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Symmetric Operators and Reproducing Kernel Hilbert Spaces

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Abstract We establish the following sufficient operator-theoretic condition for a subspace $S \subset L^2(\mathbb{R}, d\nu)$ to be a reproducing kernel Hilbert space with the Kramer sampling property. If the compression of the unitary group $U(t) := e^{itM}$ generated by the self-adjoint operator M , of multiplication by the independent variable, to S is a semigroup for $t \geq 0$, if M has a densely defined, symmetric, simple and regular restriction to S , with deficiency indices $(1, 1)$, and if ν belongs to a suitable large class of Borel measures, then S must be a reproducing kernel Hilbert space with the Kramer sampling property. Furthermore, there is an isometry which acts as multiplication by a measurable function which takes S onto a reproducing kernel Hilbert space of functions which are analytic in a region containing \mathbb{R} , and are meromorphic in \mathbb{C} . In the process of establishing this result, several new results on the spectra and spectral representations of symmetric operators are proven. It is further observed that there is a large class of de Branges functions E , for which the de Branges spaces $\mathcal{H}(E) \subset L^2(\mathbb{R}, |E(x)|^{-2}dx)$ are examples of subspaces satisfying the conditions of this result.

Keywords Self-adjoint extensions of symmetric operators · Reproducing kernel Hilbert spaces · Spectra of symmetric operators · Kramer sampling property

Mathematics Subject Classification (2000) 47B25 (symmetric and self-adjoint operators (unbounded)) · 46E22 (Hilbert spaces with reproducing kernels) · 47B32

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(operators in reproducing kernel Hilbert spaces) · 47A10 (general theory of linear operators; spectrum, resolvent)

1 Introduction

Let \mathcal{H} be a reproducing kernel Hilbert space (RKHS) of functions on a set $X \subset \mathbb{C}$, and for each $x \in X$, let $\delta_x \in \mathcal{H}$ denote the point evaluation vector at the point x , i.e. for any $\phi \in \mathcal{H}$, $\langle \phi, \delta_x \rangle = \phi(x)$ for each $x \in X$. The RKHS \mathcal{H} is said to have the Kramer sampling property if there is a countable total orthogonal subset $\{\delta_{x_n}\}_{n \in \mathbb{Z}}$ of point evaluation vectors. In this case, it follows that $1_{\mathcal{H}} = \sum_{n \in \mathbb{Z}} \langle \cdot, \delta_{x_n} \rangle \langle \delta_{x_n}, \cdot \rangle \frac{1}{\|\delta_{x_n}\|^2}$ so that for any $\phi \in \mathcal{H}$,

$$\begin{aligned} \phi(x) &= \langle \phi, \delta_x \rangle = \langle 1_{\mathcal{H}} \phi, \delta_x \rangle \\ &= \sum_{n \in \mathbb{Z}} \langle \phi, \delta_{x_n} \rangle \langle \delta_{x_n}, \delta_x \rangle \frac{1}{\delta_{x_n}(x_n)} \\ &= \sum_{n \in \mathbb{Z}} \phi(x_n) K(x, x_n), \end{aligned} \tag{1}$$

where $K(x, x_n) := \frac{\delta_{x_n}(x)}{\delta_{x_n}(x_n)}$. Elements of \mathcal{H} are said to obey a sampling formula, since they are uniquely determined and perfectly reconstructible from their ‘samples’, or values taken on the discrete set of points $\{x_n\}_{n \in \mathbb{Z}}$. The fact that any RKHS which has a total orthogonal set of point evaluation vectors obeys a sampling formula is called Kramer’s abstract sampling theorem, see [1].

The classic example of such a function space is the Paley–Wiener space $B(\Omega)$ of Ω -bandlimited functions. The space $B(\Omega)$ is that subspace of $L^2(\mathbb{R})$ which is the image of $L^2[-\Omega, \Omega]$ under the Fourier transform. The finite number $\Omega > 0$ is called the band-limit. The space $B(\Omega)$ is a RKHS with point evaluation vectors $\delta_t(x) = \frac{1}{2\Omega} \frac{\sin(\Omega(x-t))}{\Omega(x-t)}$ where $\|\delta_t\|^2 = \frac{1}{2\Omega}$ for each $t \in \mathbb{R}$. For any $\alpha \in [0, 1)$, the set of point evaluation vectors $\{\delta_{x_n(\alpha)}\}_{n \in \mathbb{Z}}$, where $x_n(\alpha) := \frac{(n+\alpha)\pi}{\Omega}$ is a total orthogonal set in $B(\Omega)$. Such reproducing kernel Hilbert spaces with the sampling property have many practical applications in a wide variety of fields including pure mathematics, communication engineering, signal processing, and more recently, mathematical physics [2–5]. In particular, the spaces of bandlimited functions are used extensively in signal processing to efficiently discretize and reconstruct continuous signals, e.g. music signals are approximated by bandlimited functions so that they can be recorded as discrete values on a CD, and then later reconstructed by the CD player with minimal error. Recently it has been observed that by considering more general RKHS with the sampling property, one may be able to increase the efficiency of signal processing including the discretization and reconstruction of certain classes of continuous signals [3, 6]. This motivates both the search for and the study of such Hilbert spaces. This paper provides tools for determining when subspaces of $L^2(\mathbb{R}, d\nu)$ are reproducing kernel Hilbert spaces with the sampling property.

It is known that if B is a symmetric operator which is simple, regular, has deficiency indices $(1, 1)$, and is densely defined on some domain $\text{Dom}(B) \subset \mathcal{H}$ of a separable Hilbert space \mathcal{H} , that there exists a unitary transformation U which maps \mathcal{H} onto a Hilbert space $\mathcal{H}' \subset L^2(\mathbb{R}, d\mu)$, which is a reproducing kernel Hilbert space of meromorphic functions with the sampling property [3, 7, 8]. Furthermore, under this unitary transformation, $M' := UBU^*$, the image of B , acts as multiplication by the independent variable on the dense domain $U\text{Dom}(B) \subset \mathcal{H}'$. This result can be thought of as a specialized spectral theorem which applies to this particular class of symmetric operators. Such a representation, (M', \mathcal{H}') of a symmetric operator B will be called a spectral representation. Since any such a symmetric operator B can be realized as multiplication by the independent variable in a RKHS with the sampling property, we will say that any symmetric operator which is closed, regular, simple, and densely defined with deficiency indices $(1, 1)$ has the sampling property. The set of all symmetric operators in \mathcal{H} with the sampling property will be denoted $\text{Sym}_1(\mathcal{H})$.

In this paper we consider the following related problem. Suppose that $S \subset L^2(\mathbb{R}, d\nu)$, ν a suitable Borel measure, is such that the operator of multiplication M by the independent variable has a symmetric restriction M_S to a dense domain in S with the sampling property. Then it seems intuitively reasonable that since any unitary transformation U which takes the multiplication operator M_S onto a spectral representation (M', \mathcal{H}') maps $M_S \in \text{Sym}_1(S)$ onto a multiplication operator $M' \in \text{Sym}_1(\mathcal{H}')$ where $S \subset L^2(\mathbb{R}, d\nu)$ and $\mathcal{H}' \subset L^2(\mathbb{R}, d\mu)$, such a U should act as multiplication by a measurable function, and that \mathcal{H} itself should be a RKHS with the sampling property. One of the main achievements of this paper is the proof of results of this nature, for a large class of measures, ν . Namely, in Sect. 5 we will prove Theorem 14, which is a slightly stronger version of the following:

Theorem 0 *Let ν be a measure on \mathbb{R} such that $d\nu := v'(x)dx$ where $v'(x) > 0$ a.e. is a measurable, locally L^1 function. Suppose that $S \subset L^2(\mathbb{R}, d\nu)$ is such that the compression of the unitary group $U(t) = e^{itM}$ to S is a semigroup for $t \geq 0$, and that M has a restriction to a dense domain in S with the sampling property. Then $U(t)$ is the minimal unitary dilation of its compression to S , and there is an isometry V which acts as multiplication by a measurable, locally L^1 function which takes S onto a certain RKHS $\mathcal{H}_z \subset L^2(\mathbb{R}, d\sigma_z)$. This RKHS \mathcal{H}_z has the sampling property and consists of certain meromorphic functions which are analytic in a region containing \mathbb{R} . The measure σ_z is equivalent to Lebesgue measure. Furthermore, if v' and $1/v'$ are both locally L^∞ functions, then S itself is a RKHS with the sampling property.*

The proof of this result will be achieved by first extending the spectral theory and the theory of spectral representations of symmetric operators with the sampling property (Sects. 2–4), followed by a straightforward application of the Nagy-Foias intertwiner version of Ando's dilation theorem for two commuting contractions (Sect. 5). In Sect. 6, Theorem 14 will be applied to provide a new proof that a large class of de Branges spaces, including the Paley–Wiener spaces of bandlimited functions are reproducing kernel Hilbert spaces with the sampling property. We begin studying the spectral theory of symmetric operators in the following section.

2 Review of Spectral Theory for Symmetric Operators with the Sampling Property

Let B be a closed, symmetric operator defined on a dense domain, $\text{Dom}(B)$, in a separable Hilbert space \mathcal{H} . Recall that the deficiency indices (n_+, n_-) of S are defined as the dimensions of the subspaces $\text{Ran}(B - \bar{z})^\perp = \text{Ker}(B^* - z)$ and $\text{Ran}(B - z)^\perp = \text{Ker}(B^* - \bar{z})$ respectively, where z belongs to the open complex upper half plane (UHP). The dimensions of these two subspaces are constant for z within the upper and lower half plane respectively [9, Sect. 78]. For $z = i$, $\mathfrak{D}_+ := \text{Ker}(B^* - i)$ and $\mathfrak{D}_- := \text{Ker}(B^* + i)$ will be called the deficiency subspaces of B , $n_\pm := \dim(\mathfrak{D}_\pm)$.

Let $\mu(z) := \frac{z-i}{z+i}$. Given any symmetric operator B , its Cayley transform is defined as $\mu(B)$. If B is self-adjoint, the continuous functional calculus implies that $\mu(B)$ is a unitary operator. More generally, if B is symmetric, then $\mu(B)$ is a partially defined transformation which is an isometry from $\text{Ran}(B + i)$ onto $\text{Ran}(B - i)$ [9, Sects. 67, 79]. Furthermore, if $V = \mu(B)$, and $\mu^{-1}(z) = i \frac{1+z}{1-z}$, then $\mu^{-1}(\mu(z)) = z$ and $B = \mu^{-1}(V)$.

The domain of the adjoint B^* of B can be decomposed as [9, p. 98]:

$$\text{Dom}(B^*) = \text{Dom}(B) \dot{+} \mathfrak{D}_+ \dot{+} \mathfrak{D}_-. \tag{2}$$

Here the linear manifolds $\text{Dom}(B)$, \mathfrak{D}_+ and \mathfrak{D}_- are non-orthogonal, linearly independent, subspaces of \mathcal{H} which are not closed in general. The notation $\dot{+}$ denotes the non-orthogonal direct sum of these linear subspaces. If B has equal deficiency indices (n, n) , then all self-adjoint extensions of B within \mathcal{H} can be obtained as follows. Append an arbitrary isometry W from \mathfrak{D}_+ onto \mathfrak{D}_- to the Cayley transform V of B to obtain a unitary extension $U_W := V \oplus W$ of V , and then take the inverse Cayley transform of this unitary extension to obtain a self-adjoint extension $B_W := \mu^{-1}(U_W)$ of B , with domain

$$\text{Dom}(B_W) = \text{Dom}(B) \dot{+} (W - 1)\mathfrak{D}_+. \tag{3}$$

2.1 Spectra of Symmetric Operators

A point $z \in \mathbb{C}$ is called a regular point for B if $B - z$ is bounded below. It is straightforward to show that every $z \in \mathbb{C} \setminus \mathbb{R}$ is automatically a regular point for any symmetric operator B . A symmetric operator B is called regular if every $z \in \mathbb{C}$ is a regular point for B . A symmetric operator B , densely defined in \mathcal{H} is called simple if there is no subspace $S \subset \mathcal{H}$ such that the restriction of B to a dense domain in S is self-adjoint. Any densely defined symmetric operator B always has a closure [9, Sect. 41], $\overline{B} = B^{**}$, and so it will be assumed that all symmetric operators in this paper are closed.

We will let $\sigma(B)$, $\sigma_p(B)$, $\sigma_c(B)$, $\sigma_r(B)$, and $\sigma_e(B)$ denote the spectrum, and the point, continuous, residual and essential spectrum of B , respectively. Recall that $\sigma(B)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $(B - \lambda)$ does not have a bounded inverse defined on all of \mathcal{H} . The point spectrum $\sigma_p(B)$ is defined as the set of all eigenvalues, $\sigma_c(B)$ is here defined as the set of all λ such that $\text{Ran}(B - \lambda)$ is not closed, $\sigma_r(B)$ is defined as the set of all λ such that $\lambda \notin \sigma_p(B)$ and $\text{Ran}(B - \lambda)$ is not dense, and

$\sigma_e(B)$ is the set of all λ such that $B - \lambda$ is not Fredholm. Recall that a closed, densely defined operator T is called Fredholm if $\text{Ran}(T)$ is closed and if the dimension of $\text{Ker}(T)$ and the co-dimension of $\text{Ran}(T)$ are both finite. If T is unbounded, we include the point at infinity as part of the essential spectrum. Clearly all the above sets are subsets of $\sigma(B)$, and $\sigma(B) = \sigma_p(B) \cup \sigma_c(B) \cup \sigma_r(B)$.

If B is symmetric, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then it is easy to see that $B - z$ is bounded below by $\frac{1}{\text{Im}(z)}$. This shows that any non-real $z \in \sigma(B)$ must belong to the residual spectrum $\sigma_r(B)$ of B . If B has finite deficiency indices, then the orthogonal complement of $\text{Ran}(B - z)$ is finite dimensional for any $z \in \mathbb{C} \setminus \mathbb{R}$, which, along with the previous observations, implies that $\sigma_e(B) \subset \mathbb{R}$.

If B has finite deficiency indices, and if the co-dimension of $\text{Ran}(B - \lambda)$ is infinite, then $\lambda \in \mathbb{R}$. Furthermore if $\lambda \in \text{Ran}(B - \lambda)^\perp$ then $\bar{\lambda}$ is an eigenvalue to B^* . This and the fact that the dimension of $\text{Dom}(B^*)$ modulo $\text{Dom}(B)$ is finite (by Eq. (2)) allows one to conclude that λ must be an eigenvalue of infinite multiplicity to B . Hence if $\lambda \in \sigma_e(B)$ then either it is an eigenvalue of infinite multiplicity or it belongs to the continuous spectrum of B . It is not difficult to show [9, Sect. 83] and [10], that if B has finite and equal deficiency indices, and if B' is any self-adjoint extension of B , then $\sigma_e(B) = \sigma_e(B')$.

Further recall the following simple facts about the spectra of symmetric operators with finite and equal deficiency indices [9, Sect. 83]:

Theorem 1 *If λ is a real point of regular type of a symmetric operator B with finite deficiency indices (n, n) , then there exists a self-adjoint extension B' of B for which λ is an eigenvalue of multiplicity n .*

If λ is a real point that is not an eigenvalue of B , then the dimension of $\text{Ker}(B^ - \lambda)$ does not exceed n .*

In summary, if B is simple, regular, and has deficiency indices $(1, 1)$, then each λ in \mathbb{C} is an eigenvalue of multiplicity one to B^* , and the spectrum of any self-adjoint extension B' of B is purely discrete, and consists of eigenvalues of multiplicity one.

The following is a minor refinement of a theorem first established in [3].

Theorem 2 *Let B be a closed symmetric operator densely defined in \mathcal{H} . If B is simple, regular and has deficiency indices $(1, 1)$, then the spectra of any one of its self-adjoint extensions consists of eigenvalues of multiplicity one with no finite accumulation point. Furthermore, the spectra of all of its self-adjoint extensions covers \mathbb{R} exactly once.*

Recall that all self-adjoint extensions of a densely defined symmetric operator with deficiency indices $(1, 1)$ can be labeled by $\alpha \in [0, 1)$. That is, B' is a self-adjoint extension of B if and only if $B' = B(\alpha)$ for some $\alpha \in [0, 1)$, where $B(\alpha)$ is defined as the inverse Cayley transform of $U(\alpha) := V \oplus e^{i2\pi\alpha} \phi_-(\cdot, \phi_+)$ on $\mathcal{H} = \text{Dom}(V) \oplus \mathcal{D}_+$. Here ϕ_\pm are fixed unit norm vectors in \mathcal{D}_\pm and V is the Cayley transform of B .

