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Lecture I

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VECTORS

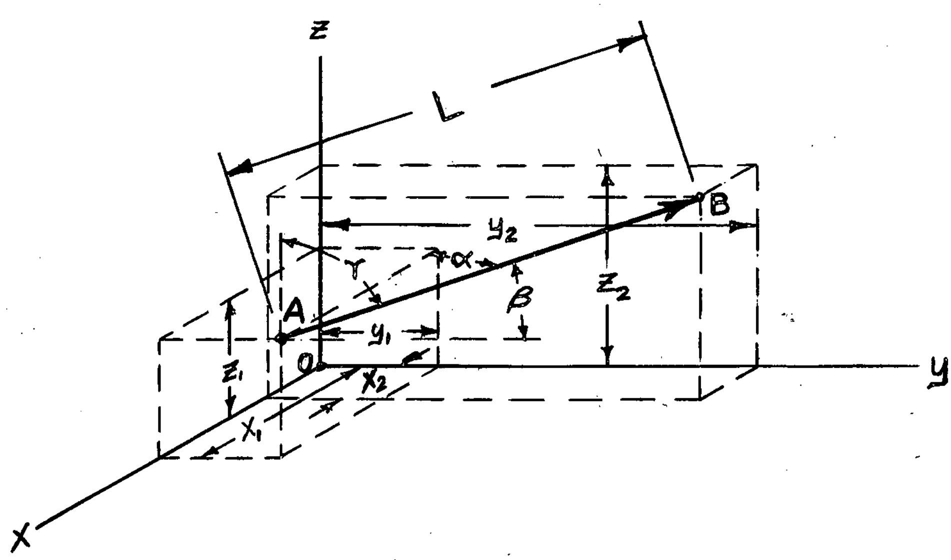
REFERENCES:

Abraham and Becker, Classical Electricity and Magnetism
Harnwell,
Marks, Lionel S., Mechanical Engineers Handbook
Synge and Griffith, Principles of Mechanics

DEFINITION OF A VECTOR

A vector is a quantity which has magnitude and direction and which adds according to the laws of the triangle.

Thus a vector can describe the motion of a particle which travels in the shortest path from point A to point B, or a vector may describe the force existing in a member of a truss under a given set of conditions. In the above example of the motion of a particle, both the magnitude and direction are given. Points A and B are separated by X units of length, and since specific points are designated, that is, points A and B, then a definite direction is also specified. If rectangular coordinates are assumed as a reference frame then point A may be specified as having the coordinates (x_1, y_1, z_1) and Point B by (x_2, y_2, z_2).



If L is the magnitude of this vector it is then equal to the quantity

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

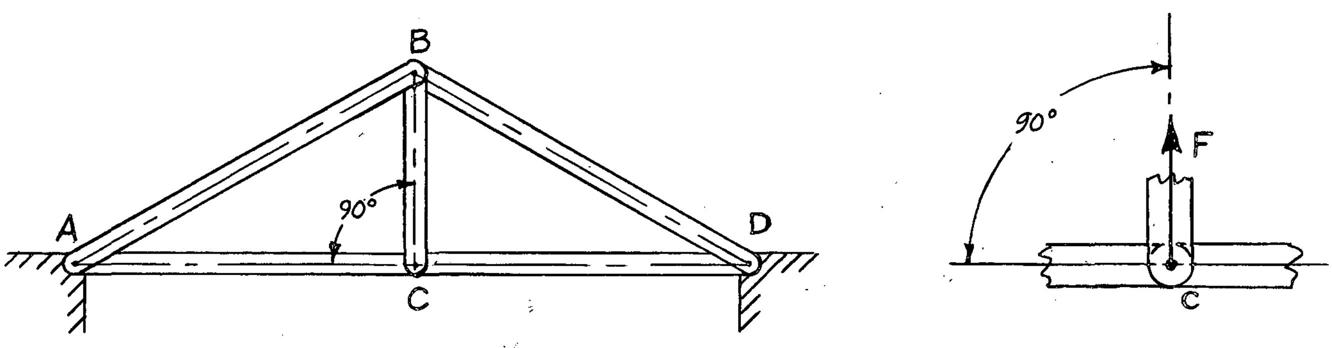
The direction of this vector can be defined by giving the angle between the vector and lines which pass through point A and which are parallel to the three coordinate axes.

