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Asymptotic Pseudotrajectories and Chain Recurrent Flows, with Applications

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We present a general framework to study compact limit sets of trajectories for a class of nonautonomous systems, including asymptotically autonomous differential equations, certain stochastic differential equations, stochastic approximation processes with decreasing gain, and fictitious plays in game theory. Such limit sets are shown to be internally chain recurrent, and conversely.

KEY WORDS: Asymptotically autonomous systems; dynamical systems; chain recurrence; game theory; reaction diffusion; stochastic approximation.

0. INTRODUCTION

A *semiflow* Φ on a metric space (M, d) is a continuous map

$$\Phi: M \times \mathbf{R}_+ \to M, \qquad (x, t) \mapsto \Phi_t(x)$$

such that

$$\Phi_0 = \text{Identity}, \qquad \Phi_{t+s} = \Phi_t \circ \Phi_s$$

for all $(t, s) \in \mathbf{R}_+ \times \mathbf{R}_+$. Replacing $\mathbf{R}_+$ by $\mathbf{R}$ defines a *flow.*

A continuous function $X: \mathbf{R}_+ \to M$ is an *asymptotic pseudotrajectory* for Φ if

$$\lim_{t \to \infty} d(X(t+T), \Phi_T(X(t))) = 0$$

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locally uniformly in $T \in \mathbf{R}^p$. Thus for each fixed $s > 0$, the curve

$$[0, s] \to M: t \mapsto X(t+T)$$

shadows the Φ-trajectory of the point $X(T)$ over the interval $[0, s]$ with arbitrary accuracy for sufficiently large T. By abuse of language we call X *precompact* if its image has compact closure in M. All our results concern precompact asymptotic pseudotrajectories.

The *limit set* $L\{X\}$ of an asymptotic pseudotrajectory X, defined in analogy to the omega limit set of a trajectory, is the set of limits of convergent sequences $X(t_k)$, $t_k \to \infty$.

A great deal of research has gone into methods for determining omega limit sets of trajectories of a flow or semiflow; this could well be considered the goal of dynamical systems theory. But asymptotic pseudotrajectories, occurring in many applications, are also important. They are curves in the state space M which differ from true trajectories in a controlled way, with errors tending to zero, and it is often possible to describe their asymptotic behavior in terms of the dynamics of Φ. As we show, asymptotic pseudotrajectories arise not only in many dynamical settings, but also in fields seemingly unrelated to differential equations, such as game theory and stochastic approximation.

In this paper we treat limit sets of asymptotic pseudotrajectories in a unified topological framework. It turns out that that with considerable generality, limits sets of precompact asymptotic pseudotrajectories are connected and internally chain recurrent. If the dynamics of the flow are not too complicated, this makes it possible to identify all possible limits sets of asymptotic pseudotrajectories and to give quantitative conditions ensuring that an asymptotic pseudotrajectory is asymptotic with a true trajectory.

The main results are stated below: proofs are postoned to Sections 7, 8, and 9. Several useful dynamical results are presented in Section 1. Applications are made to asymptotically autonomous differential equations (Section 2), reaction-diffusion systems (3), stochastic differential equations (4), stochastic approximation (5), and game theory (6).

Main Results

We use the following notation. For any function $a(q)$ defined on N (the natural numbers) or $\mathbf{R}_+$ (the set $[0,\infty)$ of nonnegative reals), and taking values in $\mathbf{R}_+$, set

$$\mathscr{R}(a) = \mathscr{R}_{q \to \infty} a(q) = \limsup_{q \to \infty} a(q)^{1/q}$$

Φ denotes a semiflow (or flow) on a metric space M, and X: $[0, \infty) \to M$ is a precompact asymptotic pseudotrajectory for Φ. The limit set of X is

$$L\{X\} = \bigcap_{t \geq 0} \overline{X[t, \infty)}$$

Theorem 0.1 characterizes $L\{X\}$ in terms of its dynamic properties. Theorem 0.3 shows that, fairly generally, every compact limit set of an asymptotic pseudotrajectory embeds dynamically as an omega limit set of some flow.

Theorem 0.4 gives a sufficient condition for X to be exponentially asymptotic with a trajectory of Φ; Corollary 0.5 states that under this condition, the limit set is not merely an internally chain recurrent set for Φ, but the omega limit set of some Φ-trajectory. In certain applications this can be used to show that $X(t)$ converges to a fixed point of Φ.

If $K \subset M$ is an invariant set ($\Phi_t = K$ for all t), we say K is *internally chain recurrent* if $\Phi | K$ is chain recurrent in the sense of Conley (1978) or, equivalently, if $\Phi | K$ has no proper attractor (see Section 1).

Theorem 0.1. *$L\{X\}$ is Φ-invariant, connected, compact, and internally chain recurrent.*

A partial converse is the following.

Theorem 0.2. *Let $L \subset M$ be a connected, compact internally chain transitive set, and assume M is locally path connected. Then there exists an asymptotic pseudotrajectory X such that $L\{X\} = L$.*

The following result generalizes theorems of Bowen (1975) and Franke and Selgrade (1976). It says that when M is locally path connected, (M, Φ) embeds equivariantly in a flow on a larger space $\hat{M}$, in such a way that every internally chain recurrent continuum in M is an omega limit set for the flow in $\hat{M}$.

Theorem 0.3. *Let M be locally connected and Φ a flow (respectively, semiflow) on M. Then there exists a flow Ψ on a metric space $\hat{M}$, a closed Ψ-invariant set $S_\Phi \subset \hat{M}$ and a homeomorphism $H: M \to S_\Phi$ such that*

(a) *S_Φ attracts all forward trajectories of Ψ.*

(b) *$H \circ \Phi_t = \Psi_t | (S_\Phi \circ H)$ for all $t \in \mathbf{R}$ (respectively, for all $t \in \mathbf{R}_+$).*

(c) *H maps each compact connected internally chain recurrent sets for Φ homeomorphically onto a compact omega limit sets for Ψ.*

The *asymptotic error rate* of X *is the number* $e(X)$ defined by

$$e(X) = \sup_{T>0} \mathscr{R}_{s \to \infty} d(X(s+T), \Phi_T X(s))$$

If $e(X) \leqslant \lambda < 1$ we call X a *λ-pseudotrajectory*.

We say that a point $u \in M$, or its Φ-orbit, *λ-shadows* X if

$$\mathscr{R}_{t \to \infty} d(\Phi_t u, X(t+t_0)) \leqslant \lambda$$

for some $t_0 \geqslant 0$. This means that as $t \to \infty$, $X(t+t_0)$ tracks the Φ-trajectory of u with error going to zero like λ^t.

To each compact Φ-invariant set K we associate a number $0 \leqslant \varepsilon(\Phi, K) \leqslant \infty$ called the *expansion rate* of Φ at K (see Section 9). When Φ is a smooth flow,

$$\varepsilon(\Phi, K) = \sup_{t>0} \min_{x \in K} \|D\Phi_{-t}(\Phi_t(x))\|^{-1/t}$$

When Φ is the gradient flow of a C^2 function u on a Riemannian manifold M, then $\varepsilon(\Phi, K) = e^{\rho}$, where ρ is the minimum eigenvalue of the Hessian of u at critical points of u in K.

For the following theorem we assume that for any $\tau > 0$ and any ball $B_\rho(x) \subset M$, there exists a common Lipschitz constant for the maps $\Phi_t | B_\rho(x)$, $0 \leqslant t \leqslant \tau$. This holds for C^1 flows, and for the solution flows of the semilinear parabolic equation in Section 3.

Theorem 0.4. *Let $K \subset M$ be a compact invariant set containing the limit set $L\{X\}$. Assume*

$$e(X) < \lambda < \min\{1, \varepsilon(\Phi, K)\}$$

Then there there is a unique Φ-orbit that λ-shadows X.

Corollary 0.5. *Under the assumptions of Theorem 0.4, $L\{X\}$ is an omega limit set of Φ.*

1. DYNAMICS OF ASYMPTOTIC PSEUDOTRAJECTORIES

Theorem 0.1 and its applications in later sections show the importance of understanding the dynamics and topology of internally chain recurrent sets (which in most dynamical settings are the same as limit sets of asymptotic pseudotrajectories). Many of the results which appear in the literature on asymptotically autonomous equations and stochastic approximation can be easily deduced from properties of chain recurrent sets.[3] While there is no general structure theory for internally chain recurrent sets, much can be said about many common situations. Several useful results are presented in this section.

We continue to assume that $X: \mathbf{R}_+ \to M$ is a precompact asymptotic pseudotrajectory for a flow or semiflow Φ in a metric space M.

Attractors

A subset $A \subset M$ is an *attractor* for Φ provided that

(i) A is nonempty, compact, and invariant ($\Phi_t A = A$); and

(ii) A has a neighborhood $W \subset M$ such that $\text{dist}(\Phi_t x, A) \to 0$ as $t \to \infty$ uniformly in $x \in W$.

The *basin* of A is the positively invariant open set comprising all points x such that $\text{dist}(\Phi_t x, A) \to 0$ as $t \to \infty$. If $A \neq M$, then A is called a *proper* attractor. A *global attractor* is an attractor whose basin is all the space M. An equilibrium (=stationary point) which is an attractor is called *asymptotically stable.*

Theorem 1.1. *Let e be a asymptotically stable equilibrium with basin of attraction W. If $X(t_k) \in W$ for some sequence $t_k \to \infty$, then $\lim_{t \to \infty} X(t) = e$.*

This was proved by Thieme (1992, Theorem 4.1) for asymptotically autonomous equations, generalizing a theorem of Markus (1956). In the context of stochastic approximations this result was proved by Kushner and Clark (1978). It is an easy consequence of Theorem 8.2 because e is the only chain recurrent point in its own basin.

More generally we have the following.

