

Lawrence Berkeley National Laboratory

Recent Work

Title

MICROSCOPIC DERIVATION OF THE LOW-LYING EXCITATION SPECTRUM OF AN INTERACTING BOSE SYSTEM

Permalink

<https://escholarship.org/uc/item/50p586cr>

Authors

Bierter, Willy
Morrison, Harry L.

Publication Date

1969-03-17

eg-Z

RECEIVED
LAWRENCE
RADIATION LABORATORY

APR 17 1969

LIBRARY AND
DOCUMENTS SECTION

MICROSCOPIC DERIVATION OF THE LOW-LYING EXCITATION
SPECTRUM OF AN INTERACTING BOSE SYSTEM

Willy Bierter and Harry L. Morrison

March 17, 1969

AEC Contract No. W-7405-eng-48

TWO-WEEK LOAN COPY

*This is a Library Circulating Copy
which may be borrowed for two weeks.
For a personal retention copy, call
Tech. Info. Division, Ext. 5545*

LAWRENCE RADIATION LABORATORY
UNIVERSITY of CALIFORNIA BERKELEY

eg-Z

DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

MICROSCOPIC DERIVATION OF THE LOW-LYING EXCITATION SPECTRUM
OF AN INTERACTING BOSE SYSTEM*

Willy Bierter

Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

and

Harry L. Morrison

Lawrence Radiation Laboratory
University of California
Livermore, California 94550

March 17, 1969

Summary. - The low-lying excitation spectrum of an interacting Bose system is derived directly from the N-body microscopic Hamiltonian expressed in terms of local densities and currents as quantum-mechanical coordinates.

Long ago Feynman (1) derived an important relationship between the elementary excitation spectrum of the low-lying states of a Bose system and its correlation function for the density fluctuations: $\omega(\underline{k}) = k^2/2mS(\underline{k})$. This result was obtained by Feynman as a consequence of a special choice of the wave function and a subsequent variational calculation. Although the range of validity of the Feynman wave function is not quite clear (2), this result is supported by the saturated sum-rule argument of Pines (3), and the hydrodynamic derivation of Pitaevskii (4). Among these derivations, that of Pitaevskii is simplest and most concise.

Recently, however, doubt has been cast on the validity of the local velocity operator upon which the Landau theory of quantum hydrodynamics is based (5). Thus the consistency of the Pitaevskii theory is likewise open to question, since its dynamical variables are those of the Landau theory. The purpose of this note is to present an N-body microscopic derivation of the Feynman result, which is in the spirit of the Pitaevskii theory, but not subject to the objectionable use of a quantum-mechanical velocity field.

As a motivational preliminary to a theory of hadron dynamics, Dashen and Sharp (6) have demonstrated that the nonrelativistic quantum mechanics of a system of N spinless identical particles may be described completely and exactly by a theory in which the dynamical variables are the local density and currents. Unlike previous theories which are based on a density description of the system (7), no

dynamical variable canonically conjugate to the density operator is introduced (8).

The expression for the Hamiltonian of a system of N identical spinless bosons (of unit mass) interacting through a local two-body potential is, in the language of second quantization, given by

$$(1) \quad H = \frac{1}{2} \int d^3x \nabla \psi^\dagger(\underline{x}) \nabla \psi(\underline{x}) , \\ + \frac{1}{2} \iint d^3x d^3y \psi^\dagger(\underline{x}) \psi^\dagger(\underline{y}) v(|\underline{x}-\underline{y}|) \psi(\underline{x}) \psi(\underline{y}) .$$

The field operators $\psi^\dagger(\underline{x})$ and $\psi(\underline{x})$ satisfy the usual canonical commutation relations:

$$(2) \quad [\psi^\dagger(\underline{x}), \psi^\dagger(\underline{y})] = [\psi(\underline{x}), \psi(\underline{y})] = 0 ,$$

$$[\psi(\underline{x}), \psi^\dagger(\underline{y})] = \delta(\underline{x} - \underline{y}) .$$

In the second-quantized formalism, the local density and current operators are given by

$$(3) \quad \rho(\underline{x}) = \psi^\dagger(\underline{x}) \psi(\underline{x}) ,$$

$$\underline{J}(\underline{x}) = \frac{1}{2i} [\psi^\dagger(\underline{x}) \nabla \psi(\underline{x}) - \nabla \psi^\dagger(\underline{x}) \psi(\underline{x})] .$$

