = p$, or $i = s$ and we have to prove that

$$(x^-_{\beta_{\lambda,\nu}-\alpha_i} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1+r})w_{\lambda_1+\nu} = 0, \quad r \in \mathbb{Z}_+.$$

But this follows from (2.2.2) since in all cases $(\lambda_1 + \nu)(h_i) = 1$, which implies $(\lambda_1 + \nu)(h_{\beta_{\lambda,\nu}-\alpha_i})/d_{\beta_{\lambda,\nu}-\alpha_i} \in \mathbb{Z}^+$, and so $r_{1,\beta_{\lambda,\nu}} - 1 = r_{1,\beta_{\lambda,\nu}-\alpha_i}$. The second relation in (2.2.1) is trivial if $r = 0$ and if $r > 0$ then it follows from (2.2.2) since

$$(h \otimes t^r)w = (x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}+r-1})w_{\lambda_1+\nu} = 0.$$

The third relation is immediate since the modules are all finite-dimensional.

Now for roots α with $d_\alpha = 2$ we need to check the relation $(x^-_\alpha \otimes t^{r'_{1,\alpha}})w = 0$, where $r'_{1,\alpha} = \left\lceil \frac{(\lambda_1 - \beta_{\lambda,\nu}+\nu)(h_\alpha)}{2} \right\rceil$. If $\beta_{\lambda,\nu}(h_\alpha) = 0$, then $\alpha + \beta_{\lambda,\nu} \notin R^+$ since $d_\alpha = 2$, and $r'_{1,\alpha} = r_{1,\alpha}$, so the relation holds.

If $\beta_{\lambda,\nu}(h_\alpha) = -1$, then $\alpha + \beta_{\lambda,\nu} \in R^+$, and $\alpha = \alpha_{i,s-1}$ for $1 \le i \le s-1$, or $\alpha = \alpha_{i,p-1}$ for $1 \le i \le p-1$. Then if $(\lambda_1 - \beta_{\lambda,\nu} + \nu)(h_\alpha)$ is odd then $r'_{1,\alpha} = r_{1,\alpha} + 1$, and $r'_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}} + 1$. If $(\lambda_1 - \beta_{\lambda,\nu} + \nu)(h_\alpha)$ is even then $r'_{1,\alpha} = r_{1,\alpha}$, and $r'_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}}$. In either case, we have

$$\begin{aligned}
&(x^-_\alpha \otimes t^{r'_{1,\alpha}})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1} \\
&= (x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})(x^-_\alpha \otimes t^{r'_{1,\alpha}})w_{\lambda_1} + (x^-_{\alpha+\beta_{\lambda,\nu}} \otimes t^{r'_{1,\alpha}+r_{1,\beta_{\lambda,\nu}}})w_{\lambda_1+\nu} = 0
\end{aligned}$$

If $\beta_{\lambda,\nu}(h_\alpha) = 1$ and $d_\alpha = 2$, then either $\alpha - \beta_{\lambda,\nu} \in R^+$ or $\beta_{\lambda,\nu} - \alpha \in R^+$. If

$\beta_{\lambda,\nu} - \alpha \in R^+$, then $\alpha = \beta_{i,p}$ with $s < i < p$, or $\alpha = \alpha_{s,j}$ with $s \le j < p-1$. If $\alpha = \beta_{i,p}$ then $r'_{1,\beta_{i,p}} = r_{1,\beta_{i,p}}$, and so the relation holds. If $\alpha = \alpha_{s,j}$ then $r'_{1,\alpha_{s,j}} = 1 = r_{1,\alpha_{s,j}} - 1$ and

$$(x^-_{s,j} \otimes t)^2 w_{\lambda_1+\nu} = 0 \in D(1, \lambda_1 + \nu),$$

and $r_{1,\alpha_{s,j}} + r_{1,\beta_{\lambda,\nu}-\alpha_{s,j}} = r_{1,\beta_{\lambda,\nu}}$. So we have

$$0 = (x^-_{\beta_{\lambda,\nu}-\alpha_{s,j}} \otimes t^{r_{1,\beta_{\lambda,\nu}-\alpha_{s,j}}})(x^-_{s,j} \otimes t)^2 = 2(x^-_{s,j} \otimes t)(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu},$$

and the relation holds.

If $\alpha - \beta_{\lambda,\nu} \in R^+$, then $\alpha = \beta_{i,p}$ with $1 \le i \le s-1$, or $\alpha = \beta_{i,s}$ with $1 \le i \le s-1$. Since $\beta_{\lambda,\nu}(h_\alpha) = 1$, if $(\lambda_1 + \nu)(h_\alpha)$ is even then we have $r'_{1,\alpha} = r_{1,\alpha}$, so the relation holds. If $(\lambda_1 + \nu)(h_\alpha)$ is odd then $r'_{1,\alpha} = r_{1,\alpha} - 1$, and $r_{1,\alpha-\beta_{\lambda,\nu}} + r_{!,\beta_{\lambda,\nu}} = r_{1,\alpha}$. Then since

$$(x^-_{\beta_{\lambda\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}})^2 w_{\lambda_1+\nu} = 0 \in D(1, \lambda_1 + \nu),$$

we have that

$$0 = (x^-_{\alpha-\beta_{\lambda,\nu}} \otimes t^{r_{1,\alpha-\beta_{\lambda,\nu}}})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu} = 2(x^-_\alpha \otimes t^{r'_{1,\alpha}})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu}$$

and the relation holds.

Lastly, for $\alpha = \beta_{\lambda,\nu}$, we have $r'_{1,\beta_{\lambda,\nu}} = r_{1,\beta_{\lambda,\nu}} - 1$, but

$$(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu} = 0 \in D(1, \lambda_1 + \nu),$$

so the relation in question holds. This finishes the proof of the existence of the inclusion map. $\square$

4.1.8 Proof of Main Inductive Step

The proof follows along in the same fashion as the Type B_n case. We restate it here for convenience. Lemma 4, Proposition 27 and the inductive hypothesis establish the following inequalities:

$$\dim D(\lambda_1+\nu, \lambda_2+\nu) \leq \dim V(\lambda_1+\nu, \lambda_2+\nu),$$

$$\dim V(\lambda_1+\nu, \lambda_2+\nu) \leq \dim D(2, 2\nu+\lambda) + \dim D(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu).$$

The inductive step follows if we prove that

$$\dim D(\lambda_1+\nu, \lambda_2+\nu) = \dim D(2, 2\nu+\lambda) + \dim D(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu). \tag{4.1.9}$$

