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Journal
Educational Studies in Mathematics, 99(3)

ISSN
0013-1954

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Publication Date
2018-11-01

DOI
10.1007/s10649-018-9845-0

Educational Studies in Mathematics
https://doi.org/10.1007/s10649-018-9845-0

Rhythmic movement as a tacit enactment goal mobilizes the emergence of mathematical structures

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Published online: 03 September 2018

Abstract

This article concerns the purpose, function, and mechanisms of students' rhythmic behaviors as they solve embodied-interaction problems, specifically problems that require assimilating quantitative information structures embedded into the environment. Analyzing multimodal data of one student tackling a bimanual interaction design for proportion, we observed the (1) evolution of coordinated movements with stable temporal–spatial qualities; (2) breakdown of this proto-rhythmic form when it failed to generalize across the problem space; (3) utilization of available resources to obtain greater specificity by way of measuring spatial spans of movements; (4) determination of an arithmetic pattern governing the sequence of spatial spans; and (5) creation of a meta-rhythmic form that reconciles continuous movement with the arithmetic pattern. The latter reconciliation selectively retired, modified, and recombined features of her previous form. Rhythmic enactment, even where it is not functionally imperative, appears to constitute a tacit adaptation goal. Its breakdown reveals latent phenomenal properties of the environment, creating opportunities for quantitative reasoning, ultimately supporting the learning of curricular content.

Keywords Embodiment · Proportion · Rhythm · Technology · Unit of measurement

1 Attending to physical movement as a characteristic of an embodiment approach to research on mathematics education

The objective of this paper is to contribute to a growing body of educational research scholarship that has been promoting the theorization of mathematics learning as a process of guided reflection on situated physical enactment (Bamberger & diSessa, 2003; Kelton & Ma, 2018; Nemirovsky & Ferrara, 2009; Radford, Arzarello, Edwards, & Sabena, 2017; Roth &

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Thom, 2009; Simmt & Kieren, 2015; Sinclair, Chorney, & Rodney, 2016). Specifically, some scholars informed by the embodiment turn in the cognitive sciences have been evaluating the thesis that individual comprehension of mathematical concepts emerges through discursive objectification of tacit sensorimotor adaptations to the social enactment of cultural practice (Abrahamson, 2009, 2014; Abrahamson & Trninic, 2015; Nemirovsky, Kelton, & Rhodehamel, 2013; Radford, 2009). Often operating in the design-based approach to educational research, these scholars of enactive mathematics have created task-based activities that offer students opportunities to (a) develop new goal-oriented sensorimotor schemes for moving effectively within the constraints of a learning environment; (b) reflect on their solutions in qualitative, pre-symbolic register; and ultimately (c) refine and consolidate the solutions via appropriating normative frames of reference from the target discipline (Abrahamson & Lindgren, 2014).

One consequence of a research focus on students' physical movement is enhancing our capacity to appreciate and investigate any rhythmic qualities these may bear. Radford (2015), who approaches mathematical thinking as "fully material, corporeal, embodied, and sensuous phenomenon" (p. 82), implicates rhythm as a central organizing principle of thinking. Radford calls for further research on the evolution of rhythm components in mathematical thinking. Here, we are interested in particular in the emergence of rhythmic qualities and their iterative adaptation to emergent environmental constraints on effective movement. Drawing on constructivist, enactivist, and coordination-dynamics literature, we believe that this process, in which physical actions fall into regulated spatial–temporal forms, is largely tacit. We maintain that this tacit process is important for educational researchers to understand, because the process is pivotal in coordinating effective enactment in new interaction environments, such as those designed to foster conceptual change.

In order to demonstrate what we mean by emergent rhythmic qualities of students' enacted solutions to physical interaction problems as well as the pedagogical potential of these rhythmic movements and their research appeal more generally, the paper will consider empirical results from implementing a design for proportion that used the Mathematics Imagery Trainer (Abrahamson & Trninic, 2015). Discussing rhythmic qualities inherent in students' physical movements within this environment, we will theorize the micro-process by which tacit phenomenal features of sensorimotor interaction emerge for conscious reflection and elaboration that in turn lead to insight and codification relevant to mathematics learning.

Our interest in the rise of latent features of an environment into a child's consciousness, as she engages in solving a situated problem, suggests the seminal work of John Mason (1989, 2010) on the role of attentional shifts during mathematics learning. Indeed, in a sense, we are hoping to extend that general research orientation, which by and large has considered pattern-finding sensory perception of *static* visual displays, such as geometrical inscriptions, so as here to foreground pro-action sensori*motor* aspects of developing competence in handling interactive *dynamical* displays. Drawing also on the work of Roth and collaborators (e.g., Bautista & Roth, 2012), we thus examine movement forms students develop, perform, refine, and articulate in the course of participating in educational activities designed for learning mathematical content. We argue for the formative role of these emergent movement forms as creating opportunities for mobilizing students' proto-mathematical reasoning and learning. Thus, whereas other researchers have argued for students' generalization processes in interactive learning environments (e.g., Leung, Baccaglini-Frank, & Mariotti, 2013), our enactivist approach seeks to characterize these processes by revealing and foregrounding tacit coordinative aspects of these processes. In particular, we demonstrate the spontaneous incorporation of regulating temporal elements in students' explorative manipulation of interactive display elements. That is, we are interested in evaluating for any pedagogical affordances inherent in the emergence and adaptation of rhythmical qualities of physical performance that we observe in students' attempts to develop goal-oriented situated competence.

More broadly, our research program looks to orchestrate the acculturation of *enactive artifacts* (Abrahamson & Trninic, 2015), that is, domain-general movement forms of cultural–historical significance. For example, our research agenda includes an interest in fostering public adoption of an enactive artifact for proportional equivalence in the form of a bimanual conceptual gesture, that is, a conventional multimodal sign bearing and evoking generic dynamical embodied meanings of mathematical nomenclature.

The paper will now continue with a literature review of research on the tacit emergence of rhythmic structure in enactive mathematics learning (Section 2), followed by the case study (methods in Section 3, results in Section 4), and ending with implications for the research and design of mathematics learning (Section 5).