Proof of Theorem 2 First, by Theorem 1, since B is regular, any self-adjoint extension $B(\alpha)$ of B has no continuous spectrum, and given any point $\lambda \in \mathbb{R}$, there is an extension $B(\alpha)$ of B for which λ is an eigenvalue of multiplicity one. Again, by Theorem 1, any $\lambda \in \mathbb{R}$ is not an eigenvalue of multiplicity greater than one for any

fixed self-adjoint extension of B . Finally, if $\lambda \in \mathbb{R}$ was an eigenvalue to two different self-adjoint extensions $B(\alpha)$ and $B(\beta)$ of B , Theorem 1 implies that any eigenvector of $B(\alpha)$ with eigenvalue λ must also be an eigenvector of $B(\beta)$ with eigenvalue λ . The Neumann formula (3) would then imply that

$$\phi_\lambda = \phi_B + c_1(e^{i2\pi\alpha}\phi_- - \phi_+) = \varphi_B + c_2(e^{i2\pi\beta}\phi_- - \phi_+) \tag{4}$$

for some non-zero $c_1, c_2 \in \mathbb{C}$ and $\phi_B, \varphi_B \in \text{Dom}(B)$ so that,

$$0 = (\phi_B - \varphi_B) + (c_1e^{i2\pi\alpha} - c_2e^{i2\pi\beta})\phi_- + (c_2 - c_1)\phi_+ \tag{5}$$

in $\text{Dom}(B^*) = \text{Dom}(B) \dot{+} \mathfrak{D}_- \dot{+} \mathfrak{D}_+$. Since these three linear manifolds are linearly independent it follows that $\phi_B = \varphi_B, c_1e^{i2\pi\alpha} = c_2e^{i2\pi\beta}$ and that $c_1 = c_2$. This shows, in particular, that $e^{i2\pi\alpha} = e^{i2\pi\beta}$. Since $\alpha, \beta \in [0, 1)$ this proves that $\alpha = \beta$ so that $B(\alpha) = B(\beta)$, contradicting the assumption that these are two different self-adjoint extensions of B .

The fact that the eigenvalues of any $B(\alpha)$ cannot have a finite accumulation point follows from the assumption that B is regular. If λ was an accumulation point of $\sigma(B(\alpha))$, then $\lambda \in \sigma_e(B(\alpha)) = \sigma_e(B)$. As remarked before this theorem, since B has finite deficiency indices, such a λ would belong to either $\sigma_p(B)$ or $\sigma_c(B)$, contradicting the regularity of B . \square

By the above theorem, given any self-adjoint extension $B(\alpha), \alpha \in [0, 1)$ of B , $\sigma(B(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{M}}$ can be arranged as a discrete strictly increasing sequence of eigenvalues with no finite accumulation point, and as α ranges in $[0, 1)$, $\sigma(B(\alpha))$ covers \mathbb{R} exactly once. Here, $\mathbb{M} = \pm\mathbb{N}$ or \mathbb{Z} , depending on whether the spectrum of $B(\alpha)$ is bounded above, below or neither bounded above nor below. It will later follow from Theorem 6 that if one self-adjoint extension of B is bounded above or below then all are.

For the rest of the paper we restrict our attention to the class of symmetric operators $\text{Sym}_1(\mathcal{H})$. Recall that $\text{Sym}_1(\mathcal{H})$ denotes the set of all symmetric operators with the sampling property in \mathcal{H} , i.e. the set of all symmetric operators which are simple, regular, densely defined in \mathcal{H} , and which have deficiency indices $(1, 1)$.

2.2 Spectral Representations of Symmetric Operators with the Sampling Property

This particular class, $\text{Sym}_1(\mathcal{H})$, of symmetric operators has been studied extensively by M.G. Krein. In this section we provide a brief outline of Krein’s theory of spectral representations of symmetric operators with the sampling property.

Given $B \in \text{Sym}_1(\mathcal{H})$, let B' be an arbitrary self-adjoint extension of B within \mathcal{H} . Throughout this paper, when we refer to a self-adjoint extension of $B \in \text{Sym}_1(\mathcal{H})$, this will always be assumed to mean a self-adjoint extension which is a densely defined operator in \mathcal{H} , not a self-adjoint extension to a larger Hilbert space. Given any fixed $z_0 \in \mathbb{C} \setminus \sigma(B')$ and non-zero $\varphi_{z_0} \in \text{Ker}(B^* - z_0)$, let

$$\varphi_z := (B' - z_0)(B' - z)^{-1}\varphi_{z_0} = \varphi_{z_0} + (z - z_0)(B' - z)^{-1}\varphi_{z_0}. \tag{6}$$

It is not difficult to prove (see, e.g. [7, p. 9]), that the operator $(B' - w)(B' - z)^{-1}$ is a bijective map from $\mathfrak{D}_w := \text{Ker}(B^* - w)$ onto $\mathfrak{D}_z = \text{Ker}(B^* - z)$. It follows that φ_z is analytic on $\mathbb{C} \setminus \sigma(B')$, that $0 \neq \varphi_z \in \mathfrak{D}_z$ for all $z \in \mathbb{C} \setminus \sigma(B')$, and it is not hard to see that φ_z has simple poles at the points of $\sigma(B') = \sigma_p(B')$.

Given some $z' \notin \sigma(B')$, choose $u \in \mathcal{H}$ such that $\langle \varphi_{z'}, u \rangle \neq 0$. It follows that the function $f(z) = \langle \varphi_z, u \rangle$ is meromorphic in \mathbb{C} , has simple poles at the points of $\sigma(B')$, and has zeroes on a countable set of points, \mathfrak{N}_u , which have no finite accumulation points in \mathbb{C} . In the terminology of Krein, the element u is called a choice of *gauge* [7]. The set \mathfrak{N}_u is clearly equal to the set of all $z \in \mathbb{C}$ for which \mathcal{H} cannot be written as a linear combination of elements of $\text{Ran}(B - z)$ and $\mathbb{C}\{u\}$.

Let $\delta_z := \frac{\varphi_z}{\langle \varphi_z, u \rangle}$. It is not difficult to see that this is a meromorphic function on \mathbb{C} with simple poles at points of the set \mathfrak{N}_u (observe that the poles of φ_z at the points of $\sigma(B')$ coincide with those of $\langle \varphi_z, u \rangle$) [8]. Furthermore, we have the following:

Lemma 1 [8, p. 5] *The vector-valued function δ_z does not depend on the choice of self-adjoint extension B' of B used to define φ_z .*

It is not hard to see, with the aid of the above lemma, that $\delta_z \in \text{Ker}(B^* - z)$ for every $z \in \mathbb{C} \setminus \mathfrak{N}_u$. In particular $0 \neq \delta_x \in \text{Ker}(B^* - x)$ for every $x \in \mathbb{R}$, such that $x \notin \mathfrak{N}_u$.

Now define a linear map Φ on \mathcal{H} which takes \mathcal{H} onto a certain vector space $\Phi[\mathcal{H}]$ of meromorphic functions as follows. If $\phi \in \mathcal{H}$, $\Phi[\phi](z) := \langle \phi, \delta_{\bar{z}} \rangle = \overline{\langle \delta_{\bar{z}}, \phi \rangle}$. For simplicity of notation, let $\hat{\phi} := \Phi[\phi]$. We will sometimes write Φ_u instead of Φ to show the dependence of Φ on the choice of gauge u . For any $\phi \in \mathcal{H}$, $\hat{\phi}$ is a meromorphic function whose poles are contained in the set \mathfrak{N}_u . Krein calls the gauge u quasi-regular if the set of all $\phi \in \mathcal{H}$ for which $\hat{\phi}$ is analytic in a region containing \mathbb{R} is dense in \mathcal{H} . In particular, if $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$ then u is a quasi-regular gauge. Krein asserts that one can always choose u so that $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$ [8, 11]. Since the original paper in which this is proven is in Russian, and the author is not aware of a translation, here is an original and simple proof of this fact:

Lemma 2 *Suppose B is a symmetric operator with the sampling property. If $0 \neq \phi_\lambda \in \mathfrak{D}_\lambda := \text{Ker}(B^* - \lambda)$ where $\lambda \in \mathbb{R}$, then $\langle \phi_\lambda, \phi_z \rangle \neq 0$ for any $0 \neq \phi_z \in \mathfrak{D}_z$, $z \in \mathbb{C} \setminus \mathbb{R}$.*

It follows, by this lemma, that given any $z \in \mathbb{C} \setminus \mathbb{R}$, any non-zero $\phi_z \in \mathfrak{D}_z$ is a quasi-regular gauge for B , and if $u = \phi_z$ is chosen as a gauge that the set of all $\hat{\phi}$ for $\phi \in \mathcal{H}$ is a set of meromorphic functions with simple poles contained in the set \mathfrak{N}_{ϕ_z} , $\mathfrak{N}_{\phi_z} \cap \mathbb{R} = \emptyset$.

Proof Suppose $\phi_\lambda \perp \phi_z$ where $0 \neq \phi_z \in \text{Ker}(B^* - z)$, $z \in \mathbb{C} \setminus \mathbb{R}$, and $\lambda \in \mathbb{R}$. Then $\phi_\lambda \in \text{Ran}(B - \bar{z})$ and $\phi_\lambda = (B - \bar{z})\phi$ for some $\phi \in \text{Dom}(B)$. Now ϕ_λ is an eigenvector with eigenvalue λ for some self-adjoint extension B' of B . Hence $(B' - \bar{z})^{-1}\phi_\lambda = (\lambda - \bar{z})^{-1}\phi_\lambda$. But $(B' - \bar{z})^{-1}\phi_\lambda = (B' - \bar{z})^{-1}(B - \bar{z})\phi = \phi$. Hence $(\lambda - \bar{z})^{-1}\phi_\lambda = \phi \in \text{Dom}(B)$, so that $\phi_\lambda \in \text{Dom}(B)$. This contradicts both the simplicity and regularity of B . \square

The set of all $\hat{\phi} \in \Phi[\mathcal{H}]$ can be made into a Hilbert space as follows. Let $\mathcal{P}(B)$ denote the convex set of all unital positive operator valued regular Borel measures, $Q(\cdot)$, on the σ -algebra of all Borel subsets of \mathbb{R} such that

$$\|B\phi\|^2 = \int_{-\infty}^{\infty} \lambda^2 \langle Q(d\lambda)\phi, \phi \rangle, \tag{7}$$

and

$$B\phi = \int_{-\infty}^{\infty} \lambda Q(d\lambda)\phi \tag{8}$$

for all $\phi \in \text{Dom}(B)$. This set can be thought of as the set of all POVM's which 'diagonalize' the symmetric operator B . For $Q \in \mathcal{P}(B)$ and a quasi-regular gauge u for B , let σ_Q be the regular Borel measure on Borel subsets Ω of \mathbb{R} defined by $\sigma_Q(\Omega) := \langle Q(\Omega)u, u \rangle$. Krein has proven the following theorem on spectral representations of the symmetric operator B [7, p. 12, 51, 55]:

Theorem 3 *Let B be a regular symmetric operator with deficiency indices $(1, 1)$. Let u be a quasi-regular gauge for B . Then, for any $Q \in \mathcal{P}(B)$, the map $\Phi, \Phi[\phi](z) = \frac{\langle \phi, \varphi_{\bar{z}} \rangle}{\langle u, \varphi_{\bar{z}} \rangle}$, is an isometry from \mathcal{H} into $L^2(\mathbb{R}, d\sigma_Q)$. The map Φ is onto if and only if Q is a projection-valued measure. Furthermore, $\Phi B \Phi^{-1}$ acts as multiplication by the independent variable on its dense domain $\Phi \text{Dom}(B) \subset \Phi \mathcal{H}$.*

Suppose now that u is a quasiregular gauge such that $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$. Then, given any $\phi \in \mathcal{H}$, $\hat{\phi} = \Phi_u[\phi]$ is analytic on a region containing \mathbb{R} . It follows that given any $z \in \mathbb{C} \setminus \mathfrak{N}_u$, $|\hat{\phi}(z)| = |\langle \phi, \delta_{\bar{z}} \rangle| \leq \|\phi\| \|\delta_{\bar{z}}\|$. This shows that point evaluation at any $z \in \mathbb{C} \setminus \mathfrak{N}_u$ is a bounded linear functional on $\Phi[\mathcal{H}]$, so that $\Phi[\mathcal{H}]$ is a reproducing kernel Hilbert space of meromorphic functions in \mathbb{C} . In fact, since δ_z is analytic on $\mathbb{C} \setminus \mathfrak{N}_u$, this shows that point evaluation is uniformly bounded on compact subsets of $\mathbb{C} \setminus \mathfrak{N}_u$. If $\mathfrak{N}_u = \emptyset$, then the functions $\hat{\phi}$ are entire, and $\Phi[\mathcal{H}]$ is a reproducing kernel Hilbert space of entire functions. Any symmetric $B \in \text{Sym}_1(\mathcal{H})$ for which there exists a gauge u such that the functions $\hat{\phi} = \Phi_u[\phi]$ are entire for all $\phi \in \mathcal{H}$ is called an entire operator. The class of entire operators has been studied extensively by Krein [7].

In [8], the authors exploit Krein's theory to show that if u is chosen so that $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$, then the reproducing kernel Hilbert spaces $\Phi_u \mathcal{H} \subset L^2(\mathbb{R}, d\sigma_Q)$ have the sampling property. Their result can be seen as a simple consequence of the following original and simple theorem.

Theorem 4 *Let \mathcal{H} be a reproducing kernel Hilbert space of functions on a set $\mathbb{S} \supset \mathbb{R}$ with positive definite kernel function (i.e., $K(x, x) = \|\delta_x\|^2 > 0$ for all $x \in \mathbb{R}$). Suppose that the operator of multiplication by the independent variable, M , belongs to $\text{Sym}_1(\mathcal{H})$, i.e it is symmetric, densely defined in \mathcal{H} , regular and simple with deficiency indices $(1, 1)$. Then \mathcal{H} has the sampling property.*

In particular, if $\sigma(M(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{M}}$ and $\delta_{\lambda_n(\alpha)}$ is the point evaluation vector at the point $\lambda_n(\alpha) \in \mathbb{R}$, then for any $\phi \in \mathcal{H}$, the vectors

$$\phi_N := \sum_{|n| \leq N; n \in \mathbb{M}} \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \delta_{\lambda_n(\alpha)} \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2} = \sum_{|n| \leq N; n \in \mathbb{M}} \phi(\lambda_n(\alpha)) \frac{\delta_{\lambda_n(\alpha)}}{\delta_{\lambda_n(\alpha)}(\lambda_n(\alpha))}$$

converge to ϕ as $N \rightarrow \infty$ both pointwise and in norm. If the map $z \mapsto \delta_z$ is continuous for $z \in \mathbb{S}$, this pointwise convergence is uniform on compact subsets of \mathbb{S} .

Here, recall that $\mathbb{M} = \pm\mathbb{N}$ or \mathbb{Z} , see the discussion following the proof of Theorem 2. Any RKHS satisfying the conditions of the above theorem actually has a $U(1)$ parameter family of total orthogonal sets of point evaluation vectors, $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{M}}$, $\alpha \in [0, 1)$. Here the $U(1)$ parameter $\alpha \in [0, 1)$ labels the $U(1)$ family $M(\alpha)$ of self-adjoint extensions of M , and the sampling lattice $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ is the spectrum of the self-adjoint extension $M(\alpha)$, (see the remarks following the statement and proof of Theorem 2). We will say that any RKHS with such a $U(1)$ family of total orthogonal sets of point evaluation vectors $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{M}}$; $\alpha \in [0, 1)$, where the sampling lattices $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ cover \mathbb{R} exactly once as α ranges in $[0, 1)$, has the $U(1)$ sampling property.

Proof By Theorem 2, the assumptions on M imply that the spectra of all self-adjoint extensions of M cover \mathbb{R} exactly once, and consist of eigenvalues of multiplicity one. Since \mathcal{H} is a reproducing kernel Hilbert space, let δ_x denote the point evaluation vector at $x \in \mathbb{R}$. Since $\delta_x \neq 0$ for any $x \in \mathbb{R}$, each δ_x is an eigenvector of M^* to eigenvalue x . To see this observe that for any $\phi \in \text{Dom}(M)$, $\langle M\phi, \delta_x \rangle = x\phi(x) = x\langle \phi, \delta_x \rangle = \langle \phi, x\delta_x \rangle$ which implies that $M^*\delta_x = x\delta_x$, by the definition of the adjoint. It follows that if $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ are the sequences of eigenvalues of the self-adjoint extensions $M(\alpha)$ of M , that $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{M}}$ is a total orthogonal set of eigenvectors to $M(\alpha)$ for each $\alpha \in [0, 1)$. This proves that \mathcal{H} has the Kramer sampling property.

Given any $\phi \in \mathcal{H}$, let

$$\phi_N = \sum_{n=-N}^N \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \delta_{\lambda_n(\alpha)} \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2}. \quad (9)$$

Clearly, $\phi_N \in \mathcal{H}$ for each $N \in \mathbb{N}$, and since $\{\frac{\delta_{\lambda_n(\alpha)}}{\|\delta_{\lambda_n(\alpha)}\|}\}_{n \in \mathbb{Z}}$ is an orthonormal basis of \mathcal{H} , ϕ_N converges to ϕ in norm.

Furthermore, for any $z \in \mathbb{S}$ and $\phi \in \mathcal{H}$,

$$\begin{aligned} |\phi(z) - \phi_N(z)| &= |\langle \phi - \phi_N, \delta_z \rangle| \\ &\leq \|\phi - \phi_N\| \|\delta_z\|. \end{aligned} \quad (10)$$

Since $\phi_N \rightarrow \phi$ in norm, it follows that the above vanishes in the limit as $N \rightarrow \infty$. If the map $z \mapsto \delta_z$ is continuous, then $\|\delta_z\|$ is uniformly bounded on compacta, so that $\phi_N \rightarrow \phi$ uniformly on compacta as well. \square

Under the same assumptions as in Theorem 3, the following is an immediate consequence of Theorem 4.