For this, we observe that Lemma 28 gives an inclusion

$$0 \to D(1, \lambda_1+\nu-\beta_{\lambda,\nu}) \otimes D(1, \lambda_2+\nu) \to D(1, \lambda_1+\nu) \otimes D(\lambda_2+\nu),$$

which sends

$$w_{\lambda_1+\nu-\beta_{\lambda,\nu}} \otimes w_{\lambda_2+\nu} \to \Big((x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu}\Big) \otimes w_{\lambda_2+\nu}.$$

Since $r_{1,\beta_{\lambda,\nu}} - 1 = r_{2,\beta_{\lambda,\nu}}$ we see that (2.2.2) gives

$$(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})\,(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu}) = \Big((x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu}\Big) \otimes w_{\lambda_2+\nu}.$$

It follows that we have an inclusion

$$\iota : D(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu) \hookrightarrow D(\lambda_1+\nu, \lambda_2+\nu)$$

and it suffices to prove that the corresponding quotient is isomorphic to $D(2, 2\nu+\lambda)$. By Lemma 8 we have the following surjective maps

$$V(\lambda_1+\nu, \lambda_2+\nu) \twoheadrightarrow D(\lambda_1+\nu, \lambda_2+\nu) \twoheadrightarrow D(2, 2\nu+\lambda).$$

These maps are all unique up to scalars and Proposition 14 shows that the kernel of the composite map is generated by the element $(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}$. Hence the kernel of

$$D(\lambda_1+\nu,\lambda_2+\nu) \twoheadrightarrow D(2,2\nu+\lambda)$$

is generated by $(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})(w_{\lambda_1+\nu} \otimes w_{\lambda_2+\nu})$. But this means that the latter kernel is precisely the image of ι and hence the corresponding quotient is isomorphic to $D(2,2\nu+\lambda)$ as needed.

4.1.9 The Graded Character

Just as in the type B_n case, in the course of proving Theorem 24 we also established that the right exact sequence in Proposition 27 is exact. Hence we get an equality of graded characters

$$\operatorname{ch}_{\mathrm{gr}} D(\lambda_1+\nu,\lambda_2+\nu) = \operatorname{ch}_{\mathrm{gr}} D(2,2\nu+\lambda) + v^{(\lambda_1+\nu)(h_{\beta_{\lambda,\nu}})-1} \operatorname{ch}_{\mathrm{gr}} D(\lambda_1-\beta_{\lambda,\nu}+\nu,\lambda_2+\nu). \tag{4.1.10}$$

Using this we can compute $\operatorname{ch}_{\mathrm{gr}} D(\lambda_1+\nu,\lambda_2+\nu)$ in terms of Demazure modules. We will compute an explicit example.

Example 29. *Let $\mathfrak{g}$ be of type C_6. Consider $(\lambda_1,\lambda_2)=(\omega_2+\omega_6,\omega_1+\omega_3)$ and $\nu=\omega_4$. Then $(p'',p',p)=(2,3,6)$ and $\min\nu=4$. Since $3<\min\nu<6$ we have $\beta_{\lambda,\nu}=\beta_{4,6}=\alpha_4+\alpha_5+\alpha_6$, and*

$$\begin{aligned}\operatorname{ch}_{\mathrm{gr}} D(\omega_2+\omega_4+\omega_6,\omega_1+\omega_3+\omega_4) &= \operatorname{ch}_{\mathrm{gr}} D(2,\omega_1+\omega_2+\omega_3+2\omega_4+\omega_6)\\ &\quad + v\operatorname{ch}_{\mathrm{gr}} D(\omega_2+\omega_3+\omega_5,\omega_1+\omega_3+\omega_4)\end{aligned}$$

So our next interlacing pair is $(\omega_2+\omega_5, \omega_1+\omega_4)$ *with* $\nu' = \omega_3$. *With an abuse of notation, we now have* $(p'', p', p) = (2, 4, 5)$, *but* $2 < \min \nu < 4$, *so this means that*

$$D(\omega_2+\omega_3+\omega_5, \omega_1+\omega_3+\omega_4) \cong D(2, \omega_1+\omega_2+2\omega_3+\omega_4+\omega_5).$$

Therefore

$$\begin{aligned}\operatorname{ch}_{\mathrm{gr}} D(\omega_2+\omega_4+\omega_6, \omega_1+\omega_3+\omega_4) &= \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+\omega_3+2\omega_4+\omega_6) \\ &\quad + v \operatorname{ch}_{\mathrm{gr}} D(2, \omega_1+\omega_2+2\omega_3+\omega_4+\omega_5).\end{aligned}$$

4.2 Proof of Proposition 25 and Proposition 27.

4.2.1 Minimality of $\beta_{\lambda,\nu}$

Recall our convention that $\alpha_{i,j} = 0$ if $i > j$, and

$$R(\lambda_1, \lambda_2, \nu) = \{\beta_{i,j} \in R^+ \,:\, (\lambda_1-\lambda_2)(h_{\beta_{i,j}}) = \pm 2,\ (\lambda_1+\nu)(h_{\beta_{i,j}}) \equiv (\lambda_2+\nu)(h_{\beta_{i,j}}) \equiv 1 \mod 2\}.$$

Note that Lemma 21 implies that $R(\lambda_1, \lambda_2, \nu) = \varnothing$ if $p'' < \min \nu \le p'$ or if $\lambda(h_{1,n-1}+h_n) \le 2$ and $\min \nu \ge p$. The next result establishes the converse.

Lemma 30. *Suppose that* $p' < \min \nu$. *If* $\lambda(h_{1,n-1}+h_n) \ge 3$ *or* $\lambda(h_{1,n-1}+h_n) \le 2$ *with* $p' < \min \nu < p$ *then* $\beta_{\lambda,\nu} \in R(\lambda_1, \lambda_2, \nu)$ *and more generally,*

$$\beta_{i,j} \in R(\lambda_1, \lambda_2, \nu) \iff \beta_{i,j} = \alpha_{i,s-1} + \alpha_{j,p-1} + \beta_{\lambda,\nu} \quad \textit{with}$$

$$(\lambda_1+\nu)(h_{i,s-1}+h_{j,p-1}) = (\lambda_2+\nu)(h_{i,s-1}+h_{j,p-1}), \quad \textit{and}$$

$$(\lambda_1+\nu)(h_{i,s-1}+h_{j,p-1}) \equiv (\lambda_2+\nu)(h_{i,s-1}+h_{j,p-1}) \equiv 0 \mod 2.$$