2 Rhythmic orientation to movement enactment as epigenetic inclination of the human neural architecture: implications for mathematics education

The phenomenon of rhythmic qualities in children's physical enactment of repetitive movement has recently been considered by a range of scholars with interest in both typical and atypical cognitive development and learning (Trninic & Saxe, 2017). The phylogenetically evolved tendency of the human cognitive architecture to enact physical movement in coordinated rhythmic structure bears developmental advantages (Kelso & Engstrom, 2006; Richmond & Zacks, 2017; Vandervert, 2016). For example, motor-action researchers Spencer, Semjen, Yang, and Ivry (2006) demonstrated the utility of rhythm in constructing and enacting a temporal event structure consisting of bimanual actions. Presumably, constructing and routinizing tightly encapsulated event structures bears pragmatic advantages by way of freeing cognitive resources during motor enactment.

Humans' epigenetic inclination to engage in regular spatial–temporal micro-routines in enacting cultural practice has drawn the attention of educational researchers with an interest in envisioning new pedagogical horizons. For example, rhythmic features of social activity in generating musical performance have been found to facilitate the learning of ratio, fractions, and proportion (Bamberger & diSessa, 2003). Abrahamson (2004) has implicated rhythmic discursive gesture as marking students' negotiation between situated enactment and normative mathematical forms. Bautista and Roth (2012), who studied grade 3 students classifying three-dimensional objects, documented the appearance of rhythmical hand movements apparently emerging from the students' dynamical haptic interaction with structural regularities in the material resources they were manipulating. They suggest that rhythm is both a resource and an outcome of engaging in geometry activities. In like vein, Sinclair et al. (2016) used rhythm as their focal analytic construct in investigating the mathematical activity of young children working with a tablet application designed for learning number. They implicate rhythmic actions as the embodied origin of cognitive structure, prior to planning and reflection.

Radford (2015) conceptualizes situated rhythmic dynamics as a constituting quality of mathematical thinking. Radford lists four aspects of rhythmic behaviors he observed in analyzing the implementation of an algebraic pattern-generating activity: *meter*, *rhythmic* *grouping*, *prolongation*, and *theme*. These spatial–temporal qualities of students' multimodal enactment emerged as the students considered and eventually constructed one inscribed shape, then another, then another.

The abovementioned spontaneous rhythmic qualities observed in empirical investigation of children's goal-oriented situated movement, we propose, can be seen as related to the cultural practices of measurement as defined by cognitive developmental psychology. Piaget, Inhelder, and Szeminska (1960) offer that "To measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of sub-division and change of position" (p. 3). When we imbue this logico–mathematical definition of measuring from Piaget et al. (1960) with the phenomenal parametrization of enacted rhythm from Radford (2015), we may speculate on the epigenesis of measuring operations as a cultural enhancement of rhythmic adaptation to contingencies of a learning environment. By this view, rhythmic enactment: (1) conserves the nature and size of the unit as it is extracted or ported (theme and prolongation) and (b) iterates the unit (rhythmic grouping and meter). This speculative epigenetic trajectory of rhythmic enactment bears implications for educational design.

As educational design researchers, we seek to both foster and understand student mathematization of movement while accomplishing situated tasks. As such, our data analysis was initially geared to characterize which temporal–spatial attributes of manual movement precipitate the emergence of mathematical structures. In the course of our analysis, a new research interest arose concerning learners' adaptive responses when encountering contexts where their rhythmic movement falls short of achieving the task objective, leading to pivotal insights bearing conceptual potential.

In our explorative study, we sought to contribute to the literature on the role of rhythmic movement in mathematics learning by investigating: (1) the emergence and adaptation of rhythmic movement through task-based interaction with instructional artifacts as well as (2) students' assimilation of quantitative frames of reference into their rhythmic enactment. We will demonstrate a case study of a student who at two milestone events of different phenomenological quality responded adaptively to experiences of enactment breakdown by modifying selective temporal–spatial attributes of her movement elements; in so doing, she subsumed an earlier set of local enactment patterns into a new global enactment pattern that led to articulating the design's targeted learning objective—proportional reasoning.

3 Method

The empirical context for this study was a design-based educational research project evaluating a new activity genre centered on an interactive technological device called the Mathematics Imagery Trainer for Proportion (MITp, Abrahamson & Trninic, 2015; see Fig. 1). K was an 11-year-old female student, one of 25 students participating voluntarily in a task-based semi-structured clinical interview (for details, see Rosen, Palatnik, & Abrahamson, 2018). The interview lasted in total 18 min, where, following a brief task introduction, K manipulated two virtual iconic images of (a) hot-air balloons (7 min), (b) cars (4 min), and (c) crosshair targets (7 min). The interview took place in our laboratory and was audio–video recorded for subsequent analysis.

We identified all events where the student expressed verbally new insight pertaining to an effective manipulation strategy. We then parsed the interview into episodes, which we

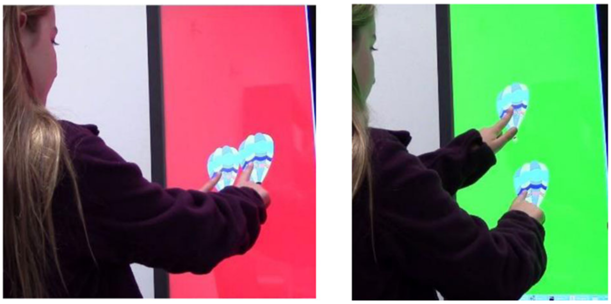

Fig. 1 The Mathematics Imagery Trainer for Proportion (MITp). The student manipulates two cursors along vertical axes, one by each hand. The task is to make the screen green and then keep it green while moving your hands. The screen will be green only when the heights of the two cursors above the screen base relate by a particular ratio unknown to the user (e.g., here 1:2). Otherwise, the screen is red

characterize as subtasks, running from each insight to the subsequent insight. Subtasks were further coded as *local* ("finding green" by placing both cursors at once at particular screen locations and leaving the hands there statically) or *global* ("keeping green" while sliding the cursors up and down the screen continuously).