Thus:

$$\cos \alpha = \frac{x_1 - x_2}{L}$$

$$\cos \beta = \frac{y_1 - y_2}{L}$$

$$\cos \gamma = \frac{z_1 - z_2}{L}$$



The force which the member B C of the pin-jointed truss illustrated, exerts on point C may be defined by a vector $\overline{F}$ whose direction is given as lying on the straight line joining B and C and whose magnitude is defined as being such that the substitution of this force for the member produces no change in the equilibrium of point "C".

In the two above examples the vectors have the following characteristics:

1. A point of origin (as point A in the first example) or a point of application (as point C in the second example).

2. A direction (in the first example defined by the three direction cosines, i.e. $\cos \alpha$, $\cos \beta$, $\cos \gamma$ and in the second example by the angle of 90°).

3. A magnitude (in the first example defined by

$$L = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$ and in the

second example as stated in discussing the example.)

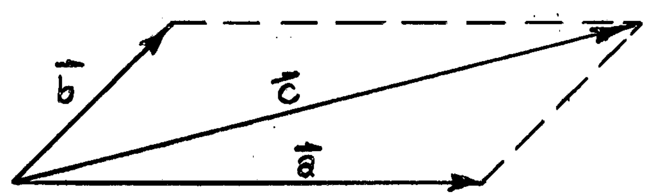
Quantities which do not satisfy these conditions and are therefore not vectors are 10 gallons of water - \$5 - 100 ft. lbs. - 40° Centigrade, etc. These quantities have magnitude but no direction is defined. Such quantities as these are known as scalar quantities.

VECTOR NOTATION

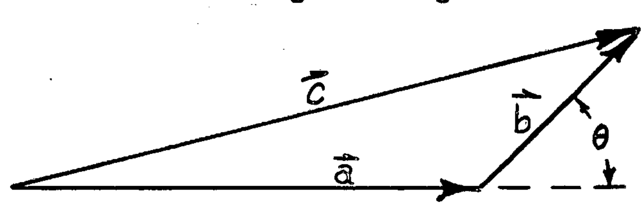
A vector is denoted in print by a bold face letter. It may also be denoted by a letter which is underlined as: $\underline{A}$ or by a letter surmounted by an arrow thus $\vec{A}$.

Vectors which are equal may have this equality shown in the usual manner thus $\vec{A} = \vec{B}$. To be equal they must have parallel directions, be of the same sense, and of equal magnitude. The sense of a vector is its direction of action, that is A.———→.B the point moves from A to B, or the force acts from A in the direction of B in the illustration.

The sum of two vectors $\vec{a}$ and $\vec{b}$ is defined as the vector represented by the diagonal of a parallelogram whose two adjacent sides are the vectors $\vec{a}$ and $\vec{b}$. Thus $\vec{c}$ is the vector sum of $\vec{a}$ and $\vec{b}$.



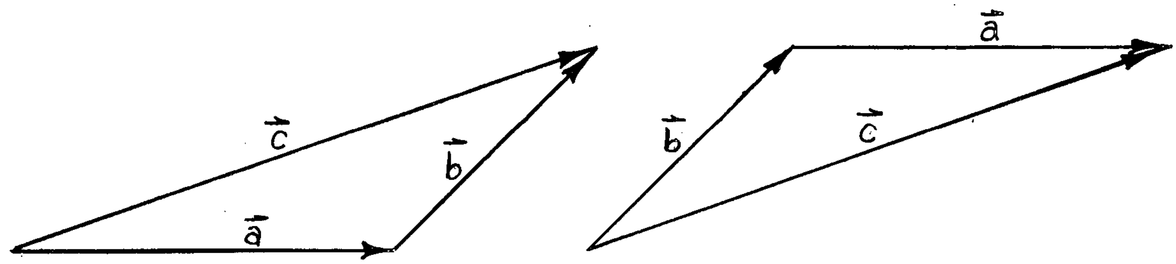
This may be shown as a triangle configuration as shown:



$$\vec{a} + \vec{b} = \vec{c} = \sqrt{a^2 + b^2 + 2\,a\,b\,\cos\theta}$$

The addition of vectors is commutative, that is it makes no difference what sequence is chosen in the addition of vectors. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