Theorem 1.2. *Let A be an attractor with basin W. If $X(t_k) \in W$ for some sequence $t_k \to \infty$, then $L\{X\} \subset A$.*

Proof. Follows from Theorem 0.1 and Lemma 8.1. **QED**

Simple Flows, Cyclic Orbit Chains, and Liapunov Functions

A flow is called *simple* if it has only a finite set of alpha and omega limit points (necessarily consisting of equilibria). This property is inherited by the restriction of Φ to invariant sets.

[3] Some of our hypotheses are stronger.

A subset $\Gamma \subset M$ is an *orbit chain* for Φ provided that for some natural number $k \geqslant 2$, Γ can be expressed as the union

$$\Gamma = \{e_1, ..., e_k\} \cup \gamma_1 \cup \cdots \cup \gamma_{k-1}$$

of equilibria $\{e_1, ..., e_k\}$ and nonsingular orbits $\gamma_1, ..., \gamma_{k-1}$ connecting them: this means that γ_i has alpha limit set $\{e_i\}$ and omega limit set $\{e_{i+1}\}$. Neither the equilibria nor the obits of the orbit chain are required to be distinct. If $e_1 = e_k$, Γ is called a *cyclic orbit chain*. A homoclinic loop is an example of a cyclic obit chain.

Concerning cyclic orbit chains, Benaïm and Hirsch (1994, Theorem 3.1) noted the following useful consequence of the important Akin–Nitecki–Shub lemma (Akin, 1993).

Proposition 1.3. *Let $L \subset M$ be a compact internally chain recurrent set. If $\Phi \mid L$ is simple, then every nonstationary point of L belongs to a cyclic orbit chain in L.*

From Theorem 0.1 we thus get the following.

Corollary 1.4. *Assume that $\Phi \mid L\{X\}$ is simple. Then every point of $L\{X\}$ is an equilibrium or belongs to a cyclic orbit chain in $L\{X\}$.*

A continuous function $V: M \to \mathbf{R}$ is called a *strict Liapounov function* for Φ if $V(\Phi_t(x))$ is strictly decreasing along nonconstant forward trajectories of Φ. It is not difficult to prove directly that a semiflow with a strict Liapunov function and isolated equilibria cannot be chain recurrent on any compact invariant set containing nonequilibrium points. This can also be deduced from Corollary 1.3, because a flow with a strict Lyapounov function and only isolated equilibria is simple and has no cyclic orbit chain. As a consequence we obtain the following.

Corollary 1.5. *Assume that Φ admits Liapunov function and that equilibria in $L\{X\}$ are isolated. Then $X(t)$ converges to an equilibrium as $t \to \infty$.*

Ball (1976) obtained essentially this result under considerably broader hypotheses on the flow; see also Artstein (1976) for further results on Liapunov functions.

Planar Systems

The following result of Benaïm and Hirsch (1964) goes far toward describing the dynamics of internally chain recurrent sets for planar flows with isolated equilibria.

Theorem 1.6. *Assume Φ is a flow defined on $\mathbf{R}^2$ with isolated equilibria. Let L be an internally chain recurrent set. Then for any $p \in L$ one of the following holds:*

(i) *p is an equilibrium.*

(ii) *p is periodic (i.e., $\Phi_T(p) = p$ for some $T > 0$).*

(iii) *There exists a cyclic orbit chain $\Gamma \subset L$ which contains p.*

Notice that this rules out trajectories in L which spiral toward a periodic obit, or even toward a cyclic orbit chain.

In view of Theorem 0.1 we obtain the following.

Corollary 1.7. *Let Φ be a flow in $\mathbf{R}^2$ with isolated equilibria. If X is a bounded asymptotic pseudotrajectory of Φ, then $L(X)$ is a connected union of equilibria, periodic orbits, and cyclic orbit chains of Φ.*

The following corollary can be seen as a Poincarè–Bendixson result for asymptotic pseudotrajectories.

Corollary 1.8. *Let Φ be a flow defined on $\mathbf{R}^2$, $K \subset \mathbf{R}^2$ a compact subset without equilibria, X an asymptotic pseudotrajectory of Φ. If there exists $T > 0$ such that $X(t) \in K$ for $t \geq T$, then $L(X)$ is either a periodic orbit or a cylinder of periodic orbits.*

Of course if $X(t)$ is an actual trajectory of Φ, the Poincaré–Bendixson theorem precludes a cylinder of periodic orbits. But this can easily occur for an asymptotic pseudotrajectory.

The following consequence of Theorem 1.6 generalizes the special case proved by Thieme (1994, Theorem 1.3) for asymptotically autonomous differential equations.

Theorem 1.9. *Let Φ be a flow defined on $\mathbf{R}^2$ and X a bounded asymptotic pseudotrajectory of Φ. If $L\{X\}$ contains a nonstationary periodic orbit as a proper subset, then it contains an annulus of periodic orbits.*

Proof. Let L be any internally chain recurrent for a planar flow, and let $P \subset L$ denote the set of nonstationary periodic points. Previously (Benaïm and Hirsch, 1994) we proved (Theorem 4.1) that each component of P which is not a single orbit is homeomorphic to an annulus. The theorem follows by applying this to $L = L\{X\}$. **QED**

The next result extends Dulac's criterion for convergence in planar flows having negative divergence.

Theorem 1.10. *Let Φ be a flow in an open set in the plane, and assume that Φ_t decreases area for $t > 0$. Then*

(a) $L\{X\}$ is a connected set of equilibria which is nowhere dense and which does not separate the plane.

(b) If Φ has at most countably many stationary points, then $L\{X\}$ consists of a single stationary point.

Proof. The proof is contained in that of Theorem 1.6 (Benaïm and Hirsch, 1994); here is a sketch. The assumption that Φ decreases area implies that no invariant continuum can separate the plane. A generalization of the Poincaré–Bendixson theorem (Hirsch and Pugh, 1988) shows that an internally chain recurrent continuum (such as $L\{X\}$), which does not separate the plane consists entirely of stationary points. Simple topological arguments complete the proof. **QED**

Remark 1.1. A result similar to Theorem can be proved for the case where Φ is a volume decreasing flow of a cooperative (= quasimonotone) vector field in $\mathbf{R}^3$, using results of Hirsch (1989).

Remark 1.12. The conclusion of Theorem 1.10(b) can also be obtained for flows of a cooperative or competitive vector field in $\mathbf{R}^2$ using results of Hirsch (1982).

Remark 1.13. The classical structural stability theorems of Andronov and Pontryagin (1937) and de Baggis (1952) imply that for generic C^1 vector fields in $\mathbf{R}^2$, $L\{X\}$ is a periodic (possibly stationary) orbit. The same applies to dynamics on compact orientable surfaces, using the general density theory of Peixoto (1962, 1973).

2. ASYMPTOTICALLY AUTONOMOUS DIFFERENTIAL EQUATIONS

Let $g: \mathbf{R}^n \to \mathbf{R}^n$ and $f: \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous maps. The ordinary differential equation

$$\frac{dx}{dt} = f(t, x) \tag{1}$$

is called *asymptotically autonomous with limit equation*

$$\frac{dx}{dt} = g(x) \tag{2}$$

if $\lim_{t \to +\infty} f(t, x) = g(x)$ locally uniformly in x.

Asymptotically autonomous equations were introduced by Markus (1956) is a seminal paper which has significantly influenced the development of the qualitative theory of nonautonomous differential equations. Several of Markus's results have been recently generalized by Thieme in a series of papers (1992, 1994). Some of Thieme's results are discussed and extended in the present paper. Closely related to our work is a paper by Mischaikow *et al.* (1995) which also considers asymptotically autonomous equation in relation to chain recurrence. Earlier work by Artstein (1976) and Ball (1976) contains results close to some of ours.

Consider the asymptotically autonomous system (1) with limit Eq. (2). When g is locally Lipschitz, a standard application of Gronwall's inequality yields proves the following.

Proposition 2.1. *Let X be any bounded solution to (1). Let Φ be the flow generated by (2). Then X is an asymptotic pseudotrajectory of Φ.*

In fact this holds more generally as follows.

Theorem 2.2. *Assume that f is continuous and that g has unique integral curves. Then the conclusion of Proposition 2.1 holds.*

This is proved as Corollary 7.3.

These asymptotic pseudotrajectories coincide with the notion of a trajectory of an "asymptotically autonomous semiflow" introduced by Thieme (1992).

In his work on planar asymptotically autonomous systems, Thieme (1992, 1994) proves a Poincaré–Bendixson type theorem similar to Theorem 1.9 and asks the following question: Let $X(t)$, $t \geqslant 0$ be a bounded solution of an asymptically autonomous planar system (1). Suppose equilibria of the planar limit Eq. (2) are isolated. Then is $L\{X\}$ necessarily the union of periodic orbits, equilibria, and orbits connecting equilibria of g?

To answer Thieme's problem we use the fact (Proposition 2.1) that $X(t)$ is an asymptotic pseudotrajectory for (2). Therefore Thieme's question is answered affirmatively by Theorem 1.6.

The following result gives a sufficient condition that a solution $X(t)$ to an asymptotically autonomous differential equation be exponentially asymptotic to some solution $\Phi_t(x)$ of the limit equation.

For any continuous function h: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ and any compact set $K \subset \mathbf{R}^n$, we define a function $t \mapsto \|h(t, \cdot)\|_K$ by

$$\|h(t, \cdot)\|_K = \sup_{x \in K} \|h(t, x)\|$$

Theorem 2.3. *Let X: $[0, \infty) \to \mathbf{R}^n$ be a bounded solution to*

$$\frac{dx}{dt} = f(t, x) \tag{3}$$

where f is continuous in (t, x) and locally Lipschitz in x. Let $K \subset \mathbf{R}^n$ be a compact positively invariant set for the solution flow Φ of

$$\frac{dx}{dt} = g(x) \tag{4}$$

Assume

$$\lim_{t \to \infty} \operatorname{dist}(X(t), K) = 0 \tag{5}$$

Let $\mu = \varepsilon(\Phi, K)$.