We applied a grounded theory micro-genetic analysis methodology to our empirical data (e.g., Goldin, 2000), focusing on the student's range of physical actions and multimodal utterance around the available media (Nemirovsky & Ferrara, 2009) as well as on the task-effectiveness of her actions. First, we attended to the student's actions that preceded her articulation of a new rule for "making green." We searched in particular for patterns in the timing and sequencing of her hand movements through space (Sinclair et al., 2016). A rudimentary choreographic notation system emerged, through our iterative analytic process, for marking the most frequently used movements (see Table 1).

Second, we analyzed how K responded to our recurring question, "How would you explain your strategy for finding green to another person?" We thus probed for an association between two facets of K's behavior: (1) apparent transformation in her explorative manipulation strategy (captured by means of the hand-movement notation rubric, see Table 1) and (2) apparent changes we observed in the succession of her multimodal discursive responses to the recurring interview probe. These transformations, we sensed, could be marking a sequence in the adaptive emergence of K's spatial–temporal micro-routine for enacting the task solution. Our analysis drew also on our earlier findings regarding K's case (Rosen et al., 2018), where we investigated for the emergence of new perceptual structures mediating her interaction with the technology.

Table 1 Notation rubric for coding the most frequently observed bimanual activity

Movement	Notation
Vertical, simultaneous, oppositional bimanual movement (e.g., the left hand moves down while the right hand moves up)	↓↑
Horizontal, simultaneous, converging bimanual movement (hands moving toward each other)	→←
Horizontal, simultaneous, diverging bimanual movement (hands moving away from each other)	←→
Vertical, simultaneous, co-oriented bimanual movement with the (e.g.) right hand going up twice as fast as the left hand	↑↑x2
Placing both fingers onto the screen	●●
Fixing the position of the hands after having moved both of them	□□

We wish to emphasize that prior to the data analysis phase of this study we had not anticipated engaging in issues of rhythm. As such, we note also that our interview protocol had not been constructed with an eye on evoking rhythmic behaviors let alone unit-based mathematization of these behaviors. Thus, we cannot at this point depict K's case as generalizing to the set of our study participants. Rather, K is perhaps a case of where this educational design genre could evolve and where we or others could adapt the activity sequence so as to encourage these observed behaviors.

4 Breakdown in rhythmic enactment elicits latent phenomenal features of a problem space: the case of K adapting simple bimanual movements to the emergent constraints of a proportionality-based interactive environment

In this section, we present results from the qualitative micro-analysis of K's hot-air balloons episode. The episode comprises K's attempts at performing a total of seven alternating local and global subtasks. As we will explain, K began with heuristic explorative hand movements that gradually took form as a locally effective, stable, and iterated enactment bearing situated temporal–spatial structure of coordinated bimanual movements (*evolution* of proto-rhythmic forms). As the episode ensues, though, this early, locally effective enactive form proves inadequate across the parametric span of the entire problem space (*breakdown* of proto-rhythmic movement form), so that K must respond to the environment's unexpected feedback by accommodating the form. In so doing, K engaged in reflective discourse with the tutor, who encouraged K to articulate how she had had to adapt her form; this intervention, in turn, led to K introduce a measuring unit and devise an arithmetic pattern governing the sequence of spatial spans (*quantification* of rhythmic movement form), which she then incorporated into a new scheme for enacting the adapted form (*creation of meta-rhythmic form* that reconciles continuous movement with the arithmetic pattern). K's multimodal actions, reactions, and utterances in addressing emergent problems in the course of attempting to satisfy the task thus made manifest her logical and increasingly quantitative reasoning.

The structure of the report, below, takes into account Mason's (2002) notion of *accounting of* (noticing of what happened), and *accounting for* (theorizing why events occur). The first two subsections present an *account of* K's episode as enfolding along the following formative events: (Section 4.1) epigenesis—*evolution and breakdown* of proto-rhythmic movement form vis-à-vis emergent problems of enactment and (Section 4.2) cultural intervention—*quantification* of movement forms and consequent *creation* of a meta-rhythmic form that reconciles continuous movement with an arithmetic pattern. A final subsection (Section 4.3) continuous movement revisited as discreetly discrete: hidden *rhythmic qualities of enacting proportion* offers summative analysis of these formative events and provides an *account for* the role rhythmic qualities of enacting proportion play in student's mathematical thinking.

4.1 Epigenesis: evolution and breakdown of rhythmic movement structures vis-à-vis emergent problems of enactment

Local subtask The tutor asked K to "find green" at any location on the screen. Very soon (01:00) she did so. K placed her fingers on the screen, laterally aligned but not contiguous (●●); she moved the fingers horizontally toward each other (→←), which does not change the feedback in this task; she moved the fingers vertically apart (↓↑) until achieving green; and then she held the fingers stationary (□□) at green. The tutor asked K to “find green” also at the top and then at the bottom of the screen. Attempting to find green at each screen location, K repeated one movement combination from the previous enactment (↑↓) while slightly altering another (→←). Asked to explain what she noticed about her hands’ positions at these three locations (middle, top, and bottom), K articulated her rule for obtaining green as “the balloons were roughly one above the other” (02:05).

Global subtask In the next subtask (02:26), the interviewer asked K to “keep green” on the screen while moving her hands from the bottom of the screen to the top of the screen. Note that this task a priori negates a direct utilization of the “↓↑” bimanual form that K had established to solve the succession of local tasks, because now her hands must necessarily move in a co-oriented form, “↑↑”. Thus, where oppositional movements might be used contextually, correctively, they cannot constitute the base form. As such, to leverage her earlier discovery, K would need to decompose that form into its elements, modify one of them (“↓↑” becomes “↑↑”), and then recompose the elements. Alternatively, she must ignore the ill-adapted oppositional movement element and attend only to its end-result spatial property (the changing extent of the interval between her hands). All this, only if K had noticed the behavior of the spatial property, which she had not. As we will see, K developed for the global task a completely different scheme.