Theorem 5 *If the quasiregular gauge u is such that $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$, then the subspaces $\Phi_u \mathcal{H} \subset L^2(\mathbb{R}, d\sigma_Q)$ are reproducing kernel Hilbert spaces with the $U(1)$ sampling property. These spaces consist of meromorphic functions with poles contained in the set \mathfrak{N}_u .*

The fact that any symmetric operator B with the sampling property is unitarily equivalent to the operator of multiplication by the independent variable in a reproducing kernel Hilbert space with the sampling property was first proven by Kempf in [3], prior to [8], and without the use of Krein’s theory of spectral representations of such symmetric operators [7]. Kempf’s approach to proving this result stems from Theorem 2. Choosing a unit norm $\delta_x \in \text{Ker}(B^* - x)$ for each $x \in \mathbb{R}$, Kempf defines, for each $\phi \in \mathcal{H}$, $\hat{\phi}(x) := \langle \phi, \delta_x \rangle$. The fact that the set of all such $\hat{\phi}$ obeys a sampling formula follows from the fact that $\{\delta_{\lambda_n(\alpha)}\}_{n \in \mathbb{M}}$ is an orthonormal basis of \mathcal{H} for each $\alpha \in [0, 1)$. Here, recall that $(\lambda_n(\alpha))_{n \in \mathbb{M}} = \sigma(B(\alpha))$. Kempf further shows that the unit norm δ_x can be chosen so that the functions ϕ_x are continuous, and indicates how the set of all $\hat{\phi}$ can be made into a reproducing kernel Hilbert space [3].

The present paper will now proceed as follows. We will extend and refine some of the methods of [3], and use Theorem 2 and the spectrum of any symmetric operator B with the sampling property to define a C^∞ -diffeomorphism $\lambda : \mathbb{R}_{\mathbb{M}} \rightarrow \mathbb{R}$ such that $\lambda(x) \in \sigma(B(x - \lfloor x \rfloor))$ for each $x \in \mathbb{R}_{\mathbb{M}}$. Here $\lfloor x \rfloor$ is the integer part of x , and $\mathbb{R}_{\mathbb{M}}$ is equal to $(0, \infty)$, $(-\infty, 0)$ or \mathbb{R} and $\mathbb{M} = \pm\mathbb{N}$ or \mathbb{Z} depending on whether the self-adjoint extensions are bounded below, above or neither bounded above nor below, respectively. This will be done in the following section, Sect. 3. This C^∞ diffeomorphism λ will be called the spectral function of $B \in \text{Sym}_1(\mathcal{H})$. In Sect. 4, the spectral function λ of $B \in \text{Sym}_1(\mathcal{H})$ will be combined with Krein’s methods of Sect. 2.2 to explicitly construct ‘spectral measures’ σ_z which are equivalent with respect to Lebesgue measure, i.e. they have the same sets of measure zero, and reproducing kernel Hilbert spaces $\mathcal{H}_z \subset L^2(\mathbb{R}, d\sigma_z) =: \mathcal{K}_z$ for each $z \in \mathbb{D}$ with the following properties. There is a unitary transformation U_z from \mathcal{H} onto \mathcal{H}_z that transforms B into an operator M'_z of multiplication by the independent variable. If M_z denotes the self-adjoint operator of multiplication by the independent variable in \mathcal{K}_z , then the unitary group $U(t) := e^{itM_z}$ is the minimal unitary dilation of its compression to \mathcal{H}_z , and if P_z denotes the projection of \mathcal{K}_z onto \mathcal{H}_z , then $P_z \mu(M_z) U_z \phi_\pm = z U_z \phi_\pm$. Here ϕ_\pm are fixed unit norm vectors in $\mathfrak{D}_\pm \subset \mathcal{H}$. These constructions will be key to the proof of the major result of this paper, Theorem 14, which provides a sufficient condition for a subspace of $L^2(\mathbb{R}, d\nu)$ to be a RKHS with the $U(1)$ sampling property.

3 The Spectral Function of a Symmetric Operator with the Sampling Property

Recall that if $B \in \text{Sym}_1(\mathcal{H})$, then the family of all self-adjoint extensions of B can be labeled by a single real parameter $\alpha \in [0, 1)$. Explicitly, fix unit norm vectors $\phi_\pm \in \mathfrak{D}_\pm$ and then define

$$U(\alpha) := V \oplus e^{i2\pi\alpha} \phi_- \langle \cdot, \phi_+ \rangle \tag{11}$$

on $\mathcal{H} := \text{Dom}(V) \oplus \mathfrak{D}_+$, where V is the Cayley transform of B . The self-adjoint extensions $B(\alpha)$ of B are defined as the inverse Cayley transforms of the $U(\alpha)$. All self-adjoint extensions of B are obtained in this manner.

Recall that the spectrum of each self-adjoint extension $B(\alpha)$ of B can be arranged as a non-decreasing sequence of eigenvalues $(\lambda_n(\alpha))_{n \in \mathbb{M}}$ where $\mathbb{M} = -\mathbb{N}, \mathbb{N}$ or \mathbb{Z} , and that the spectra $\sigma(B(\alpha))$ of the self-adjoint extensions do not intersect and cover \mathbb{R} exactly once (see Theorem 2). In fact, even more can be said [7, p. 19]:

Theorem 6 (Krein) *Let B be a closed simple symmetric operator in \mathcal{H} with deficiency indices $(1, 1)$. Suppose that the interval $I \subset \mathbb{R}$ consists of regular points of B . Then, the eigenvalues of any two self-adjoint extensions B' and B'' of B in I alternate.*

In our case, we assume B is regular so that every point in \mathbb{R} is regular for B . It follows that the eigenvalues of any two self-adjoint extensions $B(\alpha)$ and $B(\beta)$ of B alternate. That is, given any two consecutive eigenvalues, $\lambda_n(\alpha)$ and $\lambda_{n+1}(\alpha)$ of $B(\alpha)$, every other self-adjoint extension $B(\beta)$ of B , $\beta \neq \alpha$ has exactly one eigenvalue in the interval $(\lambda_n(\alpha), \lambda_{n+1}(\alpha))$. In particular, if $\sigma(B(\alpha))$ is bounded above or below, or is not bounded above or below, then the same is true of the spectrum of every other self-adjoint extension of B . This means that $\sigma(B(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{M}}$ where \mathbb{M} is equal to $\pm\mathbb{N}$ or \mathbb{Z} , and is the same for every $\alpha \in [0, 1)$.

Suppose that $\mathbb{M} = \mathbb{Z}$ and consider $\{\lambda_n(0)\}_{n \in \mathbb{Z}}$. Given any $\alpha \in (0, 1)$, define $\lambda_n(\alpha)$ to be that unique eigenvalue of $B(\alpha)$ in the interval $(\lambda_n(0), \lambda_{n+1}(0))$. Then, for each $n \in \mathbb{Z}$ the map $\lambda_n(\alpha)$ is a bijection from $[0, 1]$ onto $[\lambda_n(0), \lambda_{n+1}(0)]$, and $\sigma(B(\alpha)) = (\lambda_n(\alpha))_{n \in \mathbb{Z}}$. Using the functions $\lambda_n(\alpha)$, we can define $\hat{\lambda}, \tilde{\lambda} : \mathbb{R} \rightarrow \mathbb{R}$ by $\hat{\lambda}(x) := \lambda_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$ and $\tilde{\lambda}(x) := \lambda_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. Here $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. It follows that both $\hat{\lambda}$ and $\tilde{\lambda}$ are bijections, and that $\hat{\lambda}(x) \in \sigma(B(x))$ while $\tilde{\lambda}(x) \in \sigma(B(-x))$ for each $x \in \mathbb{R}$.

Remark 3.0.1 If $\mathbb{M} = \pm\mathbb{N}$ instead, $\tilde{\lambda}$ and $\hat{\lambda}$ can be defined analogously. For example, suppose that $\mathbb{M} = -\mathbb{N}$ so that each self-adjoint extension of B is bounded above. Fix the deficiency vectors $\phi_{\pm} \in \mathfrak{D}_{\pm}$ so that $B(0)$ is the most negative self-adjoint extension of B , and consider its eigenvalues $(\lambda_n(0))_{n=1}^{\infty}$ arranged in a strictly increasing sequence. By Theorem 6, if we define $\lambda_0(0) = +\infty$, then every self-adjoint extension $B(\alpha)$ of B , for $\alpha \in (0, 1)$ has exactly one eigenvalue which we label $\lambda_n(\alpha)$ in the interval $(\lambda_n(0), \lambda_{n+1}(0))$ for each $n \in -\mathbb{N}$. The functions $\tilde{\lambda}$ and $\hat{\lambda}$ can now be defined as above, except that in this case their domains are equal to $(-\infty, 0)$.

Our goal is to prove that each $\lambda_n(\alpha)$ is an infinitely differentiable function of α . Using this fact, a stronger version of the following proposition will be established.

Proposition 1 *Either $\hat{\lambda}$ or $\tilde{\lambda}$ is an infinitely differentiable homeomorphism of $\mathbb{R}_{\mathbb{M}}$ onto \mathbb{R} .*

Here, recall that $\mathbb{R}_{\mathbb{M}}$ is equal to $(0, \infty)$, $(-\infty, 0)$ or \mathbb{R} depending on whether $\mathbb{M} = \mathbb{N}, -\mathbb{N}$ or \mathbb{Z} , i.e. depending on whether the self-adjoint extensions of B are bounded below, above, or neither bounded above nor below. For convenience, and to simplify the presentation, we will assume for the remainder of this section that $\mathbb{M} = \mathbb{Z}$ so that $\mathbb{R}_{\mathbb{M}} = \mathbb{R}$.

3.1 The Spectral Function of a Symmetric Operator with the Sampling Property

The proof of Proposition 1 will be broken into several smaller claims.

Consider the Möbius transform $\mu(z) := \frac{z-i}{z+i}$ and its inverse $\mu^{-1}(z) := i \frac{1+z}{1-z}$. Let

$$U(z) := \mu(B(z)) = \mu(B) \oplus e^{i2\pi z} \phi_{-} \langle \cdot, \phi_{+} \rangle \tag{12}$$

for any $z \in \mathbb{C}$. For $x \in \mathbb{R}$, $U(x) = U(\lfloor x \rfloor) = U(x + k)$ for any $k \in \mathbb{Z}$ is the Cayley transform of $B(x)$. The spectral mapping theorem implies that the spectrum of $U(\alpha)$ is $(\kappa_n(\alpha))_{n \in \mathbb{Z}}$ where $\kappa_n(\alpha) := \mu(\lambda_n(\alpha))$ so that $\hat{\kappa}(x) := \mu(\hat{\lambda}(x)) = \mu(\lambda_{\lfloor x \rfloor}(x - \lfloor x \rfloor)) = \kappa_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$. Now since $\hat{\lambda}(x) = \mu^{-1}(\hat{\kappa}(x))$, it follows that $\hat{\lambda}$ will be infinitely differentiable for $x \in \mathbb{R}$ if $\hat{\kappa}$ is. Similarly, we define $\tilde{\kappa}(x) = \mu(\tilde{\lambda}(x)) = \kappa_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. Again, observe that $\hat{\kappa}(x) \in \sigma(U(x))$ and $\tilde{\kappa}(x) \in \sigma(U(-x))$ for each $x \in \mathbb{R}$. Further note that for $n \in \mathbb{Z}$, $\hat{\kappa}(n) = \kappa_n(0)$ while $\tilde{\kappa}(n) = \kappa_n(1) = \kappa_n(0)$ since $U(0) = U(1)$.

The fact that $\hat{\kappa}, \hat{\lambda}$ are continuous functions of x follows from the discreteness of the spectra of each $U(x)$, the continuity of the operator valued function $U(x)$, and Newburgh’s theorem [12]:

Theorem 7 (Newburgh) *Let \mathbb{A} be a unital Banach algebra and let $a \in \mathbb{A}$. Suppose that $\sigma(a) \subset U \cup V$ where U, V are open and disjoint, $U \cap V = \emptyset$ and $U \cap \sigma(a) \neq \emptyset$. Then there is an $\epsilon > 0$ such that $\|x - a\| < \epsilon$ implies that $\sigma(x) \cap U \neq \emptyset$.*

Notation 3.1.1 Given $z, w \in \mathbb{T}$, the unit circle in the complex plane, we will write (z, w) to denote the arc of the circle \mathbb{T} which lies between z and w , and does not include the point 1. That is (z, w) is the image of the open interval $(\mu^{-1}(z), \mu^{-1}(w)) \subset \mathbb{R}$ under the Möbius transformation μ . Similarly, if $z, w \in \mathbb{T}$, we will say that $z \leq w$ if $\mu^{-1}(z) \leq \mu^{-1}(w)$. Furthermore, we will say that $\hat{\kappa}$ or $\tilde{\kappa}$ is monotonically increasing on $(z, w) \subset \mathbb{T}$ if $\hat{\lambda}$ or $\tilde{\lambda}$ is monotonically increasing on $(\mu^{-1}(z), \mu^{-1}(w))$.

First consider the functions $\kappa_n(\alpha)$ for $\alpha \in (0, 1)$ and $n \in \mathbb{Z}$. Recall that $\kappa_n(\alpha)$ is the unique eigenvalue to $U(\alpha)$ in the open arc $(\kappa_n(0), \kappa_{n+1}(0))$.

Claim 1 For each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a continuous map from $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$.

Proof We already know that for each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a bijection from $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$. It remains to establish continuity. Choose $\alpha' \in (0, 1)$ and $n \in \mathbb{Z}$. Let $\epsilon > 0$ be arbitrary and consider $S := \mathcal{B}_\epsilon(\kappa_n(\alpha')) \cap (\kappa_n(0), \kappa_{n+1}(0))$, where $\mathcal{B}_\epsilon(x)$ denotes the open ball of radius ϵ about x . Since $U(\alpha)$ is a continuous operator-valued function of $\alpha \in (0, 1)$, it follows from Newburgh’s theorem, Theorem 7, that there is a $\delta > 0$ such that if $\alpha \in (0, 1)$ satisfies $|\alpha - \alpha'| < \delta$ then $\sigma(U(\alpha)) \cap S \neq \emptyset$. For such an α , $\sigma(U(\alpha)) \cap S = \kappa_n(\alpha)$ so that $|\kappa_n(\alpha) - \kappa_n(\alpha')| < \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves the claim. □

Claim 2 For each $n \in \mathbb{Z}$, $\kappa_n(\alpha)$ is a monotonic strictly increasing or monotonic strictly decreasing function of $\alpha \in (0, 1)$. If $\kappa_n(\alpha)$ is increasing then $\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_n(0)$ and $\lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_{n+1}(0)$. Conversely, if κ_n is decreasing then $\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_{n+1}(0)$ and $\lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0)$.

Proof Suppose that for some $n \in \mathbb{Z}$ that $\kappa_n(\alpha)$ was not monotonic. Then there would exist $\alpha_i \in (0, 1)$, $1 \leq i \leq 3$, $\alpha_1 < \alpha_2 < \alpha_3$, such that $\kappa_n(\alpha_1) < \kappa_n(\alpha_2)$ and $\kappa_n(\alpha_3) < \kappa_n(\alpha_2)$. Let $M := \max\{\kappa_n(\alpha_1), \kappa_n(\alpha_3)\}$. Since $\kappa_n(\alpha)$ is continuous on $[\alpha_1, \alpha_3]$ by Claim 1, the intermediate value theorem then implies that there exists a $c_1 \in [\alpha_1, \alpha_2)$ and a $c_2 \in (\alpha_2, \alpha_3]$ such that $\kappa(c_1) = M = \kappa(c_2)$. This contradicts the fact that $\kappa_n(\alpha)$ is injective. This proves that each $\kappa_n(\alpha)$ is either monotonically strictly increasing or decreasing.

Now suppose that $\kappa_n(\alpha)$ is monotonically decreasing on $(0, 1)$. Suppose, contrary to the claim, that

$$\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) \neq \kappa_{n+1}(0). \quad (13)$$

It follows that there is an $\epsilon > 0$ such that for each $k \in \mathbb{N}$, one can find $\alpha_k \in (0, 1)$ so that $\alpha_k \rightarrow 0$, and $\kappa_{n+1}(0) - \kappa_n(\alpha_k) > \epsilon$. Since $\kappa_n(\alpha)$ is a bijection of $(0, 1)$ onto $(\kappa_n(0), \kappa_{n+1}(0))$, it follows that there is an $\alpha' \in (0, 1)$ such that $\kappa_{n+1}(0) - \kappa_n(\alpha') < \frac{\epsilon}{2}$. It follows that $\kappa_n(\alpha') > \kappa_n(\alpha_k)$ for all $k \in \mathbb{N}$. Choosing k large enough so that $\alpha_k < \alpha'$ contradicts our assumption that κ_n is monotonically decreasing. The remainder of the claim is proved in a similar fashion. \square

Claim 3 The functions $\kappa_n(\alpha)$, $n \in \mathbb{Z}$, are either all monotonically increasing, or all monotonically decreasing.

Proof Suppose that for some fixed $m \in \mathbb{Z}$, that $\kappa_m(\alpha)$ is monotonically decreasing. Then, by Claim 2, it follows that $\lim_{\alpha \rightarrow 0^+} \kappa_m(\alpha) = \kappa_{m+1}(0)$, and $\lim_{\alpha \rightarrow 1^-} \kappa_m(\alpha) = \kappa_m(0)$. Let $\epsilon := \min\{\frac{\kappa_{m+1}(0) - \kappa_m(0)}{2}, \frac{\kappa_m(0) - \kappa_{m-1}(0)}{2}\}$. Choose $\delta > 0$ such that $\alpha \in (0, 1)$ and $\alpha < \delta$ implies that $\kappa_{m+1}(0) - \kappa_m(\alpha) < \epsilon$. By Newburgh's theorem, Theorem 7, for any sufficiently large $k \in \mathbb{N}$, there is a $\delta_k > 0$ so that $|\alpha| < \delta_k$ implies that $\sigma(U(\alpha)) \cap \mathcal{B}_{1/k}(\kappa_m(0)) \neq \emptyset$. Choose $\delta'_k := \min\{\delta, \delta_k\}$, and $K \in \mathbb{N}$ so that $\frac{1}{K} < \epsilon$. It follows that when $k > K$, if $\alpha \in (0, 1)$ and $\alpha < \delta'_k$ then $\kappa_m(\alpha) \notin \mathcal{B}_{1/k}(\kappa_m(0))$. Since $\sigma(U(\alpha)) \cap \mathcal{B}_{1/k}(\kappa_m(0)) \neq \emptyset$ it follows that for each such α , $\kappa_{m-1}(\alpha) \in \mathcal{B}_{1/k}(\kappa_m(0))$. It follows that $\lim_{\alpha \rightarrow 0^+} \kappa_{m-1}(\alpha) \neq \kappa_{m-1}(0)$. By Claim 2, it follows that $\lim_{\alpha \rightarrow 0^+} \kappa_{m-1}(\alpha) = \kappa_m(0)$, and that $\kappa_{m-1}(\alpha)$ is monotonically decreasing on $(0, 1)$.