Furthermore, if $\beta_{i,j} \in R(\lambda_1, \lambda_2, \nu)$, *and* $s > j$, *then*

$$(\lambda_1 + \nu)(h_{j,p-1}) = 0 = (\lambda_2 + \nu)(h_{j,p-1}).$$

If $j \le s$, *then* $\alpha_{i,s-1} + \alpha_{j,p-1} = \alpha_{i,p-1} + \alpha_{j,s-1}$, *and*

$$(\lambda_k + \nu)(h_{i,s-1}) \equiv (\lambda_k + \nu)(h_{j,p-1}) \equiv 0 \mod 2$$

or

$$(\lambda_k + \nu)(h_{i,p-1}) \equiv (\lambda_k + \nu)(h_{j,s-1}) \equiv 0 \mod 2$$

for $k \in \{1, 2\}$

Proof. Recall from Section 4.1.5 that $\beta_{\lambda,\nu} = \beta_{s,p}$ where $s \in \{\min \nu, p"\}$. A calculation shows that

$$(\lambda_1 - \lambda_2)(h_{\beta_{i,j}}) = \pm 2 \implies i \le s \text{ or } s < j \le p,$$

and in fact, because of our convention that $\lambda(h_p) = 1 = \lambda_1(h_p)$ we have $(\lambda_1 - \lambda_2)(h_{\beta_{i,j}}) = 2$ which shows that $\beta_{i,j} = \alpha_{i,s-1} + \alpha_{j,p-1} + \beta_\lambda$.

Since $\beta_{i,p} \in R^+$ if $i < s$ we have

$$(\lambda_1 - \lambda_2)(h_{\beta_{i,p}}) = (\lambda_1 - \lambda_2)(h_{i,s-1}) + 2$$

and Lemma 21 forces $(\lambda_1 - \lambda_2)(h_{i,s-1}) \in \{-1, 0\}$. If $j < p$ we have $\beta_{s,j} = \beta_{s,p} + \alpha_{j,p-1}$ and hence a similar argument shows that $(\lambda_1 - \lambda_2)(h_{j,p}) \in \{-1, 0\}$. Together with assumption that $(\lambda_1 - \lambda_2)(h_{\beta_{i,j}}) = \pm 2$ and $(\lambda_1 + \nu)(h_{\beta_{i,j}}) \equiv (\lambda_2 + \nu)(h_{\beta_{i,j}}) \equiv 1 \mod 2$ we get $(\lambda_1 - \lambda_2)(h_{i,s-1}) = 0 = (\lambda_1 - \lambda_2)(h_{j,p-1})$, and $(\lambda_1 + \nu)(h_{i,s-1} + h_{j,p-1}) \equiv (\lambda_2 + \nu)(h_{i,s-1} + h_{j,p-1}) \equiv 0 \mod 2$.

Now if $\beta_{i,j} \in R(\lambda_1, \lambda_2, \nu)$, and $s < j$, then $(\lambda_1+\nu)(h_{j,p-1}) = 0$. If $(\lambda_2+\nu)(h_{j,p-1} \neq 0$, then $s = p''$ and $(\lambda_2 + \nu)(h_{j,p-1} = 1$. This implies that $(\lambda_1 + \nu)(h_{i,s-1})$ is odd, which is a contradiction to that fact that $(\lambda_1 + \nu)(h_{i,s-1} + h_{j,p-1})$ is even.

If $j \leq s$, then $\alpha_{i,s-1} + \alpha_{j,p-1} = \alpha_{i,p-1} + \alpha_{j,s-1}$. Then in any case we have $(\lambda_1 + \nu)(h_{i,p-1}) = (\lambda_1 + \nu)(h_{i,s-1}) + 1$, and $(\lambda_2 + \nu)(h_{j,p-1}) = (\lambda_2 + \nu)(h_{j,s-1}) + 1$, which proves the last statement of the Lemma. □

Remark 31. *If $\beta_{i,j} \in R(\lambda_1, \lambda_2, \nu)$, Lemma 30 implies that we can always write $\beta_{i,j} = \alpha + \alpha' + \beta_{\lambda,\nu}$, where $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{1,\alpha}$, and $\max\{r_{1,\alpha'}, r_{2,\alpha'}\} = r_{1,\alpha'}$, and $r_{1,\alpha} + r_{1,\alpha'} + r_{1,\beta_{\lambda,\nu}} = r_{1,\beta_{i,j}}$.*

4.2.2 Induction Begins

Assume that $p'' < \min \nu \leq p'$ or $\lambda(h_{1,n-1} + h_n) \leq 2$ and $\min \nu \geq p$. By Lemma 21 and Lemma 30 we have $(\lambda_1-\lambda_2)(h_\alpha) \in \{-1, 0, 1\}$ or $(\lambda_1-\lambda_2)(h_{\beta_{i,j}}) = 2$ with $(\lambda_1+\nu)(h_{\beta_{i,j}}) \equiv (\lambda_1 + \nu)(h_{\beta_{i,j}}) \equiv 0 \mod 2$, and so $R(\lambda_1, \lambda_2, \nu) = \varnothing$. The defining relations give

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \cong D(2, 2\nu + \lambda_1 + \lambda_2).$$

Since the maps in Lemma 4 and Lemma 23 are unique up to scalars, it now follows that the map

$$V(\lambda_1 + \nu, \lambda_2 + \nu) \twoheadrightarrow D(\lambda_1 + \nu, \lambda_2 + \nu) \twoheadrightarrow D(2, 2\nu + \lambda_1 + \lambda_2),$$

is an isomorphism and hence all the maps are isomorphisms. Proposition 25 is proved.

4.2.3 Cyclic Kernel

We turn to the proof of Proposition 27. We have $R(\lambda_1,\lambda_2,\nu)\neq\varnothing$ and our conventions imply that $\lambda_1(h_{\beta_{s,p}})=3-\delta_{s,\min\nu}$ and $\lambda_2(h_{\beta_{s,p}})=1-\delta_{s,\min\nu}$. From Lemma 30 we actually have $\lambda_1(h_\beta)\geq\lambda_2(h_\beta)$ for all $\beta\in R(\lambda_1,\lambda_2,\nu)$. Hence for all $\beta\in R(\lambda_1,\lambda_2,\nu)$,

$$\max\{r_{1,\beta},r_{2,\beta}\}=r_{1,\beta}=s_\beta+1.$$

Furthermore, By writing

$$s_\alpha=\left\lceil\frac{(\lambda+2\nu)(h_\alpha)}{2d_\alpha}\right\rceil$$

and using Lemma 21, we have $\max\{r_{1,\alpha},r_{2,\alpha}\}=s_\alpha$ for $\alpha\notin R(\lambda_1,\lambda_2,\nu)$.