K first found green as previously (●●; →←; ↓↑; □□). She then moved her hands slowly in a fixed “one above the other” formation, thus keeping a constant interval between the left- and right index fingers (↑↑), which resulted in a red screen. (Recall that the application measures for a goal ratio, and so keeping a fixed interval between the hands while raising them, rather than increasing the interval, will inevitably violate the goal ratio, so that the screen will turn from green to red.) As she raised her hands up along the screen, K responded to the color feedback by correcting the (relative) location of her fingers so as to re-achieve a green screen (↓↑). Yet though K thus effectively was gradually increasing the interval between her hands as she raised them, which is patently clear to any observer of these data, K nevertheless explained that the hands should be in the “same position, same distance from each other” (see Reinholz, Trninic, Howison, & Abrahamson, 2010, for similar findings with this design, where students’ verbal report contradicts their actions, so that one might say that the body is at the vanguard of the student’s mathematical discovery).

Local subtask In her next attempt (03:15), K slid her fingers on the screen, enacting a more complex movement pattern (see Fig. 2).

K repeated this pattern at different screen locations: bottom, middle, and top. Her movements were slow (approximately 5 s for the whole pattern of movements at each location) but deliberate. Asked to articulate her current rule, K turned to the screen and, gesturing toward it, said: “Down here [screen bottom] my hands were really close, and then up here [screen middle] they were a little apart, and then up here [screen top] they were really apart.” We thus see that K believes that the relative *vertical* positioning of the two hands matters in this particular activity. Curiously, she nevertheless perseverates in enacting *lateral* displacement of the hands (→←;←→), perhaps because this contextually redundant movement never bears any negative consequences.

Global subtask When K was again asked to “maintain green” on the screen while moving her hands continuously (04:30), she placed both hands near the bottom of the screen and moved them both simultaneously up, vertically, with RH moving twice as fast as LH (↑↑x2). In so doing, K maintained green almost without performing any corrections. She then repeated this action, at the same pace, voicing over

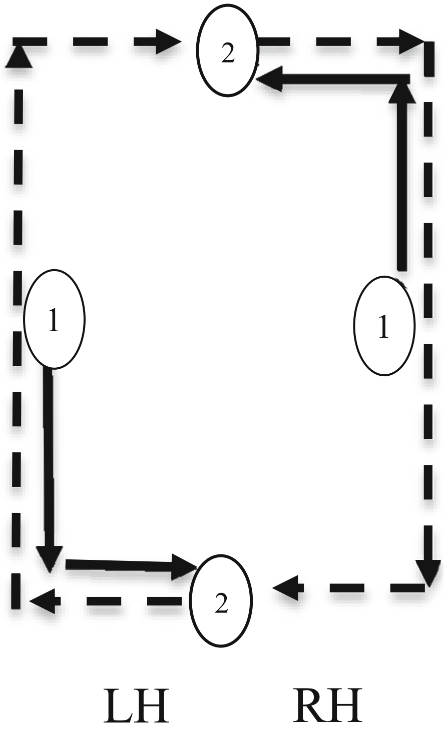

Fig. 2 K's exploratory movement pattern: ●●; ↓↑; →←; and then, in reverse: ←→; ↑↓; →←

> Well, I think one hand moves faster than the other as it goes up. My hands keep gradually farther apart [sic]. It stays green, whole way… So then eventually it's farther away from the other hand than it was at the beginning.

(For other cases of students shifting between mathematically complementary visualizations of the goal bimanual movement, see Abrahamson, Lee, Negrete, & Gutiérrez, 2014)

To summarize, K has developed two different sensorimotor schemes as her solutions to the tasks she was assigned (Fig. 3). For the local subtask of finding green at distinct screen locations, she determined the "the higher—the bigger" scheme governing the covariation of the interval's vertical location and spatial extension: "really close," "a little apart," "really apart" (see Fig. 3, local task 3), and for the global subtask of keeping green while moving her hands continuously up, she determined the "one hand faster than the other" scheme (see Fig. 3, global task 4). Note that both schemes were articulated in qualitative register.

4.2 Cultural intervention: quantification of movement forms and consequent creation of a meta-rhythmic form that reconciles continuous movement with an arithmetic pattern

In the hope of eliciting from K greater specificity on her movement rules, the interviewer chose to focus on the interval between the cursors: "Ok. Do you have any sense of…kind of…how this [the interval] is changing? How much it is changing, how much faster it is moving?"

