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Publication Date

1995-07-01

Peer reviewed

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Working Paper No. 95-238

Strict Monotonicity in Comparative Statics

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July 1995

Key words: strict monotonicity, comparative statics, single crossing property
JEL Classification: C60, C61, D11, H21

Abstract

This note provides sufficient conditions to draw strict monotone comparative statics conclusions in optimization problems. These results extend the lattice-theoretic results of Milgrom and Shannon (1994) by imposing a stronger differential version of the single crossing property and arguing from first-order conditions. We illustrate the importance of these results through examples involving holdup problems and optimal tax problems.

Thanks to Eric Emch, Steve Goldman, and Paul Milgrom for their comments. This work has been supported in part by the National Science Foundation under grant SBR-9321022.

1 Introduction

A great deal of recent research has focused on monotone methods for comparative statics (see e.g., Topkis (1978), Milgrom and Shannon (1994), Milgrom (1994) and Shannon (1995)). This literature has provided two main services. First, it clarifies what lies at the root of comparative statics results, in part by forgoing the analysis of first-order conditions and emphasizing that differentiability and convexity are not critical features of comparative statics analysis. Secondly, these monotonicity theorems are stated in a general form which abstracts from the specifics of any particular model. Such “off-the-shelf” theorems allow researchers to simply check a few conditions instead of repeating similar arguments to prove versions of these results specific to each application.

These theorems are not quite as strong as some might like, however. For example, Milgrom and Shannon (1994) show that in an optimization problem with choice variables x and exogenous parameters t , the set of maximizers will be nondecreasing in the parameters and constraints if the objective function is quasisupermodular in the choice variables x and satisfies the single crossing property in x and t . This result does not guarantee that a maximizer x^* associated with a low parameter value t^* will be smaller than a maximizer x' associated with a high parameter value t' . Indeed, x^* may actually be larger than x' . Although this cannot happen if there is a unique maximizer for one of these parameter values, or if the objective function satisfies the stronger strict single crossing property, even under these stronger conditions we cannot conclude that x' must be larger than x^* , only that it cannot be smaller.

In some problems, however, it is valuable to know that a maximizer is increasing, rather than simply nondecreasing, in response to some change in an exogenous parameter. For example, consider the following holdup problem between a buyer and a seller. A great deal of research on vertical integration and the nature of organizations has been spawned by the idea that the seller will invest inefficiently if he anticipates that he will have to negotiate with the buyer over the share of the surplus from trade he will receive. This inefficiency result is essentially an application of a strict comparative statics theorem. To see this, note that when the seller's share is 1, he invests efficiently; if his investment is increasing in his

share, then when he is held up by the buyer, i.e., when his share is less than 1, he will underinvest relative to the efficient investment level. On the other hand, if we knew merely that his investment were nondecreasing in his share, then his investment might be efficient even in the presence of holdups. Similarly, in problems involving incomplete information, such as the optimal tax problem we discuss in section 3, knowing only that agents' actions are nondecreasing in their type means that pooling and separating behavior are both possible, while knowing that their actions are increasing in their type means that types will always separate.

The theorems in this note provide conditions on the objective function and constraints in an optimization problem under which one can draw strict comparative statics conclusions. They extend the order-theoretic conclusions of Milgrom-Shannon (1994) by combining lattice-theoretic methods with differential techniques. In order to derive strict comparative statics results, we use a stronger differential version of the single crossing property and, in contrast with much of the recent work on monotone methods for comparative statics in optimization problems, we find it necessary to argue from first-order conditions.¹ Although versions of our one- and two-dimensional results exist in particular applications (see, e.g., Edlin and Reichelstein (1995, Proposition 1), Meyer and Zwiebel (1994)), these papers do not provide theorems which can be used in a variety of applications. We hope to follow the existing comparative statics literature by providing usable general theorems.

The note proceeds as follows. In section 2, we give sufficient conditions for strict monotonicity in one-dimensional choice problems, and discuss this holdup problem in more detail. Section 3 presents sufficient conditions for strict monotonicity in two-dimensional choice problems together with an application to optimal tax problems with incomplete information, and in section 4 we extend these results to general multi-dimensional choice problems.

¹In their analysis of fixed point problems, Milgrom and Roberts (1994) use similar arguments to derive strict comparative statics results.