An inspection of the defining relations of the modules shows that the kernel K of the canonical map

$$\psi_{\lambda_1+\nu,\lambda_2+\nu}:V(\lambda_1+\nu,\lambda_2+\nu)\to D(2,\lambda+\nu)\to 0,\quad w_{\lambda_1+\nu,\lambda_2+\nu}\to w_{2\nu+\lambda},$$

is generated by the elements

$$(x_\beta^-\otimes t^{r_{1,\beta}-1})w_{\lambda_1+\nu,\lambda_2+\nu},\quad \beta\in R(\lambda_1,\lambda_2,\nu).$$

The proof of Proposition 27 is now shown in two steps. The first step is to show that

$$K=\mathbf{U}(\mathfrak{g}[t])(x_{\beta_{\lambda,\nu}}^-\otimes t^{r_{1,\beta_{\lambda,\nu}}-1)})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

For this, if $\beta_{i,j}\in R(\lambda_1,\lambda_2,\nu)$, Lemma 30 implies that we can always write $\beta_{i,j}=\alpha+\alpha'+\beta_{\lambda,\nu}$, where $\max\{r_{1,\alpha},r_{2,\alpha}\}=r_{1,\alpha}$, and $\max\{r_{1,\alpha'},r_{2,\alpha'}\}=r_{1,\alpha'}$, and $r_{1,\alpha}+r_{1,\alpha'}+r_{1,\beta_{\lambda,\nu}}=r_{1,\beta_{i,j}}$. and assume that $i\leq s-1$ or $j\leq p-1$ (otherwise $\beta=\beta_{\lambda,\nu}$ and there is nothing to prove).

The defining relations

$$(x_\alpha^- \otimes t^{r_{1,\alpha}})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x_{\alpha'}^- \otimes t^{r_{1,\alpha'}})w_{\lambda_1+\nu,\lambda_2+\nu},$$

imply if $i \leq s-1$ and $j \leq p-1$

$$(x_\alpha^- \otimes t^{r_{1,\alpha}})(x_{\alpha'}^- \otimes t^{r_{1,\alpha'}})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} =$$

$$[x_\alpha^-, [x_{\alpha'}^-, x_{\beta_{\lambda,\nu}}^-]] \otimes t^{r_{1,\beta_{i,j}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} = (x_{\beta_{i,j}}^- \otimes t^{r_{1,\beta_{i,j}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

An obvious modification works if $i \geq s$ or $j \geq p$ and the first step is proved.

The next step is to show that the element $(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the defining relations of the element $w_{\lambda_1+\nu-\beta_{\lambda,\nu},\lambda_2+\nu} \in V(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu)$. This establishes the existence of the map $V(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu) \to K \to 0$.

4.2.4 Quotient of a Local Weyl Module

We prove that $(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the relations in (4.1.2) and (4.1.3) with $(\lambda_1+\nu, \lambda_2+\nu)$ replaced by $(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu)$. Since $(x_i^+ \otimes t^r)w_{\lambda_1+\nu,\lambda_2+\nu} = 0$ for $1 \leq i \leq n$ and $r \in \mathbb{Z}_+$ the first relation in (4.1.2) is immediate if $\beta_{\lambda,\nu} - \alpha_i \notin R^+$. Otherwise we must prove that

$$(\dagger) \quad (x_{\beta_{\lambda,\nu}-\alpha_i}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1+r})w_{\lambda_1+\nu,\lambda_2+\nu} = 0.$$

Since $\beta_{\lambda,\nu} = \beta_{s,p}$, it follows that either $i = p$ or $i = s$. In the first case we have $(\lambda_1+\nu)(h_p) = 1$ and $(\lambda_2+\nu)(h_p) = 0$ (see Section 4.1.5). In the second case we have $(\lambda_1+\nu)(h_s) = 1$ and $(\lambda_2+\nu)(h_s) = \delta_{s,\min\nu}$. This implies that $\max\{r_{1,\beta_{\lambda,\nu}-\alpha_i}, r_{2,\beta_{\lambda,\nu}-\alpha_i}\} = r_{1,\beta_{\lambda,\nu}-\alpha_i} = r_{1,\beta_{\lambda,\nu}}-1$, and so

$$(x_{\beta_{\lambda,\nu}-\alpha_i}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

is a defining relation and (†) follows by applying $(h \otimes t^r)$. It is trivial to see that relation (4.1.3) holds and the third is then immediate since $V(\lambda_1+\nu, \lambda_2+\nu)$ is finite-dimensional.

4.2.5 Existence of Map Onto Kernel

To complete the proof of the second step, we now show that the relations (3.1.4), (4.1.5), and (4.1.6) also hold. It is useful to note from 4.1.5 that $(\lambda_1+\nu)(h_{\beta_{\lambda,\nu}})$ is always odd, and remark 22 will also come in handy.

Set $r'_{1,\alpha} = \left\lceil \frac{(\lambda_1+\nu-\beta_{\lambda,\nu})(h_\alpha)}{d_\alpha} \right\rceil$. We start with relation (4.1.4), that is, we need to show that

$$(x_\alpha^- \otimes t^{\max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\}})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \text{ for all } \alpha \in R^+.$$

First, suppose that $\beta_{\lambda,\nu}(h_\alpha) = 0$. Then $\max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\} = \max\left\{r_{1,\alpha}, r_{2,\alpha}\right\}$, so if $\alpha + \beta_{\lambda,\nu} \notin R^+$ then the relation holds. In this case $\alpha+\beta_{\lambda,\nu} \in R^+$ only when $\alpha = \alpha_{s,p-1}$, where $s \in \{p'', \min\nu\}$ for $\beta_{\lambda,\nu} = \beta_{s,p}$. So we need to check that

$$\max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\} + r_{1,\beta_{\lambda,\nu}} - 1 \geq \max\left\{r_{1,\alpha+\beta_{\lambda,\nu}}, r_{2,\alpha+\beta_{\lambda,\nu}}\right\}$$

In this case we have $(\lambda_k+\nu)(h_{s,p-1}) = \delta_{s,p''} + \delta_{s,\min\nu} = 1$, for $k \in \{1,2\}$, so that $\max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\} = 1$. Then we have

$$1 + \left(r_{1,\beta_{\lambda,\nu}} - 1\right) = r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}} = \max\left\{r_{1,\alpha+\beta_{\lambda,\nu}}, r_{2,\alpha+\beta_{\lambda,\nu}}\right\}.$$

So in this case the relation holds.