Although $\rho(\underline{x})$ and $\underline{J}(\underline{x})$ are not canonically conjugate variables, together they provide a set of quantum-mechanical coordinates that is complete (9) and they satisfy the following algebra:

$$\begin{aligned}
 (4) \quad [\rho(\underline{x}), \rho(\underline{y})] &= 0, \\
 [\rho(\underline{x}), J_i(\underline{y})] &= -i \frac{\partial}{\partial x_i} [\delta(\underline{x} - \underline{y}) \rho(\underline{x})], \\
 [J_i(\underline{x}), J_j(\underline{y})] &= -i \frac{\partial}{\partial x_j} [\delta(\underline{x} - \underline{y}) J_i(\underline{x})] \\
 &\quad + i \frac{\partial}{\partial y_i} [\delta(\underline{x} - \underline{y}) J_j(\underline{y})].
 \end{aligned}$$

Using the crucial identities

$$\begin{aligned}
 \nabla \rho(\underline{x}) + 2i \underline{J}(\underline{x}) &= 2\psi^+(\underline{x}) \nabla \psi(\underline{x}), \\
 \nabla \rho(\underline{x}) - 2i \underline{J}(\underline{x}) &= 2\nabla \psi^+(\underline{x}) \psi(\underline{x}),
 \end{aligned}$$

one may write the Hamiltonian (1) in terms of the observable quantities $\rho(\underline{x})$ and $\underline{J}(\underline{x})$ in the form

$$\begin{aligned}
 (5) \quad H &= \frac{1}{8} \int d^3x [\nabla \rho(\underline{x}) - 2i \underline{J}(\underline{x})] \frac{1}{\rho(\underline{x})} [\nabla \rho(\underline{x}) + 2i \underline{J}(\underline{x})] \\
 &\quad + \frac{1}{2} \iint d^3x d^3y \rho(\underline{x}) V(|\underline{x}-\underline{y}|) \rho(\underline{y}).
 \end{aligned}$$

The variables $\rho(\underline{x})$ and $\underline{J}(\underline{x})$ are natural collective variables for a description of oscillatory processes in systems consisting of large numbers of interacting particles (10). In order to study the oscillation spectrum, we set $\rho(\underline{x}) = \langle \rho \rangle + \hat{\rho}(\underline{x})$, where $\langle \rho \rangle$ is the equilibrium or ground state average of $\rho(\underline{x})$ (which is a constant for

translationally invariant systems). With this substitution in (5), we expand to second order in $\hat{\rho}(\underline{x})$. One obtains

$$(6) \quad H = \frac{1}{8\langle\rho\rangle} \int d^3x [(\nabla \hat{\rho}(\underline{x}))^2 + 4(\underline{J}(\underline{x}))^2] \\ + \frac{1}{2} \iint d^3x d^3y (\langle\rho\rangle + \hat{\rho}(\underline{x})) v(|\underline{x}-\underline{y}|) (\langle\rho\rangle + \hat{\rho}(\underline{y})).$$

To eliminate $\underline{J}(\underline{x})$, we use the equation of continuity,

$\dot{\hat{\rho}}(\underline{x}) = -\nabla \cdot \underline{J}(\underline{x})$. We then expand the density fluctuation $\hat{\rho}(\underline{x})$ into Fourier components:

$$\rho(\underline{x}) = \sum_{\underline{k} \neq 0} \hat{\rho}(\underline{k}) e^{i\underline{k}\underline{x}},$$

where the $\underline{k} = 0$ term is omitted due to particle conservation. Thus the Hamiltonian becomes, to second order in $\rho(\underline{k})$,

$$(7) \quad H = \frac{1}{2} \langle\rho\rangle^2 \iint d^3x d^3y v(|\underline{x}-\underline{y}|) \\ + \frac{(2\pi)^3}{2} \sum_{\underline{k} \neq 0} \left\{ \left[\frac{k^2}{4\langle\rho\rangle} + v(\underline{k}) \right] |\hat{\rho}(\underline{k})|^2 + \frac{1}{k^2\langle\rho\rangle} |\dot{\hat{\rho}}(\underline{k})|^2 \right\}.$$

This is immediately recognized as the Hamiltonian for a system of independent harmonic oscillators with frequencies

$$(8) \quad \omega(\underline{k})^2 = k^2 \langle\rho\rangle \left(\frac{k^2}{4\langle\rho\rangle} + v(\underline{k}) \right).$$

Because the single particle kinetic energy is given by $T(\underline{k}) = k^2/2$, one can write eq. (8) as

$$(9) \quad \omega(\underline{k}) = [T^2(\underline{k}) + 2\langle\rho\rangle V(\underline{k}) T(\underline{k})]^{1/2}.$$