(a) Assume $0 \leqslant \lambda < 1$ is such that

$$\mathscr{R}_{t \to \infty} \|f(t, \cdot) - g(\cdot)\|_K \leqslant \lambda \tag{6}$$

Then $e(X) \leqslant \lambda$.

(b) Assume, also,

$$\lambda < \min(1, \mu) \tag{7}$$

Then there exists a solution u: $[0, \infty) \to K$ to (4) such that

$$\mathscr{R}_{t \to \infty} \|u(t) - X(t)\| \leqslant \lambda$$

and any two such solutions extend to the same maximally defined solution to (3).

Proof. X is an asymptotic pseudotrajectory for Φ by Proposition 2.2, whose limit set lies in K by (5). Fix a compact neighborhood N of K.

Fix any λ_1 such that $\lambda < \lambda_1$. By (6) we can then fix $t_1 > 0$ and a compact neighborhood N of K such that for $x \in N$, $t \geqslant t_1$ we have

$$\|f(t, x) - g(x)\| < \lambda_1'$$

Let $L > 0$ be a Lipschitz constant in $x \in N$ for $f(t, x)$. Let $t_1 > 0$ be so large that $X(s) \in N$ for all $s \geqslant t_0$. Then from a standard application of Gronwall's inequality, we obtain, for $t + T \geqslant t \geqslant t_0$:

$$\begin{aligned}\|X(T+t)-\Phi_T X(t)\| &\leqslant e^{LT}\int_t^{t+T}\|f(s,X(s))-g(X(s))\|\,ds\\&\leqslant e^{LT}\int_t^{t+T}\lambda_1^s\,ds\\&\leqslant C\lambda_1^t\end{aligned}$$

where $C=|\log\lambda_1|\,e^{LT}$. Therefore the error rate $e(X)$ of X is bounded above by λ_1. Since this holds for all $\lambda_1>\lambda$, part (a) follows. If $\lambda<\mu$, part (b) follows from Theorem 0.4. **QED**

An Example

The following example (Ball, 1976, p. 240) illustrates the use of these theorems. Consider asymptotically autonomous planar systems given in polar coordinates by

$$\frac{dr}{dt}=-r(r-1)^2 \tag{8}$$

$$\frac{d\theta}{dt}=\cos^2\theta+h(t) \tag{9}$$

where $h\colon \mathbf{R}\to\mathbf{R}$ is continuous and $h(t)\to 0$ as $t\to\infty$.

Notice that every solution of the limiting autonomous system

$$\frac{dr}{dt}=-r(r-1)^2 \tag{10}$$

$$\frac{d\theta}{dt}=\cos^2\theta \tag{11}$$

converges to one of the three equilibria: $r=0$, $(r,\theta)=(1,\pi/2)$, $(r,\theta)=(1,-\pi/2)$.

From Theorem 2.2 we see that any solution $X(t)=(r(t),\theta(t))$ to (8), (9) is an asymptotic pseudotrajectory of (10), (11). Therefore by Theorem 0.3, X has for its limit set a connected, compact, internally chain recurrent set for the flow Φ of (10), (11). There are four such sets: the three equilibria and the unit circle S^1.

Suppose first that

$$h(s)\geqslant 0 \tag{12}$$

$$\int_0^\infty h(s)\,ds=\infty \tag{13}$$

Then any solution to (8), (9) starting on S^1 has as limit set the whole of S^1, because

$$\theta(t) \geqslant \theta(t_0) + \int_{t_0}^{t} h(s)\, ds$$

Now in place of (12), (13), suppose that

$$\limsup_{s \to \infty} |h(s)| = \lambda < 2 \tag{14}$$

In this case the error rate $e(X)$ of any solution $X(t)$ to (8), (9) is $\leqslant \lambda$ by Theorem 2.3(a). Now the expansion rate $\varepsilon(\Phi, S^1)$ is easily computed to be 2: this follows from (i), (ii), or (iii) in Section 9, the key fact being that the derivative of $\cos^2 \theta$ is bounded in absolute value by 2. Therefore from Theorem 2.3(b) we see that (12), (13), and (14) imply that every solution to (8), (9) is asymptotic with some solution of (10), (11), and it therefore converges to one of the three equilibria of Φ.

3. REACTION DIFFUSION EQUATIONS

Let $\bar{\Omega} \subset \mathbf{R}^m$ be a smooth (i.e., C^1) compact m-dimensional submanifold with interior Ω.

Consider PDEs of the following kind, to be satisfied by continuous functions

$$u = (u_1(x, t), \ldots, u_n(x, t)), \qquad t \geqslant 0, \quad x \in \bar{\Omega}$$

with values in $\mathbf{R}^n$:

$$\frac{\partial u}{\partial t} = B\, \Delta u + g(u), \qquad t > 0 \tag{15}$$

$$\frac{\partial u}{\partial \nu} = 0 \tag{16}$$

Here Δ is the Laplacean in the spatial variable x, operating on each component of u, B is a diagonal $n \times n$ matrix with positive diagonal entries b_j, g is a smooth vector field on $\mathbf{R}^n$, and ν is the inward-pointing unit vector field normal to the boundary of Ω.

Hale (1986) and Conway *et al.* (1978), for more general equations, give conditions ensuring that solutions to (15), (16) decay to spatially homogeneous solutions, which are interpreted as trajectories of g. Hale considers solutions having initial values in the basin of an attractor for g, while Conway *et al.* take initial values to be in an invariant region for the PDE. In both cases they conclude that $u(x, t)$ is asymptotic with the solution of a certain asymptotically autonomous equation with limit equation $dy/dt = g(y)$.

We show that under slightly more stringent conditions, $u(x, t)$ is exponentially asymptotic with a trajectory of g.

Let $E \subset C(\bar{\Omega}, \mathbf{R}^n)$ be a linear subspace endowed with a norm making the inclusion continuous. A (local) semiflow S in E is a *solution semiflow* to (15), (16) in case the solution with initial value $u(x, 0) = v(x)$ is given by

$$u(x, t) = (S_t v)x$$

Hale (1986) assumes that g is C^2 and $m \leqslant 3$ and states that solutions to (15), (16) form a local semiflow $S = \{S_t\}_{t \geqslant 0}$ in the fractional power space X^α corresponding to the operator $-\Delta$ with Neumann boundary conditions with dense domain $W^{2,2}(\Omega, \mathbf{R}^n) \subset X = L^2(\Omega, \mathbf{R}^n) \subset X = L^2(\Omega, \mathbf{R}^n)$, $\frac{3}{4} < \alpha < 1$. In this case $X^\alpha \subset W^{1,2}(\Omega, \mathbf{R}^n) \cap L^\infty(\Omega, \mathbf{R}^n)$ with continuous inclusion (Henry, 1981, p. 75).

On the other hand, Conway *et al.*, allowing g to be merely C^1 and placing no restriction on m, obtain a solution semiflow S in $H^1(\Omega, \mathbf{R}^n)$.

We first consider Hale's result. Let $K \subset \mathbf{R}^n$ be a compact attractor (Section 8) for the flow Φ of the vector field g. Identifying $\mathbf{R}^n$ with the subspace of X^α comprising constant maps $\Omega \to \mathbf{R}^n$, we consider K as a compact invariant set in X^α for the solution semiflow S.

Define $b = \max\{b_1, \ldots, b_n\} > 0$. Let $\lambda > 0$ denote the smallest positive eigenvalue of $-\Delta$ operating on functions in Ω having Neumann boundary conditions.

We take the stance that Ω (and hence λ), g, and K are fixed, whereas b is a parameter governing diffusion.

Hale (1986) proves that K is an exponential attractor for S, at rate $e^{-\sigma t}$, provided b is sufficiently large. Precisely:

Theorem 3.1 (Hale, 1986). *Let $K \subset \mathbf{R}^n$ be an attractor for the flow Φ of the C^1 vector field g on $\mathbf{R}^n$. Assume $1 \leqslant m \leqslant 3$, $\frac{3}{4} < \alpha < 1$. Let $\Omega \subset \mathbf{R}^m$, $B = \mathrm{diag}\{b_1, \ldots, b_n\}$, $b > 0$, $\lambda > 0$ be as above. Given $0 < \sigma < \lambda$, there exists a number $b_* > 0$ (depending on σ, Ω, g, and K) with the following property. Assume*

$$b > b_* \tag{17}$$

Then there is a neighborhood $V \subset X^\alpha$ of K and a constant $C > 0$ such that for any $v \in V$, the solution $u(\cdot, t) = S_t v$ to (15), (16) satisfies

$$\|u(\cdot, t) - \bar{u}(t)\|_{X^\alpha} \leqslant Ce^{-\sigma t}, \qquad t \geqslant 0 \tag{18}$$

where

$$\bar{u}(t) = |\Omega|^{-1} \int_{\Omega} u(x, t)\, dx$$

Moreover, $\bar{u}(t)$ satisfies an equation

$$\frac{d\bar{u}}{dt} = g(\overline{u(t)}) + h_v(t) \tag{19}$$

where $h_v\colon \mathbf{R}_+ \to \mathbf{R}$ satisfies

$$|h_v(t)| \leqslant Ce^{-\sigma t}, \qquad t \geqslant 0 \tag{20}$$

Thus $\bar{u}(t)$ satisfies an asymptotically autonomous system (19) with limit system

$$\frac{y}{t} = g(y)$$

Fixing v and setting $f(t, y) = g(y) + h_v(t)$, we see that (20) means that

$$\mathscr{R}_{t \to \infty} \|f(t, \cdot) - g(\cdot)\|_K \leqslant e^{-\sigma} \tag{1}$$

From Theorem 2.3 we see that $\bar{u}(t)$ is an asymptotic pseudotrajectory for the flow of g, with asymptotic error rate bounded by σ. Therefore from Theorem 3.1 and the fact that we can take σ arbitrarily close to λ, we obtain the following.