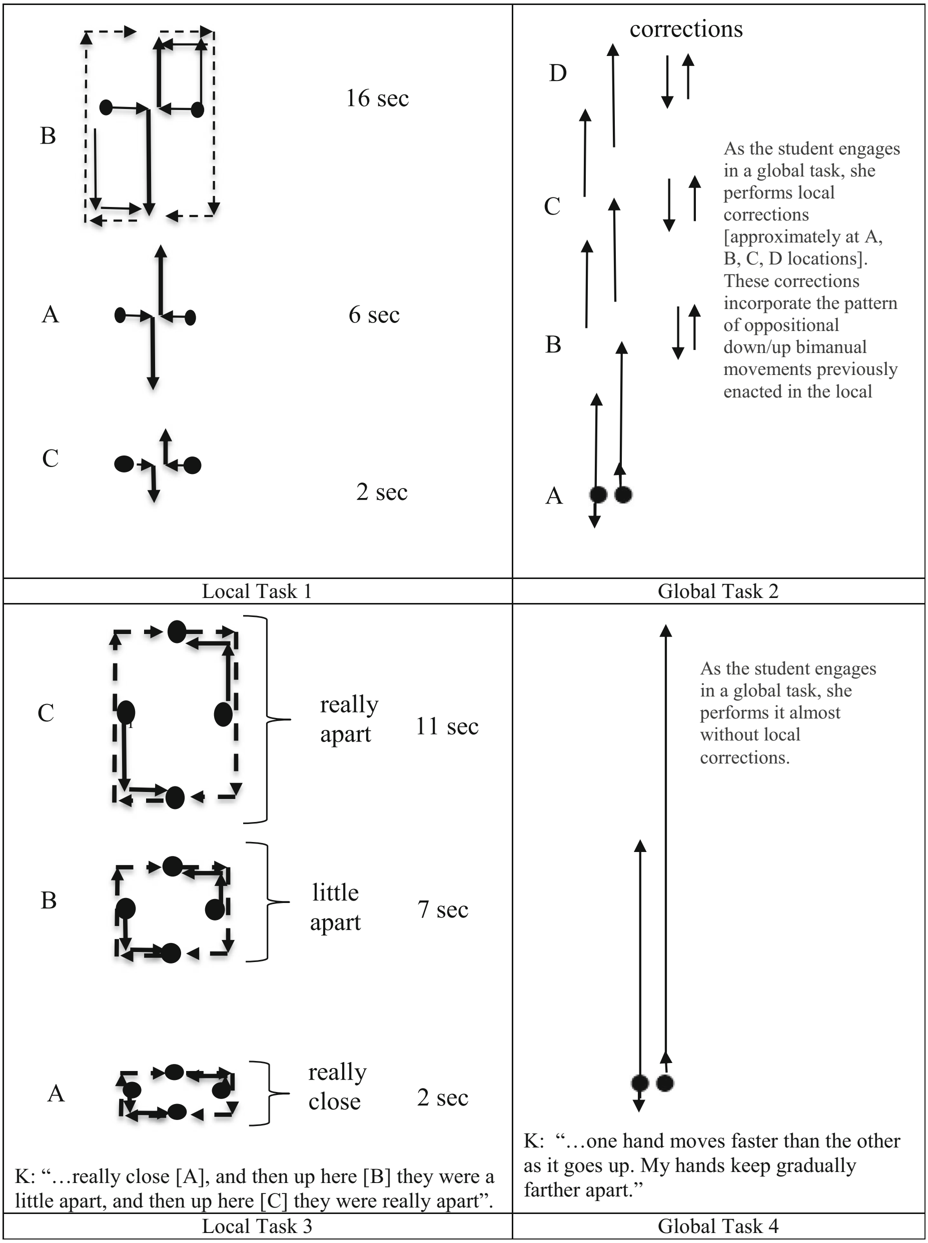

Fig. 3 Construction of two different sensorimotor schemes as solutions to the local and global tasks. 1 and 3 local tasks lead to "the higher—the bigger" scheme. 2 and 4 global tasks lead to "one hand faster than the other" scheme

Local subtask At 05:29, K responds to the question by performing a particular movement pattern at the bottom and top of the screen.[1] At each location, K laterally aligned her left- and

[1] The interviewer's question could have been interpreted as relating to a global enactment of continuous movement, yet K responded to the question by initiating local subtasks.

right-hand index fingers on the screen and then moved them away from each other along the vertical axis (●●; ↓↑) until she got green (Fig. 4, local task 5). At 5:41, K used quantitative language for the first time in the interview: "Maybe they are twice as far apart… or more… actually… four times, I don't know."

Local subtask Asked again to explain what happens at the different screen locations, K once again enacted the same movement pattern. It is as if K has quickly conjured a little empirical experiment comparing the behavior of the vertical interval at three different locations along the screen. Referring to the relative positions of the two balloon icons, as they appeared at the bottom, middle, and top of the screen, K said they were the following:

- Bottom—"… touching each other"
- Middle—"There is about a balloon between them….the length of the balloon"
- Top—"Two balloons [apart], maybe"

K summarized her observation with the following actions and articulations:

> (6:45) K: Ammm, Kind of at the bottom…, there…
> it goes zero balloons between them (bottom ●●; ↓↑),
> in the middle there is one balloon between them (middle ●●; ↓↑), and at the top,… two… balloons… between… them (top ●●; ↓↑).
> So it grows by one at a time.

Thus, K's quantitative construction of her solution enactment has itself now fallen into an articulated pattern, the arithmetic sequence of "0, 1, 2" mapped respectively onto the bottom, middle, and top of the screen. It is of note that enacting the same movement form "●●; ↓↑" lasted approximately 1 s at the bottom, approximately 2.5 s in the middle, and 4 s at the top (Fig. 4, local task 6).

The interviewer then asked K whether she now knew "how to keep the screen green."

Global subtask Without actually touching the screen, K gestured toward the screen a performance of continuous hand movements (↑↑x2). She then placed her hands on the screen and moved the virtual icons in the same manner (with one hand moving twice as fast) (Fig. 4, global task 7).

> (7:15) K: I would say, like, start at the bottom, and put them close together. And then, move one hand up faster… Wait, actually (switches hands to ↑x2↑), [inaudible] one hand up faster and, as I said, in the middle, they are separated like one balloon [inaudible] (makes a correction ↑↓), and at the top (makes a correction ↑↓) two balloons.

Thus, in the current episode, K attempted to incorporate features of the bottom, middle, and top solutions to the local subtasks into a single global enactment, where her earlier local performances become milestone goals for the global continuous movement (see Fig. 4).

Later in the course of the interview (14:00), the iconic manipulation cursors (the two hot-air balloons) were replaced with generic manipulation cursors (two crosshair targets). Working on the global subtask, K said, "One of them has to go faster, to stay green….The same as the last time. Twice as fast, maybe." K persistently articulated her new rules in quantitative register. Also, she did not use the target's height as a unit, but instead used more general terms:

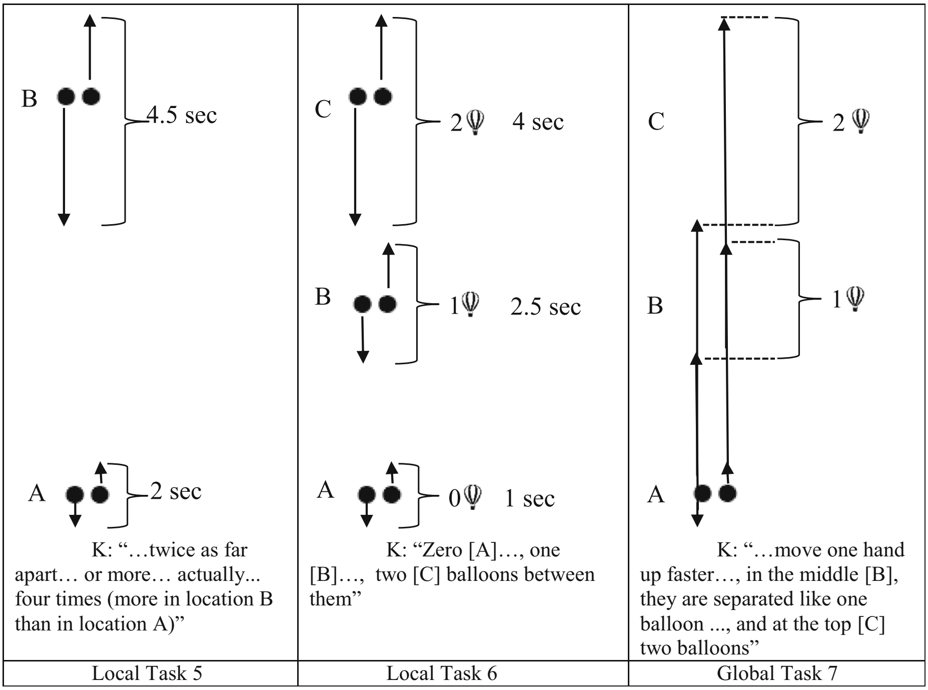

Fig. 4 Quantification of locally effective rhythmic movement forms and their subsequent adaptive recomposition into a globally effective form