Now suppose that $\beta_{\lambda,\nu}(h_\alpha) = -1$. Then we have that $\alpha = \alpha_{i,s-1}$ or $\alpha = \alpha_{i,p-1}$ for $1 \leq i \leq s-1$ or $1 \leq i \leq p-1$ respectively, and $\alpha + \beta_{\lambda,\nu} \in R^+$. By our convention on

(λ_1, λ_2) and Lemma 4.1.1, in this case we have $0 \leq (\lambda_2 - \lambda_1)(h_\alpha) \leq 1$, then we can see that

$$r'_{1,\alpha} = \max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\} \geq \max\left\{r_{1,\alpha}, r_{2,\alpha}\right\}, \quad \text{and} \quad r_{1,\alpha+\beta_{\lambda,\nu}} = \max\left\{r_{1,\alpha+\beta_{\lambda,\nu}}, r_{2,\alpha+\beta_{\lambda,\nu}}\right\}$$

so we need to check that

$$r'_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} - 1 \geq r_{1,\alpha+\beta_{\lambda,\nu}}.$$

If $(\lambda_1 + \nu)(h_\alpha)$ is odd then $r'_{1,\alpha} = r_{1,\alpha}$, and

$$(r_{1,\beta_{\lambda,\nu}} - 1) + r'_{1,\alpha} = r_{1,\alpha+\beta_{\lambda,\nu}}$$

If $(\lambda_1 + \nu)(h_\alpha)$ is even, then $r'_{1,\alpha} = r_{1,\alpha} + 1$, and

$$(r_{1,\beta_{\lambda,\nu}} - 1) + r'_{1,\alpha} = r_{1,\alpha+\beta_{\lambda,\nu}}$$

and either way, the relation holds.

Now suppose that $\beta_{\lambda,\nu}(h_\alpha) = 1$, then $\alpha + \beta_{\lambda,\nu} \notin R^+$ and either $\beta_{\lambda,\nu} - \alpha \in R^+$ or $\alpha - \beta_{\lambda,\nu} \in R^+$. Furthermore, by Lemma 3.1.1 we have that $|r_{1,\alpha} - r_{2,\alpha}| \leq 1$ for all $\alpha \in R^+$. If $r_{1,\alpha} \leq r_{2,\alpha}$, or if $d_\alpha = 2$ with $r_{1,\alpha} > r_{2,\alpha}$ and $(\lambda_1 + \nu)(h_\alpha)/2 \in \mathbb{Z}^+$, then in any case we have that $\max\{r_{1,\alpha}, r_{2,\alpha}\} = \max\left\{r'_{1,\alpha}, r_{2,\alpha}\right\}$, so the relation holds.

If $r_{1,\alpha} = r_{2,\alpha} + 1$ and $d_\alpha = 1$ or $(\lambda_1 + \nu)(h_\alpha)/d_\alpha \notin \mathbb{Z}^+$, then $r'_{1,\alpha} = r_{1,\alpha} - 1$. However, relation (4.1.5) gives that

$$(\dagger) \quad (x_\alpha^- \otimes t^{r_{1,\alpha}-1})^2 w_{\lambda_1+\nu,\lambda_2} = 0 = (x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu}.$$

Recall that $\beta_{\lambda,\nu} = \beta_{s,p}$ where $s \in \{p'', \min \nu\}$. If $\beta_{\lambda,\nu} - \alpha \in R^+$, then α is one of the following:

- $\beta_{i,p}$ $s < i < p$
- $\alpha_{s,j}$ $s \leq j < p-1$
- β_p

In any case, we have that $d_{\beta_{\lambda,\nu}-\alpha} = 2$, and since $\lambda_1(h_{\beta_{\lambda,\nu}}) = \lambda_2(h_{\beta_{\lambda,\nu}})+2$, we have

$$\max\left\{r_{1,\beta_{\lambda,\nu}-\alpha}, r_{2,\beta_{\lambda,\nu}-\alpha}\right\} = r_{1,\beta_{\lambda,\nu}-\alpha}.$$

If $d_\alpha = 2$, then since $(\lambda_1+\nu)(h_{\beta_{\lambda,\nu}})$ and $(\lambda_1+\nu)(h_\alpha)$ are both odd, we have that $(\lambda_1 + \nu)(h_{\beta_{\lambda,\nu}-\alpha})$ is even, so no matter if $d_\alpha = 1$ or if $d_\alpha = 2$, we get

$$r_{1,\beta_{\lambda,\nu}-\alpha} + r_{1,\alpha} = r_{1,\beta_{\lambda,\nu}}$$

Therefore, by † and since $\alpha + \beta_{\lambda,\nu} \notin R^+$, we have

$$\begin{aligned} 0 &= (x^-_{\beta_{\lambda,\nu}-\alpha} \otimes t^{r_{1,\beta_{\lambda,\nu}-\alpha}})(x^-_\alpha \otimes t^{r_{1,\alpha}-1})^2 w_{\lambda_1+\nu,\lambda_2} \\ &= 2(x^-_\alpha \otimes t^{r_{1,\alpha}-1})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu} \end{aligned}$$

and so the relation holds.

Now suppose that $\alpha - \beta_{\lambda,\nu} \in R^+$. Then α is one of the following:

- $\beta_{i,p}$ $1 \leq i \leq s-1$
- $\beta_{i,s}$ $1 \leq i \leq s-1$
- β_s

In any case, $d_{\alpha-\beta_{\lambda,\nu}} = 2$, and since $\lambda_1(h_{\beta_{\lambda,\nu}}) = \lambda_2(h_{\beta_{\lambda,\nu}}) + 2$, and $r_{1,\alpha} = r_{2,\alpha}+1$ by assumption, we have

$$\max\left\{r_{1,\alpha-\beta_{\lambda,\nu}}, r_{2,\alpha-\beta_{\lambda,\nu}}\right\} = r_{1,\alpha-\beta_{\lambda,\nu}}.$$

As before, since $(\lambda_1+\nu)(h_{\alpha-\beta_{\lambda,\nu}})$ is always even, then in any case we have

$$r_{1,\alpha-\beta_{\lambda,\nu}} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha}.$$

Then by † we have

$$0 = (x^-_{\alpha-\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}-\alpha}})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu,\lambda_2}$$

$$= 2(x^-_\alpha \otimes t^{r_{1,\alpha}-1})(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu}.$$

Lastly, if $\beta_{\lambda,\nu}(h_\alpha) = 2$, this implies that $\alpha = \beta_{\lambda,\nu}$, and by relation (4.1.6) we have

$$(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

which is exactly what we needed to show. Therefore, the relation (4.1.4) holds for all $\alpha \in R^+$ with weights $(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu)$ on the vector $(x^-_{\beta_{\lambda,\nu}} \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu}$