Theorem 3.2. *Assume the hypotheses of Theorem 3.1. Let $\mu = \varepsilon(\Phi, K)$ denote the expansion rate at K of the flow Φ of g. Suppose*

$$e^{-\lambda} < \mu \tag{22}$$

Then there exists $b_* > 0$, $\varepsilon > 0$, with the following property. Assume

$$b > b_*$$

Let $S_t v = u(x, t)$, $t \geqslant 0$ be a solution to (18), (19) with initial value $u(\cdot, 0) = v \in X^\alpha$ lying in the ε-neighborhood of K in X^α. Then there exists $y \in K$ and $t_0 \geqslant 0$ such that

$$\mathscr{R}_{t \to \infty} \|S_{t+t_0} v - \Phi_t y\|_{X^\alpha} \leqslant e^{-\lambda}$$

and the Φ-orbit of such a y is unique.

Remark 3.3. A similar result can be obtained for functional differential equations, using Theorem 4.1 of Hale (1986).

Remark 3.4. Hale's choice of b_* is deeply entwined with the dynamics of the PDE (15), (16); there is no obvious formula for such a b_* given solely terms of λ and b. This is why, in our statement of Theorem 3.1 and 3.2, we allow b_* to depend on all the data of the PDE.

It is interesting to contrast these results with those derived from Conway *et al.* (1978). These assume that the PDE has a compact invariant region $\Gamma \subset \mathbf{R}^n$, meaning that $S_t v(\Omega) \subset \Gamma$ provided $v(\Omega) \subset \Gamma$.

As Hale points out, an invariant region is not as robust as an attractor, and postulating one severely restrictis the class of PDEs under consideration.

Conway *et al.* take the solution flow S to be in $H^1(\Omega, \mathbf{R}^n)$. Set

$$M = \max_{y \in \Gamma} \|Dg(y)\|$$

Notice that in the following result, the data of the PDE and the ODE enter the determination of σ only in the combination $b\lambda - M$.

Theorem 3.5 (Conway* et al., *1978). *Suppose $b\lambda - M = \sigma > 0$. Then for any $v \in H^1(\Omega, \mathbf{R}^n)$ taking values in Γ, the solution $u(\cdot, t) = S_t v$ to (15), (16) with initial value v satisfies*

$$\|u(\cdot, t) - \bar{u}(t)\|_{H^1(\Omega, \mathbf{R}^n)} \leqslant Ce^{-\sigma t}, \qquad t \geqslant 0$$

where

$$\bar{u}(t) = |\Omega|^{-1} \int_\Omega u(x, t)\, dx$$

Moreover, $\bar{u}(t)$ satisfies an equation

$$\frac{d\bar{u}}{dt} = g(\overline{u(t)}) + h_v(t)$$

where $h_v\colon \mathbf{R}_+ \to \mathbf{R}$ satisfies

$$|h_v(t)| \leqslant Ce^{-\sigma t}, \qquad t \geqslant 0$$

By applying Theorem 0.4 we obtain the following.

Theorem 3.6 (Hirsch, 1994). *Let $\mu = \varepsilon(\Phi, \Gamma)$. Suppose $b\lambda - M = \sigma < \mu$. Then for any $v \in H^1(\Omega, \mathbf{R}^n)$ taking values in Γ, there exists $y \in \Gamma$ and $t_0 \geq 0$ such that*

$$\mathscr{R}_{t \to \infty} \|S_{t+t_0} v - \Phi_t y\|_{H^1(\Omega, \mathbf{R}^n)} \leq e^{-\sigma}$$

and the Φ-orbit of such a y is unique.

4. STOCHASTIC DIFFERENTIAL EQUATIONS

Let g: $\mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz map, and σ: $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^{n \times p}$ a continuous map, Lipschitz in $x \in \mathbf{R}^n$ uniformly in $t \in \mathbf{R}$, where $\mathbf{R}^{n \times p}$ denotes the space of $n \times p$ real matrices. Consider the *stochastic differential equation* in $\mathbf{R}^n$:

$$dX = g(X)\,dt + \sigma(X, t)\,dB_t \tag{23}$$

where $B = (B_t)_{0 \leq t < \infty}$ is a standard *Brownian motion* on $\mathbf{R}^p$ defined on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$.

The normal definition of (23) requires stochastic integration. For an introduction see, for example, Friedman (1975). An intuitive notion of (23) is to imagine a pollen grain in a river. The grain is subject to the current force (the "drift" given by $g(x)$) and is bombarded by water molecules (the "diffusion force" given by $\sigma(x, t)$). Solutions of (23) are (with probability one) continuous but nowhere differentiable.

Proposition 4.1. *Assume that $\|\sigma(x, t)\| \leq \varepsilon(t)$ for some nonincreasing function $\varepsilon(\cdot)$ such that*

$$\int_0^\infty \exp\left[-\frac{\beta}{\varepsilon(t)^2}\right] dt < \infty$$

for all $\beta > 0$.[4] *Then any solution of (23) is (with probability one) an asymptotic pseudotrajectory of the flow induced by g.*

The proof is given in Section 10.

Let Φ be the solution flow of $dx/dt = g(x)$; set $\mu = \varepsilon(\Phi)$. The following result gives conditions ensuring that a solution $X(t)$ be asymptotic to a trajectory of the vector field g: $\mathbf{R}^n \to \mathbf{R}^n$.

[4] For example, $\varepsilon(t) = O(1/(\log(t))^a)$ with $a > \frac{1}{2}$.

Proposition 4.2. *Let X be a solution to (23). Assume*

(a) Φ admits a global attractor A.

(b) There exist $l > 0$ such that solutions to (23) with initial value $X(0) = x_0$ satisfy $P(\sup_{t \geqslant 0} \|X(t)\| \leqslant l) = 1$.

(c) $\mathscr{R}_{t \to \infty} \varepsilon(t) < \min(1, \mu)$, where $\mu = \varepsilon(\Phi, A)$.

Then there exists a random vector $Y \in \mathbf{R}^n$ such that

(i) Almost surely $\lim_{t \to \infty} \|X(t) - \Phi_t(Y)\| = 0$.

(ii) $\mathscr{R}_{t \to \infty} \|X(t) - \Phi_t(Y)\|_2 < \min(1, \mu)$

where $\|\cdot\|_2$ denotes the L^2 norm on the space of $\mathbf{R}^n$-valued random variables.

The proof of this result is obtained by an application of Theorem 9.2 to a convenient flow in Banach space of L^2 functions on the underlying probability space. This is carried out in Section 10.

5. STOCHASTIC APPROXIMATION AND URN PROCESSES

In this section and the following we describe applications of asymptotic pseudotrajectories which a priori have nothing to do with differential equations.

Stochastic Approximation

Let h: $\mathbf{R}^n \to \mathbf{R}^n$ be a continuous function. Consider the following discrete time stochastic process:

$$x_{k+1} - x_k = \gamma_{k+1}(g(x_k) + U_{k+1}) \tag{24}$$

where $\{\gamma_k\}$ is a given sequence of nonnegative numbers such that

(a) $\sum_k \gamma_k = \infty$,

(b) $\sum_k \gamma_k^{1+\delta} < \infty$ for some $\delta > 0$.

and $\{U_k\}$ is a sequence of random variables uniformly bounded defined on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{P}, (\mathscr{F}_k)_{k \geqslant 0})$, which satsfies the so-called Robbins-Monro condition:

$$\mathbf{E}(U_{k+1} \mid \mathscr{F}_k) = 0$$

Formula (24) can be considered to be a noisy version of a variable-step size Cauchy–Euler approximation scheme for numerically solving $dx/dt = g(x)$:

$$y_{k+1} - y_k = \gamma_{k+1}\, g(x_k)$$

It is thus natural to compare the set of limit points of a sample path $\{x_k\}$ with limit sets of the vector field g. To this end, we set

$$\tau_0 = 0; \qquad \tau_k = \sum_{i=1}^{k} \gamma_i, \qquad k \geqslant 1$$

and define the *interpolated continuous process* $X: \mathbf{R}_+ \to \mathbf{R}^n$ as follows:

(i) $X(\tau_k) = x_k$,

(ii) X is affine on $[\tau_k, \tau_{k+1}]$.

The precise connection between X and trajectories of g is the following.

Proposition 5.1. *Assume the sequence $\{x_k\}$ is bounded with probability one. Assume g is Lipschitz.*[5] *Then the interpolated continuous process X is almost surely an asymptotic pseudotrajectory of the flow Φ induced by g.*

This follows from Benaïm (1996) and Benaïm and Hirsch (1993). The proof is in the same spirit as that of Proposition 4.1.

It is easy to see that because $\gamma_k \to 0$, the limit set of X coincides with the limit set $L\{x_k\}$ of $\{x_k\}$. Thus the results of Section 8 are applicable to the process $\{x_k\}$. Therefore we obtain the following.

Corollary 5.2. *Almost surely $L\{x_k\}$ is an internally chain recurrent invariant continuum for Φ.*

Generalized Polya Urn Processes

Let

$$\Delta^n = \{v \in \mathbf{R}^{n+1} : v_i \geqslant 0, \textstyle\sum v_i = 1\}$$

denote the *unit n-simplex*. Consider an urn which initially (i.e., at time $k=0$) contains $k_0 > 0$ balls of colors $1,..., n+1$. Assume that at each time step a new ball is added to the urn and its color is randomly chosen as follows.

Let $x_{k,i}$ be the proportion of balls having color i at time k and denote by $x_k \in \Delta^n$ the vector of proportions: $x_k = (x_{k,1},..., x_{k,n+1})$. The color of the ball added at time $k+1$ is chosen to be i with probability $f_i(x_k)$, where the f_i are the coodinates of a function $f: \Delta^n \to \Delta^n$.