(14:50) K(sliding one hand at a time) This one [right cursor] has to go farther up each time to keep green.
IHow much farther does it need to go?
KMmm… (sliding one hand at a time) I think about twice as far each time.

K thus articulated yet another insight, a ratio-like 1:2 rule, by which for every 1 unit you rise on the left, you rise 2 units on the right (see Abrahamson et al., 2014, on this "*a*-per-*b*" enactment rule).

We will now offer summative commentary on the earlier subsections.

4.3 Continuous movement revisited as discreetly discrete: hidden rhythmic qualities of enacting proportion

In tackling the local subtasks K explored the problem space in the following way. First, she constructed a pattern of movements at one location; next, she iterated the pattern at a sequence of locations; and then she refined the pattern. In so doing, K experimented with different combinations of the same movements (recall one of the iterated forms: ●●; ↓↑; →←; ←→; ↑↓; →←) in an apparent attempt to decide whether any elements were disruptive, inefficient, or redundant.

Applying Radford's (2015) framework for analyzing the rhythmic qualities of manual movements, we can characterize the initial result of K's exploration as a rhythmic structure in formation: patterns containing a movement (↓↑) leading to green (themes); iteration of patterns (rhythmic grouping); and constant locations at the bottom, middle, and top (a "three-syllable" meter).

Now applying the same framework to the next subtasks, we witness difference. K is not experimenting with themes anymore. On the one hand, there is the same rhythmic structure consisting of a stable and simple movement pattern: ●●; ↓↑ (a theme); iteration of this pattern (rhythmic grouping); and constant locations at the bottom, middle, and top (three-syllable meter). However, a new quality of enactment will now emerge—arrhythmic prolongation. Namely, at the three locations (bottom, middle, top), the hands must traverse increasingly greater intervals so as to achieve green. In a sense, even as K set her rhythm she recognized a destructive element in it (see Table 2).

Being engaged in reflective discourse with the tutor, K was encouraged to offer greater specificity in describing the variation she had perceived in her own enactment of the movement pattern at the three spatial localities. This specificity was achieved through measuring. K utilized the serendipitously available hot-air balloon icons themselves so as to quantify her qualitative descriptions of moving these icons (cf. Abrahamson & Trninic, 2015). A hot-air balloon icon appears to afford measuring: It has physical height, which is suitable for subtending a vertical interval; moreover, as a familiar worldly artifact, balloons prime and vectorize the student's sensory attention toward the vertical dimension, because that is a balloon's normative, anticipated, prominent trajectory (on the effect of iconic vs. generic manipulatives on students' task framing, see Rosen et al., 2018).

Eventually, K's failed attempt to enact one and the same rhythmic movement pattern at all the three localities was a phenomenological mobilizer of breakdown and insight. As a consequence, K constructed a meta-rhythmic movement form that modulated selected phenomenal elements of locally effective movement patterns into a globally effective structure (same theme, same grouping, same meter, but an arithmetic sequence of prolongations—"0…1…2…So it grows by one at a time"). K's measuring units and quantification enabled her to regain a species of globally effective rhythmic equilibrium that subsumed the locally effective schemes into enactive coherence.

Table 2 Evolution of rhythmic components leads to breakdown: surfacing of different prolongations in local tasks prevents establishment of overall rhythmic structure for global task. Disequilibrium resolved: meta-rhythmic sequential structure with prolongations lasting respectively 0, 1, and 2 temporal-duration "units"

Task	Rhythmic components			
	Theme	Grouping	Meter	Prolongation (spatial/temporal)
1	●●; →←; ↓↑; □□	Tentative grouping	Unspecified	Unclear/unclear
3	●●; ↓↑; →←; ←→; ↑↓; →←	Stable elements: ●●, ↓↑	Bottom, middle, top	Close, little apart, really apart/unclear
5	●●; ↓↑	Stable group	Bottom, top	"Twice as far apart… or more… actually… four times"/2 s at the bottom, 4.5 s at the top
6	●●; ↓↑	Stable group	Bottom, middle, top	Zero…, one…, two balloons between them/1 s at the bottom, approximately 2.5 s in the middle, and 4 s at the top

In the global subtasks, K was oriented by the tutor to move along the vertical axis. She applied corrections (↓↑) to her movement (↑↑) so as to "keep green." The higher her hands went—the bigger corrections she had to make. In her multimodal utterance to the tutor, K demonstrated that she had recognized a relationship between two complementary proto-mathematical notions of proportion: (1) as the hands rise, the distance between them increases and (2) the upper hand is rising faster than the lower hand (see Table 3). Though logically trivial, this connection, we submit, is psychologically and mathematically profound (see also Abrahamson et al., 2014).

The enactive polysemy of one and the same (bimanual) movement, where different potential meanings are phenomenologically disparate yet conceptually complementary and where reconciling these would-be conflicted meanings creates powerful learning opportunity (see also Abrahamson & Wilensky, 2007). We argue that this reconciliation required of K to decompose and relinquish selected components from a locally effective rhythmic scheme, maintaining only those elements that still obtain for the new task. K's rhythmic enactment in the set of local subtasks had distinctive features: an iterated movement ↓↑ and an arithmetic sequence of prolongations—"0…1…2." In the global task, the movement component of ↓↑ faded away as redundant, while the expression of spatial extensions in terms of units and quantities was transferred from local to global enactment (see Table 3).