Now we need to show the relations in (4.1.5) and (4.1.6) hold, i.e., for $\alpha \in R^+$ with $d_\alpha = 2$

$$(x^-_\alpha \otimes t^{s'_\alpha-1})^{m'_\alpha+1}(x^-_{\beta_{\lambda,\nu}} t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad \alpha \notin R(\lambda'_1, \lambda'_2, \nu')$$

$$(x^-_\beta \otimes t^{s'_\beta})^2 (x^-_{\beta_{\lambda,\nu}} t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu} = 0, \quad \beta \in R(\lambda'_1, \lambda'_2, \nu')$$

where s'_α and m'_α are the unique nonnegative integers satisfying

$$(\lambda - \beta_\lambda + 2\nu)(h_\alpha) = 4(s'_\alpha - 1) + m'_\alpha,$$

with $1 < m'_\alpha \leq 4$. If $s = p''$ then

$$(\lambda'_1, \lambda'_2, \nu') = (\lambda_1 - \beta_{\lambda,\nu} - \nu_0, \lambda_2 - \nu_0, \nu_0 + \nu),$$

and if $s = \min\nu$ then

$$(\lambda_1', \lambda_2', \nu') =$$

$$(\lambda_1 - \beta_{\lambda,\nu} + (1-\delta_{\min\nu,p-1})\omega_{\min\nu} - \nu_0, \lambda_2 + (1-\delta_{\min\nu,p-1})\omega_{\min\nu} - \nu_0, \nu_0 + \nu - (1-\delta_{\min\nu,p-1})\omega_{\min\nu})$$

from Lemma 26. We only have to show the relation (4.1.5) when $(\lambda - \beta_\lambda + 2\nu)(h_\alpha) > 4$, and $m_\alpha' \neq 4$. We will start with this relation.

First, if $\beta_{\lambda,\nu}(h_\alpha) = 0$, then $\alpha + \beta_{\lambda,\nu} \notin R^+$ since $d_\alpha = 2$. Then $m_\alpha' = m_\alpha$ and $s_\alpha' = s_\alpha$, so the relation holds.

Now suppose that $\beta_{\lambda,\nu}(h_\alpha) = -1$. Then we have that $\alpha = \alpha_{i,s-1}$ or $\alpha = \alpha_{i,p-1}$ for $1 \le i \le s-1$ or $1 \le i \le p-1$ respectively, and $\alpha + \beta_{\lambda,\nu} \in R^+$. By our convention on (λ_1, λ_2) and Lemma 4.1.1, in this case we have $0 \le (\lambda_2 - \lambda_1)(h_\alpha) \le 1$. Since $\lambda_1(h_{\beta_{\lambda,\nu}}) = \lambda_2(h_{\beta_{\lambda\nu}}) + 2$, we always have

$$\max\{r_{1,\alpha+\beta_{\lambda,\nu}}, r_{2,\alpha+\beta_{\lambda,\nu}}\} = r_{1,\alpha+\beta_{\lambda,\nu}}$$

Recall that we are only checking the relation when $(\lambda - \beta_{\lambda,\nu} + 2\nu)(h_\alpha) > 4$. If $(\lambda_1 + \nu)(h_\alpha) = (\lambda_2 + \nu)(h_\alpha) - 1$ with $(\lambda_1 + \nu)(h_\alpha)$ odd, then $m_\alpha' = 4$ so there is nothing to check.

If $(\lambda_1 + \nu)(h_\alpha) = (\lambda_2 + \nu)(h_\alpha) - 1$ with $(\lambda_1 + \nu)(h_\alpha)$ even, then $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{1,\alpha} + 1$ and $s_\alpha' = s_\alpha = r_{1,\alpha} + 1$ and $m_\alpha' = 2 = m_\alpha + 1$. Noting that $r_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}}$ and that

$$(x_\alpha^- \otimes t^{s_\alpha - 1})^2 w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

is a defining relation in $V(\lambda_1+\nu, \lambda_2+\nu)$, we have

$$(x_\alpha^- \otimes t^{s'_\alpha-1})^3(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} =$$

$$(x_\alpha^- \otimes t^{s'_\alpha-1})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+ 2(x_{\alpha+\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\alpha+\beta_{\lambda,\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= 0.$$

If $(\lambda_1+\nu)(h_\alpha) = (\lambda_2+\nu)(h_\alpha)$ and they are both even, then $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{1,\alpha}$ and $s'_\alpha = s_\alpha + 1 = r_{1,\alpha} + 1$ and $m'_\alpha = 1 = m_\alpha - 3$. Noting that $r_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}}$ and that

$$(x_\alpha^- \otimes t^{s_\alpha})w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

is a defining relation in $V(\lambda_1+\nu, \lambda_2+\nu)$, we have

$$(x_\alpha^- \otimes t^{s'_\alpha-1})^2(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} =$$

$$(x_\alpha^- \otimes t^{s'_\alpha-1})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha})w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+ (x_{\alpha+\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\alpha+\beta_{\lambda,\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha})$$

$$= 0.$$

If $(\lambda_1+\nu)(h_\alpha) = (\lambda_2+\nu)(h_\alpha)$ and they are both odd, then $\max\{r_{1,\alpha}, r_{2,\alpha}\} = r_{1,\alpha}$ and $s'_\alpha = s_\alpha = r_{1,\alpha}$ and $m'_\alpha = 3 = m_\alpha + 1$. Noting that $r_{1,\alpha} + r_{1,\beta_{\lambda,\nu}} = r_{1,\alpha+\beta_{\lambda,\nu}} + 1$ and that

$$(x_\alpha^- \otimes t^{s_\alpha-1})^3 w_{\lambda_1+\nu,\lambda_2+\nu} = 0$$

is a defining relation in $V(\lambda_1+\nu, \lambda_2+\nu)$, we have

$$(x_\alpha^- \otimes t^{s'_\alpha - 1})^2 (x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu} =$$

$$(x_\alpha^- \otimes t^{s'_\alpha - 1})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda-\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha - 1}) w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$+ 3(x_{\alpha+\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\alpha+\beta_{\lambda,\nu}}-1})(x_\alpha^- \otimes t^{s_\alpha-1})^3 w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= 0.$$

Now suppose that $\beta_{\lambda,\nu}(h_\alpha) = 1$, then $\alpha + \beta_{\lambda,\nu} \notin R^+$ and either $\beta_{\lambda,\nu} - \alpha \in R^+$ or $\alpha - \beta_{\lambda,\nu} \in R^+$. Recall, by Lemma 3.1.1 we have that $|r_{1,\alpha} - r_{2,\alpha}| \leq 1$ for all $\alpha \in R^+$. If $\beta_{\lambda,\nu} - \alpha \in R^+$with $d_\alpha = 2$, then α is one of the following:

- $\beta_{i,p}$ $s < i < p$
- $\alpha_{s,j}$ $s \leq j < p-1$

Recall from before that since $\lambda_1(h_{\beta_{\lambda,\nu}}) = \lambda_2(h_{\beta_{\lambda,\nu}}) + 2$, we have

$$\max\left\{r_{1,\beta_{\lambda,\nu}-\alpha}, r_{2,\beta_{\lambda,\nu}-\alpha}\right\} = r_{1,\beta_{\lambda,\nu}-\alpha}.$$

In either case we have $s'_\alpha = r_{1,\alpha} = s_\alpha$, and $m'_\alpha = m_\alpha - 1$. Then the result follows from

$$0 = (x_{\beta_{\lambda,\nu}-\alpha}^- \otimes t^{r_{1,\beta_{\lambda,\nu}-\alpha}})(x_\alpha^- \otimes t^{r_{1,\alpha}-1})^{m_\alpha+1} w_{\lambda_1+\nu,\lambda_2+\nu}$$

$$= (m_\alpha+1)(x_\alpha^- \otimes t^{s'_\alpha-1})^{m'_\alpha+1}(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1}) w_{\lambda_1+\nu,\lambda_2+\nu}.$$

For the second case we always have $(\lambda - \beta_{\lambda,\nu} + 2\nu)(h_{s,j}) \leq 1$, so there is nothing to check.