Such processes, known as *generalized Polya urns*, have been considered by Hill *et al.* Sudderth (1980), Arthur *et al.* (1983), Pemantle

[5] If g is only continuous with a unique flow, Proposition 5.1 remains true.

(1990), and Benaïm and Hirsch (1993) among others. Arthur (1988) and Auriol and Benaïm (1994) use such processes to model competing technologies.

Let

$$\gamma_i = \frac{1}{k_0 + i}$$

It is easy to verify that $\{x_k\}$ satisfies a recursion of the form (24), where $g(x) = f(x) - x$. Thus from Proposition 5.1 we obtain the following.

Corollary 5.3. *Let g be the vector field on Δ^n defined as $g(x) = f(x) - x$. Assume f is Lipschitz. Then almost certainly*

(a) the interpolated continuous process X is an asymptotic pseudotrajectory of the flow Φ induced by g;

(b) $L\{x_k\}$ is an internally chain recurrent invariant continuum for Φ.

6. FICTITIOUS PLAY IN GAME THEORY

We show here that the notion of asymptotic pseudotrajectory is well suited to analyze a class of repeated noncooperative games with infinite horizon considered by many authors including (Robinson, 1951; Smale, 1980; Cowan, 1992; Fudenberg and Kreps, 1993). More details and examples will be given elsewhere (Benaïm and Hirsch, 1994).

For notational convenience we restrict attention to a two-players game; extension to any finite number of players is straightfoward. The players are labeled $i = 1, 2$.

To player i there is associated a measure space A_i called the *action space*, an *information space* $\mathbf{R}^{n_i}$, and a measurable *observation map*

$$\Psi_i: A = A_1 \times A_2 \to \mathbf{R}^{n_i}$$

These functions may include information about the one or both players' past payoffs $P_i: A \to \mathbf{R}$ (if any), actions, etc.

The *observable space* $S_i \subset \mathbf{R}^{n_i}$ of player i is the closure of the convex hull of $\Psi_i(A)$. It contains all possible averages of observations made by player i.

The *state space* $S \subset \mathbf{R}^n \times \mathbf{R}^{n_2}$ is the closure of the convex hull of $\Psi(A) = \Psi_1(A) \times \Psi_2(A)$. Notice that $S \subset S_1 \times S_2$.

Consider now the repeated play of the game. At round t of play, player i chooses an action in $a_i(t) \in A_i$ (independently of the other player), based on her past observations, and then observes the outcome through

Ψ_i. At time $k \in \mathbf{N}$, the history of the game is represented by the sequence $\{(a_1(t), a_2(t)),\ t = 1,..., k\}$ of actions taken by both players between times 1 and k.

A behavior rule for player i gives the probability distribution of her next action based on all her past observations. We now make the key hypothesis that *player i's action probabilities at time* $k+1$ *are based* only *on the average* $\langle \Psi_i \rangle_k$ *of her past observations, where*

$$\langle \Psi_i \rangle_k = \frac{1}{k} \sum_{t=1}^{k} \Psi_i(a_1(t), a_2(t))$$

We call $\langle \Psi_i \rangle_k$ the *average observation at time k*. Denoting the set of probability measures on A_i by $\mathscr{P}(A_i)$, we define a *behavior rule* for player i as a map

$$\Pi^i: S_i \to \mathscr{P}(A_i)$$
$$x_i \mapsto \Pi^i(x_i, \cdot)$$

whose interpretation is as follows: For any measurable subset $Y \subset A_i$, $\Pi^i(x_i, Y)$ is the probability that player i chooses her action in Y at time $k+1$, given that $\langle \Psi_i \rangle_k = x_i$.

Example 6.1. Assume the action spaces are finite sets:

$$A_1 = \{1, 2,..., k_i\}, \qquad A_2 = \{1, 2,..., k_2\}$$

Suppose the observable function is given by

$$\Psi: A \to \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \qquad (l, m) \mapsto (e_m^2, e_l^1)$$

where e_j^i denotes the jth standard basis vector in $\mathbf{R}^{n_i}$. Each player observes the action of the other and chooses her next action based solely on the empirical frequencies of the *action choices* of the other in the past. In this case the observable space S_i of player i is the unit simplex of $\mathbf{R}^{n_i}$ and the state space is $S = S_1 \times S_2$. A common approach is to attempt to maximize expected payoff, assuming that the empirical frequencies will govern the other players next action. This is called *fictitious play*.

For the general case of behavior rules Π^i based on the average past observation $\langle \Psi_i \rangle_k$, it is easy to calculate the expected value $h_i(x_1, x_2)$ of $\langle \Psi_i \rangle_{k+1}$, given that $(\langle \Psi_1 \rangle_k, \langle \Psi_2 \rangle_k) = (x_1, x_2)$. We obtain

$$h_i: S_1 \times S_2 \to \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$$

$$(x_1, x_2) \mapsto \int_{A_1 \times A_2} \Psi_i(a_1, a_2)\, \Pi^1(x_1, da_1)\, \Pi^2(x_2, da_2)$$

A computation shows that the sequence of random vectors $x(k)=(x_1(k), x_2(k))$ satisfies a recursion of type (24) with $h(x)=g(x)-x$ and $\gamma_k=1/k$. We therefore obtain the following from Proposition 5.1 and Corollary 5.2.

Proposition 6.2. *Assume the vector field $g(x)=h(x)-x$ is Lipschitz on S. Then almost surely $L\{x_k\}$ is an invariant internally chain recurrent continuum for the flow induced by g.*

Example 6.3 (Smale Play). Smale (1980) proposed a solution to certain cases of the symmetrical Prisoner's Dilemma in which each player has access to the average payoffs of both the players. Thus each player's observable function is the payoff function to both players, and each player has the space of pairs of average payoffs as her observable space.

Smale considers only deterministic strategies. This fits into our framework if we choose Dirac measures for the strategies: $\Pi^i(x_i, \cdot)=\delta_{g_i(x_i)}(\cdot)$. Proposition 6.2 below has not been proved in this generality, however.

Benaïm and Hirsch (1994a) use Theorem 8.2 to prove an analogue to Theorem 1 of Smale (1980), for stochastic behavior rules yielding a Lipschitz game vector field.

7. CHARACTERIZATION OF ASYMPTOTIC PSEUDOTRAJECTORIES

Let $C^0(\mathbf{R}, M)$ denote the space of continuous M-valued functions $\mathbf{R}\to M$ endowed with the topology of uniform convergence on compact intervals. If $X\colon \mathbf{R}_+\to M$ is a continuous function, we consider X as an element of $C^0(\mathbf{R}, M)$ by setting $X(t)=X(0)$ for $t<0$. The space $C^0(\mathbf{R}, M)$ is metrizable. Indeed, a distance is given by: for all $f, g\in C^0(\mathbf{R}, M)$,

$$d(f, g)=\sum_{k\in N}\frac{1}{2^k}\min(1, d_k(f, g))$$

where $d_k(f, g)=\sup_{x\in[-k,k]} d(f(x), g(x))$.

The *translation flow* $\Theta\colon C^0(\mathbf{R}, M)\times\mathbf{R}\to C^0(\mathbf{R}, M)$ is the flow defined by

$$\Theta_t(X)(s)=X(t+s)$$

Let Φ be a flow or a semiflow on M. For each $p\in M$, the trajectory $\Phi^p\colon t\to\Phi_t p$ is an element of $C^0(\mathbf{R}, M)$ (with the convention that $\Phi^p(t)=p$ if $t<0$ and Φ is a semiflow). The set of all such Φ^p defines a subspace $\mathbf{S}_\Phi$.

It is easy to see that $\mathbf{S}_\Phi$ is a closed set invariant under Θ. Define the retraction $\hat{\Phi}$: $C^0(\mathbf{R}, M) \to \mathbf{S}_\Phi$ as

$$\hat{\Phi}(X)(t) = \Phi_t(X(0))$$

Lemma 7.1. *A continuous function $X: \mathbf{R}_+ \to M$ is an asymptotic pseudotrajectory of Φ if and only if*

$$\lim_{t \to \infty} d(\Theta^t(X), \hat{\Phi} \circ \Theta^t(X)) = 0$$

Proof. Follows from definitions. **QED**

Roughly speaking, this means that an asymptotic pseudotrajectory of Φ is a point of $C^0(\mathbf{R}_+, M)$ whose foward trajectory under Θ is attracted by $\mathbf{S}_\Phi$. We also have the following result.

Theorem 7.2. *Let X: $\mathbf{R}_+ \to M$ be a continuous function whose image has compact closure in M. Then X is an asymptotic pseudotrajectory of Φ if and only if X is uniformly continuous and every limit point*[6] *of $\{\Theta^t(X)\}$ is in $\mathbf{S}_\Phi$ (i.e., a fixed point of $\hat{\Phi}$).*

Proof. Let K denote the closure of $\{X(t)$: $t \geqslant 0\}$. Let $\varepsilon > 0$. By continuity of the flow and compactness of K, there exists $a > 0$ such that $d(\Phi_s(x), x) < \varepsilon/2$ for all $|s| \leqslant a$ uniformly in $x \in K$. Therefore there exists $T > 0$ such that $d(\Phi_s(X(t), X(t)) < \varepsilon/2$ for all $t > T$, $|s| \leqslant a$.

If now X is an asymptotic pseudotrajectory of Φ, T can be chosen large enough such that $d(\Phi_s(X(t), X(t+s)) < \varepsilon/2$ for all $t > T$, $|s| \leqslant a$. It follows that $d(X(t+s), X(t)) < \varepsilon$ for all $t > T$, $|s| \leqslant a$. This proves uniform continuity of X. On the other hand, the above discussion shows that any limit point of $\{\Theta^t(X)\}$ is a fixed point of $\hat{\Phi}$.