K had performed the task of making the screen green at several discrete positions on the screen and in so doing appeared to develop a rhythmic movement pattern of placing her hands alongside each other and then moving one hand up and the other down until she had achieved green. But when she transitioned to the next task of keeping the screen green while moving her hands continuously up the screen, this rhythmic movement pattern proved inadequate, so that K realized she was modifying the pattern along two dimensions: (1) she had to replace the downward movement with an upward movement, because she needed to raise both hands, and (2) she had to modify the spatial–temporal prolongation as corresponding to the screen location (the higher you go, the greater the spatial–temporal span). In the course of doing this work, K realized that her upper hand should move faster than the lower hand. She also attended to the spatial spans extending between her hands, and she used ad hoc units to quantify these spans. In turn, K then relied on these quantifications to create goal locations for her hands along the continuous upward movement.

Once evoked, unitization and quantification informed K's movement in the due course of the interview (from the 7th minute and on), as evident from her utterances "…twice as fast…"

Table 3 Components of local rhythmic enactment are adopted differently in global enactment: the ↓↑ movement is implicated as redundant for global enactment, whereas the expression of spatial extensions in terms of units and quantities was transferred from local to global enactment

Task	Global enactment
2	↑↑ with corrections (↓↑); "hands are in the same position, same distance from each other"
4	↑↑x2 with minor (↓↑) corrections; "one hand moves faster than the other as it goes up. My hands keep gradually farther apart."
7	↑↑x2 and ↑x2↑; "move one hand up faster…, in the middle, they are separated like one balloon…, and at the top two balloons."

or "…twice as far each time…" instead of just "faster." With the problem space now parsed by virtual units, K was able to maintain a 1:2 rhythm globally and reflect on this performance. In turn, this performance gave rise to K noticing yet a third proto-mathematical notion of proportion: the *a*-per-*b* expression of the quantitative relation between the left and right hands' respective quota along the parallel progression.

5 Rhythm makes enactment mathematizable

On the basis of our proposed interpretation for these empirical data, we are putting forth an argument for the instrumental role of individuals' rhythmic enactive forms in learning mathematics. These rhythmic enactment forms serve both initially as an efficient way of organizing one's interaction over space, time, materials, and other humans and subsequently as phenomenological mobilizers of change. Rhythmic movement is a tacit enactment goal mobilizing the emergence of mathematical structures in two interrelated ways: first, by creating temporal–spatial movement patterns and sustaining the learner's attention to these patterns as a means of organizing and regulating the enactment of a new comprehensive movement and, second, by alerting the learner's attention to latent irregularities in the enactment that result from encountering in the workspace unfamiliar information structures; these irregularities emerge as rhythmic breakdown, a breakdown that must be resolved through a new meta-pattern of enactment. As such, humans seek rhythmic structures to consolidate and simplify our natural and cultural mundane activities, thus relieving cognitive resources for coping with novelty, but we learn when our default actions fall out of rhythm upon encountering novelty. These breakdowns and their resolutions create pedagogical opportunities (Koschmann, Kuuti, & Hickman, 1998).

In the presented case, a student's gradual refinement of rhythmic structures was interwoven with refinement of her reasoning and served as a means of solving a coordination task that instantiates the calculus of proportion. In the due course of an interview, we observed a self-organizing event structuring cycle. We observed feedback loops, where at first unsystematic and explorative movements were coordinated into proto-rhythmic patterns, and those patterns in turn were iteratively repeated. Students in embodied-learning environments incline toward rhythmic enactment (with stable theme, meter, grouping, and temporal–spatial prolongation, c.f. Radford, 2015). When rhythm is disturbed, cognitive resources are mobilized to restore equilibrium, for instance by means of unitizing. The entire process was gently steered by the instructor, who took measures to orient the student toward useful regions and behaviors of the interactive problem space, in so doing applying on the student socio-epistemic pressure to reflect on selected aspects of her actions.

We note that the empirical context of our study was substantially different from previous studies probing rhythmic qualities of mathematical thinking (e.g., Bautista & Roth, 2012; Sinclair et al., 2016). Paraphrasing Radford (2015), in the current study, mathematical thinking was not only a movement of thought but an outcome of the learner's movement. In the embodied-interaction design that served as our empirical context for this study, perceptual information across the problem space became available to the student only *through* interaction;

the information was not presented a priori or simultaneously, as in traditional algebraic pattern-generating activities that present assemblies of static visual displays. Thus, tasks where information becomes available only through enactment may solicit rhythmic behavior that “carries” a working hypothesis from one location to another. The inherent ephemeral nature of movement tacitly conjures rhythmic lulling as an embodied cognitive orientation, an organizing principle and mnemonic device for gravitating toward some encapsulating action form within continuous media.

We argue that despite their ephemeral nature, students’ actions in the course of an embodied-learning mathematical activity constitute appropriate events for productive proto-mathematical reasoning and learning. In particular, quantification of rhythmic movement forms may play a bridging role between construction and refinement of local rhythmic movement structures and their further decomposition and recomposition into a globally effective rhythmic movement structure. A sequence of formative events in K’s case resonates with the findings of Spencer et al. (2006) on rhythm as a formative factor in constructing an event structure. More generally, forms that emerge through the social enactment of cultural practice mediate the development of intellectual activity (e.g., Newman, Griffin, & Cole, 1989; Radford, 2009). As such, rhythmic enactment can serve in transitioning from naïve to scientific reasoning (Abrahamson, 2004). This observation concurs with and expands findings of Bautista and Roth (2012) on bodily rhythm as a vital dimension of geometrical proficiency.

Students participating in action-based embodied design activities are challenged to perform tasks that require the coordination of continuous motor actions. To achieve such coordination in the form of a task-effective sensorimotor scheme, the students may need to develop auxiliary enactive forms, for instance rhythm, which the designer had preconceived as grounding the target mathematical concepts (Abrahamson, 2014).

We note that some students, like our case study, may be more attentive than others to ephemeral qualities of rhythmic structures. To support more students in these activities, we could look into technological resources and instructional practices for enhancing student production of rhythmic behaviors as well as for surfacing relevant features of these behaviors for their scrutiny and quantification.