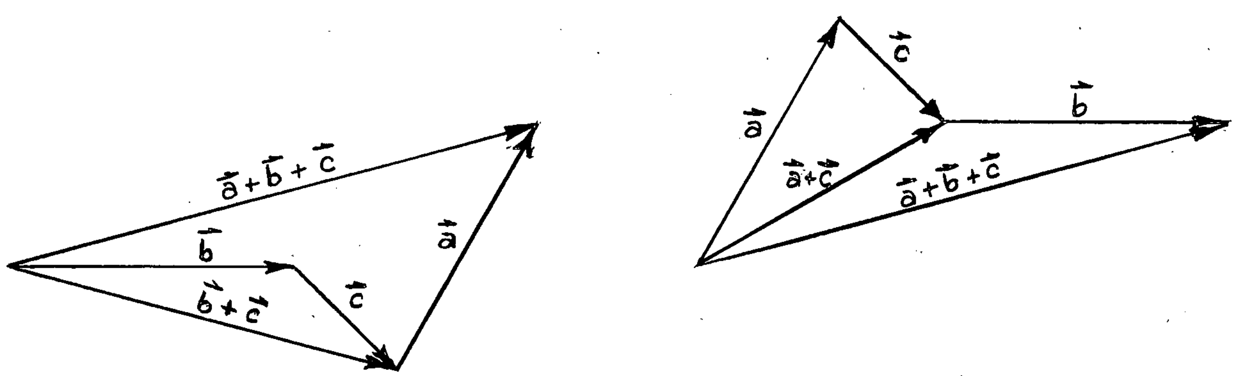
or



Vector addition is also associative. That is

$$\vec{a} + (\vec{b} + \vec{c}) = \vec{b} + (\vec{a} + \vec{c}).$$

It may be considered that $(\vec{b} + \vec{c}) + \vec{a}$ means to add vectors $\vec{b}$ and $\vec{c}$ after which vector $\vec{a}$ is added to that vector which is the sum of vectors $\vec{b} + \vec{c}$.



It is thus seen that the order in which operations are performed is of no consequence in the addition of a number of vectors.

MULTIPLICATION OF A VECTOR BY A SCALAR QUANTITY

If $\vec{a}$ is a vector and M is a scalar, the product of M $\vec{a}$ or of $\vec{a}$ M is defined as follows:

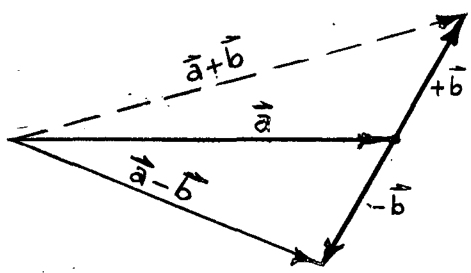
When M is positive, M $\vec{a}$ has the direction of $\vec{a}$ and a magnitude of M a; when M is negative, M $\vec{a}$ has the reverse sense of $\vec{a}$ and of magnitude of -M a.

$\vec{a}$ $\vec{a}$ $\vec{a}$ $\vec{a}$ . . . . $\vec{a}$ M times. Thus $\vec{a}$ is added M times giving the product M $\vec{a}$. Similiarly -M $\vec{a}$ is shown below:

$\vec{a}$ $\vec{a}$ $\vec{a}$ $\vec{a}$ . . . . $\vec{a}$ M times.

SUBTRACTION OF VECTORS

The subtraction of vectors is written $\vec{a} + (-\vec{b}) = \vec{c}$



NOTE: That $-\vec{b}$ has the same direction as does $+\vec{b}$ but is of the opposite sense.