Conversely, if $\{X(t)$: $t \geqslant 0\}$ is relatively compact and X is uniformly continuous, $\{\Theta^t(X)\}$ is relatively compact and equicontinuous. Hence by the Ascoli theorem, $\{\Theta^t(X)\}$ is relatively compact in $C^0(\mathbf{R}, M)$. Therefore, $\lim_{t \to \infty} d(\Theta^t(X), \hat{\Phi}(\Theta^t(X)) = 0$. **QED**

The following corollary illustrates the use of Theorem 7.2.

Corollary 7.3. *Consider the asymptotically autonomous system (1) with limit Eq. (2), but instead of assuming f and g Lipschitz, assume only that the vector field g is continuous with unique integral curves and f is continuous. Let X be a bounded solution to (1). Then X is an asymptotic pseudotrajectory of the flow induced by g.*

[6] By a limit point of $\{\Theta^t(X)\}$, we mean the limit in $C^0(\mathbf{R}_+, M)$ of a convergent sequence $\Theta^{t_k}(X)$, $t_k \to \infty$.

Proof. It is easy to see that X is uniformly continuous (in fact $X(\cdot)$ is Lipschitz because $t \mapsto f(X(t), t)$ is bounded). On the other hand, a simple computation shows that

$$\Theta^t(X) = L_g(\Theta^t(X)) + A_t$$

where $L_g\colon C^0(\mathbf{R}, \mathbf{R}^n) \to C^0(\mathbf{R}, \mathbf{R}^n)$ is the continuous function defined as

$$L_g(X)(s) = X(0) + \int_0^s g(X(u))\, du$$

and

$$A_t(s) = \int_t^{t+s} [f(X(u), u) - g(X(u))]\, du$$

Hence $\lim_{t \to \infty} A_t = 0$.

Let X^* denote a limit point of $\{\Theta^t(X)\}$. Then

$$X^* = L_g(X^*)$$

By uniqueness of integral curves, this implies

$$X^* = \Phi(X^*)$$

Therefore Theorem 7.2 shows that X is an asymptotic pseudotrajectory of Φ. QED

8. LIMIT SETS OF ASYMPTOTIC PSEUDOTRAJECTORIES

Chain Recurrence

We first briefly review the notion of chain recurrence. For the general theory we refer the reader to Bowen (1975), Conley (1978), and Akin (1993).

Let Φ be a flow or semiflow on the metric space (M, d). Let $\delta > 0$, $T > 0$. A *(δ, T)-pseudo-orbit* from $a \in M$ to $b \in M$ is a finite sequence of partial trajectories

$$\{\Phi_t(y_i)\colon 0 \leqslant t \leqslant t_i\}; \qquad i = 0, \ldots, k-1; \qquad t_i \geqslant T$$

such that

$$d(y_0, a) < \delta$$

$$d\Phi_{t_j}(y_j), y_{j+1}) < \delta, \qquad j = 0,\dots, k-1$$

$$y_k = b$$

We write $a \rightsquigarrow b$ if for every $\delta > 0$, $T > 0$, there exists a (δ, T)*-pseudo-orbit* from a to b. If $a \rightsquigarrow a$, then a is a *chain recurrent* point. If every point of M is chain recurrent, then Φ is a chain recurrent semiflow (or flow).

If $a \rightsquigarrow b$ for all $a, b \in M$, we say the flow Φ is *chain transitive*. When M is compact this is equivalent to chain recurrence plus connectedness of M.

If M is compact, chain recurrence of Φ is equivalent to the condition that there are no proper attractors.

We denote by $R(\Phi)$ the set of chain recurrent points for Φ. This is a closed invariant set which contains the nonwandering set of Φ.

Let $\Lambda \subset M$ be a nonempty invariant set. Φ is called *chain recurrent* on Λ if every point $p \in \Lambda$ is a chain recurrent point for $\Phi \mid \Lambda$, the restriction of Φ to Λ. In other words, $\Lambda = R(\Phi \mid \Lambda)$.

Conley (1978) proved that a flow Φ is chain recurrent on $R(\Phi)$ if M is compact and that Φ is chain recurrent on any compact alpha or omega limit set for Φ. Mischaikov *et al.* (1995) show that the same holds for a semiflow (with also follows from the proof of 8.2(i) below).

A compact invariant set on which Φ is chain recurrent (or chain transitive) is called an *internally chain recurrent* (or *internally chain transitive*) set.

Lemma 8.1. *If an internally chain transitive compact set K meets the basin of an attractor A, it is contained in A.*

Proof. By compactness, $K \cap A$ is nonempty, hence an attractor for the $\Phi \mid K$. Since $\Phi \mid K$ has no proper attractors, being chain transitive, it follows that $K \subset A$. **QED**

Limit Sets

Here we prove Theorems 0.1, 0.2, and 0.3. Recall that the limit set of an asymptotic pseudotrajectory X: $\mathbf{R}_+ \to M$ is

$$L\{X\} = \bigcap_{t \geq 0} \overline{X([t, \infty))}$$

Theorem 8.2

(i) Let X be a precompact asymptotic pseudotrajectory of Φ. Then $L\{X\}$ is connected, compact, and internally chain transitive.

(ii) Let $L \subset M$ be a connected, compact internally chain transitive set, and assume M is locally path connected. Then there exists an asymptotic pseudotrajectory X such that $L\{X\} = L$.

Proof. Since $\{X(t)\colon t \geqslant 0\}$ is relatively compact, Theorem 7.2 shows that $\{\Theta^t(X)\colon t \in \mathbb{R}\}$ is relatively compact in $C^0(\mathbf{R}, M)$ and $\lim_{t\to\infty} d(\Theta^t(X), \mathrm{S}_\Phi) = 0$. Therefore the omega limit set of X for Θ, denoted by $\omega_\Theta(X)$, is a nonempty compact connected subset of S_Φ, and the restriction of Θ to $\omega_\Theta(X)$ is chain recurrent.

The homeomorphism $H\colon M \to \mathrm{S}_\Phi$, defined by $H(x)(t) = \Phi_t(x)$, maps $L\{X\}$ onto $\omega_\Theta(X)$, and conjugates $\Theta \mid \mathrm{S}_\Phi$ and Φ:

$$(\Theta^t \mid \mathrm{S}_\Phi) \circ H = H \circ \Phi_t$$

where $t \geqslant 0$ for a semiflow Φ, and $t \in \mathbf{R}$ for a flow. Since chain recurrence is defined in terms of the maps Φ_t, $t \geqslant 0$, assertion (i) follows.

In order to prove (ii) we use compactness of L, and the hypothesis that M is locally path connected and compact to obtain a family of paths

$$\{J(x, y)\colon [0, 1] \to M\colon (x, y) \in L \times L\}$$

such that

$$J(x, y)\colon 0 \mapsto x, \qquad 1 \mapsto y$$

and the diameter of the image of $J(x, y)$ goes to zero with $d(x, y)$.

Fix $T > 1$. Let $\{x_n\}_{n\in\mathrm{N}}$ be a sequence of points dense in L. Since L is internally chain transitive, there exists a $(2^{-n}, T)$ pseudo-orbit from x_n to x_{n+1} in L. Putting together these pseudo-orbits we get a sequence of points $\{y_i\}_{i\in\mathrm{N}}$, $y_i \in L$, and a sequence of times $\{t_i\}_{i\in\mathrm{N}}$, $2T \geqslant t_i > T$, such that

(a) $\{y_i\}_{i\in\mathrm{N}}$ is dense in L.

(b) $d(\Phi_{t_{i+1}} y_i, y_{i+1}) \to 0$ as $i \to \infty$.

Define numbers

$$\tau_n = \sum_{i=0}^{n} t_i$$

$$s_i = \left(1 - \frac{1}{2^{i+1}}\right) t_{i+1}$$

and denote by J_i: $[0, 1] \to M$ the path $J(\Phi_{s_i} y_i, y_{i+1})$. Define Z: $\mathbf{R}_+ \to M$ as follows:

$$
\begin{aligned}
Z(t) &= \Phi_t y_0 && \text{if } 0 \leqslant t \leqslant \tau_0 \\
&= \Phi_{t-\tau_i} y_i && \text{if } \tau_i \leqslant t \leqslant \tau_i + s_i \\
&= J_i((t-\tau_i - s_i)/s_i) && \text{if } \tau_i + s_i \leqslant t \leqslant \tau_{i+1}
\end{aligned}
$$

Straightforward estimates based on (a), (b) show that Z is an asymptotic pseudotrajectory whose limit set is L. **QED**

Remark 8.3. Mischaikow *et al.* (1995) proved that any compact internally chain recurrent set for an autonomous differential equation $dx/dt = g(x)$ is the limit set of a solution of some asymptotically autonomous equation $dx/dt = h(t, x)$ having the autonomous equation as its limit.

This result can be compared with a theorem of Bowen (1975) and Franke and Selgrade (1976). Bowen proved that any compact chain transitive set for a homeomorphism can be seen as an omega limit set for an extension of the homeomorphism to some a larger space. This was extended to flows by Franke and Selgrade. As a consequence of Theorem 8.2 we get the following embedding theorem.

Corollary 8.4. *Let M be locally path connected and Φ a flow (respectively, semiflow) on M. There exists a flow Ψ on a metric space M, a closed Ψ-invariant set $\mathrm{S}_\Phi \subset M$ and a homeomorphism H: $M \to S_\Phi$ such that*

(a) *S_Φ attracts all forward trajectories of Ψ.*

(b) *$H \circ \Phi_t = \Psi_t \mid (\mathrm{S}_\Phi \circ H)$ for all $t \in \mathbf{R}$ (respectively, for all $t \in \mathbf{R}_+$).*

(c) *H maps compact connected internally Φ-chain recurrent sets onto compact omega limit sets for Ψ.*

Proof. We use the notations introduced in Section 7.