Proceeding in a similar fashion, it is not difficult to show that for every $n \in \mathbb{Z}$, $\kappa_n(z)$ is monotonically decreasing. Proving the other half of the claim is directly analogous. \square

Recall that the functions $\hat{\kappa}, \tilde{\kappa} : \mathbb{R} \rightarrow \mathbb{R}$ are defined by $\hat{\kappa}(x) := \kappa_{\lfloor x \rfloor}(x - \lfloor x \rfloor)$ and $\tilde{\kappa}(x) := \kappa_{\lfloor x \rfloor}(1 - (x - \lfloor x \rfloor))$. Recall that $\hat{\kappa}(x)$ is the unique eigenvalue to the operator $U(x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x \rfloor+1}(0)) \subset \mathbb{T}$ while $\tilde{\kappa}(x)$ is the unique eigenvalue to the operator $U(-x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x \rfloor+1}(0))$. Furthermore, recall that $\sigma(U(0)) = (\kappa_n(0))_{n \in \mathbb{Z}}$, and that $\kappa_n(1) := \kappa_n(0)$, so that $\tilde{\kappa}(n) = \kappa_n(1) = \kappa_n(0) = \hat{\kappa}(n)$, for any $n \in \mathbb{Z}$. Also, remember that $\kappa_n(0) < \kappa_{n+1}(0)$ for all $n \in \mathbb{Z}$.

Claim 4 Either $\hat{\kappa}$ or $\tilde{\kappa}$ is a homeomorphism of \mathbb{R} onto \mathbb{R} which is strictly monotonically increasing.

Proof By Claim 3, either

$$\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_n(0) \quad \text{and} \quad \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_{n+1}(0) \tag{14}$$

for all $n \in \mathbb{Z}$, or

$$\lim_{\alpha \rightarrow 0^+} \kappa_n(\alpha) = \kappa_{n+1}(0) \quad \text{and} \quad \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0) \tag{15}$$

for all $n \in \mathbb{Z}$.

In the first case where Eq. (14) holds, it is clear that $\hat{\kappa}$ satisfies the requirements of the claim. In the second case of Eq. (15), it is not difficult to verify that $\tilde{\kappa}$ satisfies the requirements of the claim since, for example,

$$\begin{aligned} \lim_{x \rightarrow n^+} \tilde{\kappa}(x) &= \lim_{x \rightarrow 0^+} \kappa_n(1 - (x - \lfloor x \rfloor)) \\ &= \lim_{\alpha \rightarrow 1^-} \kappa_n(\alpha) = \kappa_n(0) = \tilde{\kappa}(n). \end{aligned} \tag{16}$$

□

In lieu of the above result,

Definition 3.1.2 For a symmetric operator B with the sampling property, define κ to be the choice of the two functions $\hat{\kappa}, \tilde{\kappa}$ in the above claim which is continuous on all of \mathbb{R} . If $\kappa = \tilde{\kappa}$, redefine $U(x) := V \oplus e^{-i2\pi x} \langle \cdot, \phi_+ \rangle \phi_-$ so that for each $x \in \mathbb{R}$, $\kappa(x)$ is the unique eigenvalue to $U(x)$ in the arc $[\kappa_{\lfloor x \rfloor}(0), \kappa_{\lfloor x+1 \rfloor}(0))$ of the unit circle, \mathbb{T} .

The function $\kappa(x)$ will be called the **spectral function** of the isometric operator $V = \kappa(B)$, and $\lambda := \mu^{-1}(\kappa)$ will be called the spectral function of the symmetric operator B .

Remark 3.1.3 The definition of $U(x)$, and hence of $\kappa(x)$, depends on the arbitrary choice of unit norm $\phi_{\pm} \in \mathfrak{D}_{\pm}$. If $\varphi_{\pm} \in \mathfrak{D}_{\pm}$ are a different choice of deficiency vectors then, since the deficiency subspaces \mathfrak{D}_{\pm} are one dimensional, $\varphi_{\pm} = e^{i2\pi\theta_{\pm}} \phi_{\pm}$, for some $\theta_{\pm} \in [0, 1)$. If one then defines $\tilde{U}(x) := V \oplus e^{i2\pi x} \varphi_- \langle \cdot, \varphi_- \rangle$, it follows that $\tilde{\kappa}(x) = \kappa(x - \theta_+ + \theta_-)$. If one uses \tilde{U} to define a spectral function $\tilde{\kappa}$, then $\tilde{\kappa}(x) = \kappa(x - \theta_+ + \theta_-)$, for all $x \in \mathbb{R}$. For this reason, we will say that two spectral functions κ_1 and κ_2 are equivalent if there is a $c \in \mathbb{R}$ such that $\kappa_1(x) = \kappa_2(x + c)$ for all $x \in \mathbb{R}$. It is not difficult to show that $B_1, B_2 \in \mathcal{S}\text{ym}_1(\mathcal{H})$ are unitarily equivalent if and only if their spectral functions are equivalent [6, Theorem 10.3.8].

3.2 Infinite Differentiability and Analyticity of the Spectral Function

Using standard functional calculus techniques, this section will show that the functions $\kappa(x)$ and $\lambda(x)$ are infinitely differentiable.

Claim 5 Let λ be an eigenvalue of a self-adjoint extension B' of the symmetric operator B . Then there is an $\epsilon > 0$ such that $\mathcal{B}_{\epsilon}(\lambda) \cap \sigma(B(\alpha))$ contains at most one point for each $\alpha \in [0, 1)$.

Proof Choose $\epsilon > 0$ so that $\mathcal{B}_\epsilon(\lambda) \cap \sigma(B') = \{\lambda\}$. Suppose that the claim does not hold. Then there would be a sequence of values $\alpha_k \in (0, 1)$ such that for each $k \in \mathbb{N}$ there is a self-adjoint extension $B(\alpha_k)$ of B that has at least two eigenvalues in $\mathcal{B}_{1/k}(\lambda)$. Choose $K \in \mathbb{N}$ so that $\frac{1}{K} < \epsilon$. For $k > K$, the alternating eigenvalue theorem, Theorem 6, implies that each such $B(\alpha_k)$ can have at most two eigenvalues λ_k, μ_k in $\mathcal{B}_\epsilon(\lambda)$ where $\lambda_k < \lambda < \mu_k$. Otherwise B' would have more than one eigenvalue in $\mathcal{B}_\epsilon(\lambda)$, which is a contradiction. It follows that $\lambda_k \rightarrow \lambda$ and that $\mu_k \rightarrow \lambda$. Fix a self-adjoint extension $\tilde{B} \neq B'$ of B . By the alternating eigenvalue theorem, it follows that \tilde{B} has eigenvalues α_k such that $\lambda_k \leq \alpha_k \leq \mu_k$ for each $k > K$. Since $\sigma(\tilde{B})$ is closed, it follows that $\lambda \in \sigma(\tilde{B})$ which, by Theorem 2, is a contradiction. \square

The goal now is to show that κ , and hence λ , is infinitely differentiable. Fix $y \in \mathbb{R}$. We will show that $\kappa^{(k)}(y) = \frac{d^k}{dx^k} \kappa(x)|_{x=y}$ exists for any $k \in \mathbb{Z}$. Since y is arbitrary, this will establish Proposition 1.

By Claim 5, there is an $\epsilon > 0$ so that $\overline{\mathcal{B}_{2\epsilon}(\kappa(y))} \cap \sigma(U(x))$ contains at most one point for each $x \in \mathbb{R}$. Since κ is continuous, choose $\delta' > 0$ so that $|x - y| < \delta'$ implies that $\kappa(x) \in \mathcal{B}_\epsilon(\kappa(y))$. It follows that for all $|x - y| < \delta'$ that

$$\sigma(U(x)) \cap \left(\overline{\mathcal{B}_{2\epsilon}(\kappa(y))} \setminus \mathcal{B}_\epsilon(\kappa(y)) \right) = \emptyset \tag{17}$$

and that

$$\sigma(U(x)) \cap \mathcal{B}_\epsilon(\kappa(y)) = \kappa(x). \tag{18}$$

For each x such that $|x - y| < \delta'$, let $P(x)$ denote the projection onto the eigenspace of $U(x)$ to eigenvalue $\kappa(x)$. This is a one-dimensional subspace, spanned by some normalized eigenvector which we denote $\phi_{\kappa(x)}$. For each such x , the spectrum of $U(x)$ is purely discrete and is contained in the union of the open sets $V' := \mathbb{C} \setminus \overline{\mathcal{B}_{2\epsilon}(\kappa(y))}$ and $\mathcal{B}_\epsilon(\kappa(y))$. Let $S := V' \cup \mathcal{B}_\epsilon(\kappa(y))$. Then S is an open set containing the spectrum of $U(x)$ for all x such that $|x - y| < \delta'$.

Recall that the spectrum of a bounded operator is upper semi-continuous [13]:

Theorem 8 (Upper semi-continuity of the spectrum) *Let \mathbb{A} be a Banach algebra. Then if $a \in \mathbb{A}$, and U is an open set such that $\sigma(a) \subset U$, then there exists a $\delta > 0$ such that $\|b - a\| < \delta$ implies that $\sigma(b) \subset U$.*

Since $U(w)$ is an entire operator-valued function for $w \in \mathbb{C}$, it follows that there is a $\delta_1 > 0$ such that $|w - y| < \delta_1$ implies that $\sigma(U(w)) \subset S$. Choose a simple, smooth, counterclockwise contour Γ that lies in the interior of $\mathbb{C} \setminus S$, i.e., so that Γ lies between the balls of radius ϵ and 2ϵ about $\kappa(y)$. Let $\delta_2 := \min\{\delta_1, \delta'\}$. For $|w - y| < \delta_2$, the Riesz holomorphic functional calculus can be used to define the following operators $P(w)$ and $U(w)P(w)$:

$$P(w) = \frac{1}{2\pi i} \int_{\Gamma} (z - U(w))^{-1} dz, \tag{19}$$

and,

$$U(w)P(w) = \frac{1}{2\pi i} \int_{\Gamma} z(z - U(w))^{-1} dz. \tag{20}$$

It follows from the Riesz decomposition theorem that for each w such that $|w - y| < \delta_2$, the operators $P(w)$ are idempotents such that $\sigma(U(w)|_{P(w)\mathcal{H}}) = \sigma(U(w)) \cap B_\epsilon(\kappa(y))$. In particular, when $w = x \in \mathbb{R}$ so that $U(x)$ is a unitary operator, $P(w) = P(x)$ is the self-adjoint projection onto the eigenspace of $U(x)$ to eigenvalue $\kappa(x)$.

Now since $U(w) \rightarrow U(y)$ in operator norm as $w \rightarrow y$, and since the spectrum of $\sigma(U(w)) \subset S$ for all $|w - y| < \delta_2$, the following standard functional calculus result shows that $P(w) \rightarrow P(y)$ in operator norm as $x \rightarrow y$:

Proposition 2 *Let $a \in \mathbb{A}$, a unital Banach algebra, and let $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{A}$ be a sequence such that $a_n \rightarrow a$. Let $U \supset \sigma(a)$ be open, suppose that $\sigma(a_n) \subset U$ for all $n \in \mathbb{N}$, and that f is analytic on U . Then $f(a_n) \rightarrow f(a)$.*

It follows from the above proposition that $P(w) \rightarrow P(y)$ as $w \rightarrow y$. Hence, there is a $\tilde{\delta} > 0$ so that $|w - y| < \tilde{\delta}$ implies that $|\langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle| > 0$. Choose $\delta := \min\{\tilde{\delta}, \delta_2\}$.

For all $w \in \mathbb{C}$ such that $|w - y| < \delta$, define

$$\kappa(w) := \frac{\langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle}{\langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle}. \tag{21}$$

If $w = x \in \mathbb{R}$, then $U(x)P(x) = \kappa(x)P(x)$, and the above agrees with our original definition of $\kappa(x)$. Hence, this definition of $\kappa(w)$ is an extension of $\kappa(x)$ to a neighbourhood of y in the complex plane.

Using this representation of $\kappa(w)$, Eqs. (19) and (20) can now be applied to show that $\kappa(w)$ is analytic in $B_\delta(y)$, and hence is infinitely differentiable at y .

Let

$$f(w) := \langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle, \tag{22}$$

and,

$$g(w) := \langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle. \tag{23}$$

The fact that f and g are analytic functions of w for $z \in \mathbb{C} \setminus S$ will follow from the fact that $U(w)$ is an entire $B(\mathcal{H})$ -valued function.

Now $U(w)$ is clearly an entire operator-valued function of $w \in \mathbb{C}$. Indeed, if $\mathcal{U}(w) := i2\pi e^{i2\pi w} \langle \cdot, \phi_+ \rangle \phi_-$, it is easy to check that

$$\lim_{z \rightarrow 0} \left\| \frac{1}{z} (U(w + z) - U(w)) - \mathcal{U}(w) \right\| = 0, \tag{24}$$

so that $\mathcal{U}(w) = U'(w)$ for all $w \in \mathbb{C}$.

Claim 6 The operators $P(w)$ and $U(w)P(w)$ are analytic for $w \in \mathcal{B}_\delta(y)$.

The following lemma will be used in the proof of Claim (6). Its proof is straightforward, and is omitted.

Lemma 3 Let $A(z)$ be an operator-valued function that is differentiable at w . Suppose that each $A(z)$ has a bounded inverse $A(z)^{-1}$, and that $\|A(z)^{-1}\|$ is uniformly bounded for z in some neighbourhood \mathcal{N}_w of w . Then $A(z)^{-1}$ is differentiable at w , and $\frac{d}{dz}A(w)^{-1} = A(w)^{-1}A'(w)A(w)^{-1}$.

Proof of Claim 6 By the previous lemma, and the fact that for each $z \in \mathbb{C} \setminus S$ $(z - U(w))$ is an analytic function of w for $w \in \mathcal{B}_\delta(y)$, it follows that for each such z , $(z - U(w))^{-1}$ and $U(w)(z - U(w))^{-1}$ are also analytic as functions of $w \in \mathcal{B}_\delta(y)$. To show that $P(w)$ and $U(w)P(w)$ are analytic, we will use Morera’s theorem.