2 One-Dimensional Choice Problems

This section develops sufficient conditions to draw strict comparative statics conclusions in one-dimensional choice problems. As a motivating example, consider the holdup problem, as popularized by Williamson (1975) and Klein, Crawford, and Alchian (1978).² In this problem, two agents, the buyer and the seller, anticipate trading some good or service. The seller can make a relationship-specific investment to enhance the value of the trade. After this investment has been made, however, the seller must bargain with the buyer over the terms of trade. The concern is that the seller will make inefficient investment decisions due to the prospect of losing some of the social return from his investment in the bargaining process.

To formalize the holdup problem, let $t \in [0, 1]$ denote the share of the surplus from trade that the seller attains in bargaining ex post. Let $g(x)$ be an increasing function that gives the expected value of the surplus if the seller invests the amount x . The seller's problem is then

$$\begin{aligned} \max \quad & tg(x) - x \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

Note that $f(x, t) \equiv tg(x) - x$ has **strictly increasing differences** in $(x; t)$, i.e., for all $x' > x$, $f(x', t) - f(x, t)$ is increasing in t .³ Thus by Topkis (Topkis (1978, Theorem 6.3)), every selection from $\arg \max_{x \geq 0} tg(x) - x$ is non-decreasing in t (see also Rogerson (1984)). Since investment is efficient when $t = 1$, this result implies that the seller will not overinvest when $t < 1$. However, this line of analysis does not show that investment is actually inefficient when $t < 1$. On the other hand, a large body of literature on the economics of organizations has been prompted by worries that holdups, which occur when $t < 1$, *do* cause inefficient investment. What would justify such a conclusion? We could be sure that holdups would cause inefficient investment, and in fact underinvestment, if we knew that the seller's optimal

²See also Grout (1984), Rogerson (1984), Hart and Moore (1988) and Edlin and Reichelstein (1995).

³In general, we will be careful to distinguish between increasing and non-decreasing: when we say that a function $q(z)$ is increasing, we mean that if $z' > z$, then $q(z') > q(z)$. Using this definition, the terminology "strictly increasing differences" may be redundant, but it is standard in the literature.

investment level were strictly increasing in his share of surplus t .

To derive this stronger conclusion, instead of assuming that g is increasing, assume that g is differentiable and that $g'(x) > 0$ for all x . Then if a positive investment level x^* satisfies the first-order necessary conditions for the seller's problem given a sharing rule t^* , so that $t^*g'(x^*) = 1$, this investment level x^* will not satisfy the corresponding first-order condition for any other sharing rule $t \neq t^*$. Thus if efficiency requires positive investment, then holdups will cause inefficient investment, and in particular, underinvestment.

The following theorem generalizes this observation, and illustrates the role of first-order conditions in drawing conclusions about strict comparative statics.

Strict Monotonicity Theorem 1. *Let $S \subset \mathbb{R}$, $f(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $x^* \in \arg \max_{x \in S} f(x, t^*)$ and $x' \in \arg \max_{x \in S} f(x, t')$. Suppose that f is C^1 , $\frac{\partial f}{\partial x}$ is increasing in t , and $x^* \in \text{int } S$. Then $x' > x^*$ if $t' > t^*$, and $x' < x^*$ if $t' < t^*$.*

Proof: We will prove the result in the case $t' > t^*$; the other case is analogous. Since f satisfies the strict single crossing property, the Monotone Selection Theorem of Milgrom-Shannon (1994) implies that if $t' > t^*$, then $x' \geq x^*$. To see that this inequality is strict, observe first that since $x^* \in \text{int } S$ and $x^* \in \arg \max_{x \in S} f(x, t^*)$,

$$\frac{\partial f}{\partial x}(x^*, t^*) = 0.$$

Since $\frac{\partial f}{\partial x}(x^*, t)$ is increasing in t , $\frac{\partial f}{\partial x}(x^*, t') > 0$, which implies that $x^* \notin \arg \max_{x \in S} f(x, t')$. On the other hand, $x' \in \arg \max_{x \in S} f(x, t')$, so x' is different from x^* . Thus $x' > x^*$. ■

An immediate corollary of this result is that in such an optimization problem, every interior selection from the set of maximizers is strictly increasing.⁴

Corollary 1. *Let $S \subset \mathbb{R}$ and $f(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$. Suppose that f is C^1 and that $\frac{\partial f}{\partial x}$ is increasing in t . Then every interior selection $x^*(t)$ from $\arg \max_{x \in S} f(x, t)$ is strictly increasing.*

These results extend the previous weak monotonicity results, which were derived using order-theoretic techniques. To do so, we impose differentiability and a stronger differential

⁴Also note that the monotonicity results of Theorem 1 and Corollary 1 remain valid provided only that $\frac{\partial f}{\partial x}(x^*, \cdot)$ is increasing in t .