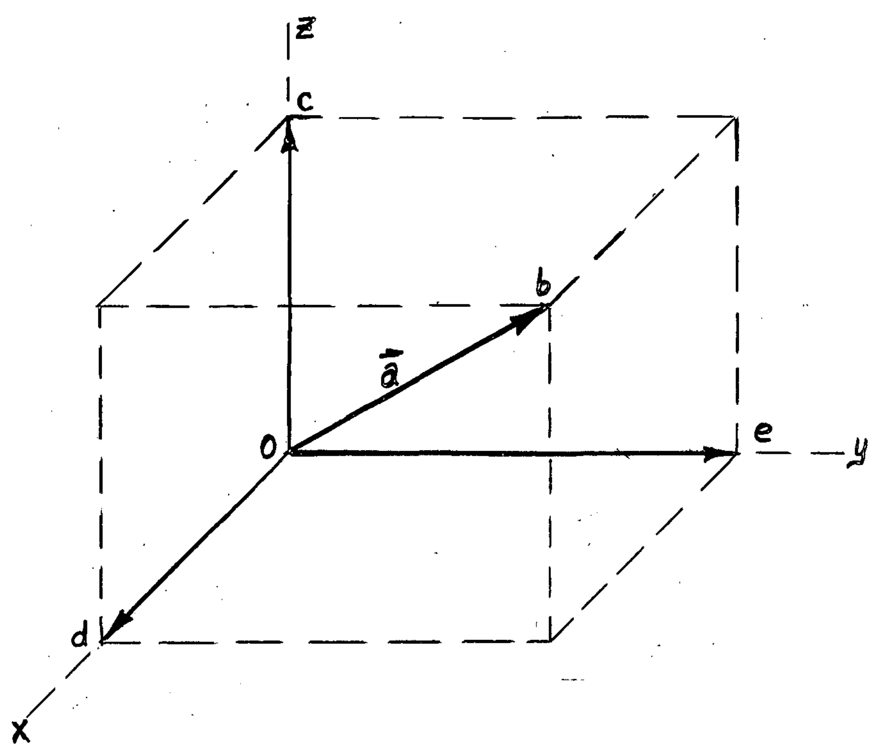
The subtraction of vectors is both commutative and associative as in the addition of vectors.

UNIT VECTOR

A unit vector has a magnitude equal to 1 and a direction. It specifies a direction. A unit vector a is denoted by $\hat{a}$.

Thus a vector $\vec{b}$ divided by the magnitude of $\vec{b}$ gives the unit vector $\hat{b}$. (The magnitude of vector b is denoted by $|\vec{b}|$.) $\frac{\vec{b}}{|\vec{b}|} = \hat{b}$

COMPONENTS OF A VECTOR

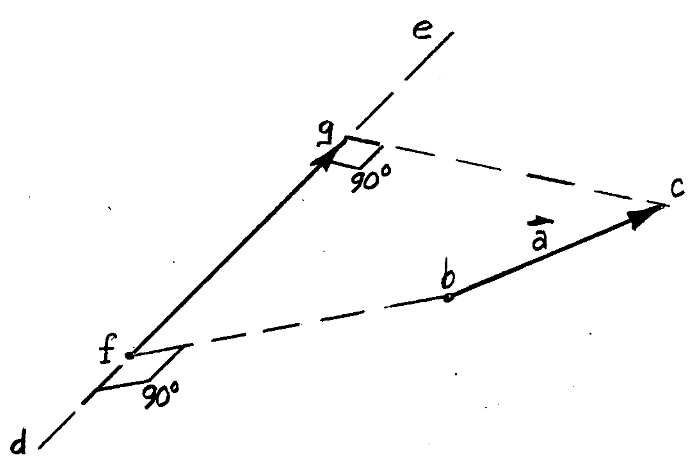


Vector $\vec{a}$ is shown starting at the origin of the three axes: x, y, and z. The three components of the vector $\vec{a}$ along the z, y, and x axes are found as follows:

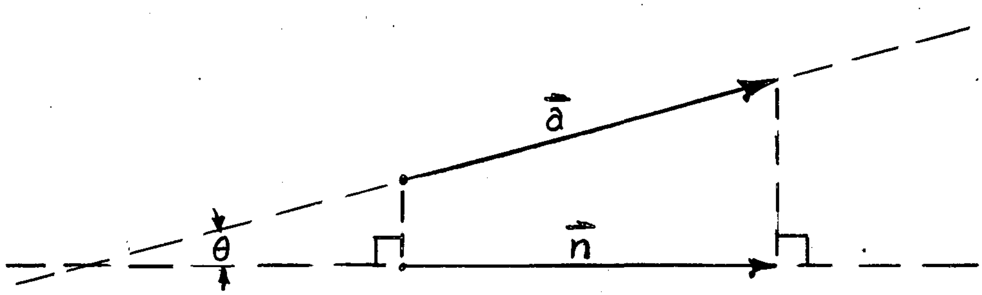
To find the z component of $\vec{a}$, pass planes through points 0 and through point b, which are perpendicular to the z axis. These planes intersect the z axis at point 0 (in this case) and at point c. Vector $\vec{0c}$ is the z component of vector $\vec{a}$. Similiarly to find the y component of $\vec{a}$ pass planes which are perpendicular to y through points 0 and b. These planes intersect y at 0 and at e. The vector $\vec{0e}$ is the y component of $\vec{a}$.