Now suppose that $\alpha - \beta_{\lambda,\nu} \in R^+$. Since $d_\alpha = 2$ then α is one of the following:

- $\beta_{i,p}$ $1 \leq i \leq s-1$
- $\beta_{i,s}$ $1 \leq i \leq s-1$

By Lemma 21, we have that $(\lambda_1 - \lambda_2)(h_\alpha) \geq 0$ for roots of these type. So in this case, $s'_\alpha = s_\alpha - 1$ if and only if $r_{1,\alpha} = r_{2,\alpha} + 1$ and $(\lambda_1 + \nu)(h_\alpha)$ is odd. But then $m'_\alpha = 4$

and there is nothing to check. Otherwise, $s'_\alpha = r_{1,\alpha} = s_\alpha$, and $m'_\alpha = m_\alpha - 1$. Then the relation follows from

$$0 = (x_\alpha^- \otimes t^{r_{1,\alpha}-1})^{m'_\alpha}(x_{\alpha-\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\alpha-\beta_{\lambda,\nu}}})(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$
$$= (m_\alpha+1)(x_\alpha^- \otimes t^{s'_\alpha-1})^{m'_\alpha+1}(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}.$$

If $\beta_{\lambda,\nu}(h_\alpha) = 2$, then $\alpha = \beta_{\lambda,\nu}$ and $s'_{\beta_{\lambda,\nu}} = r_{1,\beta_{\lambda,\nu}} - 1 = s_{\beta_{\lambda,\nu}}$. The fact that

$$(x_{\beta_{\lambda,\nu}}^- \otimes t^{s_\beta+1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0 = (x_{\beta_{\lambda,\nu}}^- \otimes t^{s_\beta})^2 w_{\lambda_1+\nu,\lambda_2+\nu}$$

by relations (4.1.4) and (4.1.6), implies that

$$0 = (x_{\beta_{\lambda,\nu}}^+ \otimes t)^{4s_{\beta_{\lambda,\nu}}-3}(x_{\beta_{\lambda,\nu}}^- \otimes 1)^{(\lambda+2\nu)(h_{\beta_{\lambda,\nu}})+1} w_{\lambda_1+\nu,\lambda_2+\nu} = (x_{\beta_{\lambda,\nu}}^- \otimes t^{s_{\beta_{\lambda,\nu}}-1})^3(x_{\beta_{\lambda,\nu}}^- \otimes t^{s_{\beta_{\lambda,\nu}}}),$$

but $s_{\beta_{\lambda,\nu}} = r_{1,\beta_{\lambda,nu}} - 1$, so this is exactly what we needed to show.

Now for relation (4.1.6). Using Lemma 7 and the fact that $\lambda_1(h_{\beta_{i,j}}) \geq \lambda_2(h_{\beta_{i,j}})$ for all $1 \leq i < j \leq n$, it is not hard to see that $\beta_{i,j} \in R(\lambda'_1, \lambda'_2, \nu')$ if and only if $\beta_{i,j} \in R(\lambda_1, \lambda_2, \nu)$ and $\beta_{\lambda,\nu}(h_{\beta_{\lambda,\nu}}) = 0$. So $\beta_{\lambda,\nu} + \beta_{i,j} \notin R^+$ and $s'_{\beta_{i,j}} = s_{\beta_{i,j}}$ implies that

$$(x_{\beta_{i,j}}^- \otimes t^{s_{\beta_{i,j}}-1})^2(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu} = 0,$$

so the relation holds.

This concludes the proof that $(x_{\beta_{\lambda,\nu}}^- \otimes t^{r_{1,\beta_{\lambda,\nu}}-1})w_{\lambda_1+\nu,\lambda_2+\nu}$ satisfies the relations in (4.1.2) and (4.1.3) with $(\lambda_1+\nu, \lambda_2+\nu)$ replaced by $(\lambda_1+\nu-\beta_{\lambda,\nu}, \lambda_2+\nu)$.

Chapter 5

Conclusion

The presentation given for the modules $D(\lambda, \mu) \subset D(1, \lambda) \otimes D(1, \mu)$ is work towards a more general conjecture that I eventually wish to prove. It has to do with the family of modules defined by Chari and Venkatesh in [11]. These modules $V(\xi)$ for $\mathfrak{g}[t]$ are quotients of local Weyl modules of a given weight $\lambda \in P^+$ that are indexed by $|R^+|$-tuples of partitions $\xi = (\xi^\alpha)_{\alpha \in R^+}$, where ξ^α is a partition of $\lambda(h_\alpha)$. For the defining relations see [11]. If we think of partitions as infinite tuples of nonnegative integers starting with the largest number with 0's in all but finitely many components. Then define the sum of two partitions by point wise addition. For two $|R^+|$-tuples of partitions ξ_1 and ξ_2, define $\xi_1 + \xi_2 = (\xi_1^\alpha + \xi_2^\alpha)_{\alpha \in R^+}$. Now we can state the conjecture.

Conjecture 32. *If v_{ξ_1} and v_{ξ_2} are the generating highest weight vectors for $V(\xi_1)$ and $V(\xi_2)$ respectively, then we have the following isomorphism of $\mathfrak{g}[t]$-modules*

$$V(\xi_1 + \xi_2) \cong \mathbf{U}(\mathfrak{g}[t])(v_{\xi_1} \otimes v_{\xi_2}) \subset V(\xi_1) \otimes V(\xi_2)$$

This is interesting because many examples of graded limits of modules for quantum affine Lie algebras have been shown to be isomorphic to these generalized Demazure modules, and one hopes that this conjecture would be a useful tool in this context.

In [11] they showed that the stable Demazure modules $D(\ell, \lambda)$ can be realized as a member of this family of $\mathfrak{g}[t]$-modules. One can show that the modules $V(\lambda, \mu)$ in this paper and in [7] are also members of this family, and in fact are evidence of this conjecture. Proving this in general will most likely take combinatorial and geometric methods, as these tools are useful with respect to Demazure modules. Although not stated or proven in this context, the conjecture is true for $\mathfrak{sl}_2$.