Let $\hat{M} = \{X \in C^0(\mathbf{R}, M);\ \lim_{t \to \infty} d(\Theta^t(X), \mathrm{S}_\Phi) = 0\}$ with the metric induced by $C^0(\mathbf{R}, M)$. Since $\hat{M}$ is Θ-invariant, we can define the flow $\Psi = \Theta \mid M$. The homeomorphism H is given by $H(x)(t) = \Phi_t(x)$. Let $L \subset M$ be a compact connected internally Φ-chain recurrent set. Theorem 8.2(ii) shows that $H(L)$ is the omega limit set for Ψ of a point $X \in \hat{M}$. **QED**

9. EXPONENTIALLY ASYMPTOTIC PSEUDOTRAJECTORIES

As the asymptotic behavior of an asymptotic pseudotrajectory X for a flow Φ is evidently related to the asymptotic behavior of Φ, it is natural to ask, under what conditions is $X(t)$ asymptotic with some trajectory $\Phi_t(x)$ —that is, when does there exist x such that that $\lim_{t\to\infty} d(X(t), \Phi_t(x)) = 0$? When this happens, the limit set $L\{X\}$ is not an arbitrary internally chain transitive set of Φ, but an omega limit set. In many cases omega limit sets are much more restricted than internally chain recurrent sets. For example, suppose that each trajectory of Φ converges to some stationary point. Then every omega limit set is a stationary point; but there may also be uncountably many homoclinic loops or, more generally, cyclic obit chains, which are necessarily internally chain transitive sets.

A sufficient condition that $L\{X\}$ be an omega limit set of Φ is that it admit a hyperbolic structure (Guckenheimer and Holmes, 1983): If Φ is a smooth flow and L is a connected hyperbolic set which is internally chain transitive, then it is known that L is an omega limit set (see Franke and Selgrade, 1976). Therefore as a consequence of Theorem 8.2(i) we have the following.

Theorem 9.1. *Let Φ be a smooth flow and X a relatively compact asymptotic pseudotrajectory fo Φ. If $L\{X\}$ is hyperbolic, then $L\{X\}$ is omega limit set.*

It is usually very difficult, however, to prove that $L\{X\}$ is hyperbolic. In the following section we give some practical conditions ensuring that an asymptotic pseudotrajectory is asymptotic with an actual trajectory at an asymptotic rate, and the obit of the latter is unique. These conditions are stated in terms of the expansion constant of a flow, which we now review.

Expansion Rates

Let (M, d) be a complete metric space. $B(x, \rho)$ or $B_\rho(x)$ denotes the closed ball centered at x with radius ρ, and $N_\delta(J)$ denotes the closed δ-neighborhood of a subset $J \subset M$.

Let $h: M_0 \to M$ be a map defined in an open subset $M_0 \subset M$. For any subset $J \subset M_0$ the *expansion constant* $EC(h, J)$ of h at J is the largest $\mu \geqslant 0$ having the following property: For any $0 \leqslant \nu < \mu$ there exists $\rho^* > 0$ such that

$$B(h(x), \nu\rho) \subset h(B(x, \rho))$$

provided $0 \leqslant \rho \leqslant \rho^*$.

If J is positively invariant [i.e., $h(J) \subset J$], it is easy to see that

$$EC(h^k, J) \geqslant EC(h, J)^k$$

Suppose M is either a Banach space or a Riemannian manifold and h is a C^1 diffeomorphisms. If $K \subset M$ is compact, then it is nt difficult to show that

$$EC(h, K) = \min_{x \in K} \|Dh(x)^{-1}\|^{-1} \tag{25}$$

where $Dh(x)$ denotes the derivative of h at $x \in M$ and $\|A\|$ denotes the operator norm of a linear operator A. In this case continuity of Dh implies

$$\begin{aligned} EC(h, K) &= \lim_{\delta \to 0+} EC(h, N_\delta(K)) \\ &= \sup_{\delta > 0} EC(h, N_\delta(K)) \end{aligned}$$

If Φ is a semiflow in the metric space M, the *expansion rate* of Φ at the compact positively invariant set K is defined as

$$\varepsilon(\Phi, K) = \sup_{t > 0} EC(\Phi_t, K)^{1/t}$$

If Φ admits a global attractor Λ, we define the *expansion rate* of Φ as

$$\varepsilon(\Phi) = \varepsilon(\Phi, \Lambda) = \varepsilon(\Phi, L)$$

where L denotes the closure of all alpha and omega limit points for Φ.

When Φ is a smooth flow,

$$D\Phi_t(x)^{-1} = D\Phi_{-t}(\Phi_t(x))$$

and according to (25) we therefore have

$$\varepsilon(\Phi, K) = \sup_{t > 0} [\min_{x \in K} \|D\Phi_{-t}(\Phi_t(x))\|^{-1/t}]$$

Several properties of the expansion rate are given by Hirsch (1994). We review some of these properties.

(i) Assume Φ is the flow generated by a smooth vector field g on $\mathbf{R}^n$. Let $\beta(g, x)$ denote the smallest eigenvalue of the symmetric matrix $\frac{1}{2}[Dg(x) + Dg(x)^T]$, where T denotes the transpose of a matrix. Let $\beta = \beta(g, K) = \min_{x \in K} \beta(g, x)$. Then

$$\varepsilon(\Phi, K) \geqslant e^\beta$$

(ii) Another estimate is obtained by noticing that

$$|\beta| \leqslant M = M(G, K) = \max_{x \in K} \|DG(x)\|$$

(using the Schwarz inequality) so that $\beta \geqslant -M$. This yields the estimate

$$\mathscr{E}(\Phi, K) \geqslant e^{-M(G,K)} \tag{26}$$

(iii) If $L \subset K$ is a compact set containing all alpha and omega limit points of K, then

$$\varepsilon(\Phi, K) = \varepsilon(\Phi, L)$$

If follows that if all forward and backward trajectories in K are attracted to hyperbolic periodic orbits (possibly stationary), and the real parts of the Floquet exponents of these periodic orbits are all $\geqslant \gamma \in \mathbf{R}$, then

$$\mathscr{E}(\Phi, K) \geqslant e^{\gamma} \tag{27}$$

(iv) If K is contained in the basin of attraction of an attractor Λ, then

$$\varepsilon(\Phi, K) = \varepsilon(\Phi, \Lambda)$$

Shadowing

We return to the map $h\colon M_0 \to M$ considered above. Let $\{y_n\}_{n \in \mathbf{N}}$ be a sequence in M.

Let $0 \leqslant \lambda < 1$. A sequence $\{y_k\}_{k \in \mathbf{N}}$ in $\mathscr{X}$ is called a *λ-pseudo-orbit* for h if

$$\mathscr{R}_{k \to \infty}\, d(y_{k+1}, h(y_k)) \leqslant \lambda$$

A point $u \in M_0$ is said to *λ-shadow* $\{y_n\}$ provided

$$\mathscr{R}_{k \to \infty}\, d(h^k(u), y_{k+m}) \leqslant \lambda$$

for some $m \geqslant 0$.

The following exponential shadowing theorem, from which all the results of this section will be deduced, is a slight variation on Theorem 3.2 of Hirsch (1994), replacing a compactness condition with uniformity assumptions. The proof is similar.

Theorem 9.2. *Let* $M_1 \subset M_0 \subset M$ *be closed sets that* $\operatorname{dist}(M_1, M \backslash M_0) > 0$. *Assume* $EC(h, M_0) = \mu > 0$. *Let* $\{y_k\}_{k \in \mathbf{N}}$ *be* λ*-pseudo-orbit for* h *in* M_1 *such that*

$$0 < \lambda < \min(1, \mu)$$

Then

(a) *There exists* $x \in M_0$ *which* λ*-shadows* $\{y_n\}_{n \in \mathbf{N}}$.

(b) *If* $x, z \in M_0$ *both* λ*-shadow* $\{y_n\}$, *then* x *and* z *belong to the same orbit of* h.

(c) *If all the* $\{y_k\}$ *lie in a closed invariant set* $J \subset M_0$, *then* $x \in J$.

Now let Φ denote a semiflow (or flow) on the complete metric space M. For Theorem 9.3 we assume that for any $\tau > 0$ and any ball $B_\rho(x) \subset M$, there exists a common Lipschitz constant for the maps $\Phi_t \mid B_\rho(x)$, $0 \leqslant t \leqslant \tau$. This holds for C^1 flows and for the solution flows of standard semilinear parabolic evolution equations.

Let $X: \mathbf{R}_+ \to M$ be a precompact asymptotic pseudotrajectory. We say $u \in M$, or its orbit, *λ-shadows X* if

$$\mathscr{R}_{t \to \infty} d(\Phi_t u, X(t + t_0)) \leqslant \lambda$$

for some $t_0 \geqslant 0$.

Define the *asymptotic error rate of X* to be

$$e(X) = \sup_{T>0} [\limsup_{s \to \infty} d(X(s+Y), \Phi_T X(s))^{1/s}]$$

If $e(X) \leqslant \lambda < \infty$, we call X a *λ-pseudotrajectory.*

Theorem 9.3. *Let* $X: \mathbf{R}^+ \to M$ *be a precompact pseudotrajectory for* Φ. *Let* $K \subset M$ *be a compact invariant set containing the limit set* $L\{X\}$. *Assume*

$$e(X) < \lambda = \min\{1, \varepsilon(\Phi, K)\} \tag{28}$$

Then there exists a unique Φ*-orbit which* λ*-shadows* X.

Proof. By the second inequality in (28), we fix $T > 0$ and a neighborhood $M_0 \subset M$ of K such that

$$EC(\Phi_T, M_0) > \lambda^T \tag{29}$$

Let $M_1 \subset \operatorname{Int} M_0$ be a compact neighborhood of K. Since $L\{X\} \subset K$, we can choose $s_0 > 0$ such that $X(t) \in M_1$ for all $t \geqslant s_0$.