Let Γ_1 be a closed, finite, straight line contour in $\mathbb{C} \setminus S$ and Γ_2 be a closed, finite straight line contour in $\mathcal{B}_\delta(y)$. That is, the curve Γ_1 is described by $\Gamma_1(r) = re^{i\alpha} + a$ for $r \in [r_1, r_2]$, $a \in \mathbb{C}$ while $\Gamma_2(s) = se^{i\beta} + c$ for $s \in [s_1, s_2]$, $c \in \mathbb{C}$. Given any $\phi, \psi \in \mathcal{H}$, consider the integral

$$\int_{\Gamma_2} \int_{\Gamma_1} \langle (z - U(w))^{-1} \phi, \psi \rangle dz dw = \int_{s_1}^{s_2} \int_{r_1}^{r_2} \langle (re^{i\alpha} + a) - U(se^{i\beta} + c)^{-1} \phi, \psi \rangle dz dw. \tag{25}$$

Since $\|(z - U(w))^{-1}\|$ is a continuous function of z and w , for $(z, w) \in \mathbb{C} \setminus S \times \mathcal{B}_\delta(y)$, it follows that $\|(z - U(w))^{-1}\| \leq M$ for all (z, w) in the compact set $\Gamma_1 \times \Gamma_2$. Hence, it follows that

$$\int_{\Gamma_2} \int_{\Gamma_1} |\langle (z - U(w))^{-1} \phi, \psi \rangle| dz dw \leq (r_2 - r_1)(s_2 - s_1)M \|\phi\| \|\psi\| < \infty. \tag{26}$$

By Fubini’s theorem [14, p. 25], it follows that we can interchange the order of integration so that

$$\int_{\Gamma_2} \int_{\Gamma_1} \langle (z - U(w))^{-1} \phi, \psi \rangle dz dw = \int_{\Gamma_1} \int_{\Gamma_2} \langle (z - U(w))^{-1} \phi, \psi \rangle dw dz. \tag{27}$$

Since any finite length contour can be approximated arbitrarily well by a finite number of straight line contours, it follows that Eq. (27) holds for all finite, closed, smooth contours $\Gamma_1 \in \mathbb{C} \setminus V$ and $\Gamma_2 \in \mathcal{B}_\delta(y)$. In particular, it follows that for any such a contour Γ_2 ,

$$\begin{aligned} \int_{\Gamma_2} P(w)dw &= \frac{1}{2\pi i} \int_{\Gamma_2} \int_{\Gamma} (z - U(w))^{-1} dz dw \\ &= \frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma_2} (z - U(w))^{-1} dw dz = 0, \end{aligned} \tag{28}$$

since for each $z \in \Gamma$, $(z - U(w))^{-1}$ is analytic in w for $w \in \mathcal{B}_\delta(y)$. It then follows from Morera’s theorem [15, p. 88], that $\langle P(w)\phi, \psi \rangle$ is an analytic function of w for any $\phi, \psi \in \mathcal{H}$. This proves that $P(w)$ is an analytic operator-valued function of w . Similar arguments show that $U(w)P(w)$ is analytic. \square

In summary, it can be concluded that both the functions $f(w) := \langle P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle$ and $g(w) := \langle U(w)P(w)\phi_{\kappa(y)}, \phi_{\kappa(y)} \rangle$ are analytic in $\mathcal{B}_\delta(y)$, and that f does not vanish on $\mathcal{B}_\delta(y)$. Since $y \in \mathbb{R}$ was arbitrary, we can immediately conclude that κ and hence λ is infinitely differentiable and has an analytic extension to some neighbourhood of any point $x \in \mathbb{R}$. The preceding analysis in this section works just as well for the cases where the self-adjoint extensions of B are bounded below or above so that $\mathbb{M} = \pm\mathbb{N}$ and λ is defined on $\mathbb{R}_\mathbb{M}$ which equals $(0, \infty)$ if $\mathbb{M} = \mathbb{N}$ and $(-\infty, 0)$ in the other case. This leads to the conclusion:

Theorem 9 *The spectral function $\lambda = \mu^{-1}(\kappa)$ of a symmetric operator B with the sampling property is a monotonically strictly increasing homeomorphism of $\mathbb{R}_\mathbb{M}$ onto \mathbb{R} , and is infinitely differentiable at any point $x \in \mathbb{R}_\mathbb{M}$. Furthermore, at any point $x \in \mathbb{R}_\mathbb{M}$, it has an analytic extension to a neighbourhood of x . Here $\mathbb{R}_\mathbb{M} = (0, \infty)$, $(-\infty, 0)$ or \mathbb{R} and $\mathbb{M} = \mathbb{N}, -\mathbb{N}$ or \mathbb{Z} , depending on whether the self-adjoint extensions of B are bounded below, above, or neither bounded above nor below.*

Remark 3.2.1 The above theorem implies, in particular, that $\lambda'(x)$ can only vanish on a countable set of points with no finite accumulation point. The results of [3, Section 7.4] further imply that $\lambda'(x)$ cannot vanish at any point at $x \in \mathbb{R}_\mathbb{M}$, as this would contradict the simplicity of B . Hence, $\lambda'(x) > 0$ for all $x \in \mathbb{R}_\mathbb{M}$, and λ is a diffeomorphism of $\mathbb{R}_\mathbb{M}$ onto \mathbb{R} .

4 A Special Class of Spectral Representations of the Symmetric Operator

Given the symmetric operator $B \in \text{Sym}_1(\mathcal{H})$ with the sampling property, and any fixed $z_0 \in \mathbb{C} \setminus \mathbb{R}$, choose as a gauge, $u = \varphi_{z_0}$. Then, as discussed in Sect. 2.2, u is a quasiregular gauge, $\mathfrak{N}_u \cap \mathbb{R} = \emptyset$, and the linear map Φ on \mathcal{H} defined by $\Phi[\phi](z) := \langle \phi, \delta_{\bar{z}} \rangle = \overline{\langle \delta_{\bar{z}}, \phi \rangle}$, where $\delta_z := \frac{\varphi_z}{\langle \varphi_z, \varphi_{z_0} \rangle}$ takes \mathcal{H} onto a vector space of meromorphic functions which are analytic in the region $\mathbb{C} \setminus \mathfrak{N}_u \supset \mathbb{R}$ and have simple poles at the points of \mathfrak{N}_u (see Sect. 2.2). For simplicity of notation, let $\hat{\phi} := \Phi[\phi]$.

Let τ denote the monotonically increasing function which is the inverse of the infinitely differentiable monotonically increasing diffeomorphism λ , the spectral function of B . Since, by Remark 3.2.1, $\lambda'(x) > 0$ for all $x \in \mathbb{R}_\mathbb{M}$, it follows that $\tau'(x) = \frac{1}{\lambda'(\tau(x))} > 0$ for all $x \in \mathbb{R}_\mathbb{M}$. In this section we will assume for convenience that the spectral function $\lambda(x)$ of B is such that $\lambda(x) \in \sigma(B(x))$ where $B(x)$ is the inverse Cayley transform of $U(x) = V \oplus e^{i2\pi x} \langle \cdot, \phi_+ \rangle \phi_-$, and that the self-adjoint extensions of B are neither bounded above nor below so that λ is a C^∞ diffeomorphism of \mathbb{R} onto \mathbb{R} . The results of this section are easily extended to the alternate cases where $\lambda(x) \in \sigma(B(-x))$, and/or where the self-adjoint extensions of B are bounded below or above so that λ is defined on $(0, \infty)$ or $(-\infty, 0)$ (see Definition 3.1.2 of Sect. 3, Remark 3.0.1 and Theorem 9).

We can endow the range of Φ with an inner product as follows. Let μ be an arbitrary positive Borel probability measure on $[0, 1]$, i.e $\mu([0, 1]) = 1$. Given any $\phi \in \mathcal{H}$,

$$\begin{aligned}
 \langle \phi, \phi \rangle &= \int_0^1 \langle \phi, \phi \rangle d\mu(\alpha) \\
 &= \int_0^1 \sum_{n \in \mathbb{Z}} \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \phi \rangle \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2} d\mu(\alpha) \\
 &= \sum_{n \in \mathbb{Z}} \int_0^1 \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \phi \rangle \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2} d\mu(\alpha) \tag{29} \\
 &= \int_{-\infty}^{\infty} \langle \phi, \delta_{\lambda(x)} \rangle \langle \delta_{\lambda(x)}, \phi \rangle \frac{1}{\|\delta_{\lambda(x)}\|^2} d\mu(x) \\
 &= \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \phi \rangle \frac{1}{\|\delta_y\|^2} d\mu(\tau(y)) \\
 &= \int_{-\infty}^{\infty} |\hat{\phi}(y)|^2 \frac{1}{\|\delta_y\|^2} d\mu(\tau(y)). \tag{30}
 \end{aligned}$$

In the above, the measure μ is extended periodically to define a measure on \mathbb{R} . The interchange of the summation and integral on line (29) can be justified using Fubini's theorem. The above shows that the linear transformation Φ defined by $\Phi[\phi](x) := \hat{\phi}(x) = \langle \phi, \delta_x \rangle$ is a unitary transformation of \mathcal{H} onto a subspace $\Phi[\mathcal{H}]$ of $L^2(\mathbb{R}, d\sigma)$ where $d\sigma(y) := \frac{1}{\|\delta_y\|^2} d\mu(\tau(y))$. Observe that $\|\delta_z\|^2 = \frac{\|\varphi_z\|^2}{|\langle \varphi_z, \varphi_{z_0} \rangle|}$ is continuous and strictly positive on \mathbb{R} . Further note that if $\mu([a, b]) = \int_a^b \mu'(x) dx$ where $\mu'(x)$ is a measurable, locally L^1 function, then $d\mu(\tau(y)) = \mu'(\tau(y))\tau'(y)dy$ is absolutely continuous with respect to Lebesgue measure. In this case it follows that σ is also absolutely continuous with respect to Lebesgue measure.

Theorem 10 *The image $\Phi[\mathcal{H}] \subset L^2(\mathbb{R}, d\sigma)$ of \mathcal{H} under Φ is a reproducing kernel Hilbert space of meromorphic functions which are analytic in the region $\mathbb{C} \setminus \mathfrak{N}_u \supset \mathbb{R}$, and $\Phi B \Phi^{-1}$ acts as multiplication by the independent variable on its dense domain in $\Phi[\mathcal{H}]$. If $(\lambda_n(\alpha))_{n \in \mathbb{Z}} = \sigma(B(\alpha))$, then for any $\hat{\phi} \in \Phi[\mathcal{H}]$, $\hat{\phi} = \sum_{n \in \mathbb{Z}} \hat{\phi}(\lambda_n(\alpha)) \frac{\delta_{\lambda_n(\alpha)}}{\|\delta_{\lambda_n(\alpha)}\|^2}$, and this series converges both in norm, and uniformly on compact subsets of $\mathbb{C} \setminus \mathfrak{N}_u$.*

This theorem shows that $\Phi[\mathcal{H}]$ has the $U(1)$ sampling property.

Proof For any $z \in \mathbb{C} \setminus \mathfrak{N}_u$, define $K_z := \Phi[\delta_z]$. Then for any $\hat{\phi} := \Phi[\phi] \in \Phi[\mathcal{H}]$, $\phi \in \mathcal{H}$,

$$\begin{aligned} \langle \hat{\phi}, K_z \rangle &= \int_{-\infty}^{\infty} \hat{\phi}(y) \overline{K_z(y)} d\sigma(y) \\ &= \int_{-\infty}^{\infty} \langle \phi, \delta_y \rangle \langle \delta_y, \delta_z \rangle \frac{1}{\|\delta_y\|^2} d\mu(\tau(y)) \\ &= \langle \phi, \delta_z \rangle = \hat{\phi}(z). \end{aligned} \tag{31}$$

This proves that $\Phi[\mathcal{H}]$ is a reproducing kernel Hilbert space. Now consider the symmetric operator $M' := \Phi B \Phi^{-1}$ with domain $\text{Dom}(M') := \Phi[\text{Dom}(B)]$. It follows that given any $\hat{\phi} = \Phi[\phi] \in \text{Dom}(M') \subset \Phi[\mathcal{H}]$ that

$$\begin{aligned} M' \hat{\phi}(y) &= \Phi[B\phi](y) = \langle B\phi, \delta_y \rangle \\ &= \langle \phi, B^* \delta_y \rangle = y \langle \phi, \delta_y \rangle = y \hat{\phi}(y). \end{aligned} \tag{32}$$

The functions $\hat{\phi} := \Phi[\phi]$ are all clearly meromorphic functions with poles contained in the set \mathfrak{N}_u . Furthermore, Since $\delta_x \neq 0$ for any $x \in \mathbb{R}$, it follows that $K(x, x) = \langle \delta_x, \delta_x \rangle > 0$ for all $x \in \mathbb{R}$ so that $\Phi[\mathcal{H}]$ has positive definite reproducing kernel. Since the map $z \mapsto K_z$ is continuous on $\mathbb{C} \setminus \mathfrak{N}_u$, and the multiplication operator $M' \in \text{Sym}_1(\mathcal{H})$, the remainder of the claim follows by Theorem 4. \square

Given any $z = re^{i\beta} \in \mathbb{D}$, define the probability measure μ_z on $[0, 1)$ by $d\mu_z(\alpha) := P_r(\beta - 2\pi\alpha)d\alpha$ where

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r \cos \theta + r^2} \tag{33}$$

is the Poisson kernel [16]. Recall that $P_r(\theta)$ is a periodic function of $\theta \in [-\pi, \pi]$, that $P_r(\theta) \geq 0$ for all $r < 1$ and $\theta \in [-\pi, \pi]$, and that given any function $f(z)$ which is harmonic in the unit disc \mathbb{D} ,

$$f(re^{i\theta}) = \int_0^{2\pi} f(e^{it}) P_r(\theta - t) \frac{dt}{2\pi} = \int_0^1 f(e^{i2\pi t}) P_r(\theta - 2\pi t) dt. \tag{34}$$

Let σ_z be the measure on \mathbb{R} defined by $d\sigma_z(x) = \frac{1}{\|\delta_x\|^2} d\mu_z(\tau(x)) = \frac{1}{\|\delta_x\|^2} P_r(\beta - 2\pi\tau(x))\tau'(x)dx$, and let \mathcal{H}_z be the image of \mathcal{H} under Φ in $L^2(\mathbb{R}, d\sigma_z) =: \mathcal{K}_z$. We will use the symbol U_z to denote the unitary transformation Φ from \mathcal{H} to \mathcal{H}_z . Since σ'_z and $1/\sigma'_z$ are both continuous on \mathbb{R} , for any $z \in \mathbb{D}$, the measures σ_z are all equivalent to Lebesgue measure, i.e. they have the same sets of measure zero as Lebesgue measure.

Theorem 11 *If M_z is the operator of multiplication by the independent variable in $L^2(\mathbb{R}, d\sigma_z)$, then the compression of e^{itM_z} , $t \geq 0$ to \mathcal{H}_z is a semi-group, and $P_{\mathcal{H}_z} \mu(M_z) U_z \phi_+ = z U_z \phi_-$. Moreover, the unitary group e^{itM_z} is the minimal unitary dilation of its compression to \mathcal{H}_z .*

In the proof of this theorem we will use the fact that if $B \in \mathcal{S}ym_1(\mathcal{H})$, then $U(z) := \mu(B) \oplus z \langle \cdot, \phi_+ \rangle \phi_-$ on $\mathcal{H} := \text{Ran}(B + i) \oplus \mathfrak{D}_+$ is an entire operator valued function, and if $B(z) := \mu^{-1}(U(z))$, then $U(z) = \mu(B(z))$. Note that this definition of $U(z)$ is different from the one used in Sect. 3.

Before proving Theorem 11, it will be convenient to first establish the following lemma.

Lemma 4 *Let B belong to $\mathcal{S}ym_1(\mathcal{H})$. Then, for any $z \in \overline{\mathbb{D}}$, the extension $B(z)$ of B is the generator of a one parameter, strongly continuous semigroup of contractions, $e^{itB(z)}$, $t \geq 0$, and for any fixed $t \geq 0$, $e^{itB(z)}$ is a $H^\infty(\mathbb{D})$ contraction-valued function.*

This lemma will be proven with a simple application of the following characterization of co-generators of contraction semigroups:

Theorem 12 (Nagy-Foias) *Given a contraction $V \in \mathcal{H}$, V is the co-generator of a contraction semigroup $V(s)$, $s \geq 0$, if and only if $1 \notin \sigma_p(V)$. In this case V and $V(s)$ determine each other by the formulas $V(s) = e_s(V)$, and $V = \lim_{s \rightarrow 0^+} \varphi_s(V(s))$ where $e_s(z) := e^{s \frac{z+1}{z-1}}$ and $\varphi_s(z) := \frac{z-1+s}{z-1-s}$.*

Proof of Lemma 4 Given $U(z) = \mu(B(z))$, $1 \notin \sigma_p(U(z))$ for any $z \in \overline{\mathbb{D}}$. To see this first consider $z \in \mathbb{T}$, the unit circle, and note that if there is a $\psi \in \mathcal{H}$ such that $U(z)\psi = \psi$, then for any $\phi \in \mathcal{H}$,

$$\langle U(z)\phi - \phi, \psi \rangle = \langle U(z)\phi, U(z)\psi \rangle - \langle \phi, \psi \rangle = 0, \tag{35}$$

since $U(z)$ is unitary for $z \in \mathbb{T}$. This shows that $\psi \perp \text{Ran}(U(z) - 1)$ which contradicts the density of $\text{Ran}(\mu(B) - 1) = \text{Dom}(B)$. If $z \notin \mathbb{T}$, note that if $U(z)\psi = \psi$, then, by the same argument as above $\psi \notin \text{Dom}(\mu(B))$, since $\mu(B)$ is an isometry from its domain to its range. Hence if $U(z)\psi = \psi$, then $\psi = \psi_1 + \psi_2$ where $\psi_1 \in \text{Dom}(\mu(B))$ and $0 \neq \psi_2 \in \mathfrak{D}_+$. Hence $U(z)\psi = V\psi_1 + z \langle \psi_2, \phi_+ \rangle \phi_-$. Since $\psi_2 \in \mathfrak{D}_+$ it follows that $\psi_2 = e^{i\theta} \|\psi_2\| \phi_+$ for some $\theta \in [0, 2\pi)$ so that $\|U(z)\psi\| = \|\psi_1\| + |z| \|\psi_2\| \neq \|\psi\| = \|\psi_1\| + \|\psi_2\|$ since $|z| < 1$.

Since $1 \notin \sigma_p(\mu(B(z)))$, and $\mu(B(z))$ is a contraction for all $z \in \overline{\mathbb{D}}$, Theorem 12 implies that $\mu(B(z))$ is the co-generator of a contraction semigroup for any $z \in \overline{\mathbb{D}}$.