Compliance with ethical standards The research program was approved by and strictly complied with the university’s Internal Review Board stipulations.

The research goals were explained to the participants verbally and in printed form. Informed consent was obtained in advance of meetings.

Conflict of interest The authors declare that they have no conflict of interest.

References

Abrahamson, D. (2004). Embodied spatial articulation: A gesture perspective on student negotiation between kinesthetic schemas and epistemic forms in learning mathematics. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 791–797). Toronto, Ontario: Preney.

Abrahamson, D. (2009). Embodied design: Constructing means for constructing meaning. *Educational Studies in Mathematics, 70*(1), 27–47.

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1–16.

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (2nd ed., pp. 358–376). Cambridge: Cambridge University Press.

Abrahamson, D., Lee, R. G., Negrete, A. G., & Gutiérrez, J. F. (2014). Coordinating visualizations of polysemous action: Values added for grounding proportion. *ZDM Mathematics Education, 46*(1), 79–93.

Abrahamson, D., & Trninic, D. (2015). Bringing forth mathematical concepts: Signifying sensorimotor enactment in fields of promoted action. *ZDM Mathematics Education, 47*(2), 295–306.

Abrahamson, D., & Wilensky, U. (2007). Learning axes and bridging tools in a technology-based design for statistics. *International Journal of Computers for Mathematical Learning, 12*(1), 23–55.

Bamberger, J., & diSessa, A. (2003). Music as embodied mathematics: A study of a mutually informing affinity. *International Journal of Computers for Mathematical Learning, 8*(2), 123–160.

Bautista, A., & Roth, W.-M. (2012). The incarnate rhythm of geometrical knowing. *The Journal of Mathematical Behavior, 31*(1), 91–104.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates.

Kelso, J. A., & Engstrom, D. A. (2006). *The complementary nature*. Cambridge, MA: MIT Press.

Kelton, M. L., & Ma, J. Y. (2018). Reconfiguring mathematical settings and activity through multi-party, whole-body collaboration. *Educational Studies in Mathematics, 98*(2), 177–196.

Koschmann, T., Kuuti, K., & Hickman, L. (1998). The concept of breakdown in Heidegger, Leont'ev, and Dewey and its implications for education. *Mind, Culture, and Activity, 5*(1), 25–41.

Leung, A., Baccaglini-Frank, A., & Mariotti, M. A. (2013). Discernment of invariants in dynamic geometry environments. *Educational Studies in Mathematics, 84*(3), 439–460.

Mason, J. (1989). Mathematical abstraction as the result of a delicate shift of attention. *For the Learning of Mathematics, 9*(2), 2–8.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge Falmer.

Mason, J. (2010). Attention and intention in learning about teaching through teaching. In R. Leikin & R. Zazkis (Eds.), *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp. 23–47). Dordrecht: Springer.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. In L. Radford, L. Edwards, & F. Arzarello (Eds.), Gestures and multimodality in the construction of mathematical meaning [special issue]. *Educational Studies in Mathematics, 70*(2), 159-174.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education, 44*(2), 372–415.

Newman, D., Griffin, P., & Cole, M. (1989). *The construction zone: Working for cognitive change in school.* New York, NY: Cambridge University Press.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry* (E.A. Lunzer, Trans.). New York: Basic.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. In L. Radford, L. Edwards, & F. Arzarello (Eds.), Gestures and multimodality in the construction of mathematical meaning [special issue]. *Educational Studies in Mathematics*, *70*(2), 111-126.

Radford, L. (2015). Rhythm as an integral part of mathematical thinking. In M. Bockarova, M. Danesi, D. Martinovic, & R. Núñez (Eds.), *Mind in mathematics: Essays on mathematical cognition and mathematical method* (pp. 68–85). Munich: LINCOM GmbH.

Radford, L., Arzarello, F., Edwards, L., & Sabena, C. (2017). The multimodal material mind: Embodiment in mathematics education. In J. Cai (Ed.), *First compendium for research in mathematics education* (pp. 700–721). Reston, VA: National Council of Teachers of Mathematics.

Reinholz, D., Trninic, D., Howison, M., & Abrahamson, D. (2010). It's not easy being green: Embodied artifacts and the guided emergence of mathematical meaning. In P. Brosnan, D. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 32)* (Vol. VI, Ch. 18: "Technology," pp. 1488–1496). Columbus, OH: PME-NA.

Richmond, L. L., & Zacks, J. M. (2017). Constructing experience: Event models from perception to action. *Trends in Cognitive Sciences, 21*(12), 962–980.

Rosen, D. M., Palatnik, A., & Abrahamson, D. (2018). A better story: An embodied-design argument for generic manipulatives. In N. Calder, N. Sinclair, & K. Larkin (Eds.), *Using mobile technologies in the learning of mathematics* (pp. 189–211). New York: Springer.

Roth, W.-M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: From classical views to a phenomenological reconceptualization. In L. Radford, L. Edwards, & F. Arzarello (Eds.), Gestures and multimodality in the construction of mathematical meaning [special issue]. *Educational Studies in Mathematics, 70*(2), 175-189.

Sinclair, N., Chorney, S., & Rodney, S. (2016). Rhythm in number: Exploring the affective, social and mathematical dimensions of using TouchCounts. *Mathematics Education Research Journal, 28*(1), 31–51.

Simmt, E., & Kieren, T. (2015). Three "moves" in enactivist research: A reflection. *ZDM Mathematics Education, 47*(2), 307–317.

Spencer, R. M. C., Semjen, A., Yang, S., & Ivry, R. B. (2006). An event-based account of coordination stability. *Psychonomic Bulletin & Review, 13*(4), 702–710.

Trninic, D. (Chair), & Saxe, G. (Discussant). (2017, June). *Embodied patterns of knowing: Investigating the role of rhythm in cognition and development*. Symposium presented at the annual meeting of the Jean Piaget Society, San Francisco, June 8–10, 2017.

Vandervert, L. (2016). The prominent role of the cerebellum in the learning, origin and advancement of culture. *Cerebellum & Ataxias, 3*(1), 10. https://doi.org/10.1186/s40673-016-0049-z