In the same manner $\vec{0d}$ is found to be the x component of $\vec{a}$. In a like manner, if given the vector $\vec{a}$ and the line d e the component $\vec{n}$ of $\vec{a}$ along line d e is found as follows:

Pass planes through point b and c which planes are perpendicular to the line d e. These planes intersect the line d e at points f and g. The vector $\vec{n}$'s magnitude is defined by the portion of line d e existing between points f and g. Its direction is that of the line d e.

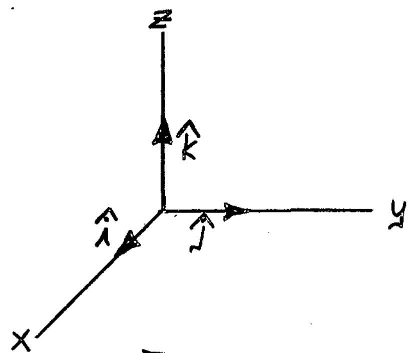


When the action line of the vector intersects the line along which the component is to be found the component is found in a similiar manner. However since the component and vector are coplanar the solution is simplified and the component $\vec{n}$ equals $|\vec{a}|$ Cos θ, where θ is the angle between the vector $\vec{a}$ and the line.

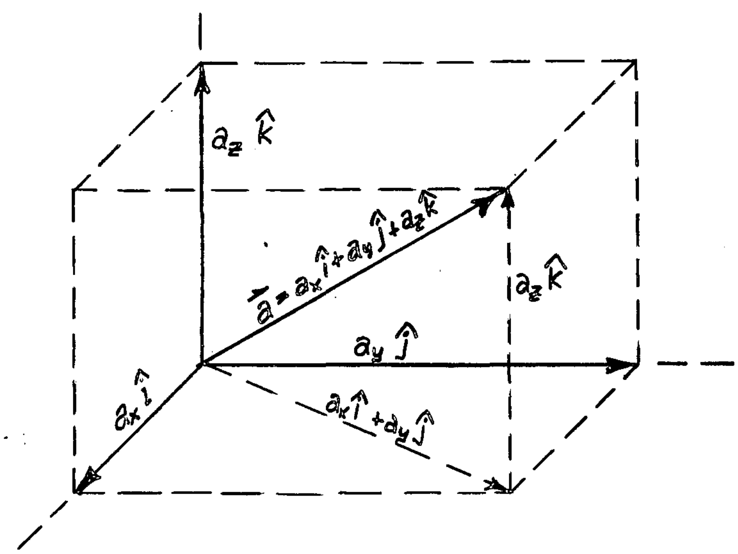


UNIT COORDINATE VECTORS

The set of vectors shown as i, j, and k are known as the unit Orthogonal Triad. These vectors are written as $\hat{i}$, $\hat{j}$, and $\hat{k}$ and have unit magnitude and a specific direction such that $\hat{i}$ lies along the x axis, $\hat{j}$ along the y axis and $\hat{k}$ along the z axis. They have a common origin O and have a positive sense.



If we have a vector $\vec{a}$ and at its origin erect cartesian coordinate axes x, y, and z, the components of $\vec{a}$ along these are a_x, a_y, and a_z.



By the addition of vectors it can be proven that $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$.
This is shown by the dashed lines in the figure.

The reduction of vectors to their three components is of use as a link between vector analysis and the usual methods of analysis.