Since $\mu^{-1}(z) = -i \frac{z+1}{z-1}$, it follows that if $V(s) = e_s(U(z))$ is the semigroup co-generated by $U(z)$, then $V(s) = e^{isB(z)}$ is generated by $iB(z)$. Finally, it follows that since $e_s(z)$ belongs to $H^\infty(\mathbb{D})$, the Hardy space on \mathbb{D} , that the operator-valued function $e_t(U(z)) = e^{itB(z)}$ is a $H^\infty(D)$ operator-valued function for each $t \geq 0$. \square

Proof of Theorem 11 Given $\hat{\phi} := U_z \phi \in \mathcal{H}_z$, and $\hat{\psi} := U_z \psi$, consider the following. Given $z := re^{i\beta}$, observe that

$$\begin{aligned}
 \langle e^{itM_z} \hat{\phi}, \hat{\psi} \rangle_{\mathcal{H}_z} &= \int_{-\infty}^{\infty} \langle \phi, \delta_x \rangle \langle \delta_x, \psi \rangle e^{itx} d\sigma_z(x) \\
 &= \int_{-\infty}^{\infty} \langle \phi, \delta_{\lambda(x)} \rangle \langle \delta_{\lambda(x)}, \psi \rangle e^{it\lambda(x)} \frac{1}{\|\delta_{\lambda(x)}\|^2} P_r(\beta - 2\pi x) dx \\
 &= \sum_{n \in \mathbb{Z}} \int_0^1 \langle \phi, \delta_{\lambda_n(\alpha)} \rangle \langle \delta_{\lambda_n(\alpha)}, \psi \rangle e^{it\lambda_n(\alpha)} \frac{1}{\|\delta_{\lambda_n(\alpha)}\|^2} P_r(\beta - 2\pi\alpha) d\alpha \\
 &= \int_0^1 \langle e^{itM'_z(e^{i2\pi\alpha})} \phi, \psi \rangle P_r(\beta - 2\pi\alpha) d\alpha. \tag{36}
 \end{aligned}$$

Here, $M'_z = U_z B U_z^{-1} \in \text{Sym}_1(\mathcal{H}_z)$ is the symmetric restriction of M_z to a dense domain $U_z \text{Dom}(B) \subset \mathcal{H}_z$ and $M'_z(e^{i2\pi\alpha})$ is that self-adjoint extension of M'_z which is the inverse Cayley transform of $U(e^{i2\pi\alpha})$, as defined after the statement of Theorem 11. Since $e^{itM'_z(w)}$ is a $H^\infty(D)$ operator-valued function of $w \in \mathbb{D}$, the above Eq. (36) evaluates to $\langle e^{itM'_z(z)} \hat{\phi}, \hat{\psi} \rangle = \langle e^{itB(z)} \phi, \psi \rangle$, $z = re^{i\beta}$, so that $P_{\mathcal{H}_z} e^{itM_z}|_{\mathcal{H}_z} = e^{itM'_z(z)}$, which, by Lemma 4, is a semi-group for $t \geq 0$, $|z| \leq 1$.

To see that e^{itM_z} is the minimal unitary dilation of its compression to \mathcal{H}_z , note that if it were not, then there would be a proper subspace $S \subset L^2(\mathbb{R}, d\sigma_z)$, containing \mathcal{H}_z such that S reduces the unitary group e^{itM_z} . Such a subspace would be invariant for the self-adjoint multiplication operator M_z so that $S = L^2(\Omega, d\sigma_z)$ where $\Omega \subset \mathbb{R}$ is some Borel subset such that $\mathbb{R} \setminus \Omega$ has non-zero Lebesgue measure. Observe here that since σ'_z and $1/\sigma'_z$ are locally L^∞ functions, that σ_z is equivalent to Lebesgue measure, *i.e.* it has the same sets of measure zero as Lebesgue measure. This would imply that there are sets of non-zero Lebesgue measure Ω , such that if $x \in \Omega$, then $\langle \phi, \delta_x \rangle = \phi(x) = 0$ for all $\phi \in \mathcal{H}_z$. This would imply that $\delta_x = 0$ for all $x \in \Omega$ which is a contradiction. Since elements of \mathcal{H}_z are meromorphic, this would also imply that $\hat{\phi} = 0$ for all $\hat{\phi} \in \mathcal{H}_z$, and hence that $\mathcal{H}_z = \{0\} = \mathcal{H}$. This would contradict our implicit assumption that the Hilbert space \mathcal{H} is non-trivial.

Similarly, it is not hard to check that $P_{\mathcal{H}_z} \mu(M_z)|_{\mathcal{H}_z} = \mu(M'_z(z))$, and we know by definition of $\mu(B(z))$ that $\mu(M'_z(z))U_z \phi_+ = zU_z \phi_-$. \square

5 A Sufficient Condition for Reproducing Kernel Hilbert Space

Now consider $\mathcal{K} := L^2(\mathbb{R}, d\nu)$ where $d\nu(x) := \nu'(x)dx$ is absolutely continuous with respect to Lebesgue measure and $\nu' > 0$ *a.e.* Further suppose that the operator M , of multiplication by the independent variable in \mathcal{K} , has a simple, regular, symmetric and

densely defined restriction M_S to a subspace $S \subset \mathcal{K}$ with deficiency indices $(1, 1)$, i.e. $M_S \in \text{Sym}_1(S)$.

In this case the compression $P_S \mu(M)|_S$ of the Cayley transform of M to S is a contractive, non-unitary extension of the Cayley transform of M_S . Define an isometric transformation V_z of $L^2(\mathbb{R}, d\sigma_z)$ onto \mathcal{K} by $V_z f(x) = \sqrt{\frac{\sigma'_z(x)}{v'(x)}} f(x)$. As in Sect. 4, let U_z denote the isometry of S onto $\mathcal{H}_z \subset L^2(\mathbb{R}, d\sigma_z)$. Let $V'_z := V_z U_z$, so that V'_z is an isometry from $S \subset \mathcal{K}$ onto another subspace $\mathcal{H}'_z := V'_z S$ in \mathcal{K} . Choose $z' \in \mathbb{D}$ such that $P_S \mu(M)\phi_+ = z'\phi_-$, and let $\mathcal{H}' := \mathcal{H}'_{z'} \subset \mathcal{K}$.

Lemma 5 $P_{\mathcal{H}'_z} e^{itM}|_{\mathcal{H}'_z}; t \geq 0$ is a contraction semi-group, its minimal unitary dilation is e^{itM} ; $t \in \mathbb{R}$, and $P_{\mathcal{H}'_z} \mu(M)V'_z \phi_+ = zV'_z \phi_-$. If $z = z'$ is chosen so that $P_S \mu(M)\phi_+ = z'\phi_-$, then the isometry $V' := V'_{z'}$ intertwines the compressions $P_S \mu(M)|_S$ and $P_{\mathcal{H}'} \mu(M)|_{\mathcal{H}'}$ of $\mu(M)$ to S and $\mathcal{H}' := \mathcal{H}'_{z'}$.

Proof The image of S under U_z is \mathcal{H}_z . Furthermore, by Theorem 11, the compression of e^{itM_z} to \mathcal{H}_z is a contraction semi-group and $P_{\mathcal{H}_z} \mu(M_z)U_z \phi_+ = zU_z \phi_-$. It is clear that the image of the unitary group e^{itM_z} under V_z is e^{itM} , so that the compression of e^{itM} ; $t \geq 0$ to the subspace $\mathcal{H}'_z := V_z \mathcal{H}_z = V'_z S$ is a contraction semi-group, and $P_{\mathcal{H}'_z} \mu(M)V'_z \phi_+ = zV'_z \phi_-$. Since e^{itM_z} is the minimal unitary dilation of its compression to \mathcal{H}_z , it further follows that e^{itM} is the minimal unitary dilation of its compression to \mathcal{H}'_z .

The compression of $\mu(M)$ to S is a contractive, non-unitary extension of the Cayley transform of M_S , the symmetric restriction of M to S , so that $P_S \mu(M)\phi_+ = z'\phi_-$ for some $z' \in \mathbb{D}$. It follows that if one chooses $z = z'$, and defines $V' := V'_{z'}$, $\mathcal{H}' := \mathcal{H}'_{z'}$ then $P_{\mathcal{H}'} \mu(M)V' \phi_+ = z'V' \phi_- = V' P_S \mu(M)\phi_+$. Furthermore, observe that V' maps M_S , the symmetric restriction of M to S onto M' , the symmetric restriction of M to \mathcal{H}' . It follows that $V' M_S \phi = M' V' \phi$ for all $\phi \in \text{Dom}(M_S)$, and hence that $V' \mu(M_S) \phi = \mu(M') V' \phi$ for all $\phi \in \text{Ran}(M_S + i)$. Since $S = \text{Ran}(M_S + i) \oplus \mathfrak{D}_+$, $\mathcal{H}' = \text{Ran}(M' + i) \oplus V' \mathfrak{D}_+$ and $P_S \mu(M)|_{\text{Ran}(M_S + i)} = \mu(M_S)$ and $P_{z'} \mu(M_{z'})|_{\text{Ran}(M' + i)} = \mu(M'_{z'})$, it follows that V' intertwines the compressions of $\mu(M)$ to S and \mathcal{H}' , as claimed. \square

Let $W := \mu(M)$ and W_S, W' be the compressions of W to S and \mathcal{H}' respectively. We have shown that there is an isometry V' such that $W' V' = V' W_S$. Recall that W is the minimal unitary dilation of W' . Below we collect the necessary facts about dilations of contractions which will be used to prove the main result of this section, Theorem 14. The following is an intertwining version of Ando's dilation theorem for two commuting contractions [17, p. 66]:

Theorem 13 (Nagy-Foias) Suppose that $T_i, i = 1, 2$ are contraction operators on \mathcal{H}_i , let U_i be their minimal unitary dilations on \mathcal{K}_i , and let P_i be the orthogonal projections of the \mathcal{K}_i onto the \mathcal{H}_i . If $V : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ intertwines T_1 and T_2 , $V T_1 = T_2 V$, then there is an R intertwining U_1 and U_2 such that $\|R\| = \|V\|$ and $V T_1^n = T_2^n V = P_2 R U_1^n|_{\mathcal{H}_1} = P_2 U_2^n R|_{\mathcal{H}_1}$.

Lemma 6 Under the same assumptions of the above theorem, if V is an isometry, then $V = R|_{\mathcal{H}_1}$ and R is an isometry from \mathcal{K}_1 onto \mathcal{K}_2 . If U_2 is also a unitary dilation

of T_1 , and if \mathcal{K}_1 is taken to be a subspace of \mathcal{K}_2 , R can be extended to a co-isometry \hat{R} on \mathcal{K}_2 such that $\hat{R}U_2 = U_2\hat{R}$, $\hat{R}|_{\mathcal{K}_1} = R$, and $\text{Ker}(\hat{R}) = \mathcal{K}_2 \ominus \mathcal{K}_1$.

Proof By the intertwiner dilation theorem, there is an $R : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ such that $RU_1 = U_2R$. Furthermore, $V = P_2R|_{\mathcal{H}_1}$. If $\phi \in \mathcal{H}_1$ has unit norm, $1 = \|V\phi\| = \|P_2R\phi\| \leq \|R\phi\| \leq \|R\|\|\phi\| = 1$ since $\|R\| = \|V\| = 1$. Hence $\|R\phi\| = \|\phi\|$ for all $\phi \in \mathcal{H}_1$. Given any $\phi \in \mathcal{H}_1$, $R\phi = V\phi + \psi$, where $\psi \in \mathcal{K}_2 \ominus \mathcal{H}_2$. Now, $\|\phi\| = \|R\phi\| = \|V\phi\| + \|\psi\| = \|\phi\| + \|\psi\|$. This shows that $\psi = 0$, so that $V = P_2R|_{\mathcal{H}_1} = R|_{\mathcal{H}_1}$. It is further straightforward to show that $R^*|_{\mathcal{H}_2} = V^*$. Since U_1 is the minimal unitary dilation of T_1 , linear combinations of vectors of the form $U_1^k\phi$, $k \in \mathbb{Z}$, $\phi \in \mathcal{H}_1$ are dense in \mathcal{K}_1 . Since R intertwines U_1 and U_2 , and $R^*R\phi = \phi$ for all $\phi \in \mathcal{H}_1$,

$$\begin{aligned} \langle RU_1^k\phi, RU_1^j\psi \rangle &= \langle RU_1^k\phi, U_2^jR\psi \rangle \\ &= \langle RU_1^{k-j}\phi, R\psi \rangle = \langle U_1^k\phi, U_1^j\psi \rangle, \end{aligned} \tag{37}$$

for all $k, j \in \mathbb{Z}$ and $\phi, \psi \in \mathcal{H}_1$. Since linear combinations of such vectors are dense in \mathcal{K}_1 , this proves that R is in fact an isometry of \mathcal{K}_1 into \mathcal{K}_2 . Similarly, it is not difficult to show that R^* is also an isometry, so that R is an isometry from \mathcal{K}_1 onto \mathcal{K}_2 .

Now suppose that U_2 is also a unitary dilation of T_1 . Then \mathcal{K}_1 can be viewed as a subspace of \mathcal{K}_2 which reduces U_2 , and $U_2|_{\mathcal{H}_1} = U_1$. Hence, if \hat{R} is defined as in the statement of the lemma, $\hat{R}U_2 = U_2\hat{R}$. □

It is not difficult to apply the above lemma to establish the main theorem of this section:

Theorem 14 *Suppose $\mathcal{K} := L^2(\mathbb{R}, d\nu)$ is such that $d\nu := \nu'(x)dx$, and $\nu'(x) > 0$ a.e. is a locally L^1 function. Further suppose that $S \subset L^2(\mathbb{R}, d\nu)$ is such that the compression of the unitary group e^{itM} to S is a semigroup for $t \geq 0$, and that M has a symmetric restriction to S with the sampling property. Then the following statements are true:*

- (i) e^{itM} is the minimal unitary dilation of its compression to S .
- (ii) There is an isometric transformation U_z which acts as multiplication by a locally L^1 function, which takes S onto a RKHS $\mathcal{H}_z \subset L^2(\mathbb{R}, d\sigma_z)$ with the $U(1)$ sampling property. \mathcal{H}_z consists of certain functions which are meromorphic in \mathbb{C} and analytic in a region containing \mathbb{R} . Here σ_z is equivalent to Lebesgue measure, and $\sigma', 1/\sigma'$ are continuous functions on \mathbb{R} .
- (iii) If ν' and $1/\nu'$ are both locally L^∞ functions, then S itself is a RKHS with the $U(1)$ sampling property and the isometry $U_z : S \rightarrow \mathcal{H}_z$ acts as multiplication by a function u_z which is locally L^∞ , and whose multiplicative inverse $1/u_z$ is also locally L^∞ .

Observe that if e^{itM} is the minimal unitary dilation of its compression to S , that this implies that the smallest invariant subspace of M containing S is all of $L^2(\mathbb{R}, d\nu)$. It then follows that if ν is equivalent to Lebesgue measure then there is no there is no Borel set of non-zero Lebesgue measure on which all elements of S vanish.

Recall that $d\sigma_z(x) = \frac{1}{\|\delta_x\|^2} P_r(\beta - 2\pi\tau(x))\tau'(x)dx$. Here, $z = re^{i\beta} \in \mathbb{D}$ is such that $P_S\mu(M)\phi_+ = z\phi_+$, τ is the inverse of the spectral function of B , and $\delta_x := \frac{\varphi_x}{\langle \varphi_x, \varphi_{z_0} \rangle}$. Further recall that $\varphi_z := (B' - z_0)(B' - z)^{-1}\varphi_{z_0}$, B' is any fixed self-adjoint extension of B , $z_0 \in \mathbb{C} \setminus \mathbb{R}$, and $0 \neq \varphi_{z_0} \in \text{Ker}(B^* - z_0)$ is fixed.

Proof As before, let $W_S := P_S\mu(M)|_S$, and $W' := P_{\mathcal{H}'}\mu(M)|_{\mathcal{H}'}$. As proven in Lemma 5, there is an isometry $V' := V_z$ intertwining the contractions W_S and W' , $W := \mu(M)$ a unitary dilation of W' to $\mathcal{K} := L^2(\mathbb{R}, d\nu)$, and by assumption W is also a unitary dilation of W_S . By the previous lemma, Lemma 6, there exists a co-isometry R on \mathcal{K} such that $R|_S = V'$ and $RW = WR$. It is not difficult to show that this implies that $R : \text{Dom}(M) \rightarrow \text{Dom}(M)$, and that $RM\phi = MR\phi$ for all $\phi \in \text{Dom}(M)$. Since M is multiplicity free, it follows that R belongs to the double commutant of $M|_{L^2(I, d\nu)}$ for any compact interval $I \subset \mathbb{R}$, and therefore must act as multiplication by a L^∞ function. It follows that R is normal, and since it is an isometry, it must in fact be unitary, and hence acts as multiplication by a L^∞ function of modulus one almost everywhere. Since $\text{Ker}(R) = \mathcal{K} \ominus \mathcal{K}_S$, where $\mathcal{K}_S \subset \mathcal{K}$ and $W|_{\mathcal{K}_S}$ is a unitary dilation of W_S , it follows that $\text{Ker}(R) = \{0\}$, and that $\mathcal{K} = \mathcal{K}_S$ so that W is also the minimal unitary dilation of W_S .

Since $V' = R|_S$, V' acts as multiplication by a L^∞ function $r(x)$ of modulus one almost everywhere with respect to Lebesgue measure. The isometry V' has the following properties. First, $V' := V_z U_z$, where $z \in \mathbb{D}$ is chosen such that $P_S\mu(M)\phi_+ = z\phi_-$, the isometry U_z takes S onto a subspace $\mathcal{H}_z \subset L^2(\mathbb{R}, d\sigma_z)$ which consists of certain meromorphic functions which have only simple poles and which are analytic on a region containing the real line. The measure σ_z is equivalent to Lebesgue measure, *i.e.* they have the same sets of measure zero, and both σ'_z and $1/\sigma'_z$ are strictly positive, continuous functions. The isometry V_z is an isometry from $L^2(\mathbb{R}, d\sigma_z)$ onto \mathcal{K} which acts as multiplication by the measurable function $v_z(x) := \sqrt{\frac{\sigma'_z(x)}{v'(x)}}$. Hence, the isometry $U_z := (V_z)^*V'$ acts as multiplication by the measurable, locally L^1 function $u_z(x) := \frac{r(x)}{v_z(x)}$. This proves statement (ii).