Thus we obtain the Bogoliubov spectrum of elementary excitations as the frequency of the oscillators. The energy of each oscillator is related to its frequency by

$$E(\underline{k}) = \omega(\underline{k}) \left(n_{\underline{k}} + \frac{1}{2} \right) \quad (n_{\underline{k}} = 0, 1, 2, \dots).$$

The ground state energy of the Bose liquid is

$$E_0 = \frac{1}{2} \langle\rho\rangle^2 \iint d^3x d^3y v(|\underline{x}-\underline{y}|) + \sum_{\underline{k}} \frac{1}{2} \omega(\underline{k}).$$

Since the mean value of the potential energy of an oscillator in a given state is half the mean value of the total energy of the oscillator in that state, we may write

$$\frac{1}{4} \omega(\underline{k}) = \frac{(2\pi)^3}{2} \left(\frac{k^2}{4\langle\rho\rangle} + v(\underline{k}) \right) \langle |\hat{\rho}(\underline{k})|^2 \rangle,$$

where the bracket denotes a ground state average. With the help of eq. (8), we immediately obtain

$$(10) \quad \omega(\underline{k}) = \frac{k^2}{2S(\underline{k})},$$

where

$$S(\underline{k}) = (2\pi)^3 \frac{\langle |\hat{\rho}(\underline{k})|^2 \rangle}{\langle \rho \rangle}$$

is the Fourier component of the density correlation function. Equation (10) is exactly Feynman's result.

Thus, we have shown that the small oscillation approximation to the Dashen-Sharp Hamiltonian (5) yields the Bogoliubov spectrum independently of any special assumptions about Bose condensation, and it further provides an alternative microscopic derivation of the Pitaevskii theory. It is interesting that the phonon spectrum, which we obtain in the limit of small k values, is independent of the presence of the Bose-condensed particles. This implies a sound velocity for these quanta which is the same above and below the λ -point. This property of the spectrum is in concert with the recent neutron scattering experiments of Woods (11) on liquid ${}^4\text{He}$. For quanta having $k < 0.38 \text{ \AA}^{-1}$, Woods observes that the sound velocity is essentially independent of temperature through the λ -point.

One of us (W.B.) is indebted to Professor Geoffrey Chew for the kind hospitality extended to him by the Theoretical Group at the Lawrence Radiation Laboratory, and a research fellowship from the "Swiss National Fund" is gratefully acknowledged.

FOOTNOTES AND REFERENCES

- * This work was supported in part by the U.S. Atomic Energy Commission.
1. R. P. FEYNMAN, Phys. Rev. 94, 262 (1954).
 2. C. C. LIN, in Liquid Helium, Varenna Lectures Course 21 (Academic Press, New York, 1963).
 3. D. PINES, Tokyo Summer Lectures in Theoretical Physics, Part I (Problem of Many Bodies), R. Kubo ed. (W. A. Benjamin, New York, 1966).
 4. L. P. PITAEVSKII, Sov. Phys. JETP 4, 439 (1957); I. M. KHALATNIKOV, Introduction to the Theory of Superfluidity (W. A. Benjamin, New York, 1965).
 5. H. FRÖHLICH, Physica 34, 847 (1967); J. C. GARRISON, J. WONG, and H. L. MORRISON, Lawrence Radiation Laboratory Report UCRL-7 1472, to be published.
 6. R. DASHEN and D. H. SHARP, Phys. Rev. 165, 1857 (1968); see also W. BIERTER and H. L. MORRISON, J. Low-Temp. Phys. 1, 65 (1969), for a discussion of the relation of this representation to the Landau theory of quantum hydrodynamics.
 7. See for example, H. M. CHAN and J. G. VALATIN, Nuovo Cimento 19, 118 (1961).
 8. The existence of such an operator is in conflict with the positivity condition on the density; see Ref. 5.
 9. Every operator that commutes with both $\rho(\underline{x})$ and $\underline{J}(\underline{x})$ is a multiple of the identity.

10. N. N. BOGOLIUBOV and D. N. ZUBAROV, Sov. Phys. JETP 1, 83 (1955).
11. A. D. B. WOODS, in Quantum Fluids, D. F. Brewer ed. (North-Holland Publishing Co., Amsterdam, 1966).

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or*
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.*

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

TECHNICAL INFORMATION DIVISION
LAWRENCE RADIATION LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720