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version of strictly increasing differences. This allows us to argue from the first-order conditions that the maximizer cannot remain unchanged when the exogenous parameters change. To see that this condition is in fact stronger than strictly increasing differences, consider a situation like that depicted in Figure 1.

Figure 1: — — — $f(x, t')$ — $f(x, t^*)$

Here, there are two values of the parameter, t^* and t' , where $t^* < t'$. The objective function $f(x, t)$ is a function such that $\frac{\partial f}{\partial x}(x, t^*) < \frac{\partial f}{\partial x}(x, t')$ for all $x \neq x^*$, and $\frac{\partial f}{\partial x}(x^*, t^*) = \frac{\partial f}{\partial x}(x^*, t') = 0$.⁵ Moreover, x^* is the unique maximizer of this function for both parameter values, even though f has strictly increasing differences in x and t . To see that f has strictly

⁵As an example of such a function, let $g(x) = 120 - 12(x - 1)^2 + 8(x - 1)^3$. Then define

$$f(x, t^*) = \begin{cases} g(x) + 2(x - 1)^4, & \text{if } x \leq 1; \\ g(x) + (x - 1)^4 & \text{if } x > 1 \end{cases}$$

and

$$f(x, t') = \begin{cases} g(x) + (x - 1)^4, & \text{if } x \leq 1; \\ g(x) + 2(x - 1)^4 & \text{if } x > 1. \end{cases}$$

increasing differences in x and t , note that for any $x' > x$,

$$\begin{aligned} f(x', t^*) - f(x, t^*) &= \int_x^{x'} \frac{\partial f}{\partial x}(x, t^*) dx \\ &< \int_x^{x'} \frac{\partial f}{\partial x}(x, t') dx \\ &= f(x', t') - f(x, t'), \end{aligned}$$

where the inequality follows since $\frac{\partial f}{\partial x}(x, t^*) < \frac{\partial f}{\partial x}(x, t')$ for all $x \neq x^*$. However, the function clearly does not satisfy the stronger condition that $\frac{\partial f}{\partial x}(x, t)$ is increasing in t for all x .

3 Two-Dimensional Choice Problems

This section provides strict monotone comparative statics results for two-dimensional choice problems. These results are useful in a wide class of problems in information economics in which agents are divided into types according to their preferences over two choice variables. In such problems, comparative statics conclusions can be drawn when the marginal rate of substitution is monotonically related to the agents' type, that is, when preferences over these choice variables satisfy the Spence-Mirrlees single crossing property (see e.g., Mirrlees (1971), Spence (1973), and Stiglitz (1987)). Milgrom and Shannon (1994) develop general comparative statics results along these lines by showing that the single crossing property generalizes the Spence-Mirrlees single crossing property when the choice space has been given a particular order.

More formally, let X denote the choice set, which is partially ordered by the order relation $\geq$. For any two elements x and y in the choice set X , let $x \vee y$ denote the least upper bound, or **join**, of x and y in X , if it exists, and let $x \wedge y$ denote the greatest lower bound, or **meet** of x and y in X , if it exists. The set X is called a **lattice** if for any such pair of elements x and y in X , $x \vee y \in X$ and $x \wedge y \in X$, and a subset S of a lattice X is a **sublattice** if S is closed under the operations meet and join. In many applications, the choice set X is $\mathbb{R}^n$ and the partial order is the usual componentwise order, so that the join of two elements is the pointwise maximum, and the meet is the pointwise minimum. In this section we will focus on problems in which the choice set is $\mathbb{R}^2$, and use the **lexicographic order**, denoted

$\geq_\ell$, in which $(x'_1, x'_2) \geq_\ell (x_1, x_2)$ if $x'_1 > x_1$, or $x'_1 = x_1$ and $x'_2 \geq x_2$.