To prove the third and final statement, consider an arbitrary $x \in \mathbb{R}$, and let K_x be the point evaluation vector in \mathcal{H}_z at the point x . Now if v' and $1/v'$ are locally L^∞ functions, it follows that $u_z(x)$ and $1/u_z(x)$ will be locally L^∞ functions. Fix a member $\tilde{u}_z(x)$ from the equivalence class of $u_z(x)$ which obeys $\|u_z \cdot \chi_I\|_\infty \geq |\tilde{u}_z(x)| \geq \|\chi_I/\tilde{u}_z\|_\infty^{-1}$ for all x in every interval $I \subset \mathbb{R}$, and define $\delta_x := 1/\tilde{u}_z(x)U_z^*K_x \in S$. Here, χ_I denotes the characteristic function of the interval I . Clearly, $\delta_x \neq 0$ is an element of S with finite norm. Then for any $\phi \in S$,

$$\langle \phi, \delta_x \rangle = \langle \phi, (\overline{\tilde{u}_z(x)})^{-1}U_z^*K_x \rangle \tag{38}$$

$$\begin{aligned} &= \frac{1}{\tilde{u}_z(x)} \langle U_z\phi, K_x \rangle \\ &= \frac{1}{\tilde{u}_z(x)} (U_z\phi)(x) = \phi(x) \text{ a.e.} \end{aligned} \tag{39}$$

Hence, identifying ϕ with that member of its equivalence class which is actually equal to $\langle \phi, \delta_x \rangle$ everywhere, we see that S is a reproducing kernel Hilbert space with point evaluation vectors $\delta_x := (\widetilde{u}_z(x))^{-1} U_z^* K_x$. The fact that the subspace S has the $U(1)$ sampling property now follows from the fact that \mathcal{H}_z has the $U(1)$ sampling property, as proven in Theorem 10. \square

The above result allows one to apply a purely operator theoretic condition to show that certain subspaces are reproducing kernel Hilbert spaces.

6 De Branges Spaces

In this section we show by concrete example that there is a large class of subspaces which satisfy the assumptions of Theorem 14.

6.1 The Example of $B(\Omega)$

The Paley-Wiener space $B(\Omega) \subset L^2(\mathbb{R})$, $\Omega > 0$, is an example of a subspace satisfying the conditions of Theorem 14. The image of $B(\Omega)$ under the unitary Fourier transform is $L^2[-\Omega, \Omega]$. It is easy to verify that $D := i \frac{d}{dx}$ defines a symmetric derivative operator defined on the dense domain

$$\text{Dom}(D) := \{f \in L^2[-\Omega, \Omega] \mid f \in AC[-\Omega, \Omega]; f' \in L^2[-\Omega, \Omega]; f(\pm\Omega) = 0\}, \tag{40}$$

where $AC[-\Omega, \Omega]$ denotes the set of all absolutely continuous functions on $[-\Omega, \Omega]$. It is further not difficult to prove that D is closed and has deficiency indices $(1, 1)$ [9, Section 49].

The operator D is both simple and regular. This will be proven by showing that the minimum uncertainty of D is bounded below. Here, the uncertainty, $\Delta S[\phi]$, of a symmetric operator S with respect to a unit-length vector $\phi \in \text{Dom}(S)$ is defined by $\Delta S[\phi] := \sqrt{\langle S\phi, S\phi \rangle - \langle S\phi, \phi \rangle^2}$. The overall lower bound on the uncertainty of S will be denoted by $\Delta S := \inf_{\phi \in \text{Dom}(S), \|\phi\|=1} \Delta S[\phi]$.

Consider the multiplication operator \widetilde{M} on $L^2[-\Omega, \Omega]$. This is a bounded, self-adjoint operator defined on the whole space. It is a simple algebraic exercise to prove the following lower bound on the product of the uncertainties for two symmetric operators S and T for unit norm vectors $\phi \in \text{Dom}(T) \cap \text{Dom}(S)$:

$$\Delta S[\phi] \Delta T[\phi] \geq \frac{1}{2} |\langle S\phi, T\phi \rangle - \langle T\phi, S\phi \rangle|. \tag{41}$$

This above inequality is often referred to as the Heisenberg uncertainty relation. Observe that \widetilde{M} maps $\text{Dom}(D)$ into itself since it preserves the boundary conditions, the function $f(x) = x$ a.e. is absolutely continuous, and the product of any two absolutely continuous functions is itself absolutely continuous [18, p. 337]. It is clear that

for all unit length vectors ϕ , $\Delta\tilde{M}[\phi] \leq \|\tilde{M}\| = \Omega$ so for all unit length $\phi \in \text{Dom}(D)$, it follows that

$$\Delta D[\phi] \geq \frac{1}{2\Omega} |\langle \phi, (D\tilde{M} - \tilde{M}D)\phi \rangle| = \frac{1}{2\Omega} > 0. \quad (42)$$

This shows that $\Delta D \geq \frac{1}{2\Omega} > 0$. It follows that the symmetric operator D can have no eigenvalues and no continuous spectrum on the real line as otherwise there would be unit length vectors $\phi \in \text{Dom}(D)$ for which $\Delta D[\phi]$ is either 0 or arbitrarily small. This shows that $D - \lambda$ is bounded below for any $\lambda \in \mathbb{R}$ so that D is regular. Furthermore, D must also be simple. Otherwise, if there were a subspace S of $L^2[-\Omega, \Omega]$ such that the restriction of D to S was self-adjoint, then D would have eigenvalues or continuous spectra. In conclusion, $D \in \text{Sym}_1(L^2[-\Omega, \Omega])$, and its Fourier transform $M' \in \text{Sym}_1(B(\Omega))$ is a symmetric restriction of the self-adjoint operator M of multiplication by the independent variable in $L^2(\mathbb{R})$ to a dense domain in $B(\Omega)$. For a more detailed study on the relationship between minimum uncertainty and the spectra of symmetric operators see [10, 19].

Also note that if \tilde{D} denotes the self-adjoint derivative operator $\tilde{D} := i \frac{d}{dx}$ on its dense domain in $L^2(\mathbb{R})$, that $B(\Omega) := \chi_{[-\Omega, \Omega]}(\tilde{D})$ is an invariant subspace of \tilde{D} . It follows that $B(\Omega)$ is invariant under translations, since the unitary group $e^{it\tilde{D}}$ generates translations, and hence that there is no Borel subset $\Omega \subset \mathbb{R}$ of non-zero measure on which all elements of $B(\Omega)$ vanish. To prove that $B(\Omega)$ satisfies the assumptions of Theorem 14, it remains to prove that the compression of e^{itM} to $B(\Omega)$ is a semigroup for $t \geq 0$. By Fourier transform, this is equivalent to proving that the compression of $e^{it\tilde{D}}$ to $L^2[-\Omega, \Omega]$ is a semigroup for $t \geq 0$. It is indeed easily verified that $V(t) = P_{L^2[-\Omega, \Omega]} e^{it\tilde{D}}|_{L^2[-\Omega, \Omega]}$, is a semigroup. This semigroup is in fact a semigroup of partial isometries, known as a semigroup of truncated shifts [20]. In conclusion, since $B(\Omega) \subset L^2(\mathbb{R})$ satisfies the conditions of Theorem 14, it follows that $B(\Omega)$ must be a reproducing kernel Hilbert space with the $U(1)$ sampling property.

The Paley-Wiener space of Ω -bandlimited functions is an example of a de Branges space. In fact, it appears that $B(\Omega)$ is the canonical example that de Branges generalized to arrive at his theory of Hilbert spaces of entire functions [21, p. 50]. While it is already known that any de Branges space $\mathcal{H}(E)$ is a reproducing kernel Hilbert spaces of entire functions, and that a large class of de Branges spaces have the $U(1)$ sampling property, it is of interest to see whether they satisfy the conditions of Theorem 14, so that these special properties can be seen as a consequence of Theorem 14.

6.2 Review of De Branges Spaces

It will be necessary to introduce a few concepts from complex function theory. Given a region $\Omega \subset \mathbb{C}$, let $\text{Hol}(\Omega)$ denote the set of functions which are holomorphic in Ω .

Definition 6.2.1 A function f , holomorphic in a region Ω is said to be of bounded type in that region if there exist functions $p, q \in \text{Hol}(\Omega)$ such that $f(z) = \frac{p(z)}{q(z)}$, $q \neq 0$, and q, p are bounded in Ω .

If f is analytic in the upper half plane (UHP), then the mean type $h[f]$ of f can be defined by

$$h[f] := \limsup_{y \rightarrow \infty} \frac{1}{y} \ln |f(iy)|. \tag{43}$$

Mean type for functions analytic in the lower half plane is defined analogously. The notion of mean type is a measure of growth in the upper half plane, and is clearly a generalization of the notion of exponential type to functions analytic in the upper half plane.

Given an entire function f , let f^* denote the entire function defined by $f^*(z) := \overline{f(\bar{z})}$. An entire function E is called a de Branges function if it obeys $|E(x - iy)| < |E(x + iy)|$ for all $y > 0$. This inequality implies, in particular, that E has no zeroes in the upper half plane. Given such a function E , the de Branges space $\mathcal{H}(E)$ is defined as the set of all entire functions F such that F/E and F^*/E are of bounded type and non-positive mean type in the upper half plane, and which are square integrable with respect to the norm generated by the inner product:

$$\langle F, G \rangle := \int_{-\infty}^{\infty} F(t) \overline{G(t)} \frac{1}{|E(t)|^2} dt. \tag{44}$$

The space $\mathcal{H}(E)$ is complete with respect to this inner product [21, p. 53].

Let $A := \frac{1}{2}(E + E^*)$ and $B := \frac{i}{2}(E - E^*)$. Then the following theorem shows that $\mathcal{H}(E)$ is a reproducing kernel Hilbert space whose reproducing kernel can be expressed in terms of E and E^* [21, p. 50].

Theorem 15 (*de Branges*) *Given any entire function E such that $|E(x - iy)| < |E(x + iy)|$ for $y > 0$, let $K(w, z) := \frac{B(z)A(w) - A(z)B(w)}{\pi(z - \bar{w})}$. Then K_w , where $K_w(z) := K(w, z)$, belongs to $\mathcal{H}(E)$ for every $w \in \mathbb{C}$ and $F(w) = \langle F, K_w \rangle$ for any $F \in \mathcal{H}(E)$.*

Note that $B(\Omega)$ is the de Branges space defined by the function $E(z) := e^{-i\Omega z}$. It is a known fact that many de Branges spaces have the $U(1)$ sampling property. The following equivalent axiomatic definition of de Branges spaces makes this fact more apparent [21, pp. 56–57].

Theorem 16 *A Hilbert space of entire functions \mathcal{H} is isometrically equivalent to a de Branges space $\mathcal{H}(E)$ if and only if the following three axioms are satisfied:*

- (A1) *Point evaluation at every $z \in \mathbb{C} \setminus \mathbb{R}$ is a bounded linear functional.*
- (A2) *If $F \in \mathcal{H}$, then $F^* \in \mathcal{H}$, and $\|F\| = \|F^*\|$.*
- (A3) *If $F \in \mathcal{H}$ and $F(w) = 0$ for some $w \in \mathbb{C} \setminus \mathbb{R}$, then $G(z) := F(z) \frac{z - \bar{w}}{z - w} \in \mathcal{H}$, and $\|G\| = \|F\|$.*

Notice that axiom (A3) immediately implies that we can define multiplication by the function $\mu_w(z) := \frac{z - \bar{w}}{z - w}$ on a certain subspace of a de Branges space for any $w \in \mathbb{C} \setminus \mathbb{R}$, and that this resulting multiplication operator V_w is an isometry from its domain onto its range. It is not difficult to further prove the following

Theorem 17 *Let \mathcal{H} be any Hilbert space of entire functions satisfying the axioms (A1), (A2), and (A3) of Theorem 16. Then multiplication by z is a closed, symmetric operator in \mathcal{H} with deficiency indices (1, 1).*

De Branges leaves the above result as an exercise in his textbook [21]. We provide the proof here for the convenience of the reader.

Proof Let V_w denote the operator of multiplication by the function $\mu_w(z) := \frac{z-\bar{w}}{z-w}$. Then, by assumption, V_w is defined on the subspace $\text{Dom}(V_w)$ of all $F \in \mathcal{H}$ for which $F(w) = 0$. Property (A1) implies that $\text{Dom}(V_w)$ is closed, and (A3) implies that V_w is an isometry from its domain onto its range, $\text{Ran}(V_w)$. It will now be shown that $n := \dim(\text{Dom}(V_w)^\perp) = 1$ for any $w \in \mathbb{C} \setminus \mathbb{R}$.

If $n > 1$, then there exist 2 linearly independent functions F_1 and F_2 which are orthogonal to $\text{Dom}(V_w)$, and hence do not vanish at w . But then $F := F_1 - \frac{F_1(w)}{F_2(w)} F_2$ must belong to $\text{Dom}(V_w)^\perp$ since it is a subspace, and yet $F(w) = 0$ which means $F \in \text{Dom}(V_w)$. Hence $F = 0$ so that F_1 and F_2 are linearly dependent. This proves that $n \leq 1$.

If $G \in \text{Dom}(V_w)$ then G has a zero of finite order k at w . By property (A3), $V_w^k G \in \mathcal{H}(E)$ is non-zero, and has no zero at w . Hence, $V_w^k G \notin \text{Dom}(V_w)$. Since $\text{Dom}(V_w)$ is closed this means that $\text{Dom}(V_w)^\perp$ is non-empty so that $n > 0$. We conclude that $n = 1$.

Let M denote the operator which acts as multiplication by z . Then for any $w \in \mathbb{C} \setminus \mathbb{R}$ we have that $M = (\bar{w}V_w - w)(V_w - 1)^{-1}$ where $\text{Dom}(M) := \text{Ran}(V_w - 1)$. As observed previously, (A1) implies that $\text{Dom}(V_w)$ is closed, so that V_w is a closed linear transformation. It is straightforward to show that this implies that M is closed.

Now observe that the range of V_w is equal to the domain of $V_{\bar{w}}$. To see this note that if F is in the range of V_w then $F(\bar{w}) = 0$ so that $F \in \text{Dom}(V_{\bar{w}})$ and that $V_{\bar{w}}$ is just the inverse of V_w . Furthermore, it is elementary to check that $\text{Ran}(M - w) = \text{Ran}(V_w) = \text{Dom}(V_{\bar{w}})$ and that $\text{Ran}(M - \bar{w}) = \text{Dom}(V_w)$. By the previous arguments, $\dim(\text{Ran}(M - w)^\perp) = \dim(\text{Ran}(M - \bar{w})^\perp) = 1$ so that M has deficiency indices $(1, 1)$. \square

Let $\mathcal{H}(E)$ be a de Branges space, and let M denote the operator of multiplication by the independent variable, defined in the proof of the previous theorem.

Theorem 18 *The symmetric multiplication operator M in $\mathcal{H}(E)$ is both simple and regular.*

Although the following proof is new, this theorem follows immediately from more powerful results of [21]. The proof will make use of the following lemma [21, Problem 44, p. 52].

Lemma 7 *If E is a de Branges function then $E = SE_0$ where E_0 is a de Branges function with no real zeroes, $S = S^*$, and multiplication by S is a unitary transformation of $\mathcal{H}(E_0)$ onto $\mathcal{H}(E)$.*

Proof of Theorem 18 If λ is an eigenvalue of M , it must be a finite real value. If λ is such an eigenvalue then $\mu = \frac{\lambda - \bar{w}}{\lambda - w}$ is an eigenvalue of V_w which lies on the unit circle. Let $F \neq 0$ be the corresponding eigenfunction. Then $V_w^k F = \mu^k F$, so that $0 \neq V_w^k F \in \text{Dom}(V_w)$ for every $k \in \mathbb{N}$. This implies that F has a zero of infinite order at w , which is impossible as $F \neq 0$ is entire.

Suppose that $\lambda \in \sigma_c(M)$. Since M is symmetric it follows that $\lambda \in \mathbb{R}$. Assume that $E(\lambda) \neq 0$. Then since $\lambda \in \sigma_c(M)$, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \text{Dom}(M)$ such that $\|f_n\| = 1$ and $(M - \lambda)f_n \rightarrow 0$. Now f_n is a bounded sequence, and so it has a weakly convergent subsequence $(g_k = f_{n_k})_{k=1}^\infty, g_k \xrightarrow{w} g$.

It will now be shown that $\|g\| = 1$. To see this, first choose $B > 0$ arbitrary. For any $\epsilon > 0$, there is a $N \in \mathbb{N}$ such that $n > N$ implies that $\|g_n\|_{\mathbb{R} \setminus [\lambda - B, \lambda + B]}^2 := \|g_n\|^2 - \int_{[\lambda - B, \lambda + B]} \left| \frac{g_n(x)}{E(x)} \right|^2 dx < \epsilon$. If this were not true, then there would be an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there is an $n > N$ for which $\|g_n\|_{\mathbb{R} \setminus [\lambda - B, \lambda + B]}^2 > \epsilon$. For any such n ,

$$\|(M - \lambda)g_n\|^2 \geq \int_{-\infty}^{\lambda - B} (x - \lambda)^2 \left| \frac{g_n(x)}{E(x)} \right|^2 dx + \int_{\lambda + B}^\infty (x - \lambda)^2 \left| \frac{g_n(x)}{E(x)} \right|^2 dx \geq B^2\epsilon. \tag{45}$$

This would contradict the fact that $(M - \lambda)g_n \rightarrow 0$.

Since $g_n \xrightarrow{w} g, |g_n(z)| = |\langle g_n, K_z \rangle| \leq \|g_n\| \|K_z\| = \|K_z\|$, and K_z is an anti-analytic vector-valued function in \mathbb{C} , it follows that the sequence $(g_n)_{n \in \mathbb{N}}$ is uniformly bounded on compacta. Since the g_n are entire, it is easy to see that this implies that this sequence is uniformly equicontinuous on any compact $K \subset \mathbb{C}$. It follows that given any compact $K \subset \mathbb{C}$, there is a subsequence $(h_k := g_{n_k})_{k \in \mathbb{N}}$, which converges uniformly on K .