The magnitude of vector $\vec{a}$ is expressed in terms of its scalar components by the equation:

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$a_x = |\vec{a}| \text{ Cos (of the angle between } \vec{a} \text{ and x axis)}$$

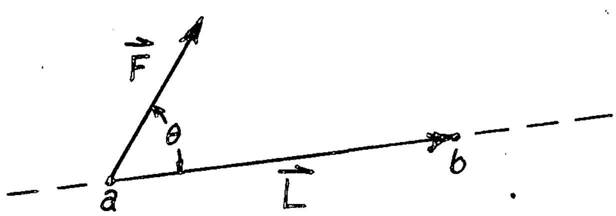
$$= |\vec{a}| \text{ Cos } (a, x)$$

$$a_y = |\vec{a}| \text{ Cos } (a, y)$$

$$a_z = |\vec{a}| \text{ Cos } (a, z)$$

MULTIPLICATION OF VECTORS

There are two vector products which are of interest. The scalar product of two vectors is the first considered. An example of a scalar product of vector quantities is work. If a force $\vec{F}$ has a point of application on a line and this force moves along the line a vector distance $\vec{L}$ then the work done by $\vec{F}$ is the scalar product $|\vec{F}|\,|\vec{L}|$ Cos Θ, where Θ is the angle between $\vec{F}$ and $\vec{L}$.



W = work = $|\vec{F}||\vec{L}|$ Cos Θ. This is scalar product.

Scalar product is denoted by a dot, thusly $\vec{a} \circ \vec{b}$

Scalar products are independent of the order in which the product is taken. Thus $\vec{a} \circ \vec{b} = \vec{b} \circ \vec{a}$ or $\vec{a} \circ (\vec{b} + \vec{c}) = \vec{a} \circ \vec{b} + \vec{a} \circ \vec{c}$

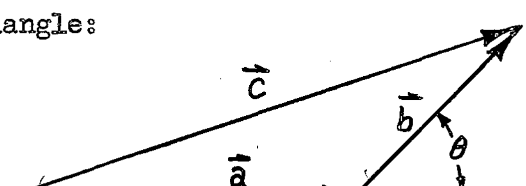
Scalar products follow the rules of algebra. That is

$$(\vec{a} + \vec{b}) \circ (\vec{a} + \vec{b}) = \vec{c} \circ \vec{c}$$

$$\vec{a}^2 + \vec{b}^2 + 2\vec{a} \circ \vec{b} = \vec{c}^2 \qquad \vec{a} \circ \vec{b} = ab \text{ Cos } \Theta$$

$$a^2 + b^2 + 2ab \text{ Cos } \Theta = c^2$$

This is the case of a triangle:



If the scalar product $\vec{a} \cdot \vec{b} = 0$ then the vectors are perpendicular or one vector is zero.

If the scalar product $\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|$ the vectors are parallel.

If the scalar product $\vec{a} \cdot \vec{b} = -(|\vec{a}|\,|\vec{b}|)$ the vectors are in opposition (that is they have opposite sense to each other).

It follows therefore that:

$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$, these sets of unit vectors are parallel.

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{i}\cdot\hat{k} = 0$, these sets of unit vectors are $\perp$.

Now given two vectors $\vec{a}$ and $\vec{b}$ who have rectangular components a_x, a_y, and a_z, and b_x, b_y, and b_z, then

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$$

then:

$$\vec{a} \cdot \vec{b} = a_xb_x\,\hat{i}\cdot\hat{i} + a_yb_x\,\hat{i}\cdot\hat{j} + a_zb_x\,\hat{i}\cdot\hat{k} + a_xb_y\,\hat{i}\cdot\hat{j} + a_yb_y\,\hat{j}\cdot\hat{j} + a_zb_y\,\hat{i}\cdot\hat{k} +$$

$$a_xb_z\,\hat{i}\cdot\hat{k} + a_yb_z\,\hat{j}\cdot\hat{k} + a_zb_z\,\hat{k}\cdot\hat{k} \quad (a_yb_x\,\hat{i}\cdot\hat{j} = 0 \text{ since } \hat{i}\cdot\hat{j} = 0) \text{ therefore}$$