When the choice space is one-dimensional, the main property driving monotone comparative statics conclusions is the single crossing property. If T is a partially ordered set representing the parameters in the problem, then $f : X \times T \rightarrow \mathbb{R}$ satisfies the **single crossing property** in $(x; t)$ if for all $x' > x^*$, whenever $f(x', t^*) \geq f(x^*, t^*)$, then $f(x', t') \geq f(x^*, t')$ for all $t' > t^*$; and whenever $f(x', t^*) > f(x^*, t^*)$, then $f(x', t') > f(x^*, t')$ for all $t' > t^*$. The function satisfies the **strict single crossing property** in $(x; t)$ if for all $x' > x^*$, whenever $f(x', t^*) \geq f(x^*, t^*)$ then $f(x', t') > f(x^*, t')$ for all $t' > t^*$.

When the choice set is multi-dimensional, monotone comparative statics results require some additional restrictions on the interactions between the choice variables, such as quasisupermodularity. A function $f : X \rightarrow \mathbb{R}$ is **quasisupermodular** if for every $x, y \in X$, $f(x) \geq (>) f(x \wedge y) \Rightarrow f(x \vee y) \geq (>) f(y)$. Using these concepts, Milgrom and Shannon (1994) show that the set of solutions to the optimization problem $\max_{x \in S} f(x, t)$ is nondecreasing in (t, S) if and only if f is quasisupermodular in x and satisfies the single crossing property in $(x; t)$; of course for this to make sense we must first explicitly define an order on the subsets of a set. The order Milgrom and Shannon use is the **strong set order**, denoted $\leq_s$. Under this order, a set Y is larger than a set Z , denoted $Y \geq_s Z$, if for each $y \in Y$ and $z \in Z$, then $y \vee z \in Y$ and $y \wedge z \in Z$. Using this order, to say that the set $M(t, S) \equiv \arg \max_{x \in S} f(x, t)$ is nondecreasing in (t, S) is to say that if $(t', S') \geq (t^*, S^*)$, so that $t' \geq t^*$ and $S' \geq_s S^*$, then $M(t', S') \geq_s M(t^*, S^*)$. As noted in the introduction, this conclusion does not rule out the possibility that there is a decreasing selection from the set of solutions, i.e., that for some $(t', S') > (t^*, S^*)$, there exist $x' \in M(t', S')$ and $x^* \in M(t^*, S^*)$ such that $x' < x^*$, although every selection from the set of maximizers will be nondecreasing if this optimization problem has a unique solution for all parameter values, or if the objective function is quasisupermodular in x and satisfies the stronger strict single crossing property in $(x; t)$. Even these stronger assumptions, however, do not guarantee that any selection from the set of maximizers is increasing. The following proposition combines differential techniques with lattice-theoretic techniques in order to draw strict comparative statics conclusions in such problems.

Strict Monotonicity Theorem 2. Let $f : \mathbb{R}^2 \times T \rightarrow \mathbb{R}$, $G : \mathbb{R}^2 \rightarrow \mathbb{R}$, $x^* \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t^*)$ and $x' \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$. Suppose that f and G are C^1 , $\frac{\partial f}{\partial x_2} \neq 0$ and has the same sign for every (x_1, x_2, t) , $-\frac{\partial f}{\partial x_1} / \frac{\partial f}{\partial x_2}$ is increasing in t , and x^* is interior, i.e., $x_1^*, x_2^* > 0$. Then $x'_1 > x_1^*$ if $t' > t^*$ and $x'_1 < x_1^*$ if $t' < t^*$.

Proof: Again we prove only the case where $t' > t^*$. By Theorem 3 of Milgrom-Shannon (1994), $f(x, t)$ satisfies the strict single crossing property in $(x; t)$ with respect to the lexicographic order. Since the lexicographic order is a complete order, f is quasisupermodular in x . For the same reason, $S = \{x \in \mathbb{R}^2 : x \geq 0 \text{ and } G(x) = 0\}$ is a sublattice of $\mathbb{R}^2$ under this order. The Monotone Selection Theorem of Milgrom-Shannon (1994) then implies that if $t' > t^*$, then $x' \geq_\ell x^*$, and in particular $x'_1 \geq x_1^*$.⁶ To see that this inequality is strict, observe first that since x^* is strictly positive, $\frac{\partial f}{\partial x_2} \neq 0$, and $x^* \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t^*)$,

$$\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}(x^*, t^*) = \frac{\frac{\partial G}{\partial x_1}}{\frac{\partial G}{\partial x_2}}(x^*).$$

