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1.) Introduction

In UCRL 17795, it was shown that curves in the x,y plane having reflection symmetry about the positive diagonal are invariant under the transformation:

$$x' = y$$
$$y' = -x+f(y), \qquad (1).$$

where f(y) is the sum of the two values of x corresponding to the given y. It is required that there be just two values, but the two branches on which they occur are not required to have a common analytic form. An example given was the pair of rectangular hyperbolas $y = 1 - a/(x+1)$ and $y = -1 + a/(1-x)$, with $f(y) = 2ay/(1-y^2)$, mentioned in paragraph 3 and illustrated in Fig. 1. The question whether there are other invariant curves belonging to the same f(y) was left open.

This question was answered by John M. Greene in a letter to L. Jackson Laslett (March 8, 1968). He pointed out that all curves of the form $(1-x^2)(1-y^2)+2axy=\text{const.}$ are such invariants. If the constant has the value $2a-a^2$, the equation factors into two equations representing the rectangular hyperbolas, which are now seen to be simply the separatrices of a family of invariant curves. In the course of checking the invariance of "Greene's function" by the methods of UCRL 17795, I found that it is a special case of a broader class, which can be called "double quadratic" curves.

2.) "Double quadratic" curves

Any equation which is quadratic in x can be solved explicitly for x. If x and y occur in it symmetrically, it represents a curve with the required symmetry about the positive diagonal. The most general equation with these properties is:

$$Ax^2y^2 + B(x^2y+xy^2) + C(x^2+y^2) + Dxy + E(x+y) + F = 0, \qquad (2).$$

whose solution is:

$$x = \frac{1}{2(Ay^2+By+C)}\left[-(By^2+Dy+E) \pm \sqrt{(By^2+Dy+E)^2 - 4(Ay^2+By+C)(Cy^2+Ey+F)}\right] \qquad (3).$$

The sum of the two values of x gives f(y):

$$f(y) = -\frac{By^2 + Dy + E}{Ay^2 + By + C} \qquad (4).$$

Since f(y) does not depend on F, all members of the family generated by giving different values to F are invariant under the transformation (1), with f(y) given by (4).

We thus have the remarkable result that an f(y) which is the ratio of any two quadratic functions of y leads to a family of invariant curves, with the single restriction that the coefficients of y^2 in the numerator and of y in the denominator must be of equal magnitude and opposite in sign.

The first order fixed points, if they exist, are at f(y) = 2y, and are therefore the solutions of:

$$2\,Ay^3 + 3\,By^2 + (2C+D)y + E = 0 \qquad (5).$$

The number of parameters in (4) is easily reduced; E can be eliminated by a coordinate displacement along the positive diagonal, either A or B can be made equal to D or E by a change of scale, and any one of the remaining parameters can be set equal to unity. Thus we have a two-parameter system. Some interesting cases are:

(1) $A = 1,\ B = 0,\ C = -1,\ D = 2a,\ E = 0,\ F = c.$

$$x^2y^2 - x^2 - y^2 + 2a\,x\,y + c = 0 . \qquad \text{("Greene's function")}$$

$$f(y) = \frac{2\,a\,y}{1 - y^2}$$

The first order fixed points are at $y = 0,\ \pm\sqrt{1-a}$.

The separatrices are displaced rectangular hyperbolas, as pointed out above.

(2) $A = 1,\ B = 0,\ C = 1,\ D = -2a,\ E = 0,\ F = c.$

$$x^2y^2 + x^2 + y^2 + 2a\,x\,y + c = 0.$$

$$f(y) = \frac{2\,a\,y}{1 + y^2}$$

The first order fixed points are at $y = 0,\ \pm\sqrt{a-1}$.

The separatrix is the curve given by setting c=0.

In cases (1) and (2), if a is negative, the curve is rotated by 90°, and the first order fixed points (except the one at $x = 0$) become second order fixed points. (See paragraph 6 and Fig. 3b of UCRL 17795)

(3) $A = 0,\ B = 1,\ C = -1,\ D = 0,\ E = 0,\ F = c.$

$$x^2y + xy^2 - x^2 - y^2 + c = 0.$$

$$f(y) = \frac{y^2}{1 - y}$$

The first order fixed points are at $y = 0, \frac{2}{3}$.

The separatrices are the curve given by setting $c = \frac{8}{27}$, the line $x+y+2 = 0$, and the curve $xy - x - y + 2 = 0$.

(I thank Dr. Laslett for finding the last two of these.)

(4) $A = 1,\ B = -2,\ C = 1,\ D = 0,\ E = 0,\ F = c.$

$$x^2y^2 - 2\,(x^2y+xy^2) + x^2+y^2 + c = 0.$$

$$f(y) = \frac{2y^2}{(1-y)^2}$$

The first order fixed points are at $y = 0, \frac{1}{2}(3 \pm \sqrt{5})$.