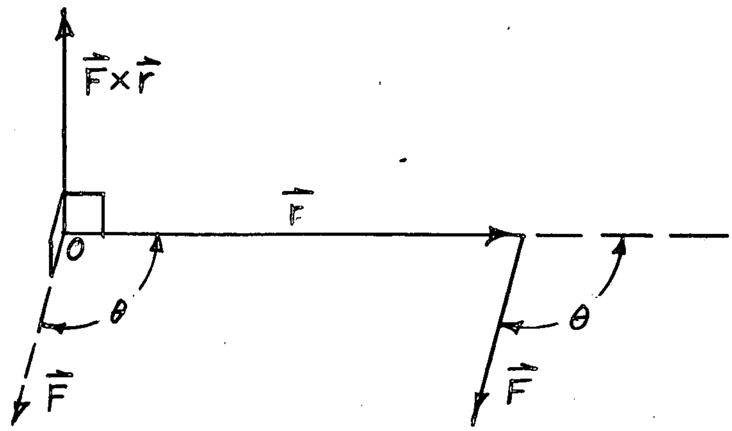
$$\vec{a} \cdot \vec{b} = a_xb_x + a_yb_y + a_zb_z$$

In cartesion coordinates if $a_xb_x + a_yb_y + a_zb_z = 0$ the vectors $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\text{Cos}\ (\vec{a} \cdot \vec{b}) = \frac{a_xb_x + a_yb_y + a_zb_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}\ \sqrt{b_x^2 + b_y^2 + b_z^2}}$$

The vector product is the second type of vector multiplication. The vector product is denoted by a cross, thus $\vec{a} \times \vec{b}$, and is a vector whose magnitude is given by ab sin θ where θ is the angle between a and b and whose direction is perpendicular to the plane of a and b, and having a sense such that with the vector grasped in the right hand in such a way as to turn a toward b the thumb points in the direction of the vector $\vec{a} \times \vec{b}$.

An example of a vector product is a torque. Point 0 is given and force $\vec{F}$.



$\vec{F} \times \vec{r}$ is known as a pseudo-vector because an arbitrary rule determines the sense of the moment vector. Torque about point O of $\vec{F}$ force is:

$$\text{Torque} = |\vec{F}|\,|\vec{r}|\sin\theta$$

Because of the question of determining the sense of the vector product, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$. However, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. The order in which the vectors are considered is therefore important.

The vector product $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ or

$(\vec{a}+\vec{b}) \times (\vec{c}+\vec{d}) = \vec{a} \times \vec{c} + \vec{a} \times \vec{d} + \vec{b} \times \vec{c} + \vec{b} \times \vec{d}$ provided that the order of the factors in each product is preserved.

In rectangular coordinates there is a simple rule for the cross products. The terms are arranged in a determinant thus:

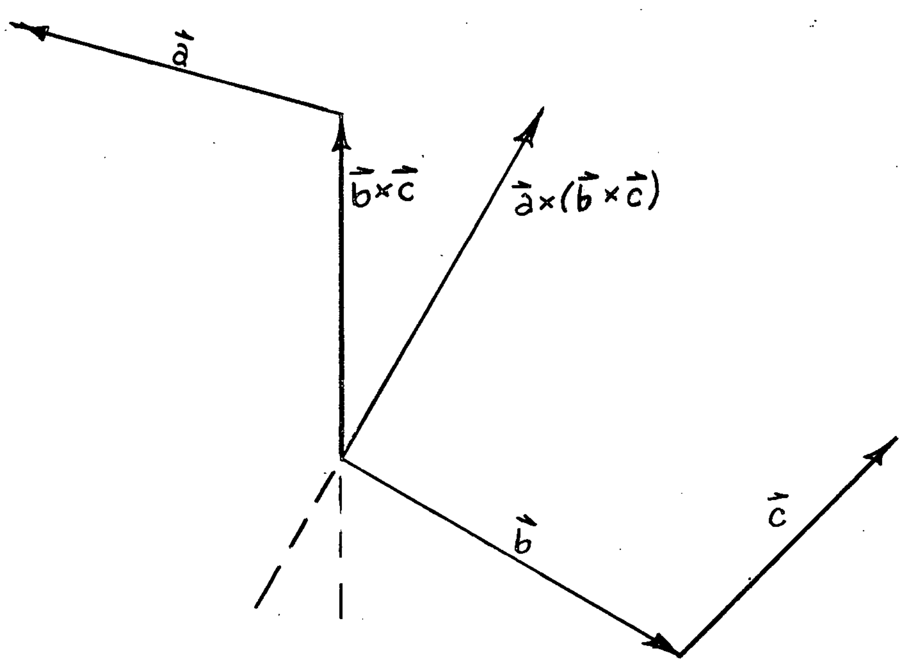
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}\,(A_yB_z - B_yA_z) + \hat{j}\,(A_zB_x - A_xB_z) + \hat{k}\,(A_xB_y - A_yB_x)$$

For the product of three vectors the following hold:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c}$$

If however, only two letters are interchanged as $a \cdot (c \times b)$ the sign changes.