Since $\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}(x^*, t)$ is increasing in t ,

$$\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}(x^*, t') > \frac{\frac{\partial G}{\partial x_1}}{\frac{\partial G}{\partial x_2}}(x^*),$$

which implies that $x^* \notin \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$. On the other hand, $x' \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$, so x' is different from x^* , and thus $x' > x^*$. Furthermore, we claim that in fact $x'_1 > x_1^*$. If instead $x'_1 = x_1^*$, then $x'_2 > x_2^*$. This is impossible since the constraint set does not depend on t , and the rankings of types t^* and t' are identical over (x_1^*, x_2^*) and (x_1^*, x'_2) when $x_1^* = x'_1$ because $\frac{\partial f}{\partial x_2}$ has the same sign for all t . ■

Note that if in addition to the other assumptions of the theorem, $\frac{\partial f}{\partial x_1} > 0$ and $\frac{\partial f}{\partial x_2} < 0$ for all $(x, t) \in \mathbb{R}^2 \times T$, then we can also conclude that $x'_2 > x_2^*$ if $t' > t^*$ and $x'_2 < x_2^*$ if $t' < t^*$.

⁶The Monotone Selection Theorem applies to optimization problems of the form $\max_{x \in S(t)} f(x, t)$, and requires that f be quasisupermodular in x , satisfy the strict single crossing property in $(x; t)$, and that the constraint $S(t)$ be nondecreasing in the strong set order. When the constraint set is independent of t , as it is here, we must show that $S \leq_s S$; this is true if and only if S is a sublattice.

Again an immediate corollary of this result is that in such an optimization problem, if $x^*(t)$ is an interior selection from the set of maximizers, then $x_1^*(t)$ is strictly increasing.⁷

Corollary 2. *Let $f : \mathbb{R}^2 \times T \rightarrow \mathbb{R}$ and $G : \mathbb{R}^2 \rightarrow \mathbb{R}$. Suppose that f and G are C^1 , $\frac{\partial f}{\partial x_2} \neq 0$ and has the same sign for all $(x_1, x_2, t) \in \mathbb{R}^2 \times T$, and that $-\frac{\partial f}{\partial x_1} / \frac{\partial f}{\partial x_2}$ is increasing in t . If $x^*(t)$ is a selection from $\arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t)$ such that $x^*(t)$ is interior for each t , then $x_1^*(t)$ is strictly increasing.*

Again, note that if in addition to the other assumptions of the theorem, $\frac{\partial f}{\partial x_1} > 0$ and $\frac{\partial f}{\partial x_2} < 0$ for all $(x, t) \in \mathbb{R}^2 \times T$, then we can also conclude that $x_2^*(t)$ is strictly increasing for any interior selection $x^*(t)$ from the set of maximizers.

As in the previous section, these results extend the analogous weak monotonicity results by imposing differentiability and a stronger differential version of the strict single crossing property, which is just the strict Spence-Mirrlees condition. Although Milgrom and Shannon (1994, Theorem 3) claim that these conditions are equivalent, the strict Spence-Mirrlees condition is actually stronger, and this additional strength is what allows us to draw stronger comparative statics conclusions.

To see why the strict Spence-Mirrlees condition is not equivalent to the strict single crossing property, consider a situation as in the previous section in which there are two parameter values, t^* and t' , where $t^* < t'$, and let $f(x_1, t)$ be a function such that $\frac{\partial f}{\partial x_1}(x_1, t^*) < \frac{\partial f}{\partial x_1}(x_1, t')$ for all $x_1 \neq x_1^*$, and $\frac{\partial f}{\partial x_1}(x_1^*, t^*) = \frac{\partial f}{\partial x_1}(x_1^*, t') = 0$. Then the function $g(x, t) \equiv f(x_1, t) + x_2$ does not satisfy the strict Spence-Mirrlees condition, because

$$\begin{aligned} \frac{\frac{\partial g}{\partial x_1}(x_1^*, x_2, t^*)}{\frac{\partial g}{\partial x_2}(x_1^*, x_2, t^*)} &= \frac{\partial f}{\partial x_1}(x_1^*, t^*) \\ &= \frac{\partial f}{\partial x_1}(x_1^*, t') = \frac{\frac{\partial g}{\partial x_1}(x_1^*, x_2, t')}{\frac{\partial g}{\partial x_2}(x_1^*, x_2, t')}. \end{aligned}$$