By the first inequality in (28), there exists $s_1 > s_0$ such that if $s \geqslant s_1$, then

$$d(X(s+T), \Phi_t X(s)) < \lambda^s \tag{30}$$

Now set $h = \Phi_T$, $y_k = X(s_0 + kT)$. Then for all $k \in \mathbf{N}$,

$$\begin{aligned} d(y_{k+1}, h(y_k)) &= d(X([s_0 + kT] + T), \Phi_T X(s_0 + kT)) \\ &< \lambda^{s_0 + kT} \end{aligned}$$

by (30). Therefore

$$\begin{aligned} \limsup_{k \to \infty} d(y_{k+1}, h(y_k))^{1/k} &\leqslant \lambda^T \\ &< EC(\Phi_T, M_0) \end{aligned}$$

by (29).

Therefore Theorem 9.2 implies that there exists which λ^T shadows $\{u_k\}_{k \in \mathbf{N}}$ for the map Φ_T. The uniform Lipschitz constants for the maps $\Phi_t | V$, $0 \leqslant t \leqslant T$ and continuity of Φ can now be used to show that X is λ-shadowed by u. The uniqueness statement follows from Theorem 9.2(b). **QED**

10. PROOF OF PROPOSITIONS 3.7 AND 3.8

We assume given a complete probability space $(\Omega, \mathscr{F}, \mathscr{P})$. We let E denote the mathematical expectation defined by $\mathbf{E}(X) = \int_\Omega X(\omega)\, dP(\omega)$ for any integrable function X. We let L^p, $p > 0$, denote the space of $\mathbf{R}^n$ valued random variables X such that $\mathbf{E}(\|X\|^p) < \infty$. L^p is a Banach space for the norm $\|X\|_p = \mathbf{E}(\|X\|^p)^{1/p}$.

Proof of Proposition 3.7

We have

$$\begin{aligned} \Phi_h(X(t)) - X(t+h) = &\int_0^h [g(\Phi_s(X(t))) - g(X(t+s))]\, ds \\ &+ \int_t^{t+h} \sigma(X(s), s)\, dB_s \end{aligned}$$

Let L be the Lipschitz constant of g. From Gronwall's inequality we get

$$\sup_{0 \leqslant h \leqslant T} \|\Phi_h(X(t)) - X(t+h)\| \leqslant e^{LT} \sup_{0 \leqslant h \leqslant T} \|Z_{t+h} - Z_t\| \tag{31}$$

where Z is the continuous time martingale:

$$Z_t = \int_0^t \sigma(X(s), s)\, dB$$

To estimate $\|Z_{t+h} - Z_t\|$ it suffices to estimate each component of the vector $Z_{t+h} - Z_t$. Therefore if suffices to consider the case where $n = 1$. We shall use the following inequality (Friedman, 1975, p. 93, Theorem 7.5).

Lemma 10.1 (Exponential Martingale Inequality). *Let $\alpha > 0$, $\beta > 0$. Then*

$$P\left\{\sup_{0 \leqslant h \leqslant T} \left[\int_t^{t+h} \sigma(X(s), s)\, dB_s - \alpha/2 \int_t^{t+h} \sigma(X(s), s)^2\, ds\right] > \beta\right\} \leqslant e^{-\alpha\beta}$$

Using this inequality, we get

$$\begin{aligned} &P\{\sup_{0 \leqslant h \leqslant T} [Z_{t+h} - Z_t] > \beta\} \\ &\quad \leqslant P\left\{\sup_{0 \leqslant h \leqslant T} \left[\int_t^{t+h} \sigma(X(s), s)\, dB_s - \alpha/2 \int_t^{t+h} \sigma(X(s), s)^2\, ds\right]\right. \\ &\quad \left. > \beta - \alpha/2 \int_t^{t+T} \varepsilon^2(s)\, ds\right\} \\ &\quad \leqslant \exp\left(-\alpha\beta + \alpha^2/2 \int_t^{t+T} \varepsilon^2(s)\, ds)\right) \end{aligned}$$

Choosing $\alpha = \beta/(\int_t^{t+T} \varepsilon^2(s)\, ds)$ gives

$$P\{\sup_{0 \leqslant h \leqslant T} [Z_{t+h} - Z_t] > \beta\} \leqslant \exp\left(-\frac{\beta^2}{\int_t^{t+T} \varepsilon^2(s)\, ds}\right)$$

Therefore

$$\begin{aligned} &P\{\sup_{0 \leqslant h \leqslant T} |Z_{t+h} - Z_t| > \beta\} \\ &\quad \leqslant 2 \exp\left(-\frac{\beta^2}{\int_t^{t+T} \varepsilon^2(s)\, ds}\right) \leqslant 2 \exp\left(-\frac{\beta^2}{T\varepsilon^2(t+T)}\right) \end{aligned} \tag{32}$$

Since by assumption $\sum_{k \geqslant 0} \exp(-\beta^2/T\varepsilon^2(kT)) < \infty$, the Borel–Cantelli lemma implies

$$\lim_{k \to \infty} \sup_{0 \leqslant h \leqslant T} |Z_{kT+h} - Z_{kT}| = 0$$

Let $k \in \mathbf{N}$, $kT \leqslant t < (k+1)T$, and $0 \leqslant h \leqslant T$. We have $Z_{t+h} - Z_r = (Z_{t+h} - Z_{kT}) - (Z_t - Z_{kT})$. Then

$$\sup_{0 \leqslant h \leqslant T} |Z_{t+h} - Z_t| \leqslant 2 \sup_{0 \leqslant h \leqslant 2T} |Z_{kT+h} - Z_{kT}|$$

Thus,

$$\lim_{t \to \infty} (\sup_{0 \leqslant h \leqslant T} |Z_{t+h} - Z_t|) = 0$$

It then follows from (31) that X is an asymptotic pseudotrajectory of Φ. QED

Proof of Proposition 3.8

Let Λ be a global attractor for g. Let $r > l$ be such that $\Lambda \subset B(0, r)$, β a smooth bump function which is 1 on $B(0, r+1)$ and zero outside $B(0, r+2)$.

First, by multiplying g by β we assume that g is zero outside $B(0, r+2)$. We let Φ denote the flow of $g(x)$. Since every Φ-trajectory spends all but a finite amount of time in any neighborhood of $\Lambda \cup (\mathbf{R}^n - B(0, r+2))$, we have

$$\varepsilon(\Phi, \mathbf{R}^n) \geqslant \min(\varepsilon(\Phi, \mathbf{R}^n - B(0, r+2)), \varepsilon(\Phi, \Lambda)) = \min(1, \mu)$$

As Φ is assumed to be the identity outside $B(0, r+2)$, the image by Φ_t of any L^p random variable is in L^p. Therefore Φ induces a flow Φ^p on L^p defined as $\Phi_t^p(X) = \Phi_t \circ X$.

Lemma 10.2. *Let $\eta > 0$. There exists $T > 0$ such that*

$$EC(\Phi_T^p, L^p)^{1/T} \geqslant \min(1, \mu) - \eta$$

Proof. As $\varepsilon(\Phi, \mathbf{R}^n) = \sup_{t>0} EC(\Phi_t, \mathbf{R}^n)^{1/t}$, there exists $T > 0$ such that

$$EC(\Phi_T, \mathbf{R}^n)^{1/T} \geqslant \min(1, \mu) - \eta$$

Thus

$$\min_{x \in \mathbf{R}^n} \|D\Psi_{-T}(x)\|^{-1/T} \geqslant \min(1, \mu) - \eta \tag{33}$$

From (33) we deduce that

$$\|\Phi_{-T}(x) - \Phi_{-T}(y)\| \leqslant \frac{1}{(\min(1, \mu) - \eta)^T} \|x - y\|$$

for all $x, y \in \mathbf{R}^n$. Therefore by integration we get

$$\|\Phi^p_{-T}(X) - \Phi^p_{-T}(Y)\|_p \leqslant \frac{1}{(\min(1,\mu)-\eta)^T} \|X - Y\|_p$$

and consequently,

$$EC(\Phi^p_T, L^p)^{1/T} \geqslant \min(1,\mu) \qquad \text{QED}$$

Let now X denote a solution of (23) with initial condition $X(0) = x_0$. Set $\rho = \mathscr{R}_{t \to \infty} \varepsilon(t)$. By assumption (c) of Proposition 4.2, there exists $\eta > 0$ such that $\rho < \min(1,\mu) - \eta$. Choose T as in Lemma 1.2.

From inequality (31) and Doob's inequality for continuous martingales (Friedman, 1975, p. 87, Corollary 6.4), we obtain

$$\mathbf{E}(\sup_{0 \leqslant h \leqslant T} \|\Phi_h(X(t)) - X(t+h)\|^2) \leqslant e^{LT} C \mathbf{E} \left(\int_t^{t+T} \varepsilon(s)^2 \, ds \right) \tag{34}$$

$$\leqslant e^{LT} c T \varepsilon^2(t+T) \tag{35}$$

for some constant $c > 0$.

Define $y_n = X((nT)$ and $h = \Phi^2_T$. Using Lemma 10.2 and inequality (35), we obtain

$$\mathscr{R}_{n \to \infty} \|(y_{n+1} - h(y_n)\|_2 \leqslant \rho^T \leqslant (\min(1,\mu) - \eta)^T \leqslant EC(h, L^2) \tag{36}$$

It then follows from Theorem 9.2 that there exists a random variabe $Y \in L^2$ such that

$$R_{n \to \infty} \|y_n - h^n(Y)\|_2 \leqslant \min(1,\mu)^T$$

Therefore, using the continuity of the flow and inequality (35), we obtain

$$R_{t \to \infty} \|X(t) - \Phi_t(Y)\|_2 \leqslant \min(1,\mu)$$

This proves assertion (ii) of Proposition 4.2. As $R_{t \to \infty} \|\Phi_t(Y) - X(t)\|_2 < 1$, the Bore–Cantelli lemma implies the almost sure convergence of $\Phi_t(Y) - X(t)$ to zero, proving assertion (i).

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