Since $E(\lambda) \neq 0$ and E is entire, choose δ small enough so that $E \neq 0$ on $[\lambda - \delta, \lambda + \delta]$. Then $1/E$ is continuous on $[\lambda - \delta, \lambda + \delta]$, and, by the above arguments, it follows that one can find a subsequence $(h_n)_{n \in \mathbb{N}}$ of the $(g_k)_{k \in \mathbb{N}}$ such that h_n/E converges to g/E uniformly on this interval.

Given any $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $n > N$ implies that $\|h_n\|_{\mathbb{R} \setminus [\lambda - \delta, \lambda + \delta]} < \frac{\epsilon}{2}$. Now choose $N' \in \mathbb{N}$ such that $n > N'$ implies that $|(h_n(x) - g(x))/E(x)| \leq \frac{\epsilon}{4\delta}$ for all $x \in [\lambda - \delta, \lambda + \delta]$. It follows that for any $n > M := \max\{N, N'\}$ we have that

$$\begin{aligned} \|g\| &\geq \|g\|_{[\lambda - \delta, \lambda + \delta]} = \|h_n + (g - h_n)\|_{[\lambda - \delta, \lambda + \delta]} \\ &\geq \|h_n\|_{[\lambda - \delta, \lambda + \delta]} - \|g - h_n\|_{[\lambda - \delta, \lambda + \delta]} \\ &> \left(1 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} = 1 - \epsilon \end{aligned} \tag{46}$$

Since $\epsilon > 0$ was arbitrary, it follows that $\|g\| \geq 1$. Conversely, $|\langle h_n, g \rangle| \leq \|g_n\| \|g\| = \|g\|$ and $|\langle h_n, g \rangle| \rightarrow \|g\|^2$, so that $\|g\|^2 \leq \|g\|$. This implies that $\|g\| \leq 1$ and hence that $\|g\| = 1$.

The fact that $\|g\| = 1$ means that h_n actually converges strongly to g . This follows because

$$\begin{aligned} \|h_n - g\|^2 &= \langle h_n - g, h_n - g \rangle \\ &= \|h_n\|^2 + \|g\|^2 - 2\text{Re}(\langle h_n, g \rangle) \\ &= 2(1 - \text{Re}(\langle h_n, g \rangle)) \rightarrow 0. \end{aligned} \tag{47}$$

Since $h_n \rightarrow g \neq 0, (M - \lambda)h_n \rightarrow 0$, and M is a closed operator, it follows that $g \in \text{Dom}(M)$ and $(M - \lambda)g = 0$. In other words, g is actually an eigenvector of M

to eigenvalue λ . We have already proven that this is not possible. We conclude that $\lambda \notin \sigma_c(M)$.

By Lemma 7, the multiplication operator M in $\mathcal{H}(E)$ is unitarily equivalent to the operator M_0 of multiplication by z in some $\mathcal{H}(E_0)$ where E_0 is a de Branges function with no real zeroes. By the above arguments M_0 is simple and regular, and hence so is M . \square

In conclusion, the operator M , of multiplication by z in $\mathcal{H}(E)$ is always simple, regular, symmetric, and closed with deficiency indices $(1, 1)$. Since $\mathcal{H}(E)$ is a RKHS, in order to prove that it has the $U(1)$ sampling property, it remains to show that M is densely defined so that Theorem 4 can be applied. De Branges has characterized exactly when this happens [21, p. 84]:

Theorem 19 *A necessary and sufficient condition for a function $S \in \mathcal{H}(E)$ to be orthogonal to the domain $\text{Dom}(M)$ of the operator of multiplication by the independent variable in $\mathcal{H}(E)$ is that $S = aE + bE^*$ for some $a, b \in \mathbb{C}$. In particular, if no such function belongs to $\mathcal{H}(E)$, then M is densely defined.*

In summary, if $aE + bE^* \notin \mathcal{H}(E)$ for any $a, b \in \mathbb{C}$, then $M \in \text{Sym}_1(\mathcal{H}(E))$, and $\mathcal{H}(E)$ has the $U(1)$ sampling property as described in Theorem 4.

6.3 Proof that $\mathcal{H}(E)$ has the Semigroup Property

The purpose of this subsection is to show that if $\mathcal{H}(E)$ is any de Branges space, and if \tilde{M} is the operator of multiplication by the independent variable in $L^2(\mathbb{R}, |E(x)|^{-2} dx)$, then the compression of the unitary group $U(t) := e^{it\tilde{M}}$ to $\mathcal{H}(E)$ is a semi-group for $t \geq 0$, and $U(t)$ is the minimal unitary dilation of this semi-group.

In this subsection we assume that $\mathcal{H}(E)$ is a de Branges space such that E has no real zeroes, and such that no linear combination of E and E^* belongs to $\mathcal{H}(E)$. As discussed in the previous subsection, for such an E , the operator of multiplication by the independent variable has the sampling property, $M \in \text{Sym}_1(\mathcal{H}(E))$. Also, for such an E , the measure $|E(x)|^{-2} dx$ is clearly such that $|E(x)|^2$ and $|E(x)|^{-2}$ are locally L^∞ . Once it is established that $U(t) := e^{it\tilde{M}}$ is a unitary dilation of its compression, $V(t) := P_{\mathcal{H}(E)} U(t)|_{\mathcal{H}(E)}$; $t \geq 0$, to $\mathcal{H}(E)$, and that $V(t)$ is a semigroup, Theorem 14 will imply that any such $\mathcal{H}(E)$ must be a RKHS with the $U(1)$ sampling property.

Theorem 20 *Suppose that $\mathcal{H}(E)$ is a de Branges space, \tilde{M} is the self-adjoint operator of multiplication by the independent variable in $L^2(\mathbb{R}, |E(x)|^{-2} dx) \supset \mathcal{H}(E)$, $U(t) := e^{it\tilde{M}}$ is the unitary group generated by \tilde{M} , and P is the projector of $L^2(\mathbb{R}, |E(x)|^{-2} dx)$ onto $\mathcal{H}(E)$. Then $V(t) := P U(t)|_{\mathcal{H}(E)}$ is a contraction semigroup for $t \geq 0$, and $U(t)$ is a unitary dilation of $V(t)$. If E has no real zeroes then $U(t)$ is the minimal unitary dilation of $V(t)$.*

The proof of this theorem will rely on the following generalization of Cauchy's integral theorem to functions of bounded type and non-positive mean type in the upper half plane [21, p. 32].

Theorem 21 *Let f be a function which is analytic, and of bounded type and non-positive mean type in the upper half-plane. Further suppose that f has a continuous extension to \mathbb{R} and that $f \in L^2(\mathbb{R})$. Then, if z belongs to the upper half plane,*

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (t - z)^{-1} f(t) dt, \tag{48}$$

and

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (t - \bar{z})^{-1} f(t) dt. \tag{49}$$

Proof of Theorem 20 Let $\rho(z) := P(\tilde{M} - z)^{-1}|_{\mathcal{H}(E)}$. To prove the claim of the theorem, it is sufficient to show that $\rho(z)$ obeys the first resolvent formula,

$$\frac{1}{z - w} (\rho(z) - \rho(w)) = \rho(z)\rho(w), \tag{50}$$

for all $z, w \in LHP$. This fact and elementary holomorphic functional calculus techniques will then establish the theorem. The fact that $\rho(z)$ obeys the first resolvent formula can be straightforwardly proven by direct computation as follows.

Given $F \in \mathcal{H}(E)$, $w \in \mathbb{C}$, and $z_1 \in LHP$,

$$\begin{aligned} \rho(z_1)F(w) &= \langle \rho(z_1)F, K_w \rangle = \langle (\tilde{M} - z_1)^{-1}F, K_w \rangle \\ &= \int_{-\infty}^{\infty} \frac{F(x)}{x - z_1} \overline{K_w(x)} \frac{1}{|E(x)|^2} dx. \end{aligned} \tag{51}$$

Let $G(z) := \frac{F(z)}{z - z'}$ for some $z' \in LHP$. Then it follows that $\frac{G(z)}{E(z)}$ is analytic in UHP , and is of bounded type and non-positive mean type in the UHP since F/E is. Furthermore G/E belongs to $L^2(\mathbb{R})$, and has a continuous extension to \mathbb{R} (see the proof of Theorem 19 in [21] to see that F/E is continuous at the zeroes of E on \mathbb{R}). By Cauchy's theorem in the upper half plane, Theorem 21, it follows that if $z \in UHP$,

$$\frac{1}{z - z'} \frac{F(z)}{E(z)} = \frac{G(z)}{E(z)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G(t)}{E(t)} \frac{1}{t - z} dt. \tag{52}$$

Now consider $G^*(z) = \frac{F^*(z)}{z - z'}$. Since F^*/E is of bounded type and non-positive mean type in UHP , it follows again by Cauchy's theorem that

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{G^*(t)}{E(t)} \frac{1}{t - \bar{z}} dt &= \frac{1}{2\pi i(\bar{z} - z')} \int_{-\infty}^{\infty} \frac{F^*(t)}{E(t)} \left(\frac{1}{t - z'} - \frac{1}{t - \bar{z}} \right) dt \\ &= \frac{1}{\bar{z} - z'} \frac{F^*(\bar{z}')}{E(z')}. \end{aligned} \tag{53}$$

Now, for $t \in \mathbb{R}$, $K_w(t) = \frac{E(t)\overline{E(w)} - \overline{E(t)}E(\overline{w})}{2\pi i(\overline{w}-t)}$. Hence for $w \in UHP$,

$$\begin{aligned} \langle G, K_w \rangle &= \frac{E^*(w)}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{w-t} \frac{G(t)}{E(t)} dt - \frac{E(w)}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{w-t} \frac{G(t)}{E(t)} dt \\ &= \frac{E(w)}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{t-w} \frac{G(t)}{E(t)} dt + E^*(w) \overline{\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{t-\overline{w}} \frac{G^*(t)}{E(t)} dt} \\ &= E(w) \frac{G(w)}{E(w)} + E^*(w) \overline{\frac{1}{\overline{w}-\overline{z}} \frac{F^*(\overline{z})}{E(\overline{z})}} \\ &= \frac{F(w)}{w-z} - \frac{E^*(w)}{E^*(z)} \frac{F(z)}{w-z}. \end{aligned} \quad (54)$$

In summary, for $z \in LHP$, $w \in UHP$,

$$\rho(z)F(w) = \frac{1}{w-z} \left(F(w) - \frac{E^*(w)}{E^*(z)} F(z) \right). \quad (55)$$

It follows that for $w \in UHP$ and $z_1, z_2 \in LHP$,

$$\begin{aligned} \left(\frac{1}{z_1 - z_2} (\rho(z_1) - \rho(z_2)) F \right) (w) &= \frac{1}{(w - z_1)(w - z_2)} F(w) \\ &\quad - \frac{E^*(w)}{z_1 - z_2} \left(\frac{1}{w - z_1} \frac{F(z_1)}{E^*(z_1)} - \frac{1}{w - z_2} \frac{F(z_2)}{E^*(z_2)} \right). \end{aligned} \quad (56)$$

Let $H(w) := \rho(z_2)F(w)$. Since $H \in \mathcal{H}(E)$, we can iterate equation (55) to calculate

$$\begin{aligned} \rho(z_1)\rho(z_2)F(w) &= \rho(z_1)H(w) = \frac{1}{w - z_1} \left(H(w) - \frac{E^*(w)}{E^*(z_1)} H(z_1) \right) \\ &= \frac{1}{w - z_1} \left[\frac{1}{w - z_2} F(w) - \frac{E^*(w)}{E^*(z_2)} \frac{F(z_2)}{w - z_2} \right. \\ &\quad \left. - \frac{E^*(w)}{E^*(z_1)} \frac{1}{z_1 - z_2} \left(F(z_1) - \frac{E^*(z_1)}{E^*(z_2)} F(z_2) \right) \right] \\ &= \frac{F(w)}{(w - z_1)(w - z_2)} + E^*(w) \frac{F(z_2)}{E^*(z_2)} \left(\frac{1}{(w - z_1)(z_1 - z_2)} \right. \\ &\quad \left. - \frac{1}{(w - z_2)(w - z_1)} \right) - E^*(w) \frac{F(z_1)}{E^*(z_1)} \frac{1}{(w - z_1)(z_1 - z_2)} \\ &= \frac{1}{(w - z_1)(w - z_2)} F(w) - \frac{E^*(w)}{z_1 - z_2} \left(\frac{1}{w - z_1} \frac{F(z_1)}{E^*(z_1)} \right. \\ &\quad \left. - \frac{1}{w - z_2} \frac{F(z_2)}{E^*(z_2)} \right) \\ &= \left(\frac{1}{z_1 - z_2} (\rho(z_1) - \rho(z_2)) F \right) (w). \end{aligned} \quad (57)$$

Since elements of $\mathcal{H}(E)$ are entire functions, this proves that

$$\frac{1}{z-w}(\rho(z) - \rho(w)) = \rho(z)\rho(w), \quad (58)$$

for all $z, w \in LHP$.

Let Γ be a straight line contour parallel to the real axis in the LHP that runs from left to right. The formula

$$U(t) = e^{it\tilde{M}} = \frac{1}{2\pi i} \int_{\Gamma} (z - \tilde{M})^{-1} e^{itz} dz, \quad (59)$$

follows from the holomorphic functional calculus for closed operators whose spectrum is confined to a sector of the complex plane [22] (see also Theorem 1.15 of [23]). Using this, and the fact that $\rho(z)$ obeys the first resolvent formula in the LHP , it is straightforward to show that $V(t) := PU(t)|_{\mathcal{H}(E)}$ is a semi-group for $t \geq 0$.

Now suppose that E has no real zeroes. The fact that $U(t)$ is the minimal unitary dilation of $V(t)$ follows from the fact that $\mathcal{H}(E)$ consists of entire functions, and the assumption that E has no real zeroes, so that the measure defined by $\frac{1}{|E(t)|^2} dt$ is equivalent to Lebesgue measure. Since there is no Borel subset of \mathbb{R} of non-zero measure on which all elements of $\mathcal{H}(E)$ vanish, the smallest invariant subspace for \tilde{M} containing $\mathcal{H}(E)$ is all of $L^2(\mathbb{R}, |E(x)|^{-2} dx)$, and $U(t)$ is the minimal unitary dilation of $V(t)$. \square

Let E be a de Branges function with no real zeroes such that there is no linear combination of E and E^* belonging to $\mathcal{H}(E)$. By the above theorem, Theorems 20, 19, and 18, it follows that the de Branges space $\mathcal{H}(E)$ satisfies the conditions of Theorem 14, and hence is a reproducing kernel Hilbert space with the $U(1)$ sampling property.

Since we assume that $E(x) \neq 0$ for any $x \in \mathbb{R}$, the fact that $K(x, x) > 0$ for all $x \in \mathbb{R}$ for the de Branges space $\mathcal{H}(E)$ follows from [21, Problem 45, p. 52]. Every space $\mathcal{H}(E)$ is already known to be RKHS, so the fact that the class of de Branges spaces we are considering have the $U(1)$ sampling property also follows straightforwardly from Theorem 4.

7 Outlook

Theorem 14 provides a sufficient condition for a subspace of $L^2(\mathbb{R}, d\nu)$ to be a RKHS with the $U(1)$ sampling property. It seems natural to expect that Theorem 14 should have a generalization to the case of symmetric multiplication operators with higher and finite equal deficiency indices (n, n) , on spaces of n -component vector functions on \mathbb{R} which are square integrable with respect to a $n \times n$ matrix valued measure. It will be interesting to see whether the results of Sects. 2–4 can be generalized to apply to such operators.

The Paley-Wiener spaces $B(\Omega)$ of bandlimited functions can be seen as invariant subspaces of the self-adjoint second derivative operator $D := -\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$,

$B(\Omega) = \chi_{[0, \Omega^2]}(D)$. Here, $\chi_{[0, \Omega^2]}$ denotes the characteristic function of the interval $[0, \Omega^2]$. The author is currently trying to determine whether the invariant subspaces $B(p, q, \Omega) := \chi_{[0, \Omega^2]}(D_{pq})$ of more general second order Sturm-Liouville differential operators $D_{pq} := -\frac{d}{dx} \left(p(x) \frac{d}{dx} \cdot \right) + q(x)$ which are essentially self-adjoint on a dense domain in $L^2(\mathbb{R})$ are also RKHS with the $U(1)$ sampling property, or even de Branges spaces, if p, q are chosen suitably. For example, if $p = 1$, and $q = q^*$ is an entire function where $q(x) \geq 0$ for all $x \in \mathbb{R}$, it is not difficult to show that the subspace $B(p, q, \Omega)$ is a RKHS of entire functions [6]. The author is currently investigating whether it can be shown that the operator M of multiplication by the independent variable in $L^2(\mathbb{R})$ has a symmetric restriction M' to a dense domain in $B(p, q, \Omega)$ for suitable p, q so that Theorems 14 or 4 can be applied to prove that $B(p, q, \Omega)$ are indeed reproducing kernel Hilbert spaces with the $U(1)$ sampling property. Ultimately, it would be useful to fully characterize all subspaces of $L^2(\mathbb{R}, d\nu)$ which have the properties described in Theorem 14. Such a characterization would be both of theoretical interest, as well as of practical interest for applications including signal processing.

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