$\vec{a} \times (\vec{b} \times \vec{c})$ triple product lies in the plane of $\vec{b}$ and $\vec{c}$.

DIFFERENTIATION OF A VECTOR BY A SCALAR

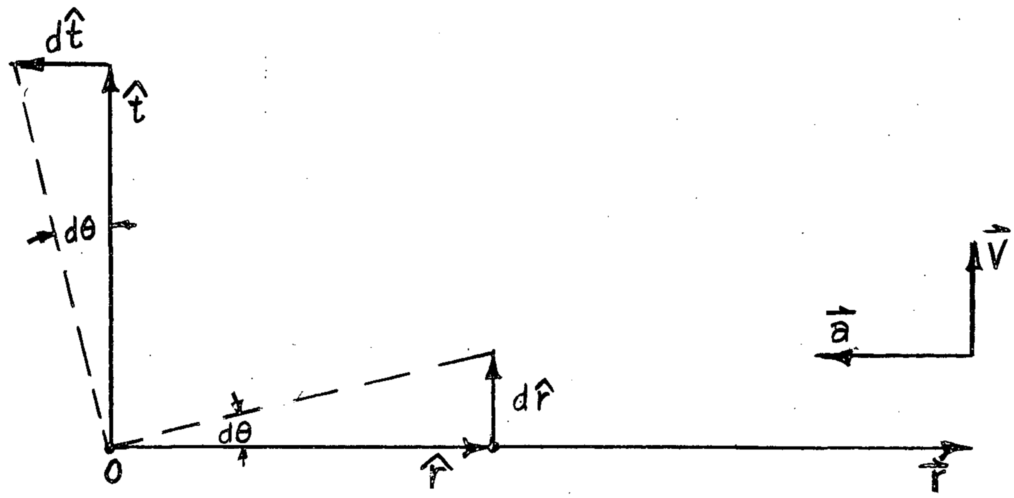
The differentiation of a vector by a scalar can be shown by the example of the rate of change of a vector with time.

The notation is the same as for all derivatives. The derivative of $\vec{a}$ in respect to time t is written $\frac{d\vec{a}}{dt}$.

If $\vec{a} = F(t)$ is the equation of a vector $\vec{a}$ in respect to time t, a scalar quantity, then

$$\frac{d\vec{a}}{dt} = \left. \frac{F(t+\Delta t) - F(t)}{\Delta t} \right|_{\text{limit as } \Delta t \to 0}$$

Consider a particle B moving with uniform circular motion about point O, its position vector being $\vec{r}$. Take a radial unit vector $\hat{r}$ parallel with $\vec{r}$, and a tangential unit vector $\hat{t}$ perpendicular to $\vec{r}$.



The differential of a unit vector is perpendicular to the unit vector, and is a measure of change in direction only. In rotating the unit vectors through an infinitesimal angle dθ, it can be seen that

$$d\hat{r} = (1)\,(d\theta)\,(\hat{t}) = d\theta\ \hat{t}$$

$$d\hat{t} = (1)\,(d\theta)\,(-\hat{r}) = d\theta\,(-\hat{r})$$

in which the unit vectors $\hat{t}$ and $(-\hat{r})$ specify direction, and 1 is the magnitude of the unit vector.

The velocity of B is given by the vector

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\,(r\,\hat{r})}{dt} = r\,\frac{d\,\hat{r}}{dt} = r\,\frac{d\theta}{dt}\,\hat{t} = r\omega\,\hat{t},$$

in which $\hat{t}$ shows that the velocity is in the direction of $\hat{t}$.

The acceleration of B is given by the vector

$$\vec{a} = \frac{d\,\vec{V}}{dt} = \frac{d\,(r\omega\hat{t})}{dt} = r\omega\,\frac{d\,\hat{t}}{dt} = r\omega\,\frac{d\theta}{dt}(-\hat{r})$$

$$= r\,\omega^2\,(-\hat{r})$$

in which $(-\hat{r})$ shows that the acceleration is in the opposite direction to $\hat{r}$.

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