Nonetheless, g does satisfy the strict single crossing property with respect to the lexicographic order on the choice space. To see this, suppose that $x' >_\ell x$ and $g(x', t^*) \geq g(x, t^*)$. Then either $x'_1 > x_1$, or $x'_1 = x_1$ and $x'_2 > x_2$. If $x'_1 = x_1$, then

$$g(x', t') = f(x'_1, t') + x'_2 = f(x_1, t') + x'_2$$

⁷As in section 2, these results also hold provided just that $-\frac{\partial f}{\partial x_1} / \frac{\partial f}{\partial x_2}(x^*, \cdot)$ is increasing in t .

$$> f(x_1, t') + x_2 = g(x, t').$$

If $x'_1 > x_1$, then since $g(x', t^*) \geq g(x, t^*)$, we know that

$$\begin{aligned} x_2 - x'_2 &\leq f(x'_1, t^*) - f(x_1, t^*) \\ &< f(x'_1, t') - f(x_1, t') \end{aligned}$$

since $f(x_1, t)$ has strictly increasing differences. Thus $g(x', t') > g(x, t')$ in either case, which shows that g satisfies the strict single crossing property. The fact that g does not satisfy the strict Spence-Mirrlees condition means we cannot conclude from the first-order conditions that the maximizer will be increasing, and in fact that is not true here since the optimal value of x_1 is the same for both parameter values.

As an example of these results, consider the optimal tax problem described in Stiglitz (1987). Taxpayers are divided into two types, those of high ability whose pre-tax wage rate is w^H , and those of low ability, whose pre-tax wage rate is w^L , where $w^H > w^L$. Both types have the same endowment of labor/leisure, $\bar{\ell}$, and the same preferences over leisure (ℓ) and consumption (c), represented by the C^1 utility function $u(c, \ell)$, where $\frac{\partial u}{\partial c} > 0$ and $\frac{\partial u}{\partial \ell} > 0$ for all (c, ℓ) . The different wage rates, however, induce different preferences over consumption and pre-tax income $y^i \equiv w^i(\bar{\ell} - \ell_i)$, where $i = H, L$. Let these preferences be denoted $v(c, y, w) \equiv u(c, \bar{\ell} - \frac{y}{w})$. The government cannot observe workers' types but can observe their pre-tax income, and sets a tax schedule $t(y)$. Workers then solve the problem

$$\begin{aligned} \max \quad & v(c, y, w) \\ \text{s.t.} \quad & c = y - t(y). \end{aligned}$$

If consumption is a normal good, then $-\frac{\partial v}{\partial c} / \frac{\partial v}{\partial y}$ is increasing in w . Moreover, $\frac{\partial v}{\partial y} = -\frac{1}{w} \frac{\partial u}{\partial \ell}$, which is negative for all (c, y, w) . So if the government sets a C^1 tax schedule and one type consumes a positive amount, then the high type will always consume more than the low type, and have a higher pre-tax income. Therefore in equilibrium, if there are only two types of workers, then the workers reveal their abilities rather than mimicking each other. With more than two types of workers, as Stiglitz (1987) points out, it may be optimal to have partially pooling equilibria. By this argument, however, partial pooling cannot be accomplished with

a differentiable tax structure; to achieve partial pooling the government must impose a tax schedule with kinks.

4 Multi-Dimensional Choice Problems

This section extends the strict monotonicity results of the previous sections to multi-dimensional choice problems. In the previous sections, what allowed us to move from a weak monotonicity conclusion to a strict monotonicity conclusion was that the marginal rate of substitution changed when the type t changed, which guarantees that an interior solution to the optimization problem for one type will not be a solution for any other type. Since the single crossing property is a multi-dimensional analogue of the Spence-Mirrlees condition, a similar condition requiring that the type t have an effect on marginal rates of substitution together with the single crossing property should allow us to move from a weak monotonicity conclusion to a strict monotonicity conclusion in multi-dimensional choice problems. This observation motivates the following definition.