When $\mathfrak{g}$ is simply laced, it is known that $D(1, \lambda) \cong W_{\text{loc}}(\lambda)$. In [8] Chari and Loktev exhibited a basis for these local Weyl modules. One would hope to use these to construct a basis for the generalized Demazure module $D(\lambda, \mu)$, and there seems to be some deep combinatorics involved in this problem. Similarly, it would be nice to use the rich theory of characters of Demazure modules to develop a general character formula for these generalized Demazure modules.

Bibliography

[1] R. Biswal, V. Chari, P. Shereen and J. Wand, *Macdonald Polynomials and level two Demazure modules for affine* $\mathfrak{sl}_{n+1}$, (2019) arXiv:1910.05848.

[2] M. Brito, and F. Pereira, *Graded Limits of Simple Tensor Product of Kirillov–Reshetikhin Modules for* $U_q(\tilde{\mathfrak{sl}}_{n+1})$, Comm. in Algebra, 44, **10**, (2016), 4504-4518.

[3] M. Brito, and V. Chari, *Tensor products and q-characters of HL-modules and monoidal categorifications*, Journal de l'École Polytechnique — Mathématiques, **6**, (2019), 581-619.

[4] M. Brito, V. Chari, and A. Moura, *Demazure modules of level two and prime representations of quantum affine* sl_{n+1}, J. Inst. Math. Jussieu, 31 pages, 2015.

[5] V. Chari, *On the fermionic formula and the Kirillov-Reshetikhin conjecture*, Int. Math. Res. Notices **12** (2001), 629–654.

[6] V. Chari, *Braid group actions and tensor products*, Int. Math. Res. Notices (2002), 357–382.

[7] V. Chari, J. Davis, and R. Moruzzi, *Generalized Demazure Modules and Prime Representations in Type* D_n, (2019) arXiv:1911.07155.

[8] V. Chari and S. Loktev, *Weyl, Demazure and fusion modules for the current algebra of* sl_{r+1}, Adv. Math. **207** (2006), no.2, 928-960

[9] V. Chari and A. Moura, *The restricted Kirillov–Reshetikhin modules for the current and twisted current algebras,* Comm. Math. Phys. **266** (2006), no. 2, 431–454.

[10] V. Chari, A. Pressley, *Weyl Modules for Classical and Quantum Affine Algebras* Reprsent Theory, **5** (2001), 191-223.

[11] V. Chari and R. Venkatesh, *Demazure modules, fusion products and Q-systems*, Comm. Math. Phy. **333** (2), (2015), 799–830,

[12] P. Di Francesco and R. Kedem, *Proof of the combinatorial Kirillov-Reshetikhin conjecture*, Int. Math. Res. Not. IMRN **7** (2008).

[13] E. Frenkel and E. Mukhin, *Combinatorics of q-Characters of Finite-Dimensional Representations of Quantum Affine Algebras*, Comm. Math. Phy. **216**, no. 1, (2001), pp 23-57.

[14] E. Frenkel and E. Mukhin, *The Hopf algebra* $RepU_q(gl_\infty)$, Selecta Math. (N.S.) **8**, no. 4, (2002), 537–635.

[15] E. Frenkel and N. Reshetikhin, *The q-Characters of Representations of Quantum Affine Algebras and Deformations of W-Algebras*, Recent Developments in Quantum Affine Algebras and related topics, Cont. Math., vol. 248, (1999), 163–205.

[16] G. Fourier, P. Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math., 211 (2)(2007), pp. 566-593.

[17] N. Fujita, E. Lee and D.Y. Suh *varieties and applications to representations*, 1805.01664, 2018.

[18] H. Garland, *The arithmetic theory of loop algebras*, J. Algebra **53** (1978), 480–551.

[19] D. Hernandez, *The Kirillov-Reshetikhin conjecture and solutions of T-systems*, J. Reine Angew. Math. **596**, (2006), 63–87/

[20] D. Hernandez and B. Leclerc, *Cluster algebras and quantum affine algebras*, Duke Math. J. **154** (2010), 265–341, DOI 10.1215/00127094-2010-040.

[21] D. Hernandez and B. Leclerc, *Monoidal categorifications of cluster algebras of type A and D*, Symmetries, Integrable Systems and Representations, Springer Proceedings in Mathematics & Statistics 40 (2013), 175–193.

[22] A. Joseph, *Modules with a Demazure flag, in Studies in Lie Theory*, Progr. Math., Vol. 243, Birkhäuser Boston, Boston, MA, 2006, 131–169.

[23] S. Kumar, *Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture,* Invent. Math. **93** (1988), 117-130.

[24] V. Lakshmibai, P. Littelmann, and P. Magyar, *Standard monomial theory for Bott–Samelson varieties*, Compos. Math., 130(3):293–318, 2002.

[25] J.R. Li and K. Naoi, *Graded limits of minimal affinizations over the quantum loop algebra of type* G_2, Algebras and Representation Theory 19 **4** (2016), 957-973.

[26] O. Mathieu, *Construction d'un groupe de Kac-Moody et applications*, Compositio Math. **69** (1989), no. 1, 37–60.

[27] A. Moura, *Restricted limits of minimal affinizations*, Pacific J. Math. **244** (2010), 359–397.

[28] A. Moura and F. Pereira, *Graded limits of minimal affinizations and beyond: the multiplicity free case for type* E_6, Algebra and Discrete Mathematics **12** (2011), 69-115.

[29] H. Nakajima, *Quiver Varieties and t-Analogs of q-Characters of Quantum Affine Algebras*, Ann. of Math. **160** (2004), 1057 - 1097.

[30] H. Nakajima, *t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7**, (2003), 259–274.

[31] H. Nakajima, Quiver varieties and cluster algebras, Kyoto J. Math.**51**, (2011), 71-126.

[32] K. Naoi, *Fusion products of Kirillov-Reshetikhin modules and the $X = M$ conjecture*, Adv. Math. **231** (2012), 1546–1571.

[33] K. Naoi, *Weyl modules, Demazure modules and finite crystals for non-simply laced type*, Advances in Mathematics **229**, (2012), no. 2, 875-934.

[34] K. Naoi, *Demazure modules and graded limits of minimal affinizations*, Represent. Theory **17** (2013), 524–556.

[35] K. Naoi, *Graded limits of minimal affinizations in type D*, SIGMA **10** (2014), 047, 20 pages.

[36] K. Naoi, *Defining relations of fusion products and Schur positivity*, Toyama Mathmatical Journal, **37** (2015), 87-106.

[37] B. Ravinder, *Generalized Demazure modules and fusion products*, J. Algebra, **476** (2017), 186–215.