Definition. Let $f : X \times T \rightarrow \mathbb{R}$ be a C^1 function. Then f **depends critically** on the variable t if for every $x \in X$ and for every $t', t'' \in T$ such that $t' \neq t''$, there exist i, j such that

$$\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}(x, t') \neq \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}(x, t'').$$

If the objective function depends critically on the exogenous parameters t , then the same sort of argument used in the previous sections should allow us to conclude that an interior solution for one parameter value cannot be a solution for any other parameter value. As long as the underlying order on the choice set is one in which the equivalence classes contain a single element, this conclusion means that the solution must be strictly increasing in the parameter. Then if X is a partially ordered set with the partial order $\geq$, and the equivalence classes under $\geq$ are singletons, we will say that $\geq$ is a **proper partial order**, and that X is **properly partially ordered**. For example, $\mathbb{R}^n$ with the usual order satisfies this condition, as does $\mathbb{R}^2$ with the lexicographic order.⁸ Using these notions, the results of the previous

⁸A preference order on $\mathbb{R}^n$ is an example of a partial order which is not necessarily proper.

sections can be extended to an arbitrary choice set.

Theorem 3. Let X be a properly partially ordered lattice, T be a partially ordered set, $f : X \times T \rightarrow \mathbb{R}$ and $G : X \rightarrow \mathbb{R}$. Let $x^* \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t^*)$ and $x' \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$, where x^* is interior. Suppose that f and G are C^1 , f satisfies the strict single crossing property in (x, t) , is quasisupermodular in x , and depends critically on t , and that $S \equiv \{x \in X : G(x) = 0, x \geq 0\}$ is a sublattice. Then $x' > x^*$ if $t' > t^*$ and $x' < x^*$ if $t' < t^*$.

Proof: Again, consider the case where $t' > t^*$. By the Monotone Selection Theorem of Milgrom-Shannon (1994), $x' \geq x^*$. To see that this inequality is strict, observe first that since x^* is strictly positive, f depends critically on t and $x^* \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t^*)$, there exist

i and j such that

$$\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}(x^*, t^*) = \frac{\frac{\partial G}{\partial x_i}}{\frac{\partial G}{\partial x_j}}(x^*),$$

and such that

$$\frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}(x^*, t') > \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}(x^*, t^*).$$

This implies that $x^* \notin \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$. On the other hand, $x' \in \arg \max_{\substack{G(x)=0 \\ x \geq 0}} f(x, t')$, so x' is different from x^* . Since $\geq$ is a proper partial order, $x' \neq x^*$, which implies that $x' > x^*$. ■

So far, we have concentrated on optimization problems in which the objective function depends on some exogenous parameters but the constraint set is independent of these parameters. In a number of important problems in economics, however, such as utility maximization and expenditure minimization, the constraint set depends on parameters fundamental to the model. If the constraint depends critically on the exogenous parameters and the objective function is independent of these parameters, then symmetric arguments to those we used before allow us to extend weak monotonicity results to strict monotonicity results.

Theorem 4. Let X be a properly partially ordered lattice, T be a partially ordered set, $f : X \rightarrow \mathbb{R}$ and $G : X \times T \rightarrow \mathbb{R}$. Let $x^* \in \arg \max_{\substack{G(x, t^*)=0 \\ x \geq 0}} f(x)$ and $x' \in \arg \max_{\substack{G(x, t')=0 \\ x \geq 0}} f(x)$, where x^* is interior. Suppose that f and G are C^1 , f is quasisupermodular in x , and G depends critically on t . Moreover, suppose that $S(t) \equiv \{x \in X : G(x, t) = 0, x \geq 0\}$ is

sublattice-valued and nondecreasing in the strong set order. Then $x' > x^*$ if $t' > t^*$ and $x' < x^*$ if $t' < t^*$.

Proof: The argument is exactly the same as the argument proving Theorem 3, with the roles of f and G interchanged. ■

Finally, the same ideas can be used to give a strict monotonicity result for optimization problems in which both the objective function and the constraint set depend on these exogenous parameters. The proof is left to the reader.

Theorem 5. *Let $f : X \times T \rightarrow \mathbb{R}$, where X is a properly partially ordered set, f satisfies the strict single crossing property in $(x; t)$ and is quasisupermodular in X . Suppose f and G are C^1 , and that $\forall x \in X$ there exists i, j such that $\frac{\partial f}{\partial x_i} / \frac{\partial f}{\partial x_j}$ is increasing in t and $\frac{\partial G}{\partial x_i} / \frac{\partial G}{\partial x_j}$ is decreasing in t . Moreover, suppose that $S(t) = \{x \in X : G(x, t) = 0\}$ is sublattice-valued and nondecreasing in the strong set order. Then every interior selection $x^*(t)$ from $\arg \max_{x \in S(t)} f(x)$ is strictly increasing.*

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